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Abstract

The concept of practical fixed-time input-to-state stability for neutral time-delay systems with exogenous perturbations is introduced. Lyapunov-Krasovskii theorems are formulated in explicit and implicit ways. Further, the problem of static nonlinear output-feedback stabilization of a linear system with parametric uncertainties, external bounded state and output disturbances by using artificial delays is considered. The constructive control design consists in solving linear matrix inequalities with only four tuning parameters to be chosen. It is shown both, theoretically and numerically, that the system governed by the proposed controller converges faster to the given invariant set than in the case of using its linear counterpart.

Key words: neutral time-delay systems, implicit Lyapunov-Krasovskii functional, nonlinear control, delay-induced output feedback, practical input-to-state stability, linear matrix inequalities

1 Introduction

Stabilization of dynamical systems with a faster than exponential rate of convergence has become one of the main trends in modern control theory [21,24]. Frequently, such an approach allows systems to be stabilized at the origin in a finite time. For example, for homogeneous autonomous systems, a special class of nonlinear ones, the type of convergence is defined by their degree of homogeneity [1]. For perturbed systems this concept can be extended to non-asymptotic input-to-state stability (ISS) [14] when the steady-state error is upper bounded by the norm of external disturbance. In [2] robustness of homogeneous systems with respect to bounded exogenous perturbations was studied.

However, finite-time stabilization is hard to obtain for time-delay systems [7,22]. For instance, to ensure such a property the delays have to diminish proportionally to the norm of the state vector and vanish at the origin, or time-delay terms have to be multiplied by the instantaneous state vector. But in many applications it is sufficient to stabilize a system in finite time only in the vicinity of the origin, the radius of which depends on the time delay and external perturbations, and following [5] such a problem is investigated in this work. In [5] the homogeneity theory was extended to neutral type systems and it was shown how the convergence can be accelerated by selecting a non-zero degree of homogeneity. Nevertheless, it is worth mentioning that for linear systems any stable set is reachable in a finite time also and the settling time can be reduced by feedback gains increasing. But differently from the delay-free case, this approach has limited use for time-delay systems: for any given delay h sufficiently large gains make the closed-loop system unstable, which motivated [5].

Stability analysis of time-delay systems could be done by using different methods [8,12,13,18]. For example, in [17] Hurwitz stability of transcendental polynomials has been studied. However, such an approach is difficult to use for the synthesis of control systems with delays

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due to its complexity. Another conventional tools are Lyapunov-Krasovskii [20] or Lyapunov-Razumikhin [26] methods. They impose restrictions on the time derivative of an auxiliary functional or function, respectively, with respect to the differential equation of the system. Being well-developed for analysis, both of them do not provide a constructive way for control design in the nonlinear case. On the contrary, their implicit extensions are free of such a drawback: all stability conditions can be checked directly by analyzing some algebraic equations, which implicitly define Lyapunov functionals (functions) [25]. Moreover, control parameters can be obtained by solving a system of linear matrix inequalities (LMIs).

The goal of this work is to extend the exponential ISS concept for neutral time-delay systems to its fixed-time analog. Both, Lyapunov-Krasovskii theorem and its implicit counterpart, are introduced. Then the proposed approach is applied to static nonlinear output-feedback stabilization of a non-delayed linear system in the controllable canonical form with parametric uncertainties, external bounded state and output disturbances. To this end, the unmeasured states are approximated by finite differences [3,9,10], i.e., an artificial delay is induced. In [27] it was shown that in this case closed-loop system has a neutral time-delay representation. Moreover, since no observers/predictors are introduced, the control law is static, which essentially simplifies its practical implementation. Differently from [5], in this paper 1) the homogeneity is not used to prove non-asymptotic rate of convergence, 2) the designed control system is practically fixed-time stable and 3) feedback gains are explicitly calculated. In a conference version of this paper [23] fixed-time stability has not been considered and the influence of parametric uncertainties, external state and output disturbances has not been studied.

The outline of this work is as follows. Notation and auxiliary lemmas are given in Section 2. Practical fixed-time ISS concept of neutral time-delay systems and Lyapunov-Krasovskii theorems are introduced in Section 3. Output stabilization of a linear perturbed system is considered in Section 4. Results of numerical simulations and comparison with a linear analog of the proposed controller are discussed in Section 5. Finally, all the proofs can be found in the Appendices.

2 Preliminaries

2.1 Notation

1) Sets: Denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of natural and real numbers, respectively, $\mathbb{R}_+^\star = \{x : x > 0\}$, $\mathbb{R}_+ = \mathbb{R}_+^\star \cup 0$. A series of natural numbers up to n is defined as $\overline{1,n}$.

2) Spaces: $\mathcal{L}_\infty^m$ is the space of Lebesgue measurable essentially bounded functions $d : [0, +\infty) \to \mathbb{R}^m$ with the norm $\|d\|_\infty := \operatorname{ess\,sup}_{t\in[0,+\infty)} \|d(t)\| < +\infty$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. For $\hbar > 0$ denote the space of Lebesgue square integrable functions $\chi : [-\hbar, 0] \to \mathbb{R}^n$ with the norm $\|\chi\|_2 := \sqrt{\int_{-\hbar}^0 \|\chi(\tau)\|^2 d\tau} < +\infty$ by $L_\hbar^2$. The Banach space $\mathbb{W}_\hbar^1$ of absolutely continuous functions $\chi : [-\hbar, 0] \to \mathbb{R}^n$ has the norm $\|\chi\|_\mathbb{W} := \max_{\tau\in[-\hbar,0]} \|\chi(\tau)\| + \|\dot\chi\|_2$. $\mathbb{W}_\hbar^{1,0} = \{\chi \in \mathbb{W}_\hbar^1 : \chi(0) = 0\}$ is a subspace of $\mathbb{W}_\hbar^1$.

3) Matrices: For symmetric matrices $P \in \mathbb{R}^{n\times n}$ notations $P \succ 0$ ($P \prec 0$) and $P \succcurlyeq 0$ ($P \preccurlyeq 0$) mean that P is positive (negative) definite and semidefinite, respectively. The minimal and maximal eigenvalues of a symmetric matrix are symbolized by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$. Block diagonal matrices are indicated as $\operatorname{diag}\{\lambda_i\}_{j=1}^n$ or $\operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_i \in \mathbb{R}^{n_i\times n_i}$. Identity and zero $n \times n$ matrices are marked as I_n and O_n, respectively. A zero column is denoted by $o_n \in \mathbb{R}^{n\times 1}$.

4) Functions: Denote by C^i a class of i times continuously differentiable functions $\mathbb{R}_+^\star \to \mathbb{R}$.

2.2 Comparison functions

A continuous function $w : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the *class* $\mathcal{K}$ if it is strictly increasing on $\mathbb{R}_+^\star$ and $w(0) = 0$; if additionally it is unbounded then w belongs to $\mathcal{K}_\infty$. A continuous function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *generalized class-*$\mathcal{K}$ *function* ($\mathcal{GK}$ function) if it is strictly increasing on $(s_0, +\infty)$ and $w(s) = 0$ for all $s \in [0, s_0]$ for some $s_0 \in \mathbb{R}_+$. A function $\nu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a *generalized class-*$\mathcal{KL}$ *function* ($\mathcal{GKL}$ function) if for each fixed $t \geq 0$ the function $\nu(\cdot, t)$ is a class-$\mathcal{GK}$ function, and for each fixed $p \geq 0$ the function $\nu(p, \cdot)$ is continuous, strictly decreasing and there exists some $\bar T(p) \in \mathbb{R}_+$ such that $\nu(p, t) \to 0$ as $t \to \bar T$.

Definition 1 ([25]) *A function $q : \mathbb{R}_+^{\star 2} \to \mathbb{R}, (\rho, s) \mapsto q(\rho, s)$ is said to be of the class $\mathcal{IK}_\infty$ if and only if: 1) q is continuous on $\mathbb{R}_+^{\star 2}$; 2) for any $s \in \mathbb{R}_+^\star$ there exists $\rho \in \mathbb{R}_+^\star$ such that $q(\rho, s) = 0$; 3) for any fixed $s \in \mathbb{R}_+^\star$ the function $q(\cdot, s)$ is strictly decreasing on $\mathbb{R}_+^\star$; 4) for any fixed $\rho \in \mathbb{R}_+^\star$ the function $q(\rho, \cdot)$ is strictly increasing on $\mathbb{R}_+^\star$; 5) $\lim\limits_{s\to 0^+} \rho = 0$, $\lim\limits_{\rho\to 0^+} s = 0$ and $\lim\limits_{s\to\infty} \rho = \infty$ for all $(\rho, s) \in \Gamma = \{(\rho, s) \in \mathbb{R}_+^{\star 2} : q(\rho, s) = 0\}$.*

In other words, Definition 1 states that there exists a unique function $\rho \in \mathcal{K}_\infty$ such that $q(\rho(s), s) = 0$ for all $s \in \mathbb{R}_+^\star$.

2.3 Auxiliary lemmas

Lemma 1 (Jensen's inequality [28]) *Let $\phi : [a, b] \to \mathbb{R}$ and $\varpi, \vartheta : [a, b] \to [0, \infty)$ be such that integration concerned is well-defined. Then:*

$$\Big(\int_a^b \vartheta(s)\phi(s)ds\Big)^2 \leq \int_a^b \frac{\vartheta(s)}{\varpi(s)}ds \int_a^b \varpi(s)\vartheta(s)\phi^2(s)ds.$$

Lemma 2 ([21]) *For $\forall s \in [0, \bar s]$, $\beta \in \mathbb{R}_+^\star \setminus \{1\}$ the function $g_\beta(s) := |s^\beta - s|$ admits the following estimate*

$$\max_{s\in[0,\bar s]} g_\beta(s) \leq \bar g(\bar s, \beta) := \max\{g_\beta(\beta^{1/(1-\beta)}), g_\beta(\bar s)\}.$$

3 Input-to-state stability of neutral systems

Consider a functional differential equation of neutral type with external disturbance:

$$\begin{cases} \dot{x}(t) = f(x_t, \dot{x}_t, d(t)), & t > 0, \\ x(\tau) = \Phi(\tau), & \tau \in [-\hbar, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the instantaneous state; $x_t \in \mathbb{W}_\hbar^1$ is the state function defined by $x_t(\tau) := x(t+\tau)$, $\tau \in [-\hbar, 0]$ with $\dot{x}_t \in L_\hbar^2$; $d(t) \in \mathbb{R}^m$ is the external disturbance, $d \in \mathcal{L}_\infty^m$. The continuous operator $f : \mathbb{W}_\hbar^1 \times L_\hbar^2 \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz in the second variable with a constant smaller than one, ensuring forward uniqueness and existence of the system solutions at least locally in time [19]. Assume that the origin is an equilibrium point of the system (1), i.e., $f(0,0,0) = 0$. A solution of the system (1) with the initial function $\Phi \in \mathbb{W}_\hbar^1$ is denoted by $x(t, \Phi, d) \in \mathbb{R}^n$ or $x_t(\Phi, d) \in \mathbb{W}_\hbar^1$.

Following [14], we present the concept of practical fixed-time ISS stability of neutral time-delay systems with external inputs.

Definition 2 *The system* (1) *is called* (γ, κ)*-practically locally fixed-time ISS*[1]*, if there exist a constant* $v \geq 0$ *and functions* $w \in \mathcal{K}$, $\nu \in \mathcal{GKL}$ *with the settling time estimate* $T := \sup_{p<\gamma} \bar{T}(p) < +\infty$ *such that:*

$$\|x(t, \Phi, d)\| \leq \nu(\|\Phi\|_{\mathbb{W}}, t) + v + w(\|d\|_\infty), \ \forall t \geq 0, \tag{2}$$

for all $\Phi \in \mathcal{X} := \{\Phi \in \mathbb{W}_\hbar^1 : \|\Phi\|_{\mathbb{W}} < \gamma\}$ *and* $d \in \mathcal{D} := \{d \in \mathcal{L}_\infty^m : \|d\|_\infty < \kappa\}$.

If $v = 0$*, then system* (1) *is called* (γ, κ)*-locally fixed-time ISS. If additionally* $\gamma = \kappa = +\infty$*, then system* (1) *is called fixed-time ISS.*

The following theorem (see the proof in Appendix A) provides sufficient conditions to check (γ, κ)-practical local fixed-time ISS property (2) by using Lyapunov-Krasovskii functionals. For any $\chi \in \mathbb{W}_\hbar^1$ and $d \in \mathbb{R}^m$ we define the upper-right Dini derivative of functional $V_k : \mathbb{W}_\hbar^1 \to \mathbb{R}_+$, $k = 1, 2$, with respect to equation (1) as follows:

$$D^+ V_k(\chi, d) := \limsup_{\Delta t \to 0^+} \frac{V_k(x_{\Delta t}(\chi, \tilde{d})) - V_k(\chi)}{\Delta t},$$

where $x_{\Delta t}(\chi, \tilde{d})$ is the solution of (1) with initial conditions $\chi \in \mathbb{W}_\hbar^1$ and the input $\tilde{d} = d$ for all $t \in [0, \Delta t)$.

Theorem 1 *Let there exist constants* $\bar{v} \in [0, 1)$, $\bar{\gamma} > 1$, $\mu_1 \in (-1, 0)$, $\mu_2 > 0$, $\theta_k > 0$*, functions* $\rho_{1,k}, \rho_{2,k} \in \mathcal{K}_\infty$, $\bar{w} \in \mathcal{K}$ *and continuous functionals* $V_k : \mathbb{W}_\hbar^1 \to \mathbb{R}_+$, $k = 1, 2$*, such that for all* $\chi \in \mathbb{W}_\hbar^1$ *and* $d \in \mathcal{D}$:

$$\rho_{1,k}(\|\chi(0)\|) \leq V_k(\chi) \leq \rho_{2,k}(\|\chi\|_{\mathbb{W}}); \tag{3a}$$

$$V_1(\chi) \leq 1 \Leftrightarrow V_2(\chi) \leq 1; \tag{3b}$$

$$\begin{aligned} &\max\{\bar{v}, \bar{w}(\|d\|_\infty)\} < V_1 \leq 1 \Rightarrow \\ &\quad D^+ V_1(\chi, d) \leq -\theta_1 V_1^{1+\mu_1}(\chi); \end{aligned} \tag{3c}$$

$$\begin{aligned} &\max\{1, \bar{w}(\|d\|_\infty)\} < V_2 < \bar{\gamma} \Rightarrow \\ &\quad D^+ V_2(\chi, d) \leq -\theta_2 V_2^{1+\mu_2}(\chi). \end{aligned} \tag{3d}$$

Then the system (1) *is* (γ, κ)*-practically locally fixed-time ISS* (2) *with* γ, κ, T, v, $w(s)$ *and* $\nu(p, t)$ *given by*

$$\begin{aligned} &\gamma = \tilde{\rho}_{2,2}(\bar{\gamma}), \quad \kappa = \tilde{w}(\bar{\gamma}), \quad T = \tfrac{1}{\mu_2 \theta_2} + \tfrac{1}{|\mu_1|\theta_1}, \\ &v = \tilde{\rho}_{1,1}(\bar{v}), \quad w(s) = \begin{cases} \tilde{\rho}_{1,1}(\bar{w}(s)), & \textit{if } \bar{w}(s) < 1, \\ \tilde{\rho}_{1,2}(\bar{w}(s)), & \textit{if } \bar{w}(s) \geq 1, \end{cases} \\ &\nu(p, t) = \begin{cases} \nu_2(p, t), & t \in [0, T_2(p)), \\ \nu_1(p, t), & t \in [T_2(p), T_2(p) + T_1(p)), \\ 0, & t \geq T_2(p) + T_1(p), \end{cases} \end{aligned} \tag{4}$$

where functions $\tilde{w}$, $\tilde{\rho}_{1,1}$, $\tilde{\rho}_{1,2}$ *and* $\tilde{\rho}_{2,2}$ *are inverse of* $\bar{w}$, $\rho_{1,1}$, $\rho_{1,2}$ *and* $\rho_{2,2}$*, respectively, and*

$$\begin{aligned} \nu_1(p, t) &= \tilde{\rho}_{1,1}((\mu_1 \theta_1 (t - T_2(p) - T_1(p)))^{-1/\mu_1}), \\ \nu_2(p, t) &= \tilde{\rho}_{1,2}((\mu_2 \theta_2 (t - T_2(p)) + 1)^{-1/\mu_2}), \\ T_1(p) &= \max\{0, (\min\{1, \rho_{2,1}^{-\mu_1}(p)\})/(-\mu_1 \theta_1)\}, \\ T_2(p) &= \max\{0, (1 - \rho_{2,2}^{-\mu_2}(p))/(\mu_2 \theta_2)\}. \end{aligned}$$

One can see that conditions (3c) and (3d) in general are hard to check, especially in a control design scenario. As it has been shown in [25], this problem can be overcome by defining functionals V_k in Theorem 1 implicitly. To this end, we first need to introduce Fréchet derivatives.

Definition 3 *An operator* $\mathcal{F} : \mathbb{U} \to \mathbb{V}$ *is called Fréchet differentiable at* $\chi \in \mathbb{U}$ *if there exists a bounded linear operator* $D\mathcal{F}_\chi : \mathbb{U} \to \mathbb{V}$ *such that:*

$$\lim_{\Delta\chi \to 0} \frac{\|\mathcal{F}(\chi + \Delta\chi) - \mathcal{F}(\chi) - D\mathcal{F}_\chi(\Delta\chi)\|_{\mathbb{V}}}{\|\Delta\chi\|_{\mathbb{U}}} = 0,$$

where $\|\cdot\|_{\mathbb{U}}$ *and* $\|\cdot\|_{\mathbb{V}}$ *are norms in the Banach spaces* $\mathbb{U}$ *and* $\mathbb{V}$*, respectively.*

Denote by $Q'_{V,k}(V_k, \chi)$ and $Q'_{t,k}(V_k, \chi, d)$ derivatives of functions $V_k \mapsto Q_k(V_k, \chi)$ and $t \mapsto Q_k(V_k, x_t(\chi, d))$, where $x(t)$ satisfies (1) with initial conditions $\Phi = \chi$, respectively.

Theorem 2 *Let there exist constants* $\bar{v} \in [0, 1)$, $\bar{\gamma} > 1$, $\mu_1 \in (-1, 0)$, $\mu_2 > 0$, $\theta_k > 0$*, functions* $q_{1,k}, q_{2,k} \in \mathcal{IK}_\infty$, $\bar{w} \in \mathcal{K}$ *and continuous functionals* $Q_k : \mathbb{R}_+^\star \times \mathbb{W}_\hbar^1 \to \mathbb{R}$, $k = 1, 2$ *such that:*

C1) $Q_k(V_k, \chi)$ *are continuously Fréchet differentiable for all* $V_k \in \mathbb{R}_+^\star$ *and* $\chi \in \mathbb{W}_\hbar^1$;

C2) for any $\chi \in \mathbb{W}_\hbar^1$ *there exist* $V_k \in \mathbb{R}_+^\star$ *such that* $Q_k(V_k, \chi) = 0$;

C3) $Q'_{V,k}(V_k, \chi) < 0$ *for all* $(V_k, \chi) \in \Omega_k = \{(V_k, \chi) \in \mathbb{R}_+^\star \times \mathbb{W}_\hbar^1 : Q_k(V_k, \chi) = 0\}$;

C4) for all $(V_k, \chi) \in \Omega_k$ *and* $d \in \mathcal{D}$:

$$\begin{aligned} q_{1,k}(V_k, \|\chi(0)\|) &\leq Q_k(V_k, \chi), \ \forall \chi \in \mathbb{W}_\hbar^1 \setminus \mathbb{W}_\hbar^{1,0}, \\ Q_k(V_k, \chi) &\leq q_{2,k}(V_k, \|\chi\|_{\mathbb{W}}), \ \forall \chi \in \mathbb{W}_\hbar^1 \setminus \{0\}; \end{aligned} \tag{5a}$$

$$Q_1(1, \chi) = Q_2(1, \chi); \tag{5b}$$

[1] Hereinafter, ISS also stands for "*input-to-state stable*".

$$\max\{\bar{v}, \bar{w}(\|d\|_\infty)\} < V_1 \le 1 \Rightarrow \\ Q'_{t,1}(V_1,\chi,d) \le \theta_1 V_1^{1+\mu_1} Q'_{V,1}(V_1,\chi); \tag{5c}$$

$$\max\{1, \bar{w}(\|d\|_\infty)\} < V_2 < \bar{\gamma} \Rightarrow \\ Q'_{t,2}(V_2,\chi,d) \le \theta_2 V_2^{1+\mu_2} Q'_{V,2}(V_2,\chi). \tag{5d}$$

Then the system (1) *is* (γ,κ)*-practically locally fixed-time ISS* (2) *with* γ, κ, T, v, $w(s)$ *and* $\nu(p,t)$ *given by* (4)*, where functions* $\rho_{i,k}(s)$ *implicitly defined by* $q_{i,k}(\rho_{i,k}(s),s)=0$*, respectively,* $i,k=1,2$.

The proof of Theorem 2 can be found in Appendix B.

Despite the seeming complexity of conditions (5c) and (5d), in the next section we will show how Theorem 2 can be successfully applied to design a control law.

4 Nonlinear delay-induced control

4.1 Problem statement

Consider a system in the controllable canonical form with a relative degree $n \ge 2$, matched parametric uncertainties, state disturbances and output perturbations:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\big(u(t) + d_1(t) + ax(t)\big), \\ y(t) = Cx(t) + d_2(t), \end{cases} \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}$ is the control input; $y(t) \in \mathbb{R}$ is the output available for measurements; $d_1(t) \in \mathbb{R}$ and $d_2(t) \in \mathbb{R}$ are the external state and output disturbances, respectively, $d = [d_1, d_2]^\top \in \mathcal{D} := \{d \in \mathcal{L}^2_\infty : \|d\|_\infty < \kappa\}$; $a \in \mathbb{R}^{1\times n}$ is the vector of unknown coefficients such that $aa^\top \le \epsilon$;

$$A = \begin{bmatrix} o_{n-1} & I_{n-1} \\ 0 & o_{n-1}^\top \end{bmatrix}, \quad B = \begin{bmatrix} o_{n-1} \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & o_{n-1}^\top \end{bmatrix}.$$

Note that all linear single-input single-output controllable systems with a relative degree n can be rewritten in the canonical form (6) by applying a linear coordinate transformation. Moreover, for many nonlinear systems, such as a pendulum ($n=2$), a magnetic suspension system ($n=3$) or a single link manipulator with flexible joints and negligible damping ($n=4$), there is a change of variables that transforms them into the form (6) [16].

The goal is to design a static output-feedback control practically stabilizing the system (6) with the rate of convergence faster than exponential.

4.2 Control design

Inspired by [21], we will define a nonlinear control law in the following form:

$$u(\tilde{y}) = \textstyle\sum_{j=1}^n K_j \lceil \tilde{y}_j \rfloor^{\alpha_j(\|\tilde{y}\|)}, \tag{7a}$$

$$\alpha_j(\|\tilde{y}\|) = \begin{cases} \frac{1}{r_{2,j}}, & \text{if } \|\tilde{y}\| \ge b_2, \\ \frac{1}{r_{1,j}}, & \text{if } \|\tilde{y}\| \le b_1, \\ \frac{r_{1,j}-r_{2,j}}{r_{1,j}r_{2,j}} \frac{\|\tilde{y}\|-b_1}{b_2-b_1} + \frac{1}{r_{1,j}}, & \text{otherwise}, \end{cases} \tag{7b}$$

$$r_{k,j}(\mu_k) = 1 - (n+1-j)\mu_k, \quad k = 1,2, \tag{7c}$$

where $\tilde{y} \in \mathbb{R}^n$ with $\tilde{y}_1(t) := y(t)$, $\tilde{y}_{i+1}(t)$ is the approximation of the i-th output derivative $y^{(i)}(t)$, $i=\overline{1,n-1}$, $\mu_1 = -\mu$ and $\mu_2 = \mu$ are degrees of nonlinearity with $\mu \in (0, 1/n)$, $b_1 > 1$ and $b_2 > b_1$ are switch thresholds, $K_j < 0$, $j = \overline{1,n}$ are feedback gains, $K := [K_1, \ldots, K_n]$, $\lceil\cdot\rfloor^\alpha := \mathrm{sign}(\cdot)|\cdot|^\alpha$ is the signed power.

Instead of introducing a state observer, in this paper, we approximate the output derivatives by finite differences $\tilde{y}_{i+1}(t) \approx y^{(i)}(t)$, $i = \overline{1,n-1}$:

$$\begin{aligned} \tilde{y}_{i+1}(t) &:= \tfrac{\tilde{y}_i(t) - \tilde{y}_i(t-h)}{h} \\ &= \tfrac{1}{h^i} \textstyle\sum_{s=0}^i (-1)^s \tfrac{i!}{s!(i-s)!} y(t-sh), \end{aligned} \tag{8}$$

where $h > 0$ is a time delay. Since the value of $y(t-sh)$ is undefined for $t \in [0, sh)$, then we set it equal to $y(0)$.

Selection of approximation (8) follows from the well-known fact: if $h \to 0$ then $\tilde{y}_{i+1}(t) \to y^{(i)}(t)$. It is worth noting that the proposed scheme is similar to a high-gain observer [16], since only for sufficiently small delays $h > 0$ derivative estimates $\tilde{y}_{i+1}(t)$ can be used in stabilizing feedback [9,10]. But differently from the conventional observer-based control, approximation (8) is fully static and, therefore, easy to implement. Nevertheless, to apply Theorem 2, first we have to present $\tilde{y}_{i+1}(t)$ in a different form.

Proposition 1 ([27]) *If* $y \in C^i$ *and* $y^{(i)}$ *is absolutely continuous,* $i \in \mathbb{N}$*, then* $\tilde{y}_{i+1}(t)$ *defined in* (8) *satisfies:*

$$\tilde{y}_{i+1}(t) = y^{(i)}(t) - \textstyle\int_{t-ih}^t \varphi_i\big(\tfrac{t-s}{h}\big) y^{(i+1)}(s)ds, \tag{9}$$

where $\varphi_1(\xi) := 1 - \xi$ *and for* $i \in \mathbb{N} \setminus \{1\}$*:*

$$\varphi_i(\xi) := \begin{cases} \int_0^\xi \varphi_{i-1}(\lambda)d\lambda + 1 - \xi, & \xi \in [0,1], \\ \int_{\xi-1}^\xi \varphi_{i-1}(\lambda)d\lambda, & \xi \in (1, i-1), \\ \int_{\xi-1}^{i-1} \varphi_{i-1}(\lambda)d\lambda, & \xi \in [i-1, i]. \end{cases} \tag{10}$$

Since $x_1 \in C^n$, $x_1^{(n)}$ is absolutely continuous and approximation (8) is linear, it follows from (9) that $\tilde{y}_{i+1}(t) = x_{i+1}(t) + \delta_i(t) + \tilde{d}_{2,i+1}(t)$, $i = \overline{1,n-1}$, where

$$\delta_i(t) := -\textstyle\int_{t-ih}^t \varphi_i\big(\tfrac{t-s}{h}\big)\dot{x}_{i+1}(s)ds, \tag{11a}$$

$$\tilde{d}_{2,i+1}(t) := \tfrac{\tilde{d}_{2,i}(t) - \tilde{d}_{2,i}(t-h)}{h} \tag{11b}$$

with $\tilde{d}_{2,1}(t) := d_2(t)$. Therefore, the closed-loop system (6), (7) is in the form (1) and Theorem 2 can be applied. To this end, introduce two implicit Lyapunov-Krasovskii functionals (ILKFs) $Q_k(V_k,\chi)$, $k=1,2$, by the equality:

$$\begin{aligned} Q_k(V_k,\chi) &:= -1 + \chi^\top(0)\Lambda_{V_k}^{-r_k} P \Lambda_{V_k}^{-r_k}\chi(0) \\ &+ \textstyle\sum_{i=1}^{n-1} \tfrac{i}{2S_i} V_k^{-2r_{k,i+2}+\mu_k} \int_{-ih}^0 \psi_i\big(\tfrac{-\tau}{h}\big)\dot{\chi}_{i+1}^2(\tau)d\tau, \end{aligned} \tag{12}$$

where $P = P^\top \succ 0$, $\Lambda_{V_k}^{-r_k} := \mathrm{diag}\{V_k^{-r_{k,j}}\}_{j=1}^n$, $S_i > 0$, $i = \overline{1,n-1}$. Note that in a linear case ($\mu_k = 0$), equation $Q_k(V_k,\chi) = 0$ defines a Lyapunov-Krasovskii functional $V_k(\chi) = \sqrt{\chi^\top(0)P\chi(0) + \sum_{i=1}^{n-1}\frac{i}{2S_i}\int_{-ih}^0 \psi_i(\frac{-\tau}{h})\dot{\chi}_{i+1}^2(\tau)d\tau}$.

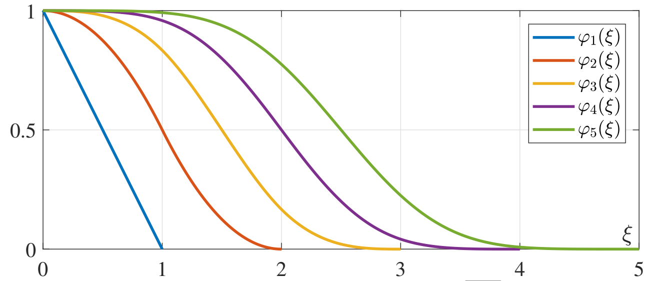


Fig. 1. Plots of $\varphi_i(\xi)$ for $i = \overline{1,5}$

For the following Lyapunov-Krasovskii analysis we will utilize some characteristics of the functions $\varphi_i(\xi)$ (see Fig. 1) and their integrals $\psi_i(\xi) := \int_\xi^i \varphi_i(\lambda)d\lambda$ which are summarized below (see the proof in Appendix C).

Proposition 2 *The functions $\varphi_i(\xi)$ defined in (10) and their integrals $\psi_i(\xi)$ possess the following properties:*

P1) $\varphi_i'(\xi) < 0$ *on* $\xi \in (0, i)$*;*
P2) $0 \le \varphi_i(\xi) \le 1$ *for all* $\xi \in [0, i]$*;*
P3) $\varphi_i(\xi) + \varphi_i(i - \xi) = 1$ *for all* $\xi \in [0, i]$*;*
P4) $\varphi_i''(\xi) < 0$ *on* $\xi \in (0, i/2)$ *and* $\varphi_i''(\xi) > 0$ *on* $\xi \in (i/2, i)$ *for* $i \ge 2$*;*
P5) $\psi_i(0) = i/2$ *and* $\psi_i(i) = 0$*;*
P6) $\psi_i(\xi) \le (i/2)\varphi_i(\xi)$ *for all* $\xi \in [0, i]$*;*
P7) for all $i \in \mathbb{N}$ *the following integral is well-defined:*

$$\zeta_i := \int_0^i \psi_i^{-1}(\xi)\varphi_i^2(\xi)d\xi. \tag{13}$$

Remark 1 *It is worth mentioning that parameters ζ_i are independent of time delay $h > 0$ and, thus, can be calculated in advance. For example, direct computation of ζ_1 gives a quite simple result: $\zeta_1 = 2$. The other values of ζ_i can be found by numerical integration (see Table 1).*

Now we are ready to present the restrictions on constructive selection of adjustable parameters μ, h, b_1 and b_2 such that Theorem 2 holds for ILKFs (12) with respect to the system (6), (7) (see the proof in Appendix D).

Theorem 3 *Given $\epsilon > 0$, let there exist $\mu \in (0, 1/n)$, $h > 0$, $b_1 > 1$, $b_2 > b_1$ such that the system of LMIs:*

$$0 \prec XH_{r_k} + H_{r_k}X \preccurlyeq 2\omega_k X, \tag{14a}$$

$$\max\{\|\sigma\|, b_0\}I_n \preccurlyeq X \preccurlyeq I_n/2, \tag{14b}$$

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & Y^\top \\ * & \Xi_{22} & \Xi_{12}^\top B \\ * & * & -\frac{4S_{n-1}}{(n-1)^2} \end{bmatrix} \preccurlyeq 0, \quad \begin{bmatrix} Z & X \\ * & M \end{bmatrix} \succcurlyeq 0, \quad \begin{bmatrix} N & N\varrho \\ * & X\varrho \end{bmatrix} \succcurlyeq 0, \tag{14c}$$

where $H_{r_k} := \operatorname{diag}\{r_{k,j}\}_{j=1}^n$, $\omega_1 := 1 + (n+1)\mu$, $\omega_2 := 1$, $\varrho := h^{1-\sqrt{\mu}}$, $b_0 := b_1^2 - 1$, $\|\sigma\| := \max_{k=1,2}\|\sigma_k\|$, $\sigma_{1,j} := \bar{g}(b_1, 1/r_{1,j})$, $\sigma_{2,j} := \bar{g}(b_1, 1/r_{2,j}) + \bar{g}(b_2^{1/r_{2,j}}, r_{2,j}/r_{1,j})$,

$$\Xi_{11} := XA^\top + Y^\top B^\top + AX + BY + Z + \tfrac{2}{n-1}X,$$

$$\Xi_{12} := \left[BY\sqrt{\varrho},\ BY\sqrt{\|\sigma\|},\ BY\sqrt{b_0},\ B,\ B\sqrt{\epsilon}\right],$$

$$\Xi_{22} := -\tfrac{1}{n-1}\operatorname{diag}\left\{(1-\varrho)X,\ \tfrac{1}{2}X,\ \tfrac{1}{2}X,\ \tfrac{c_1}{4},\ \tfrac{1}{2}\right\},$$

$$M := \operatorname{diag}\left\{c_2, c_3, \tfrac{4S_i}{i^2}\right\}_{i=1}^{n-2},\ N := \tfrac{1}{n-1}\operatorname{diag}\left\{c_4, S_i\right\}_{i=1}^{n-1},$$

Table 1. Values of ζ_i for $i = \overline{1,5}$.

i	1	2	3	4	5
ζ_i	2	2.1577	2.3282	2.4614	2.5680

is feasible for some $c_l > 0$, $0 < S_i \le \frac{i}{4\zeta_i \varrho}$, $X = X^\top \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{1\times n}$, $Z = Z^\top \in \mathbb{R}^{n\times n}$, *with ζ_i and $r_{k,j}$ defined by* (13) *and* (7c)*, respectively,* $i = \overline{1, n-1}$, $j = \overline{1, n}$, $k = 1, 2$, $l = \overline{1, 4}$.

Then the closed-loop system (6), (7) *with* $K = YX^{-1}$ *is* (γ, κ)*-practically locally fixed-time ISS* (2) *with* γ, κ, T, v *and* $w(s)$ *given by*

$$\gamma = \frac{h^{-r_{2,1}/\sqrt{\mu}}}{\sqrt{\max\{\|\sigma\|^{-1},\ h^{\sqrt{\mu}}S_0\}}}, \quad \kappa = \frac{h^{-r_{2,1}/\sqrt{\mu}}}{\eta},$$

$$T = \frac{1}{4(n-1)\mu}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right), \quad v = \frac{h^{r_{1,n}/\sqrt{\mu}}}{\sqrt{2}},$$

$$w(s) = \frac{1}{\sqrt{2}}\begin{cases} (\eta s)^{r_{1,n}/r_{1,1}}, & \text{if } \eta s < 1, \\ (\eta s)^{r_{2,n}/r_{2,1}}, & \text{if } \eta s > 1, \end{cases}$$

where $S_0 := \max\limits_{i=\overline{1,n-1}} \frac{i^2}{4S_i}$, $\eta := \sqrt{\max\{c_1, \frac{(2/h)^{2n}-1}{(2/h)^2-1}/b_0^2\}}$.

Let us give some comments on the choice of tuning parameters. Firstly, LMIs (14) are always feasible provided ϵ, μ, h, b_1 and b_2 are sufficiently small. Obviously, this is true for $\epsilon = \mu = h = 0$ and $b_1 = b_2 = 1$. Indeed, taking into account that in this case $\|\sigma\| = \varrho = b_0 = 0$ and $r_{k,j} = \omega_k = 1$, one can see that LMIs (14) hold for some $0 \prec X \preccurlyeq I_n/2$, Y, $Z \succcurlyeq 0$ and sufficiently large c_l, S_i. Clearly, LMIs (14) remain feasible for some positive nonzero ϵ, μ, h and $1 < b_1 < b_2$ since $r_{k,j}$, ω_k, $\sigma_{k,j}$, ρ and b_0 are continuous functions of μ, h, b_1 and b_2.

Secondly, it follows from Theorem 3 that the settling time T is inversely proportional to parameter μ. Thus, the best strategy of parameter tuning consists in maximizing μ, for which LMIs (14) are feasible for given ϵ. On the other hand, note that $\gamma = \gamma(h)$ and $v = v(h)$ are the functions of the time delay h for the fixed nonlinear degree μ. Obviously, γ (v) can be enlarged (decreased) by reducing time delay h and in the limit case: $\gamma \to +\infty$ ($v \to 0^+$) as $h \to 0^+$. However, in practice, time delay h cannot be chosen arbitrarily small due to related implementation problems.

Remark 2 *Note that, similar to high-gain observers, approximation* (8) *is sensitive to high-frequency output perturbations [15]. In order to show this, let us assume that $d_2(t)$ is a Lipschitz continuous function of time, i.e., there exists a positive constant L such that* $|d_2(t_1) - d_2(t_2)| \le L\|d_2\|_\infty |t_1 - t_2|$ *for all* $t_1, t_2 \in \mathbb{R}$*. Taking into account* (11b)*, it can be shown that in this case* $\eta = \sqrt{\max\{c_1, (1 + L^2\frac{(2/h)^{2(n-1)}-1}{(2/h)^2-1})/b_0^2\}}$*, which coincides with the one given in Theorem 3 if $hL = 2$. Thus, the slower the output disturbance d_2 changes (the smaller L), the smaller the steady-state error. Nevertheless, the problem of making approximation* (8) *more robust to high-frequency output perturbations (e.g., by introducing low-pass filters [11,15]) is out of the scope of this work.*

Let us show what is the main advantage of the proposed control law (7) compared to its linear analog ($\mu = 0$) with the same gains K.

Proposition 3 *Let the conditions of Theorem 3 be fulfilled. Then there are* $h_0 \in (0, h]$ *and* $\gamma_0 \in (0, \gamma(h_0)]$ *such that for all* $\Phi \in \mathcal{X}_0 := \{\mathcal{X} : \|\Phi\|_{\mathbb{W}} \geq \gamma_0\}$ *and* $d \in \mathcal{D}$ *the system* (6), (7) *with time delay* h_0 *converges faster to the set* $\mathcal{A} := \{x \in \mathbb{R}^n : \|x(t, \Phi, d)\| \leq v + w(\|d\|_\infty)\}$ *than its linear counterpart* ($\mu = 0$)*.*

The proof of Proposition 3 is given in Appendix E.

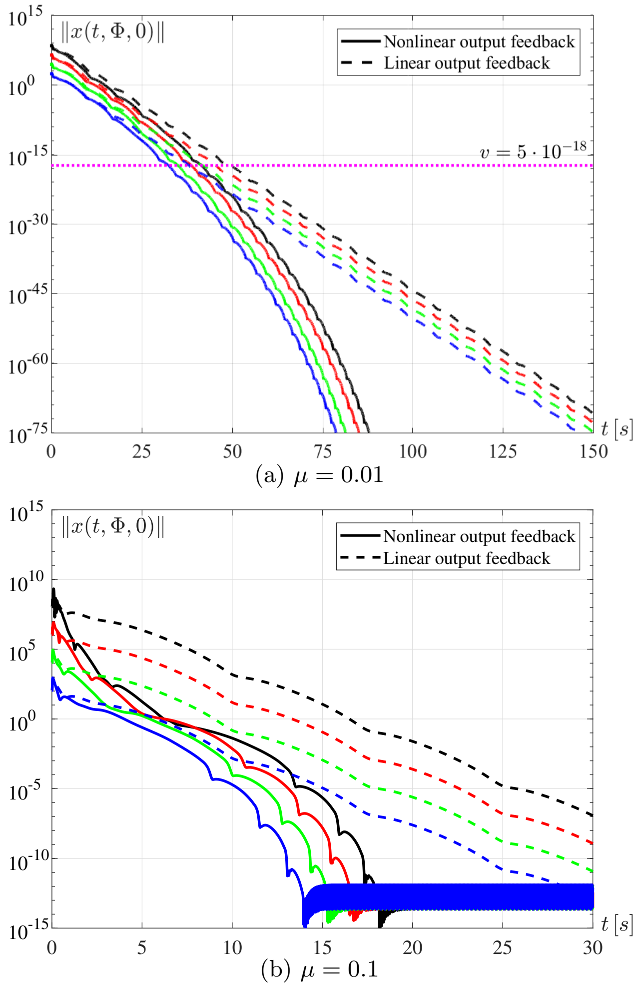
In other words, for sufficiently large initial conditions or sufficiently small perturbations the proposed control system always converges faster to the vicinity of the origin than its linear analog.

5 Numerical simulations

Let $n = 3$ and $\epsilon = 0.05$. Then LMIs (14) are feasible for $\mu = 0.01$, $h = 0.02$, $b_1 = 1.001$ and $b_2 = 1.1$. Therefore, $K = [-3.11, -5.95, -4.14]$, $\gamma = 1.25 \cdot 10^{15}$, $v = 5 \cdot 10^{-18}$, $\eta = 5 \cdot 10^6$ and $\kappa = 6 \cdot 10^9$. For further comparison we set $a = [1, 1, 1] \cdot 0.125$ such that $aa^\top = 0.047 < \epsilon$. The numerical simulation of the closed-loop system (1), (7) has been done in MATLAB Simulink by using the explicit Euler method with a state-dependent step [6]. The basic and minimum discretization steps, the maximum number of iterations and the homogeneous norm have been defined as $\Delta t_0 = 10^{-2}$, $\Delta t_{\min} = 10^{-4}$, $N_{\max} = 2 \cdot 10^4$ and $\|x\|_{hom} := (\sum_{j=1}^{n} |x_j|^{\alpha_j/\alpha_1})^{\alpha_1}$, respectively.

First, we will show that the proposed control scheme (6), (7) is indeed (γ, κ)-practically fixed-time stable. To this end, we will compare it with its linear analog ($\mu = 0$) when $\|d\|_\infty = 0$. Choosing initial conditions as $\Phi(\tau) = [0, 1, -0.5] \cdot 10^{8-2i}$, $i = \overline{0, 3}$ for all $\tau \in [-0.04, 0]$, we guarantee that $\|\Phi\|_{\mathbb{W}} < \gamma$. The norm of the solutions $x(t, \Phi, 0)$ is depicted in Figure 2a in the logarithmic scale, where solid lines corresponds to the proposed control law (7) and dashed ones represent its linear counterpart ($\mu = 0$). The dotted magenta line defines the radius of the set $\mathcal{A}$. The results illustrate Proposition 3: the solutions of the nonlinear system (6), (7) converge faster to the set $\mathcal{A}$ than its linear analog. However, the superiority of the proposed control over its linear counterpart is not so evident due to the smallness of μ. Recall that this parameter should be chosen as large as possible to ensure the feasibility of LMIs (14). Since our Lyapunov analysis is rather conservative, one might expect that the closed-loop system (6), (7) remain fixed-time stable even for larger μ. To demonstrate this, we chose $\mu = 0.1$ and kept other control parameters the same. The results of this numerical comparison are depicted in Figure 2b. Clearly, the proposed control significantly does outperform the linear one.

Now we compare performance of the proposed control system (6), (7) with its linear counterpart in the presence of the state disturbance $d_1(t) = \cos(t)$ and the output perturbation $d_2(t) = 0.1 \sin(10t)$. As a result,



(a) $\mu = 0.01$

(b) $\mu = 0.1$

Fig. 2. The norm of the solutions $x(t, \Phi, 0)$ (disturbance-free case)

$w(\|d\|_\infty) = (\eta\sqrt{1.01})^{r_{2,n}/r_{2,1}} = 7 \cdot 10^6$. The norm of the solutions $x(t, \Phi, d)$ is depicted in Figure 3a in the logarithmic scale, where the initial conditions Φ are chosen the same as for the disturbance-free case. Again the obtained results go with Proposition 3. As well as in the disturbance-free case, for larger values of μ the difference between nonlinear and linear approaches becomes clearer (see Fig. 3b).

6 Conclusion

The paper introduces the concept of practical fixed-time input-to-state stability for neutral time-delay systems with exogenous perturbations. Related Lyapunov-Krasovskii theorems have been formulated explicitly and implicitly. The latter has been applied to solve the problem of static output-feedback delay-induced stabilization of a linear system in the controllable canonical form with parametric uncertainties, bounded state and output disturbances. The control design consists in solving linear matrix inequalities with only four tuning parameters to be chosen. It has been shown that for sufficiently

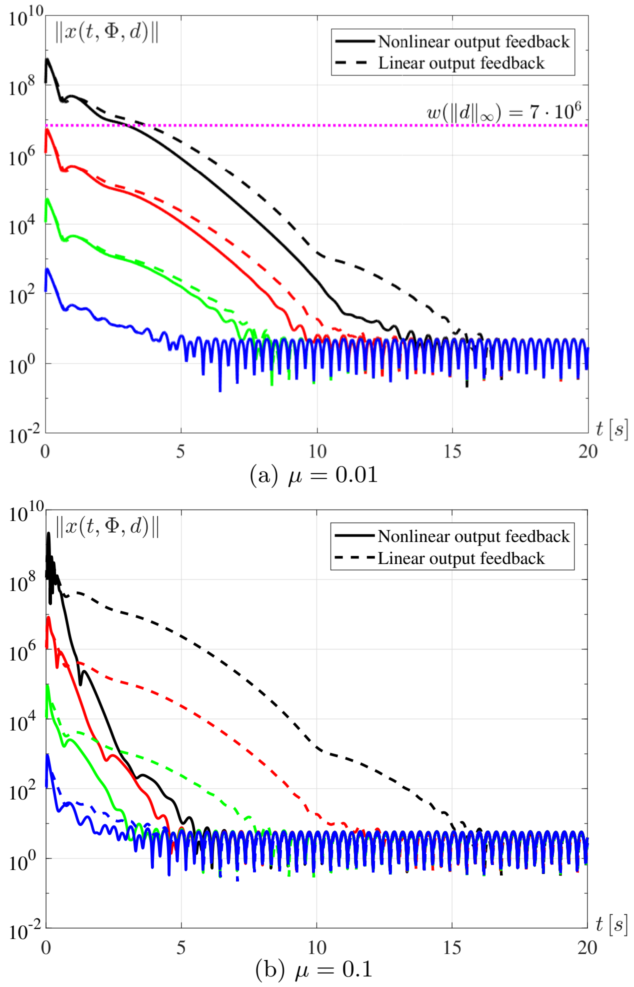


Fig. 3. The norm of the solutions $x(t, \Phi, d)$ (disturbed case)

large initial conditions or sufficiently small perturbations the proposed control scheme converges to the stable set faster than its linear counterpart. The numerical simulation has verified the theoretical results. One of the directions for future research may be the search for new, less conservative LMI constraints on nonlinear degree μ and time delay h.

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A Proof of Theorem 1

Let $x_t = \chi$, satisfying (1). If $\bar{w}(\|d\|_\infty) < 1$, then applying the Comparison Lemma (Lemma 3 in [22]) to the function $\bar{V}_2(t) := V_2(x_t)$ from (3d) on interval $t \in [0, T_2)$, where $T_2 = \inf\{t \geq 0 : \bar{V}_2(t) \leq 1\}$, we get $\bar{V}_2(t) \leq (\mu_2\theta_2 t + \bar{V}_2^{-\mu_2}(0))^{-1/\mu_2}$. Obviously, $T_2 \leq (1 - \bar{V}_2^{-\mu_2}(0))/(\mu_2\theta_2)$. Hence, if $\rho_{2,2}(\|\Phi\|_{\mathbb{W}}) \leq 1$, then (3a) implies $\bar{V}_2(t) \leq 1$ and $T_2 = 0$. Otherwise, $\|x(t)\| \leq \tilde{\rho}_{1,2}(\bar{V}_2(t))$ for $t \in [0, T_2)$ and $\bar{V}_2(0) \leq \rho_{2,2}(\|\Phi\|_{\mathbb{W}})$ due to (3a). On the other hand, if $\bar{w}(\|d\|_\infty) \geq 1$, then there exists a moment of time $T_2' \in [0, T_2)$ such that $\bar{V}_2(t) \leq \bar{w}(\|d\|_\infty)$ for $t \geq T_2'$. Thus, one can conclude that $\|x(t)\| \leq \nu_2(\|\Phi\|_{\mathbb{W}}, t) + w(\|d\|_\infty)$ for all $t \in [0, T_2)$. Moreover, $\bar{V}_2(0) < \bar{\gamma}$ if $\|\Phi\|_{\mathbb{W}} < \tilde{\rho}_{2,2}(\bar{\gamma})$.

If $\bar{w}(\|d\|_\infty) < 1$, then (3b) implies $\bar{V}_1(t) := V_1(x_t) \leq 1$ for $t \geq T_2$. Assume first that $\max\{\bar{v}, \bar{w}(\|d\|_\infty)\} = 0$. Applying the Comparison Lemma to the function $\bar{V}_1(t)$ from (3c) on interval $t \in [T_2, T_2 + T_1)$, where $T_2 + T_1 = \inf\{t \geq 0 : \bar{V}_1(t) = 0\}$, we get $\bar{V}_1(t) \leq (\mu_1\theta_1(t - T_2) + \bar{V}_1^{-\mu_1}(T_2))^{-1/\mu_1}$. It is clear that $T_1 \leq \bar{V}_1^{-\mu_1}(T_2)/(-\mu_1\theta_1)$, where $\bar{V}_1(T_2) \leq 1$ if $T_2 > 0$ or $\bar{V}_1(T_2) \leq \rho_{2,1}(\|\Phi\|_{\mathbb{W}})$ if $T_2 = 0$. Hence, $\|x(t)\| \leq \tilde{\rho}_{1,1}(\bar{V}_1(t)) \leq \nu_1(\|\Phi\|_{\mathbb{W}}, t)$ for $t \in [T_2, T_2 + T_1)$ and $\|x(t)\| = 0$ for $t \geq T_2 + T_1$ due to (3a). Now assume that $0 < \max\{\bar{v}, \bar{w}(\|d\|_\infty)\} < 1$. Then there exists a moment of time $T_1' \in [T_2, T_2 + T_1)$ such that $\bar{V}_1(t) \leq \max\{\bar{v}, \bar{w}(\|d\|_\infty)\}$ for $t \geq T_1'$. Thus, $\|x(t)\| \leq \nu_1(\|\Phi\|_{\mathbb{W}}, t) + v + w(\|d\|_\infty)$ for all $t \geq T_2$. □

B Proof of Theorem 2

In order to prove the theorem it is sufficient to show that there exist functionals $V_k : \mathbb{W}^1_\hbar \to \mathbb{R}_+$, satisfying conditions of Theorem 1. Indeed, C2) and C3) guarantee existence of unique functionals $V_k : \mathbb{W}^1_\hbar \setminus \{0\} \to \mathbb{R}^\star_+$ such that $Q_k(V_k(\chi), \chi) = 0$ for any $\chi \in \mathbb{W}^1_\hbar \setminus \{0\}$. Moreover, Theorem 1 from [25] and C1) guarantee that functionals V_k are continuously Fréchet differentiable on $\mathbb{W}^1_\hbar \setminus \{0\}$.

From (5a) it follows that $q_{1,k}(V_k(\chi), \|\chi(0)\|) \leq Q_k(V_k(\chi), \chi) = 0 = q_{1,k}(\rho_{1,k}(\|\chi(0)\|), \|\chi(0)\|)$ for all $\chi \in \mathbb{W}^1_\hbar \setminus \mathbb{W}^{1,0}_\hbar$ and $q_{2,k}(\rho_{2,k}(\|\chi\|_{\mathbb{W}}), \|\chi\|_{\mathbb{W}}) = 0 = Q_k(V_k(\chi), \chi) \leq q_{2,k}(V_k(\chi), \|\chi\|_{\mathbb{W}})$ for all $\chi \in \mathbb{W}^1_\hbar \setminus \{0\}$. Due to properties of $\mathcal{IK}_\infty$ functions, the obtained inequalities imply $\rho_{1,k}(\|\chi(0)\|) \leq V_k(\chi)$ for all $\chi \in \mathbb{W}^1_\hbar \setminus \mathbb{W}^{1,0}_\hbar$ and $V_k(\chi) \leq \rho_{2,k}(\|\chi\|_{\mathbb{W}})$ for all $\chi \in \mathbb{W}^1_\hbar \setminus \{0\}$. Thus, the functional $V_k(\chi)$ can be extended by continuity to $\mathbb{W}^1_\hbar$ as follows $V(0) = 0$. Taking into account that $0 = \rho_{1,k}(\|\chi(0)\|) < V_k(\chi)$ for all $\chi \in \mathbb{W}^{1,0}_\hbar \setminus \{0\}$, we finally derive condition (3a).

Conditions (5b) and C3) guarantee that (3b) holds. Indeed, if $V_1(\chi) \leq 1$, then $Q_2(1, \chi) = Q_1(1, \chi) \leq Q_1(V_1(\chi), \chi) = 0 = Q_2(V_2(\chi), \chi)$ and, consequently, $V_2(\chi) \leq 1$.

Let $x_t = \chi$ be a solution of (1). Consider the functions $\bar{V}_k(t) := V_k(x_t)$, $\bar{Q}_k(V_k, t) := Q_k(V_k, x_t)$ and $\frac{d}{dt}\bar{V}_k(t) := \frac{d}{dt}V_k(x_t)$. Clearly, $\bar{Q}_k(\bar{V}_k(t), t) = 0$ for all $t \geq 0$ such that $x_t \neq 0$. Then the implicit function theorem [4, p. 221] for Euclidean spaces and (5c), (5d) imply that $\frac{d}{dt}\bar{V}_k(t) = -\bar{Q}'_{t,k}(V_k, t)/\bar{Q}'_{V,k}(V_k, t) \leq -\theta_k \bar{V}_k^{1+\mu_k}(t)$. Thus, all steps of the proof of Theorem 1 can be repeated. □

C Proof of Proposition 2

P1)–P2) First, it is clear to see that $\varphi_1'(\xi) = -1 < 0$ for all $\xi \in [0, 1]$. Differentiating (10) with respect to ξ, we obtain:

$$\varphi_i'(\xi) := \begin{cases} \varphi_{i-1}(\xi) - 1, & \xi \in [0, 1], \\ \varphi_{i-1}(\xi) - \varphi_{i-1}(\xi - 1), & \xi \in (1, i-1), \\ -\varphi_{i-1}(\xi - 1), & \xi \in [i-1, i]. \end{cases} \quad \text{(C.1)}$$

Obviously, using induction, one can prove that $\varphi_i'(\xi) < 0$ on $\xi \in (0, i)$ for $i \geq 2$. Indeed, if $\varphi_{i-1}'(\xi) < 0$ on $\xi \in (0, i-1)$, then $\varphi_{i-1}(\xi)$ is strictly decreasing. Then taking into account that (10) implies $\varphi_i(0) = 1$ and $\varphi_i(i) = 0$, we finish the proof.

P3) Property 4) from Proposition 2 in [27] postulates that $\bar{\varphi}_i(h\xi) + \bar{\varphi}_i(h(i - \xi)) = 1$, where functions $\bar{\varphi}_i(h\xi)$ are such that $\bar{\varphi}_i(h\xi) = \varphi_i(\xi)$. Thus, $\varphi_i(\xi) + \varphi_i(i - \xi) = 1$.

P4) Differentiating (C.1) with respect to ξ, we get:

$$\varphi_i''(\xi) := \begin{cases} \varphi_{i-1}'(\xi), & \xi \in [0, 1], \\ \varphi_{i-1}'(\xi) - \varphi_{i-1}'(\xi - 1), & \xi \in (1, i-1), \\ -\varphi_{i-1}'(\xi - 1), & \xi \in [i-1, i]. \end{cases}$$

For $i = 2$ it is obvious that $\varphi_2''(\xi) < 0$ on $\xi \in [0, 1)$ and $\varphi_2''(\xi) > 0$ on $\xi \in (1, 2]$, since $\varphi_1'(\xi) = -1 < 0$ for all $\xi \in [0, 1]$. Moreover, $\varphi_2'(\xi)$ is strictly decreasing and strictly increasing on corresponding intervals.

Applying property P1) for $i > 2$, it is sufficient to prove by using induction that function $\varphi_i''(\xi)$ has the unique zero at $\xi_{0i} = i/2$. Indeed, $\varphi_{i-1}''((i-1)/2) = 0$ implies that $\varphi_{i-1}'(\xi)$ is strictly decreasing on $\xi \in (0, (i-1)/2)$ and strictly increasing on $\xi \in ((i-1)/2, 1)$. Therefore, $\varphi_i''(\xi)$ has the only one zero on $\xi \in (0, i)$. Let us show that $\xi_{0i} = i/2$. Using (C.1), the condition $\varphi_i''(i/2) = \varphi_{i-1}'(i/2) - \varphi_{i-1}'(i/2 - 1) = 0$ can be equivalently rewritten as

$$2\varphi_1(1/2) = 1 \qquad \text{for } i=3,$$
$$2\varphi_{i-2}\big(\tfrac{i-2}{2}\big) = \varphi_{i-2}\big(\tfrac{i}{2}\big) + \varphi_{i-2}\big(\tfrac{i-4}{2}\big) \quad \text{for } i \geq 4,$$

since $3/2 \in [1,2]$ and $(3/2-1) \in [0,1]$ for $i=3$, $i/2$ and $(i/2-1) \in [1, i-1]$ for $i \geq 4$. Applying property P3), one can see that these relations hold and, therefore, $\xi_{0i} = i/2$ is the unique inflection point of $\varphi_i(\xi)$ for $i > 2$.

P5) It is obvious that $\psi_i(i) = 0$. Then using the change of variable $\tilde{\lambda} = i - \lambda$ and property P3), we obtain $\psi_i(0) = \int_0^{i/2} \varphi_i(\lambda)d\lambda + \int_{i/2}^{i} \varphi_i(\tilde{\lambda})d\tilde{\lambda} = \int_0^{i/2} \varphi_i(\lambda)d\lambda + \int_0^{i/2} \varphi_i(i-\lambda)d\lambda = \int_0^{i/2} [\varphi_i(\lambda) + \varphi_i(i-\lambda)]d\lambda = i/2$.

P6) Function $\psi_i(\xi)$ could be rewritten as follows:

$$\psi_i(\xi) = \begin{cases} \frac{i}{2} - \int_0^{\xi} \varphi_i(\lambda)d\lambda, & \xi \in [0, i/2], \\ \int_{\xi}^{i} \varphi_i(\lambda)d\lambda, & \xi \in [i/2, i]. \end{cases}$$

Taking into account property P4), integral terms can be estimated by the area of a trapezoid from below and a triangle from above, respectively:

$$\psi_i(\xi) \leq \begin{cases} i/2 - \xi[1+\varphi_i(\xi)]/2, & \xi \in [0, i/2], \\ (i-\xi)\varphi_i(\xi)/2, & \xi \in [i/2, i]. \end{cases}$$

Since $i - \xi \leq i\varphi_i(\xi)$ for $[0, i/2]$ and $i - \xi \leq i$ for all $\xi \in [0, i]$, we conclude the proof.

P7) Since function $\tilde{\varphi}_i(\xi) := \psi_i^{-1}(\xi)\varphi_i^2(\xi)$ is continuous on $\xi \in [0, i)$, it is sufficient to prove that $\tilde{\varphi}_i(i+0^-) < \infty$. Indeed, applying L'Hôpital's rule, we get $\tilde{\varphi}_i(i+0^-) = -2\varphi_i'(i)$. From (C.1) it follows that $\varphi_i'(i) = \varphi_{i-1}(i-1) = 0$. Therefore, $\zeta_i = \int_0^i \tilde{\varphi}_i(\xi)d\xi$ is well-defined and function $\tilde{\varphi}_i(\xi)$ can be prolonged to $\xi = i$ by defining $\tilde{\varphi}_i(i) = 0$. $\square$

D Proof of Theorem 3

Let us show that ILKFs (12) satisfy all conditions of Theorem 2.

D.1 Proof of conditions C1)–C3), (5a) *and* (5b)

The functionals $Q_k(V_k, \chi)$ defined by (12) are continuously Fréchet differentiable on $\mathbb{R}_+^\star \times \mathbb{W}_\hbar^1$, where $\hbar = (n-1)h$. Indeed, the following operators

$$DQ_{k,V}(\Delta V_k) := -(\Delta V_k/V_k)\Big(\chi(0)^\top \Lambda_{V_k}^{-r_k} D_k \Lambda_{V_k}^{-r_k}\chi(0) - \textstyle\sum_{i=1}^{n-1} \frac{i m_k(i)}{2S_i} V_k^{-2r_{k,i+2}+\mu_k} \int_{-ih}^{0} \psi_i(\frac{-\tau}{h})\dot{\chi}_{i+1}^2(\tau)d\tau\Big),$$

$$DQ_{k,\chi}(\Delta\chi) := 2\chi^\top(0)\Lambda_{V_k}^{-r_k} P \Lambda_{V_k}^{-r_k}\Delta\chi(0) + \textstyle\sum_{i=1}^{n-1} \frac{i}{S_i} V_k^{-2r_{k,i+2}+\mu_k} \int_{-ih}^{0} \psi_i(\frac{-\tau}{h})\dot{\chi}_{i+1}(\tau)\frac{d\Delta\chi_{i+1}(\tau)}{d\tau}d\tau,$$

where $\Delta V_k \in \mathbb{R}_+^\star$, $\Delta\chi \in \mathbb{W}_\hbar^1$, $D_k := H_{r_k}P + PH_{r_k}$ and $m_k(i) := 2r_{k,i+2} - \mu_k$, $i = \overline{1, n-1}$, are continuous partial Fréchet derivatives of function $V_k \mapsto Q_k(V_k, \chi)$ and functional $\chi \mapsto Q_k(V_k, \chi)$, respectively, for all $V_k \in \mathbb{R}_+^\star$ and $\chi \in \mathbb{W}_\hbar^1$.

Since $P \succ 0$, then the following inequalities

$$\frac{\lambda_{\min}(P)\|\chi(0)\|^2}{\max\{V_k^{2-2\mu_k}, V_k^{2-2n\mu_k}\}} - 1 \leq Q_k(V_k, \chi) \leq \frac{\lambda_{\max}(P)\|\chi(0)\|^2 + \sum_{i=1}^{n-1} \frac{i^2}{4S_i} V_k^{-\mu_k} \int_{-(n-1)h}^{0} |\dot{\chi}_{i+1}(\tau)|^2 d\tau}{\min\{V_k^{2-2\mu_k}, V_k^{2-2n\mu_k}\}} - 1$$

hold for all $V_k \in \mathbb{R}_+^\star$ and $\chi \in \mathbb{W}_\hbar^1$. Hence, it is easy to see that for any $\chi \in \mathbb{W}_\hbar^1$ there exist $V_k \in \mathbb{R}_+^\star$ such that $Q_k(V_k, \chi) = 0$. Taking into account (14b), introduce the functions $q_{1,k}, q_{2,k} \in \mathcal{IK}_\infty$ by the formulas

$$q_{1,k}(\rho_{1,k}, s) = \frac{2s^2}{\max\{\rho_{1,k}^{2-2\mu_k}, \rho_{1,k}^{2-2n\mu_k}\}} - 1,$$
$$q_{2,k}(\rho_{2,k}, s) = \frac{\max\{\frac{1}{\|\sigma\|}, \rho_{2,k}^{-\mu_k} \max_{i=\overline{1,n-1}} \frac{i^2}{4S_i}\}s^2}{\min\{\rho_{2,k}^{2-2\mu_k}, \rho_{2,k}^{2-2n\mu_k}\}} - 1,$$

where $\rho_{1,k}, \rho_{2,k}, s \in \mathbb{R}_+^\star$. The obtained estimates also guarantee that $q_{1,k}(V_k, \|\chi(0)\|) \leq Q(V_k, \chi) \leq q_{2,k}(V_k, \|\chi\|_{\mathbb{W}})$ for all $V_k \in \mathbb{R}_+^\star$ and $\chi \in \mathbb{W}_\hbar^1$. Moreover, condition (5b) obviously holds.

One can see that $m_k(i) \leq 2\omega_k$ and $0 \prec D_k \preccurlyeq 2\omega_k P$ due to (14a). Taking into account that by definition $Q'_{V,k}(V_k, \chi)\Delta V_k = DQ_{k,V}(\Delta V_k)$, we conclude that

$$-2\omega_k \leq V_k Q'_{V,k}(V_k, \chi) < 0, \quad \forall (V_k, \chi) \in \Omega_k. \tag{D.1}$$

Therefore, the condition C3) of Theorem 2 hold. $\square$

D.2 Proof of conditions (5c) *and* (5d)

If $x_t = x_t(\Phi, d)$ is the solution of the system (6), (7), then using property P5), we obtain

$$Q'_{t,k}(V_k, x_t, d) = R_{1,k} + R_{2,k} + R_{3,k}, \tag{D.2}$$

where

$$R_{1,k} := 2x^\top \Lambda_{V_k}^{-r_k} P \Lambda_{V_k}^{-r_k} f(x_t, \dot{x}_t, d),$$
$$R_{2,k} := V_k^{\mu_k} \sum_{i=1}^{n-1} \tfrac{i^2}{4S_i} V_k^{-2r_{k,i+2}} \dot{x}_{i+1}^2(t),$$
$$R_{3,k} := -\sum_{i=1}^{n-1} \tfrac{i}{2hS_i} V_k^{-2r_{k,i+2}+\mu_k} \int_{t-ih}^{t} \varphi_i(\tfrac{t-s}{h})\dot{x}_{i+1}^2(s)ds.$$

Taking into account that $\Lambda_{V_k}^{-r_k} A = V_k^{\mu_k} A \Lambda_{V_k}^{-r_k}$, $\Lambda_{V_k}^{-r_k} B = V_k^{-r_{k,n}} B = V_k^{-1+\mu_k} B$ and (11), $R_{1,k}$ could be rewritten as follows:

$$R_{1,k} = 2V_k^{\mu_k} x^\top \Lambda_{V_k}^{-r_k} P\big((A+BK)\Lambda_{V_k}^{-r_k} x + BK(d_{h,k} + d_{\mu,k} + d_{y,k}) + BV_k^{-1}(d_1 + ax)\big),$$

where disturbance terms $d_{h,k} := \Lambda_{V_k}^{-r_k}[0,\ \delta_1,\ \ldots,\ \delta_{n-1}]^\top$, $d_{\mu,k} := V_k^{-1}\big[\lceil \tilde{y}_1 \rfloor^{\alpha_1(\|\tilde{y}\|)},\ \ldots,\ \lceil \tilde{y}_n \rfloor^{\alpha_n(\|\tilde{y}\|)}\big]^\top - \Lambda_{V_k}^{-r_k}\tilde{y}$ and $d_{y,k} := \Lambda_{V_k}^{-r_k}[\tilde{d}_{2,1},\ \tilde{d}_{2,2},\ \ldots,\ \tilde{d}_{2,n}]^\top$ represent finite-difference approximation error, nonlinear deviation of feedback and presence of the output perturbation, respectively.

Since $c_2, c_3 > 0$, then $R_{2,k}$ has the following estimate:

$$R_{2,k} \leq V_k^{\mu_k} x^\top \Lambda_{V_k}^{-r_k} M^{-1} \Lambda_{V_k}^{-r_k} x + V_k^{\mu_k} \tfrac{(n-1)^2}{4S_{n-1}} (V_k^{-1}\dot{x}_n)^2.$$

Note that $V_k^{-1}\dot{x}_n = \Theta z_k$ with $\Theta := [Y,\ B^\top \Xi_{12}]$ and

$$z_k = \left[x^\top \Lambda_{V_k}^{-r_k} P,\ \frac{d_{h,k}^\top P}{\sqrt{\varrho}},\ \frac{d_{\mu,k}^\top P}{\sqrt{\|\sigma\|}},\ \frac{d_{y,k}^\top P}{\sqrt{b_0}},\ \frac{d_1}{V_k},\ \frac{x^\top a^\top}{V_k\sqrt{\varepsilon}}\right]^\top.$$

Term $R_{3,k}$ either can be upper-bounded by using P6):

$$R_{3,k} \le -2h^{-1}(1 - x^\top \Lambda_{V_k}^{-r_k} P \Lambda_{V_k}^{-r_k} x)/(n-1) \quad \text{(D.3)}$$

or by applying Lemma 1 with $\vartheta = \varphi_i$, $\phi = \dot{x}_{i+1}$, $\varpi = 1$ and noting that $d_{h,k}^\top C^\top C d_{h,k} = 0$:

$$R_{3,k} \le -V_k^{-\mu_k} h^{-2} d_{h,k}^\top N^{-1} d_{h,k}/(n-1). \quad \text{(D.4)}$$

Adding and subtracting corresponding terms to (D.2) to construct a quadratic form with respect to the vector z_k and matrix $\Xi := \left[\begin{smallmatrix}\Xi_{11} & \Xi_{12}\\ * & \Xi_{22}\end{smallmatrix}\right]$, we obtain

$$\begin{aligned}
&Q'_{t,k}(V_k, x_t, d) \le V_k^{\mu_k} z_k^\top \Big(\Xi + \Theta^\top \tfrac{(n-1)^2}{4S_{n-1}}\Theta\Big) z_k\\
&+ V_k^{\mu_k} x^\top \Lambda_{V_k}^{-r_k}\Big(M^{-1} - PZP\Big)\Lambda_{V_k}^{-r_k} x\\
&+ \Big(\varrho R_{3,k} + V_k^{\mu_k}\tfrac{2}{n-1}(1 - x^\top \Lambda_{V_k}^{-r_k} P \Lambda_{V_k}^{-r_k} x)\Big)\\
&+ (1-\varrho)\Big(R_{3,k} + V_k^{\mu_k}\tfrac{1}{(n-1)\varrho} d_{h,k}^\top P d_{h,k}\Big) \qquad \text{(D.5)}\\
&+ V_k^{\mu_k}\tfrac{1}{4(n-1)}\Big(c_1 (V_k^{-1} d_1)^2 + \tfrac{2}{\epsilon}(V_k^{-1} a x)^2 - 2\Big)\\
&+ V_k^{\mu}\tfrac{1}{2(n-1)}\Big(\tfrac{1}{b_0} d_{y,k}^\top P d_{y,k} + \tfrac{1}{\|\sigma\|} d_{\mu,k}^\top P d_{\mu,k} - 2\Big)\\
&- V_k^{\mu_k}\tfrac{1}{2(n-1)}.
\end{aligned}$$

Let us show that first six terms in (D.5) are nonpositive for all $(V_k, x_t) \in \Omega_k$ such that $V_1 \in (\max\{\bar{v}, \bar{w}(\|d\|_\infty)\}, 1]$ and $V_2 \in (\max\{1, \bar{w}(\|d\|_\infty)\}, \bar{\gamma})$. Firstly, applying Schur complement to the first and the second terms, it is easy to see that they are not positive due to (14c).

Secondly, it follows from (D.3) that the third term in (D.5) is negative if $\varrho/h > V_k^{\mu_k}$, i.e. if $\bar{v} = \bar{\gamma}^{-1} = h^{1/\sqrt{\mu}}$. Taking into account that (14c) implies $N^{-1} \succcurlyeq \varrho P$, it is obvious that the fourth term is negative due to (D.4).

Thirdly, for all $(V_k, x_t) \in \Omega_k$ ILKFs (12) and (14b) imply $(V_k^{-1} a x)^2 \le (\epsilon/2)\max\{V_k^{2(r_{k,1}-1)},\ V_k^{2(r_{k,n}-1)}\}$. Since $r_{1,1} > r_{1,n} > 1$ and $r_{2,1} < r_{2,n} < 1$, then it is clear that $(V_k^{-1} a x)^2 \le \epsilon/2$ for $V_1 \le 1$ and $V_2 > 1$. Moreover, one can see that $c_1 (V_k^{-1} d_1)^2 \le \eta^2 (V_k^{-1} d)^2 \le 1$ if $\bar{w}(s) \ge \eta s$. Thus, the fifth term is also negative.

Finally, taking into account (11b), it can be proven that $|\tilde{d}_{2,j}| \le (2/h)^{j-1}\|d_2\|_\infty$ for $j = \overline{1,n}$. As a result, $\|d_{y,k}\|^2 \le \frac{(2/h)^{2n}-1}{(2/h)^2-1}\|d\|_\infty^2 \max\{V_k^{-2r_{k,1}},\ V_k^{-2r_{k,n}}\}$ and $d_{y,k}^\top P d_{y,k} \le b_0$ if $\bar{w}(s) \ge \max\{(\eta s)^{1/r_{1,1}}, (\eta s)^{1/r_{2,1}}\}$. Then assuming that $\|d_{\mu,k}\|^2 \le \|\sigma_k\|^2$ (see the proof in the next subsection), it is clear that the sixth term in (D.5) is negative due to (14c).

Since (D.1) implies that $-1 \le V_k \frac{1}{2\omega_k} Q'_{V,k}(V_k, x_t)$, one can conclude that conditions (5c) and (5d) are proven with $\bar{v} = \bar{\gamma}^{-1} = h^{1/\sqrt{\mu}}$, $\bar{w}(s) = \max\{(\eta s)^{1/r_{1,1}}, (\eta s)^{1/r_{2,1}}\}$ and $\theta_k^{-1} = 4(n-1)\omega_k$. Taking into account formulas of $q_{1,k}$ and $q_{2,k}$, $k = 1, 2$ parameters γ, κ, T, v and function $w \in \mathcal{K}$ can be easily calculated using (4). □

D.3 Proof of the estimate $\|d_{\mu,k}\|^2 \le \|\sigma_k\|^2$

The disturbance term $\|d_{\mu,k}\|^2$ can be rewritten as:

$$\begin{aligned}\|d_{\mu,k}\|^2 &= \textstyle\sum_{j=1}^n \Big(\big(V_k^{-1}|\tilde{y}_j|^{1/r_{k,j}} - |V_k^{-r_{k,j}}\tilde{y}_j|\big)\\ &\quad + V_k^{-1}\big(|\tilde{y}_j|^{\alpha_j(\|\tilde{y}\|)} - |\tilde{y}_j|^{1/r_{k,j}}\big)\Big)^2.\end{aligned} \quad \text{(D.6)}$$

First, applying Lemma 1 with $\vartheta = \varphi_i$, $\phi = \dot{x}_{i+1}$, $\varpi = \psi_i/\varphi_i$ to (12) for all $(V_k, x_t) \in \Omega_k$ such that $V_k^{-\mu_k}\varrho/h > 1$, we deduce that:

$$x^\top \Lambda_{V_k}^{-r_k} P \Lambda_{V_k}^{-r_k} x + \textstyle\sum_{i=1}^{n-1} \frac{i}{2\varrho\zeta_i S_i}\big(V_k^{-r_{k,i+1}}\delta_i\big)^2 \le 1.$$

Due to (14b) and upper bound on S_i the following holds:

$$2V_k^{-2r_{k,1}} x_1^2 + 2\textstyle\sum_{i=1}^{n-1} V_k^{-2r_{k,i+1}}(x_{i+1}^2 + \delta_i^2) \le 1.$$

Then $\|\Lambda_{V_k}^{-r_k}\tilde{y}\|^2 \le 1 + 2\|d_{y,k}\|^2 \le 1 + b_0 = b_1^2$. So it follows that $\|\tilde{y}\| \le b_1 \max\{V_k^{r_{k,1}}, V_k^{r_{k,n}}\}$ and $|V_k^{-r_{k,j}}\tilde{y}_j| \le b_1$. Thus, applying Lemma 2, the first term in (D.6) can be bounded as follows:

$$|V_k^{-r_{k,j}}\tilde{y}_j|^{1/r_{k,j}} - |V_k^{-r_{k,j}}\tilde{y}_j| \le \bar{g}(b_1, 1/r_{k,j}). \quad \text{(D.7)}$$

Since $V_1 \le 1$ implies that $\|\tilde{y}\| \le b_1$, we deduce that $\alpha_j(\|\tilde{y}\|) = 1/r_{1,j}$ for $V_1 \le 1$. Therefore, the second term in (D.6) is zero and $\|d_{\mu,1}\|^2 \le \|\sigma_1\|^2$.

On the other hand, if $|\tilde{y}_j| \ge b_2 > b_1$ for all $j = \overline{1,n}$, then $\|\tilde{y}\| \ge b_2$ and $\alpha_j(\|\tilde{y}\|) = 1/r_{2,j}$. Thus, the second term in (D.6) for $V_2 > 1$ and $|\tilde{y}_j| \le b_2$ could be estimated as:

$$\begin{aligned}&V_2^{-1}(|\tilde{y}_j|^{\alpha_j(\|\tilde{y}\|)} - |\tilde{y}_j|^{1/r_{2,j}})\\ &\le \max_{|\tilde{y}|\in[0,b_2]} \big||\tilde{y}_j|^{1/r_{1,j}} - |\tilde{y}_j|^{1/r_{2,j}}\big| = \bar{g}(b_2^{1/r_{2,j}}, \tfrac{r_{2,j}}{r_{1,j}}).\end{aligned}$$

Taking into account (D.7), one can finally conclude that $\|d_{\mu,2}\|^2 \le \|\sigma_2\|^2$. □

E Proof of Proposition 3

It is a well-known fact [27] that the system (6), (7) with $\mu = 0$ is exponentially ISS with a decay rate $\beta \in (0, \beta_0)$, where $\beta_0 > 0$ is the decay rate of the corresponding state-feedback control, i.e. for all $\Phi \in \mathbb{W}_h^1$ and $d \in \tilde{\mathcal{D}} := \{d \in \mathcal{L}_\infty^m : \|d\|_\infty < \tilde{\kappa}\}$ there exists a constant $c_0 > 0$ and a function $\tilde{w} \in \mathcal{K}$ such that

$$\|x(t, \Phi, d)\| \le c_0 \|\Phi\|_{\mathbb{W}} e^{-\beta t} + \tilde{w}(\|d\|_\infty), \quad \forall t \ge 0.$$

Define by T_0 the moment of time when the system (6), (7) with time delay h_0 and $\mu = 0$ reaches the set $\mathcal{A}$, i.e. $T_0 = \inf\{t \ge 0 : \|x(t, \Phi, d)\| \le v + w(\|d\|_\infty)\}$. Obviously, $T_0 \ge \max\{0, \beta_0^{-1}\ln(c_0\|\Phi\|_{\mathbb{W}}/(v + w(\kappa)))\}$ if $v + w(\|d\|_\infty) \ge \tilde{w}(\|d\|_\infty)$. Otherwise, the set $\mathcal{A}$ is unreachable. Therefore, it is easy to see that $T \le T_0$ if

$$\|\Phi\|_{\mathbb{W}} \ge \gamma_0 := e^{\beta_0 \frac{2 + (n+1)\mu}{4(n-1)(1 + (n+1)\mu)\mu}}(v + w(\kappa))/c_0.$$

Clearly, there is a small enough h_0 such that $\gamma \ge \gamma_0$. □