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Abstract

We give experimental evidence that the only known upper bound on the diameter of the flip graph of a hyperbolic surface recently proven by Despré, Schlenker, and Teillaud (SoCG'20), is largely overestimated. To this aim, we develop an experimental framework for the storage of triangulations of hyperbolic surfaces and modifications through twists. We show that the computations with algebraic numbers can be overcome, and we propose ways to generate surfaces that are meaningful for the experiments.

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The source code is available at
https://members.loria.fr/Monique.Teillaud/Exp-hyperb-flips/

1 Introduction

It was recently proven that the geometric flip graph of a closed oriented hyperbolic surface is connected [7]. A Delaunay flip algorithm can thus transform any input geometric triangulation T, i.e., a triangulation whose edges are embedded as geodesic segments only intersecting at common endpoints, into a Delaunay triangulation. This is particularly useful in practice as a crucial preprocessing step to computing Delaunay triangulations on a surface: it transforms a "bad" representation of a surface, e.g., by a very elongated fundamental domain, to a "nice" representation by a Delaunay triangulation with only one vertex. Inserting a lot of points would rather be done by Bowyer's incremental algorithm [8, 6], inspired from previous work in the flat case [11].

The authors prove an upper bound on the number of flips: $C_h \cdot \Delta(T)^{6g-4} \cdot n^2$, where C_h is a constant, $\Delta(T)$ is the diameter of T, g is the genus of the surface, and n is the number of vertices [7]. The diameter $\Delta(T)$ is the smallest diameter of a fundamental domain that is the union of lifts of the triangles of T in $\mathbb{H}$. If T is a triangulation of a genus two surface with

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only one vertex then $\Delta(T)$ and the diameter of any other such domain differ by a constant factor at most. In the experiments, we will thus use the domain that naturally appears.

In this paper, we experimentally study the dependence of the number of flips on $\Delta(T)$ (Section 5), for surfaces of genus two. We suspect that the factor $\Delta(T)^{6g-4}$ is largely overestimated. We focus on triangulations having only one vertex, both because the dependence on the number of vertices is clear, and because we are motivated by the abovementioned preprocessing aspect of the algorithm.

Our setup for experiments relies on the representation of genus two surfaces by octagons in $\mathbb{H}$ (Section 2.3). We obtain input triangulations with a large diameter by twisting the abovementioned octagons (Section 4). The data structure we use offers a representation of a triangulation that intrinsically lies on the surface (Section 3).

2 Background

2.1 Hyperbolic surfaces

Consider a closed oriented hyperbolic surface $\mathcal{S}$ (i.e., a connected compact oriented surface without boundary) of genus 2 and the underlying topological surface S_2. Given a hyperbolic structure h on $\mathcal{S}$, associated to a metric of constant curvature -1, the surface $\mathcal{S} = (S_2, h)$ is isometric to the quotient $\mathbb{H}/G$, where $\mathbb{H}$ is the hyperbolic plane and G is a (non-Abelian) discrete subgroup of the isometry group of $\mathbb{H}$ isomorphic to the fundamental group $\pi_1(S_2)$.

The universal cover of $\mathcal{S}$ is isometric to $\mathbb{H}$ equipped with a projection $\rho : \mathbb{H} \to \mathcal{S}$ that is a local isometry. The group G acts on $\mathbb{H}$, so that for any $p \in \mathcal{S}$, $\rho^{-1}(p)$ is an orbit under the action of G. A lift $\widetilde{p}$ of a point $p \in \mathcal{S}$ is one of the elements of the orbit $\rho^{-1}(p)$.

We use the Poincaré disk model, in which $\mathbb{H}$ is represented as the open unit disk $\mathbb{D} \subset \mathbb{C}$.

2.2 Triangulations and flips on hyperbolic surfaces

We call *triangulation* T of a hyperbolic surface $\mathcal{S}$ any geodesic embedding of an undirected graph with a finite number of vertices onto $\mathcal{S}$ such that each resulting face is homeomorphic to an open disk and is bounded by exactly three distinct edge-embeddings. The lift $\widetilde{T}$ of T is the (infinite) triangulation of $\mathbb{H}$ whose vertices and edges are the lifts of the vertices and the edges of T. A Delaunay triangulation T of $\mathcal{S}$ is a triangulation whose lift $\widetilde{T}$ is a Delaunay triangulation in $\mathbb{H}$; for each face t of T and any of its lifts $\widetilde{t}$, the open disk in $\mathbb{D}$ circumscribing $\widetilde{t}$ contains no vertex of $\widetilde{T}$.

Lifting an edge e of T to some $\widetilde{e}$, together with the two triangles incident to $\widetilde{e}$ in the lifted triangulation $\widetilde{T}$, we say that e is *Delaunay-flippable* if the open disks of these triangles contain the fourth vertex of the quadrilateral they form. In this case, the geodesic segment $\widetilde{e}'$ that is the other diagonal of the quadrilateral is contained in it. The *Delaunay flip* of e in T consists in replacing $\widetilde{e}$ by $\widetilde{e}'$ and projecting it back to $\mathcal{S}$ by ρ. A *Delaunay flip algorithm* takes as input a triangulation of $\mathcal{S}$ and flips Delaunay-flippable edges (in any order) until there is none left. Such an algorithm terminates and outputs a Delaunay triangulation [7].

2.3 Admissible loosely-symmetric octagons

We use a slight extension of a set of parameters introduced by Aigon-Dupuy, Buser *et al.* [1], who proved that any closed hyperbolic surface of genus 2 has a fundamental domain that is an octagon in $\mathbb{D}$. This versatile representation allows us to easily construct and manipulate surfaces in our experiments.

We say that a hyperbolic octagon P is *loosely-symmetric* if the opposite sides of P are isometric and the opposite interior angles of P are equal. If moreover the hyperbolic area of P is 4π then P is *admissible*. Clearly, the symmetric octagons introduced by Aigon-Dupuy, Buser *et al.* [1] are loosely-symmetric and the notions of admissibility coincide. Identifying the opposite sides of an admissible loosely-symmetric octagon gives a closed hyperbolic surface of genus 2 [3, Theorem 1.3.5]. Each such surface can be obtained this way [1]. We refer to the full paper [5, Section 3.2] for details and computations.

3 Data structure

Though an *ad hoc* data structure was previously proposed for flipping triangulations [7], we choose to use *combinatorial maps* [10, Section 3.3], which are commonly used to represent graphs embedded on a surface. The data structure we use offers a representation of the triangulation that intrinsically lies on the surface, while the earlier data structure [7, Section 4.1] stuck to specific representatives of all vertices and faces of the lifted triangulation.

For our experiments, we use the flexible implementation of combinatorial maps that is publicly available in CGAL [4]. The *dart* (or *flag*) is the central object in a combinatorial map: it gives access to all incidence relations of an edge of the graph (see Figure 1).

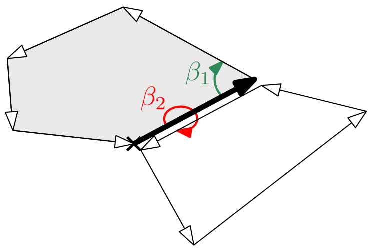

Figure 1 A dart in a combinatorial map (bold).

The geometric information for the triangulation is stored as a cross-ratio for each edge. Recall that the cross-ratio of four pairwise-distinct points in $\mathbb{H}$ represented by $z_1, z_2, z_3, z_4 \in \mathbb{D}$ is the complex $[z_1, z_2, z_3, z_4] = \dfrac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)}$ [2]. Let $\operatorname{Im}[\cdot]$ denote the imaginary part of a complex. Cross-ratios are suitable for a flip algorithm, due to their well-known property: assuming that z_1, z_2, z_3, z_4 are counterclockwise, $\operatorname{Im}[z_1, z_2, z_3, z_4] > 0$ if and only if z_4 lies in the open disk circumscribing (z_1, z_2, z_3).

Given an edge e of a triangulation T of $\mathcal{S}$ we consider a lift $\widetilde{e} = (\widetilde{u_1}, \widetilde{u_3})$ of e in $\mathbb{D}$ and the other vertices $\widetilde{u_2}$ and $\widetilde{u_4}$ of the two faces incident to $\widetilde{e}$ in $\widetilde{T}$, numbering vertices counterclockwise. The cross-ratio $\mathcal{R}_T(e)$ is defined as $[\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3}, \widetilde{u_4}]$; it is independent of the choice of the lift of e, as the cross-ratio is invariant under orientation preserving isometries of $\mathbb{D}$. An edge e of T is Delaunay-flippable if and only if $\operatorname{Im}[\mathcal{R}_T(e)] > 0$.

Note that in our experiments, the lifts in $\mathbb{D}$ are only used to initialize the cross-ratios of a given input triangulation T; they are ignored during the flips, thus preserving the property that the data structure only considers the embedding of the triangulation on the surface. However, in order to be able to recover a lift in $\mathbb{D}$ in the end, e.g., for drawing a representation in $\mathbb{D}$ of the final Delaunay triangulation, we need to maintain an *anchor* during flips. The anchor $A = (\delta, a_1, a_2, a_3)$ consists in some dart δ, chosen arbitrarily, together with a triple (a_1, a_2, a_3) of points in $\mathbb{D}$ that are the vertices of a lift of the face containing δ.

A triangulation T is thus represented by (M, F, A), where M is the combinatorial map,

F maps edges of M to their cross-ratios, and A is the anchor. We refer to the full paper [5, Section 3.4] for details on how to update the data structure during a flip.

4 Generating input for the experiments

We generate input for the Delaunay flips algorithms by triangulating admissible loosely-symmetric octagons. An algorithm [5, Appendix C.2] produces such octagons whose vertices are represented in $\mathbb{D}$ by complex numbers with rational real and imaginary parts. We proved a density result [5, Theorem 2] on such rational coordinates allowing us to run experiments using rational numbers only. This is crucial as computing with algebraic numbers is problematic in practice [5, Section 4.1]. For the experiments in Section 5, we need to generate surfaces with large diameter. Our attempts to directly compute such surfaces, taking the diameter as a parameter, were not conclusive. An effective approach consists in starting by generating octagons with a small diameter, then we twist them many times to obtain octagons with a very large diameter. This way we will also study the dependency of the number of flips on those twists.

4.1 Twisting admissible loosely-symmetric octagons

Given $j \geq 3$ and $z_0, \ldots, z_j \in \mathbb{D}$ in geodesically convex position, $G[z_0, \ldots, z_j]$ denotes the hyperbolic polygon whose vertices are $z_1, \ldots, z_j$. Let $G[z_0, \ldots, z_7]$ be an admissible loosely-symmetric octagon. We will consider the Dehn twists [9] along the axes of its side-pairings, as follows (see Figure 2). For every $k \in \{0, \ldots, 7\}$ let τ_k be the orientation preserving isometry of $\mathbb{D}$ satisfying $\tau_k(z_{k+5}) = z_k$ and $\tau_k(z_{k+4}) = z_{k+1}$. Let $t \in \{0, \ldots, 7\}$. For $k \in \{0, \ldots, 7\}$ we set

$$z'_k = \begin{cases} \tau_t(z_k) & \text{if } k - t \in \{1, 2, 3, 4\} \mod 8, \\ z_k & \text{otherwise.} \end{cases}$$

The polygon $G[z'_0, \ldots, z'_7]$ is an admissible loosely-symmetric octagon defining a surface isometric to the one defined by $G[z_0, \ldots, z_7]$ [5, Section 3.2]. We say that $(z'_0, \ldots, z'_7)$ is obtained by t-*twisting* $(z_0, \ldots, z_7)$.

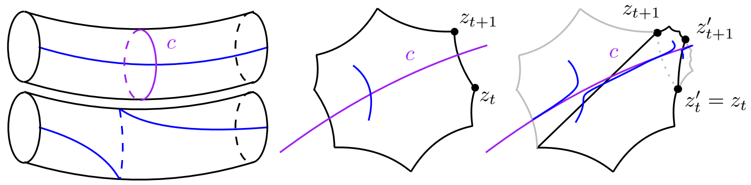

Figure 2 (Left) A Dehn twist along the curve c modifies the blue curve as shown. (Right) A t-twist on an admissible loosely-symmetric octagon.

For a word $t = t_1 \ldots t_m$, we define the t-twist as the composition of the t_k-twists, $k = 1, \ldots, m$, in this order. We pick $t_1, \ldots, t_m$ in $\{0, \ldots, 3\}^m$ instead of $\{0, \ldots, 7\}^m$ to consider the generators without their inverses and quickly obtain large diameters. Indeed for $k \in \mathbb{Z}$ we have $\tau_{k+4} = \tau_k^{-1}$ so a $(t+4)$-twist is the inverse of a t-twist (the indices are modulo 8).

4.2 Generating triangulations

We generate a large number of triangulations having a large diameter following three steps. In the full paper [5, Section 5] a fourth step (**[step 2]**) enables to generate input triangulations with more than one vertex. This step is omitted here as not used in this version.

[step 1] We construct an initial admissible symmetric octagon O.
[step 3] We choose $m \geq 0$ and a sequence $t = t_1 \dots t_m$ of twists.
[step 4] We construct an admissible loosely-symmetric octagon O' by t-twisting O and build the input triangulation T by first cutting O' into 5 triangles and then identifying the edges of the resulting triangulation that correspond to opposite sides of O'.

We refer to the full paper [5, Section 5] for details on these steps. The triangulation T of the hyperbolic surface defined by O has 1 vertex, 9 edges and 5 faces.

We will study two kinds of twists sequences (step 3) in Section 5:

- A *power sequence* is represented by a word u^m for some $u \in \{0, \dots, 3\}$.
- In a *random sequence*, $t_1, \dots, t_m$ are chosen uniformly and independently in $\{0, \dots, 3\}$.

Section 5 will refer to the above three steps. Before doing any experiment Step 1 was applied a thousand times to construct octagons $Q_1, \dots, Q_{1,000}$; the experiments consider the first n_q octagons. We also constructed (for step 3) some 10,000 random sequences of twists noted $S_1, \dots, S_{10,000}$, each of length 10, of which some of the experiments will use the first n_s sequences. The values of n_q, n_s will be specified in the description of each experiment.

5 Exploring the relationship between number of flips and diameter

As recalled in Section 2.2, a Delaunay flip algorithm can flip Delaunay-flippable edges in any order. In the full paper [5, Section 6] we studied various orders, and observed that the number of flips obtained by the *naive* strategy is close to the minimum: we choose the first Delaunay-flippable edge given by the iterator `DartRange::iterator` of the CGAL combinatorial map. As it runs much faster than all other strategies, we stick to it.

Two sets of experiments will be carried out: experiments I and J use power sequences while experiments K, L, and M use random sequences. We use the notations of Section 4.2.

Experiments I and J are parameterized by the number n_q of octagons: $n_q = 1$ in I and $n_q = 1,000$ in J. We perform step 4 with $O = Q_k$ and $t_1 \dots t_m = u^{3l}$ for $k \in \{1, \dots, n_q\}$, $u \in \{0, 1, 2, 3\}$, $l \in \{0, \dots, 50\}$ and we compute the approximate hyperbolic diameter $\varnothing_{k,l,u}$ of O'. We run the Delaunay flip algorithm, counting the number $\alpha_{k,l,u}$ of flips that were needed by the algorithm to terminate. Figure 3 shows the result.

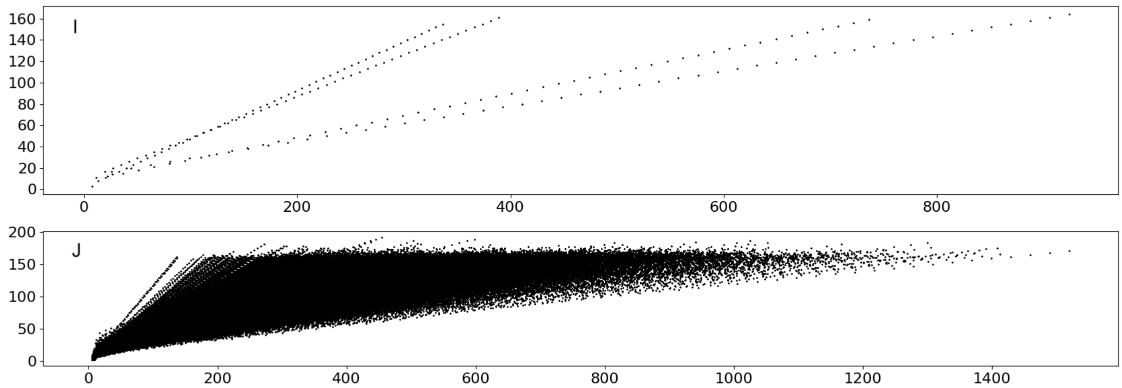

Figure 3 Experiments I and J: number of flips $\alpha_{k,l,u}$ with respect to the (approximate) diameter $\varnothing_{k,l,u}$, $k \in \{1, \dots, n_q\}, l \in \{0, \dots, 50\}, u \in \{0, 1, 2, 3\}$

For experiments K, L, and M the values of (n_q, n_s) are respectively $(1, 10.000)$, $(10, 1.000)$ and $(1.000, 100)$. We first construct the set X containing the 11 prefixes of S_k (including the empty sequence) for every $k \in \{1, \ldots, n_s\}$. Then for every $k \in \{1, \ldots, n_q\}$ and every $s \in X$, we perform step 4 with $O = Q_k$, and $t_1 \ldots t_m = s$. We compute the approximate hyperbolic diameter $\varnothing_{k,s}$ of O'. We run the Delaunay flip algorithm and count the number $\alpha_{k,s}$ of flips that were needed by the algorithm to terminate. Figure 4 shows $\alpha_{k,s}$ as a function of $10\ln(\varnothing_{k,s})$ for $k \in \{1, \ldots, n_q\}, s \in X$.

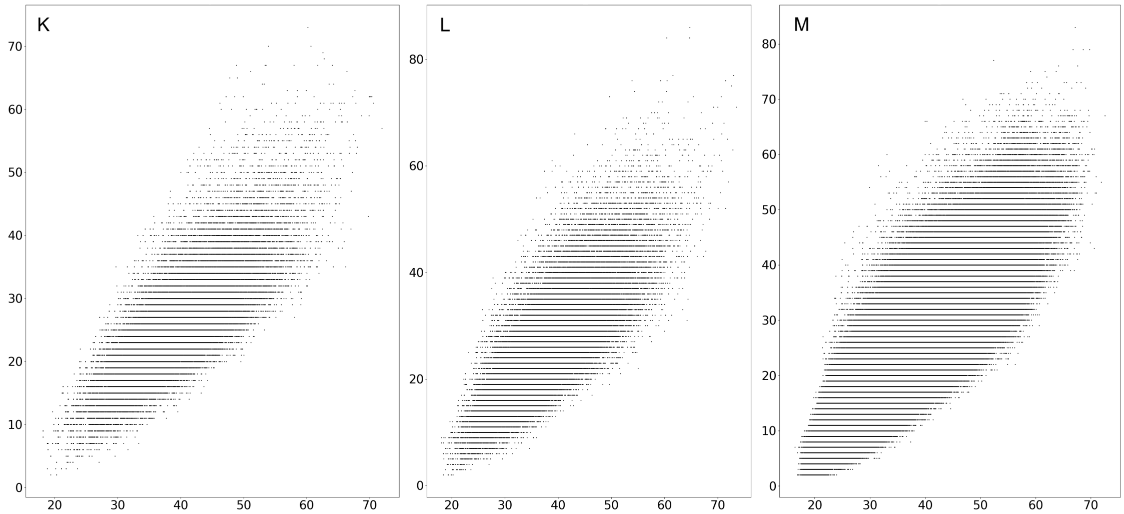

Figure 4 Experiments K, L, and M: number of flips $\alpha_{k,s}$ with respect to $10\ln(\varnothing_{k,s})$, $k \in \{1, \ldots, n_q\}, s \in X$; the maximum diameter is about 1500

Our experiments show that controlling the sequence of twists actually allows us to control the number of flips needed by the flip algorithm. Indeed, in the case of power sequences, we observe that the number of flips is linear in the diameter of the input triangulation: Delaunay flips untwist the triangulation by performing a constant number of flips per iteration of the twist. However, for random sequences, we observe that the number of flips is logarithmic in the diameter of the input triangulation. These results can be interpreted using insights on the mapping class group [5, Section 7.4].

In light of our results, we conjecture that the complexity of the Delaunay flip algorithm is worst-case linear in the diameter of the triangulation, and logarithmic on average.

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