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Graph polynomials and group coloring of graphs[☆]

Bartłomiej Bosek^a, Jarosław Grytczuk^b, Grzegorz Gutowski^a,
Oriol Serra^c, Mariusz Zając^b

^a Institute of Theoretical Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland

^b Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland

^c Department of Mathematics, Universitat Politècnica de Catalunya, Barcelona, Spain



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ABSTRACT

Let Γ be an Abelian group and let G be a simple graph. We say that G is Γ -colorable if for some fixed orientation of G and every edge labeling $\ell : E(G) \rightarrow \Gamma$, there exists a vertex coloring c by the elements of Γ such that $c(y) - c(x) \neq \ell(e)$, for every edge $e = xy$ (oriented from x to y).

Langhede and Thomassen proved recently that every planar graph on n vertices has at least $2^{n/9}$ different \mathbb{Z}_5 -colorings. By using a different approach based on graph polynomials, we extend this result to K_5 -minor-free graphs in the more general setting of *field coloring*. More specifically, we prove that every such graph on n vertices is \mathbb{F} -5-choosable, whenever \mathbb{F} is an arbitrary field with at least 5 elements. Moreover, the number of colorings (for every list assignment) is at least $5^{n/4}$.

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1. Introduction

Let Γ be an Abelian group and let G be a simple graph. We say that G is Γ -colorable if for some orientation of G and every edge labeling ℓ by the elements of Γ there exists a vertex coloring c by

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E-mail addresses: bartlomiej.bosek@uj.edu.pl (B. Bosek), j.grytczuk@mini.pw.edu.pl (J. Grytczuk), grzegorz.gutowski@uj.edu.pl (G. Gutowski), oriol.serra@upc.edu (O. Serra), m.zajac@mini.pw.edu.pl (M. Zając).

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the elements of Γ such that $c(y) - c(x) \neq \ell(e)$, for every edge $e = xy$ (oriented from x towards y). This notion was introduced by Jaeger, Linial, Payane and Tarsi [9] as a dual concept to *group connectivity*.

Answering a question posed in [9], Lai and Zhang [11] proved that every planar graph is \mathbb{Z}_5 -colorable. Recently, Langhede and Thomassen [12] strengthened this result by proving that the number of \mathbb{Z}_5 -colorings of every planar graph on n vertices (for any fixed edge labeling) is at least $2^{n/9}$. The proof is elementary but quite involved.

In this paper we further extend these results by using the polynomial method. It is convenient to introduce a slightly more general setting. Let \mathbb{F} be an arbitrary field and let G be a simple graph. Suppose that each edge $e = xy$ of G is assigned a triple $(a_e, b_e, c_e) \in \mathbb{F}^3$, with $a_e, b_e \neq 0$. We say that G is \mathbb{F} -colorable if for every such edge labeling there exists a vertex coloring f by the elements of \mathbb{F} such that $a_e f(x) + b_e f(y) + c_e \neq 0$, for every edge $e = xy$. Clearly, \mathbb{F} -colorability of a graph implies its Γ -colorability, where Γ is the additive group of the field \mathbb{F} .

Define a graph G to be \mathbb{F} - k -choosable if it is \mathbb{F} -colorable from arbitrary lists of elements of \mathbb{F} , each of size k , assigned to the vertices.

Theorem 1. *Let G be a graph on n vertices without a minor of K_5 . Let \mathbb{F} be an arbitrary field with at least 5 elements. Then G is \mathbb{F} -5-choosable. Moreover, the number of colorings (for any fixed list assignment and any fixed edge labeling) is at least $5^{n/4}$.*

The proof is based on the method of graph polynomials. We use two tools, the Combinatorial Nullstellensatz of Alon [2] and a result of Alon and Füredi [3] concerning the number of non-zero values of a polynomial evaluated at all points of a multidimensional grid.

Notice that Theorem 1 only covers Abelian groups that are additive groups of a field. For a general Abelian group Γ of order at least 5, Chuang, Lai, Omid, Wang, and Zakeri proved in [6] by elementary methods that every K_5 -minor free graph is Γ -5-choosable. Theorem 1 does not extend this result, but in the overlapping cases guarantees a stronger conclusion, and also easily implies it in the case of arbitrary cyclic groups Γ (Theorem 10).

2. The results

2.1. Graph polynomials

Let G be a simple graph on the set of vertices $V(G) = \{x_1, x_2, \dots, x_n\}$. Let P_G be the graph polynomial of G , defined by

$$P_G(x_1, x_2, \dots, x_n) = \prod_{x_i x_j \in E(G), i < j} (x_i - x_j). \tag{2.1}$$

We identify symbols denoting vertices of G with variables of P_G . We may consider P_G as a polynomial over an arbitrary field \mathbb{F} .

In the process of expanding the polynomial P_G , one creates monomials by picking one variable from each factor $(x_i - x_j)$. Thus, every monomial corresponds to the unique orientation of G obtained by directing the edge $x_i x_j$ towards the picked variable. Thus, the degrees of the variables in the monomial coincide with the in-degrees of the vertices in the corresponding orientation.

Let \mathcal{M}_G denote the multi-set of all monomials arising in this way. So, the cardinality of \mathcal{M}_G is equal to 2^m , where $m = |E(G)|$, and the multiplicity of each monomial M is equal to the number of orientations of G sharing the same in-degree sequence (corresponding to the degrees of the variables in M). The sign of a monomial $M \in \mathcal{M}_G$ is the product of signs of all variables picked to form M . The coefficient of a monomial M in P_G , denoted as $c_M(P_G)$, is the sum of signs of all copies of M in \mathcal{M}_G . A monomial M is called non-vanishing in P_G if $c_M(P_G) \neq 0$.

Suppose now that each edge $e = x_i x_j$ of a graph G (oriented so that $i < j$) is assigned an arbitrary pair (a_e, b_e) of non-zero elements of \mathbb{F} . We say that the edges of G are decorated with pairs (a, b) , and we define the corresponding decorated graph polynomial D_G in which every factor $(x_i - x_j)$ corresponding to the edge $e = x_i x_j$ decorated with (a_e, b_e) is substituted with $(a_e x_i + b_e x_j)$:

$$D_G(x_1, x_2, \dots, x_n) = \prod_{e=x_i x_j \in E(G), i < j} (a_e x_i + b_e x_j). \tag{2.2}$$

Of course, different decorations may give different polynomials, but we denote the whole family of them with the same symbol D_G , hoping that this ambiguity will not cause too much confusion.

2.2. Combinatorial Nullstellensatz

For a monomial M , let $\deg_{x_i}(M)$ denote the degree of the variable x_i in M . The *total degree* of the monomial M is the sum $\sum_{i=1}^n \deg_{x_i}(M)$. In a graph polynomial each monomial has the same total degree equal to the number of edges of G . Recall that the *degree* of a polynomial is the maximum of total degrees of its non-vanishing monomials.

We will use the following famous theorem of Alon [2].

Theorem 2 (Combinatorial Nullstellensatz, [2]). *Let P be a polynomial in $\mathbb{F}[x_1, x_2, \dots, x_n]$, where \mathbb{F} is an arbitrary field of coefficients. Suppose that there is a non-vanishing monomial $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ in P whose total degree is equal to the degree of P . Then, for arbitrary sets $A_i \subseteq \mathbb{F}$, with $|A_i| = k_i + 1$, there exist elements $a_i \in A_i$ such that*

$$P(a_1, a_2, \dots, a_n) \neq 0.$$

In view of this theorem it is convenient to denote by $A_{\mathbb{F}}(P)$ the least integer k such that the polynomial P has a non-vanishing monomial M whose degree is equal to the degree of P and satisfying $\deg_{x_i}(M) \leq k$, for each $i = 1, 2, \dots, n$.

Let D_1, D_2 be any two orientations of the graph G sharing the same in-degree sequence. Observe that the set of edges oriented differently in D_1 than in D_2 is an Eulerian subgraph of both D_1 and D_2 . Thus, every monomial corresponding to an acyclic orientation of G is of multiplicity exactly 1 in \mathcal{M}_G . Such monomials are non-vanishing in P_G , and $A_{\mathbb{F}}(P_G)$ is well defined for every graph G .

For the decorated graph polynomial D_G , which denotes a collection of polynomials, we define $A_{\mathbb{F}}(D_G)$ as the least number k such that $A_{\mathbb{F}}(P) \leq k$ holds for every P in D_G . It is not hard to demonstrate that for every graph G we have

$$A_{\mathbb{F}}(P_G) \leq A_{\mathbb{F}}(D_G) \leq \text{col}(G) - 1,$$

where $\text{col}(G)$ is the *coloring number* of G , defined as the least k such that the vertices of G can be linearly ordered so that each vertex v has at most $k - 1$ neighbors that precede v in the ordering.

2.3. Field coloring and graph polynomials

Let \mathbb{F} be any field and let ℓ be any edge labeling of a graph G by the elements of \mathbb{F} . Let D_G be a decorated graph polynomial with some fixed decoration over \mathbb{F} . Consider the polynomial $D_{G,\ell}$ defined as:

$$D_{G,\ell}(x_1, x_2, \dots, x_n) = \prod_{e=x_i x_j \in E(G)} (a_e x_i + b_e x_j + \ell(e)). \tag{2.3}$$

Our basic observation is formulated as follows.

Theorem 3. *Let G be any simple graph, with a graph polynomial P_G , and let \mathbb{F} be an arbitrary field. Let ℓ be any edge labeling of G by the elements of \mathbb{F} . Then $A_{\mathbb{F}}(D_{G,\ell}) = A_{\mathbb{F}}(D_G)$.*

Proof. First notice that the polynomial $D_{G,\ell}$ can be written as:

$$D_{G,\ell} = D_G + Q, \tag{2.4}$$

where Q is a polynomial of degree strictly smaller than the degree of D_G . Indeed, we obtain the summand D_G by choosing the whole expression $(a_e x_i + b_e x_j)$ in every factor of $D_{G,\ell}$. In every other case, the formed monomial uses at least one constant $\ell(e)$, hence the total degree of such monomial is strictly smaller than the number of edges of G , which is equal to the degree of D_G . This means that every non-vanishing monomial of D_G does not vanish in $D_{G,\ell}$. This completes the proof. \square

2.4. Field coloring of planar graphs

In [21] Zhu proved that every planar graph G satisfies $A_{\mathbb{Q}}(P_G) \leq 4$, though the same proof works for an arbitrary field. This constitutes an algebraic analog of the famous result of Thomassen [16] on 5-choosability of planar graphs. We will derive below a slightly stronger statement.

Theorem 4. *Let G be a planar graph and let D_G be its decorated graph polynomial over an arbitrary field \mathbb{F} . Then $A_{\mathbb{F}}(D_G) \leq 4$.*

By Theorems 3 and 4 we get immediately the following result.

Corollary 1. *Every planar graph is \mathbb{F} -5-choosable, where \mathbb{F} is an arbitrary field with at least 5 elements.*

The original proof from [21] is by induction with the same scenario as in Thomassen’s famous proof from [16], except for one unexpected twist. We will give below a purely algebraic proof along similar lines of the following more general statement, stressing the fact that it works for decorated polynomials over an arbitrary field (which is crucial for our applications).

Theorem 5. *Let G be a near-triangulation and let $e = xy$ be an arbitrary edge of the boundary cycle of G . Then a decorated graph polynomial D_{G-e} over an arbitrary field \mathbb{F} contains a non-vanishing monomial M satisfying the following conditions:*

- (i) $\deg_x(M) = \deg_y(M) = 0$,
- (ii) $\deg_v(M) \leq 2$, for every boundary vertex v ,
- (iii) $\deg_u(M) \leq 4$, for every interior vertex u .

Before we present the proof, let us comment on how Theorem 4 is derived from Theorem 5. We can choose any planar drawing of a planar graph G , and any edge $e = xy$ of the boundary cycle of the drawing. Let M be the non-vanishing monomial in D_{G-e} given by Theorem 5, and observe that the monomial Mx does not vanish in D_G and certifies that $A_{\mathbb{F}}(D_G) \leq 4$.

Proof of Theorem 5. Let us denote $P = D_{G-e}$, and call a non-vanishing monomial M in P satisfying conditions (i)–(iii), a nice monomial for (G, e) . We use induction on the number of vertices of G . It is easy to check that $G = K_3$, a complete graph on vertex set $\{x, y, v\}$, satisfies the assertion. In this case we have $D_{G-e} = (a_1x + b_1v)(a_2y + b_2v)$, and the only nice monomial is $M = v^2$, whose coefficient is $b_1b_2 \neq 0$. Hence, M is non-vanishing.

We distinguish two cases.

Case 1. The boundary cycle of G has a chord.

Suppose first that G has a chord $f = wz$. This chord splits G into two subgraphs G_1 and G_2 . We assume that e is in G_1 , while f belongs to both subgraphs. By induction we assume that both graphs contain nice non-vanishing monomials M_1 and M_2 for (G_1, e) and (G_2, f) , respectively.

Let us denote $P_1 = D_{G_1-e}$, and $P_2 = D_{G_2-f}$. Then we have $P = P_1P_2$, and we see that the monomial $M = M_1M_2$ appears in the expansion of P . We claim that M is a nice monomial for (G, e) . It is easy to see that M satisfies conditions (i)–(iii). To see that it is non-vanishing, notice that $M = M_1M_2$ is the only way of expressing the monomial M as a product of two monomials from P_1 and P_2 , respectively. This is because the only common variables of P_1 and P_2 are w and z , and they do not occur in M_2 . This shows that

$$c_M(P) = c_{M_1}(P_1) \cdot c_{M_2}(P_2) \neq 0,$$

confirming that M is non-vanishing in the polynomial P .

Case 2. The boundary cycle of G has no chord.

Suppose that there is no chord in G . Let $v \neq x$ be the neighbor of y on the boundary of G . Let t be the other neighbor of v on the outer face, and let x_1, x_2, \dots, x_k be the neighbors of v lying in the interior of G . Let $G' = G - v$.

By the inductive assumption, there is a nice monomial M' for (G', e) in the graph polynomial $P' = D_{G'-e}$. So, we have $c_{M'}(P') \neq 0$.

Subcase 1. The boundary face is a triangle.

Suppose first that $t = x$, which means that the boundary face is a triangle. In this case the nice monomial M' has the form

$$M' = Yx_1^{r_1}x_2^{r_2} \dots x_k^{r_k},$$

where $r_i \leq 2$ for each $i = 1, 2, \dots, k$, and Y is a monomial consisting of the rest of the variables. Since M' is nice for (G', e) , the monomial Y does not contain variables x and y , and each other variable z in Y satisfies $\deg_z(Y) \leq 4$.

First notice that $P = P'Q$, where

$$Q = (av + bx)(cv + dy)(a_1v + b_1x_1)(a_2v + b_2x_2) \dots (a_kv + b_kx_k).$$

In the expansion of Q we get the monomial

$$N = v^2x_1 \dots x_k,$$

whose coefficient is $c_N(Q) = acb_1 \dots b_k \neq 0$. Hence, in the expansion of the product $P'Q$ we get the monomial $M = M'N$, which can be written as

$$M = Yx_1^{r_1+1}x_2^{r_2+1} \dots x_k^{r_k+1}v^2.$$

Clearly M satisfies conditions (i)–(iii). We claim that it is also non-vanishing in P . More specifically, we claim that there is only one way of expressing M as a product $M = AB$ of two monomials, with A from P' and B from Q . Indeed, to get v^2 in B we have to use the first two factors of Q , since otherwise we have x or y in M . This implies that $B = N$ and $A = M'$. Thus

$$c_M(P) = c_{M'}(P') \cdot c_N(Q) \neq 0,$$

which shows that M is non-vanishing in P .

For the remaining subcases, we assume that $t \neq x$.

Subcase 2. There is a special monomial.

Suppose first that there exists a non-vanishing *special* monomial S in P' which satisfies all conditions (i)–(iii), except that $\deg_t(S) \leq 1$ and $\deg_{x_i}(S) \leq 3$ for at most one i . We may assume without loss of generality that $i = 1$. So, we assume that $c_S(P') \neq 0$.

This special monomial S can be written in the form

$$S = Zx_1^{s_1}x_2^{s_2} \dots x_k^{s_k}t^s,$$

where $s \leq 1$, $s_1 \leq 3$, $s_i \leq 2$ for $i = 2, 3, \dots, k$, and Z is some monomial consisting of the rest of the variables. As before we may write the polynomial $P = D_{G-e}$ as the product $P = P'Q$ with

$$Q = (av + by)(cv + dt)(a_1v + b_1x_1) \dots (a_kv + b_kx_k).$$

In the expansion of Q we get the monomial

$$N' = vtx_1 \dots x_k,$$

with coefficient $c_{N'}(Q) = adb_1 \dots b_k \neq 0$. Hence, in the expansion of the product $P'Q$ we get the monomial $M = SN'$, which can be written as

$$M = ZX_1^{s_1+1}x_2^{s_2+1} \dots x_k^{s_k+1}t^{s+1}v.$$

Clearly, M satisfies conditions (i)–(iii). Also, as in the previous case, the splitting $M = SN'$ is unique. Indeed, assume that $M = AB$ is any decomposition of M into the product of monomials from P' and Q , respectively. To get v in B we have to use the first factor of Q , since otherwise we have y in M . This already implies that $B = N'$ and $A = S$. Hence,

$$c_M(P) = c_S(P') \cdot c_{N'}(Q) \neq 0,$$

so, M is non-vanishing in P .

Subcase 3. There is no special monomial.

Finally, assume that there is no special monomial in P' . However, by inductive assumption, there is still a nice non-vanishing monomial M' in the graph polynomial $P' = D_{G'-e}$. This monomial can be written now as

$$M' = Yx_1^{r_1}x_2^{r_2} \dots x_k^{r_k}t^r,$$

where $r \leq 2$, $r_i \leq 2$, for all $i = 1, 2, \dots, k$, and Y is a monomial consisting of the rest of the variables.

As in Subcase 1, we have $P = P'Q$, where

$$Q = (av + by)(cv + dt)(a_1v + b_1x_1)(a_2v + b_2x_2) \dots (a_kv + b_kx_k).$$

In the expansion of Q we get the monomial

$$N = v^2x_1 \dots x_k,$$

with $c_N(Q) = acb_1 \dots b_k \neq 0$. Hence, in the expansion of the product $P'Q$ we get the monomial $M = M'N$, which can be written as

$$M = Yx_1^{r_1+1}x_2^{r_2+1} \dots x_k^{r_k+1}t^r v^2.$$

Clearly M satisfies conditions (i)–(iii). We claim that there is only one way of expressing M as a product of two monomials, $M = AB$, with A from P' and B from Q . Indeed, to get v^2 in B we have to choose variable v exactly twice from the factors of Q . The first choice must be from the first factor, otherwise y appears in M . The second choice must be from the second factor, since otherwise the variable t appears in B , while some x_i is missing. Then, in order to get $M = AB$, we would have to have t^{r-1} and $x_i^{r_i+1}$ in the monomial A . But then A is a special monomial in P' , contrary to our assumption. Hence, we must have $B = N$ and $A = M'$. Thus

$$c_M(P) = c_{M'}(P') \cdot c_N(Q) \neq 0,$$

which demonstrates that M is non-vanishing in P .

The proof is complete. \square

In [8] Grytczuk and Zhu proved that every planar graph G contains a matching S such that $A_{\mathbb{F}}(P_{G-S}) \leq 3$. The proof is similar to the above and can be easily modified to give the following result.

Theorem 6. Every planar graph G contains a matching S such that $A_{\mathbb{F}}(D_{G-S}) \leq 3$, for an arbitrary field \mathbb{F} . Thus, $G - S$ is \mathbb{F} -4-choosable, and in particular, $\mathbb{Z}_2 \times \mathbb{Z}_2$ -colorable.

2.5. K_5 -minor-free graphs

To extend the above results to graphs without a K_5 -minor we will use the well-known characterization theorem of Wagner [18]. A similar approach was taken by Abe, Kim, and Ozeki [1] in an extension of the result of Zhu [21] to graphs with no K_5 -minor.

Recall that a k -clique-sum of two graphs is a new graph obtained by gluing the two graphs along a clique of size k in each of them, and possibly deleting some edges of the clique. Recall also that the Wagner graph V_8 is the graph obtained from the cycle C_8 by adding four edges joining antipodal pairs of vertices.

Theorem 7 (Wagner, [18]). *Every edge-maximal graph without a K_5 -minor can be built recursively from planar triangulations and the graph V_8 by clique-sums with cliques on at most 3 vertices.*

We need the following result.

Theorem 8. *Let G be a plane triangulation and let T be any triangle in G . Then the decorated graph polynomial $D_{G-E(T)}$ over an arbitrary field \mathbb{F} contains a non-vanishing monomial N such that $\deg_u(N) = 0$, for every vertex $u \in V(T)$, and $\deg_w(N) \leq 4$, for all other vertices.*

Proof. Let $V(T) = \{x, y, v\}$ and denote $e = xy$. Suppose first that T is a facial triangle. We may assume that T is the outer face of G . By Theorem 5 we know that there is a non-vanishing monomial M in D_{G-e} such that $\deg_x(M) = \deg_y(M) = 0$, $\deg_v(M) \leq 2$, and $\deg_w(M) \leq 4$, for all other variables. In order to form this monomial we have to pick v exactly twice; once from each factor, $(ax + bv)$ and $(cy + dv)$, since we can choose neither x , nor y . Thus, when we delete the corresponding two edges xv and yv from the graph $G - e$, we must have a non-vanishing monomial $N = M/v^2$ in the polynomial $D_{G-E(T)}$.

If T is not a facial triangle, then we may split the triangulation G into two sub-triangulations, G_1 and G_2 , lying inside and outside the triangle T , respectively. Then we may apply the same argument to each sub-triangulation separately to get the desired monomials N_1 and N_2 in polynomials $D_{G_1-E(T)}$ and $D_{G_2-E(T)}$, respectively. Clearly, we have

$$D_{G-E(T)} = D_{G_1-E(T)}D_{G_2-E(T)},$$

and it is easy to see that $N = N_1N_2$ is a non-vanishing monomial in $D_{G-E(T)}$ satisfying the assertion of the theorem. \square

Now we may give the proof of the aforementioned extension of Theorem 4.

Theorem 9. *Let G be a graph without a K_5 -minor and let D_G be its decorated graph polynomial over an arbitrary field \mathbb{F} . Then $A_{\mathbb{F}}(D_G) \leq 4$.*

Proof. Let G be an edge-maximal K_5 -minor-free graph. We proceed by induction on the number of terms in a clique-sum giving G . So, suppose that G is k -clique-sum, $k \leq 3$, of two graphs H and F , where H is a clique-sum with a smaller number of terms, while F is a plane triangulation or $F = V_8$. Assume by induction that $A_{\mathbb{F}}(D_H) \leq 4$, and let M be the monomial witnessing this inequality with coefficient $c_M(D_H) \neq 0$.

In the triangulation case, let $\{x, y, z\}$ be the three vertices of the common triangle T in H and F . Let N be a monomial in $D_{F-E(T)}$ guaranteed by Theorem 8 with coefficient $c_N(D_{F-E(T)}) \neq 0$. We claim that the monomial MN occurs in the polynomial D_G with coefficient

$$c_{MN}(D_G) = c_M(D_H) \cdot c_N(D_{F-E(T)}).$$

Indeed, we have an obvious equality $D_G = D_H D_{F-E(T)}$ and the only common variables of the two polynomial factors are x, y, z , none of which appears in the monomial N .

If $F = V_8$, then the reasoning is similar. Notice that V_8 is triangle-free, so the clique-sum can be made on one vertex or one edge. Suppose it is the latter situation (the former is even easier). Let x, y be the two common vertices of H and F . It is enough to notice that for every edge $e = xy$ of V_8 there is an acyclic orientation of $V_8 - e$ with in-degrees of both vertices x and y equal to 0. The monomial J corresponding to this orientation has a non-zero coefficient, the variables x and y do not occur in J , while other variables have degrees at most 3. Thus, as before we have

$$c_{MJ}(D_G) = c_M(D_H) \cdot c_J(D_{F-e}) \neq 0.$$

This completes the proof. \square

By Theorems 9 and 3 we get immediately the following results, whose special case extends \mathbb{Z}_5 -colorability of planar graphs.

Corollary 2. *Let \mathbb{F} be an arbitrary field with at least 5 elements. Then every graph G without a K_5 -minor is \mathbb{F} -5-choosable.*

2.6. Extending to cyclic group choosability

The polynomial method is tied to an ambient field. One way of extending the potential results to cyclic groups is to use cyclic subgroups of multiplicative groups of fields with appropriate order and to express the conditions on the coloring using multiplication instead of addition. By this trick we get the following result.

Theorem 10. *Let Γ be an arbitrary cyclic group of order at least 5. Then every K_5 -minor free graph G is Γ -5-choosable.*

Proof. Let Γ be a (multiplicative) cyclic group of order $m \geq 5$. Let p be a prime number such that $\gcd(m, p) = 1$. Consider the field $F = \mathbb{F}_{p^{\phi(m)}}$. Its multiplicative group F^* is the cyclic group of order $p^{\phi(m)} - 1$. Since m divides $p^{\phi(m)} - 1$ (by Euler’s theorem), Γ is a subgroup of F^* .

Let ℓ be a labeling of the edges of a graph G by the elements of Γ . Denote by $\ell_{ij} = \ell(x_i x_j)$ the label of an edge $x_i x_j$ in G . For any fixed orientation \vec{G} of G , let d_i be the in-degree of the vertex x_i . Consider now the following function

$$f_G(x_1, x_2, \dots, x_n) = \prod_{(x_i, x_j) \in E(\vec{G})} (x_i x_j^{-1} - \ell_{ij}) = \frac{1}{\prod_{i=1}^n x_i^{d_i}} \prod_{(x_i, x_j) \in E(\vec{G})} (x_i - \ell_{ij} x_j).$$

By Theorem 9, the polynomial

$$P_G(x_1, x_2, \dots, x_n) = \prod_{(x_i, x_j) \in E(\vec{G})} (x_i - \ell_{ij} x_j)$$

has a nonvanishing monomial of degree at most four. It follows that, for every collection of sets $A_1, A_2, \dots, A_n \subseteq \Gamma \subseteq F^*$, each with cardinality at least five, there is a point (a_1, a_2, \dots, a_n) in $A_1 \times A_2 \times \dots \times A_n$ where P_G is not vanishing, which implies that $f_G(a_1, a_2, \dots, a_n) \neq 0$. It follows that (a_1, a_2, \dots, a_n) is a Γ -coloring of G for the labeling ℓ . \square

2.7. The number of colorings

In this section we prove the second part of Theorem 1. Our main tool is the following general result of Alon and Füredi [3].

Theorem 11 (Alon and Füredi, [3]). *Let \mathbb{F} be an arbitrary field, let A_1, A_2, \dots, A_n be any non-empty subsets of \mathbb{F} , and let $B = A_1 \times A_2 \times \dots \times A_n$. Suppose that $P(x_1, x_2, \dots, x_n)$ is a polynomial over \mathbb{F} that does not vanish on all of B . Then, the number of points in B for which P has a non-zero value is at least $\min \prod_{i=1}^n q_i$, where the minimum is taken over all integers q_i such that $1 \leq q_i \leq |A_i|$ and $\sum_{i=1}^n q_i \geq \sum_{i=1}^n |A_i| - \deg P$.*

For a convenient use of this result, and for the sake of completeness, we will prove a slightly weaker statement by an argument resembling a beautiful proof of Combinatorial Nullstellensatz, due to Michałek [13]. A similar approach was taken by Bishnoi, Clark, Potukuchi, and Schmitt [4] to get some generalization of the Alon-Füredi theorem.

We need a simple technical lemma.

Lemma 1. *Let a_1, a_2, \dots, a_n be positive integers, with $\max a_i = t \geq 2$ and $\sum_{i=1}^n a_i = S$. Then*

$$A = \prod_{i=1}^n a_i \geq t^{\frac{S-n}{t-1}}. \tag{2.5}$$

Proof. The proof is by induction on n . For $n = 1$ we have $A = a_1 = t$ and $S = a_1 = t$, hence we get equality in (2.5). For $n \geq 2$, let $a_{i_0} = \min a_i = m$. Then, by the inductive assumption, we have

$$\frac{A}{a_{i_0}} = \frac{A}{m} \geq t^{\frac{(S-m)-(n-1)}{t-1}}.$$

Observe that for every $x \in [1, t]$ we have

$$x \geq t^{\frac{x-1}{t-1}}. \tag{2.6}$$

Indeed, it is not hard to check that the function $f(x) = t^{(x-1)/(t-1)}$ is convex, with $f(1) = 1$ and $f(t) = t$. Hence, taking $x = m$, we may write

$$A \geq m \cdot t^{\frac{(S-m)-(n-1)}{t-1}} \geq t^{\frac{m-1}{t-1}} \cdot t^{\frac{(S-m)-(n-1)}{t-1}} = t^{\frac{S-n}{t-1}},$$

as asserted. \square

Theorem 12. *Let \mathbb{F} be an arbitrary field, and let A_1, A_2, \dots, A_n be any non-empty subsets of \mathbb{F} , with $S = \sum_{i=1}^n |A_i|$ and $t = \max |A_i|$. Let $B = A_1 \times A_2 \times \dots \times A_n$ and suppose that $P(x_1, x_2, \dots, x_n)$ is a polynomial over \mathbb{F} of degree $\deg P = d$, that does not vanish on all of B . Then, the number of points in B for which P has a non-zero value is at least*

$$t^{\frac{S-n-d}{t-1}},$$

provided that $S \geq n + d$ and $t \geq 2$.

Proof. The proof is by induction on d and n . If $d = 0$, then P equals some non-zero constant $c \in \mathbb{F}$, and therefore all points of B are non-vanishing for P . There are exactly $\prod_{i=1}^n |A_i|$ of them, so, the assertion follows from Lemma 1 by putting $a_i = |A_i|$.

For $n = 1$ and arbitrary $d \geq 1$, we know that the number of roots of a polynomial P is at most d . Hence, the number of elements of A_1 for which P is non-zero is at least $t - d$. So, by the assumption that $t \geq d + 1$ and the inequality (2.6), we have

$$t - d \geq t^{\frac{t-1-d}{t-1}} = t^{\frac{S-1-d}{t-1}},$$

as $S = t$ in this case.

Assume now that $d \geq 1$ and $n \geq 2$. Let $|A_1| = j$, and assume, without loss of generality, that there is another set A_i , with $|A_i| = t$. Let $b \in A_1$ be any element, and let us divide the polynomial P by $(x_1 - b)$:

$$P = (x_1 - b)Q_b + R_b.$$

Observe that $\deg Q_b = \deg P - 1 = d - 1$, $\deg R_b \leq d$, and that the polynomial R_b does not contain the variable x_1 .

Suppose first that the polynomial R_b vanishes at all points of the grid $A_2 \times \dots \times A_n$. This implies that $j \geq 2$, since otherwise, the polynomial P would vanish over the whole grid B , contrary to the assumption. Furthermore, each non-vanishing point of P in B is at the same time a non-vanishing point of Q_b in the grid $(A_1 - b) \times A_2 \times \dots \times A_n$, and vice versa. Thus, by the inductive assumption we get that P has at least

$$t^{\frac{(S-1)-n-(d-1)}{t-1}} = t^{\frac{S-n-d}{t-1}}$$

non-vanishing points in B .

Finally, suppose that for every $b \in A_1$, the polynomial R_b has some non-vanishing points in the grid $A_2 \times \dots \times A_n$. Then each such point can be extended to a non-vanishing point of P by setting $x_1 = b$. By the inductive assumption on n , the number of such points for each R_b is at least

$$t^{\frac{(S-j)-(n-1)-d}{t-1}}.$$

Hence, the total number of non-vanishing points of P in the grid B is at least

$$j \cdot t^{\frac{(S-j)-(n-1)-d}{t-1}} \geq t^{\frac{j-1}{t-1}} \cdot t^{\frac{(S-j)-(n-1)-d}{t-1}} = t^{\frac{S-n-d}{t-1}},$$

by the inequality (2.6). The proof is complete. \square

The above result and Theorem 9 easily imply the following corollary.

Corollary 3. *Let G be a graph on n vertices with no K_5 -minor. Let \mathbb{F} be an arbitrary field with at least 5 elements. Then for every assignment of lists of size 5 from \mathbb{F} to the vertices of G and every edge labeling ℓ , there are at least $5^{n/4}$ different \mathbb{F} -colorings of G from these lists.*

Proof. Let G be a graph satisfying assumptions of the corollary and let P_G be its polynomial. Let $A_i \subseteq \mathbb{F}$ be lists of size 5 assigned to the vertices of G . So, keeping notation from [Theorem 12](#), we have $t = 5$ and $S = 5n$. Since the number of edges of G is at most $3n - 6$ (by [Theorem 7](#)), we have $d = \deg P_G \leq 3n - 6$. Hence, the number of \mathbb{F} -colorings is at least

$$t^{(S-n-d)/(t-1)} \geq 5^{(5n-n-(3n-6))/4} = 5^{(n+6)/4} > 5^{n/4},$$

as asserted. \square

3. Discussion

Let us conclude the paper with some further observations and possible extensions of our results.

First, let us point on other improvements of existing results one may easily get by using graph polynomials. For instance, in [\[17\]](#) Thomassen proved that every planar graph on n vertices has at least $2^{n/9}$ colorings from arbitrary lists of size 5. By a direct application of the Alon-Füredi theorem one may improve this bound to $5^{n/4}$, arguing in the same way as in the proof of [Corollary 3](#). A similar improvement can be made for planar graphs of girth at least 5. Thomassen proved that every such graph G with n vertices has at least $2^{n/10000}$ colorings from lists of size 3, while the Alon-Füredi theorem gives $3^{n/6}$.

Another striking consequence can be formulated in connection to the famous theorem of Brooks [\[5\]](#) which asserts that every connected graph G of maximum degree Δ is Δ -colorable, except for the two cases, when G is a clique or an odd cycle (see [\[19\]](#) for a short proof of some more general versions). By [Theorem 12](#) we get that the number of colorings in this case is at least

$$(\sqrt{\Delta})^{n(1-\frac{1}{\Delta-1})}.$$

By a different argument based on the probabilistic method, this can be improved to roughly

$$(\Delta/e)^{n(1-\frac{\log \Delta}{\Delta})},$$

as noticed by Alon (personal communication).

Another advantage of using graph polynomials is that any upper bound on $A_{\mathbb{F}}(P_G)$ not only gives an upper bound for the choosability of G , but also for its *game* variant, known as *paintability* or *online choosability*. Roughly speaking, a graph G is *k-paintable*, if Alice, a color-blind person, can color the vertices of G from any lists of size k in a sequence of rounds; in each round a new color is highlighted in the lists and Alice must decide which vertices will be painted by this color. This idea was introduced independently by Schauz [\[15\]](#) and Zhu [\[20\]](#). It is clear that every *k-paintable* graph is *k-choosable*, but not the other way around. In [\[14\]](#) Schauz proved that every graph G satisfying $A_{\mathbb{F}}(G) \leq k - 1$ is *k-paintable* (see [\[7\]](#) for a different, purely algebraic proof). Hence, all our results for field choosability transfer to field paintability in a direct way.

Finally, let us mention about the intriguing conjecture relating group coloring and list coloring of graphs, stated by Král', Pangrác, and Voss in [\[10\]](#). They suspect that if a graph G is Γ -colorable for every Abelian group of order at least k , then G is *k-choosable*. We offer the following, somewhat provocative, strengthening of this statement.

Conjecture 1. *If a graph G is Γ -colorable for every Abelian group Γ of order at least k , then $A_{\mathbb{Q}}(P_G) \leq k - 1$. This would imply that G is *k-paintable* and *k-choosable*.*

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