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Scouting Knots Are Not the Same Knots When Knotted

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SCOUTING KNOTS ARE NOT THE SAME KNOTS WHEN KNOTTED

ANDRES L. SANCHEZ

55 Pages

Knots are used everyday by all people, mathematicians and non-mathematicians alike. There are certain subsets of people that use knots more frequently than others such as sailors or members of the Scouting Movement.

A mathematical knot is a subset of 3-space that is homeomorphic to the unit circle. A knot used in non-mathematical settings is generally considered when two strings, etc. are wrapped around each other, with the ends potentially hanging. We will call these practical knots. We introduce and explain through examples 1) how mathematical knots differ from practical knots; 2) how mathematical knots differ from other mathematical knots via methods of deformation, colorability, and polynomials; 3) apply mathematical methods of knot determinations to practical knots to see how practical knots might behave as mathematical knots.

KEYWORDS: Knot Theory, Knots, Invariants, Topology

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SCOUTING KNOTS ARE NOT THE SAME KNOTS WHEN KNOTTED

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A.L.S.

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TABLES

Over, under, around the tree; swing past the knothole, pull and see.

CHAPTER I: SCOUTING KNOTS

History

The Scouting Movement was founded in 1908 in the United Kingdom by Lord Robert Baden-Powell. Baden-Powell was a lieutenant general in the British Army, and took it upon himself to write a book for boys about reconnaissance and scouting. This first book was called *Scouting for Boys*. It is now regarded as the first Scouting handbook and part of the start of the Scouting Movement.

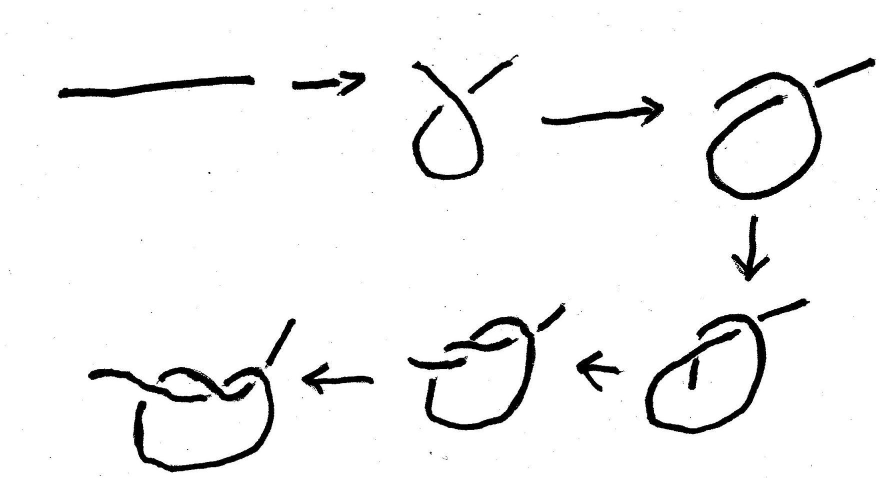
In the third chapter of the book titled "Camp Life," we are first introduced to the importance of knots and knot tying. Baden-Powell stresses how being able to tie knots can be literally life-saving skills through previous life experiences. He goes on to list useful knots, lashings, and splices with their pictures and uses.

He does not give a clear definition of a knot, lashing, or splice, rather just presuming what we all know: a knot is some type of complication in a rope, a lashing is when you bind together two or more poles with a rope, and a splice intertwines the threads a rope or multiple ropes together. He lists eleven knots, three lashings, and three splices. Some of the useful knots he lists are the Square Knot, Clove Hitch, Two Half Hitches, and Bowline.

In this paper, we will examine these knots from Baden-Powell's original *Scouting for Boys* as well as the Overhand Knot and Granny Knot.

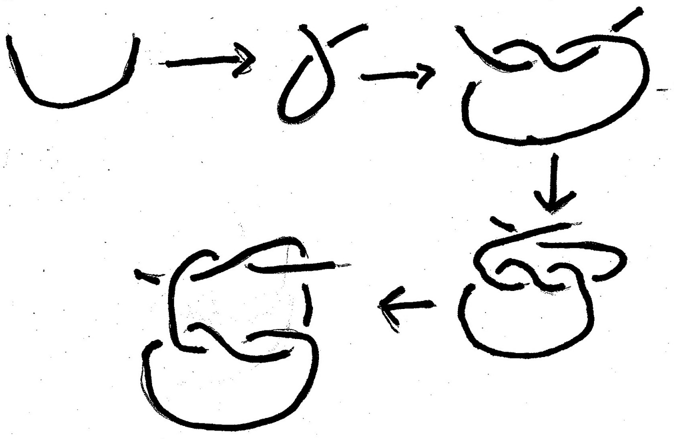
Diagrams of Construction

The Overhand Knot is the simplest practical knot. It requires two strands of rope to be tied. These strands can either be from the same rope or from two different ropes. When tied with the two ends of a single rope, the knot can act as a stopper not in the middle of the rope. First is the diagram of tying with two ends of the same rope:

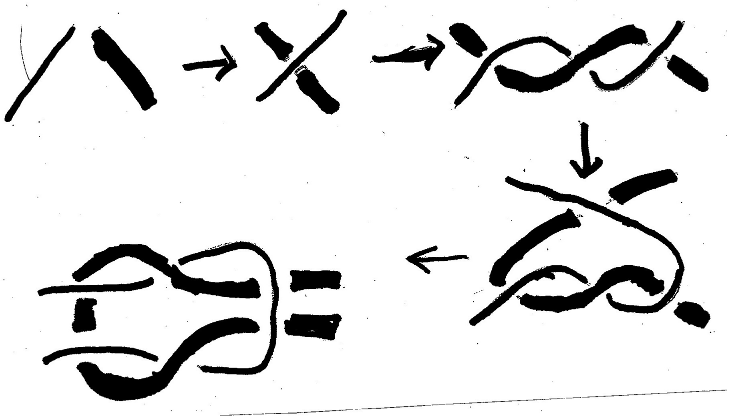


Here is a diagram tying with two strands of two different ropes:

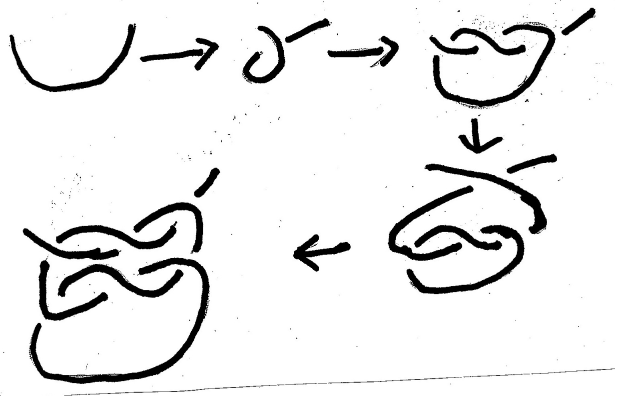
The Square Knot is one of the most important knots in Scouting. This knot combines two ropes together. This can also be tied with either a single rope or two ropes. This is the diagram for tying the square knot with the strands of a single rope:



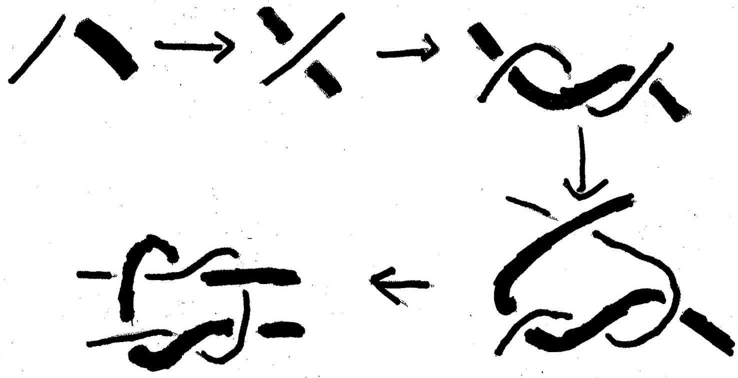
Here is a diagram tying with two different ropes.



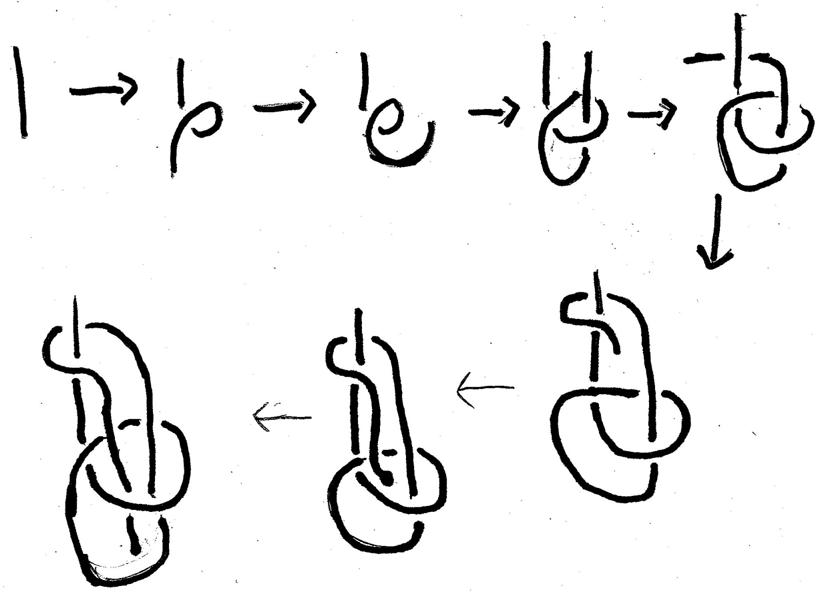
The Granny Knot is usually the result of a mis-tied square knot. This knot combines two ropes together. This can also be tied with either a single rope or two ropes. This is the diagram for tying the granny knot with the strands of a single rope:



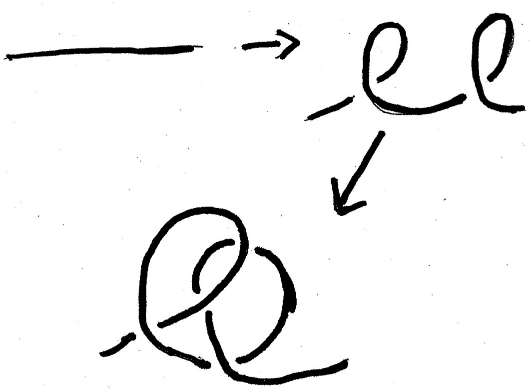
Here is a diagram tying with two different ropes.



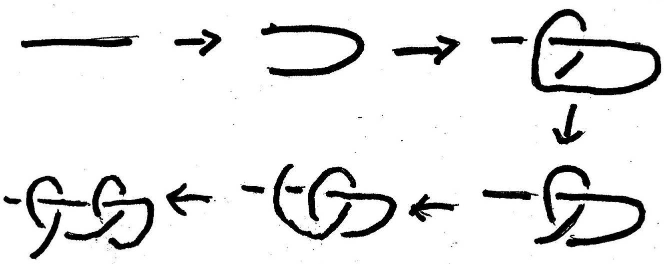
The Bowline is a knot that forms a loop that will not slip when pressure is applied. This is tied on a single rope:



The Clove Hitch can be used to start most lashings.



Two Half Hitches form a loop that can be adjusted to be larger or smaller. It is usually attached to a pole.

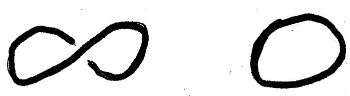


CHAPTER II: MATHEMATICAL KNOTS

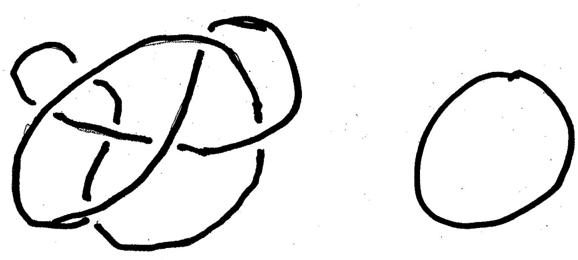
History

The study of knots did not arise from mathematics, but rather from chemistry. In the late 1800s, it was believed that a substance called aether pervaded all of space. In his study of aether, Lord Kelvin hypothesized that atoms were knots in this aether. Thus, different knots would then correspond to different elements. This led to an attempt to list all possible knots by physicist Peter Tait. If Tait could list all knots, then he would essentially be creating a list of elements. However, Kelvin's hypothesis was later proven to be false by the 1887 Michelson-Morley experiment which showed there was no aether. The interest in knots by chemists waned and wouldn't be reinvigorated until the 1980s when knots were used to model molecules. However, mathematicians stuck with knots and built out the mathematical theory of them.

Tait viewed two knots as equivalent if one could be deformed into the other. For now, let's say a knot is simply a loop of rope. In some cases this is easy to see how two knots can be equivalent.

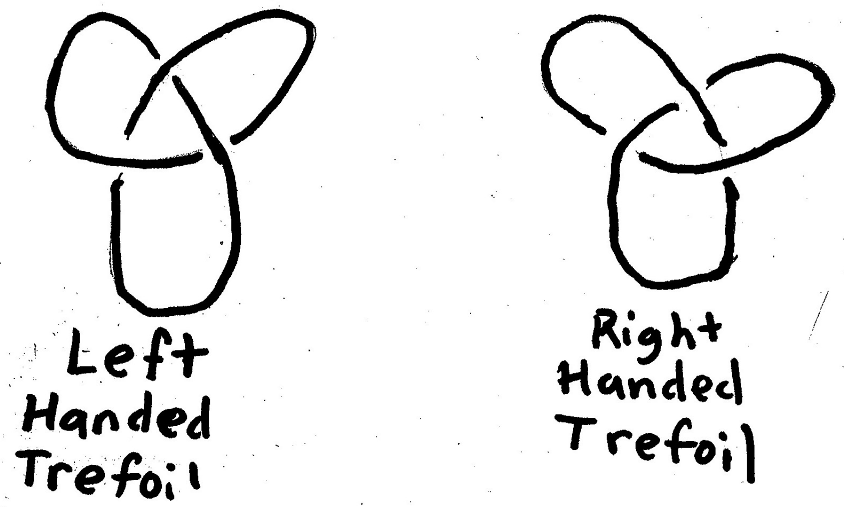


But, as knots become more twisted, it becomes harder to convince oneself that two knots are or are not equivalent.



These knots may look like they will never be deformed to each other, and perhaps after hours of trying, you become convinced they cannot be deformed into each other. But, maybe you are just missing a simple manipulation that will unlock the whole mystery. As Tait was compiling his list of knots, the question of whether two knots were distinct from each other became the main problem introduced by Tait and one of the key motivations behind knot theory.

In the early 1900s, the development of the field of topology became an important turning point in work on knots. With topology now providing firm mathematical grounding, mathematicians were able to define the objects of knot theory precisely, and start to prove theorems about them. This led to algebraic methods being introduced and providing means to prove that knots were actually distinct. In 1914, Max Dehn proved for the first time that a left-handed and right-handed trefoil knot are distinct.



In 1926, Kurt Reidemeister proved that if we have two projections of the same knot, we can deform one projection into the other by series of moves, appropriately called Reidemeister moves. In 1928, James Alexander discovered the first knot polynomial, now called the Alexander polynomial. This invariant was an important and impactful development, delivering if one knot can be deformed into another knot then both knots will have the same polynomial.

Reidemeister published the first book about knots, *Knotentheorie*, in 1932. By this time, knot theory was fairly well-developed. Soon after *Knotentheorie*, Herbert Seifert showed that if a knot is the boundary of a surface in 3-space, then that surface can be used to study the knot; he also presented an algorithm to construct a surface bounded by any given knot. This approach laid the foundation for the use of geometric methods to be used in knot theory. Up until that point, the study of knots was dominated by algebra and combinatorics. In 1947, Horst Schubert used geometric methods to prove that any knot can be decomposed uniquely as the connected sum of prime knots. A knot is prime if it cannot be decomposed as a connected sum of nontrivial knots.

While most of the work in knot theory up to that point dealt with distinguishing knots from each other, the idea of proving that one knot *can* be deformed into another was largely untouched. In 1957, Christos Papakyriakopoulos proved Dehn's Lemma, which roughly stated that if a knot were indistinguishable from the trivial knot using algebraic methods, then the knot was in fact trivial. Papakyriakopoulos's proof became the centerpiece of major developments in knot theory. In 1968, Friedhelm Waldhausen proved that two knots are equivalent if and only if certain algebraic data associated to the knots are the same. The interplay between algebra and geometry was essential to this work.

The late 1950s through the 1970s showed extensive study of classical knot invariants, and, in particular, how properties of the knot were reflected in the invariants. At the same time, the study of higher dimensional knots also became an area of interest. In 1960, the subject of higher dimensional knots had little more than a sparse collection of examples. By 1970, it had become a robust area of topology.

Since 1970, knot theory has seen great progression. John Conway introduced new combinatorial methods which, when combined with works by Vaughan Jones, led to new families of invariants. William Thurston introduced new geometric methods using hyperbolic geometry. In 1988, Cameron Gordon and John Luecke proved that knots with equivalent complements are themselves equivalent. After Jones's discovery of the Jones polynomial in 1984, other knot polynomials were discovered such as the bracket polynomial, HOMFLY polynomial, and the Kauffman polynomial. Most recently scientists have found applications of knot theory to problems in biology and chemistry examining molecules. Knot theory has also found its place in physics.

Knot theory remains a robust field of study to this day, with wide-ranging applications and many questions still left to be answered.

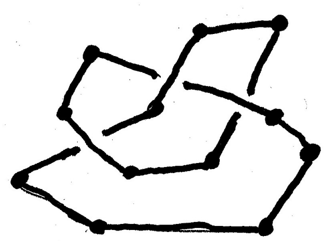
Definitions and Beginning Invariants

Imagine a piece of rope that we have tied into a knot in some way. Now we connect the ends of the rope together. A better image to use might be an extension cord, where you can plug the end of the extension cord back into itself. Now, we want to disregard the thickness of that rope or extension cord. This twisted up object that we now hold can be thought of as a mathematical knot. This is a good simple, intuitive image to hold onto as we move forward with defining a knot.

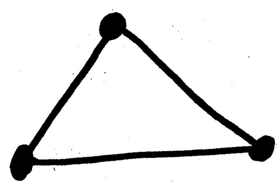
For any two distinct points in 3-space, p and q, let $[p, q]$ denote the line segment joining them. For an ordered set of distinct points, $(p_1, p_2, ..., p_3)$, the union of the segments $[p_1, p_2], [p_2, p_3], ..., [p_{n-1}, p_n]$ and $[p_n, p_1]$ is called a closed polygonal curve. If each segment intersects exactly two other segments, intersecting each only at an endpoint, then the curve is said to be simple.

Definition 1. A knot is a simple closed polygonal curve in $\mathbb{R}^3$.

The figure below shows the simplest nontrivial knot, the trefoil, drawn as a polygonal curve.



The unknot, or trivial knot, is defined to be the knot determined by three noncollinear points. The figure below shows the unknot as a polygonal curve.



Knots are usually thought of as smooth curves and not as the above drawings with distinct segments. We can view smooth knots as polygonal knots constructed from a very large number of segments. The formal way to reconcile this problem is to study the relationship between polygonal and differential knots. This will not be covered in this paper. Thus, for aesthetic and traditional reasons, all knots will be drawn as smooth knots. However, we note that all the knots could be drawn polygonally.

Since knots are defined to be a subset of 3-space, there are some clarifications that must be made. Different choices of ordered sets of points can define the same knot. For example, cyclicly permuting the order of the points does not alter the underlying knot. Further, if three consecutive points are collinear, then eliminating the middle point does not change the underlying knot.

Definition 2. If the ordered set $(p_1, p_2, ..., p_n)$ defines a knot, and no proper ordered subset defines the same knot, the elements of the set p_i are called the vertices of the knot.

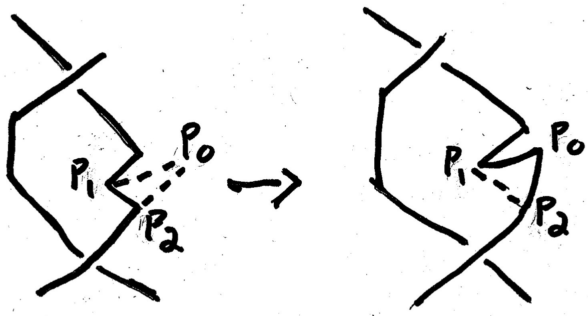
In the study of knots, it is inevitable that one will come across links as well.

Definition 3. A link is the finite union of disjoint knots. (In particular, a knot is a link with one component.) The unlink is the union of unknots all lying in a plane.

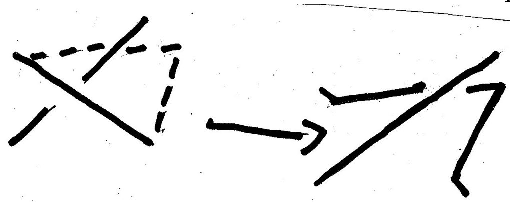
We next want to examine how we deform knots.

Definition 4. A knot J is called an elementary deformation of the knot K if one of the two knots is determined by a sequence of points $(p_1, p_2, ..., p_n)$ and the other is determined by the sequence $(p_0, p_1, p_2, ..., p_n)$, where (1) p_0 is a point which is not collinear with p_1 and p_2, and (2) the triangle spanned by (p_0, p_1, p_n) intersects the knot determined by $(p_1, p_2, ..., p_n)$ only in the segment $[p_1, p_n]$.

In this case, a triangle is a flat surface bounded by the edges $[p_0, p_1]$, $[p_1, p_n]$, and $[p_n, p_0]$. It is defined formally as $T = \{xp_0 + yp_1 + zp_n | 0 \leq x, y, z, \text{ and } x + y + z = 1\}$. The second condition is to ensure the deformation of the knot does not cross itself. The below figure shows an example of an elementary deformation.



Below shows a deformation which is not permitted.



Knots K and J are called equivalent if K can be changed into J by performing a series of elementary deformations. More precisely:

Definition 5. Knots K and J are called equivalent if there is a sequence of knots $K = K_0, K_1, ..., K_n = J$, with each K_{i+1} an elementary deformation of K_i for i greater than 0.

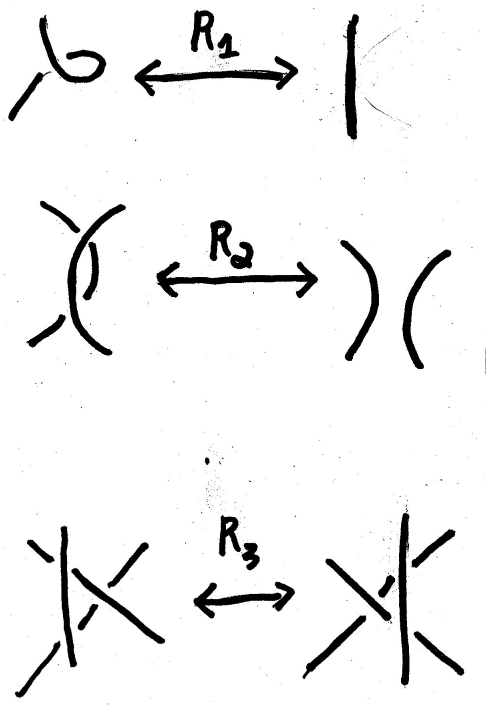
Knots can also be oriented, or in other words, given a sense of direction.

Definition 6. An oriented knot consists of a knot and ordering of its vertices. The ordering must be chosen so that it determines the original knot. Two orderings are considered equivalent if they differ by a cyclic permutation.

Orientation of a knot is usually represented by placing an arrow on its diagram.

Definition 7. Oriented knots are called oriented equivalent if there is a sequence of elementary deformations carrying one oriented knot to the other.

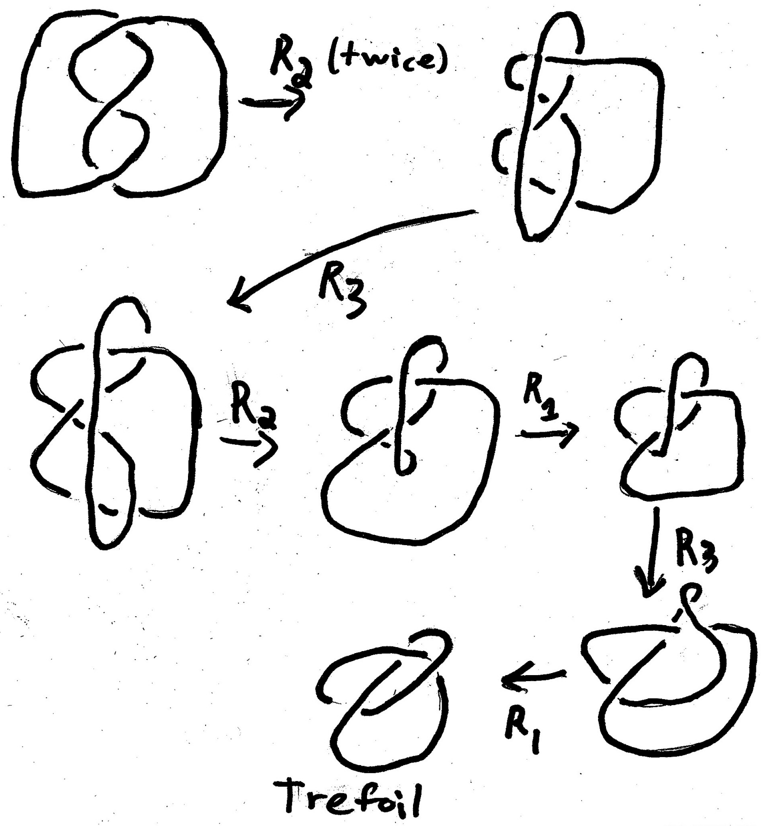
Now we will describe the Reidemeister moves that were mentioned earlier. Reidemeister proved that all such changes of knot or link diagrams can be obtained by performing three basic motions applied to small portions of the diagrams near crossings, along with simple deformations in the plane, called plane isotopies, which do not change any of the crossings of diagrams. The three Reidemeister moves are denoted by R_1, R_2, and R_3 and are illustrated below.



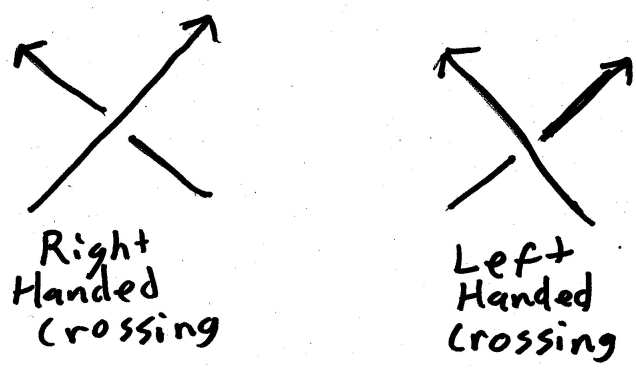
These three Reidemeister moves, six moves if considering both directions, can be performed on a knot diagram without altering the corresponding knot. This leads to our first theorem.

Theorem 8. If two knots (or links) are equivalent, their diagrams are related by a sequence of Reidemeister moves.

Here is an example of Reidemeister moves converting one diagram into another.



We now look at oriented links. A link is oriented if each of its component knots is oriented. Of particular interest when considering a diagram for an oriented link of two components are the crossing points at which the two components meet. We will call a crossing point right-handed if an observer stationed on the overpassing arc and facing in the direction of that arc observes the underpassing arc's direction as from right to left. Otherwise, we will call the crossing point left-handed.

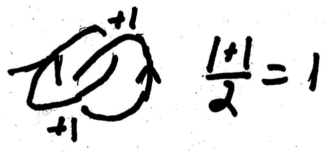


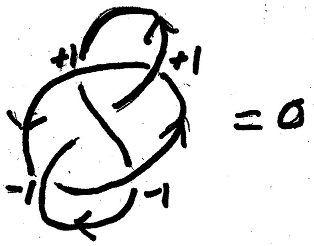
These types of crossings are involved in the definition of the linking number of a link of two components.

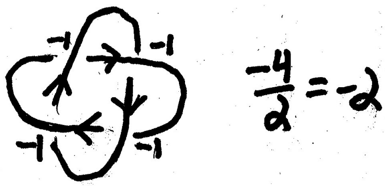
Definition 9. The linking number of an oriented link of two components is computed according to the following steps, applied to a diagram of the link:

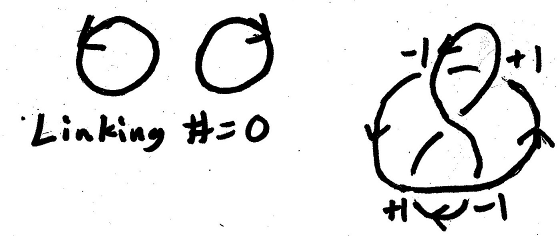
(a) To each crossing point at which the components meet assign the integer +1 if the crossing point is right-handed and the integer -1 if the crossing point is left-handed.

(b) The linking number of the oriented link is the sum of all the +1's and -1's divided by 2, unless there are no crossing points of the two components, in which case the linking number is 0.



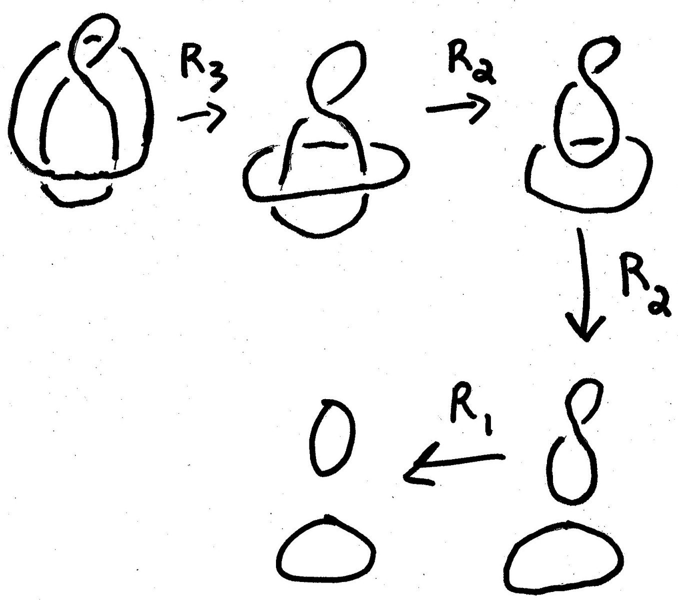






$$\frac{(+1-1+1-1)}{2}=0$$

which is also justified by Reidemeister moves of unoriented links



The previous example of the linking number of the unlink suggests it doesn't matter which diagram for an oriented link is used to compute the linking number. Reidemeister moves help to confirm this feeling.

Proposition 10. It turns out that if one link diagram for an oriented link is changed into another diagram for another oriented link by any Reidemeister move, the linking number does not change.

Another conclusion we can draw is that the absolute value of the linking numbers of two equivalent oriented links will be equal. After all, two equivalent oriented links can be considered to have the same diagram, possibly differing only in the directions of the components. Such a difference can account for either exactly the same sets of left-handed or right-handed crossings where components meet or an exchange of those types of crossings. So linking numbers are either equal or negatives of each other. What we have discovered, thus, is an invariant of unoriented links of two components.

Theorem 11. If two equivalent (unoriented links) of two components are each oriented in any way, then the absolute values of their linking numbers will be equal.

This result is also useful to us in the contrapositive.

Theorem 12. If two (unoriented) links of two components are oriented in any way and the absolute values of their linking numbers are not equal, then the two links are not equivalent.

We can now state this.

Theorem 13. There exists links of two components that are not equivalent to the unlink.

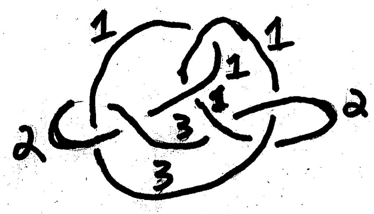
We can be confident in this because any link of two components with a nonzero linking number for some orientations cannot be equivalent to the unlink. This is a result that we have known intuitively all along.

While the linking number now officially allows us to say that there are different links of two components, we have not yet officially proven that there are any knots that cannot be unknotted. The linking number does not apply to knots since a knot has only one component. We will now talk about another invariant that will help distinguish knots from each other.

Definition 14. We will say that a knot or link is tricolorable if, given one of its diagrams and given a set of three colors, each arc of the diagram can be assigned one of the three colors in such a way that

(a) at least two of the colors are used, and

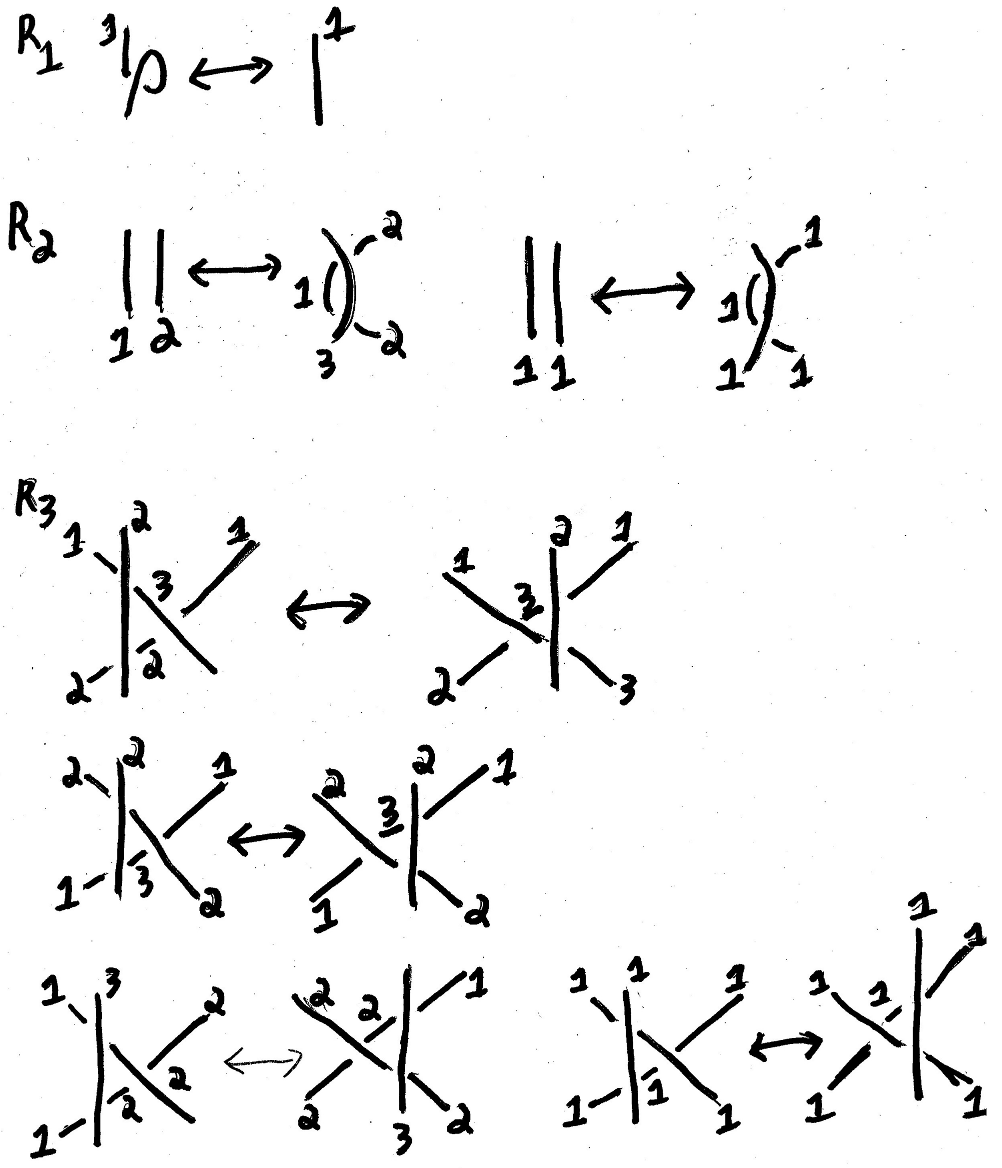
(b) if two different colors appear at any crossing, then so does the third color.



We would like to know that the property of tricolorability is an invariant for links. That is, we want to prove that if two links are equivalent and one of them is tricolorable, then so is the other.

Theorem 15. If two links are equivalent and one of them is tricolorable, then so is the other.

Proof. To see this, again we will look at Reidemeister moves. It is sufficient to show that if a Reidemeister move is performed on the tricolorable link, then the resultant link will also be tricolorable.



□

As in the case of the absolute value of a linking number for links of two components, it is again convenient to point out that the contrapositive form of this result will be useful in distinguishing knots and links.

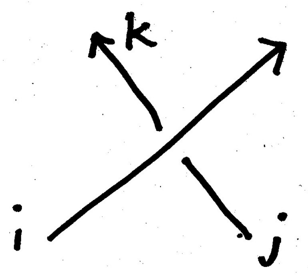
Theorem 16. If a first link is tricolorable and a second link is not tricolorable, then the two links are not equivalent.

Polynomials

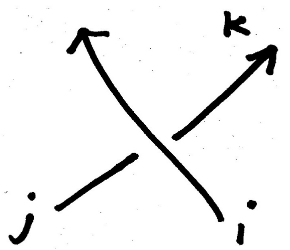
In the previous section, we looked at some invariants of knots and links. In this section, we will look at several polynomial invariants of knots and links. The first of which is the Alexander polynomial discovered by James Alexander in 1928. This was the first knot polynomial discovered and another one wouldn't be discovered until nearly sixty years later.

We denote the Alexander polynomial $\Delta(t)$. To compute the Alexander polynomial of a knot, first pick an oriented diagram for a knot K. Number the arcs of the diagram, and separately number the crossings. Next, define an $n \times n$ matrix, where n is the number of crossings (and arcs) in the diagram, according to the following procedure:

If the crossing numbered l is right-handed with arc i passing over arcs j and k, as illustrated below, enter a $1-t$ in column i of row l, enter a -1 in column j of that row, and enter a t in column k of that row.



If the crossing is left-handed, as illustrated below, enter a $1-t$ in column i of row l, enter a t in column j and enter a -1 in column k of row l. All of the remaining entries of row l are 0.



We now have a new definition.

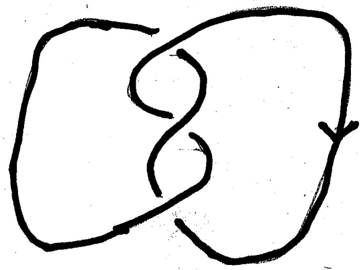
Definition 17. The $(n-1) \times (n-1)$ matrix obtained by removing the last row and column from the $n \times n$ matrix just described is called an Alexander matrix of K. The determinant of the Alexander matrix is called the Alexander polynomial of K. (The determinant of a 0×0 matrix is defined to be 1.)

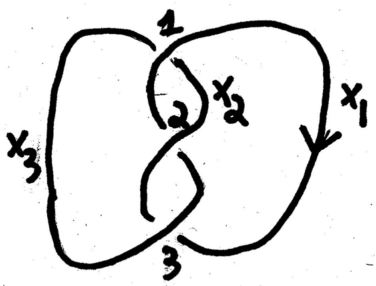
An observation to be made is that this polynomial depends upon the choice of the diagram as well as other choices involved in its description. This leads to a new theorem.

Theorem 18. If the Alexander polynomial for a knot is computed using two different sets of choices for diagrams and labelings, the two polynomials will differ by a multiple of $\pm t^k$, for some integer k.

We will work through a few examples to see how to compute Alexander polynomials.

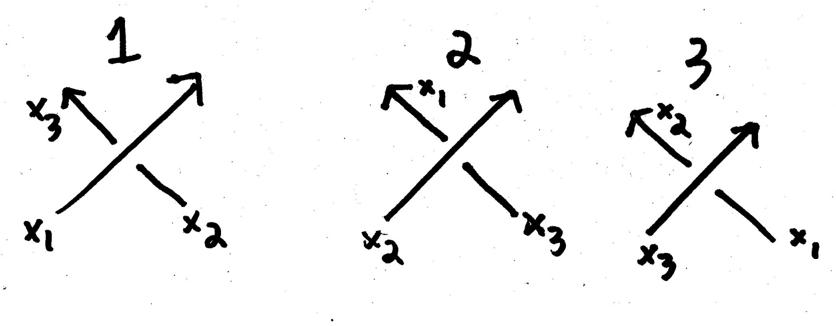
Example. We are given a trefoil knot with an orientation.





Next we will label the arcs and crossings.

The arcs will be x_1, x_2, x_3 and the crossings will be 1, 2, 3. We examine the individual crossings and whether they are left- or right-handed.



All three crossings are right-handed. We next set up a matrix according to the process stated above, with the row labeled for crossings 1, 2, 3 and the columns labeled with x_1, x_2, x_3

$$\begin{pmatrix} 1-t & -1 & t \\ t & 1-t & -1 \\ -1 & t & 1-t \end{pmatrix}$$

We delete the last row and column to get the Alexander matrix:

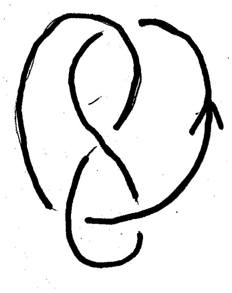
$$\begin{pmatrix} 1-t & -1 \\ t & 1-t \end{pmatrix}$$

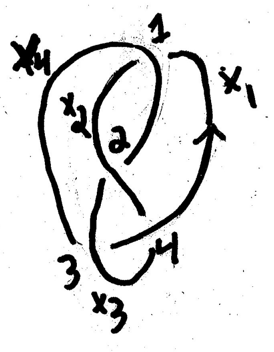
We calculate the determinant of the Alexander matrix to get the Alexander polynomial.

$$\det\begin{pmatrix} 1-t & -1 \\ t & 1-t \end{pmatrix} = (1-t)^2 - (-1)(t) = 1 - 2t + t^2 + t = t^2 - t + 1$$

Thus, the Alexander polynomial of the trefoil knot is $\Delta(t) = t^2 - t + 1$.

Example. We will determine the Alexander polynomial of a figure eight knot. We label the arcs and crossings. The arcs will be x_1, x_2, x_3, x_4 and the crossings will be 1, 2, 3, 4.





We examine the individual crossings and whether they are left- or right-handed. All four crossings are right-handed. We next set up a matrix according to the process stated above, with the row labeled for crossings 1, 2, 3, 4 and the columns labeled with

x_1, x_2, x_3, x_4

$$\begin{pmatrix} -1 & t & 0 & 1-t \\ 0 & 1-t & -1 & t \\ -1 & 0 & 1-t & t \\ 1-t & t & -1 & 0 \end{pmatrix}$$

We delete the last row and column to get the Alexander matrix:

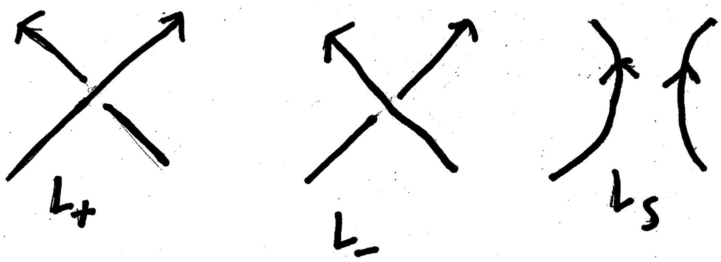
$$\begin{pmatrix} -1 & t & 0 \\ 0 & 1-t & -1 \\ -1 & 0 & 1-t \end{pmatrix}$$

We calculate the determinant of the Alexander matrix to get the Alexander polynomial.

$$\det \begin{pmatrix} -1 & t & 0 \\ 0 & 1-t & -1 \\ -1 & 0 & 1-t \end{pmatrix} = -t^2 + 3t - 1$$

Thus, the Alexander polynomial of the figure eight knot is $\Delta(t) = -t^2 + 3t - 1$.

In Alexander's original paper in 1928, he made an observation that went unexploited for forty years. Given an oriented link L, focus on a particular crossing. If that crossing is changed from right to left or vice versa, a new link diagram results. The crossed could also be smoothed to obtain yet another link diagram. The smoothing process removes small sections of the arcs that pass over and under, and replaces them with a new pair of arcs that do not cross. There is only one way of doing this while maintaining the orientation of the original diagram. Hence, for a given diagram and crossing, there are a total of three associated diagrams, corresponding to links denoted L_+, L_-, and L_s. These are illustrated below. As one will see, L_+ denotes a right-handed crossing, L_- a left-handed crossing. Please note, either L_+ or L_- will be the original link L.



In 1969, John Conway developed these ideas of Alexander further and created what is known as the Conway polynomial, or sometimes the Alexander-Conway polynomial. For an oriented link L, the variable of the polynomial will be taken to be the letter z and the Conway polynomial itself will be denoted by $\nabla_L(z)$. The definition is "implicit" in the sense that the polynomials associated with the three general links L_+, L_-, L_s are related by an equation. Then, given the polynomial associated with the unknot, specific polynomials for other more complicated oriented links may be explicitly determined from this relation in a recursive manner.

Definition 19. (a) The Conway polynomial for the unknot U is defined by $\nabla_U(z) = 1$. (b) For the links L_+, L_-, L_s, related as described above, the Conway polynomials for L_+, L_-, L_s satisfy $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_s}(z)$.

As you can see, this definition does not directly tell us how to compute the polynomial $\nabla_L(z)$ for a given oriented link L. Thus the question arises of whether or not such a polynomial can always be found. It is the case that such polynomials always exists. This was proven by Alexander, and Conway proved it using this modified version of Alexander's polynomial.

Example. Look at the following illustration.

As we can see, both L_+ and L_- are the unknot. Thus, $\nabla_{L_+}(z) = \nabla_{L_-}(z) = 1$. Thus, the equation $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_s}(z)$ yields $1 - 1 = -z\nabla_{L_s}(z)$. Then, $\nabla_{L_s}(z) = 0$. In this case, L_s is an oriented unlink of two components. Thus, we can conclude that such a link has Conway polynomial 0.

Theorem 20. The Conway polynomial of an oriented link remains unchanged if the diagram used to compute its polynomial is modified by any Reidemeister move. Therefore, the Conway polynomial of an oriented link depends only on the oriented link and not on the particular diagram used to compute the polynomial.

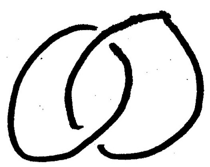
Two oriented links may be equivalent as links, without regard to their orientations. However, "equivalence as oriented links" requires more.

Definition 21. Two oriented links are equivalent as oriented links if there is a continuous deformation in 3-space from one to the other that preserves the direction of the components.

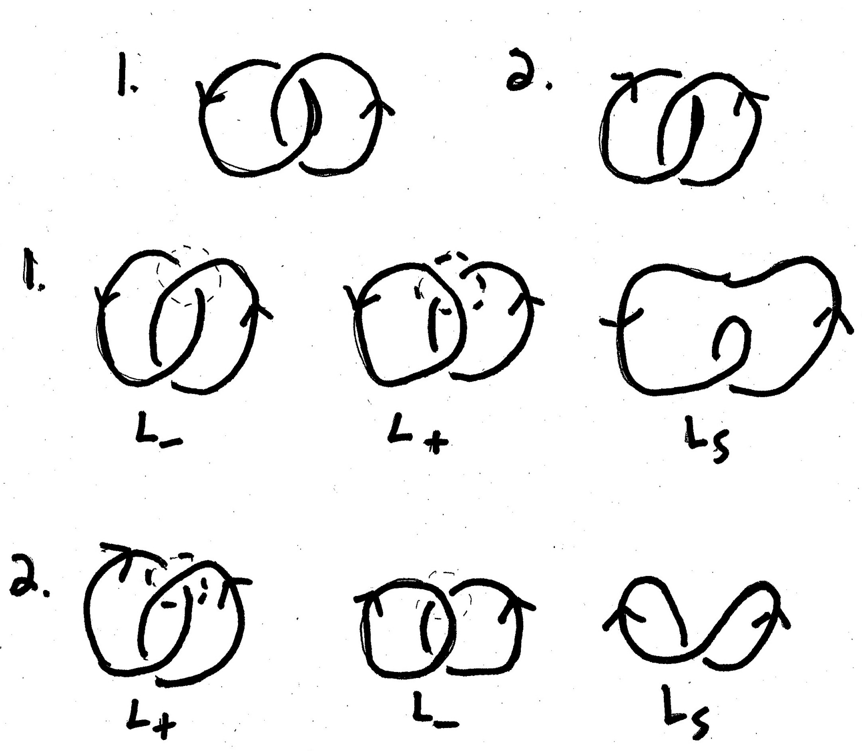
Thus, two oriented links that are equivalent as oriented links are automatically equivalent links since they are able to be deformed into each other in 3-space. Also, two oriented links that are equivalent as oriented links will possess a common oriented link diagram. This leads to our next theorem.

Theorem 22. If two oriented links are equivalent as oriented links, then they have the same Conway polynomial. Or, in contrapositive form: If two oriented links have different Conway polynomials, then they are not equivalent as oriented links.

We will look at an example to demonstrate the results. In this example, we have two equivalent links (unoriented).



Now, when we put the orientation on them, we see different results.

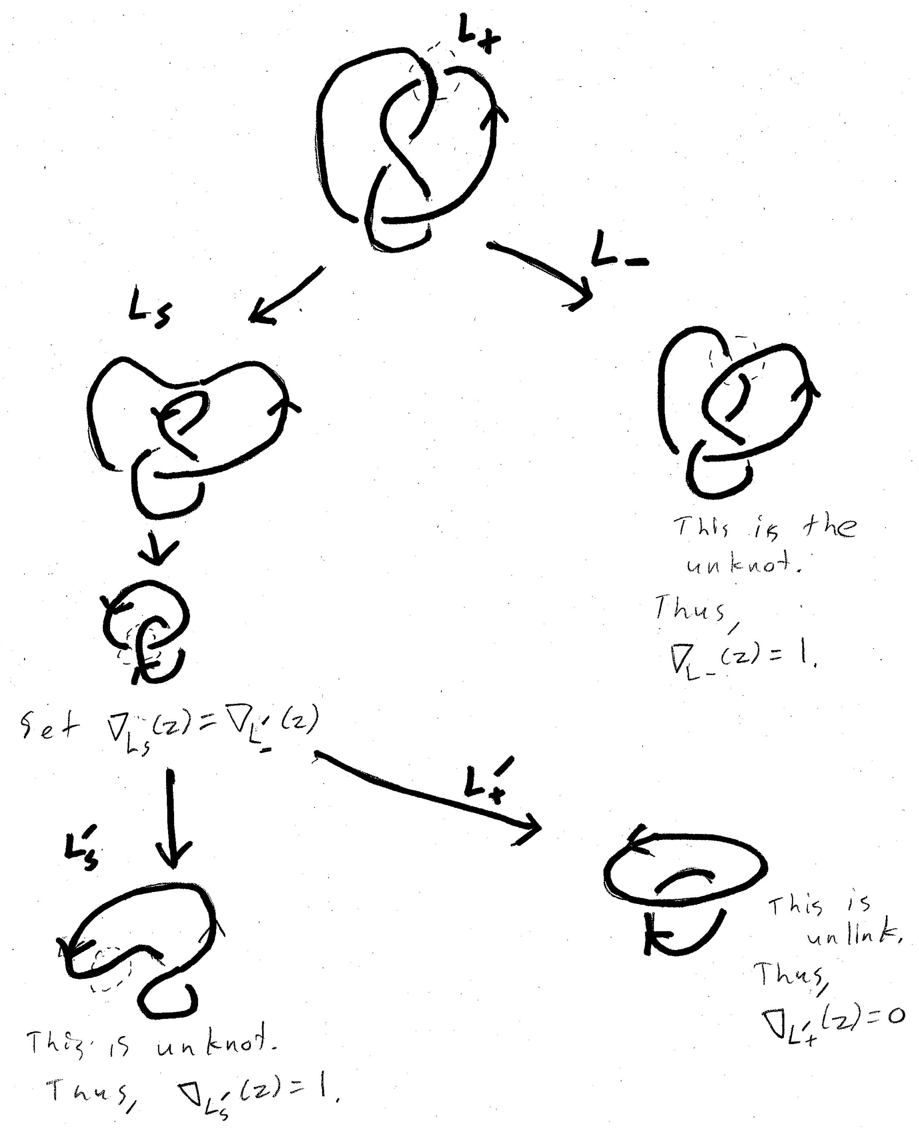


The link on the left is left-handed and appears as L_-. Its L_+ is the unlink of two components. L_s is the unknot. Thus, $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_s}(z)$ yields $0 - \nabla_{L_-} = -z \cdot 1 \implies \nabla_{L_-}(z) = z$. The link on the right is right-handed and appears as L_+. Its L_- is the unlink of two components. L_s is the unknot. Thus, $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_s}(z)$ yields $\nabla_{L_+} - 0 = -z \cdot 1 \implies \nabla_{L_-}(z) = -z$. So, these two different orientations of the same link do not yield the same Conway polynomial. Thus, as per the theorem, they are not equivalent as oriented links.

If we restrict our attention to Conway polynomials for knots, the question of which way the knot is oriented loses substance, as each crossing of a knot diagram is either left-handed or right-handed without regard to how the diagram is oriented. It follows that we can discuss the Conway polynomial as an invariant of unoriented knots.

Theorem 23. If two knots have different Conway polynomials, then they are not equivalent.

We want to return to the relationship between the Alexander polynomial and the Conway polynomial. We will calculate the Conway polynomial of the figure 8 knot.

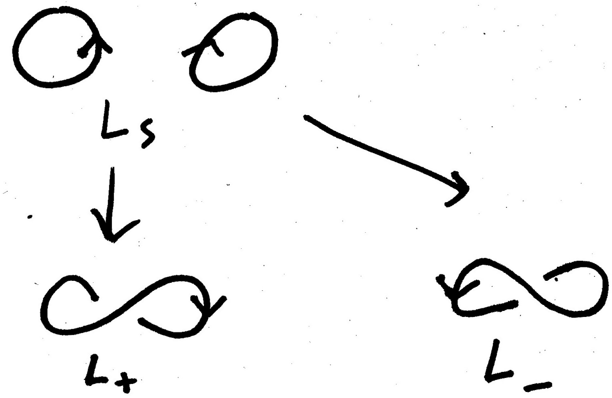


First, $\nabla_{L'_-}(z) = \nabla_{L'_+}(z) + z\nabla_{L'_s}(z) = 0 + z \cdot 1 = z = \nabla_{L_s}(z)$. With $\nabla_{L_s}(z) = z$, we have $\nabla_{L_+}(z) = \nabla_{L_-}(z) - z\nabla_{L_s}(z) = 1 - z(z) = 1 - z^2$. Therefore, the Conway polynomial of the figure eight knot is $1 - z^2$.

However, the Alexander polynomial has a conversion to the Conway polynomial: $\Delta_K(t) = \nabla_K(t^{1/2} - t^{-1/2})$. So when we use a figure eight knot: $\Delta_{4_1}(t) = 1 - (t^{1/2} - t^{-1/2})^2 = 1 - (t + t^{-1} - 2) = -t + 3 - t^{-1}$ Now, we can have Alexander polynomials that are different by a factor of $\pm t^k, k \in \mathbb{Z}$. We can scale $(-t + 3 - t^{-1})$ by t and we have what we calculated earlier for the Alexander polynomial: $(-t^2 + 3t - 1)$.

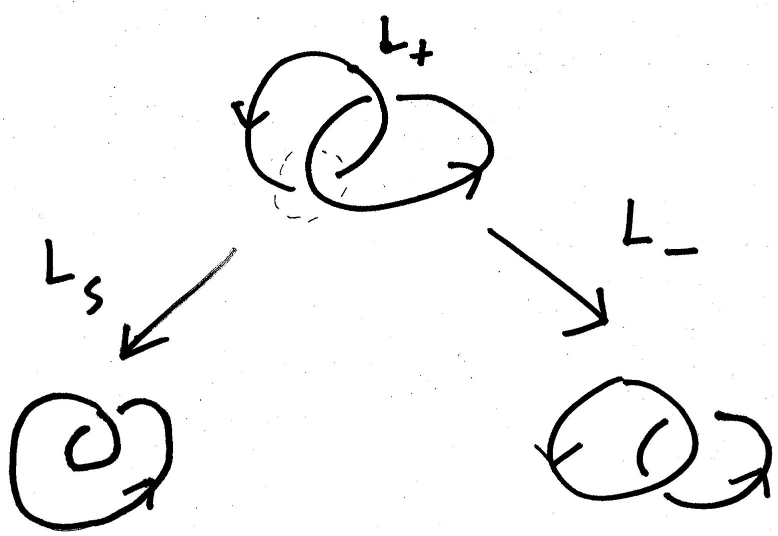
In the 1980s, a new knot polynomial was discovered by Vaughan Jones. Similar to how the Conway polynomial is defined, the Jones polynomial also uses L_+, L_-, L_s for a right-handed crossing, left-handed crossing, and smoothed out crossing. For the Jones polynomial, we start with two rules. 1) V(unknot)=1; 2) $t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_s) = 0$.

With these two rules, we can look at an example.



Both L_+ and L_- are the unknot. Thus, $V(L_+) = V(L_-) = 1$. Now, plugging into equation 2, we get $t^{-1}(1) - t(1) + (t^{-1/2} - t^{1/2})V(L_s) = 0 \implies V(L_s) = -(t^{-1/2} + t^{1/2})$. Therefore, the jones polynomial on the unlink of two components is $-(t^{-1/2} + t^{1/2})$.

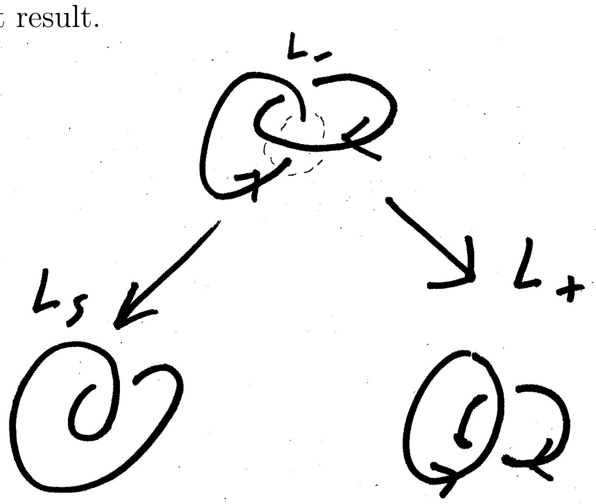
We will now look at another example of a link where the two components are interlocked.



L_s is the unknot; thus $V(L_s) = 1$. L_- is the unlink of two components which we saw in the previous example; thus $V(L_-) = -(t^{-1/2} + t^{1/2})$. Then,

$t^{-1}(L_+) - t \cdot (-(t^{-1/2} + t^{1/2})) + (t^{-1/2} - t^{1/2}) = 0 \implies V(L_+) = -t^{5/2} - t^{1/2}$.

Using the same link, but changing the orientation of one of the components yields a different result.



L_s is the unknot; thus $V(L_s) = 1$. L_+ is the unlink of two components which we saw in a previous example; thus $V(L_+) = -(t^{-1/2} + t^{1/2})$. Then,
$t^{-1} \cdot (-(t^{-1/2} + t^{1/2})) - t \cdot V(L_-) + (t^{-1/2} - t^{1/2}) = 0 \implies V(L_-) = -t^{-5/2} - t^{-1/2}$.

As we can see, changing the orientation of the link changes the Jones polynomial. This differs from the Alexander polynomial, which does not differentiate between orientations of knots. We can see this by looking at the polynomials of a trefoil:

The Conway polynomial of a right-handed trefoil is $\nabla(z) = 1 + z^2$ and a left-handed trefoil is $\nabla(z) = 1 + z^2$.

The Jones polynomial of a right-handed trefoil is $V(t) = -t^4 + t^3 + t$ and a left-handed trefoil is $V(t) = -t^{-4} + t^{-3} + t^{-1}$.

So, in a sense, we can say that the Jones polynomial is "stronger" than the Alexander and Conway polynomials.

We now know that if two knots are equivalent, then their Jones polynomials will be equivalent. However, if two knots have the same Jones polynomial, will the knots be equivalent? The answer to that is no.

The last knot polynomial that we will mention is the HOMFLY polynomial. Shortly after Jones discovered the Jones polynomial, a group of six mathematicians discovered this new polynomial. The HOMFLY polynomial is named after the initials discovers:

Jim Hoste, Adrian Ocneanu, Kenneth Millet, Peter Freyd, William Lickorish, and David Yetter. This HOMFLY polynomial differs from the Alexander and Jones polynomial in that it is a polynomial in two variables. The HOMFLY is a generalization of both the Alexander and Jones polynomials, as both of those can be obtained from the HOMFLY with proper substitutions.

Like the previous polynomials, it makes use of knot crossings L_+, L_-, L_s. The HOMFLY polynomial is defined with two rules: 1) P(unknot)=1; 2) $\alpha P(L_+) - \alpha^{-1} P(L_-) = zP(L_s)$. It generalizes the Alexander and Jones polynomials in the following way: $\Delta(t) = P(\alpha = 1, z = t^{1/2} - t^{-1/2}), V(t) = P(\alpha = t^{-1}, z = t^{1/2} - t^{-1/2})$.

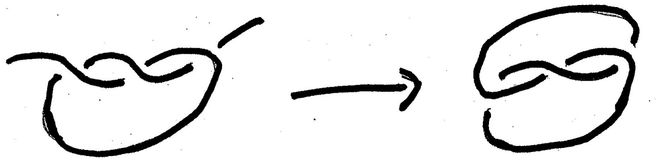
CHAPTER III: SCOUTING KNOTS AS MATHEMATICAL KNOTS

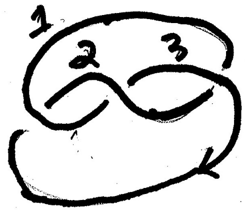
In this chapter, we will now use the tools laid out in the previous chapter to calculate some invariants of common Scouting knots-turned-mathematical knots. In particular, we will look at the Overhand knot (both with one and two ropes), Square knot (one and two ropes), Granny knot (one and two ropes), Bowline, Clove Hitch, and Two Half Hitches. We will determine their linking number (if applicable), tricolorability, Alexander Polynomial (if applicable), Conway Polynomial, and Jones Polynomial. After all the polynomials have been determined, we will examine which mathematical knots are the equivalent to the Scouting knots-turned-mathematical knots.

The first thing that we have to do for each Scouting knot is turn it into a mathematical knot. We do this by simply connecting the loose ends together to close the loop. For the single rope knots, this is easy. For the double rope knots, we will go into further discussion as there are multiple ways to close the loops.

Overhand Knot (Single Rope)

The first knot we will look at will be the Overhand Knot (Single Rope). First we close the loop of the overhand knot. As we can see, this results in a trefoil knot. We can see that it is tricolorable. We then put some orientation on the knot so we can begin calculating the Alexander, Conway, and Jones polynomials. We also labeled the diagram for use in calculating the Alexander polynomial.





tricolorable - yes

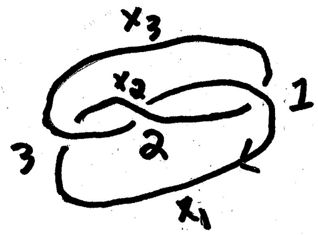
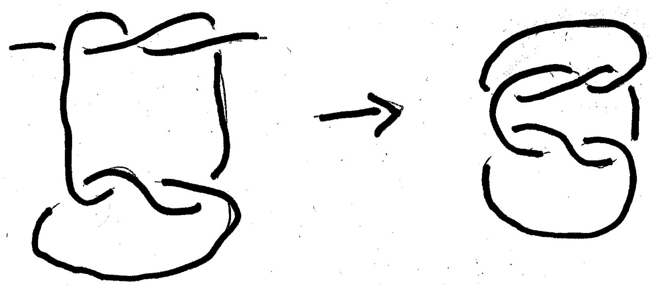


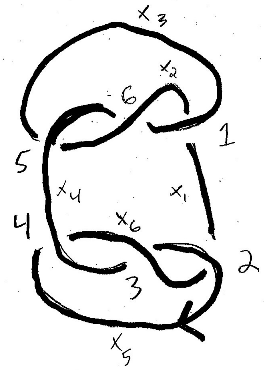
Diagram for Alexander

The results we attain after calculation are that the Alexander polynomial $\Delta(t) = t^2 - 2 + 1$; the Conway polynomial $\nabla(z) = z^2 + 1$. To check that the Conway and Alexander polynomial match, we can plug $t^{1/2} - t^{-1/2}$ into the Conway polynomial we just determined. Thus, $(t^{1/2} - t^{-1/2})^2 + 1 = t - 1 + t^{-1}$. We can multiply this by t and then we get $t^2 - t + 1$ which is equal to what we calculated for the Alexander polynomial. Finally, our Jones polynomial is $V(t) = -t^4 + t^3 + t$.

Square Knot (Single Rope)

First we close the loop of the square knot. We can see that it is tricolorable. We then put some orientation on the knot so we can begin calculating the Alexander, Conway, and Jones polynomials. We also labeled the diagram for use in calculating the Alexander polynomial.

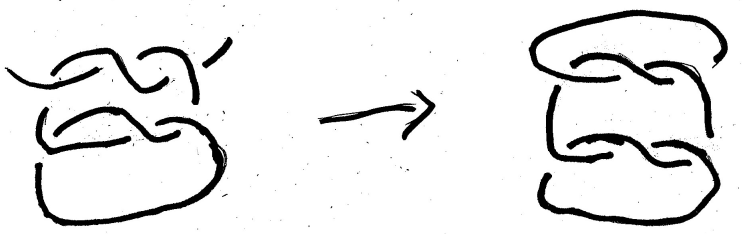


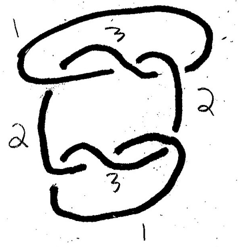


The results we attain after calculation are that the Alexander polynomial $\Delta(t) = -t^5 + 2t^4 - 3t^3 + 2t^2 - t$; the Conway polynomial $\nabla(z) = z^4 + 2z^2 + 1$. To check that the Conway and Alexander polynomial match, we can plug $t^{1/2} - t^{-1/2}$ into the Conway polynomial we just determined. Thus, $(t^{1/2} - t^{-1/2})^4 + 2(t^{1/2} - t^{-1/2})^2 + 1 = t^2 + t^{-2} - 2t - 2t^{-1} + 3$. We can multiply this by $-t^3$ and then we get $-t^5 + 2t^4 - 3t^3 + 2t^2 - t$ which is equal to what we calculated for the Alexander polynomial. Finally, our Jones polynomial is $V(t) = -t^3 - t^{-3} + t^2 - t^{-2} - t + t^{-1} + 3$.

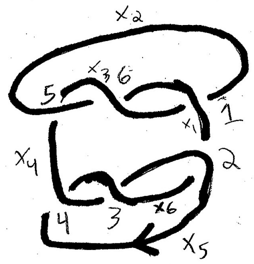
Granny Knot (Single Rope)

First we close the loop of the granny knot. We can see that it is tricolorable. We then put some orientation on the knot so we can begin calculating the Alexander, Conway, and Jones polynomials. We also labeled the diagram for use in calculating the Alexander polynomial.





Alexander

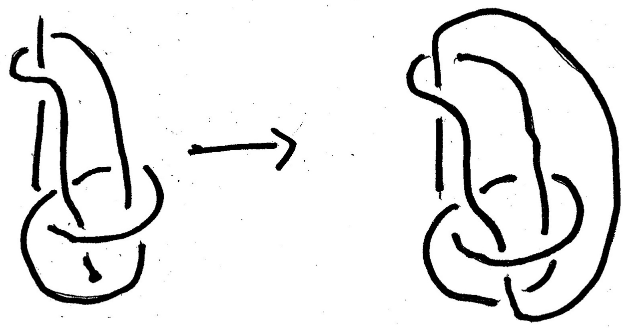


The results we attain after calculation are that the Alexander polynomial $\Delta(t) = -t^5 + 2t^4 - 3t^t3 + 2t^2 - t$; the Conway polynomial $\nabla(z) = z^4 + 2z^2 + 1$. To check that the Conway and Alexander polynomial match, we can plug $t^{1/2} - t^{-1/2}$ into the Conway polynomial we just determined. Thus, $(t^{1/2} - t^{-1/2})^4 + 2(t^{1/2} - t^{-1/2})^2 + 1 = t^2 + t^{-2} - 2t - 2t^{-1} + 3$. We can multiply this by $-t^3$ and then we get $-t^5 + 2t^4 - 3t^3 + 2t^2 - t$ which is equal to what we calculated for the Alexander polynomial. Finally, our Jones polynomial is $V(t) = -t^8 - 2t^7 + t^6 - 2t^5 + 2t^4 + t^2$.

We need to stop here and make note of something very interesting. For our previous two knows, Square and Granny, they produced the same Conway polynomial. This is curious because to the basic eye, the knots clearly *look* different. In terms of practicality, tying the two knots are different as well. So, it made us scratch our heads when the Conway polynomials were the same for both. However, when we look to the Jones polynomial, we can see that they are clearly two different knots. The two Jones polynomials are not scalable to each other, so we can rule out that they are equivalent up to scaling. Thus, the Jones polynomial works here as the stronger polynomial, reassuring our intuition that these are two different knots.

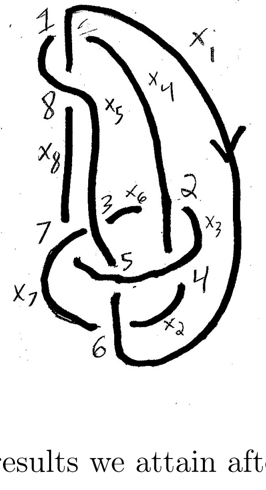
Bowline

First we close the loop of the bowline. The bowline is not tricolorable. We then put some orientation on the knot so we can begin calculating the Alexander, Conway, and Jones polynomials. We also labeled the diagram for use in calculating the Alexander polynomial.



Not tricolorable

Alexander

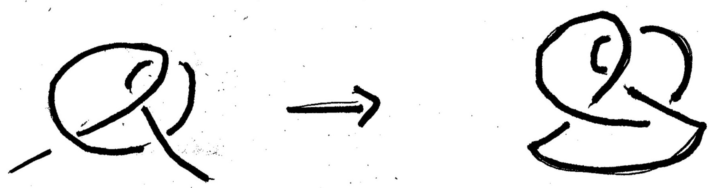


The results we attain after calculation are that the Alexander polynomial

$\Delta(t) = -t^6 + 3t^5 - 5t^4 + 3t^3 - t^2$; the Conway polynomial $\nabla(z) = z^4 + z^2 + 1$. To check that the Conway and Alexander polynomial match, we can plug $t^{1/2} - t^{-1/2}$ into the Conway polynomial we just determined. Thus,
$(t^{1/2} - t^{-1/2})^4 + (t^{1/2} - t^{-1/2})^2 + 1 = t^2 + t^{-2} - 3t - 3t^{-1} + 5$. We can multiply this by $-t^4$ and then we get $-t^6 + 3t^5 - 5t^4 + 3t^3 - t^2$ which is equal to what we calculated for the Alexander polynomial. Finally, our Jones polynomial is
$V(t) = -t^3 - t^{-3} + 2t^2 + 2t^{-2} - 2t - 2t^{-1} + 3$.

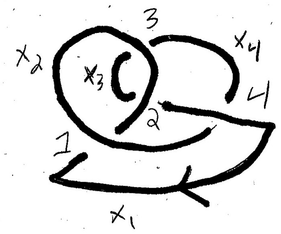
Clove Hitch

First we close the loop of the clove hitch. The clove hitch is not tricolorable. We then put some orientation on the knot so we can begin calculating the Alexander, Conway, and Jones polynomials. We also labeled the diagram for use in calculating the Alexander polynomial.



not tricolorable

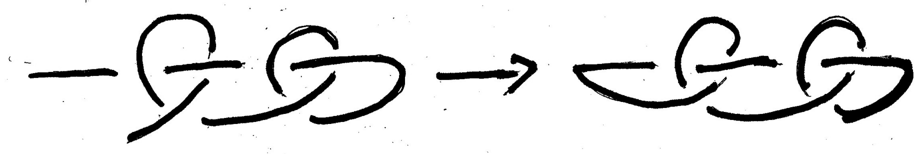
Alexander

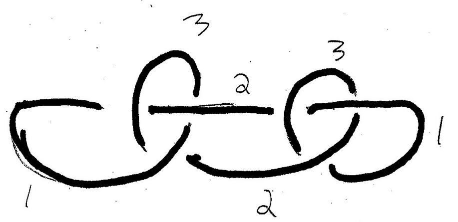


The results we attain after calculation are that the Alexander polynomial $\Delta(t) = t$; the Conway polynomial $\nabla(z) = 1$. To check that the Conway and Alexander polynomial match, we can plug $t^{1/2} - t^{-1/2}$ into the Conway polynomial we just determined. In this case, it just returns 1. We can multiply this by t and then we get t which is equal to what we calculated for the Alexander polynomial. This makes sense that since $\Delta(t)$ can be scaled by t, we can have $\Delta(t) = 1$, which makes sense since $\nabla(unknot) = 1$, and the clove hitch is equivalent to the unknot, as demonstrated in the last illustration. Finally, our Jones polynomial, when calculated, is $V(t) = 1$ which is what we get by definition for $V(unknot)$.

Two Half Hitches

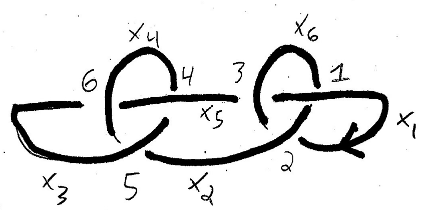
First we close the loop of the two half hitches. The two half hitches is tricolorable. We then put some orientation on the knot so we can begin calculating the Alexander, Conway, and Jones polynomials. We also labeled the diagram for use in calculating the Alexander polynomial.





tricolorable - yes

Alexander

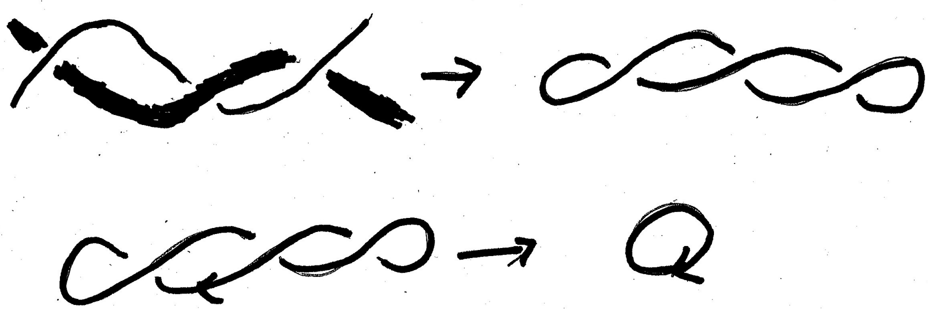


The results we attain after calculation are that the Alexander polynomial $\Delta(t) = t^5 - 2t^4 + 3t^3 - 2t^2 + t$; the Conway polynomial $\nabla(z) = z^4 + z^2 + 1$. To check that the Conway and Alexander polynomial match, we can plug $t^{1/2} - t^{-1/2}$ into the Conway polynomial we just determined. Thus, $(t^{1/2} - t^{-1/2})^4 + 2(t^{1/2} - t^{-1/2})^2 + 1 = t^2 + t^{-2} - 2t - 2t^{-1} + 1$. We can multiply this by t^3 and then we get $t^5 - 2t^4 + 3t^3 - 2t^2 + t$ which is equal to what we calculated for the Alexander polynomial. Finally, our Jones polynomial is $V(t) = -t^{-8} - 2t^{-7} + t^{-6} - 2t^{-5} + 2t^{-4} + t^{-2}$.

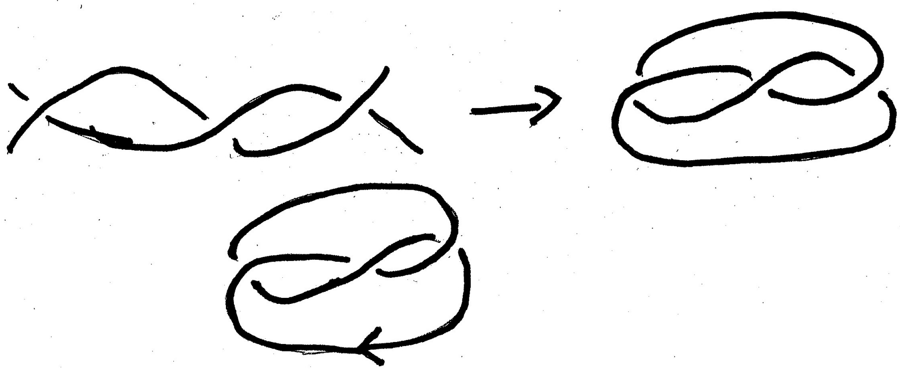
The two half hitches also displays the same Conway polynomial as the square and granny knots. In this case, the two half-hitches Jones polynomial differs from the square knot Jones polynomials. However, the Jones polynomial of the two half hitches looks eerily similar to the Jones polynomial of the granny knot. This is because they are related to each by substituting t^{-1} into either polynomial and you will get the other. This goes to show that these diagrams we chose for each knot is the reverse orientation of the other.

Overhand Knot (Two Ropes)

With the diagram of the overhand knot (two ropes), we have four loose ends that need to be attached in order to close the loop. We will go through four ways to close the loop and then look at the same invariants we looked at previously.

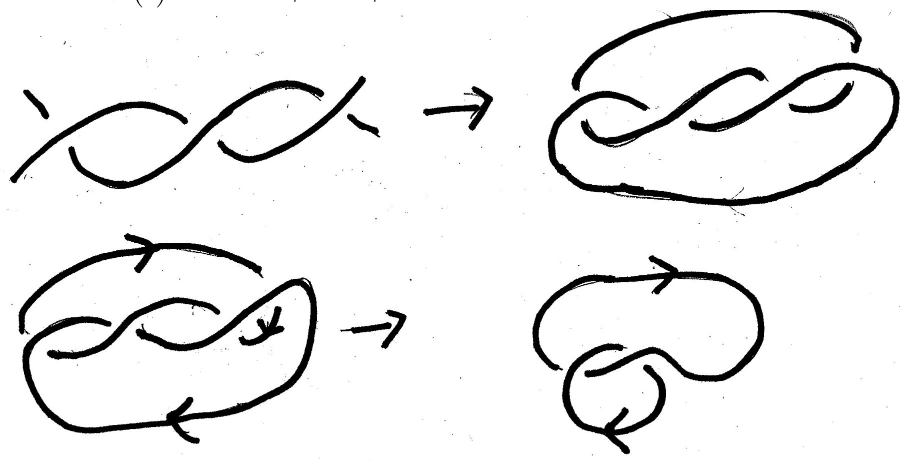


1) Once the loose ends are connected, we can see that this new knot is equivalent to the unknot. Thus, it is not tricolorable; the Alexander polynomial $\Delta(t) = 1$; Conway polynomial $\nabla(z) = 1$; Jones polynomial $V(t) = 1$.

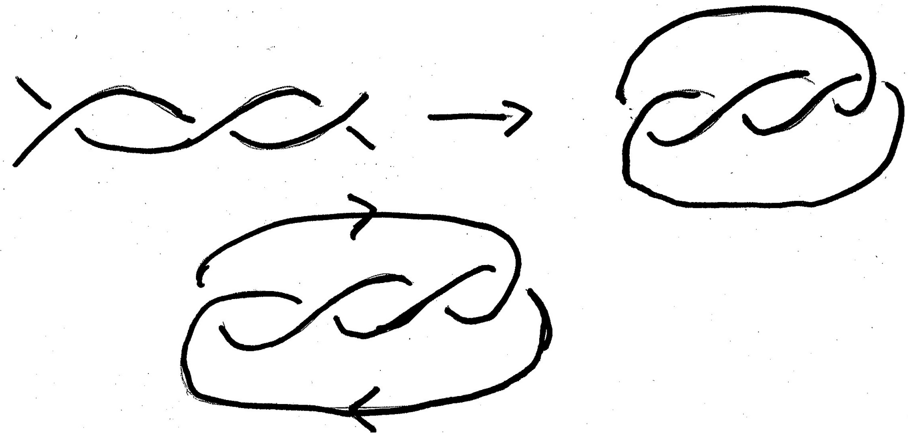


2) Once the loose ends are connected, we put some orientation on the knot and we can see that this new knot is the left-handed trefoil. Thus, it is tricolorable; the Alexander polynomial $\Delta(t) = t^2 - t + 1$; Conway polynomial $\nabla(z) = z^2 + 1$; Jones

polynomial $V(t) = -t^{-4} + t^{-3} + t^{-1}$.



3) Once the loose ends are connected, we put some orientation on the knot and we can see that we have a link of two components. We can use a Reidemeister move to simplify the diagram such that we have a simple link of two unknots. The linking number is $\frac{-1-1}{2} = -1$; Conway polynomial $\nabla(z) = -z$; Jones polynomial $V(t) = -t^{5/2} - t^{1/2}$.

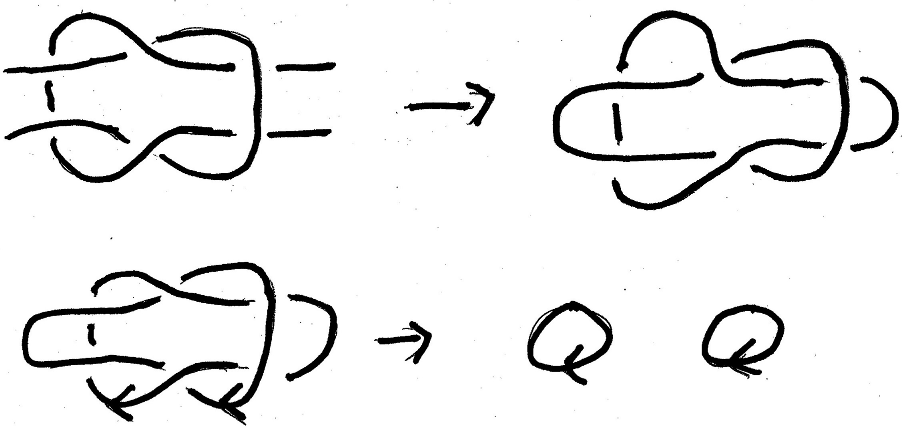


4) Once the loose ends are connected, we put some orientation on the knot and we can see that we have a link of two components.The linking number is $\frac{-1-1-1-1}{2} = -2$; Conway polynomial $\nabla(z) = -2z$; Jones polynomial $V(t) = -t^{9/2} - t^{5/2} + t^{3/2} - t^{1/2}$.

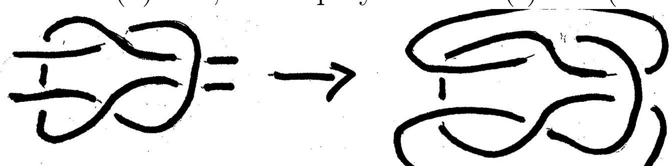
After looking at these four knots and links that result from connecting the loose ends of a overhand knot tied with two ropes, we can see that connecting the loose ends in different ways produces different mathematical knots, even though they all result from the same Scouting knot.

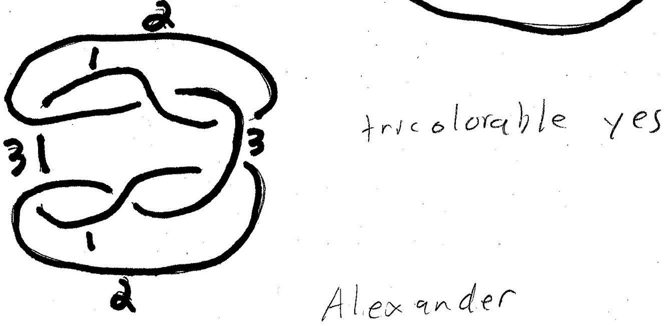
Square Knot (Two Ropes)

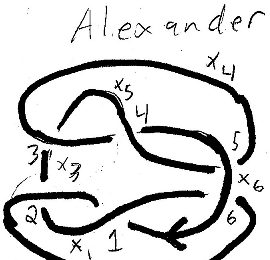
With the diagram of the square knot (two ropes), we have four loose ends that need to be attached in order to close the loop. We will go through four ways to close the loop and then look at the same invariants we looked at previously.



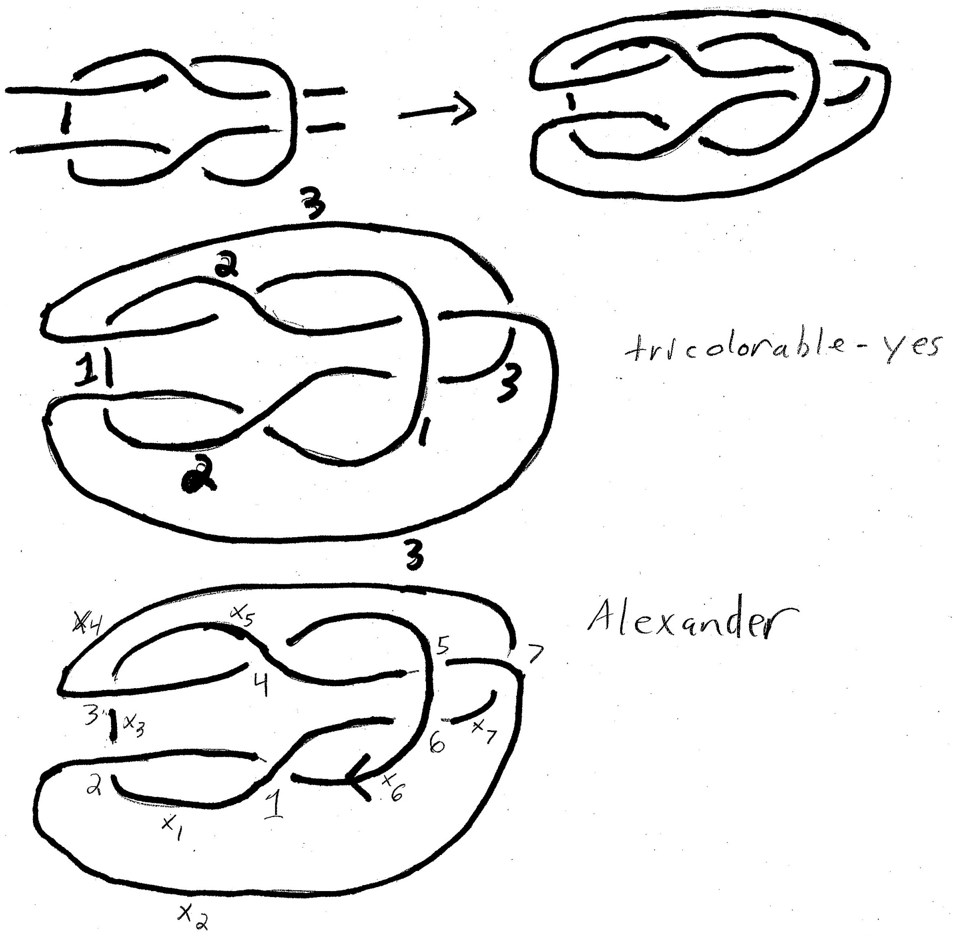
1) Once the loose ends are connected, we can see that this new knot is equivalent to the unlink of two components. Thus, its linking number is 0; it is tricolorable; Conway polynomial $\nabla(z) = 0$; Jones polynomial $V(t) = -(t^{-1/2} + t^{1/2})$.



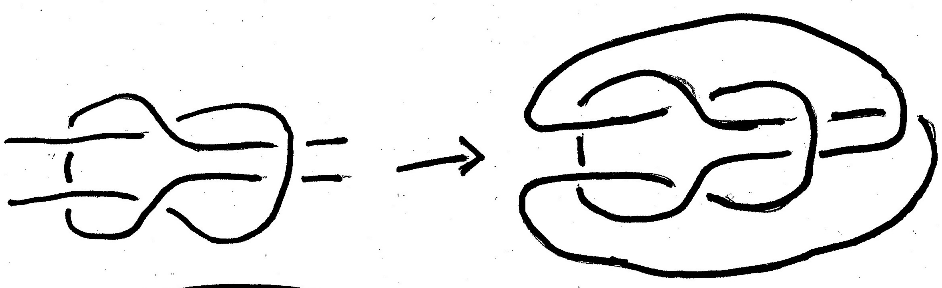


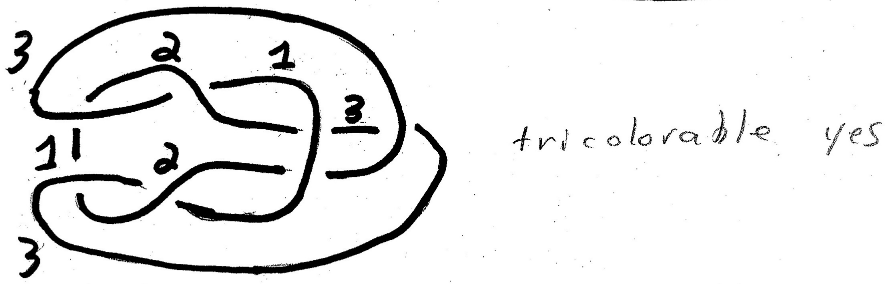


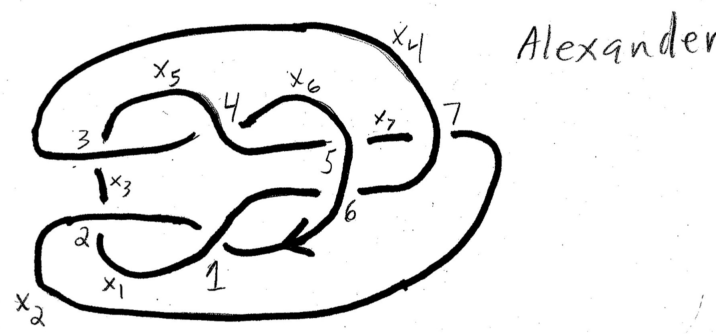
2) Once the loose ends are connected, we put some orientation on the knot. It is tricolorable; the Alexander polynomial $\Delta(t) = -t^4 + 2t^3 - 3t^2 + 2t - 1$; Conway polynomial $\nabla(z) = z^4 + 2z^2 + 1$; Jones polynomial
$V(t) = -t^3 - t^{-3} + t^2 + t^{-2} - t - t^{-1} + 3$.



3) Once the loose ends are connected, we put some orientation on the knot. It is tricolorable; the Alexander polynomial $\Delta(t) = -2t^4 + 5t^3 - 2t^2$; Conway polynomial $\nabla(z) = -2z^2 + 1$; Jones polynomial $V(t) = t^{-4} - t^{-3} + t^2 + t^{-2} - t - 2t^{-1} + 2$.





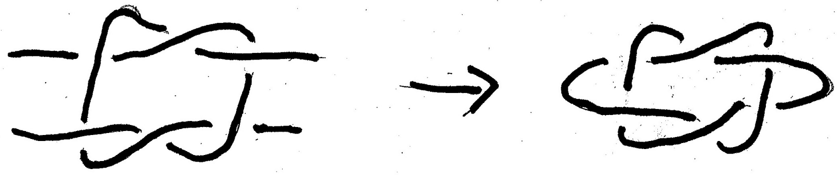


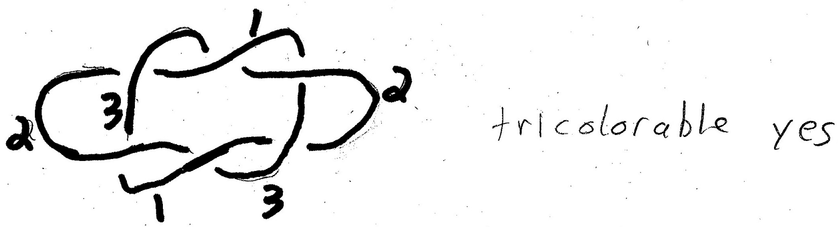
4) Once the loose ends are connected, we put some orientation on the knot. It is tricolorable; the Alexander polynomial $\Delta(t) = 2t^4 - 5t^3 + 2t^2$; Conway polynomial $\nabla(z) = 1 - 2z^2$; Jones polynomial $V(t) = t^4 - t^3 + t^2 + t^{-2} - 2t - t^{-1} + 2$.

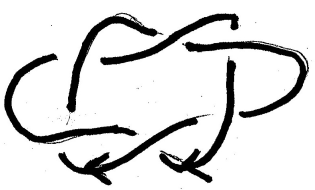
Variations 3 and 4 have reverse orientations of each other if we look at their Jones polynomials. Other than that, all the other variations are different from each other, even though they all result from the same Scouting knot.

Granny Knot (Two Ropes)

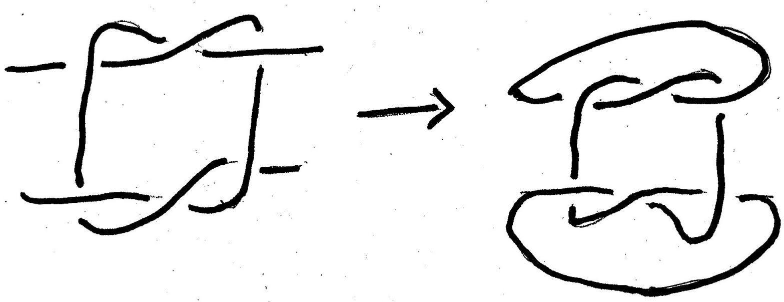
With the diagram of the granny knot (two ropes), we have four loose ends that need to be attached in order to close the loop. We will go through four ways to close the loop and then look at the same invariants we looked at previously.

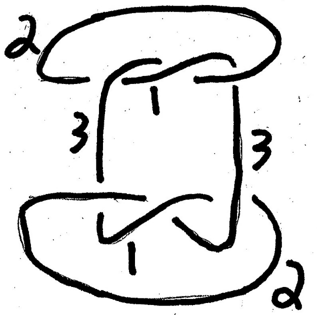




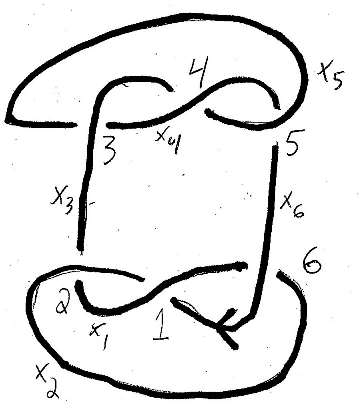


1) Once the loose ends are connected, we put some orientation on the knot. It's linking number is $\frac{-1-1-1-1-1-1}{2} = -3$; it is tricolorable; Conway polynomial $\nabla(z) = z^5 + 4z^3 + 3z$; Jones polynomial $V(t) = -t^{-9/2} - t^{-5/2} - t^{-17/2} + t^{-15/2} - t^{-13/2} + t^{-11/2}$.



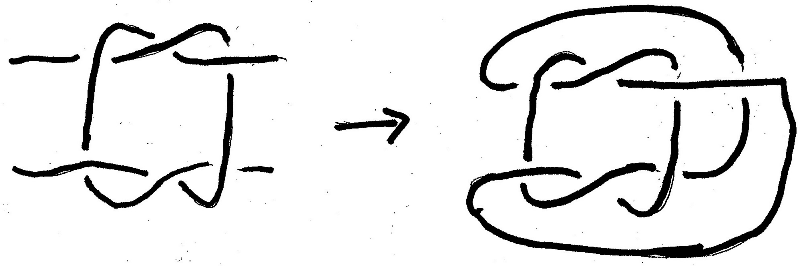


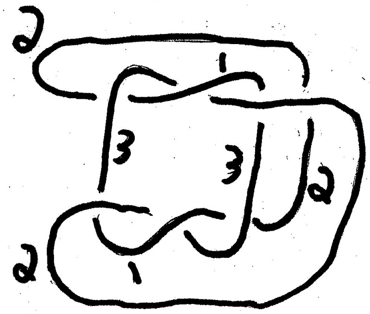
tricolorable - yes



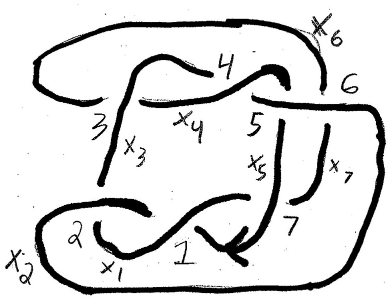
Alexander

2) Once the loose ends are connected, we put some orientation on the knot. It is tricolorable; the Alexander polynomial $\Delta(t) = t^4 - 2t^3 + 3t^2 - 2t + 1$; Conway polynomial $\nabla(z) = z^4 + 2z^2 + 1$; Jones polynomial $V(t) = t^{-8} - 2t^{-7} + t^{-6} - 2t^{-5} + 2t^{-4} + t^{-2}$.



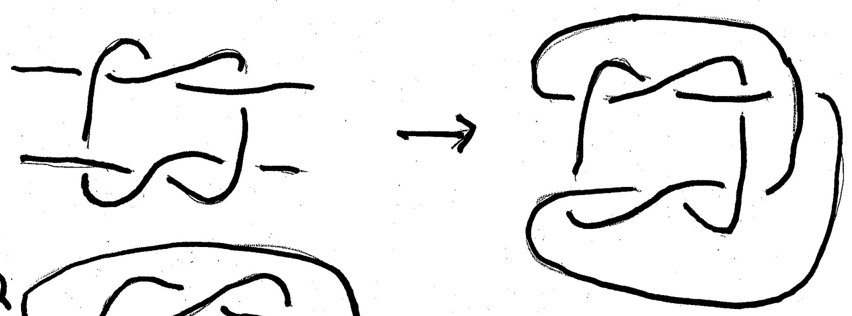


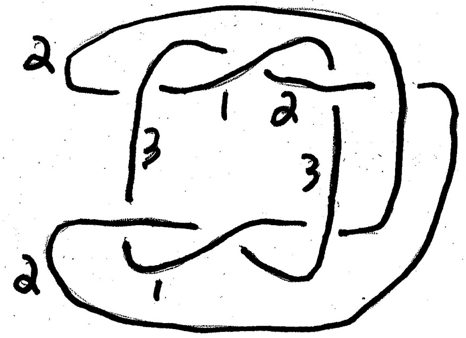
tricolorable – yes



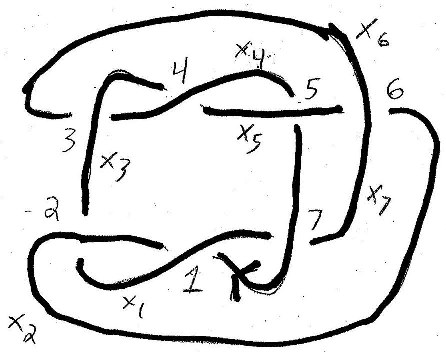
Alexander

3) Once the loose ends are connected, we put some orientation on the knot. It is tricolorable; the Alexander polynomial $\Delta(t) = -t^4 + t^3 - t^2$; Conway polynomial $\nabla(z) = -z^2 + 1$; Jones polynomial $V(t) = -t^4 + t^3 + t$.





tricolorable - yes



Alexander

4) Once the loose ends are connected, we put some orientation on the knot. It is tricolorable; the Alexander polynomial $\Delta(t) = 4t^4 - 7t^3 + 4t^2$; Conway polynomial $\nabla(z) = 4z^2 + 1$; Jones polynomial $V(t) = -t^8 + t^7 - 2t^6 + 3t^5 - 2t^4 + 3t^3 - 2t^2 + t$.

Results

We are going to list all of the results from the previous calculations in a table considering the Scouting knot, linking number, tricolorability, Alexander Polynomial, Conway polynomial, Jones polynomial, and then other known mathematical knots with the same polynomials.

Table 1: Results

Beginning of Table 1						
Scouting Knot	Linking Number	Tricolorable	Alexander Polynomial	Conway Polynomial	Jones Polynomial	Mathematical Knots
Overhand (Single Rope)	N/A	Yes	$t^2 - t + 1$	$z^2 + 1$	$-t^4 + t^3 + t$	(Right-Handed) Trefoil 3_1
Square Knot (Single Rope)	N/A	Yes	$-t^5 + 2t^4 - 3t^3 + 2t^2 - t$	$z^4 + 2z^2 + 1$	$-t^3 - t^{-3} + t^2 - t^{-2} - t - t^{-1} + 3$	Connected Sum of (Left- and Right-Handed) Trefoils
Granny Knot (Single Rope)	N/A	Yes	$-t^5 + 2t^4 - 3t^3 + 2t^2 - t$	$z^4 + 2z^2 + 1$	$t^8 - 2t^7 + t^6 - 2t^5 + 2t^4 + t^2$	Connected Sum of (Two Right-Handed) Trefoils
Bowline	N/A	No	$-t^6 + 3t^5 - 5t^4 + 3t^3 - t^2$	$z^4 + z^2 + 1$	$-t^3 - t^{-3} + 2t^2 + 2t^{-2} - 2t - 2t^{-1} + 3$	6_3
Clove Hitch	N/A	No	t	1	1	Unknot
Continuation of Table 1						

Scouting Knot	Linking Number	Tricolorable	Alexander Polynomial	Conway Polynomial	Jones Polynomial	Mathematical knots
Two Half Hitches	N/A	Yes	$t^5 - 2t^4 + 3t^3 - 2t^2 + t$	$z^4 + 2z^2 + 1$	$t^{-8} - 2t^{-7} + t^{-6} - 2t^{-5} + 2t^{-4} + t^{-2}$	Connected Sum of (Two Left-Handed) Trefoils
Overhand Knot (Two Ropes - Variation 1)	N/A	No	1	1	1	Unknot
Overhand Knot (Two Ropes - Variation 2)	N/A	Yes	$t^2 - t + 1$	$z^2 + 1$	$-t^{-4} + t^{-3} + t^{-1}$	(Left-Handed) Trefoil 3_1
Overhand Knot (Two Ropes - Variation 3)	-1	No	N/A	$-z$	$-t^{5/2} - t^{1/2}$	(Right-Handed) Hopf Link 2^2_1
Overhand Knot (Two Ropes - Variation 4)	-2	No	N/A	$-2z$	$-t^{9/2} - t^{5/2} + t^{3/2} - t^{1/2}$	Solomon Knot 4^2_1

Continuation of Table 1

Scouting Knot	Linking Number	Tricolorable	Alexander Polynomial	Conway Polynomial	Jones Polynomial	Mathematical knots
Square Knot (Two Ropes - Variation 1)	0	Yes	N/A	0	$-(t^{-1/2} + t^{1/2})$	Unlink of Two Components
Square Knot (Two Ropes - Variation 2)	N/A	Yes	$-t^4 + 2t^3 - 3t^2 + 2t - 1$	$z^4 + 2z^2 + 1$	$-t^3 - t^{-3} + t^2 + t^{-2} - t - t^{-1} + 3$	Connected Sum of (Left- and Right-Handed) Trefoils
Square Knot (Two Ropes - Variation 3)	N/A	Yes	$-2t^4 + 5t^3 - 2t^2$	$-2z^2 + 1$	$t^{-4} - t^{-3} + t^2 + t^{-2} - t - 2t^{-1} + 2$	6_1
Square Knot (Two Ropes - Variation 4)	N/A	Yes	$2t^4 - 5t^3 + 2t^2$	$-2z^2 + 1$	$t^4 - t^3 + t^2 + t^{-2} - 2t - t^{-1} + 2$	6_1
Granny Knot (Two Ropes - Variation 1)	-3	Yes	N/A	$z^5 + 4z^3 + 3z$	$-t^{-9/2} - t^{-5/2} - t^{-17/2} + t^{-15/2} - t^{-13/2} + t^{-11/2}$	6_1^2

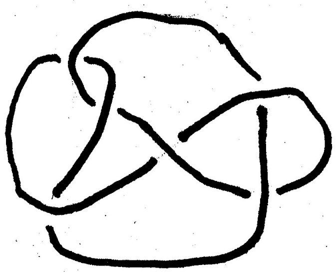
Continuation of Table 1

Scouting Knot	Linking Number	Tricolorable	Alexander Polynomial	Conway Polynomial	Jones Polynomial	Mathematical knots
Granny Knot (Two Ropes - Variation 2)	N/A	Yes	$t^4 - 2t^3 + 3t^2 - 2t + 1$	$z^4 + 2z^2 + 1$	$t^{-8} - 2t^{-7} + t^{-6} - 2t^{-5} + 2t^{-4} + t^{-2}$	Connected Sum of Two (Left-Handed) Trefoils
Granny Knot (Two Ropes - Variation 3)	N/A	Yes	$-t^4 + t^3 - t^2$	$z^2 + 1$	$-t^4 + t^3 + t$	(Right-Handed) Trefoil 3_1
Granny Knot (Two Ropes - Variation 4)	N/A	Yes	$4t^4 - 7t^3 + 4t^2$	$4z^2 + 1$	$-t^8 + t^7 - 2t^6 + 3t^5 - 2t^4 + 3t^3 - 2t^2 + t$	7_4
End of Table 1						

Discussion

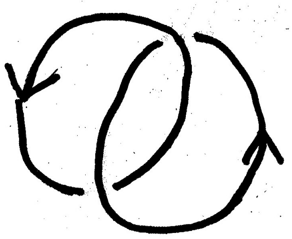
Some observations we made about our knots were

- Both the Clove Hitch and the Overhand Knot (Two Ropes, Variation 1) were the unknot.
- Bowline was the knot 6_3.

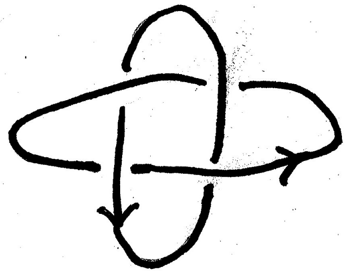


The bowline can be taught with a "story" to help people learn to tie the knot. It is usually a difficult knot to teach younger Scouts. The story is that the rabbit comes out of the hole, around the tree, then back in the hole. Looking at the picture, it's hard to see where the hole or tree is in the bowline. This does not look like the knot we are taught in Scouting.

- Overhand Knot (Two Ropes, V. 3) was the (Right-handed) Hopf Link 2_1^2.

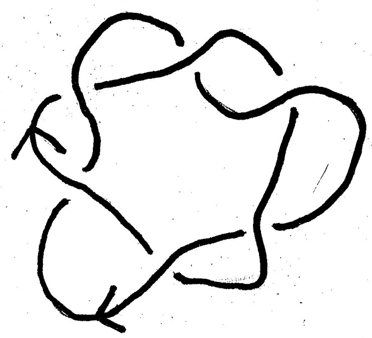


- Overhand Knot (Two Ropes, V. 4) was the Solomon's Knot 4_1^2.

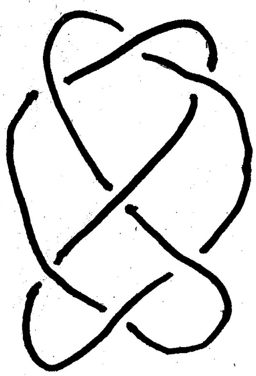


An interesting thing about the Solomon's Knot is that it is not a knot at all. It is a link of two components.

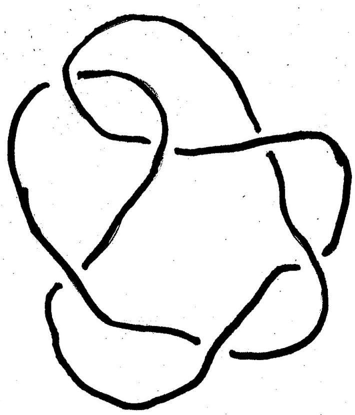
- Granny Knot (Two Ropes, V. 1) was the link 6_1^2.

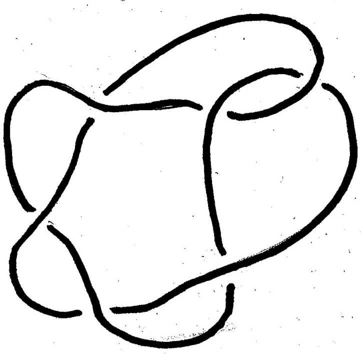


- Granny Knot (Two Ropes, V. 4) was the knot 7_4.



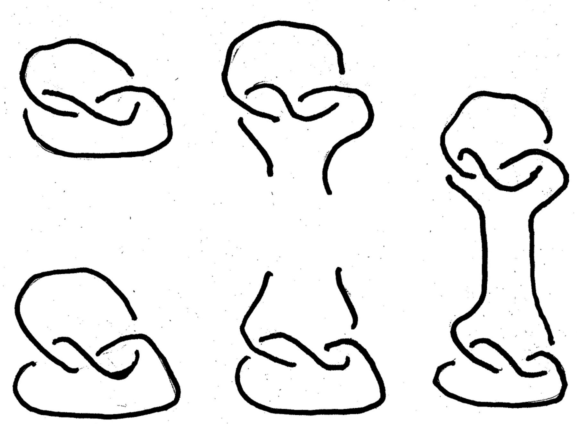
- Square Knot (Two Ropes, V. 3) was the knot 6_1, seen below on the left..
- Square Knot (Two Ropes, V. 4) was the knot 6_1, seen below on the right.. The interesting thing about these two different variations of the Square Knot with Two Ropes is that they are mirrors of each other. The only difference between the two diagrams is that there is a crossing that is switched so that the overcrossing in the first diagram is the undercrossing in the second diagram and vice versa.





- Overhand Knot (Single Rope) was the (Right-handed) Trefoil.
- Overhand Knot (Two Ropes, V. 2) was the (Left-handed) Trefoil.
- Granny Knot (Two Ropes, V. 3) was the (Right-handed) Trefoil.
- All three knots had the same Conway polynomial, and the Jones of the Overhand Knot (Single) and Granny Knot (Two Ropes, V. 3) were the same, and the inverse of the Overhand (Two Ropes, V. 2) This was surprising since the two overhand knots build up the granny knot. So, we did not expect them the overhand knot (single rope) to be the same knot as the granny knot (two ropes - v. 3). We further did not expect them to be the inverse of the overhand knot (two ropes - v. 2).

We need to pause and define a new concept. A Connected Sum of two knots can be a way of combining two knots together to make a new knot. You start with any two knots; remove a small segment from each knot; connect the loose ends without introducing any new crossings.



- Square Knot (Single Rope) was the Connected Sum of Left- and Right-handed trefoils.
- Granny Knot (Single Rope) was the Connected Sum of two Right-handed trefoils.
- Two Half Hitches was the Connected Sum of two Left-handed Trefoils.

- Square Knot (Two Ropes, V. 2) was the Connected Sum of Left- and Right-handed trefoils.
- Granny Knot (Two Ropes, V. 2) was the Connected Sum of two Left-handed trefoils.

There were genuine moments of surprise and discovery with these last five knots. First off, all five knots have the same Conway polynomial. This is surprising because while the two square knots and the granny knot are similar in their tying and use, the two half hitches is not even related to the square or granny knot. The square and granny knot are used to combine two ropes together; the two half hitches is used to tie a rope to a pole. Their constructions look completely different as well. So, it was very surprising to see that the two half hitches had the same Conway polynomial as the other four knots, and the same Jones polynomial as the granny knot (two ropes, v. 2).

In the process of calculating these polynomials, we calculated the square knot (single rope) first then the granny knot (single rope) directly after. It would be appropriate to say we were shocked and a bit scared when the Conway polynomials came back identical. The square knot and the granny knot, while related, are decidedly two different knots in practice. They are tied differently, they look different, and their uses are different. So, needless to say, we were concerned. However, once we calculated the Jones polynomial of the granny knot (single rope) and realized it is decidedly different than the Jones polynomial of the square knot (single rope), even up to $\pm t^{m/2}, m \in \mathbb{Z}$ equivalence, then we knew that the knots were different. They don't even have the same amount of terms, with the square knot having six terms and the granny knot having seven terms. These are not the same knot.

We also observed that the knots that have connected sum of same-handed trefoils have inverse Jones polynomials of each other (due to handedness) ($V(t) = t^8 - 2t^7 + t^6 - 2t^5 + 2t^4 + t^2$) and the knots with connected sum of Left- and Right-handedness were completely different ($V(t) = -t^3 - t^{-3} + t^2 - t^{-2} - t - t^{-1} + 3$)!

We wondered why these knots were so related to each other? How come the granny knot, square knot, and two half hitches all have the same Conway polynomial and shared Jones polynomials? How are they all connected sums of trefoils? We realized that since the construction of these knots as practical knots is built on overhand knots. The square knot and granny knot are two overhand knots tied one right after the other. The two half hitches is basically overhand knots tied around a pole. And we learned through out previous calculations that the overhand knot is a trefoil knots when the ends are connected. Thus, the practical knots being built on overhand knots lead to the mathematical knots being built on trefoils. This is why these knots as mathematical knots are all connected sums of trefoils.

Conclusion

Our goal with this thesis was to explore some basic knot theory concepts and invariants, in particular the Alexander, Conway, and Jones polynomial. We then looked at some practical knots often used in the Scouting movement. These are knots that people tie in the real world with a specific use and purpose. What happens when we turn these practical knots with their loose ends into mathematical knots? Do the knots behave in unexpected ways? We were surprised and delighted to see the relationships between different knots that have completely different uses and methods of tying. Some knots that are never even thought to be related in the real world, but share Conway polynomials in the mathematical world. The Jones polynomial showed its strength, demonstrating how it can distinguish knots that the Alexander and Conway polynomials couldn't. There's no question that practical knots and mathematical knots are different in fundamental ways. While we can use the practical knots to tie two ropes together, or attach a rope to a post, or create a loop in a rope that won't budge, in mathematics, we can take those exact same knots, connect their loose ends, and build new knots in our minds.

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