On the Projective Ricci Curvature

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Abstract

The notion of the Ricci curvature is defined for sprays on a manifold. With a volume form on a manifold, every spray can be deformed to a projective spray. The Ricci curvature of a projective spray is called the projective Ricci curvature. In this paper, we introduce the notion of projectively Ricci-flat sprays. We establish a global rigidity result for projectively Ricci-flat sprays with nonnegative Ricci curvature. Then we study and characterize projectively Ricci-flat Randers metrics.

Keywords: spray, Finsler metric, Randers metric, projective Ricci curvature.

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1 Introduction

In Finsler geometry, there are many important Riemannian quantities such as the Riemann curvature and the Ricci curvature, etc. and non-Riemannian quantities such as the Berwald curvature and the S-curvature, etc. The Ricci curvature is defined as the trace of the Riemann curvature. Together with the S-curvature, the Ricci curvature plays an important role in Finsler geometry. The volume of geodesic balls can be controlled by the lower bounds of the Ricci curvature and the S-curvature ([3][4]). The volume of geodesic balls can be also controlled by a single bound of the N-Ricci curvature which is the combination of the Ricci curvature and the S-curvature ([1]).

Let $\mathbf{G}$ be a spray on an n-dimensional manifold M. Given a volume form dV on M, we can construct a new spray by

$$\hat{\mathbf{G}} := \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y}.$$

The spray $\hat{\mathbf{G}}$ is called the projective spray of $(\mathbf{G}, dV)$. The projective Ricci curvature of $(\mathbf{G}, dV)$ is defined as the Ricci curvature of $\hat{\mathbf{G}}$, namely,

$$\mathbf{PRic}_{(G,dV)} := \mathbf{Ric}_{\hat{G}}. \tag{1 .1}$$

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By a simple computation, we have the following formula for the projective Ricci curvature:

$$\mathbf{PRic}_{(G,dV)} = \mathbf{Ric} + (n-1)\Big\{ \frac{\mathbf{S}_{|0}}{n+1} + \Big[\frac{\mathbf{S}}{n+1}\Big]^2 \Big\}. \qquad (1.2)$$

where $\mathbf{Ric} = \mathbf{Ric}_G$ is the Ricci curvature of the spray $\mathbf{G}$, $\mathbf{S} = \mathbf{S}_{(G,dV)}$ is the S-curvature of $(\mathbf{G}, dV)$ and $\mathbf{S}_{|0}$ is the covariant derivative of $\mathbf{S}$ along a geodesic of $\mathbf{G}$. It is known that $\hat{\mathbf{G}}$ remains unchanged under a projective change of $\mathbf{G}$ with dV fixed, thus $\mathbf{PRic}_{(\mathbf{G},dV)} = \mathbf{Ric}_{\hat{\mathbf{G}}}$ is a projective invariant of $(\mathbf{G}, dV)$ ([2]). We make the following

Definition 1.1 *A spray $\mathbf{G}$ on an n-dimensional manifold M is said to be projectively Ricci-flat if there is a volume form dV on M such that*

$$\mathbf{PRic}_{(G,dV)} = 0.$$

A Finsler metric F on M is said to be projectively Ricci-flat if the induced spray $\mathbf{G} = \mathbf{G}_F$ is projectively Ricci-flat.

It is easy to see that every projectively flat spray on R^n is projectively Ricci-flat. For projectively equivalent sprays, if one of them is projectively Ricci-flat, then so is the other.

We prove the following

Theorem 1.2 *Let $\mathbf{G}$ be a spray on an n-dimensional manifold M. $\mathbf{G}$ is projectively Ricci-flat if and only if there are a volume form dV and a scalar function f on M such that*

$$\mathbf{Ric}_G = -(n-1)\Big\{ \Xi_{|0} + \Xi^2 \Big\}, \qquad (1.3)$$

where "$|$" is the horizontal covariant derivative with respect to $\mathbf{G}$, $f_0 = f_{x^m} y^m$, $\Xi := \frac{\mathbf{S}}{n+1} - f_0$ and $\mathbf{S} = \mathbf{S}_{(G,dV)}$.

If the spray satisfies that $\mathbf{S}_{(G,dV)} = (n+1)\phi_0$ for some scalar function ϕ on M, then $\Xi = 0$ for $f = \phi$. We obtain the following

Corollary 1.3 *Let $\mathbf{G}$ be a spray on a manifold M. Suppose that $\mathbf{Ric}_G = 0$ and $\mathbf{S}_{(G,dV)} = (n+1)\phi_0$ for some volume form dV and scalar function ϕ on M, then $\mathbf{G}$ is projectively Ricci-flat.*

The condition on the S-curvature, $\mathbf{S}_{(G,dV)} = (n+1)\phi_0$ being an exact 1-form, is actually a condition on the spray, independent of the choice of a particular volume form dV. Sprays with this property have vanishing χ-curvature $\chi = 0$ ([5], Proposition 4.1). We have the following rigidity theorem:

Theorem 1.4 *Let $\mathbf{G}$ be a complete spray on an n-manifold M. Suppose that $\mathbf{G}$ is projectively Ricci-flat. If the Ricci curvature $\mathbf{Ric}_G \geq 0$, then for any volume form dV on M, the S-curvature $\mathbf{S} = (n+1)\phi_0$ for some scalar function ϕ on M. In this case, $\mathbf{Ric}_G = 0$.*

To have a better understanding on projectively Ricci-flat Finsler metrics, we consider a Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form with $\|\beta\|_\alpha < 1$. Put

$$s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where ";" denotes the covariant derivative with respect to Levi-Civita connection of α. Clearly, β is closed if and only if $s_{ij} = 0$. We prove the following

Theorem 1.5 *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. F is projectively Ricci-flat if and only if there is a scalar function h on M such that*

$$\begin{aligned}
&\mathbf{Ric}_\alpha = 2s_{0m}s^m{}_0 + \alpha^2 s^i{}_j s^j{}_i - (n-1)[h_{0;0} + (h_0)^2] \\
&s^m{}_{0;m} = (n-1)h_{x^m} s^m{}_0,
\end{aligned}$$

where $\mathbf{Ric}_\alpha$ denotes the Ricci curvature of α.

In general, projectively Ricci-flat Randers metrics are not Ricci-flat, and the S-curvature is not almost isotropic.

In [7], Cheng-Shen-Ma study the projective Ricci curvature **PRic**. They derive a formula for the projective Ricci curvature of a Randers metric with respect to the Busemann-Hausdorff volume form dV_{BH}. By this formula, they characterize Randers metrics with $\mathbf{RRic} = 0$ with respect to dV_{BH}. We should point out that the projective Ricci-flatness of Randers metrics defined in [7] is slightly different from ours. Thus the statement of Theorem 1.1 in [7] is slightly different from Theorem 1.5.

2 Preliminaries

Let M be an n-dimensional manifold and TM the corresponding tangent bundle. We denote by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle. Local coordinates on the base manifold M will be denoted by (x^i), while the induced local coordinates on TM or TM_0 will be denoted by (x^i, y^i). We call (x^i, y^i) the standard local coordinate system in TM.

A spray $\mathbf{G}$ on M is a smooth vector field on TM_0 expressed in a standard local coordinate system (x^i, y^i) in TM as follows

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{2 .4}$$

where $G^i = G^i(x, y)$ are the local functions on TM satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$ for $\forall \lambda > 0$. Put

$$N^i_j = \frac{\partial G^i}{\partial y^j}.$$

These are called the *nonlinear connection coefficients* of $\mathbf{G}$. Set $\omega^i = dx^i$ and $\omega^{n+i} := dy^i + N^i_j dx^j$. The connection 1-forms of Berwald connection are given by

$$\omega^i{}_j := \Gamma^i_{jk} dx^k,$$

where

$$\Gamma^i_{jk} = \frac{\partial N^i_j}{\partial y^k} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}.$$

We have

$$d\omega^i = \omega^j \wedge \omega^i{}_j.$$

The curvature 2-forms of the Berwald connection are defined by

$$\Omega^i_j = d\omega^i_j - \omega^l_j \wedge \omega^i_l. \qquad (2\ .5)$$

Express

$$\Omega^i_j = \frac{1}{2} R_j{}^i{}_{kl} \omega^k \wedge \omega^l - B_j{}^i{}_{kl} \omega \wedge \omega^{n+l}. \qquad (2\ .6)$$

The two curvature tensors $R_j{}^i{}_{kl}$ and $B_j{}^i{}_{kl}$ are called *Riemann curvature tensor* and *Berwald curvature tensor*, respectively.

The Riemann curvature $\mathbf{R}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces which is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^s} \frac{\partial G^s}{\partial y^k}.$$

Without much difficulty, one can show that

$$R_j{}^i{}_{kl} = \frac{1}{3}(R^i{}_{k\cdot l} - R^i{}_{l\cdot k})_{\cdot j} \qquad (2\ .7)$$

and

$$R^i{}_k = y^j R_j{}^i{}_{kl} y^l. \qquad (2\ .8)$$

Here and hereafter, notation " $\cdot$ " denotes the vertical derivatives with respect to y. For instance, $f_{\cdot k} = \frac{\partial f}{\partial y^k}$, $\ f_{\cdot k \cdot l} = \frac{\partial^2 f}{\partial y^k \partial y^l}$, etc. By (2 .7) and (2 .8), we see that the two curvature tensors $R^i{}_k$ and $R_j{}^i{}_{kl}$ can represent each other. For this reason, they are all called *Riemann curvature tensor* if there is no confusion.

The well-known Ricci curvature is defined by

$$\mathbf{Ric} := R^m_{\ m} = y^j R_j{}^m{}_{ml} y^l. \qquad (2\ .9)$$

Every Finsler metric F on a manifold induces a spray $\mathbf{G}_F = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ with the *geodesic coefficients*

$$G^i = \frac{1}{4} g^{il} \{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}, \qquad (2\ .10)$$

where $g^{ij} = (g_{ij})^{-1}$. Therefore, every Finsler space is a special spray space.

Let $\mathbf{G}$ be a spray on an n-manifold M and $dV = \sigma dx^1 \cdots dx^n$ a arbitrary volume form. The S-curvature $\mathbf{S} = \mathbf{S}_{(G,dV)}$ is defined by

$$\mathbf{S} := \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln \sigma). \tag{2 .11}$$

Using the S-curvature $\mathbf{S} = \mathbf{S}_{(G,dV)}$, one can modify the spray $\mathbf{G}$ to

$$\hat{\mathbf{G}} = \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y},$$

where $\mathbf{Y} = y^i \frac{\partial}{\partial y^i}$ is the canonical vertical vector field. In local coordinates

$$\hat{G^i} = G^i - \frac{\mathbf{S}}{n+1} y^i.$$

3 Projective Ricci curvature

Let $\mathbf{G}$ be a spray on an n-dimensional manifold M. Let dV and $d\tilde{V}$ be volume forms with $dV = e^{-(n+1)f} d\tilde{V}$, where $f = f(x)$ is a scalar function on M. The S-curvatures $\mathbf{S} = \mathbf{S}_{(G,dV)}$ and $\tilde{\mathbf{S}} = \mathbf{S}_{(G,d\tilde{V})}$ are related by

$$\mathbf{S} = \tilde{\mathbf{S}} + (n+1) f_0, \tag{3 .12}$$

where $f_0 := f_{x^m}(x) y^m$. By (3 .12), we see that $\mathbf{S}_{(G,dV)} = (n+1)\phi_0$ if and only if $\mathbf{S}_{(G,d\tilde{V})} = (n+1)\tilde{\phi}_0$ with $\phi_0 = \tilde{\phi}_0 + f_0$.

By (1 .2), we have

$$\mathbf{PRic}_{(G,dV)} = \mathbf{Ric}_G + (n-1)\Big\{ \Big[\frac{\mathbf{S}}{n+1}\Big]^2 + \frac{\mathbf{S}_{|0}}{n+1} \Big\}, \tag{3 .13}$$

$$\mathbf{PRic}_{(G,d\tilde{V})} = \mathbf{Ric}_G + (n-1)\Big\{ \Big[\frac{\tilde{\mathbf{S}}}{n+1}\Big]^2 + \frac{\tilde{\mathbf{S}}_{|0}}{n+1} \Big\}, \tag{3 .14}$$

where "|" is the horizontal covariant derivative with respect to $\mathbf{G}$. It follows from (3 .13) and (3 .14) that

$$\mathbf{PRic}_{(\mathbf{G},d\tilde{V})} = \mathbf{PRic}_{(\mathbf{G},dV)} - (n-1)\Big\{ f_{0|0} - f_0^2 + \frac{2}{n+1} f_0 \mathbf{S} \Big\}, \tag{3 .15}$$

Theorem 3.1 *Let $\mathbf{G}$ be a spray on an n-dimensional manifold M. The following are equivalent:*

(a) $\mathbf{G}$ *is projectively Ricci-flat,*

(b) for any volume form dV on M there is a scalar function f on M such that

$$\mathbf{PRic}_{(G,dV)} = (n-1)\Big\{ f_{0|0} - f_0^2 + \frac{2}{n+1} f_0 \mathbf{S} \Big\}, \tag{3 .16}$$

(c) for any volume form dV on M there is a scalar function f on M such that

$$\mathbf{Ric}_G = -(n-1)\Big\{\Xi_{|0} + \Xi^2\Big\}, \tag{3 .17}$$

where "$|$" is the horizontal covariant derivative with respect to $\mathbf{G}$, $f_0 = f_{x^m} y^m$, $\Xi := \frac{\mathbf{S}}{n+1} - f_0$ and $\mathbf{S} = \mathbf{S}_{(G,dV)}$.

Proof: (a) $\Rightarrow$ (b). Assume that for some volume form $d\tilde{V}$,

$$\mathbf{PRic}_{(G,d\tilde{V})} = 0.$$

Then by (3 .15), for any volume form $dV = e^{-(n+1)f} d\tilde{V}$,

$$\mathbf{PRic}_{(G,dV)} = (n-1)\Big\{f_{0|0} - f_0^2 + \frac{2}{n+1} f_0 \mathbf{S}\Big\}, \tag{3 .18}$$

(b) $\Rightarrow$ (c). It follows from (3 .13) and (3 .18) that

$$\begin{aligned}\mathbf{Ric}_G &= (n-1)\Big\{f_{0|0} - f_0^2 + \frac{2}{n+1} f_0 \mathbf{S}\Big\} - (n-1)\Big\{\Big[\frac{\mathbf{S}}{n+1}\Big]^2 + \frac{\mathbf{S}_{|0}}{n+1}\Big\}\\ &= -(n-1)\Big\{\Xi_{|0} + \Xi^2\Big\},\end{aligned}$$

where $\Xi = \frac{\mathbf{S}}{n+1} - f_0$.

(c) $\Rightarrow$ (a). Let dV be an arbitrary volume form on M. There is a scalar function f on M such that (3 .17) holds.

$$\mathbf{Ric} = -(n-1)\Big\{\Xi_{|0} + \Xi^2\Big\},$$

where $\Xi = \frac{\mathbf{S}}{n+1} - f_0$ and $\mathbf{S} = \mathbf{S}_{(G,dV)}$. Let $d\tilde{V} = e^{(n+1)f} dV$, we have

$$\tilde{\mathbf{S}} = \mathbf{S} - (n+1) f_0 = (n+1)\Xi.$$

By (3 .14), we get

$$\mathbf{PRic}_{(G,d\tilde{V})} = \mathbf{Ric}_G + (n-1)\Big\{\Xi^2 + \Xi_{|0}\Big\} = 0.$$

Q.E.D.

As we have mentioned in the introduction, if a spray $\mathbf{G}$ on an n-dimensional manifold M satisfies that $\mathbf{Ric}_G = 0$ and $\mathbf{S}_{(G,dV)} = (n+1)\phi_0$ for some volume form dV and scalar function ϕ on M, then (3 .17) holds. Thus $\mathbf{G}$ is projectively Ricci-flat.

Proof of Theorem 1.4: Let dV be an arbitrary volume form on M. By Theorem 3.1, there is a scalar function f on M such that

$$\mathbf{Ric}_G = -(n-1)\Big\{\Xi_{|0} + \Xi^2\Big\}, \tag{3 .19}$$

where $\Xi := \frac{\mathbf{S}}{n+1} - f_0$ and $\mathbf{S} = \mathbf{S}_{(G,dV)}$. Assume that $\Xi = \Xi(x,y) \neq 0$ for some non-zero $y \in T_xM$. Let $c(t)$ be the geodesic with $c(0) = x$ and $c'(0) = y$. By assumption on the completeness, we may assume that c is defined on $R = (-\infty, \infty)$. Let $\Xi(t) := \Xi(c(t), c'(t))$. Then $\Xi'(t) = \Xi_{|0}(c(t), c'(t))$. It follows from (3 .19) that

$$\Xi'(t) + \Xi(t)^2 \leq 0.$$

Case 1: $\Xi(0) < 0$. Let $r := \sup\{b > 0 \mid \Xi(t) < 0,\ 0 \leq t < b\} \leq +\infty$. Observe that for $0 \leq t < r$,

$$\Big[\frac{1}{\Xi(t)}\Big]' \geq 1.$$

$$\frac{1}{\Xi(t)} - \frac{1}{\Xi(0)} \geq t.$$

$$0 > \frac{1}{\Xi(t)} \geq \frac{1+\Xi(0)t}{\Xi(0)}.$$

This implies that $1 + \Xi(0)t > 0$ for $0 \leq t < r$. Thus $r \leq -1/\Xi(0)$ is finite. Then $\Xi(r) = 0$ by the definition of r. Observe that

$$\Xi(t) \leq \frac{\Xi(0)}{1+\Xi(0)t} \leq \Xi(0) < 0, \qquad 0 \leq t < r.$$

Thus $\Xi(r) \leq \Xi(0) < 0$. This is impossible.
Case 2: $\Xi(0) > 0$. Let $r = \sup\{b > 0 \mid \Xi(t) > 0,\ -b < t \leq 0\}$. By a similar argument for $\Xi(t)$ on $(-r, 0]$, we can show that this is impossible.

Therefore $\Xi = 0$. Then it follows from (3 .19) that $\mathbf{Ric}_G = 0$. Q.E.D.

Below is a specific non-trivial example.

Example 3.2 *Let* $\alpha_1 = \sqrt{a_{ij}y^iy^j}$ *and* $\alpha_2 = \sqrt{\bar{a}_{ij}y^iy^j}$ *be two Ricci-flat Riemannian metrics on the manifolds* M_1 *and* M_2*, respectively. Consider the following 4-th root metric*

$$F := \sqrt[4]{\alpha_1^4 + 2c\alpha_1^2\alpha_2^2 + \alpha_2^4},$$

where $0 < c \leq 1$ *is a constant. It is a Riemannian metric when* $c = 1$*. This is a Ricci-flat* ($\mathbf{Ric}_F = 0$) *and Berwald metric on* $M := M_1 \times M_2$*. Thus for the Busemann-Hausdorff volume form* $dV = dV_F$*, the S-curvature* $\mathbf{S}_{(F,dV)} = 0$*. Therefore* F *is projectively Ricci-flat.*

4 Randers metrics

In this section, we will derive the equivalent conditions for a Randers metric satisfying equation(3 .17).

Let $F=\alpha+\beta$ be a Randers metric on an n-dimensional manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta=b_iy^i$. Put

$$\begin{aligned} r_{ij}: &= \frac{1}{2}(b_{i;j}+b_{j;i}), \quad s_{ij}:=\frac{1}{2}(b_{i;j}-b_{j;i}),\\ t_{ij}: &= s_{il}s^l{}_j, \quad s_i:=b^js_{ji}, \quad t_i:=b^jt_{ji}=s_ms^m{}_i, \end{aligned}$$

where ";" denotes the covariant derivative with respect to Levi-Civita connection of α. Through out this paper, we will use "0" to denote the contraction with y^i. For example, $s^m{}_0=s^m{}_jy^j$, $t_{00}=t_{ij}y^iy^j$, $r_{00}=r_{ij}y^iy^j$. The geodesic coefficients of $\mathbf{G}=\mathbf{G}_F$ are given by

$$G^i=\tilde{G}^i+Py^i, \quad \tilde{G}^i=\bar{G}^i+\alpha s^i{}_0, \tag{4 .20}$$

where $\bar{G}^i$ denotes the spray coefficients of α and $P=\frac{r_{00}-2\alpha s_0}{2F}$ ([8]). Clearly, $\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i}$ and $\tilde{\mathbf{G}}=y^i\frac{\partial}{\partial x^i}-2\tilde{G}^i\frac{\partial}{\partial y^i}$ in (4 .20) are projectively equivalent. Recall that the projective spray $\hat{\mathbf{G}}=\mathbf{G}+\frac{2\mathbf{S}}{n+1}\mathbf{Y}$ is projectively invariant for a fixed volume form $dV=\sigma(x)dx^1\cdots dx^n$. Thus the projective Ricci curvature of $(\mathbf{G},dV)$ is given by

$$\mathbf{PRic}_{(G,dV)}=\mathbf{PRic}_{(\tilde{\mathbf{G}},dV)}=\mathbf{Ric}_{\tilde{\mathbf{G}}}+(n-1)\Big\{\frac{\tilde{\mathbf{S}}_{|0}}{n+1}+\Big[\frac{\tilde{\mathbf{S}}}{n+1}\Big]^2\Big\}, \tag{4 .21}$$

where $\tilde{\mathbf{S}}=\mathbf{S}_{(\tilde{G},dV)}$ is the S-curvature of $(\tilde{\mathbf{G}},dV)$ and "|" denotes horizontal covariant derivative with respect to $\tilde{\mathbf{G}}$.

Proof of Theorem 1.5: We shall calculate each term on the right side of (4 .21) as follows. Firstly, we have known that

$$\mathbf{Ric}_{\tilde{\mathbf{G}}}=\mathrm{Ric}_\alpha+(2\alpha s^m{}_{0;m}-2t_{00}-\alpha^2t^m{}_m), \tag{4 .22}$$

where ";" denotes the covariant derivative with respect to Levi-Civita connection of α and Ric_α is the Ricci curvature of α. From $\tilde{G}^i=\bar{G}^i+\alpha s^i{}_0$, it follows that

$$\begin{aligned} \tilde{\mathbf{S}} &= \frac{\partial\tilde{G}^m}{\partial y^m}-y^m\frac{\partial}{\partial x^m}(\ln\sigma)\\ &= \frac{\partial\bar{G}^m}{\partial y^m}-y^m\frac{\partial}{\partial x^m}(\ln\sigma_\alpha)+y^m\frac{\partial}{\partial x^m}\Big(\ln\frac{\sigma_\alpha}{\sigma}\Big)\\ &= y^m\frac{\partial}{\partial x^m}\Big(\ln\frac{\sigma_\alpha}{\sigma}\Big). \end{aligned} \tag{4 .23}$$

Here we have used $\frac{\partial\bar{G}^m}{\partial y^m}-y^m\frac{\partial}{\partial x^m}\ln\sigma_\alpha=0$ and $\frac{\partial}{\partial y^m}(\alpha s^m_0)=0$, where $\sigma_\alpha=\sqrt{\det(a_{ij}(x))}$. Denote by $\mu=\frac{1}{n+1}\ln\frac{\sigma_\alpha}{\sigma}$ and $\mu_0=\mu_{x^m}y^m=\frac{\partial\mu}{\partial x^m}y^m$. Then (4 .23) can be rewritten by $\tilde{\mathbf{S}}=(n+1)\mu_0$. Thus we have

$$\begin{aligned} \tilde{\mathbf{S}}_{|0} &= \tilde{\mathbf{S}}_{;0}-2\alpha s^m_0\tilde{\mathbf{S}}_{\cdot m}\\ &= (n+1)(\mu_{0;0}-2\alpha s^m_0\mu_{x^m}). \end{aligned} \tag{4 .24}$$

and

$$\tilde{\mathbf{S}}^2 = (n+1)^2\mu_0^2. \qquad (4\ .25)$$

Plugging (4 .22), (4 .24) and (4 .25) into (4 .21) yields

$$\begin{aligned} \mathbf{PRic}_{(G,dV)} &= \mathrm{Ric}_\alpha + (2\alpha s^m{}_{0;m} - 2t_{00} - \alpha^2 t^m{}_m) \\ &+ (n-1)(\mu_{0;0} - 2\alpha s^m{}_0 \mu_{x^m}) + (n-1)\mu_0^2. \qquad (4\ .26) \end{aligned}$$

On the other hand, by (4 .20), the S-curvature of (F, dV) is given by

$$\begin{aligned} \mathbf{S} &= \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln\sigma) \\ &= \frac{\partial \bar{G}^m}{\partial y^m} + (n+1)P - y^m \frac{\partial}{\partial x^m}(\ln\sigma) \\ &= y^m \frac{\partial}{\partial x^m}(\ln\sigma_\alpha) + (n+1)P - y^m \frac{\partial}{\partial x^m}(\ln\sigma) \\ &= (n+1)(\mu_0 + P), \qquad (4\ .27) \end{aligned}$$

where $P = \frac{r_{00} - 2\alpha s_0}{2F}$. Hence by (4 .26) and (4 .27), equation (3 .16) is equivalent to

$$\begin{aligned} & \mathrm{Ric}_\alpha + (2\alpha s^m{}_{0;m} - 2t_{00} - \alpha^2 t^m{}_m) \\ + \ & (n-1)\{\mu_{0;0} - 2\alpha s^m{}_0 \mu_{x^m}\} + (n-1)\mu_0^2 \\ = \ & (n-1)\{-f_0^2 + f_{0;0} - 2Pf_0 - 2\alpha s^m{}_0 f_{x^m} + \frac{2}{n+1} f_0[(n+1)(\mu_0+P)]\} \\ = \ & (n-1)\{-f_0^2 + f_{0;0} - 2\alpha s^m{}_0 f_{x^m} + 2f_0\mu_0\}. \end{aligned}$$

The above equation is actually in the form of

$$A + \alpha B = 0,$$

where A and B are polynomial in y. We see that this equation is valid if and only if $A = B = 0$. Then we obtain

$$\begin{aligned} \mathrm{Ric}_\alpha &= 2t_{00} + \alpha^2 t^m_m - (n-1)\{\mu_{0;0} - f_{0;0} + (\mu_0 - f_0)^2\}, \\ s^m{}_{0;m} &= (n-1)s^m{}_0\{\mu_{x^m} - f_{x^m}\}. \end{aligned}$$

Letting $h := \mu - f$, we complete the proof of Theorem 1.5. Q.E.D.

5 The Riemannian case

Let $F = \alpha$ be a Riemann metric and $dV = e^{-(n+1)f} dV_\alpha$ be a volume form on M. Then $\mathbf{S}_{(G,dV)} = (n+1)f_{;0}$. By (1 .2), we have

$$\mathbf{PRic}_{(G,dV)} = \mathbf{Ric}_\alpha + (n-1)\Big\{f_{0;0} + f_{;0}^2\Big\}.$$

Then Theorem 1.5 reduces to the following

Corollary 5.1 *Let (M, α) be an n-dimensional Riemannian manifold and $dV = e^{-(n+1)f}dV_\alpha$ a volume form on M. Then $\mathbf{PRic}_{(G,dV)} = 0$ if and only if*

$$\mathbf{Ric}_\alpha = -(n-1)\{f_{0;0} + f_{;0}^2\},$$

This leads to a new notion of *weighted Ricci curvature* $\mathbf{Ric}_\alpha^f$ on a Riemannian n-manifold (M, α) with a scalar function $f = f(x)$ on M:

$$\mathbf{Ric}_\alpha^f := \mathbf{Ric}_\alpha + (n-1)\{f_{0;0} + f_{;0}^2\}. \qquad (5\ .28)$$

This weighted Ricci curvature and its relationship with other geometric quantities of (α, f) deserves further study.

Example 5.2 *Let $(\mathbb{S}^n, \alpha)$ be a Riemannian n-sphere with constant sectional curvature $\mathbf{K}_\alpha = 1$. Recall that in this case, there exists a scalar function ϕ on M such that $\phi_{0;0} = -\phi\alpha^2$, where ";" is covariant derivative with respect to the Riemann metric α. Thus, the Ricci curvature $\mathbf{Ric}_\alpha$ of α satisfies*

$$\mathbf{Ric}_\alpha = (n-1)\alpha^2 = -(n-1)\frac{\phi_{0;0}}{\phi}. \qquad (5\ .29)$$

Set $f = \ln|\phi|$ which is only defined on $\{\phi \neq 0\}$. Then (5 .29) can be rewritten by

$$\mathbf{Ric}_\alpha + (n-1)\{f_0^2 + f_{0;0}\} = 0.$$

It is the case in Corollary 5.1.

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