HERMITE-PADÉ APPROXIMATION AND SIMULTANEOUS QUADRATURE FORMULAS

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Abstract. We study Hermite-Padé approximation of so called Nikishin systems of functions. In particular, the set of multi-indices for which normality is known to take place is considerably enlarged as well as the sequences of multi-indices for which convergence of the corresponding simultaneous rational approximants takes place. These results are applied to the study of the convergence properties of simultaneous quadrature rules of a given function with respect to different weights.

1. Introduction

Let $S = (s_1, \dots , s_m)$ be a system of finite Borel measures with constant sign and compact support $\mathrm{supp}(s_k) \subset \mathbb{R}, k = 1, \dots , m$, contained in the real line consisting of infinitely many points. In [1], it is claimed that some applications in computer graphics illuminating bodies require the simultaneous evaluation of the integrals $\int f(x)ds_k(x), k = 1, \dots , m$. For this purpose, the author proposes a numerical scheme of m quadrature rules all of which have the same set of nodes.

Let N distinct points $x_1, \dots , x_N$ be given which lie in $\mathrm{Co}(\cup_{k=1}^m(\mathrm{supp}(s_k)))$, the smallest interval containing the union of the supports of the measures in the system S. We say that we have an interpolatory type simultaneous scheme of quadrature rules for S of order N if

$$\int p(x)ds_k(x) = \sum_{j=1}^{N} \lambda_{k,j}p(x_j)\,, \quad k = 1, \dots , m\,, \tag{1}$$

for all $p \in \mathcal{P}_{N-1}$, the vector space of all polynomials of degree at most $N-1$, with coefficients $\lambda_{k,j}$ appropriately chosen.

Set $Q(x) = \prod_{j=1}^N (x - x_j)$. For $p \in \mathcal{P}_{N-1}$, from Lagrange's interpolation formula we have

$$p(x) = \sum_{j=1}^{N} \frac{Q(x)p(x_j)}{Q'(x_j)(x-x_j)}\,.$$

Integrating with respect to s_k one has

$$\int p(x)ds_k(x) = \sum_{j=1}^{N} p(x_j)\int \frac{Q(x)ds_k(x)}{Q'(x_j)(x-x_j)} = \sum_{j=1}^{N} \lambda_{k,j}p(x_j)\,, \quad k = 1, \dots , m\,,$$

with

$$\lambda_{k,j} = \int \frac{Q(x)ds_k(x)}{Q'(x_j)(x-x_j)}\,.$$

Therefore, given any system of distinct points $x_1, \dots , x_N$, such a simultaneous scheme of quadrature rules is always attainable.

The problem consists in the study of the convergence properties of such a scheme of simultaneous quadrature rules for a large class of functions f; for example, continuous on $\mathrm{Co}(\cup_{k=1}^m(\mathrm{supp}(s_k)))$ or analytic on a neighborhood of this set. That is, we would like to have

$$\lim_{N\to\infty} \sum_{j=1}^{N} \lambda_{N,k,j} f(x_{N,j}) = \int f(x)ds_k(x)\,, \quad k = 1, \dots , m\,,$$

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where $\{x_{N,j}\}, j = 1, \ldots, N, N \in \mathbb{N}$ is a triangular scheme of nodes contained in $\mathrm{Co}(\cup_{k=1}^m(\mathrm{supp}(s_k)))$ and f is in a sufficiently general class of functions.

Another question of equal importance is connected with the stability of the numerical procedure. For this, it is desirable to have that $\sup_{N\in\mathbb{N}}\sum_{j=1}^N |\lambda_{N,k,j}| < \infty, k = 1, \ldots, m$, or still more convenient that for each k and N the coefficients $\lambda_{N,k,j}, j = 1, \ldots, N$, preserve the same sign. In this case, from the quadrature rule, taking $p \equiv 1$, we have

$$|s_k| = |\int ds_k(x)| = |\sum_{N,k,j} \lambda_{N,k,j}| = \sum_{N,k,j} |\lambda_{N,k,j}|\,.$$

As in Gauss-Jacobi quadrature rules one may ask if the nodes $x_1, \ldots, x_N$, may be taken so that the quadrature formulas are exact in a class of polynomials as large as possible hoping to get automatically coefficients of equal sign. Unlike the case when $m = 1$, we shall see that in general this problem is not well posed in the sense that it may not have a solution or it may have infinitely many. The existence of solution may require nodes of multiplicity greater than 1 or that the nodes lie outside $\mathrm{Co}(\cup_{k=1}^m(\mathrm{supp}(s_k)))$.

In this paper we give several results of general nature concerning Gauss–Jacobi type simultaneous quadrature rules, their connection with Hermite-Padé approximation, their convergence properties, and rate of convergence. This is done in section 2. In section 3, we emphasize on the case when the measures in S are interlinked in a special way. More exactly, when they form what is called a Nikishin system of measures (see Definition 3 below).

2. Some general results.

As above, let $S = (s_1, \ldots, s_m)$ be a system of finite Borel measures with constant sign and compact support $\mathrm{supp}(s_k) \subset \mathbb{R}, k = 1, \ldots, m$, consisting of infinitely many points. Let $\widehat{S} = (\widehat{s}_1, \ldots, \widehat{s}_m)$ be the corresponding system of Markov functions; that is,

$$\widehat{s}_k(z) = \int \frac{ds_k(x)}{z - x}, \qquad k = 1, \ldots, m\,.$$

We define the simultaneous Hermite-Padé approximant of $\widehat{S}$ with respect to the multi-index $n = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$ as a vector rational function $R_n = (\frac{P_{n,1}}{Q_n}, \ldots, \frac{P_{n,m}}{Q_n})$ with common denominator Q_n that satisfies

i) $\deg Q_n \leq |n| = n_1 + \cdots + n_m, \quad Q_n \not\equiv 0\,,$

ii) $(Q_n\widehat{s}_k - P_{n,k})(z) = \mathcal{O}\left(\frac{1}{z^{n_k+1}}\right), \quad z \to \infty\,, \quad k = 1, \ldots, m\,.$

Integrating along a closed path with winding number 1 for all its interior points which surrounds $\mathrm{supp}(s_k)$ and using Fubini's theorem, it is easy to verify that Q_n fulfills the following system of orthogonality relations

$$0 = \int x^\nu Q_n(x) ds_k(x)\,, \quad \nu = 0, \ldots, n_k - 1\,, \quad k = 1, \ldots, m\,. \tag{2}$$

It is said that Q_n is a multi-orthogonal polynomial of S relative to the multi-index n. In the sequel, we assume that Q_n is monic. In general, the polynomial Q_n is not uniquely determined.

Let E be a subset of the real line $\mathbb{R}$. By $\mathrm{Co}(E)$ we denote the smallest interval which contains E. The interior of an interval of the real line refers to its interior in the euclidean topology of $\mathbb{R}$.

Definition 1. We say that a multi-index n is weakly normal for the system S if Q_n is determined uniquely. A multi-index n is said to be normal if any non trivial solution Q_n of (2) satisfies $\deg Q_n = |n|$. If Q_n has exactly $|n|$ simple zeros and they all lie in the interior of $\mathrm{Co}(\cup_{j=1}^m \mathrm{supp}(s_j))$ the index is called strongly normal. When all the indices are weakly normal, normal, or strongly normal the system S is said to be weakly perfect, perfect, or strongly perfect respectively.

Normality of indices plays a crucial role in applications to number theory and Hermite-Padé approximation. Obviously, strong normality implies normality, and it is not hard to prove that normality implies weak normality (see Lemma 1 in [8] where you can also find further discussions on the subject).

From ii) it is obvious that $P_{n,k}$ is the polynomial part of $Q_n\widehat{s}_k$. Therefore, given Q_n, the polynomial $P_{n,k}$ is uniquely determined. For a moment, set

$$P_{n,k}(z) = \int \frac{Q_n(z) - Q_n(x)}{z - x} ds_k(x), \quad k = 1, \dots, m\,.$$

Using (2) it is straightforward that ii) takes place; thus, this polynomial is in fact the one defined above. Therefore, if n is weakly normal the polynomials $P_{n,k}$ (and consequently R_n) are also uniquely determined. If the index n is strongly normal then

$$\frac{P_{n,k}(z)}{Q_n(z)} = \sum_{j=1}^{|n|} \frac{\lambda_{n,k,j}}{z - x_{n,j}}\,, \quad k = 1, \dots, m\,, \tag{3}$$

where $Q_n(z) = \prod_{j=1}^{|n|}(z - x_{n,j})$ and

$$\lambda_{n,k,j} = \lim_{z\to x_{n,j}} \frac{z - x_{n,j}}{Q_n(z)} \int \frac{Q_n(z) - Q_n(x)}{z - x} ds_k(x) = \int \frac{Q_n(x)ds_k(x)}{Q_n'(x_{n,j})(x - x_{n,j})}\,. \tag{4}$$

Definition 2. The numbers defined by (4) will be called Nikishin– Christoffel coefficients.

Lemma 1. *Let n be strongly normal for the system $S = (s_1, \dots, s_m)$. Then, for each $k = 1, \dots, m$*

$$\int p(x)ds_k(x) = \sum_{j=1}^{|n|} \lambda_{n,k,j}p(x_{n,j})\,, \quad p \in \mathcal{P}_{|n|+n_k-1}\,,$$

where $\mathcal{P}_N$ denotes the vector space of all polynomials of degree at most N.

Proof. Fix $k \in \{1, \dots, m\}$ and assume that $p \in \mathcal{P}_{|n|+n_k-1}$. Let

$$\ell(x) = \sum_{j=1}^{|n|} \frac{Q_n(x)p(x_{n,j})}{Q_n'(x_{n,j})(x - x_{n,j})}$$

be the Lagrange polynomial of degree $|n| - 1$ that interpolates p at the zeros of Q_n. By the definition of ℓ it follows that

$$p(x) - \ell(x) = Q_n(x)q(x)$$

where $q \in \mathcal{P}_{n_k-1}$. Therefore, from (2) and (4), we have

$$0 = \int (p - \ell)(x)ds_k(x) = \int p(x)ds_k(x) - \sum_{j=1}^{|n|} p(x_{n,j}) \int \frac{Q_n(x)ds_k(x)}{Q_n'(x_{n,j})(x - x_{n,j})} =$$

$$\int p(x)ds_k(x) - \sum_{j=1}^{|n|} \lambda_{n,k,j}p(x_{n,j})\,,$$

which is what we needed to prove. □

Remark . In the case of normal indices, for which the zeros are not necessarily distinct, one can obtain a similar quadrature formula exact for all $p \in \mathcal{P}_{|n|+n_k-1}$ but on the right hand appear all the derivatives of the polynomial up to the multiplicity of the corresponding zero of Q_n minus one.

Notice that in Lemma 1 we have exactness with respect to each measure at least of order $|n|$. Therefore, all such simultaneous quadrature rules are of interpolatory type. In terms of the Nikishin–Christoffel coefficients, we distinguish several cases.

Let $\Lambda \subset \mathbb{Z}_+^m$ be a sequence of distinct strongly normal multi–indices and $k \in \{1, \dots, m\}$ fixed.

A) For each $n \in \Lambda$ all $\lambda_{n,k,j}, j = 1, \dots, |n|$, have the same sign.

B)

$$\sup_{n\in\Lambda} \sum_{j=1}^{|n|} |\lambda_{n,k,j}| \le C < \infty\,.$$

C)

$$\sum_{j=1}^{|n|} |\lambda_{n,k,j}| \leq C|n|^{\alpha} < \infty\,, \quad \alpha \in (0,+\infty)\,, \quad n \in \Lambda\,.$$

D)

$$\sum_{j=1}^{|n|} |\lambda_{n,k,j}| \leq C|n|^{\alpha(n)} < \infty\,, \quad \lim_{n\in\Lambda} \alpha(n)\log|n|/|n| = 0\,.$$

It is obvious that $A) \Rightarrow B) \Rightarrow C) \Rightarrow D)$. Depending on whether one has A), B), C), or D) one can prove that

$$\lim_{n\in\Lambda} \sum_{j=1}^{|n|} \lambda_{n,k,j} f(x_{n,j}) = \int f(x) ds_k(x)\,, \tag{5}$$

for different classes of functions f.

We denote $\mathrm{Lip}_\beta([a,b]), 0 \leq \beta \leq 1$, the class of all complex valued functions f defined on the interval $[a,b] \subset \mathbb{R}$ such that

$$|f(x) - f(y)| \leq C|x-y|^{\beta}\,, \quad x, y \in [a,b]\,.$$

We say that $f \in \mathrm{Lip}_\beta([a,b]), 1 < \beta < \infty$, if the $[\beta]$th derivative of f exists and is in $\mathrm{Lip}_{\beta-[\beta]}([a,b])$, where $[\beta]$ denotes the integer part of β. The next lemma summarizes some known results which allow to deduce convergence of the quadrature rule. For completeness we include some proofs. More on sufficient conditions for the convergence of interpolatory quadrature rules see may be found in [5] and the references therein.

Lemma 2. *Let S be a system of measures and $\Lambda \subset \mathbb{Z}_+^m$ a sequence of distinct strongly normal multi–indices. Set $\Delta = \mathrm{Co}(\cup_{k'=1}^m \mathrm{supp}(s_{k'}))$. Then:*

- A) *implies* (5) *for all Riemann integrable functions f on Δ.*
- B) *implies* (5) *for all continuous functions f on Δ.*
- C) *implies* (5) *for all $f \in Lip_\beta(\Delta), \beta > \alpha$. Moreover,*

$$\left| \int f(x) ds_k(x) - \sum_{j=1}^{|n|} \lambda_{n,k,j} f(x_{n,j}) \right| = \mathcal{O}\left(\frac{1}{|n|^{\beta-\alpha}}\right)\,. \tag{6}$$

- D) *implies that*

$$\limsup_{n\in\Lambda} \left\| \widehat{s}_k - \frac{P_{n,k}}{Q_n} \right\|_K^{1/|n|} \leq \|\varphi\|_K\,, \quad K \subset \overline{\mathbb{C}} \setminus \Delta\,, \tag{7}$$

where $\|\cdot\|_K$ denotes the sup norm on the compact set K and φ denotes the conformal representation of $\overline{\mathbb{C}} \setminus \Delta$ onto $\{w : |w| < 1\}$ such that $\varphi(\infty) = 0$ and $\varphi'(\infty) > 0$. If f is analytic on a neighborhood V of $\Delta\,(f \in H(V))$, then (7) *implies*

$$\lim_{n\in\Lambda} |\int f(x) ds_k(x) - \sum_{j=1}^{|n|} \lambda_{n,k,j} f(x_{n,j})|^{1/|n|} \leq \rho_V\,, \tag{8}$$

where $\rho_V = \inf\{\rho : \gamma_\rho \subset V\}$ and $\gamma_\rho = \{z : |\varphi(z)| = \rho\}, 0 < \rho < 1$.

Proof. The first two statements are classical and contained, for example, in Theorems 15.2.2 and 15.2.1, respectively, of [18]. The third is also fairly well known. Notice that for each $p \in \mathcal{P}_{|n|-1}$, using the quadrature formula, we obtain

$$|\int f(x) ds_{k'}(x) - \sum_{j=1}^{|n|} \lambda_{n,k,j} f(x_{n,j})| \leq \int |f(x) - p(x)||ds_k(x)| + \sum_{j=1}^{|n|} |\lambda_{n,k,j}||f(x_{n,j}) - p(x_{n,j})|\,.$$

Therefore,

$$|\int f(x) ds_{k'}(x) - \sum_{j=1}^{|n|} \lambda_{n,k,j} f(x_{n,j})| \leq (|s_k| + C|n|^{\alpha})\, E_{|n|-1}(f)\,.$$

From Jackson's Theorem (see page 147 in [4]), we have that if $f \in \mathrm{Lip}_\beta(\Delta)$ then $E_{|n|-1}(f) \leq C_1/|n|^\beta$ where C_1 does not depend on $n \in \Lambda$. From this follows (5) for this class of functions when $\beta > \alpha$ with the given estimate for the error.

The last statement is familiar to specialists in Padé approximation. Let us prove (7). Since n is strongly normal, from ii) we have that

$$\widehat{s}_k - \frac{P_{n,k}}{Q_n} = \mathcal{O}\left(\frac{1}{z^{|n|+1}}\right), \quad z \to \infty,$$

and

$$\frac{\widehat{s}_k - \frac{P_{n,k}}{Q_n}}{\varphi^{|n|+1}} \in H(\overline{\mathbb{C}} \setminus \Delta).$$

Set $\gamma_\rho = \{z : |\varphi(z)| = \rho\}, 0 < \rho < 1$. Using D), it follows that

$$\left\|\widehat{s}_k - \frac{P_{n,k}}{Q_n}\right\|_{\gamma_\rho} \leq C_\rho |n|^{\alpha(n)},$$

where C_ρ is a constant which depends on the curve γ_ρ but not on n. Therefore,

$$\left|\frac{(\widehat{s}_k - \frac{P_{n,k}}{Q_n})(z)}{\varphi^{|n|+1}(z)}\right| \leq \frac{C_\rho |n|^{\alpha(n)}}{d(\gamma_\rho)\rho^{|n|+1}}, \quad z \in \gamma_\rho,$$

with $d(\gamma_\rho) = \inf\{|z - x| : z \in \gamma_\rho, x \in \Delta\}$. By the maximum principle

$$\left|(\widehat{s}_k - \frac{P_{n,k}}{Q_n})(z)\right| \leq \frac{C_\rho |n|^{\alpha(n)}}{d(\gamma_\rho)} \left(\frac{|\varphi(z)|}{\rho}\right)^{|n|+1}, \quad z \in \mathrm{Ext}(\gamma_\rho),$$

where $\mathrm{Ext}(\gamma_\rho)$ denotes the unbounded connected component of the complement of γ_ρ. Fix a compact set $K \subset \overline{\mathbb{C}} \setminus \Delta$ and take ρ sufficiently close to 1 so that $K \subset \mathrm{Ext}(\gamma_\rho)$. It follows that

$$\left\|\widehat{s}_k - \frac{P_{n,k}}{Q_n}\right\|_K \leq \frac{C_\rho |n|^{\alpha(n)}}{d(\gamma_\rho)} \left(\frac{\|\varphi\|_K}{\rho}\right)^{|n|+1}.$$

Thus, using the assumption on the sequence of numbers $\{\alpha(n)\}$, it follows that

$$\limsup_{n \in \Lambda} \left\|\widehat{s}_k - \frac{P_{n,k}}{Q_n}\right\|_K^{1/|n|} \leq \left(\frac{\|\varphi\|_K}{\rho}\right),$$

and letting $\rho \to 1$, we find that

$$\limsup_{n \in \Lambda} \left\|\widehat{s}_k - \frac{P_{n,k}}{Q_n}\right\|_K^{1/|n|} \leq \|\varphi\|_K.$$

To conclude let us show that (7) implies (8). Using (3), Cauchy's integral formula, and Fubini's Theorem, it follows that

$$\int f(x) ds_k(x) - \sum_{j=1}^{|n|} \lambda_{n,k,j} f(x_{n,j}) = \frac{1}{2\pi i} \int \int_{\gamma_\rho} \frac{f(z)}{z - x} dz ds_k(x) - \sum_{j=1}^{|n|} \lambda_{n,k,j} \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{z - x_{n,j}} dz =$$

$$\frac{1}{2\pi i} \int_{\gamma_\rho} f(z)(\widehat{s}_k - \frac{P_{n,k}}{Q_n})(z) dz.$$

Therefore,

$$\left|\int f(x) ds_k(x) - \sum_{j=1}^{|n|} \lambda_{n,k,j} f(x_{n,j})\right| \leq C \|f\|_{\gamma_\rho} \|\widehat{s}_k - \frac{P_{n,k}}{Q_n}\|_{\gamma_\rho},$$

where C denotes the length of γ_ρ divided by 2π. This inequality and (7) immediately give (8). $\square$

Remark . In the first three statements of Lemma 2 the assumption on f may have been given on $\mathrm{Co}(\mathrm{supp}(s_k))$ instead of all Δ. This is so because any function Riemann integrable, continuous, or Lip_β on $\mathrm{Co}(\mathrm{supp}(s_k))$ may be extended within the same class respectively to Δ. In this case, the quadrature formula applied to a function defined on $\mathrm{Co}(\mathrm{supp}(s_k))$ must be understood as its application to any of its extensions to Δ pertaining to the same class. Since the integral depends only on the values of the function on $\mathrm{Co}(\mathrm{supp}(s_k))$ this means that the nodes lying in $\Delta \setminus \mathrm{Co}(\mathrm{supp}(s_k))$ give no contribution to the approximate evaluation of the integral. From the practical point of view it is better to think that the function is extended with value zero outside of $\mathrm{Co}(\mathrm{supp}(s_k))$ though this extension does not necessarily preserve the class in the second and third cases. Concerning the statements following assumption D) one cannot say the same because analytic functions cannot be extended at will. Nevertheless, we point out that in the proof we only use that V is a neighborhood of an interval $[a, b]$ containing the zeros of the polynomials Q_n and the support of the measure s_k. Therefore in relations (7) and (8) one can substitute Δ by $[a, b]$. These remarks will be used in the statement and proof of Theorem 1 below without special notice.

In general, it is difficult to guarantee strong normality of a multi–index and even then it is more complicated to verify one of the conditions A)-D). For the moment, we will restrict our attention to a sufficiently general system of measures and a special selection of multi–indices for which strong normality and some of the conditions A)-D) are fulfilled.

Let σ be a finite positive Borel measure supported on a compact subset of $\mathbb{R}$ and $S = (s_1, \ldots , s_m)$ be such that $ds_k(x) = w_k(x)d\sigma(x), w_k \in L_1(\sigma), k \in \{1, \ldots , m\}$, where each w_k preserves sign on $\mathrm{supp}(\sigma)$. Whenever it is convenient we use the differential notation of a measure. Let $\Lambda_k \subset \mathbb{Z}_+^m$ be the sequence of multi–indices of the form $\widetilde{N} = (0, \ldots , 0, N, 0, \ldots , 0), N \in \mathbb{Z}_+$, and the number N is placed in the kth component of the multi–index. We have

Theorem 1. *Let S and Λ_k be as indicated above. All multi–indices in Λ_k are strongly normal. For the component k,* A) *takes place. Consequently,* (5) *holds for all bounded Riemann integrable function f on* $\mathrm{Co}(\mathrm{supp}(s_k))$ *and if $f \in Lip_\beta(\mathrm{Co}(\mathrm{supp}(s_k))), \beta > 0$, then*

$$\left| \int f(x)ds_k(x) - \sum_{j=1}^{N} \lambda_{\widetilde{N},k,j} f(x_{\widetilde{N},j}) \right| = \mathcal{O}\left(\frac{1}{N^\beta}\right) . \tag{9}$$

If for some $k' \in \{1, \ldots , m\}$, we have that

$$C_{k,k'} := \left(\int \frac{|w_{k'}(x)|^2}{|w_k(x)|} d\sigma(x) \right)^{1/2} < \infty \, , \tag{10}$$

then

$$\sum_{j=1}^{N} |\lambda_{\widetilde{N},k',j}| \le C_{k,k'} \sqrt{|s_k|} \, , \quad \widetilde{N} \in \Lambda_k \, , \tag{11}$$

and for all $f \in Lip_\beta(\mathrm{Co}(\mathrm{supp}(s_{k'}))), \beta > 0$,

$$\left| \int f(x)ds_{k'}(x) - \sum_{j=1}^{N} \lambda_{\widetilde{N},k',j} f(x_{\widetilde{N},j}) \right| = \mathcal{O}\left(\frac{1}{N^\beta}\right) . \tag{12}$$

We also have

$$\limsup_{N\to\infty} \left\| \widehat{s}_k - \frac{P_{\widetilde{N},k}}{Q_{\widetilde{N}}} \right\|_K^{1/2N} \le \|\varphi\|_K \, , \quad K \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s_k)) \, , \tag{13}$$

and

$$\limsup_{N\to\infty} \left\| \widehat{s}_{k'} - \frac{P_{\widetilde{N},k'}}{Q_{\widetilde{N}}} \right\|_K^{1/N} \le \|\varphi\|_K \, , \quad K \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s_k)) \, , \tag{14}$$

where φ denotes the conformal representation of $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s_k))$ onto $\{w : |w| < 1\}$ such that $\varphi(\infty) = 0$ and $\varphi'(\infty) > 0$. If f is analytic on a neighborhood V of $\mathrm{Co}(\mathrm{supp}(\sigma))$ $(f \in H(V))$, then

$$\lim_{N\to\infty} \left| \int f(x)ds_k(x) - \sum_{j=1}^{N} \lambda_{\widetilde{N},k,j} f(x_{\widetilde{N},j}) \right|^{1/2N} \leq \rho_V \, , \tag{15}$$

and

$$\lim_{N\to\infty} \left| \int f(x)ds_{k'}(x) - \sum_{j=1}^{N} \lambda_{\widetilde{N},k',j} f(x_{\widetilde{N},j}) \right|^{1/N} \leq \rho_V \, , \tag{16}$$

where $\rho_V = \inf\{\rho : \gamma_\rho \subset V\}$ and $\gamma_\rho = \{z : |\varphi(z)| = \rho\}, 0 < \rho < 1$.

Proof. We only need to prove that for the index k, property A) takes place and that for an index k' for which (10) holds (11) takes place and then make use of Lemma 2.

Fix $\widetilde{N} \in \Lambda_k$. From i) and (2) we have that $Q_{\widetilde{N}}$ is the Nth orthogonal polynomial with respect to the measure s_k. Therefore, $Q_{\widetilde{N}}$ has exactly $|\widetilde{N}| = N$ simple zeros in the interior of $\mathrm{Co}(\mathrm{supp}(s_k))$ as needed to affirm that n is strongly normal.

Let $j \in \{1, \ldots , N\}$ be fixed. Taking $p(x) = (Q_{\widetilde{N}}(x)/Q'_{\widetilde{N}}(x_{\widetilde{N},j})(x - x_{\widetilde{N},j}))^2$ in Lemma 1 one sees that

$$\lambda_{\widetilde{N},k,j} = \int \left(\frac{Q_{\widetilde{N}}(x)}{Q'_{\widetilde{N}}(x_{\widetilde{N},j})(x - x_{\widetilde{N},j})} \right)^2 ds_k(x) \, .$$

Therefore, all $\lambda_{\widetilde{N},k,j}, j = 1, \ldots , N$, have the same sign as the measure s_k. The convergence of the corresponding quadrature for all Riemann integrable functions follows from the first assertion of Lemma 2 and (9) is a consequence of the third statement in Lemma 2.

That (10) implies (11) is a slight generalization of a result due to Sloan and Smith in [17], Theorem 1, (they only consider weights). For completeness we include a proof.

Take $k' \in \{1, \ldots , m\}$ such that (10) takes place. Using (4), it follows that

$$\lambda_{\widetilde{N},k',j} = \int \frac{Q_{\widetilde{N}}(x)}{Q'_{\widetilde{N}}(x_{\widetilde{N},j})(x - x_{\widetilde{N},j})} \frac{w_{k'}(x)}{w_k(x)} w_k(x) d\sigma(x) \, .$$

Write $w_{k'}/w_k = S_{N-1} + R_{N-1}$ where S_{N-1} denotes the Nth partial sum of the Fourier expansion of $w_{k'}/w_k$ in the orthogonal system given by $\{Q_{\widetilde{N}}\}, N \in \mathbb{Z}_+$. From (10) we have that the function $w_{k'}/w_k$ is square integrable with respect to the measure $w_k d\sigma$; therefore, its Fourier series converges to the function in $L_2(w_k d\sigma)$. Using this and the previous formula it follows that

$$\lambda_{\widetilde{N},k',j} = \int \frac{Q_{\widetilde{N}}(x)}{Q'_{\widetilde{N}}(x_{\widetilde{N},j})(x - x_{\widetilde{N},j})} S_{N-1}(x) w_k(x) d\sigma(x) \, .$$

Since S_{N-1} is a polynomial of degree at most $N - 1$, from the orthogonality properties of $Q_{\widetilde{N}}$, we obtain

$$\lambda_{\widetilde{N},k',j} = S_{N-1}(x_{\widetilde{N},j}) \int \frac{Q_{\widetilde{N}}(x)}{Q'_{\widetilde{N}}(x_{\widetilde{N},j})(x - x_{\widetilde{N},j})} w_k(x) d\sigma(x) = \lambda_{\widetilde{N},k,j} S_{N-1}(x_{\widetilde{N},j}) \, .$$

Using the Gauss-Jacobi formula satisfied by the kth component, and the Cauchy-Schwartz and Bessel inequalities, we obtain

$$\sum_{j=1}^{N} |\lambda_{\widetilde{N},k',j}| = \sum_{j=1}^{N} \lambda_{\widetilde{N},k,j} |S_{N-1}(x_{\widetilde{N},j})| \leq \left(\sum_{j=1}^{N} \lambda_{\widetilde{N},k,j} \right)^{1/2} \left(\sum_{j=1}^{N} \lambda_{\widetilde{N},k,j} S^2_{N-1}(x_{\widetilde{N},j}) \right)^{1/2} =$$

$$\sqrt{|s_k|} \left(\int S^2_{N-1}(x) w_k(x) d\sigma(x) \right)^{1/2} \leq C_{k,k'} \sqrt{|s_k|}$$

as we needed to prove.

Now, (12), (14), and (16) are direct consequences of (6), (7), and (8) respectively taking into consideration that all the zeros of $Q_{\widetilde{N}}$ lie on $\mathrm{Co}(\mathrm{supp}(s_k))$ and that from (10) $\mathrm{supp}(s_{k'}) \subset \mathrm{supp}(s_k)$. To prove (13) and (15) one follows the same scheme noticing that for the index k one has

$$\widehat{s}_k - \frac{P_{\widetilde{N},k}}{Q_{\widetilde{N}}} = \mathcal{O}\left(\frac{1}{z^{2N+1}}\right), \quad z \to \infty .$$

With this we conclude the proof of this theorem. □

Given the way in which the nodes are chosen it is possible to prove that (10) implies convergence of the quadrature rule corresponding to the component k' for all Riemann-integrable functions f on Δ. For details see [17]. We wish to point out that condition (10), used here to derive (11), was also employed in [12] and [13] in the study of the convergence of interpolatory quadrature rules for complex weights and quadrature rules exact for rational functions with prescribed poles.

3. Nikishin systems.

In order to study more general classes of indices for which strong normality and convergence of the simultaneous quadrature rules take place, we further restrict the class of systems of measures under consideration.

Nikishin systems of measures were introduced in [16]. For them a large class of indices are known to be strongly normal. Such systems are defined as follows. We adopt the notation introduced in [11] which is clarifying.

Let σ_1 and σ_2 be two measures supported on $\mathbb{R}$ and let Δ_1, Δ_2 denote the smallest intervals containing $\mathrm{supp}(\sigma_1)$ and $\mathrm{supp}(\sigma_2)$ respectively. We write $\Delta_i = \mathrm{Co}(\mathrm{supp}(\sigma_i))$. Assume that $\Delta_1 \cap \Delta_2 = \emptyset$ and define

$$\langle \sigma_1, \sigma_2 \rangle (x) = \int \frac{d\sigma_2(t)}{x-t} d\sigma_1(x) = \widehat{\sigma}_2(x) d\sigma_1(x) .$$

Therefore, $\langle \sigma_1, \sigma_2 \rangle$ is a measure with constant sign and support equal to that of σ_1.

Definition 3. For a system of closed intervals $\Delta_1, \ldots, \Delta_m$ contained in $\mathbb{R}$ satisfying $\Delta_{j-1} \cap \Delta_j = \emptyset, j = 2, \ldots, m$, and finite Borel measures $\sigma_1, \ldots, \sigma_m$ with constant sign and $\mathrm{Co}(\mathrm{supp}(\sigma_j)) = \Delta_j$, we define by induction

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_j \rangle = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_j \rangle \rangle, \quad j = 2, \ldots, m .$$

We say that $S = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, where

$$s_1 = \langle \sigma_1 \rangle = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \ldots, s_m = \langle \sigma_1, \ldots, \sigma_m \rangle ,$$

is the Nikishin system of measures generated by $(\sigma_1, \ldots, \sigma_m)$.

Remark . All the results that follow hold true if in the definition of a Nikishin system we only require that the interior (in $\mathbb{R}$) of $\Delta_{j-1} \cap \Delta_j, j = 2, \ldots, m$, be empty as long as the corresponding measures $s_j, j = 1, \ldots, m$, are all finite. This allows consecutive intervals Δ_j to have a common end point. We restrict generality in order to simplify the arguments in the proofs.

Notice that all the measures in a Nikishin system have the same support, namely $\mathrm{supp}(\sigma_1)$. For Nikishin systems of measures all multi-indices n satisfying $1 \leq i < j \leq m \Rightarrow n_j \leq n_i + 1$ are known to be strongly normal. This result was originally proved in [6]. More recently, an extension for so called generalized Nikishin systems was given in [11]. When $m = 2$, from the results in [3] it follows that the system is strongly perfect (a detailed proof may be found in [6]). In [2], the authors were able to include in the set of strongly normal indices all those for which there do not exist $1 \leq i < j < k \leq m$ such that $n_i < n_j < n_k$. This special class of multi–indices will be denoted $\mathbb{Z}_+^m(*)$ in the sequel. For $m = 3$, in [8] the authors prove that the system is strongly perfect.

In [16] the numbers $\lambda_{n,k,j}$ were introduced for the study of the convergence properties of the Hermite–Padé approximants of a Nikishin system of two functions. Let us denote

$$F_{n,k}(z) = (Q_n \widehat{s}_k - P_{n,k})(z) , \quad k = 1, \ldots, m .$$

In [3] (see Lemmas 4-6), it was proved that the functions $F_{n,k}$ satisfy certain orthogonality relations on the second interval $\Delta_2 = \mathrm{Co}(\mathrm{supp}(\sigma_2))$. The following lemma summarizes these results and we refer to the original source for the proof. We wish to stress that the range of degrees for which (20) and (21) below are indicated here to hold is a bit larger than in the statement of the original

Lemma 6 in [3]. Nevertheless, the proof is exactly the same. In that paper the authors were not concerned with the signs of the Nikishin–Christoffel coefficients; therefore, they slightly simplified the statement in favor of brevity. Before going on with the lemma we need some additional notation.

Let $1 \leq i \leq j \leq m$. Set

$$s_{i,j} = \langle \sigma_i, \ldots, \sigma_j \rangle, \quad (s_{j,j} = \sigma_j).$$

It is well known (see Appendix in [15]) that there exists a first degree polynomial $\mathcal{L}_{i,j}$ and a finite positive Borel measure $\tau_{i,j}, \mathrm{Co}(\mathrm{supp}(\tau_{i,j})) \subset \mathrm{Co}(\mathrm{supp}(s_{i,j}))$ such that

$$\frac{1}{\widehat{s}_{i,j}(z)} = \mathcal{L}_{i,j}(z) + \widehat{\tau}_{i,j}(z) .$$

We associate to each function $F_{n,k}, k = 1, \ldots, m$, a Nikishin system of $m-1$ measures $S^k = (s_2^k, \ldots, s_m^k) = \mathcal{N}(\sigma_2^k, \ldots, \sigma_m^k)$ whose generating measures satisfy $\mathrm{supp}(\sigma_j^k) \subset \mathrm{Co}(\mathrm{supp}(\sigma_j))$ and do not depend on n. We preserve the notation introduced above meaning that $s_j^k = \langle \sigma_2^k, \ldots, \sigma_j^k \rangle, j = 2, \ldots, m$. In particular, all the measures of these m Nikishin systems have their support contained in Δ_2. The expression of the generating measures will be given in the lemma.

Lemma 3. *Let $n = (n_1, \ldots, n_m)$ be a multi-index. With the function $F_{n,1}$ we associate the Nikishin system*

$$S^1 = (s_2^1, \ldots, s_m^1) = (d\sigma_2, w_3^1 d\sigma_2, \ldots, w_m^1 d\sigma_2) = \mathcal{N}(\sigma_2, \ldots, \sigma_m)$$

with respect to which the following orthogonality relations hold

$$\int (h_j F_{n,1})(x) ds_j^1(x) = 0 , \quad \deg h_j \leq \min(n_1, n_j - 1) , \quad j = 2, \ldots, m . \tag{17}$$

With $F_{n,2}$ we associate

$$S^2 = (s_2^2, \ldots, s_m^2) = (d\tau_{2,2}, w_3^2 d\tau_{2,2}, \ldots, w_m^2 d\tau_{2,2}) = \mathcal{N}(\tau_{2,2}, \widehat{\sigma}_2 d\sigma_3, \sigma_4, \ldots, \sigma_m)$$

with respect to which we have

$$\int (h_2 F_{n,2})(x) ds_2^2(x) = 0 , \quad \deg h_2 \leq \min(n_1 - 1, n_2 - 2) , \tag{18}$$

and

$$\int (h_j F_{n,2})(x) ds_j^2(x) = 0 , \quad \deg h_j \leq \min(n_2 - 1, n_j - 1) , \quad j = 3, \ldots, m . \tag{19}$$

Finally, for each $k, 3 \leq k \leq m$, the function $F_{n,k}$ is associated with the Nikishin system

$$S^k = (s_2^k, \ldots, s_m^k) = (\tau_{2,k}, w_3^k d\tau_{2,k}, \ldots, w_m^k d\tau_{2,k}) =$$
$$\mathcal{N}(\tau_{2,k}, \widehat{s}_{2,k} d\tau_{3,k}, \ldots, \widehat{s}_{k-1,k} d\tau_{k,k}, \widehat{s}_{k,k} d\sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_m)$$

which satisfies

$$\int (h_j F_{n,k})(x) ds_j^k(x) = 0 , \quad \deg h_j \leq \min(n_1 - 1, \ldots, n_{j-1} - 1, n_k - 2) , \quad j = 2, \ldots, k , \tag{20}$$

and

$$\int (h_j F_{n,k})(x) ds_j^k(x) = 0 , \quad \deg h_j \leq \min(n_k - 1, n_j - 1) , \quad j = k+1, \ldots, m . \tag{21}$$

The next lemma is Theorem 3.1.3 in [7], where the proof may be followed. There, it is used to obtain a result similar to Lemma 3 stated above.

Lemma 4. *Let $S^1 = (s_2^1, \ldots, s_m^1) = \mathcal{N}(\sigma_2, \ldots, \sigma_m)$ and $k \in \{2, \ldots, m\}$ be fixed. Then, the following formulas take place.*

$$\frac{1}{\widehat{s}_k^1(z)} = \mathcal{L}_k(z) + \widehat{s}_2^k(z) , \tag{22}$$

$$\frac{\widehat{s}_j^1(z)}{\widehat{s}_k^1(z)} = a_j + \widehat{s}_{j+1}^k(z) + c_j \widehat{s}_j^k(z) , \quad j = 2, \ldots, k-1 , \tag{23}$$

and

$$\frac{\widehat{s}_j^1(z)}{\widehat{s}_k^1(z)} = a_j + \widehat{s}_j^k(z)\,, \quad j = k+1, \ldots, m\,, \tag{24}$$

where a_j and c_j denote certain constants, $\mathcal{L}_k$ is a first degree polynomial, and the measures s_j^k are as defined in Lemma 3.

Definition 4. Let $w_j, j = 1, \ldots, m$, be continuous functions with constant sign on an interval $[a, b]$ of the real line. It is said that $(w_1, \ldots, w_m)$ forms an AT system for the index $n = (n_1, \ldots, n_m)$ on $[a, b]$ if no matter what polynomials $h_1, \ldots, h_m$ one chooses with $\deg h_j \leq n_j - 1, j = 1, \ldots, m$, not all identically equal to zero, the function

$$\mathcal{H}_n(x) = \mathcal{H}_n(h_1, \ldots, h_m; x) = h_1(x)w_1(x) + \cdots + h_m(x)w_m(x)\,.$$

has at most $|n| - 1$ zeros on $[a, b]$ ($\deg h_j \leq -1$ forces $h_j \equiv 0$). The system $(w_1, \ldots, w_m)$ forms an AT system on $[a, b]$ if it is an AT system on that interval for all $n \in \mathbb{Z}_+^m$.

Theorem 2. *Let $S^1 = (s_2^1, \ldots, s_m^1) = \mathcal{N}(\sigma_2, \ldots, \sigma_m)$ be an arbitrary Nikishin system of $m - 1$ measures and let $n = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(*)$ (the class of all multi–indices such that there do not exist $1 \leq i < j < k \leq m$ such that $n_i < n_j < n_k$). Then, the system of functions $(1, \widehat{s}_2^1, \ldots, \widehat{s}_m^1)$ forms an AT system for the index n on any interval $[a, b]$ disjoint from* $\mathrm{Co}(\mathrm{supp}(\sigma_2))$.

Proof. We will proceed by induction on $m \in \mathbb{N}$ which represents the number of functions in $(1, \widehat{s}_2^1, \ldots, \widehat{s}_m^1)$. For $m = 1$ the system of functions reduces to 1 and $n \in \mathbb{Z}_+(*) = \mathbb{Z}_+$ may be any non-negative integer. This case is trivial because any polynomial of degree $\leq n - 1$ can have at most $n - 1$ zeros in the whole complex plane unless it is identically equal to zero. Let us assume that the statement is true for $m - 1, m \geq 2$, and let us show that it also holds for m.

Suppose that $(1, \widehat{s}_2^1, \ldots, \widehat{s}_m^1)$ is not an AT system for an index $n \in \mathbb{Z}_+^m(*)$ on an interval $[a, b]$ disjoint from $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. Then there exist polynomials $h_{n_i}, \deg h_{n_i} \leq n_i - 1, i = 1, \ldots, m$, not all identically equal to zero, such that $\mathcal{H}_n = h_{n_1} + h_{n_2}\widehat{s}_2^1 + \ldots + h_{n_m}\widehat{s}_m^1$ has at least $|n|$ zeros on $[a, b]$ counting multiplicities. Let $W_n, \deg W_n \geq |n|$, be a monic polynomial whose zeros are zeros of $\mathcal{H}_n$ lying on $[a, b]$. Therefore,

$$\frac{\mathcal{H}_n(z)}{W_n(z)} = \mathcal{O}\left(\frac{1}{z^{|n|-M}}\right) \in H(\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_2)))\,, \quad z \to \infty\,, \tag{25}$$

where $M = \max\{n_1 - 1, n_2 - 2, \ldots, n_m - 2\}$.

Assume that $M = n_1 - 1$. From (25) we have that

$$\frac{z^\nu \mathcal{H}_n(z)}{W_n(z)} = \mathcal{O}\left(\frac{1}{z^2}\right)\,, \quad z \to \infty\,, \quad \nu = 0, \ldots, |n| - n_1 - 1\,.$$

Let Γ be a closed integration path with winding number 1 for all its interior points such that $\mathrm{Co}(\mathrm{supp}(\sigma_2)) \subset \mathrm{Int}(\Gamma)$ and $[a, b] \subset \mathrm{Ext}(\Gamma)$. Here, and in the following, $\mathrm{Int}(\Gamma)$ and $\mathrm{Ext}(\Gamma)$ denote, the bounded and unbounded connected components, respectively, in which Γ divides the complex plane. From Cauchy's Theorem, it follows that

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu \mathcal{H}_n(z)}{W_n(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu(h_{n_2}\widehat{s}_2^1 + \ldots + h_{n_m}\widehat{s}_m^1)(z)}{W_n(z)}dz\,, \quad \nu = 0, \ldots, |n| - n_1 - 1\,.$$

Substituting $\widehat{s}_2^1, \ldots, \widehat{s}_m^1$ by their integral expressions, using Fubini's Theorem, and Cauchy's integral formula, we obtain ($w_j^1 = \widehat{s}_{3,j}\,, j = 3, \ldots, m$, if $m \geq 3$)

$$0 = \int \frac{x^\nu(h_{n_2} + h_{n_3}w_3^1 + \ldots + h_{n_m}w_m^1)(x)}{W_n(x)}d\sigma_2(x)\,, \quad \nu = 0, \ldots, |n| - n_1 - 1\,.$$

Since $d\sigma_2(x)/W_n(x)$ is a measure with constant sign on $\mathrm{supp}\,\sigma_2$, it follows that $h_{n_2} + h_{n_3}w_3^1 + \ldots + h_{n_m}w_m^1$ must have at least $|n| - n_1$ changes of sign on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. According to our induction hypothesis the system $(1, w_3^1, \ldots, w_m^1)$ forms an AT system on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ for the index $(n_2, \ldots, n_m) \in \mathbb{Z}_+^{m-1}(*)$ since $(w_3^1, \ldots, w_m^1)$ is a Nikishin system supported on $\mathrm{Co}(\mathrm{supp}(\sigma_3))$ which is disjoint from $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ (if $m = 2$ the system of functions reduces again to 1 and the conclusion is trivial). Therefore, $h_{n_2} + h_{n_3}w_3^1 + \ldots + h_{n_m}w_m^1$ cannot change signs more than $|n| - n_1 - 1$ times on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ and we arrive to a contradiction.

Let us consider the case when $M = n_k - 2\,, k \in \{2,\dots,m\}$. In case that this is true for several k, we choose the smallest one. Notice that with this selection and using that $n \in \mathbb{Z}_+^m(*)$, it follows that

$$n_1 \geq n_2 \geq \cdots \geq n_{k-1} \tag{26}$$

(this is the only place in the proof where we use that $n \in \mathbb{Z}_+^m(*)$). From (25) we have

$$\frac{z^\nu \mathcal{H}_n(z)}{(\widehat{s}_k^1 W_n)(z)} = \mathcal{O}\left(\frac{1}{z^2}\right)\,, \quad z \to \infty\,, \quad \nu = 0,\dots,|n| - n_k - 1\,.$$

Let Γ be as before. From Cauchy's Theorem

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (h_{n_1} + h_{n_2}\widehat{s}_2^1 + \ldots + h_{n_m}\widehat{s}_m^1)(z)}{(\widehat{s}_k^1 W_n)(z)} dz\,, \quad \nu = 0,\dots,|n| - n_k - 1\,.$$

Using Lemma 4 in the previous relation and Cauchy's Theorem, it follows that

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (h_{n_1}(\mathcal{L}_k + \widehat{s}_2^k))(z)}{W_n(z)} dz + \sum_{j=2}^{k-1}\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (h_{n_j}(a_j + \widehat{s}_{j+1}^k + c_j \widehat{s}_j^k))(z)}{W_n(z)} dz +$$

$$\sum_{j=k+1}^{m}\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (h_{n_j}(a_j + \widehat{s}_j^k))(z)}{W_n(z)} dz =$$

$$\sum_{j=2}^{k-1}\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu ((h_{n_{j-1}} + c_j h_{n_j})\widehat{s}_j^k)(z)}{W_n(z)} dz + \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (h_{n_{k-1}}\widehat{s}_k^k)(z)}{W_n(z)} dz +$$

$$\sum_{j=k+1}^{m}\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (h_{n_j}\widehat{s}_j^k)(z)}{W_n(z)} dz\,, \quad \nu = 0,\dots,|n| - n_k - 1\,.$$

Substituting $\widehat{s}_2^k,\dots,\widehat{s}_m^k$ by their integral expressions, using Fubini's Theorem, and Cauchy's integral formula, we obtain (for the definition of the functions $w_j^k, j = 3,\dots,m$, look back to Lemma 3 and set $w_2^k \equiv 1$)

$$0 = \int \frac{x^\nu (\sum_{j=2}^{k-1}(h_{n_{j-1}} + c_j h_{n_j})w_j^k + h_{n_{k-1}}w_k^k + \sum_{j=k+1}^m h_{n_j}w_j^k)(x)}{W_n(x)} d\tau_{2,k}(x)\,,$$

for each $\nu = 0,\dots,|n| - n_k - 1$. Since $d\tau_{2,k}(x)/W_n(x)$ is a measure with constant sign on $\operatorname{supp}\sigma_2$, it follows that

$$\widetilde{\mathcal{H}}_n = \sum_{j=2}^{k-1}(h_{n_{j-1}} + c_j h_{n_j})w_j^k + h_{n_{k-1}}w_k^k + \sum_{j=k+1}^{m} h_{n_j}w_j^k \tag{27}$$

must have at least $|n| - n_k$ changes of sign on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$.

For $k = 2$, $\sum_{j=2}^{k-1}$ is an empty sum and $\widetilde{\mathcal{H}}_n$ reduces to $h_{n_1} + \sum_{j=3}^m h_{n_j}w_j^2$. Since $(1, w_3^2,\dots,w_m^2)$ forms an AT system on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ for the index $(n_1, n_3,\dots,n_m) \in \mathbb{Z}_+^{m-1}(*)$ we readily arrive to a contradiction (if $m = 2$ the system of functions reduces to 1 and the conclusion is trivial).

For $n \in \mathbb{Z}_+^m(*)$ and $k \geq 3$, on account of (26), $\deg h_{n_{j-1}} + c_j h_{n_j} \leq n_{j-1} - 1, j = 2,\dots,k-1$. According to our induction hypothesis the system $(1, w_3^k,\dots,w_m^k)$ forms an AT system on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ for the index $(n_1,\dots,n_{j-1},n_{j+1},\dots,n_m) \in \mathbb{Z}_+^{m-1}(*)$ since $(w_3^k,\dots,w_m^k)$ is a Nikishin system supported on $\mathrm{Co}(\mathrm{supp}(\sigma_3))$ which is disjoint from $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. Therefore, $\widetilde{\mathcal{H}}_n$ can change signs on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ at most $|n| - n_k - 1$ times. With this contradiction we conclude the proof. □

Previously, it was known that $(1, \widehat{s}_2^1,\dots,\widehat{s}_m^1)$ forms an AT system for all multi-indices $n \in \mathbb{Z}_+$ such that $i < j$ implies that $n_j \leq n_i + 1$. It is easy to check that this class of multi-indices is strictly contained in $\mathbb{Z}_+^m(*)$. In fact, the existence of $i < j < k$ such that $n_i < n_j < n_k$ implies that $n_k > n_i + 1$. On the other hand, it is easy to find multi–indices in $\mathbb{Z}_+^m(*)$ for which $n_j > n_i + 1$ with $i < j$. In [8] it was proved that $(1, \widehat{s}_2^1, \widehat{s}_3^1)$ is an AT system on any interval disjoint from $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ (for all multi-indices $n \in \mathbb{Z}_+^3$). It is not known whether or not this property extends for $m > 3$.

We are ready for the proof of the following result.

THEOREM 3. *Let $S = (s_1,\ldots,s_m) = \mathcal{N}(\sigma_1,\ldots,\sigma_m)$ be an arbitrary Nikishin system of m measures and let $n = (n_1,\ldots,n_m) \in \mathbb{Z}_+^m(*)$. We set $k = 1$ if $n_1 - 1 = M = \max\{n_1 - 1, n_2 - 2,\ldots,n_m - 2\}$ or k is the first index in $\{2,\ldots,m\}$ such that $n_k - 2 = M$. There exists a monic polynomial $W_{n,k}$ of degree $|n| - n_k$ whose zeros are simple and lie in the interior of* $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ *such that*

$$0 = \int x^\nu Q_n(x) \frac{ds_k(x)}{W_{n,k}(x)}\,, \quad \nu = 0,1,\ldots,|n|-1\,. \tag{28}$$

Therefore, Q_n has exactly $|n|$ simple zeros in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. *All indices in $\mathbb{Z}_+^m(*)$ are strongly normal. We have the remainder formula*

$$(\widehat{s}_k - \frac{P_{n,k}}{Q_n})(z) = \frac{W_{n,k}(z)}{(QQ_n)(z)} \int \frac{(QQ_n)(x)}{W_{n,k}(x)} \frac{ds_k(x)}{z-x}\,, \tag{29}$$

where Q denotes an arbitrary polynomial of degree $\leq |n|$. Taking $Q = Q_n$ in (29), it follows that $F_{n,k}/W_{n,k}$ has no zeros in $\mathbb{C} \setminus$ $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. *In particular, this function has constant sign on* $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. *Finally,*

$$\int \frac{p(x)}{W_{n,k}(x)} ds_k(x) = \sum_{j=1}^{|n|} \lambda_{n,k,j} \frac{p(x_{n,j})}{W_{n,k}(x_{n,j})}\,, \quad p \in \mathcal{P}_{2|n|-1}\,, \tag{30}$$

and

$$\lambda_{n,k,j} = W_{n,k}(x_{n,j}) \int \left(\frac{Q_n(x)}{Q_n'(x_{n,j})(x - x_{n,j})} \right)^2 \frac{ds_k(x)}{W_{n,k}(x)}\,, \quad j = 1,\ldots,|n|\,. \tag{31}$$

Therefore, all the Nikishin–Christoffel coefficients associated with $P_{n,k}/Q_n$ have the same sign as the measure s_k and

$$\sum_{j=1}^{|n|} |\lambda_{n,k,j}| = |s_k|\,. \tag{32}$$

Proof. If $k = 1$, from (17) and the assumption on the multi–index n, it follows that

$$0 = \int (h_j F_{n,1})(x) ds_j^1(x)\,, \quad \deg h_j \leq n_j - 1\,, \quad j = 2,\ldots,m\,.$$

For $k = 2$, using (18)-(19), and the assumption on the multi–index n, it follows that

$$0 = \int (h_2 F_{n,2})(x) ds_2^2(x)\,, \quad \deg h_2 \leq n_1 - 1\,,$$

and

$$0 = \int (h_j F_{n,2})(x) ds_j^2(x)\,, \quad \deg h_j \leq n_j - 1\,, \quad j = 3,\ldots,m\,.$$

Finally, if $k \in \{3,\ldots,m\}$ from (20)-(21) and the assumption on the multi–index n, it follows that

$$0 = \int (h_j F_{n,k})(x) ds_j^k(x)\,, \quad \deg h_j \leq n_{j-1} - 1\,, \quad j = 2,\ldots,k$$

and

$$0 = \int (h_j F_{n,k})(x) ds_j^k(x)\,, \quad \deg h_j \leq n_j - 1\,, \quad j = k+1,\ldots,m\,.$$

In any case, we have that

$$0 = \int F_{n,k}(x)(h_2 + h_3 w_3^k + \cdots + h_m w_m^k(x) d\tau_{2,k}(x)\,, \tag{33}$$

where $\deg h_j \leq n_{j-1} - 1\,, 2 \leq j \leq k$, and $\deg h_j \leq n_j - 1, k < j \leq m$.

Denote by $n(k)$ the multi-index in $\mathbb{Z}_+^{m-1}(*)$ obtained from n deleting its kth component. By Lemma 4, the assumption on n, and the selection of k we know that the system $(1, w_3^k,\ldots,w_m^k)$ forms an AT system on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ for the multi–index $n(k) = (n_1,\ldots,n_{k-1},n_k,\ldots,n_m)$. Using (33), it follows that $F_{n,k}$ has at least $|n| - n_k$ sign changes on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ (later, when we obtain (29), we see that in fact it has exactly that many sign changes). This means that $P_{n,k}/Q_n$ is the

$|n|$th Padé approximant that interpolates $\widehat{s}_k$, $|n|+n_k+1$ times at $z=\infty$ and (at least) $|n|-n_k$ times at the points where $F_{n,k}$ equals zero on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. All the assertions of the theorem are direct consequences of this fact (see [10]). For convenience of the reader we proceed with the proof.

Select $|n|-n_k$ simple zeros of $F_{n,k}$ in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ and take these points as the zeros of the polynomial $W_{n,k}$. Since $\deg W_{n,k} \geq |n|-n_k$, from ii)

$$\frac{z^\nu F_{n,k}}{W_{n,k}} = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1)))\,, \quad z \to \infty\,, \quad \nu = 0,\dots,|n|-1.$$

Let Γ be a closed integration path with winding number 1 for all its interior points such that $\mathrm{Co}(\mathrm{supp}(\sigma_1)) \subset \mathrm{Int}(\Gamma)$ and $\mathrm{Co}(\mathrm{supp}(\sigma_2)) \subset \mathrm{Ext}(\Gamma)$. By Cauchy's Theorem, Fubini's Theorem and, Cauchy's Integral Formula, we obtain

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu F_{n,k}(z)}{W_{n,k}(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (Q_n\widehat{s}_k)(z)}{W_{n,k}(z)}dz = \int x^\nu Q_n(x)\frac{d\sigma_k(x)}{W_{n,k}(x)}\,, \quad \nu = 0,\dots,|n|-1\,,$$

as claimed in (28). Hence, Q_n has exactly $|n|$ simple zeros in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. Since each $n \in \mathbb{Z}_+^m(*)$ has a component k as indicated in the statement of the theorem, all such indices are strongly normal.

Take $Q \in \mathcal{P}_{|n|}$. From ii)

$$\frac{QF_{n,k}}{W_{n,k}} = \mathcal{O}\left(\frac{1}{z}\right) \in H(\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1)))\,.$$

By Cauchy's Integral Formula, Cauchy's Theorem, and Fubini's Theorem, we obtain that

$$\frac{QF_{n,k}(z)}{W_{n,k}(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{(QF_{n,k})(\zeta)}{W_{n,k}(\zeta)}\frac{d\zeta}{z-\zeta} = \frac{1}{2\pi i}\int_\Gamma \frac{(QQ_n\widehat{s}_k)(\zeta)}{W_{n,k}(\zeta)}\frac{d\zeta}{z-\zeta} = \int \frac{(QQ_n)(x)}{W_{n,k}(x)}\frac{ds_k(x)}{z-x}\,,$$

which is equivalent to (29).

Notice that for any $p \in \mathcal{P}_{2|n|-1}$, using ii)

$$\frac{p}{W_{n,k}}\left(\widehat{s}_k - \frac{P_{n,k}}{Q_n}\right) = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1)))\,, \quad z \to \infty\,.$$

Using the integral expression of $\widehat{s}_k$, the partial fraction decomposition (3) of $P_{n,k}/Q_n$, Cauchy's Theorem, Fubini's Theorem, and Cauchy's Integral Formula, we have

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{p(z)}{W_{n,k}(z)}\left(\int \frac{ds_k(x)}{z-x} - \sum_{j=1}^{|n|}\frac{\lambda_{n,k,j}}{z-x_{n,j}}\right)dz = \int \frac{p(x)}{W_{n,k}(x)}ds_k(x) - \sum_{j=1}^{|n|}\lambda_{n,k,j}\frac{p(x_{n,j})}{W_{n,k}(x_{n,j})}\,,$$

which is (30). Taking $p = (Q_n(x)/Q_n'(x_{n,j})(x-x_{n,j}))^2$ in (30), we obtain (31) and this obviously implies that the coefficients $\lambda_{n,k,j}$ have the same sign as s_k. Using this and Lemma 1 with $p \equiv 1$ we obtain (32). The proof is complete. □

From Theorems 2 and 3 we can deduce some interlacing properties of zeros. For this we need one more property relative to orthogonal polynomials with respect to a Markov system of functions. A system of N real continuous functions $\{u_1,\dots,u_n\}$ is said to form a Markov system on an interval (a,b) if there do not exist constants $c_1,\dots,c_N$, not all identically equal to zero, such that

$$\sum_{j=1}^N c_j u_j$$

has more than $N-1$ zeros on (a,b) (for more details on Markov systems see [15]). The next lemma is a reformulation of the Theorem appearing in [14]. There, it is stated for polynomials orthogonal to a Markov system with respect to the Lebesgue measure. Here, we state it for an arbitrary Borel measure supported on an interval of the real line. For this more general case, the proof is basically the same except for some minor details.

Lemma 5. *Let σ be a finite Borel measure with constant sign supported on an interval of the real line. Let $\{u_1,\dots,u_N\}$ be a Markov system of functions on $\mathrm{Co}(\mathrm{supp}(\sigma))$. Let p_N be a polynomial of degree $\leq N$ not identically equal to zero such that*

$$0 = \int u_j(x)p_N(x)d\sigma(x)\,, \quad j = 1,\dots,N\,.$$

Then $\deg p_N = N$ *and the zeros of* p_N *are simple and lie in the interior of* $\mathrm{Co}(\mathrm{supp}(\sigma))$*. Assume that* p_{N+1} *is a polynomial of degree* $N+1$ *with real distinct zeros which satisfies*

$$0 = \int u_j(x) p_{N+1}(x) d\sigma(x)\,, \quad j = 1, \ldots, N\,.$$

Then between any two consecutive zeros of p_{N+1} *lies a zero of* p_N*.*

Proof. Set

$$M_n(t_1, \ldots, t_N) = \begin{vmatrix} u_1(t_1) & u_2(t_1) & \cdots & u_N(t_1) \\ u_1(t_2) & u_2(t_2) & \cdots & u_N(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(t_N) & u_2(t_N) & \cdots & u_N(t_N) \end{vmatrix},$$

and

$$V_{N+1}(t, t_1, \ldots, t_N) = \begin{vmatrix} t^N & t^{N-1} & \cdots & 1 \\ t_1^N & t_1^{N-1} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ t_N^N & t_N^{N-1} & \cdots & 1 \end{vmatrix}.$$

Let $[a,b] = \mathrm{Co}(\mathrm{supp}(\sigma)), C = [a,b]^N$, and $T = \{(t_1, t_2, \ldots, t_N) : a \leq t_1 < t_2 < \cdots < t_N \leq b\}$.

That p_N has exactly N simple zeros in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma))$ is a direct consequence of $\{u_1, \ldots, u_N\}$ being a Markov system on that set. From this property it is also easy to see that p_N is uniquely determined by the orthogonality relations. Take p_N with leading coefficient equal to 1. Then, there exists $\lambda \neq 0$ such that

$$p_N(t) = \lambda \begin{vmatrix} t^N & t^{N-1} & \cdots & 1 \\ \int t_1^N u_1(t_1) d\sigma(t_1) & \int t_1^{N-1} u_1(t_1) d\sigma(t_1) & \cdots & \int u_1(t_1) d\sigma(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \int t_N^N u_N(t_N) d\sigma(t_N) & \int t_1^{N-1} u_N(t_N) d\sigma(t_N) & \cdots & \int u_N(t_N) d\sigma(t_N) \end{vmatrix},$$

since the polynomial defined by the determinant satisfies the same system of orthogonality relations and is not identically equal to zero. Hence,

$$p_N(t) = \lambda \int_C u_1(t_1) u_2(t_2) \cdots u_N(t_N) V_{N+1}(t, t_1, \ldots, t_N) d\sigma(t_1) \cdots d\sigma(t_N)\,.$$

Taking into consideration that $V_{N+1}(t, t_1, \ldots, t_N) = 0$ whenever $t_i = t_j, 1 \leq i, j \leq N$, from the integral above we obtain that

$$p_N(t) = \lambda \int_T \sum u_1(t_{i_1}) u_2(t_{i_2}) \cdots u_N(t_{i_N}) V_{N+1}(t, t_{i_1}, \ldots, t_{i_N}) d\sigma(t_1) \cdots d\sigma(t_N)\,,$$

where the sum extends over all $N!$ permutations of $(1, 2, \ldots, N)$. Rearranging the rows in the determinant defining $V_{N+1}(t, t_{i_1}, \ldots, t_{i_N})$ so as to get the common factor $V_{N+1}(t, t_1, \ldots, t_N)$ in the sum above and using the definition of a determinant, it follows that

$$p_N(t) = \lambda \int_T M_N(t_1, \ldots, t_N) V_{N+1}(t, t_1, \ldots, t_N) d\sigma(t_1) \cdots d\sigma(t_N) =$$

$$\lambda \int_T M_N(t_1, \ldots, t_N) V_N(t_1, \ldots, t_N) P_N(t) d\sigma(t_1) \cdots d\sigma(t_N)\,,$$

where $P_N(t) = \prod_{j=1}^N (t - t_j)$, since $V_{N+1}(t, t_1, \ldots, t_N) = V_N(t_1, \ldots, t_N) P_N(t)$. This integral representation is the main ingredient in the proof.

Let us write $p_{N+1}(x) = \prod_{j=1}^{N+1} (x - x_j)$. The rest of the proof reduces to showing that

$$p'_{N+1}(x_j) \int_T M_N(t_1, \ldots, t_N) V_N(t_1, \ldots, t_N) P_N(x_j) d\sigma(t_1) \cdots d\sigma(t_N) =$$

$$\int_T M_N(t_1, \ldots, t_N) V_N(t_1, \ldots, t_N) P_N^2(x_j) d\sigma(t_1) \cdots d\sigma(t_N)\,, \quad j = 1, \ldots, N+1\,.$$

To this end you can follow the same arguments used in [14] pages 88-90. Once this is proved, on account of the integral representation for p_N and the fact that $M_N(t_1, \ldots, t_N) V_N(t_1, \ldots, t_N)$ has constant sign on T we deduce that $p'_{N+1}(x_j)$ and $p_N(x_j)$ either have the same sign for $j =$

$1,\ldots,N+1$ or have opposite signs at all these points. From Bolzano's Theorem we conclude that the interlacing property indeed holds. □

Now, we can state the following.

COROLLARY 1. *Let $S=(s_1,\ldots,s_m)=\mathcal{N}(\sigma_1,\ldots,\sigma_m)$ be an arbitrary Nikishin system of m measures. Let $n\in\mathbb{Z}_+^m(*)$, and k be as indicated in Theorem 3 then between any two consecutive zeros of Q_n lies a zero of $P_{n,k}$. Let us denote by n_+ the vector which is obtained adding 1 to one component of n and let Q_{n_+} be the multiple orthogonal polynomials corresponding to n_+. Assume that $n_+\in\mathbb{Z}_+^m(*)$, then between any two consecutive zeros of Q_{n_+} lies a zero of Q_n.*

Proof. From Theorem 3, we know that the coefficients $\lambda_{n,k,j}, j=1,\ldots,|n|$, all have the same sign. Let $x_{n,j}<x_{n,j+1}$ be two consecutive zeros of Q_n. Using (3), taking limit from the right at $x_{n,j}$ and from the left at $x_{n,j+1}$ one obtains infinities with different sign. Therefore, $P_{n,k}$ must have an intermediate zero.

From the definition of Q_n and Q_{n_+}, we have that both of these polynomials are orthogonal to the system of functions

$$1,\ldots,x^{n_1-1},\widehat{s}_2^1,\ldots,x^{n_2-1}\widehat{s}_2^1,\ldots,\widehat{s}_m^1,\ldots,x^{n_m-1}\widehat{s}_m^1 .$$

relative to the measure σ_1. According to Theorem 2, S^1 forms an AT system for the index $n\in\mathbb{Z}_+^m(*)$ on any interval $[a,b]$ disjoint from $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. In particular, this implies that the functions with respect to which Q_n and Q_{n_+} are orthogonal form a Markov system on the interval $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. On the other hand, Theorem 3 asserts that Q_n and Q_{n_+} have exactly $|n|$ and $|n_+|$ simple zeros, respectively, contained in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. From Lemma 5 it follows that between any two consecutive zeros of Q_{n_+} lies a zero of Q_n. □

From Theorem 3 we obtain the following consequence which generalizes Corollary 2 in [3].

COROLLARY 2. *Let $S=(s_1,\ldots,s_m)=\mathcal{N}(\sigma_1,\ldots,\sigma_m)$ be an arbitrary Nikishin system of m measures. Let $\Lambda\subset\mathbb{Z}_+^m(*)$ be an infinite sequence of distinct multi–indices such that for all $n\in\Lambda$ the kth component is as it was chosen in Theorem 3. Then, for each $n\in\Lambda$ the coefficients $\lambda_{n,k,j}, j=1,\ldots,|n|$, preserve the same sign. For each compact set $K\subset\overline{\mathbb{C}}\setminus\mathrm{Co}(\mathrm{supp}(\sigma_1))$, there exists $\kappa(K)<1$ such that*

$$\limsup_{n\in\Lambda}\left\|\widehat{s}_k-\frac{P_{n,k}}{Q_n}\right\|_K^{1/2|n|}\le\kappa(K)\,, \tag{34}$$

where $\|\cdot\|_K$ denotes the sup-norm on K,

$$\kappa(K)=\sup\{\|\varphi_t\|_K : t\in\mathrm{Co}(\mathrm{supp}(\sigma_2))\cup\{\infty\}\}\,,$$

and φ_t denotes the conformal representation of $\overline{\mathbb{C}}\setminus\mathrm{Co}(\mathrm{supp}(\sigma_1))$ onto the open unit disk such that $\varphi_t(t)=0$ and $\varphi_t'(t)>0$. For each bounded Riemann integrable function f on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$

$$\lim_{n\in\Lambda}\sum_{j=1}^{|n|}\lambda_{n,k,j}f(x_{n,j})=\int f(x)ds_k(x)\,, \tag{35}$$

and if $f\in Lip_\beta(\Delta),\beta>0$, then

$$\left|\int f(x)ds_k(x)-\sum_{j=1}^{|n|}\lambda_{n,k,j}f(x_{n,j})\right|=\mathcal{O}\left(\frac{1}{|n|^\beta}\right)\,. \tag{36}$$

Finally, if $f\in H(V)$, where V is a neighborhood of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$, then

$$\lim_{n\in\Lambda}|\int f(x)ds_k(x)-\sum_{j=1}^{|n|}\lambda_{n,k,j}f(x_{n,j})|^{1/2|n|}\le\kappa_V\,, \tag{37}$$

where $\kappa_V=\inf\{\kappa(\gamma_\rho):\gamma_\rho\subset V\}$ and $\gamma_\rho=\{z:|\varphi_\infty(z)|=\rho\},0<\rho<1$. If $k\in\{2,\ldots,m\}$ and $n_1+1=n_k$ for all $n\in\Lambda$ then $(34)-(37)$ also hold for the first component.

Proof. That for each $n \in \Lambda$ and k as stated above the Nikishin-Christoffel coefficients preserve the same sign is a consequence of the last statement in Theorem 3. Using (3) and (32), we have that for each compact set $K \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))$

$$\left\| \frac{P_{n,k}(z)}{Q_n(z)} \right\|_K \leq \frac{|s_k|}{d(K)} ,$$

where $|s_k| = |\int ds_k(x)|$ and $d(K) = \inf\{|z-x| : z \in K, x \in \mathrm{Co}(\mathrm{supp}(\sigma_1))\} > 0$. Therefore the family of functions $\{\widehat{s}_k - \frac{P_{n,k}}{Q_n}\}, n \in \Lambda$, is uniformly bounded on each compact subset K of $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))$ by $2|s_k|/d(K)$.

Take $\gamma_\rho, 0 < \rho < 1$, so that $\mathrm{Co}(\mathrm{supp}(\sigma_2)) \subset \mathrm{Ext}(\gamma_\rho)$. Set $W_{n,k}(z) = \prod_{j=1}^{|n|-n_k}(z - y_{n,j})$, where $W_{n,k}$ is the polynomial given in Theorem 3. Then

$$\left\| \frac{\widehat{s}_k - \frac{P_{n,k}}{Q_n}}{\varphi_\infty^{|n|+n_k+1} \prod_{j=1}^{|n|-n_k} \varphi_{y_{n,j}}} \right\|_{\gamma_\rho} \leq \frac{2|s_k|}{d(\gamma_\rho)\delta(\gamma_\rho)^{2|n|+1}} ,$$

where

$$\delta(\gamma_\rho) = \inf\{|\varphi_t(z)| : z \in \gamma_\rho,\, t \in \mathrm{Co}(\mathrm{supp}(\sigma_2)) \cup \{\infty\}\} .$$

Considered as a function of the two variables z and t, it is easy to verify that $|\varphi_t(z)|$ is continuous in $\overline{\mathbb{C}}^2$. Hence $\delta(\gamma_\rho) > 0$ since $\gamma_\rho \cap \mathrm{Co}(\mathrm{supp}(\sigma_2)) = \emptyset$. Fix a compact set $K \in \overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))$ and take ρ sufficiently close to 1 so that $K \subset \mathrm{Ext}(\gamma_\rho)$. Since the function under the norm sign is analytic in $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))$, from the Maximum Principle it follows that the same bound holds for all $z \in K$. Consequently,

$$\left\| \widehat{s}_k - \frac{P_{n,k}}{Q_n} \right\|_K \leq \frac{2|s_k|}{d(\gamma_\rho)\delta(\gamma_\rho)^{2|n|+1}} \|\varphi_\infty^{|n|+n_k+1} \prod_{j=1}^{|n|-n_k} \varphi_{y_{n,j}}\|_K \leq \frac{2|s_k|}{d(\gamma_\rho)} \left(\frac{\kappa(K)}{\delta(\gamma_\rho)} \right)^{2|n|+1} .$$

Therefore,

$$\limsup_{n \in \Lambda} \left\| \widehat{s}_k - \frac{P_{n,k}}{Q_n} \right\|_K^{1/2|n|} \leq \frac{\kappa(K)}{\delta(\gamma_\rho)}$$

Because of the continuity of $|\varphi_t(z)|$ in $\overline{\mathbb{C}}^2$, $\lim_{\rho \to 1} \delta(\gamma_\rho) = 1$ and (34) follows. That $\kappa(K) < 1$ is also a consequence of the continuity of $|\varphi_t(z)|$ in $\overline{\mathbb{C}}^2$.

Formulas (35) and (36) are consequences of the first and third statements of Lemma 2. Formula (37) is derived following the same scheme as for proving (8) taking into consideration that here we have the more precise estimate given by (34).

Concerning the last statement, we only comment that in that case both indices 1 and k satisfy the conditions of Theorem 3 for all indices in Λ. The existence of such sequences of multi–indices is guaranteed by the sequence $\{(N, \ldots, N, N+1, \ldots, N+1)\}, N \in \mathbb{Z}_+$, where the jump in value is produced in the kth component. Other less trivial examples of such sequences are easy to construct from elements in $\mathbb{Z}_+^m(*)$. □

Unfortunately, it is not possible to have more than two components $k \in \{1, \ldots, m\}$ satisfying the conditions of Theorem 3, and if there are two, one of them must be the first one. But there are other means of obtaining (34) for more than two components.

Let $n \in \mathbb{Z}_+^m$ and $k \in \{1, 2, \ldots, m\}$. We denote by $n^k = (n_1^k, \ldots, n_m^k) \in \mathbb{Z}_+^m$ the vector whose components are defined as follows. For $k = 1$

$$n_j^1 = \begin{cases} n_1 & , \quad j = 1 , \\ \min\{n_1 + 1, n_j\} & , \quad 2 \leq j \leq m . \end{cases}$$

If $k \in \{2, \ldots, m\}$

$$n_j^k = \begin{cases} \min\{n_1, \ldots, n_j, n_k - 1\} & , \quad 1 \leq j < k , \\ \min\{n_k, n_j\} & , \quad k \leq j \leq m . \end{cases}$$

Obviously, $n - n^k \in \mathbb{Z}_+^m$ and $n \in \mathbb{Z}_+^m(*)$ implies that $n^k \in \mathbb{Z}_+^m(*)$. As before $|n - n^k| = \sum_{j=1}^m (n_j - n_j^k) = |n| - |n^k|$. Notice that if $n \in \mathbb{Z}_+^m(*)$ and k is as defined in Theorem 3, then $n = n^k$ and $|n - n^k| = 0$.

Theorem 4. *Let $S = (s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ be an arbitrary Nikishin system of m measures and let $n = (n_1, \dots, n_m) \in \mathbb{Z}_+^m$. Assume that $n^k(k) \in \mathbb{Z}_+^{m-1}(*), k \in \{1, \dots, m\}$, where $n^k(k)$ is the vector obtained deleting from n^k its kth component. Then, there exists a monic polynomial $W_{n,k}$ of degree $|n^k| - n_k = |n^k(k)|$ whose zeros are simple and lie in the interior of* $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ *such that*

$$0 = \int x^\nu Q_n(x) \frac{ds_k(x)}{W_{n,k}(x)}, \quad \nu = 0, 1, \dots, |n^k| - 1. \tag{38}$$

Therefore, Q_n has at least $|n^k|$ simple zeros in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. *We have the remainder formula*

$$(\widehat{s}_k - \frac{P_{n,k}}{Q_n})(z) = \frac{W_{n,k}(z)}{(QQ_n)(z)} \int \frac{(QQ_n)(x)}{W_{n,k}(x)} \frac{ds_k(x)}{z - x}, \tag{39}$$

where Q denotes an arbitrary polynomial of degree $\leq |n^k|$. Additionally, let us assume that the multi-index n is strongly normal (for example, $n \in \mathbb{Z}_+^m(*)$). Then

$$\int \frac{p(x)}{W_{n,k}(x)} ds_k(x) = \sum_{j=1}^{|n|} \lambda_{n,k,j} \frac{p(x_{n,j})}{W_{n,k}(x_{n,j})}, \quad p \in \mathcal{P}_{|n|+|n^k|-1}, \tag{40}$$

and at least $(|n| + |n^k|)/2$ Nikishin–Christoffel coefficients associated with $P_{n,k}/Q_n$ have the same sign as the measure s_k.

Proof. The proof is similar to that of Theorem 3 so we only outline the main ingredients. From the definition of n^k and using Lemma 3, instead of (33) we get

$$0 = \int F_{n,k}(x)(h_2 + h_3 w_3^k + \cdots + h_m w_m^k(x) d\tau_{2,k}(x), \tag{41}$$

where $\deg h_j \leq n_{j-1}^k - 1, 2 \leq j \leq k$, and $\deg h_j \leq n_j^k - 1, k < j \leq m$.

By Theorem 2 and the assumption on $n^k(k)$, we know that the system $(1, w_3^k, \dots, w_m^k)$ forms an AT system on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ for the multi–index $n^k(k)$. Using (41), it follows that $F_{n,k}$ has at least $|n^k| - n_k$ sign changes on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. On the other hand, the number of such sign changes must be finite since $F_{n,k} \not\equiv 0$. Select $|n^k| - n_k$ distinct zeros of $F_{n,k}$ on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ and take $W_{n,k}$ as the monic polynomial with a zero at each one of those points. Since $\deg W_{n,k} = |n| - n_k$, from ii)

$$\frac{z^\nu F_{n,k}}{W_{n,k}} = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))), \quad z \to \infty, \quad \nu = 0, \dots, |n^k| - 1.$$

Now, (38) is obtained as in the proof of (28).

Take $Q \in \mathcal{P}_{|n^k|}$. From ii)

$$\frac{QF_{n,k}}{W_{n,k}} = \mathcal{O}\left(\frac{1}{z}\right) \in H(\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))), \quad z \to \infty,$$

and (39) is obtained using the same arguments as for (29).

If the multi-index n is strongly normal, from (39) one sees that for any $p \in \mathcal{P}_{|n|+|n^k|-1}$

$$\frac{p}{W_{n,k}} \left(\widehat{s}_k - \frac{P_{n,k}}{Q_n}\right) = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))), \quad z \to \infty.$$

Using the integral expression of $\widehat{s}_k$ and the partial fraction decomposition (3) of $P_{n,k}/Q_n$, (40) is obtained as in proving (30).

Let κ_n be the total number of indices j such that the sign of $\lambda_{n,k,j}$ coincides with the sign of the measure s_k. Take $p = \prod'(x - x_{n,j})^2$ where $\prod'$ denotes the product over all indices j such that the sign of $\lambda_{n,k,j}$ coincides with the sign of the measure s_k. Let us suppose that $\deg p = 2\kappa_n \leq |n| + |n^k| - 1$. We can substitute this p in (40). On the other hand, it is easy to see that

$$\mathrm{sg}\left(\int \frac{p(x)}{W_{n,k}(x)} ds_k(x)\right) \neq \mathrm{sg}\left(\sum_{j=1}^{|n|} \lambda_{n,k,j} \frac{p(x_{n,j})}{W_{n,k}(x_{n,j})}\right),$$

where sg$(\cdot)$ denotes the sign of $(\cdot)$, because in the sum all terms cancel out except those which have different sign with respect to the sign of the integral. This contradiction means that $2\kappa_n \geq |n|+|n^k|$ which is equivalent to the last assertion of the theorem. $\Box$

Now we can state the following

Corollary 3. *Let $S = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be an arbitrary Nikishin system of m measures. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be an infinite sequence of distinct multi–indices such that for all $n \in \Lambda$ and k' fixed, $2 \leq k' < m$, we have that $n_1 = n_2 = \cdots = n_{k'-1}$ and $n_{k'} = n_{k'+1} = n_1 + 1$. Then, for $k = 1, k', k'+1$ and each $n \in \Lambda$ the coefficients $\lambda_{n,k,j}, j = 1, \ldots, |n|$, preserve the same sign. Consequently, for $k = 1, k', k'+1$, (34) $-$ (37) hold true.*

Proof. It is easy to verify that the components $k = 1, k'$ satisfy the assumptions of Theorem 3 and for them Corollary 2 is applicable. For $k = k'+1$ notice that $|n^k| = |n| - 1$. Using the last statement of Theorem 4, we obtain that for each $n \in \Lambda$ at least $(|n|+|n^k|)/2 = |n| - 1/2$ coefficients $\lambda_{n,k,j}, j = 1, \ldots, |n|$ must have the same sign; that is, all of them have the same sign since this number is an integer. From this point on we can follow the scheme of the proof of Corollary 2. $\Box$

Remark . The type of indices used in Corollary 3 are the only ones for which we can prove the sign preserving property for three components. For example, when $m = 4$ according to Theorem 4 the indices of the form $(n_1, n_1+1, n_1+1, n_1+1)$ may have one negative Christoffel–Nikishin coefficient for $k = 4$ and those of the form (n_1, n_1, n_1+1, n_1+1) may have a negative coefficient for $k = 2$ and it is not hard to see that these are the best possible choices. Of course, Theorem 4 only gives a sufficient condition for the sign preserving property. It would be interesting to see if it is possible or not to have this property for more than three components with appropriately chosen multi–indices.

Despite of what was said above, we can prove convergence of the simultaneous quadrature rule for all the components in the class of analytic functions on a neighborhood of Co(supp(σ_1)) when the indices are such that the orthogonality conditions are nearly equally distributed between all the measures.

Theorem 5. *Let $S = (s_1, \ldots, s_m)$ be a Nikishin system of measures. Let Λ be an infinite sequence of distinct multi-indices such that there exists a constant $c > 0$ for which for all $n \in \Lambda$ and all $k = 1, \ldots, m$, we have $n_k \geq \frac{|n|}{m} - c$ and all indices in Λ are strongly normal (for example, $\Lambda \subset \mathbb{Z}_+^m(*)$). Then, for each f analytic on a neighborhood V of* Co(supp(s_1)) *and each $k \in \{1, \ldots, m\}$,* (34) *and* (37) *take place.*

Proof. Under the assumption that $n_k \geq \frac{|n|}{m} - c, k = 1, \ldots, m, n \in \Lambda$, it follows from Theorem 1 in [3] that for each $k = 1, \ldots, m$

$$\lim_{n \in \Lambda} \frac{P_{n,k}}{Q_n} = \widehat{s}_k \,, \quad K \subset \overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(s_1)) \,,$$

in (logarithmic) capacity on each compact subset K contained in the indicated region. Since all the indices in Λ are strongly normal, the zeros of Q_n lie in Co(supp(s_1)) and using Lemma 1 in [9] it follows that in fact convergence takes place uniformly on each such compact subset. In particular, we have that the sequence $\left\{\frac{P_{n,k}}{Q_n}\right\}_{n \in \Lambda}$ is uniformly bounded on each compact subset of $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))$. From this point on we can use the arguments employed in proving (34) and (37) in Corollary 2. $\Box$

Remark . For multi–indices satisfying the conditions of Theorem 5 it is not difficult to show using Theorem 4 that for all $k = 1, \ldots, m$ the sign preserving property of the Nikishin–Christoffel coefficients is nearly satisfied. By this we mean that for all such multi–indices and all $k = 1, \ldots, m$, either $|n| - C$ of the Nikishin–Christoffel coefficients are positive or $|n| - C$ of them are negative, where C is a constant independent of n. For details see [3]. It would be interesting to prove that for such multi–indices condition B) is satisfied for all $k = 1, \ldots, m$.

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