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Numerical analysis of a dual-phase-lag model with microtemperatures

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A B S T R A C T

In the last twenty years, the analysis of problems involving dual-phase-lag models has received an increasing attention. In this work, we consider the coupling between one of these models and the microtemperatures effects. In order to overcome the infinite speed paradox, two relaxation parameters are introduced for each evolution equation related to the temperature and the microtemperatures, leading to a system of linear hyperbolic partial differential equations. Its variational formulation is written in terms of the temperature acceleration and the microtemperatures acceleration. An energy decay property is proved. Next, fully discrete approximations are introduced by using the finite element method and the Euler scheme, proving a stability property and a discrete version of the energy decay, obtaining a priori error estimates and performing one- and two-dimensional numerical simulations.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the thermal domain, assumed to be bounded and with a boundary, denoted by $\Gamma = \partial\Omega$, being Lipschitz. Let us define $[0, T]$, $T > 0$, the time interval of interest. Moreover, let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, in most of the expressions we do not indicate the dependence of the functions on $\boldsymbol{x}$ and t, and a subscript after a comma, under a variable, represents its spatial derivative with respect to the prescribed variable, that is $f_{i,j} = \dfrac{\partial f_i}{\partial x_j}$. The time derivatives are represented as a dot for the first order and two dots for the second order, over each variable. Finally, as usual the repeated index notation is used for the summation, and indices i, j are assumed to vary between 1 and d.

Fourier law is the most usual constitutive assumption for the definition of the heat flux vector. It postulates that the heat flux is proportional to the gradient of temperature. If you consider this law with the usual energy equation

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$$a\dot{\theta} = q_{i,i}, \quad a > 0, \tag{1}$$

where θ is the temperature and q_i denotes the heat flux vector, we obtain that the thermal waves propagate instantaneously. This is a paradox of the theory and it violates the causality principle. To overcome this difficulty, many scientists have tried to propose alternative theories where the thermal waves propagate with finite velocity. It seems that the most accepted theory satisfying this property is the one proposed by Cattaneo [11]. What they propose can be seen as the introduction of a relaxation parameter in the Fourier law. The combination of this constitutive law with the energy equation (1) brings us to a hyperbolic damped equation, and therefore the thermal waves travel with finite speed. However, it is worth saying that this alternative proposition brings to other serious problems from the physical point of view as we can find in the references [2,8,27,32,44]. Another proposition which has deserved much attention is the one introduced by Green and Naghdi at the end of the last century [18,19]. Anyway, we also note that the so-called type III Green and Naghdi theory also violates causality principle.

We here want to focus our attention on the proposition of Tzou [41]. The basic idea in this theory is to introduce two delay parameters in the heat flux vector and the gradient of temperature in the form:

$$q_i(\boldsymbol{x}, t + \tau_1) = \kappa\theta_{,i}(\boldsymbol{x}, t + \tau_2), \tag{2}$$

where τ_1 and τ_2 are the delay parameters. It is proved that, from a mathematical point of view, this law leads us to an ill-posed problem [15] in the modern sense of this expression [7,30]; however, in the case that we substitute the heat flux and the gradient of temperature by suitable Taylor approximations, we obtain a very big family of constitutive laws which can be compatible with the classical energy equation. Furthermore, Cattaneo law can be obtained as a particular case of this theory. In this paper, we consider the case when we approximate the heat flux vector by a second-order approximation and the gradient of temperature by a first-order approximation. That is, we obtain

$$q_i + \tau_1\dot{q}_i + \frac{\tau_1^2}{2}\ddot{q}_i = \kappa(\theta_{,i} + \tau_2\dot{\theta}_{,i}). \tag{3}$$

The combination of this constitutive equation with the energy equation has received big attention in the last twenty-five years (we may recall [4,31,33,34] among others). Although this theory can be compatible (for a suitable family of the delay parameters) with the second law of thermodynamics [17], we note that several physical anomalies have been pointed out in several recent contributions [38–40,43].

On the other side, a big interest has been developed over the past and current centuries concerning materials with microstructure [1,22–24,36,37,42]; that is, materials which, at a microscopic level, can also produce several deformations and not only at a macroscopic level. Porous materials, polar materials or microstretch materials are examples of them. Another relevant example of this kind of materials is when we assume that in the microstructure [16,20] can be considered *microtemperatures*.

To clarify the meaning of the microtemperatures, we recall that, in the case of materials with microstructure, it is assumed the existence of microelements which can be considered as a solid with temperatures. Moreover, if we suppose the existence of microtemperatures and we denote by $\boldsymbol{x}$ the center of mass of a microelement in the reference configuration and by $\tilde{\theta}$ the absolute temperature, we can consider the approximation

$$\tilde{\theta}(\boldsymbol{x}', t) = \tilde{\theta}(\boldsymbol{x}, t) + T_i(x_i' - x_i) + O(d^2),$$

where $O(d^2)$ represents a term of order two in a diameter d of the microelement. These functions T_j are called the microtemperatures and they represent the variation of the temperature inside the microelement.

A lot of interest has been developed for this kind of materials in view of their applicability. The first contributions regarding microtemperatures described the behavior by means of a system of parabolic equations. Therefore, in this theory the microtemperatures waves also propagate instantaneously. If we want to overcome again this paradox, it seems suitable to introduce another relaxation parameters. In this paper, we aim to consider the case when the microtemperatures are also described by means of a dual-phase-lag theory. The basic equations for the heat conduction with microtemperatures in the case of centrosymmetric materials can be obtained from [21,22] (among others) in the case that we do not take into account the mechanical effects by means of the evolution equations:

$$\rho\dot{\eta} = q_{i,i}, \quad \rho\dot{\varepsilon}_i = q_{ij,j} + q_i - Q_i,$$

and the constitutive equations:

$$\begin{aligned}
&\rho\eta = a\theta,\\
&\rho\varepsilon_i = -bT_i,\\
&q_i = \kappa\theta_{,i} + \kappa_1 T_i,\\
&q_{ij} = -\kappa_4 T_{r,r}\delta_{ij} - \kappa_5 T_{j,i} - \kappa_6 T_{i,j},\\
&Q_i = (\kappa - \kappa_3)\theta_{,i} + (\kappa_1 - \kappa_2)T_i,
\end{aligned}$$

where we have assumed that the reference temperature is uniformly equal to one in order to simplify the calculations, and vector Q_i is the microheat flux average, q_{ij} is the first moment heat flux tensor, η is the entropy and ε_i is the first moment of the energy vector. The number of publications dealing with the study of elastic materials with microtemperatures is really huge (see, for instance [3,5,6,9,12,13,21,25,26,29,35]).

To consider the natural counterpart to this problem for the dual-phase-lag theory, we need to propose the following constitutive equations[1]:

$$q_i + \tau_1 \dot{q}_i + \frac{\tau_1^2}{2}\ddot{q}_i = \kappa\theta_{,i} + \kappa_1 T_i + \tau_2(\kappa\dot{\theta}_{,i} + \kappa_1 \dot{T}_i),$$
$$q_{ij} + \tau_1 \dot{q}_{ij} + \frac{\tau_1^2}{2}\ddot{q}_{ij} = -\kappa_4 T_{r,r}\delta_{ij} - \kappa_5 T_{j,i} - \kappa_6 T_{ij} + \tau_2(-\kappa_4 \dot{T}_{r,r}\delta_{ij} - \kappa_5 \dot{T}_{j,i} - \kappa_6 \dot{T}_{ij}),$$
$$Q_i + \tau_1 \dot{Q}_i + \frac{\tau_1^2}{2}\ddot{Q}_i = (\kappa-\kappa_3)\theta_{,i} + (\kappa_1-\kappa_2)T_i + \tau_2(\kappa-\kappa_3)\dot{\theta}_{,i} + (\kappa_1-\kappa_2)\dot{T}_i.$$

If we substitute our constitutive equations into the evolution equations we get the following linear system:

$$a\left(\frac{\tau_1^2}{2}\dddot{\theta} + \tau_1\ddot{\theta} + \dot{\theta}\right) = \kappa(\theta_{,ii} + \tau_2\dot{\theta}_{,ii}) + \kappa_1(T_{i,i} + \tau_2\dot{T}_{i,i}),$$
$$b\left(\frac{\tau_1^2}{2}\dddot{T}_i + \tau_1\ddot{T}_i + \dot{T}_i\right) = \kappa_6(T_{i,jj} + \tau_2\dot{T}_{i,jj}) + (\kappa_4+\kappa_5)(T_{j,ji} + \tau_2\dot{T}_{j,ji})$$
$$-\kappa_2(T_i + \tau_2\dot{T}_i) - \kappa_3(\theta_{,i} + \tau_2\dot{\theta}_{,i}).$$

This is the linear system we want to analyze in this paper.

Hence, in this work we continue the research started in [28], defining in Section 2 a variational formulation of the problem and proving that the energy of the system decays. Then, in Section 3 we introduce a fully discrete approximation by using the finite element method and the implicit Euler scheme, we show a discrete stability property and that the discrete energy also decays, and we obtain a priori error estimates, from which the linear convergence of the approximation is derived under suitable regularity conditions. Finally, in Section 4 we perform some numerical simulations in one and two dimensions.

2. Thermal and variational formulations: an energy decay property

We recall that $\theta(\boldsymbol{x},t)$ and $\boldsymbol{T}(\boldsymbol{x},t) = (T_i(\boldsymbol{x},t))_{i=1}^d$ denote the temperature of the body and the microtemperatures at point $\boldsymbol{x}\in\overline{\Omega}$ and time $t\in[0,T]$.

Therefore, the following thermal problem is considered.

Problem P. *Find the temperature $\theta:\overline{\Omega}\times[0,T]\to\mathbb{R}$ and the microtemperatures $\boldsymbol{T}:\overline{\Omega}\times[0,T]\to\mathbb{R}^d$ such that, for $i,j=1,\dots,d$,*

$$a\left(\frac{\tau_1^2}{2}\dddot{\theta} + \tau_1\ddot{\theta} + \dot{\theta}\right) = \kappa(\theta_{,ii} + \tau_2\dot{\theta}_{,ii}) + \kappa_1(T_{i,i} + \tau_2\dot{T}_{i,i}) \quad \text{in} \quad \Omega\times(0,T),$$
$$b\left(\frac{\tau_1^2}{2}\dddot{T}_i + \tau_1\ddot{T}_i + \dot{T}_i\right) = \kappa_6(T_{i,jj} + \tau_2\dot{T}_{i,jj}) + (\kappa_4+\kappa_5)(T_{j,ji} + \tau_2\dot{T}_{j,ji})$$
$$-\kappa_2(T_i + \tau_2\dot{T}_i) - \kappa_3(\theta_{,i} + \tau_2\dot{\theta}_{,i}) \quad \text{in} \quad \Omega\times(0,T),$$
$$\theta(\boldsymbol{x},t) = T_i(\boldsymbol{x},t) = 0 \quad \text{for a.e.} \quad \boldsymbol{x}\in\Gamma,\ t\in[0,T],$$
$$\theta(\boldsymbol{x},0) = \theta^0(\boldsymbol{x}), \quad \dot{\theta}(\boldsymbol{x},0) = \zeta^0(\boldsymbol{x}), \quad \ddot{\theta}(\boldsymbol{x},0) = \xi^0(\boldsymbol{x}) \quad \text{for a.e.} \quad \boldsymbol{x}\in\Omega,$$
$$T_i(\boldsymbol{x},0) = T_i^0(\boldsymbol{x}), \quad \dot{T}_i(\boldsymbol{x},0) = e_i^0(\boldsymbol{x}), \quad \ddot{T}_i(\boldsymbol{x},0) = w_i^0(\boldsymbol{x}) \quad \text{for a.e.}\ \boldsymbol{x}\in\Omega,$$

where θ^0, ζ^0, ξ^0, $\boldsymbol{T}^0 = (T_i^0)_{i=1}^d$, $\boldsymbol{e}^0 = (e_i^0)_{i=1}^d$ and $\boldsymbol{w}^0 = (w_i^0)_{i=1}^d$ are given initial conditions.

In order to obtain the variational formulation of Problem P, let $Y = L^2(\Omega)$, $H = [L^2(\Omega)]^d$ and $Q = [L^2(\Omega)]^{d\times d}$, and denote by $(\cdot,\cdot)_Y$, $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$, $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational spaces E and V as follows,

[1] A general proposition for this system would change the parameter τ_1 on the left-hand side of these equations, and a similar thing could be done with the parameter τ_2 on the right-hand side; however we try here a first contribution on this line to the problem, and we consider the easiest problem assuming that the same relaxation parameters apply to the macroscopic and microscopic aspects of the heat conducting material.

$$E = \{r \in H^1(\Omega)\,;\ r = 0 \quad \text{on} \quad \Gamma\},$$
$$V = \{\boldsymbol{z} \in [H^1(\Omega)]^d\,;\ \boldsymbol{z} = \boldsymbol{0} \quad \text{on} \quad \Gamma\},$$

with scalar products $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_V$, respectively, with norms $\|\cdot\|_E$ and $\|\cdot\|_V$.

By using Green's formula and the prescribed boundary conditions, we write the variational formulation of Problem P in terms of the thermal velocity $\zeta = \dot{\theta}$, the thermal acceleration $\xi = \ddot{\theta}$, the microtemperatures velocity $\boldsymbol{e} = \dot{\boldsymbol{T}}$ and the microtemperatures acceleration $\boldsymbol{w} = \ddot{\boldsymbol{T}}$.

Problem VP. *Find the temperature acceleration $\xi : [0,T] \to E$ and the microtemperatures acceleration $\boldsymbol{w} : [0,T] \to V$ such that $\xi(0) = \xi^0$, $\boldsymbol{w}(0) = \boldsymbol{w}^0$, and, for a.e. $t \in (0,T)$,*

$$\begin{aligned} &a\left(\frac{\tau_1^2}{2}\dot{\xi}(t) + \tau_1\xi(t) + \zeta(t), r\right)_Y + \kappa\,(\nabla(\theta(t) + \tau_2\zeta(t)), \nabla r)_H \\ &\quad = \kappa_1\,(\mathrm{div}\,\boldsymbol{T}(t) + \tau_2 \mathrm{div}\,\boldsymbol{e}(t), r)_Y \quad \forall r \in E, \end{aligned} \tag{4}$$

$$\begin{aligned} &b\left(\frac{\tau_1^2}{2}\dot{\boldsymbol{w}}(t) + \tau_1\boldsymbol{w}(t) + \boldsymbol{e}(t), \boldsymbol{z}\right)_H + \kappa_6\,(\nabla\boldsymbol{T}(t) + \tau_2\nabla\boldsymbol{e}(t), \nabla\boldsymbol{z})_Q \\ &+(\kappa_4+\kappa_5)\,(\mathrm{div}\,\boldsymbol{T}(t) + \tau_2\mathrm{div}\,\boldsymbol{e}(t), \mathrm{div}\,\boldsymbol{z})_Y + \kappa_2(\boldsymbol{T}(t) + \tau_2\boldsymbol{e}(t), \boldsymbol{z})_H \\ &\quad = -\kappa_3(\nabla\theta(t) + \tau_2\nabla\zeta(t), \boldsymbol{z})_H \quad \forall \boldsymbol{z} \in V, \end{aligned} \tag{5}$$

where the temperature speed ζ, the temperature θ, the microtemperatures speed $\boldsymbol{e}$ and the microtemperatures $\boldsymbol{T}$ are obtained from the respective equations:

$$\begin{aligned} &\zeta(t) = \int_0^t \xi(s)\,ds + \zeta^0, \quad \theta(t) = \int_0^t \zeta(s)\,ds + \theta^0, \\ &\boldsymbol{e}(t) = \int_0^t \boldsymbol{w}(s)\,ds + \boldsymbol{e}^0, \quad \boldsymbol{T}(t) = \int_0^t \boldsymbol{e}(s)\,ds + \boldsymbol{T}^0. \end{aligned} \tag{6}$$

In this paper, we will assume that[2]

$$\begin{aligned} &a > 0, \quad b > 0, \quad \tau_2 > \tau_1/2, \quad \kappa > 0, \quad \kappa\kappa_2 > \kappa_1\kappa_3 > 0, \\ &\kappa_6 > 0, \quad \kappa_4 + \kappa_5 > 0. \end{aligned} \tag{7}$$

We note that it is natural to assume that the thermal capacity a is strictly positive, as well as to assume that b is also positive. These two conditions are the usual ones in the study of heat conducting materials with microtemperatures. The condition on the delay parameters is the usual one in the study of dual-phase-lag materials (see [31,33]). It is also known that the axioms of thermomechanics implies that the thermal conductivity κ is positive, and a similar point of view can be used to assume that $\kappa_6 > 0$ and $\kappa\kappa_2 > \kappa_1\kappa_3$. We also note that the Onsager postulate implies that $\kappa_1 = \kappa_3$. Hence, the assumption $\kappa_1\kappa_3 > 0$ contains the class of materials satisfying this postulate. The condition $\kappa_4 + \kappa_5 > 0$ is a mathematical assumption we propose here to overcome the difficulties of the mathematical analysis.

We will work in this paper assuming that κ_1 and κ_3 are both positive but will remark later that the analysis can be also extended to the case that these constitutive coefficients are both negative.

Proceeding as in [28], we could prove that Problem VP has a unique solution. So, we have the following.

Theorem 2.1. *Let the assumptions (7) hold. Therefore, Problem VP has a unique solution with the following regularity:*

$$\theta \in C^2([0,T]; Y) \cap C^1([0,T]; E), \quad \boldsymbol{T} \in C^2([0,T]; H) \cap C^1([0,T]; V).$$

In order to simplify the calculations, in what follows we also impose the following conditions on the constitutive coefficients:

[2] The assumption concerning the relations between the two delay parameters allows the possibility that $\tau_2 > \tau_1$. It seems that, in this case, the heat flux is determined by the temperature gradient at some future time, which could be an anomaly of the model. Nevertheless, we accept this possibility between the delay parameters because the mathematical analysis can be used even in this case. At the same time, when we use equation (3), the meaning is different from equation (2).

$$\tau_1 > 0, \quad \kappa_1 > 0, \quad \kappa_3 > 0. \tag{8}$$

Let us define the energy $E(t)$ by

$$E(t) = \frac{1}{2}\left[\kappa_3 a \tau_1 \left\|\zeta(t) + \frac{\tau_1}{2}\xi(t)\right\|_Y^2 + \kappa_1 b \tau_1 \left\|\boldsymbol{e}(t) + \frac{\tau_1}{2}\boldsymbol{w}(t)\right\|_H^2 + \kappa_3 a \frac{\tau_1}{2}\|\zeta(t)\|_Y^2 \right.$$
$$\left. + \kappa_1 b \frac{\tau_1}{2}\|\boldsymbol{e}(t)\|_H^2 + B(t) + F(t), \right]$$

where

$$B(t) = \kappa_3\kappa\left[\|\nabla\theta(t)\|_H^2 + \tau_1(\nabla\theta(t), \nabla\zeta(t))_H + \tau_2\frac{\tau_1}{2}\|\nabla\zeta(t)\|_H^2\right]$$
$$+\kappa_1\kappa_2\left[\|\boldsymbol{T}(t)\|_H^2 + \tau_1(\boldsymbol{T}(t), \boldsymbol{e}(t))_H + \tau_2\frac{\tau_1}{2}\|\boldsymbol{e}(t)\|_H^2\right]$$
$$+\kappa_1\kappa_3\left[2(\boldsymbol{T}(t), \nabla\theta(t))_H + \tau_1[(\boldsymbol{T}(t), \nabla\zeta(t))_H + (\nabla\theta(t), \boldsymbol{e}(t))_H] + \tau_2\tau_1(\boldsymbol{e}(t), \nabla\zeta(t))_H\right]$$

and

$$F(t) = \kappa_1\kappa_6\left[\|\nabla\boldsymbol{T}(t)\|_Q^2 + \tau_1(\nabla\boldsymbol{T}(t), \nabla\boldsymbol{e}(t))_Q + \tau_2\frac{\tau_1}{2}\|\nabla\boldsymbol{e}(t)\|_Q^2\right]$$
$$+\kappa_1(\kappa_4+\kappa_5)\left[\|\mathrm{div}\,\boldsymbol{T}(t)\|_Y^2 + \tau_1(\mathrm{div}\,\boldsymbol{T}(t), \mathrm{div}\,\boldsymbol{e}(t))_Y + \tau_2\frac{\tau_1}{2}\|\mathrm{div}\,\boldsymbol{e}(t)\|_Y^2\right].$$

Note that, if $a, b, \kappa_3, \kappa_1, \kappa_4+\kappa_5$, $\tau_2 - \tau_1/2 > 0$ and $\kappa\kappa_2 - \kappa_1\kappa_3 > 0$, then $E(t) \geq 0$ because the matrices

$$A = \begin{pmatrix} \kappa_3\kappa & \kappa_3\kappa\frac{\tau_1}{2} & \kappa_1\kappa_3 & \kappa_3\kappa_1\frac{\tau_1}{2} \\ \kappa_3\kappa\frac{\tau_1}{2} & \kappa_3\kappa\tau_2\frac{\tau_1}{2} & \kappa_3\kappa_1\frac{\tau_1}{2} & \kappa_3\kappa_1\tau_2\frac{\tau_1}{2} \\ \kappa_1\kappa_3 & \kappa_3\kappa_1\frac{\tau_1}{2} & \kappa_1\kappa_2 & \kappa_1\kappa_2\frac{\tau_1}{2} \\ \kappa_3\kappa_1\frac{\tau_1}{2} & \kappa_3\kappa_1\tau_2\frac{\tau_1}{2} & \kappa_1\kappa_2\frac{\tau_1}{2} & \kappa_1\kappa_2\tau_2\frac{\tau_1}{2} \end{pmatrix}$$

and

$$C = \begin{pmatrix} \kappa_6 & \kappa_6\frac{\tau_1}{2} & 0 & 0 \\ \kappa_6\frac{\tau_1}{2} & \kappa_6\tau_2\frac{\tau_1}{2} & 0 & 0 \\ 0 & 0 & \kappa_4+\kappa_5 & (\kappa_4+\kappa_5)\frac{\tau_1}{2} \\ 0 & 0 & (\kappa_4+\kappa_5)\frac{\tau_1}{2} & (\kappa_4+\kappa_5)\tau_2\frac{\tau_1}{2} \end{pmatrix}$$

are positive definite. For matrix C it is straightforward, and for matrix A it is found since the determinants of the minor leading matrices are positive:

$$\det A_2 = (\kappa_3\kappa)^2\frac{\tau_1}{2}\left(\tau_2 - \frac{\tau_1}{2}\right) > 0,$$
$$\det A_3 = \kappa_3^2\kappa\kappa_1(\kappa\kappa_2 - \kappa_1\kappa_3)\frac{\tau_1}{2}\left(\tau_2 - \frac{\tau_1}{2}\right) > 0,$$
$$\det A = \left[\kappa_3\kappa_1(\kappa\kappa_2 - \kappa_1\kappa_3)\frac{\tau_1}{2}\left(\tau_2 - \frac{\tau_1}{2}\right)\right]^2 > 0.$$

Now, we obtain the following energy decay property.

Theorem 2.2. *Suppose that assumptions (7) and (8) still hold. Thus, the energy $E(t)$ decays, i.e.*

$$\frac{d}{dt}E(t) \leq 0.$$

Proof. In this proof, we will remove the time t in all the variables for the sake of simplicity in the writing.

Taking $r = \kappa_3(\zeta + \frac{\tau_1}{2}\xi)$ in variational equation (4) and $\boldsymbol{z} = \kappa_1(\boldsymbol{e} + \frac{\tau_1}{2}\boldsymbol{w})$ in variational equation (5) we obtain

$$\kappa_3 a\left[\|\zeta\|_Y^2 + \frac{\tau_1}{4}\frac{d}{dt}\|\zeta\|_Y^2 + \frac{\tau_1}{2}\frac{d}{dt}\|\zeta + \frac{\tau_1}{2}\xi\|_Y^2\right] + \kappa_3\frac{\kappa}{2}\frac{d}{dt}\|\nabla\theta\|_H^2$$
$$+\kappa_3\kappa\left[\frac{\tau_1}{2}(\nabla\theta, \nabla\xi)_H + \tau_2\|\nabla\zeta\|_H^2 + \tau_2\frac{\tau_1}{4}\frac{d}{dt}\|\nabla\zeta\|_H^2\right]$$
$$= \kappa_3\kappa_1\left(\mathrm{div}\,\boldsymbol{T} + \tau_2\mathrm{div}\,\boldsymbol{e}, \zeta + \frac{\tau_1}{2}\xi\right)_Y$$
$$= -\kappa_3\kappa_1\left(\boldsymbol{T} + \tau_2\boldsymbol{e}, \nabla\zeta + \frac{\tau_1}{2}\nabla\xi\right)_H, \tag{9}$$

and

$$\kappa_1 b\left[\|\boldsymbol{e}\|_H^2+\frac{\tau_1}{4}\frac{d}{dt}\|\boldsymbol{e}\|_H^2+\frac{\tau_1}{2}\frac{d}{dt}\|\boldsymbol{e}+\frac{\tau_1}{2}\boldsymbol{w}\|_H^2\right]+\kappa_1\kappa_6\frac{d}{dt}\|\nabla\boldsymbol{T}\|_Q^2$$
$$+\kappa_1\kappa_6\left[\frac{\tau_1}{2}(\nabla\boldsymbol{T},\nabla\boldsymbol{w})_Q+\tau_2\|\nabla\boldsymbol{e}\|_Q^2+\tau_2\frac{\tau_1}{4}\frac{d}{dt}\|\nabla\boldsymbol{e}\|_Q^2\right]$$
$$+\kappa_1(\kappa_4+\kappa_5)\left[\frac{1}{2}\frac{d}{dt}\|\operatorname{div}\boldsymbol{T}\|_Y^2+\frac{\tau_1}{2}(\operatorname{div}\boldsymbol{T},\operatorname{div}\boldsymbol{w})_Y+\tau_2\|\operatorname{div}\boldsymbol{e}\|_Y^2\right]$$
$$+\kappa_1(\kappa_4+\kappa_5)\tau_2\frac{\tau_1}{4}\frac{d}{dt}\|\operatorname{div}\boldsymbol{e}\|_Y^2$$
$$+\kappa_1\kappa_2\left[\frac{1}{2}\frac{d}{dt}\|\boldsymbol{T}\|_H^2+\frac{\tau_1}{2}(\boldsymbol{T},\boldsymbol{w})_H+\tau_2\|\boldsymbol{e}\|_H^2+\tau_2\frac{\tau_1}{4}\frac{d}{dt}\|\boldsymbol{e}\|_H^2\right]$$
$$=-\kappa_1\kappa_3(\nabla\theta+\tau_2\nabla\zeta,\boldsymbol{e}+\frac{\tau_1}{2}\boldsymbol{w})_H. \tag{10}$$

Observing that

$$(\nabla\theta,\nabla\xi)_H=\frac{d}{dt}(\nabla\theta,\nabla\zeta)_H-\|\nabla\zeta\|_H^2,$$
$$(\nabla\boldsymbol{T},\nabla\boldsymbol{w})_Q=\frac{d}{dt}(\nabla\boldsymbol{T},\nabla\boldsymbol{e})_Q-\|\nabla\boldsymbol{e}\|_Q^2,$$
$$(\operatorname{div}\boldsymbol{T},\operatorname{div}\boldsymbol{w})_Y=\frac{d}{dt}(\operatorname{div}\boldsymbol{T},\operatorname{div}\boldsymbol{e})_Y-\|\operatorname{div}\boldsymbol{e}\|_Y^2,$$
$$(\boldsymbol{T},\boldsymbol{w})_H=\frac{d}{dt}(\boldsymbol{T},\boldsymbol{e})_H-\|\boldsymbol{e}\|_H^2,$$

and that

$$-\kappa_3\kappa_1\left\{\left(\boldsymbol{T}+\tau_2\boldsymbol{e},\nabla\zeta+\frac{\tau_1}{2}\nabla\xi\right)_H+(\nabla\theta+\tau_2\nabla\zeta,\boldsymbol{e}+\frac{\tau_1}{2}\boldsymbol{w})_H\right\}$$
$$=-\kappa_3\kappa_1\left\{\frac{d}{dt}(\boldsymbol{T},\nabla\Theta)_H+2\tau_2(\boldsymbol{e},\nabla\zeta)_H+\frac{\tau_1}{2}\left[\frac{d}{dt}(\boldsymbol{T},\nabla\zeta)_H-(\boldsymbol{e},\nabla\zeta)_H\right]\right.$$
$$\left.+\frac{\tau_1}{2}\left[\frac{d}{dt}(\nabla\theta,\boldsymbol{e})_H-(\nabla\zeta,\boldsymbol{e})_H\right]+\tau_2\frac{\tau_1}{2}\frac{d}{dt}(\boldsymbol{e},\nabla\zeta)_H\right\},$$

we find, after adding (9) and (10),

$$\kappa_3 a\frac{\tau_1}{2}\frac{d}{dt}\left[\frac{1}{2}\|\zeta\|_Y^2+\|\zeta+\frac{\tau_1}{2}\xi\|_Y^2\right]+\kappa_1 b\frac{\tau_1}{2}\frac{d}{dt}\left[\frac{1}{2}\|\boldsymbol{e}\|_H^2+\|\boldsymbol{e}+\frac{\tau_1}{2}\boldsymbol{w}\|_H^2\right]$$
$$+\frac{1}{2}\frac{d}{dt}(B+F)+\kappa_3 a\|\zeta\|_Y^2+\kappa_1 b\|\boldsymbol{e}\|_H^2+\kappa_3\kappa\left(\tau_2-\frac{\tau_1}{2}\right)\|\nabla\zeta\|_H^2$$
$$+2\kappa_3\kappa_1\left(\tau_2-\frac{\tau_1}{2}\right)(\boldsymbol{e},\nabla\zeta)_H+\kappa_1\kappa_2\left(\tau_2-\frac{\tau_1}{2}\right)\|\boldsymbol{e}\|_H^2$$
$$+\kappa_1\kappa_6\left(\tau_2-\frac{\tau_1}{2}\right)\|\nabla\boldsymbol{e}\|_Q^2+\kappa_1(\kappa_4+\kappa_5)\left(\tau_2-\frac{\tau_1}{2}\right)\|\operatorname{div}\boldsymbol{e}\|_Y^2=0.$$

Taking into account that $\tau_2-\tau_1/2>0$, $\kappa\kappa_2-\kappa_3\kappa_1>0$, and that

$$\det\begin{pmatrix}\kappa_3\kappa & \kappa_3\kappa_1\\ \kappa_3\kappa_1 & \kappa_1\kappa_2\end{pmatrix}=\kappa_3\kappa_1(\kappa\kappa_2-\kappa_3\kappa_1)>0,$$

the result follows. □

Remark 2.3. Even if we have imposed in conditions (8) that coefficients κ_1 and κ_3 are positive, in fact we only need that $\kappa_1\kappa_3>0$. In case that they are both negative, we should replace κ_1 (resp. κ_3) by $-\kappa_1$ (resp. $-\kappa_3$) in the definition of the energy $E(t)$.

3. Fully discrete approximations: an a priori error analysis

In this section, we introduce a finite element algorithm for approximating solutions to variational problem VP. This is done in two steps. First, we construct the finite element spaces E^h and V^h to approximate the variational spaces E and V, respectively, given by

$$E^h = \{r^h \in C(\overline{\Omega})\,;\; r^h_{|Tr} \in P_1(Tr) \quad \forall\, Tr \in \mathcal{T}^h, \quad r^h = 0 \text{ on } \Gamma\}, \tag{11}$$

$$V^h = \{\boldsymbol{z}^h \in [C(\overline{\Omega})]^d\,;\; \boldsymbol{z}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall\, Tr \in \mathcal{T}^h, \quad \boldsymbol{z}^h = \boldsymbol{0} \text{ on } \Gamma\}, \tag{12}$$

where $\overline{\Omega}$ is assumed to be a polyhedral domain, $\mathcal{T}^h$ denotes a triangulation of $\overline{\Omega}$, and $P_1(Tr)$ represents the space of polynomials of global degree less or equal to 1 in Tr. Here, $h > 0$ denotes the spatial discretization parameter.

Secondly, the time derivatives are discretized by using a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, and let k be the time step size, $k = T/N$. Moreover, for a continuous function $f(t)$ we denote $f_n = f(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Using the implicit Euler scheme, the fully discrete approximation of Problem VP is the following.

Problem VP^{hk}. *Find the discrete temperature acceleration $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset E^h$ and the discrete microtemperatures acceleration $\boldsymbol{w}^{hk} = \{\boldsymbol{w}_n^{hk}\}_{n=0}^N \subset V^h$ such that $\xi_0^{hk} = \xi^{0h}$ and $\boldsymbol{w}_0^{hk} = \boldsymbol{w}^{0h}$ and, for $n = 1, \ldots, N$,*

$$\begin{aligned} &a\left(\frac{\tau_1^2}{2}\delta\xi_n^{hk} + \tau_1\xi_n^{hk} + \zeta_n^{hk}, r^h\right)_Y + \kappa\left(\nabla(\theta_n^{hk} + \tau_2\zeta_n^{hk}), \nabla r^h\right)_H \\ &\quad = \kappa_1\left(\operatorname{div}\boldsymbol{T}_n^{hk} + \tau_2\operatorname{div}\boldsymbol{e}_n^{hk}, r^h\right)_Y \quad \forall r^h \in E^h, \end{aligned} \tag{13}$$

$$\begin{aligned} &b\left(\frac{\tau_1^2}{2}\delta\boldsymbol{w}_n^{hk} + \tau_1\boldsymbol{w}_n^{hk} + \boldsymbol{e}_n^{hk}, \boldsymbol{z}^h\right)_H + \kappa_6\left(\nabla\boldsymbol{T}_n^{hk} + \tau_2\nabla\boldsymbol{e}_n^{hk}, \nabla\boldsymbol{z}^h\right)_Q \\ &\quad + (\kappa_4 + \kappa_5)\left(\operatorname{div}\boldsymbol{T}_n^{hk} + \tau_2\operatorname{div}\boldsymbol{e}_n^{hk}, \operatorname{div}\boldsymbol{z}^h\right)_Y + \kappa_2(\boldsymbol{T}_n^{hk} + \tau_2\boldsymbol{e}_n^{hk}, \boldsymbol{z}^h)_H \\ &= -\kappa_3(\nabla(\theta_n^{hk} + \tau_2\zeta_n^{hk}), \boldsymbol{z}^h)_H \quad \forall \boldsymbol{z}^h \in V^h, \end{aligned} \tag{14}$$

where the discrete temperature speed ζ_n^{hk}, the discrete temperature θ_n^{hk}, the discrete microtemperatures speed $\boldsymbol{e}_n^{hk}$ and the discrete microtemperatures $\boldsymbol{T}_n^{hk}$ are then recovered from the relations:

$$\begin{aligned} &\zeta_n^{hk} = k\sum_{j=1}^n \xi_j^{hk} + \zeta^{0h}, \quad \theta_n^{hk} = k\sum_{j=1}^n \zeta_j^{hk} + \theta^{0h}, \\ &\boldsymbol{e}_n^{hk} = k\sum_{j=1}^n \boldsymbol{w}_j^{hk} + \boldsymbol{e}^{0h}, \quad \boldsymbol{T}_n^{hk} = k\sum_{j=1}^n \boldsymbol{e}_j^{hk} + \boldsymbol{T}^{0h}. \end{aligned} \tag{15}$$

Here, $\boldsymbol{w}^{0h}$, $\boldsymbol{e}^{0h}$, $\boldsymbol{T}^{0h}$, ξ^{0h}, ζ^{0h} and θ^{0h} are approximations of the initial conditions $\boldsymbol{w}^0$, $\boldsymbol{e}^0$, $\boldsymbol{T}^0$, ξ^0, ζ^0, and θ^0 defined as

$$\begin{aligned} &\boldsymbol{w}^{0h} = \mathcal{P}_1^h\boldsymbol{w}^0, \quad \boldsymbol{e}^{0h} = \mathcal{P}_1^h\boldsymbol{e}^0, \quad \boldsymbol{T}^{0h} = \mathcal{P}_1^h\boldsymbol{T}^0, \\ &\xi^{0h} = \mathcal{P}_2^h\xi^0, \quad \zeta^{0h} = \mathcal{P}_2^h\zeta^0, \quad \theta^{0h} = \mathcal{P}_2^h\theta^0, \end{aligned}$$

where $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ are the classical finite element interpolation operators over V^h and E^h, respectively (see, e.g., [14]).

Using the classical Lax-Milgram lemma we can easily prove that discrete problem VP^{hk} has a unique solution. So, the rest of this section is devoted to show a discrete stability property and a priori error estimates.

We have the following stability result.

Lemma 3.1. *Let the assumptions of Theorem 2.1 hold. Under the additional assumptions (8), it follows that the sequences $\{\theta^{hk}, \zeta^{hk}, \xi^{hk}, \boldsymbol{T}^{hk}, \boldsymbol{e}^{hk}, \boldsymbol{w}^{hk}\}$ generated by Problem VP^{hk} satisfy the stability estimate:*

$$\|\xi_n^{hk}\|_Y^2 + \|\theta_n^{hk}\|_E^2 + \|\zeta_n^{hk}\|_E^2 + \|\boldsymbol{w}_n^{hk}\|_H^2 + \|\boldsymbol{e}_n^{hk}\|_V^2 + \|\boldsymbol{T}_n^{hk}\|_V^2 \le C,$$

where C is a positive constant which is independent of the discretization parameters h and k.

Proof. Taking $r^h = \xi_n^{hk}$ as a test function in variational equation (13) we find that

$$\begin{aligned} &a\left(\frac{\tau_1^2}{2}\delta\xi_n^{hk} + \tau_1\xi_n^{hk} + \zeta_n^{hk}, \xi_n^{hk}\right)_Y + \kappa\left(\nabla(\theta_n^{hk} + \tau_2\zeta_n^{hk}), \nabla\xi_n^{hk}\right)_H \\ &\quad = \kappa_1\left(\operatorname{div}\boldsymbol{T}_n^{hk} + \tau_2\operatorname{div}\boldsymbol{e}_n^{hk}, \xi_n^{hk}\right)_Y, \end{aligned}$$

and therefore, keeping in mind that

$$\left(\delta\xi_n^{hk},\xi_n^{hk}\right)_Y \geq \frac{1}{2k}\left\{\|\xi_n^{hk}\|_Y^2-\|\xi_{n-1}^{hk}\|_Y^2\right\},$$
$$\left(\nabla\zeta_n^{hk},\nabla\xi_n^{hk}\right)_H \geq \frac{1}{2k}\left\{\|\nabla\zeta_n^{hk}\|_H^2-\|\nabla\zeta_{n-1}^{hk}\|_H^2\right\},$$

it follows that

$$\begin{aligned}&\frac{1}{2k}\left\{\|\xi_n^{hk}\|_Y^2-\|\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{1}{2k}\left\{\|\nabla\zeta_n^{hk}\|_H^2-\|\nabla\zeta_{n-1}^{hk}\|_H^2\right\}+(\nabla\theta_n^{hk},\nabla\xi_n^{hk})_H\\ &\quad\leq C\left(\|\xi_n^{hk}\|_Y^2+\|\nabla\theta_n^{hk}\|_H^2+\|\operatorname{div}\boldsymbol{T}_n^{hk}\|_Y^2+\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_Y^2\right).\end{aligned}\tag{16}$$

Now, taking $\boldsymbol{z}^h=\boldsymbol{w}_n^{hk}$ as a test function in variational equation (14) we have

$$\begin{aligned}&b\left(\frac{\tau_1^2}{2}\delta\boldsymbol{w}_n^{hk}+\tau_1\boldsymbol{w}_n^{hk}+\boldsymbol{e}_n^{hk},\boldsymbol{w}_n^{hk}\right)_H+\kappa_6\left(\nabla\boldsymbol{T}_n^{hk}+\tau_2\nabla\boldsymbol{e}_n^{hk},\nabla\boldsymbol{w}_n^{hk}\right)_Q\\ &\quad+(\kappa_4+\kappa_5)\left(\operatorname{div}\boldsymbol{T}_n^{hk}+\tau_2\operatorname{div}\boldsymbol{e}_n^{hk},\operatorname{div}\boldsymbol{w}_n^{hk}\right)_Y+\kappa_2(\boldsymbol{T}_n^{hk}+\tau_2\boldsymbol{e}_n^{hk},\boldsymbol{w}_n^{hk})_H\\ &\quad=-\kappa_3(\nabla(\theta_n^{hk}+\tau_2\zeta_n^{hk}),\boldsymbol{w}_n^{hk})_H.\end{aligned}$$

Using the following estimates:

$$\left(\delta\boldsymbol{w}_n^{hk},\boldsymbol{w}_n^{hk}\right)_H\geq\frac{1}{2k}\left\{\|\boldsymbol{w}_n^{hk}\|_H^2-\|\boldsymbol{w}_{n-1}^{hk}\|_H^2\right\},$$
$$\left(\nabla\boldsymbol{e}_n^{hk},\nabla\boldsymbol{w}_n^{hk}\right)_Q\geq\frac{1}{2k}\left\{\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{e}_{n-1}^{hk}\|_Q^2\right\},$$
$$\left(\operatorname{div}\boldsymbol{e}_n^{hk},\operatorname{div}\boldsymbol{w}_n^{hk}\right)_Y\geq\frac{1}{2k}\left\{\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{e}_{n-1}^{hk}\|_Y^2\right\},$$
$$\left(\boldsymbol{e}_n^{hk},\boldsymbol{w}_n^{hk}\right)_H\geq\frac{1}{2k}\left\{\|\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}^{hk}\|_H^2\right\},$$

we find that

$$\begin{aligned}&\frac{1}{2k}\left\{\|\boldsymbol{w}_n^{hk}\|_H^2-\|\boldsymbol{w}_{n-1}^{hk}\|_H^2\right\}+\frac{1}{2k}\left\{\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{e}_{n-1}^{hk}\|_Q^2\right\}+(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{w}_n^{hk})_Q\\ &\quad+\frac{1}{2k}\left\{\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{e}_{n-1}^{hk}\|_Y^2\right\}+(\operatorname{div}\boldsymbol{T}_n^{hk},\operatorname{div}\boldsymbol{w}_n^{hk})_Y+(\boldsymbol{T}_n^{hk},\boldsymbol{w}_n^{hk})_H\\ &\quad\leq C\left(\|\boldsymbol{e}_n^{hk}\|_H^2+\|\nabla\theta_n^{hk}\|_H^2+\|\nabla\zeta_n^{hk}\|_H^2+\|\boldsymbol{w}_n^{hk}\|_H^2\right).\end{aligned}\tag{17}$$

Combining estimates (16) and (17) it leads

$$\begin{aligned}&\frac{1}{2k}\left\{\|\xi_n^{hk}\|_Y^2-\|\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{1}{2k}\left\{\|\nabla\zeta_n^{hk}\|_H^2-\|\nabla\zeta_{n-1}^{hk}\|_H^2\right\}+(\nabla\theta_n^{hk},\nabla\xi_n^{hk})_H\\ &\quad+\frac{1}{2k}\left\{\|\boldsymbol{w}_n^{hk}\|_H^2-\|\boldsymbol{w}_{n-1}^{hk}\|_H^2\right\}+\frac{1}{2k}\left\{\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{e}_{n-1}^{hk}\|_Q^2\right\}+(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{w}_n^{hk})_Q\\ &\quad+\frac{1}{2k}\left\{\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{e}_{n-1}^{hk}\|_Y^2\right\}+(\operatorname{div}\boldsymbol{T}_n^{hk},\operatorname{div}\boldsymbol{w}_n^{hk})_Y+(\boldsymbol{T}_n^{hk},\boldsymbol{w}_n^{hk})_H\\ &\quad\leq C\Big(\|\xi_n^{hk}\|_Y^2+\|\nabla\theta_n^{hk}\|_H^2+\|\operatorname{div}\boldsymbol{T}_n^{hk}\|_Y^2+\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_Y^2+\|\boldsymbol{e}_n^{hk}\|_H^2\\ &\quad+\|\nabla\zeta_n^{hk}\|_H^2+\|\boldsymbol{w}_n^{hk}\|_H^2\Big).\end{aligned}$$

Multiplying the above estimates by k and summing up the resulting equations until n, we find that

$$\begin{aligned}&\|\xi_n^{hk}\|_Y^2+\|\nabla\zeta_n^{hk}\|_H^2+k\sum_{j=1}^n(\nabla\theta_j^{hk},\nabla\xi_j^{hk})_H+\|\boldsymbol{w}_n^{hk}\|_H^2+\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2+\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_Y^2\\ &\quad+k\sum_{j=1}^n(\nabla\boldsymbol{T}_j^{hk},\nabla\boldsymbol{w}_j^{hk})_Q+k\sum_{j=1}^n(\operatorname{div}\boldsymbol{T}_j^{hk},\operatorname{div}\boldsymbol{w}_j^{hk})_Y+k\sum_{j=1}^n(\boldsymbol{T}_j^{hk},\boldsymbol{w}_j^{hk})_H\end{aligned}$$

$$\leq Ck \sum_{j=1}^{n} \Big(\|\xi_j^{hk}\|_Y^2 + \|\nabla\theta_j^{hk}\|_H^2 + \|\mathrm{div}\, \boldsymbol{T}_j^{hk}\|_Y^2 + \|\mathrm{div}\, \boldsymbol{e}_j^{hk}\|_Y^2 + \|\boldsymbol{e}_j^{hk}\|_H^2$$
$$+ \|\nabla\zeta_j^{hk}\|_H^2 + \|\boldsymbol{w}_j^{hk}\|_H^2 \Big).$$

Now, taking into account that

$$k\sum_{j=1}^{n} (\nabla\theta_j^{hk}, \nabla\xi_j^{hk})_H = \sum_{j=1}^{n} (\nabla\theta_j^{hk}, \nabla\zeta_j^{hk} - \nabla\zeta_{j-1}^{hk})_H$$
$$= (\nabla\theta_n^{hk}, \nabla\zeta_n^{hk})_H + \sum_{j=1}^{n-1} (\nabla(\theta_j^{hk} - \theta_{j+1}^{hk}), \nabla\zeta_j^{hk})_H + (\nabla\theta_1^{hk}, \nabla\zeta_0^{h})_H,$$
$$\sum_{j=1}^{n-1} (\nabla(\theta_j^{hk} - \theta_{j+1}^{hk}), \nabla\zeta_j^{hk})_H \leq Ck \sum_{j=1}^{n-1} \|\nabla\zeta_j^{hk}\|_H^2 + \frac{C}{k}\sum_{j=1}^{n-1} \|\nabla(\theta_j^{hk} - \theta_{j+1}^{hk})\|_H^2$$
$$\leq Ck \sum_{j=1}^{n} \|\nabla\zeta_j^{hk}\|_H^2,$$

where similar estimates can be found for the terms $k\sum_{j=1}^{n}(\boldsymbol{T}_j^{hk}, \boldsymbol{w}_j^{hk})_H$, $k\sum_{j=1}^{n}(\nabla\boldsymbol{T}_j^{hk}, \nabla\boldsymbol{w}_j^{hk})_Q$ and $k\sum_{j=1}^{n}(\mathrm{div}\, \boldsymbol{T}_j^{hk}, \mathrm{div}\, \boldsymbol{w}_j^{hk})_Y$, applying a discrete version of Gronwall's inequality (see [10]) we obtain the desired stability estimates. □

The result that follows is a discrete version of the energy decay property that holds for the continuous problem.

Theorem 3.2. *Let the assumptions of Lemma 3.1 P hold and define the discrete energy E_n^{hk} as*

$$E_n^{hk} = \frac{1}{2}\Big[\kappa_3 a\tau_1 \Big\|\zeta_n^{hk} + \frac{\tau_1}{2}\xi_n^{hk}\Big\|_Y^2 + \kappa_1 b\tau_1 \Big\|\boldsymbol{e}_n^{hk} + \frac{\tau_1}{2}\boldsymbol{w}_n^{hk}\Big\|_H^2 + \kappa_3 a\frac{\tau_1}{2}\|\zeta_n^{hk}\|_Y^2$$
$$+\kappa_1 b\tfrac{\tau_1}{2}\|\boldsymbol{e}_n^{hk}\|_H^2 + B_n^{hk} + F_n^{hk}\Big] \quad \text{for} \quad n = 0, \ldots, N,$$

with

$$B_n^{hk} = \kappa_3\kappa\Big[\|\nabla\theta_n^{hk}\|_H^2 + \tau_1(\nabla\theta_n^{hk}, \nabla\zeta_n^{hk})_H + \tau_2\frac{\tau_1}{2}\|\nabla\zeta_n^{hk}\|_H^2\Big]$$
$$+\kappa_1\kappa_2\Big[\|\boldsymbol{T}_n^{hk}\|_H^2 + \tau_1(\boldsymbol{T}_n^{hk}, \boldsymbol{e}_n^{hk})_H + \tau_2\frac{\tau_1}{2}\|\boldsymbol{e}_n^{hk}\|_H^2\Big]$$
$$+\kappa_1\kappa_3\Big[2(\boldsymbol{T}_n^{hk}, \nabla\theta_n^{hk})_H + \tau_1[(\boldsymbol{T}_n^{hk}, \nabla\zeta_n^{hk})_H + (\nabla\theta_n^{hk}, \boldsymbol{e}_n^{hk})_H]$$
$$+\tau_2\tau_1(\boldsymbol{e}_n^{hk}, \nabla\zeta_n^{hk})_H\Big]$$

and

$$F_n^{hk} = \kappa_1\kappa_6\Big[\|\nabla\boldsymbol{T}_n^{hk}\|_Q^2 + \tau_1(\nabla\boldsymbol{T}_n^{hk}, \nabla\boldsymbol{e}_n^{hk})_Q + \tau_2\frac{\tau_1}{2}\|\nabla\boldsymbol{e}^{hk}\|_Q^2\Big]$$
$$+\kappa_1(\kappa_4+\kappa_5)\Big[\|\mathrm{div}\, \boldsymbol{T}_n^{hk}\|_Y^2 + \tau_1(\mathrm{div}\, \boldsymbol{T}_n^{hk}, \mathrm{div}\, \boldsymbol{e}_n^{hk})_Y + \tau_2\frac{\tau_1}{2}\|\mathrm{div}\, \boldsymbol{e}_n^{hk}\|_Y^2\Big].$$

Then, under the same assumptions in Theorem 2.2 the discrete energy decays:

$$\frac{E_n^{hk} - E_{n-1}^{hk}}{k} \leq 0 \quad \text{for} \quad n = 1, \ldots, N.$$

Proof. In this proof, we will use some auxiliary real numbers $\gamma_1 < 2$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$.

Choosing $r^h = \kappa_3(\zeta_n^{hk} + \frac{\tau_1}{2}\xi_n^{hk})$ and $\boldsymbol{z}^h = \kappa_1(\boldsymbol{e}_n^{hk} + \frac{\tau_1}{2}\boldsymbol{w}_n^{hk})$ in discrete variational equations (13) and (14), respectively, it results that

$$
\begin{aligned}
&\kappa_3 a\left[\|\zeta_n^{hk}\|_Y^2+\frac{\tau_1}{4k}\Big(\|\zeta_n^{hk}\|_Y^2-\|\zeta_{n-1}^{hk}\|_Y^2\Big)\right]\\
&+\kappa_3 a\frac{\tau_1}{2k}\Big(\Big\|\zeta_n^{hk}+\frac{\tau_1}{2}\xi_n^{hk}\Big\|_Y^2-\Big\|\zeta_{n-1}^{hk}+\frac{\tau_1}{2}\xi_{n-1}^{hk}\Big\|_Y^2\Big)\\
&+\kappa_3\kappa\frac{1}{2k}\Big(\|\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk}\|_H^2+\|\nabla\theta_n^{hk}\|_H^2-\|\nabla\theta_{n-1}^{hk}\|_H^2\Big)\\
&+\kappa_3\kappa\frac{\tau_1}{2}\left(\nabla\theta_n^{hk},\nabla\xi_n^{hk}\right)_H+\kappa_3\kappa\tau_2\|\nabla\zeta_n^{hk}\|_H^2\\
&+\kappa_3\kappa\tau_2\frac{\tau_1}{4k}\Big(\|\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk}\|_H^2+\|\nabla\zeta_n^{hk}\|_H^2-\|\nabla\zeta_{n-1}^{hk}\|_H^2\Big)\\
&\le\kappa_3\kappa_1\left(\operatorname{div}\boldsymbol{T}_n^{hk}+\tau_2\operatorname{div}\boldsymbol{e}_n^{hk},\zeta_n^{hk}+\frac{\tau_1}{2}\xi_n^{hk}\right)_Y\\
&=-\kappa_3\kappa_1\left(\boldsymbol{T}_n^{hk}+\tau_2\boldsymbol{e}_n^{hk},\nabla\zeta_n^{hk}+\frac{\tau_1}{2}\nabla\xi_n^{hk}\right)_H
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
&\kappa_1 b\left[\|\boldsymbol{e}_n^{hk}\|_H^2+\frac{\tau_1}{4k}\Big(\|\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big)\right]\\
&+\kappa_1 b\frac{\tau_1}{2k}\Big(\Big\|\boldsymbol{e}_n^{hk}+\frac{\tau_1}{2}\boldsymbol{w}_n^{hk}\Big\|_Y^2-\Big\|\boldsymbol{e}_{n-1}^{hk}+\frac{\tau_1}{2}\boldsymbol{w}_{n-1}^{hk}\Big\|_Y^2\Big)\\
&+\kappa_1\kappa_6\frac{1}{2k}\Big(\|\nabla\boldsymbol{T}_n^{hk}-\nabla\boldsymbol{T}_{n-1}^{hk}\|_Q^2+\|\nabla\boldsymbol{T}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{T}_{n-1}^{hk}\|_Q^2\Big)\\
&+\kappa_1\kappa_6\frac{\tau_1}{2}(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{w}_n^{hk})_Q+\kappa_1\kappa_6\tau_2\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2\\
&+\kappa_1\kappa_6\tau_2\frac{\tau_1}{4k}\Big(\|\nabla\boldsymbol{e}_n^{hk}-\nabla\boldsymbol{e}_{n-1}^{hk}\|_Q^2+\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{e}_{n-1}^{hk}\|_Q^2\Big)\\
&+\kappa_1(\kappa_4+\kappa_5)\frac{1}{2k}\Big(\|\operatorname{div}(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk})\|_Y^2+\|\operatorname{div}\boldsymbol{T}_n^{hk}\|_Y^2-\|\operatorname{div}\boldsymbol{T}_{n-1}^{hk}\|_Y^2\Big)\\
&+\kappa_1(\kappa_4+\kappa_5)\frac{\tau_1}{2}\left(\operatorname{div}\boldsymbol{T}_n^{hk},\operatorname{div}\boldsymbol{w}_n^{hk}\right)_Y+\kappa_1(\kappa_4+\kappa_5)\tau_2\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_Y^2\\
&+\kappa_1(\kappa_4+\kappa_5)\tau_2\frac{\tau_1}{4k}\Big(\|\operatorname{div}(\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk})\|_H^2+\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_H^2-\|\operatorname{div}\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big)\\
&+\kappa_1\kappa_2\frac{1}{2k}\Big(\|\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk}\|_H^2+\|\boldsymbol{T}_n^{hk}\|_H^2-\|\boldsymbol{T}_{n-1}^{hk}\|_H^2\Big)+\kappa_1\kappa_2\frac{\tau_1}{2}(\boldsymbol{T}_n^{hk},\boldsymbol{w}_n^{hk})_H\\
&+\kappa_1\kappa_2\tau_2\|\boldsymbol{e}_n^{hk}\|_H^2+\kappa_1\kappa_2\tau_2\frac{\tau_1}{4k}\Big(\|\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk}\|_H^2+\|\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big)\\
&\le-\kappa_1\kappa_3\left(\nabla\theta_n^{hk}+\tau_2\nabla\zeta_n^{hk},\boldsymbol{e}_n^{hk}+\frac{\tau_1}{2}\boldsymbol{w}_n^{hk}\right)_H.
\end{aligned}
\tag{19}
$$

Using the relations:

$$
\begin{aligned}
(\nabla\theta_n^{hk},\nabla\xi_n^{hk})_H&=\frac{1}{k}\Big[(\nabla\theta_n^{hk},\nabla\zeta_n^{hk})_H-(\nabla\theta_{n-1}^{hk},\nabla\zeta_{n-1}^{hk})_H\Big]\\
&\quad-\|\nabla\zeta_n^{hk}\|_H^2+\frac{1}{k}(\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk},\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk})_H,\\
(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{w}_n^{hk})_Q&=\frac{1}{k}\Big[(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{e}_n^{hk})_Q-(\nabla\boldsymbol{T}_{n-1}^{hk},\nabla\boldsymbol{e}_{n-1}^{hk})_Q\Big]\\
&\quad-\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2+\frac{1}{k}(\nabla\boldsymbol{T}_n^{hk}-\nabla\boldsymbol{T}_{n-1}^{hk},\nabla\boldsymbol{e}_n^{hk}-\nabla\boldsymbol{e}_{n-1}^{hk})_Q,\\
(\operatorname{div}\boldsymbol{T}_n^{hk},\operatorname{div}\boldsymbol{w}_n^{hk})_Y&=\frac{1}{k}\Big[(\operatorname{div}\boldsymbol{T}_n^{hk},\operatorname{div}\boldsymbol{e}_n^{hk})_Y-(\operatorname{div}\boldsymbol{T}_{n-1}^{hk},\operatorname{div}\boldsymbol{e}_{n-1}^{hk})_Y\Big]\\
&\quad-\|\operatorname{div}\boldsymbol{e}_n^{hk}\|_Y^2+\frac{1}{k}(\operatorname{div}\boldsymbol{T}_n^{hk}-\operatorname{div}\boldsymbol{T}_{n-1}^{hk},\operatorname{div}\boldsymbol{e}_n^{hk}-\operatorname{div}\boldsymbol{e}_{n-1}^{hk})_Y,\\
(\boldsymbol{T}_n^{hk},\boldsymbol{w}_n^{hk})_H&=\frac{1}{k}\Big[(\boldsymbol{T}_n^{hk},\boldsymbol{e}_n^{hk})_H-(\boldsymbol{T}_{n-1}^{hk},\boldsymbol{e}_{n-1}^{hk})_H\Big]\\
&\quad-\|\boldsymbol{e}_n^{hk}\|_H^2+\frac{1}{k}(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk},\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk})_H,
\end{aligned}
$$

noting that

$$\Big(\boldsymbol{T}_n^{hk}+\tau_2\boldsymbol{e}_n^{hk},\nabla\zeta_n^{hk}+\frac{\tau_1}{2}\nabla\xi_n^{hk}\Big)_H+\Big(\nabla\theta_n^{hk}+\tau_2\nabla\zeta_n^{hk},\boldsymbol{e}_n^{hk}+\frac{\tau_1}{2}\boldsymbol{w}_n^{hk}\Big)_H$$
$$=(\boldsymbol{T}_n^{hk},\nabla\zeta_n^{hk})_H+\frac{\tau_1}{2}(\boldsymbol{T}_n^{hk},\nabla\xi_n^{hk})_H+2\tau_2(\boldsymbol{e}_n^{hk},\nabla\zeta_n^{hk})_H$$
$$+\tau_2\frac{\tau_1}{2}(\boldsymbol{e}_n^{hk},\nabla\xi_n^{hk})_H+(\nabla\theta_n^{hk},\boldsymbol{e}_n^{hk})_H+\frac{\tau_1}{2}(\nabla\theta_n^{hk},\boldsymbol{w}_n^{hk})_H$$
$$+\tau_2\frac{\tau_1}{2}(\nabla\zeta_n^{hk},\boldsymbol{w}_n^{hk})_H$$
$$=\frac{1}{k}\Big[(\boldsymbol{T}_n^{hk},\nabla\theta_n^{hk})_H-(\boldsymbol{T}_{n-1}^{hk},\nabla\theta_{n-1}^{hk})_H+(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk},\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk})_H\Big]$$
$$+\tau_2\frac{\tau_1}{2k}\Big[(\boldsymbol{e}_n^{hk},\nabla\zeta_n^{hk})_H-(\boldsymbol{e}_{n-1}^{hk},\nabla\zeta_{n-1}^{hk})_H\Big]$$
$$+\tau_2\frac{\tau_1}{2k}\Big[(\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk},\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk})_H\Big]$$
$$+\frac{\tau_1}{2k}\Big[(\boldsymbol{T}_n^{hk},\nabla\zeta_n^{hk})_H-(\boldsymbol{T}_{n-1}^{hk},\nabla\zeta_{n-1}^{hk})_H\Big]$$
$$+\frac{\tau_1}{2k}\Big[(\nabla\theta_n^{hk},\boldsymbol{e}_n^{hk})_H-(\nabla\theta_{n-1}^{hk},\boldsymbol{e}_{n-1}^{hk})_H\Big]-2\Big(\tau_2-\frac{\tau_1}{2}\Big)(\boldsymbol{e}_n^{hk},\nabla\zeta_n^{hk})_H$$
$$+\frac{\tau_1}{2k}\Big[(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk},\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk})_H+(\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk},\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk})_H\Big],$$

and taking into account the estimates:

$$\kappa_3\kappa\frac{\tau_1}{2k}(\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk},\nabla\zeta_n^{hk}-\nabla\zeta_n^{hk})_H$$
$$\le\kappa_3\kappa\Big[\frac{\gamma_1}{4k}\left\|\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk}\right\|_H^2+\frac{\tau_1^2}{4k\gamma_1}\left\|\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk}\right\|_H^2\Big],$$
$$\kappa_1\kappa_6\frac{\tau_1}{2k}(\nabla\boldsymbol{T}_n^{hk}-\nabla\boldsymbol{T}_{n-1}^{hk},\nabla\boldsymbol{e}_n^{hk}-\nabla\boldsymbol{e}_{n-1}^{hk})_Q$$
$$\le\kappa_1\kappa_6\Big[\frac{1}{2k}\|\nabla\boldsymbol{T}_n^{hk}-\nabla\boldsymbol{T}_{n-1}^{hk}\|_Q^2+\frac{\tau_1^2}{8k}\|\nabla\boldsymbol{e}_n^{hk}-\nabla\boldsymbol{e}_{n-1}^{hk}\|_Q^2\Big],$$
$$\kappa_1(\kappa_4+\kappa_5)\frac{\tau_1}{2k}(\operatorname{div}\boldsymbol{T}_n^{hk}-\operatorname{div}\boldsymbol{T}_{n-1}^{hk},\operatorname{div}\boldsymbol{e}_n^{hk}-\operatorname{div}\boldsymbol{e}_{n-1}^{hk})_Y$$
$$\le\kappa_1(\kappa_4+\kappa_5)\Big[\frac{1}{2k}\|\operatorname{div}\boldsymbol{T}_n^{hk}-\operatorname{div}\boldsymbol{T}_{n-1}^{hk}\|_Y^2+\frac{\tau_1^2}{8k}\|\operatorname{div}\boldsymbol{e}_n^{hk}-\operatorname{div}\boldsymbol{e}_{n-1}^{hk}\|_Q^2\Big],$$
$$\kappa_1\kappa_2\frac{\tau_1}{2k}(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk},\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk})_H$$
$$\le\kappa_1\kappa_2\Big[\frac{\gamma_1}{4k}\|\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk}\|_H^2+\frac{\tau_1^2}{4k\gamma_1}\|\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big],$$
$$\kappa_1\kappa_3\frac{1}{k}(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk},\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk})_H$$
$$\le\kappa_1\kappa_3\Big[\frac{\alpha_1}{2k}\|\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk}\|_H^2+\frac{1}{2k\alpha_1}\|\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk}\|_H^2\Big],$$
$$\kappa_1\kappa_3\frac{\tau_1}{2k}(\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk},\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk})_H$$
$$\le\kappa_1\kappa_3\Big[\frac{\alpha_2}{4k}\|\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk}\|_H^2+\frac{\tau_1^2}{4\alpha_2 k}\|\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk}\|_H^2\Big],$$
$$\kappa_1\kappa_3\frac{\tau_1}{2k}(\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk},\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk})_H$$
$$\le\kappa_1\kappa_3\Big[\frac{1}{4k\alpha_3}\|\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk}\|_H^2+\frac{\tau_1^2\alpha_3}{4k}\|\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big],$$
$$\kappa_1\kappa_3\tau_2\frac{\tau_1}{2k}(\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk},\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk})_H$$
$$\le\kappa_1\kappa_3\tau_2\tau_1\Big[\frac{1}{4k\alpha_4}\|\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk}\|_H^2+\frac{\alpha_4}{4k}\|\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk}\|_H^2\Big],$$

recalling that $\tau_2>\tau_1/2$, from (18)-(19) we find that

$$
\begin{aligned}
&\kappa_3 a\frac{\tau_1}{2k}\left[\frac{1}{2}\Big(\|\zeta_n^{hk}\|_Y^2-\|\zeta_{n-1}^{hk}\|_Y^2\Big)+\left\|\zeta_n^{hk}+\frac{\tau_1}{2}\xi_n^{hk}\right\|_Y^2-\left\|\zeta_{n-1}^{hk}+\frac{\tau_1}{2}\xi_{n-1}^{hk}\right\|_Y^2\right]\\
&+\kappa_3\kappa\frac{1}{2k}\left[\|\nabla\theta_n^{hk}\|_H^2-\|\nabla\theta_{n-1}^{hk}\|_H^2+\tau_1\Big((\nabla\theta_n^{hk},\nabla\zeta_n^{hk})_H-(\nabla\theta_{n-1}^{hk},\nabla\zeta_{n-1}^{hk})_H\Big)\right]\\
&+\kappa_3\kappa\tau_2\frac{\tau_1}{4k}\Big[\|\nabla\zeta_n^{hk}\|_H^2-\|\nabla\zeta_{n-1}^{hk}\|_H^2\Big]\\
&+\kappa_1 b\frac{\tau_1}{2k}\left[\frac{1}{2}\Big(\|\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big)+\left\|\boldsymbol{e}_n^{hk}+\frac{\tau_1}{2}\boldsymbol{w}_n^{hk}\right\|_H^2-\left\|\boldsymbol{e}_{n-1}^{hk}+\frac{\tau_1}{2}\boldsymbol{w}_{n-1}^{hk}\right\|_H^2\right]\\
&+\kappa_1\kappa_6\frac{1}{2k}\left[\|\nabla\boldsymbol{T}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{T}_{n-1}^{hk}\|_Q^2+\tau_1\Big((\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{e}_n^{hk})_Q-(\nabla\boldsymbol{T}_{n-1}^{hk},\nabla\boldsymbol{e}_{n-1}^{hk})_Q\Big)\right]\\
&+\kappa_1\kappa_6\tau_2\frac{\tau_1}{4k}\Big[\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{e}_{n-1}^{hk}\|_Q^2\Big]\\
&+\kappa_1(\kappa_4+\kappa_5)\frac{1}{2k}\left[\|\mathrm{div}\,\boldsymbol{T}_n^{hk}\|_Y^2-\|\mathrm{div}\,\boldsymbol{T}_{n-1}^{hk}\|_Y^2\Big)\right]\\
&+\kappa_1(\kappa_4+\kappa_5)\frac{\tau_1}{2k}\Big[\Big(\mathrm{div}\,\boldsymbol{T}_n^{hk},\mathrm{div}\,\boldsymbol{e}_n^{hk}\Big)_Y-\Big(\mathrm{div}\,\boldsymbol{T}_{n-1}^{hk},\mathrm{div}\,\boldsymbol{e}_{n-1}^{hk}\Big)_Y\Big]\\
&+\kappa_1(\kappa_4+\kappa_5)\tau_2\frac{\tau_1}{4k}\Big[\|\mathrm{div}\,\boldsymbol{e}_n^{hk}\|_Y^2-\|\mathrm{div}\,\boldsymbol{e}_{n-1}^{hk}\|_Y^2\Big]\\
&+\kappa_1\kappa_2\frac{1}{2k}\Big[\|\boldsymbol{T}_n^{hk}\|_H^2-\|\boldsymbol{T}_{n-1}^{hk}\|_H^2+\Big((\boldsymbol{T}_n^{hk},\boldsymbol{e}_n^{hk})_H-(\boldsymbol{T}_{n-1}^{hk},\boldsymbol{e}_{n_1}^{hk})_H\Big)\Big]\\
&+\kappa_1\kappa_2\tau_2\frac{\tau_1}{4k}\Big[\|\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big]\\
&+\kappa_1\kappa_3\frac{1}{k}\Big[(\boldsymbol{T}_n^{hk},\nabla\theta_n^{hk})_H-(\boldsymbol{T}_{n-1}^{hk},\nabla\theta_{n-1}^{hk})_H\Big]\\
&+\kappa_1\kappa_3\frac{\tau_1}{2k}\Big[(\boldsymbol{T}_n^{hk},\nabla\zeta_n^{hk})_H-(\boldsymbol{T}_{n-1}^{hk},\nabla\zeta_{n-1}^{hk})_H+(\nabla\theta_n^{hk},\boldsymbol{e}_n^{hk})_H-(\nabla\theta_{n-1}^{hk},\boldsymbol{e}_{n-1}^{hk})_H\Big]\\
&+\tau_2\frac{\tau_1}{2k}\Big[(\boldsymbol{e}_n^{hk},\nabla\zeta_n^{hk})_H-(\boldsymbol{e}_{n-1}^{hk},\nabla\zeta_{n-1}^{hk})_H\Big]\\
&+\kappa_3\frac{1}{2k}\Big(\kappa-\kappa\frac{\gamma_1}{2}-\kappa_1\frac{1}{\alpha_1}-\kappa_1\frac{1}{2\alpha_3}\Big)\|\nabla\theta_n^{hk}-\nabla\theta_{n-1}^{hk}\|_H^2\\
&+\kappa_3\frac{\tau_1}{4k}\Big(\kappa\tau_2-\kappa\frac{\tau_1}{\gamma_1}-\kappa_1\frac{\tau_1}{\alpha_2}-\kappa_1\tau_2\alpha_4\Big)\|\nabla\zeta_n^{hk}-\nabla\zeta_{n-1}^{hk}\|_H^2\\
&+\kappa_1\frac{1}{2k}\Big(\kappa_2-\kappa_2\frac{\gamma_1}{2}-\kappa_3\alpha_1-\kappa_3\frac{\alpha_2}{2}\Big)\|\boldsymbol{T}_n^{hk}-\boldsymbol{T}_{n-1}^{hk}\|_H^2\\
&+\kappa_1\frac{\tau_1}{4k}\Big(\kappa_2\tau_2-\kappa_2\frac{\tau_1}{\gamma_1}-\kappa_3\alpha_3\tau_1-\kappa_3\frac{\tau_2}{\alpha_4}\Big)\|\boldsymbol{e}_n^{hk}-\boldsymbol{e}_{n-1}^{hk}\|_H^2+\kappa_2 a\|\zeta_n^{hk}\|_Y^2+\kappa_1 b\|\boldsymbol{e}_n^{hk}\|_H^2\\
&+\kappa_3\kappa\left(\tau_2-\frac{\tau_1}{2}\right)\|\nabla\zeta_n^{hk}\|_H^2+2\kappa_1\kappa_3\left(\tau_2-\frac{\tau_1}{2}\right)(\boldsymbol{e}_n^{hk},\nabla\zeta_n^{hk})_H\\
&+\kappa_1\kappa_2\left(\tau_2-\frac{\tau_1}{2}\right)\|\boldsymbol{e}_n^{hk}\|_H^2\\
&+\kappa_1\kappa_6\left(\tau_2-\frac{\tau_1}{2}\right)\|\nabla\boldsymbol{e}_n^{hk}\|_Q^2+\kappa_1(\kappa_4+\kappa_5)\left(\tau_2-\frac{\tau_1}{2}\right)\|\mathrm{div}\,\boldsymbol{e}_n^{hk}\|_Y^2\le 0.
\end{aligned}
$$

Now, the result follows if

$$
\begin{aligned}
&\kappa\left(1-\frac{\gamma_1}{2}\right)-\kappa_1\left(\frac{1}{\alpha_1}+\frac{1}{2\alpha_3}\right)\ge 0,\\
&\kappa\left(\tau_2-\frac{\tau_1}{\gamma_1}\right)-\kappa_1\left(\frac{\tau_1}{\alpha_2}+\tau_2\alpha_4\right)\ge 0,\\
&\kappa_2\left(1-\frac{\gamma_1}{2}\right)-\kappa_3\left(\alpha_1+\frac{\alpha_2}{2}\right)\ge 0,\\
&\kappa_2\left(\tau_2-\frac{\tau_1}{\gamma_1}\right)-\kappa_3\left(\alpha_3\tau_1+\frac{\tau_2}{\alpha_4}\right)\ge 0.\quad\square
\end{aligned}
$$

Next, we obtain the following a priori error estimates result.

Theorem 3.3. *Under the assumptions of Lemma 3.1, if we denote by $(\xi, \boldsymbol{w})$ the solution to problem VP and by $(\xi^{hk}, \boldsymbol{w}^{hk})$ the solution to problem VP^{hk}, then we have the following a priori error estimates, for all $r^h = \{r_n^h\}_{j=0}^N \subset E^h$ and $\boldsymbol{z}^h = \{\boldsymbol{z}_n^h\}_{j=0}^N \subset V^h$,*

$$\begin{aligned}
&\max_{0\le n\le N} \Big\{\|\xi_n - \xi_n^{hk}\|_Y^2 + \|\zeta_n - \zeta_n^{hk}\|_E^2 + \|\theta_n - \theta_n^{hk}\|_E^2 + \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 \\
&\quad + \|\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}\|_V^2 + \|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_V^2\Big\} \\
&\quad \le Ck \sum_{j=1}^N \Big(\|\dot{\xi}_j - \delta\xi_j\|_Y^2 + \|\dot{\zeta}_j - \delta\zeta_j\|_E^2 + \|\xi_j - r_j^h\|_E^2 + \|\dot{\boldsymbol{w}}_j - \delta\boldsymbol{w}_j\|_H^2 + \|\dot{\boldsymbol{e}}_j - \delta\boldsymbol{e}_j\|_V^2 \\
&\quad + \|\boldsymbol{w}_j - \boldsymbol{z}_j^h\|_V^2 + \|\dot{\theta}_j - \delta\theta_j\|_E^2 + \|\dot{\boldsymbol{T}}_j - \delta\boldsymbol{T}_j\|_V^2 + I_j^2 + J_j^2\Big) + C \max_{0\le n\le N} \|\xi_n - r_n^h\|_Y^2 \\
&\quad + \frac{C}{k}\sum_{j=1}^{N-1}\Big(\|\xi_j - r_j^h - (\xi_{j+1} - r_{j+1}^h)\|_Y^2 + \|\boldsymbol{w}_j - \boldsymbol{z}_j^h - (\boldsymbol{w}_{j+1} - \boldsymbol{z}_{j+1}^h)\|_Y^2\Big) \\
&\quad + C \max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{z}_n^h\|_H^2 + C\Big(\|\xi^0 - \xi^{0h}\|_Y^2 + \|\zeta^0 - \zeta^{0h}\|_E^2 + \|\boldsymbol{w}^0 - \boldsymbol{w}^{0h}\|_H^2 \\
&\quad + \|\boldsymbol{e}^0 - \boldsymbol{e}^{0h}\|_V^2 + \|\theta^0 - \theta^{0h}\|_E^2 + \|\boldsymbol{T}^0 - \boldsymbol{T}^{0h}\|_V^2\Big),
\end{aligned}$$

where C is again a positive constant which is independent of the discretization parameters h and k, and the integration errors I_n and J_n are given by

$$I_n = \Big\|\int_0^{t_n} \zeta(s)\,ds - k\sum_{j=1}^n \zeta_j\Big\|_E, \quad J_n = \Big\|\int_0^{t_n} \boldsymbol{e}(s)\,ds - k\sum_{j=1}^n \boldsymbol{e}_j\Big\|_V. \tag{20}$$

Proof. First, we obtain the error estimates on the temperature acceleration. Then, we subtract variational equation (4) at time $t = t_n$ for a test function $r = r^h \in E^h \subset E$ and discrete variational equation (13) to obtain, for all $r^h \in E^h$,

$$\begin{aligned}
&a\left(\frac{\tau_1^2}{2}(\dot{\xi}_n - \delta\xi_n^{hk}) + \tau_1(\xi_n - \xi_n^{hk}) + \zeta_n - \zeta_n^{hk}, r^h\right)_Y + \kappa\left(\nabla(\theta_n - \theta_n^{hk} + \tau_2(\zeta_n - \zeta_n^{hk})), \nabla r^h\right)_H \\
&\quad -\kappa_1\left(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}) + \tau_2 \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), r^h\right)_Y = 0,
\end{aligned}$$

and so, we have, for all $r^h \in E^h$,

$$\begin{aligned}
&a\left(\frac{\tau_1^2}{2}(\dot{\xi}_n - \delta\xi_n^{hk}) + \tau_1(\xi_n - \xi_n^{hk}) + \zeta_n - \zeta_n^{hk}, \xi_n - \xi_n^{hk}\right)_Y \\
&\quad +\kappa\left(\nabla(\theta_n - \theta_n^{hk} + \tau_2(\zeta_n - \zeta_n^{hk})), \nabla(\xi_n - \xi_n^{hk})\right)_H \\
&\quad -\kappa_1\left(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}) + \tau_2 \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \xi_n - \xi_n^{hk}\right)_Y \\
&= a\left(\frac{\tau_1^2}{2}(\dot{\xi}_n - \delta\xi_n^{hk}) + \tau_1(\xi_n - \xi_n^{hk}) + \zeta_n - \zeta_n^{hk}, \xi_n - r^h\right)_Y \\
&\quad +\kappa\left(\nabla(\theta_n - \theta_n^{hk} + \tau_2(\zeta_n - \zeta_n^{hk})), \nabla(\xi_n - r^h)\right)_H \\
&\quad -\kappa_1\left(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}) + \tau_2 \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \xi_n - r^h\right)_Y.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&\left(\delta\xi_n - \delta\xi_n^{hk}, \xi_n - \xi_n^{hk}\right)_Y \ge \frac{1}{2k}\Big\{\|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2\Big\}, \\
&\left(\zeta_n - \zeta_n^{hk}, \xi_n - \xi_n^{hk}\right)_Y \ge \left(\zeta_n - \zeta_n^{hk}, \dot{\xi}_n - \delta\zeta_n\right)_Y + \frac{1}{2k}\Big\{\|\zeta_n - \zeta_n^{hk}\|_Y^2 - \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2\Big\}, \\
&\left(\nabla(\zeta_n - \zeta_n^{hk}), \nabla(\xi_n - \xi_n^{hk})\right)_H \ge \left(\nabla(\zeta_n - \zeta_n^{hk}), \nabla(\dot{\zeta}_n - \delta\zeta_n)\right)_H \\
&\quad + \frac{1}{2k}\Big\{\|\nabla(\zeta_n - \zeta_n^{hk})\|_H^2 - \|\nabla(\zeta_{n-1} - \zeta_{n-1}^{hk})\|_H^2\Big\},
\end{aligned}$$

it follows that, for all $r^h \in E^h$,

$$
\begin{aligned}
&\frac{1}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{1}{2k}\left\{\|\zeta_n-\zeta_n^{hk}\|_Y^2-\|\zeta_{n-1}-\zeta_{n-1}^{hk}\|_Y^2\right\}\\
&\quad+\frac{1}{2k}\left\{\|\nabla(\zeta_n-\zeta_n^{hk})\|_H^2-\|\nabla(\zeta_{n-1}-\zeta_{n-1}^{hk})\|_H^2\right\}+\left(\nabla(\theta_n-\theta_n^{hk}),\nabla(\xi_n-\xi_n^{hk})\right)_H\\
&\le C\Big(\|\dot\xi_n-\delta\xi_n\|_Y^2+\|\dot\zeta_n-\delta\zeta_n\|_Y^2+\|\nabla(\dot\zeta_n-\delta\zeta_n)\|_H^2+\|\xi_n-\xi_n^{hk}\|_Y^2\\
&\quad+\|\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Y^2+\|\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2+\|\xi_n-r^h\|_E^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2\\
&\quad+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\nabla(\zeta_n-\zeta_n^{hk})\|_H^2+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-r^h)\Big). \qquad (21)
\end{aligned}
$$

Now, we obtain the estimates for the microtemperatures acceleration. Subtracting variational equation (5) at time $t=t_n$ for a test function $\boldsymbol{z}=\boldsymbol{z}^h\in V^h\subset V$ and discrete variational equation (14) we have, for all $\boldsymbol{z}^h\in V^h$,

$$
\begin{aligned}
&b\left(\frac{\tau_1^2}{2}\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n^{hk}+\tau_1(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})+\boldsymbol{e}_n-\boldsymbol{e}_n^{hk},\boldsymbol{z}^h\right)_H\\
&\quad+\kappa_6\left(\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})+\tau_2\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\nabla\boldsymbol{z}^h\right)_Q+\kappa_2(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}+\tau_2(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\boldsymbol{z}^h)_H\\
&\quad+(\kappa_4+\kappa_5)\left(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})+\tau_2\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\mathrm{div}\,\boldsymbol{z}^h\right)_Y\\
&\quad+\kappa_3(\nabla(\theta_n-\theta_n^{hk}+\tau_2(\zeta_n-\zeta_n^{hk})),\boldsymbol{z}^h)_H=0.
\end{aligned}
$$

Therefore, we find that, for all $\boldsymbol{z}^h\in V^h$,

$$
\begin{aligned}
&b\left(\frac{\tau_1^2}{2}\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n^{hk}+\tau_1(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})+\boldsymbol{e}_n-\boldsymbol{e}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\right)_H\\
&\quad+\kappa_6\left(\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})+\tau_2\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\right)_Q\\
&\quad+\kappa_2(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}+\tau_2(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})_H\\
&\quad+(\kappa_4+\kappa_5)\left(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})+\tau_2\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\right)_Y\\
&\quad+\kappa_3(\nabla(\theta_n-\theta_n^{hk}+\tau_2(\zeta_n-\zeta_n^{hk})),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})_H\\
&=b\left(\frac{\tau_1^2}{2}\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n^{hk}+\tau_1(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})+\boldsymbol{e}_n-\boldsymbol{e}_n^{hk},\boldsymbol{w}_n-\boldsymbol{z}^h\right)_H\\
&\quad+\kappa_6\left(\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})+\tau_2\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{z}^h)\right)_Q\\
&\quad+\kappa_2(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}+\tau_2(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\boldsymbol{w}_n-\boldsymbol{z}^h)_H\\
&\quad+(\kappa_4+\kappa_5)\left(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})+\tau_2\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{z}^h)\right)_Y\\
&\quad+\kappa_3(\nabla(\theta_n-\theta_n^{hk}+\tau_2(\zeta_n-\zeta_n^{hk})),\boldsymbol{w}_n-\boldsymbol{z}^h)_H.
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&\left(\delta\boldsymbol{w}_n-\delta\boldsymbol{w}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\right)_H\ge\frac{1}{2k}\left\{\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2-\|\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}\|_H^2\right\},\\
&\left(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\right)_H\ge\left(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk},\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n\right)_H+\frac{1}{2k}\left\{\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk}\|_H^2\right\},\\
&\left(\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\right)_Q\ge\left(\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\nabla(\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n)\right)_Q\\
&\qquad+\frac{1}{2k}\left\{\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk})\|_Q^2\right\},\\
&\left(\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\right)_Y\ge\left(\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\mathrm{div}\,(\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n)\right)_Y\\
&\qquad+\frac{1}{2k}\left\{\|\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2-\|\mathrm{div}\,(\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk})\|_Y^2\right\},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\frac{1}{2k}\left\{\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2-\|\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}\|_H^2\right\}+\left(\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\right)_Q\\
&\quad+\frac{1}{2k}\left\{\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk}\|_H^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2k}\Big\{\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk})\|_Q^2\Big\}\\
&+\frac{1}{2k}\Big\{\|\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2-\|\mathrm{div}\,(\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk})\|_Y^2\Big\}\\
&+\Big(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\Big)_Y\\
\le{}& C\Big(\|\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n\|_H^2+\|\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n\|_V^2+\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2+\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2\\
&+\|\boldsymbol{w}_n-\boldsymbol{z}^h\|_V^2+\|\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Q^2+\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2+\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_H^2\\
&+\|\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Y^2+\|\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2\\
&+\|\nabla(\zeta_n-\zeta_n^{hk})\|_H^2+(\delta\boldsymbol{w}_n-\delta\boldsymbol{w}_n^{hk},\boldsymbol{w}_n-\boldsymbol{z}^h)_H\Big).
\end{aligned}
\tag{22}
$$

Combining estimates (21) and (22) we find that

$$
\begin{aligned}
&\frac{1}{2k}\Big\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\Big\}+\frac{1}{2k}\Big\{\|\zeta_n-\zeta_n^{hk}\|_Y^2-\|\zeta_{n-1}-\zeta_{n-1}^{hk}\|_Y^2\Big\}\\
&+\frac{1}{2k}\Big\{\|\nabla(\zeta_n-\zeta_n^{hk})\|_H^2-\|\nabla(\zeta_{n-1}-\zeta_{n-1}^{hk})\|_H^2\Big\}+\Big(\nabla(\theta_n-\theta_n^{hk}),\nabla(\xi_n-\xi_n^{hk})\Big)_H\\
&+\frac{1}{2k}\Big\{\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2-\|\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}\|_H^2\Big\}+\frac{1}{2k}\Big\{\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big\}\\
&+\frac{1}{2k}\Big\{\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk})\|_Q^2\Big\}+\Big(\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\Big)_Q\\
&+\frac{1}{2k}\Big\{\|\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2-\|\mathrm{div}\,(\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk})\|_Y^2\Big\}+\Big(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\Big)_Y\\
\le{}& C\Big(\|\dot{\xi}_n-\delta\xi_n\|_Y^2+\|\dot{\zeta}_n-\delta\zeta_n\|_Y^2+\|\nabla(\dot{\zeta}_n-\delta\zeta_n)\|_H^2+\|\xi_n-\xi_n^{hk}\|_Y^2\\
&+\|\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Y^2+\|\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2+\|\xi_n-r^h\|_E^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2\\
&+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\nabla(\zeta_n-\zeta_n^{hk})\|_H^2+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-r^h)+\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2\\
&+\|\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n\|_H^2+\|\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n\|_V^2+\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2+\|\boldsymbol{w}_n-\boldsymbol{z}^h\|_V^2\\
&+\|\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Q^2+\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2+\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_H^2\\
&+(\delta\boldsymbol{w}_n-\delta\boldsymbol{w}_n^{hk},\boldsymbol{w}_n-\boldsymbol{z}^h)_H\Big).
\end{aligned}
$$

Multiplying the above estimates by k and summing up to n, we obtain

$$
\begin{aligned}
&\|\xi_n-\xi_n^{hk}\|_Y^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2+\|\nabla(\zeta_n-\zeta_n^{hk})\|_H^2+\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2+\|\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2\\
&+k\sum_{j=1}^n\Big(\nabla(\theta_j-\theta_j^{hk}),\nabla(\xi_j-\xi_j^{hk})\Big)_H+\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2+\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2\\
&+k\sum_{j=1}^n\Big[\Big(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\nabla(\boldsymbol{w}_j-\boldsymbol{w}_j^{hk})\Big)_Q+\Big(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\mathrm{div}\,(\boldsymbol{w}_j-\boldsymbol{w}_j^{hk})\Big)_Y\Big]\\
\le{}& Ck\sum_{j=1}^n\Big(\|\dot{\xi}_j-\delta\xi_j\|_Y^2+\|\dot{\zeta}_j-\delta\zeta_j\|_Y^2+\|\nabla(\dot{\zeta}_j-\delta\zeta_j)\|_H^2+\|\xi_j-\xi_j^{hk}\|_Y^2\\
&+\|\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk})\|_Y^2+\|\mathrm{div}\,(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk})\|_Y^2+\|\xi_j-r_j^h\|_E^2+\|\zeta_j-\zeta_j^{hk}\|_Y^2\\
&+\|\nabla(\theta_j-\theta_j^{hk})\|_H^2+\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-r_j^h)_Y+\|\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}\|_H^2\\
&+\|\dot{\boldsymbol{w}}_j-\delta\boldsymbol{w}_j\|_H^2+\|\dot{\boldsymbol{e}}_j-\delta\boldsymbol{e}_j\|_V^2+\|\boldsymbol{w}_j-\boldsymbol{w}_j^{hk}\|_H^2+\|\boldsymbol{w}_j-\boldsymbol{z}_j^h\|_V^2\\
&+\|\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk})\|_Q^2+\|\nabla(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk})\|_Q^2+\|\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}\|_H^2\\
&+(\delta\boldsymbol{w}_j-\delta\boldsymbol{w}_j^{hk},\boldsymbol{w}_j-\boldsymbol{z}_j^h)_H\Big)+C\Big(\|\xi^0-\xi^{0h}\|_Y^2+\|\zeta^0-\zeta^{0h}\|_E^2+\|\boldsymbol{w}^0-\boldsymbol{w}^{0h}\|_H^2\\
&+\|\boldsymbol{e}^0-\boldsymbol{e}^{0h}\|_V^2+\|\theta^0-\theta^{0h}\|_E^2+\|\boldsymbol{T}^0-\boldsymbol{T}^{0h}\|_V^2\Big).
\end{aligned}
$$

Now, keeping in mind that

Table 1
Example 1: Numerical errors for some h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.806641	0.803351	0.801443	0.800822	0.800515	0.800331	0.800270
$1/2^4$	0.403253	0.399611	0.397524	0.396861	0.396539	0.396351	0.396289
$1/2^5$	0.204708	0.200785	0.198535	0.197815	0.197469	0.197271	0.197207
$1/2^6$	0.106477	0.102182	0.099804	0.099046	0.098677	0.098462	0.098394
$1/2^7$	0.058209	0.053222	0.050643	0.049852	0.049470	0.049246	0.049173
$1/2^8$	0.035334	0.029124	0.026158	0.025311	0.024914	0.024684	0.024609
$1/2^9$	0.025596	0.017693	0.014037	0.013078	0.012653	0.012415	0.012339
$1/2^{10}$	0.022185	0.012822	0.008184	0.007020	0.006539	0.006285	0.006207
$1/2^{11}$	0.021198	0.011113	0.005573	0.004093	0.003510	0.003226	0.003142
$1/2^{12}$	0.020939	0.010619	0.004591	0.002788	0.002047	0.001705	0.001613
$1/2^{13}$	0.020874	0.010489	0.004291	0.002297	0.001394	0.000961	0.000852

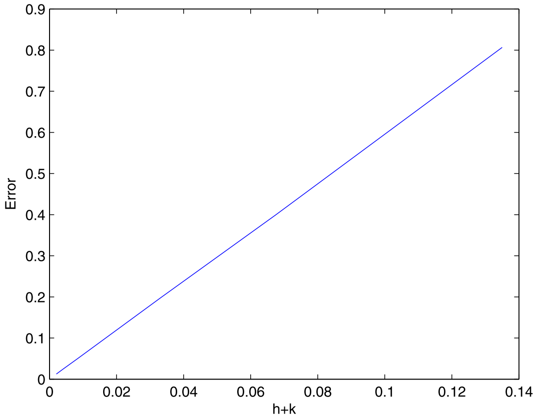

Fig. 1. Example 1: Asymptotic constant error.

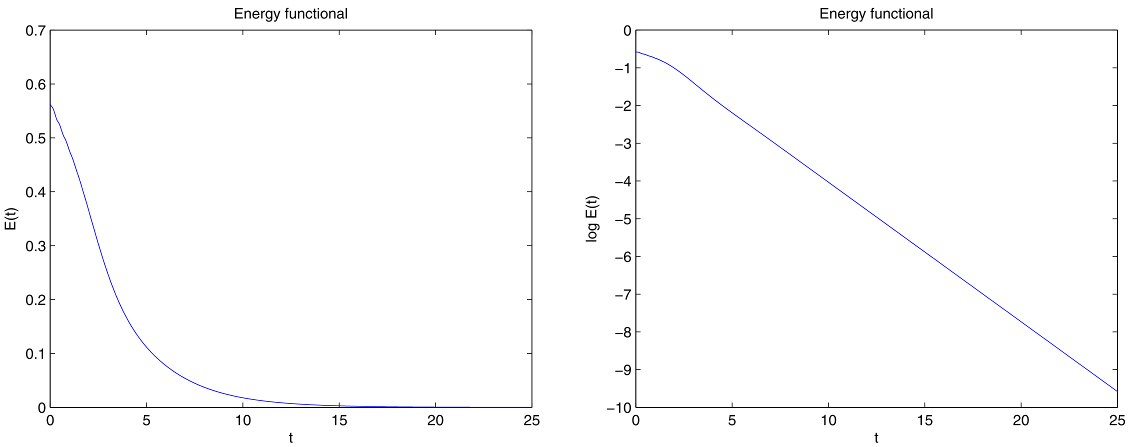

Fig. 2. Example 1: Evolution in time of the discrete energy (natural and semi-log scales).

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta\xi_j-\delta\xi_j^{hk},\xi_j-r_j^h)_Y&=\sum_{j=1}^{n}(\xi_j-\xi_j^{hk}-(\xi_{j-1}-\xi_{j-1}^{hk}),\xi_j-r_j^h)_Y\\
&=(\xi_n-\xi_n^{hk},\xi_n-r_n^h)_Y+(\xi^{0h}-\xi^0,\xi_1-r_1^h)_Y\\
&\quad+\sum_{j=1}^{n-1}(\xi_j-\xi_j^{hk},\xi_j-r_j^h-(\xi_{j+1}-r_{j+1}^h))_Y,
\end{aligned}
$$

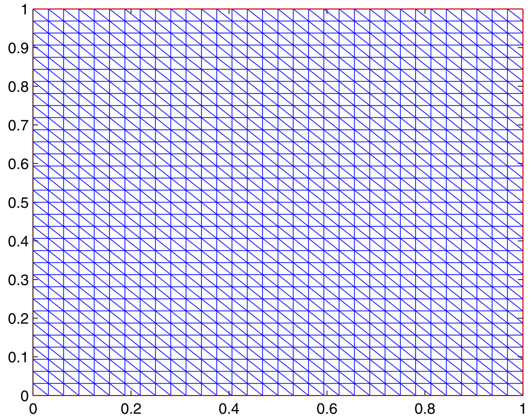

Fig. 3. Example 2: Finite element mesh.

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta \boldsymbol{w}_j-\delta \boldsymbol{w}_j^{hk},\boldsymbol{w}_j-\boldsymbol{z}_j^h)_H &= \sum_{j=1}^{n}(\boldsymbol{w}_j-\boldsymbol{w}_j^{hk}-(\boldsymbol{w}_{j-1}-\boldsymbol{w}_{j-1}^{hk}),\boldsymbol{w}_j-\boldsymbol{z}_j^h)_H\\
&= (\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{w}_n-\boldsymbol{z}_n^h)_H+(\boldsymbol{w}^{0h}-\boldsymbol{w}^0,\boldsymbol{w}_1-\boldsymbol{z}_1^h)_H\\
&\quad +\sum_{j=1}^{n-1}(\boldsymbol{w}_j-\boldsymbol{w}_j^{hk},\boldsymbol{w}_j-\boldsymbol{z}_j^h-(\boldsymbol{w}_{j+1}-\boldsymbol{z}_{j+1}^h))_H,
\end{aligned}
$$

$$
\begin{aligned}
k\sum_{j=1}^{n}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\nabla(\delta\boldsymbol{e}_j-\delta\boldsymbol{e}_j^{hk}))_Q &= \sum_{j=1}^{n}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\nabla(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}-(\boldsymbol{e}_{j-1}-\boldsymbol{e}_{j-1}^{hk}))_Q\\
&= (\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})_Q+(\nabla(\boldsymbol{T}_1^{hk}-\boldsymbol{T}_1),\nabla(\boldsymbol{e}^0-\boldsymbol{e}^{0h}))_Q\\
&\quad +\sum_{j=1}^{n-1}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}-(\boldsymbol{T}_{j+1}-\boldsymbol{T}_{j+1}^{hk})),\nabla(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}))_Q,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{n-1}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}-(\boldsymbol{T}_{j+1}-\boldsymbol{T}_{j+1}^{hk})),\nabla(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}))_Q\\
&\qquad \le C\Big(k\sum_{j=1}^{n}\|\nabla(\dot{\boldsymbol{T}}_j-\delta\boldsymbol{T}_j)\|_Q^2+k\sum_{j=1}^{n}\|\nabla(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk})\|_Q^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
k\sum_{j=1}^{n}(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\mathrm{div}\,(\delta\boldsymbol{e}_j-\delta\boldsymbol{e}_j^{hk}))_Y &= \sum_{j=1}^{n}(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\mathrm{div}\,(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}-(\boldsymbol{e}_{j-1}-\boldsymbol{e}_{j-1}^{hk}))_Y\\
&= (\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\mathrm{div}\,(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}))_Y+(\mathrm{div}\,(\boldsymbol{T}_1^{hk}-\boldsymbol{T}_1),\mathrm{div}\,(\boldsymbol{e}^0-\boldsymbol{e}^{0h}))_Y\\
&\quad +\sum_{j=1}^{n-1}(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}-(\boldsymbol{T}_{j+1}-\boldsymbol{T}_{j+1}^{hk})),\mathrm{div}\,(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}))_Y,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{n-1}(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}-(\boldsymbol{T}_{j+1}-\boldsymbol{T}_{j+1}^{hk})),\mathrm{div}\,(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}))_Y\\
&\qquad \le C\Big(k\sum_{j=1}^{n}\|\mathrm{div}\,(\dot{\boldsymbol{T}}_j-\delta\boldsymbol{T}_j)\|_Y^2+k\sum_{j=1}^{n}\|\mathrm{div}\,(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk})\|_Y^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
k\sum_{j=1}^{n}(\nabla(\theta_j-\theta_j^{hk}),\nabla(\delta\zeta_j-\delta\zeta_j^{hk}))_H &= \sum_{j=1}^{n}(\nabla(\theta_j-\theta_j^{hk}),\nabla(\zeta_j-\zeta_j^{hk}-(\zeta_{j-1}-\zeta_{j-1}^{hk}))_H\\
&= (\nabla(\theta_n-\theta_n^{hk}),\nabla(\zeta_n-\zeta_n^{hk}))_H+(\nabla(\theta_1^{hk}-\theta_1),\nabla(\zeta^0-\zeta^{0h}))_H
\end{aligned}
$$

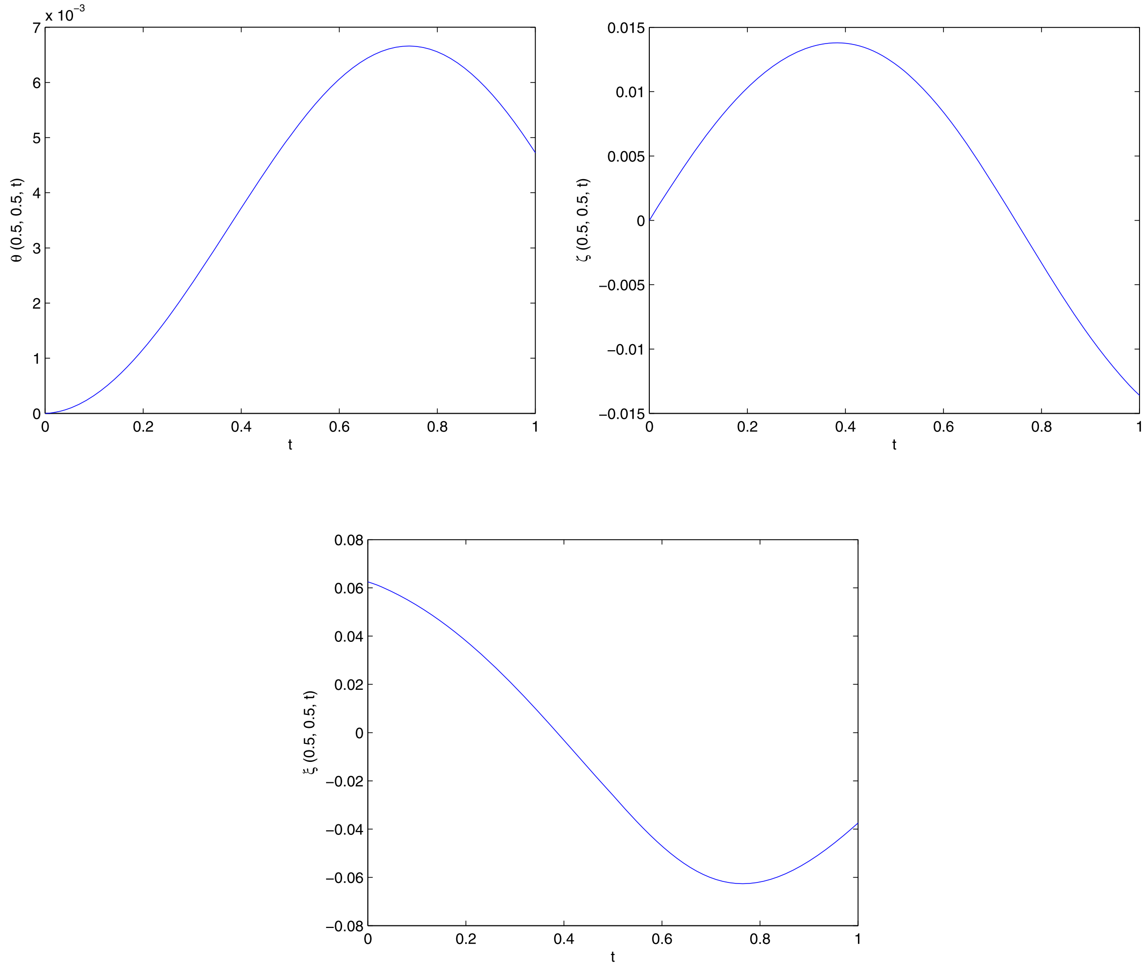

Fig. 4. Example 2: Evolution in time of the temperature, the temperature speed and the temperature acceleration at point $(0.5, 0.5)$.

$$+\sum_{j=1}^{n-1}(\nabla(\theta_j-\theta_j^{hk}-(\theta_{j+1}-\theta_{j+1}^{hk})),\nabla(\zeta_j-\zeta_j^{hk}))_H,$$

$$\sum_{j=1}^{n-1}(\nabla(\theta_j-\theta_j^{hk}-(\theta_{j+1}-\theta_{j+1}^{hk})),\nabla(\zeta_j-\zeta_j^{hk}))_H$$

$$\leq C\Big(k\sum_{j=1}^{n}\|\nabla(\dot{\theta}_j-\delta\theta_j)\|_H^2+k\sum_{j=1}^{n}\|\nabla(\zeta_j-\zeta_j^{hk})\|_H^2\Big),$$

$$\|\theta_n-\theta_n^{hk}\|_E^2\leq C(\|\theta^0-\theta^{0h}\|_E^2+I_n^2+\sum_{j=1}^{n}\|\zeta_n-\zeta_n^{hk}\|_E^2),$$

$$\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_V^2\leq C(\|\boldsymbol{T}^0-\boldsymbol{T}^{0h}\|_V^2+J_n^2+\sum_{j=1}^{n}\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_V^2),$$

where I_n and J_n are the integration errors defined in (20), applying a discrete version of Gronwall's inequality (see again [10]), we conclude the a priori error estimates. □

The error estimates shown in Theorem 3.3 can be used to obtain the convergence order of the approximations given by Problem VP^{hk}. As an example, if we assume suitable additional regularity conditions we have the following result which states the linear convergence of the algorithm.

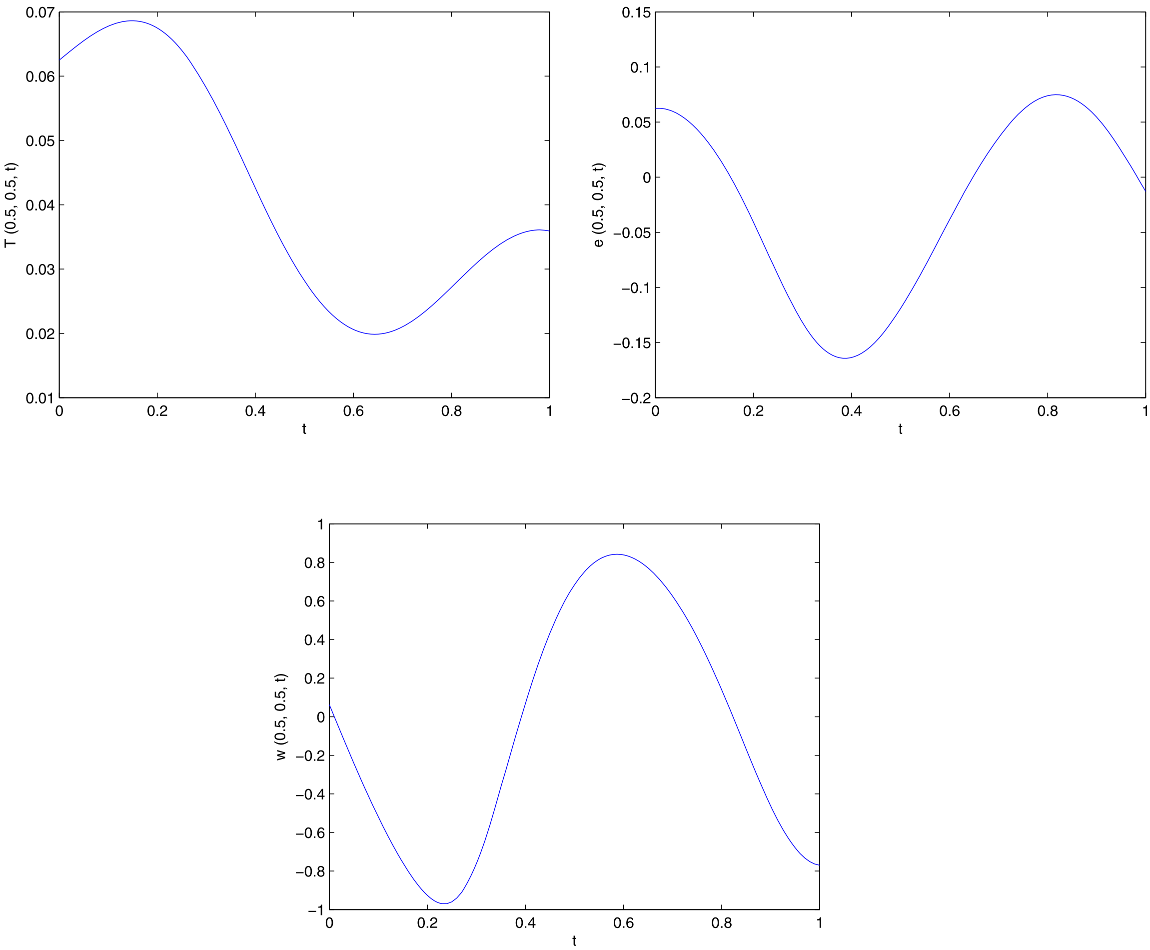

Fig. 5. Example 2: Evolution in time of the microtemperatures, the microtemperatures speed and the microtemperatures acceleration at point $(0.5, 0.5)$.

Corollary 3.4. *Let the assumptions of Theorem 3.3 hold. Therefore, if we assume the following additional regularity:*

$$\theta \in H^3(0,T;Y) \cap W^{2,\infty}(0,T;H^2(\Omega)) \cap H^2(0,T;H^1(\Omega)),$$
$$\boldsymbol{T} \in H^3(0,T;H) \cap W^{2,\infty}(0,T;[H^2(\Omega)]^d) \cap H^2(0,T;[H^1(\Omega)]^d),$$

it follows that the approximations obtained by Problem VP^{hk} are linearly convergent; that is, there exists a positive constant C, independent of the discretization parameters h and k, such that

$$\max_{0\le n\le N} \Big\{ \|\xi_n - \xi_n^{hk}\|_Y + \|\zeta_n - \zeta_n^{hk}\|_E + \|\theta_n - \theta_n^{hk}\|_E + \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H$$
$$+ \|\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}\|_V + \|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_V \Big\} \le C(h+k).$$

4. Numerical results

In this section, we describe the numerical scheme implemented in MATLAB for solving Problem VP, and we show some numerical examples to demonstrate the accuracy of the approximation and the behavior of the solution.

We solve the following linear problem, for all $r^h \in E^h$ and $\boldsymbol{z}^h \in V^h$.

$$a\left(\frac{\tau_1^2}{2}\xi_n^{hk} + \tau_1 k \xi_n^{hk} + k^2 \xi_n^{hk}, r^h\right)_Y + \kappa k \left(\nabla(k^2\xi_n^{hk} + \tau_2 k \xi_n^{hk}), \nabla r^h\right)_H$$
$$-\kappa_1 k \left((k^2 + \tau_2 k)\mathrm{div}\, \boldsymbol{w}_n^{hk}, r^h\right)_Y$$

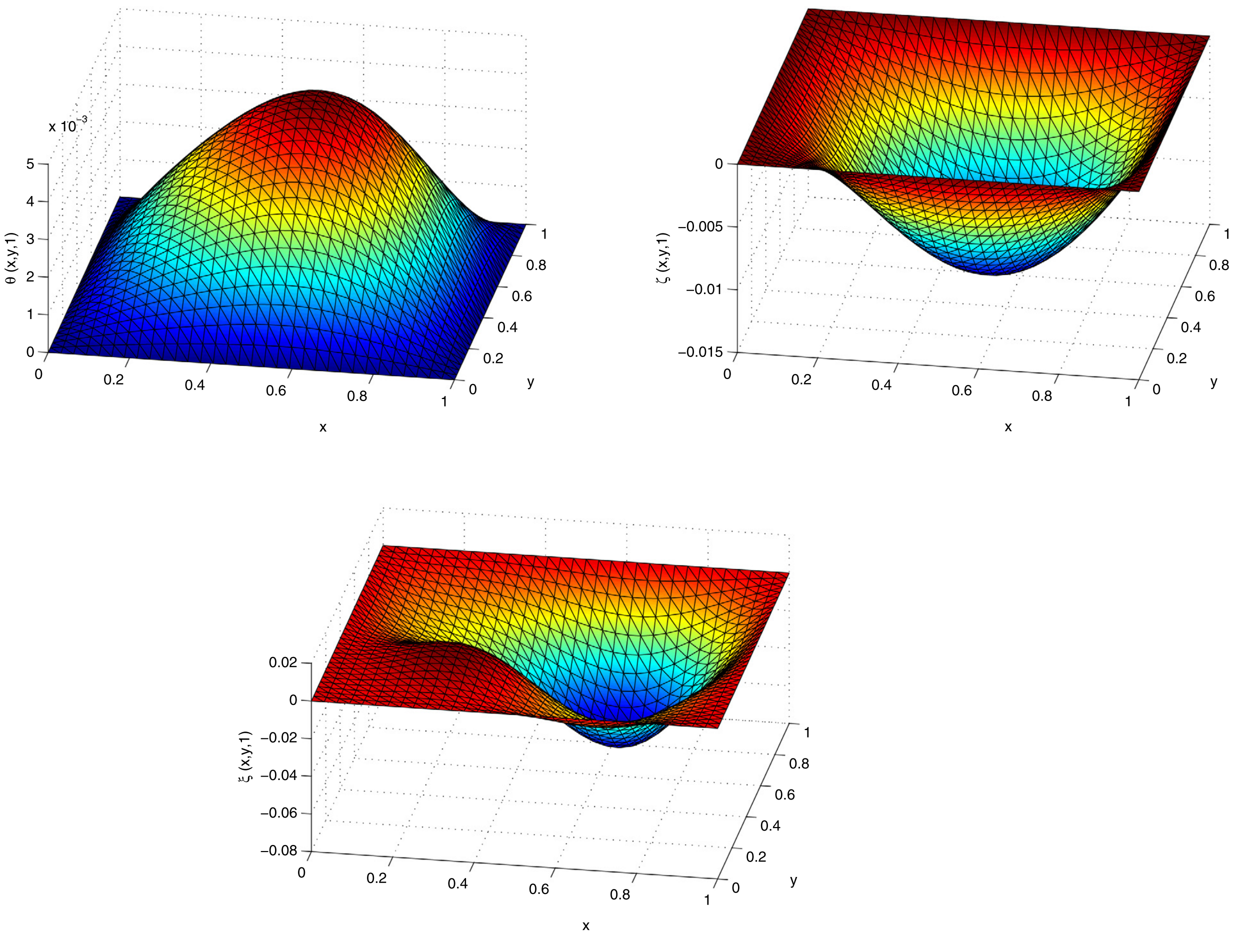

Fig. 6. Example 2: Temperature, temperature speed and temperature acceleration at final time.

$$
\begin{aligned}
&= a\left(\frac{\tau_1^2}{2}\xi_{n-1}^{hk} - k\zeta_{n-1}^{hk}, r^h\right)_Y - \kappa k\left(\nabla(\theta_{n-1}^{hk} + k\zeta_{n-1}^{hk} + \tau_2\zeta_{n-1}^{hk}), \nabla r^h\right)_H \\
&\quad + \kappa_1 k\left(\operatorname{div}\boldsymbol{T}_{n-1}^{hk} + (\tau_2 + k)\operatorname{div}\boldsymbol{e}_{n-1}^{hk}, r^h\right)_Y \quad \forall r^h \in E^h, \\
&b\left(\frac{\tau_1^2}{2}\boldsymbol{w}_n^{hk} + \tau_1 k\boldsymbol{w}_n^{hk} + k^2\boldsymbol{w}_n^{hk}, \boldsymbol{z}^h\right)_H + \kappa_6 k\left(\nabla(k^2\boldsymbol{w}_n^{hk}) + \tau_2\nabla(k\boldsymbol{w}_n^{hk}), \nabla\boldsymbol{z}^h\right)_Q \\
&\quad + (\kappa_4 + \kappa_5)k\left(k^2\operatorname{div}\boldsymbol{w}_n^{hk} + \tau_2 k\operatorname{div}\boldsymbol{w}_n^{hk}, \operatorname{div}\boldsymbol{z}^h\right)_Y \\
&\quad + \kappa_2 k(k^2\boldsymbol{w}_n^{hk} + \tau_2 k\boldsymbol{w}_n^{hk}, \boldsymbol{z}^h)_H + \kappa_3 k((k^2 + \tau_2 k)\nabla\xi_n^{hk}, \boldsymbol{z}^h)_H \\
&= b\left(\frac{\tau_1^2}{2}\boldsymbol{w}_{n-1}^{hk} - k\boldsymbol{e}_{n-1}^{hk}, \boldsymbol{z}^h\right)_H - \kappa_6 k\left(\nabla(\boldsymbol{T}_{n-1}^{hk} + k\boldsymbol{e}_{n-1}^{hk}) + \tau_2\nabla\boldsymbol{e}_{n-1}^{hk}, \nabla\boldsymbol{z}^h\right)_Q \\
&\quad - \kappa_2 k(\boldsymbol{T}_{n-1}^{hk} + k\boldsymbol{e}_{n-1}^{hk} + \tau_2\boldsymbol{e}_{n-1}^{hk}, \boldsymbol{z}^h)_H - \kappa_3 k(\nabla(\theta_{n-1}^{hk} + (\tau_2 + k)\zeta_{n-1}^{hk}), \boldsymbol{z}^h)_H \\
&\quad - (\kappa_4 + \kappa_5)k\left(\operatorname{div}(\boldsymbol{T}_{n-1}^{hk} + k\boldsymbol{e}_{n-1}^{hk}) + \tau_2\operatorname{div}\boldsymbol{e}_{n-1}^{hk}, \operatorname{div}\boldsymbol{z}^h\right)_Y \quad \forall \boldsymbol{z}^h \in V^h,
\end{aligned}
$$

where the discrete temperature speed ζ_n^{hk}, the discrete temperature θ_n^{hk}, the discrete microtemperatures speed $\boldsymbol{e}_n^{hk}$ and the discrete microtemperatures $\boldsymbol{T}_n^{hk}$ are then recovered from the relations:

$$
\begin{aligned}
&\zeta_n^{hk} = k\xi_n^{hk} + \zeta_{n-1}^{hk}, \quad \theta_n^{hk} = k\zeta_n^{hk} + \theta_{n-1}^{hk}, \\
&\boldsymbol{e}_n^{hk} = k\boldsymbol{w}_n^{hk} + \boldsymbol{e}_{n-1}^{hk}, \quad \boldsymbol{T}_n^{hk} = k\boldsymbol{e}_n^{hk} + \boldsymbol{T}_{n-1}^{hk}.
\end{aligned} \tag{23}
$$

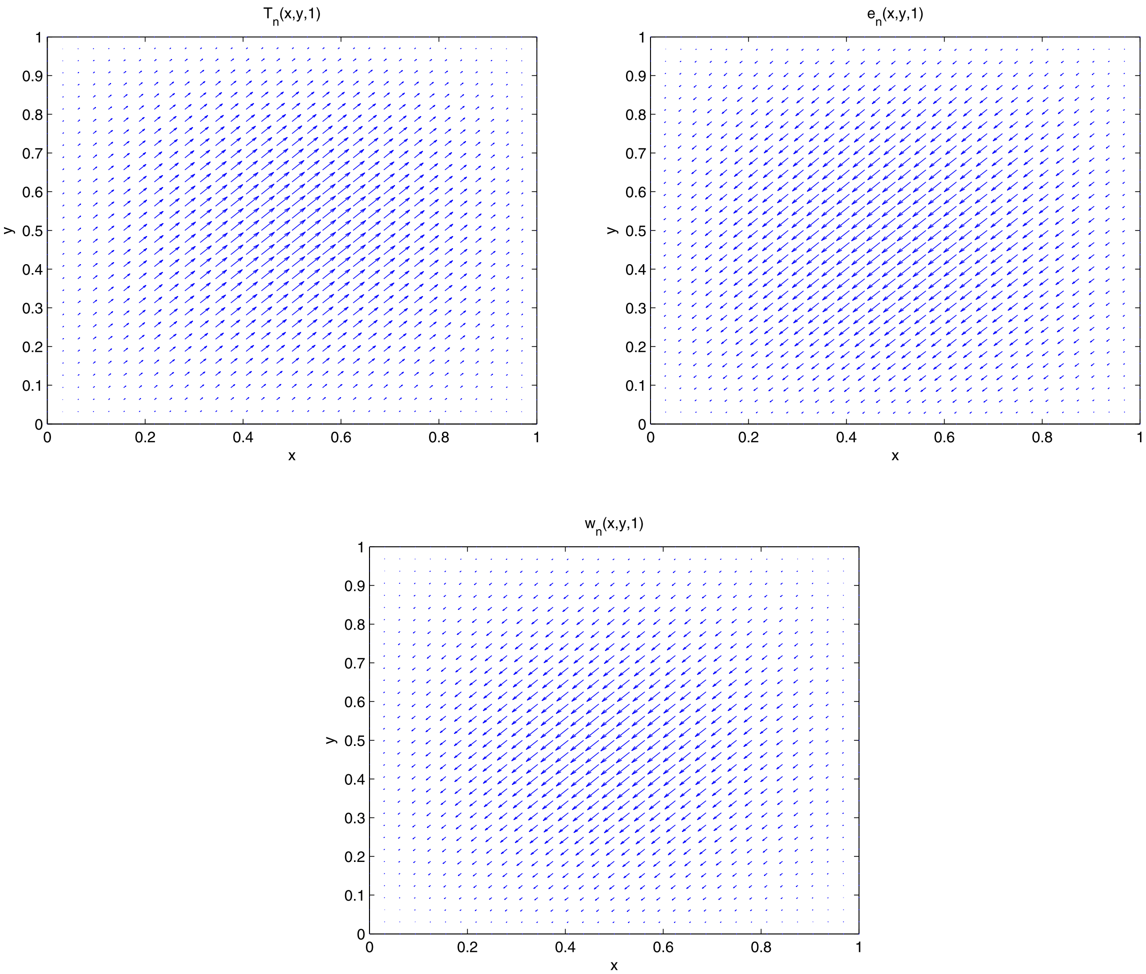

Fig. 7. Example 2: Microtemperatures, microtemperatures speed and microtemperatures acceleration at final time.

This numerical scheme was implemented on a 3.2 GHz PC using MATLAB, and a typical run ($h = k = 0.01$) took about 0.75 seconds of CPU time.

4.1. First example: numerical convergence

As an academical example, in order to show the accuracy of the approximations the following simpler problem is considered. We solve Problem P with the following data:

$$\Omega = (0,1), \quad T = 1, \quad a = 1, \quad b = 1, \quad \tau_1 = 1, \quad \tau_2 = 1, \quad \kappa = 2, \quad \kappa_1 = 1,$$
$$\kappa_2 = 1, \quad \kappa_3 = 1, \quad \kappa_4 = 1, \quad \kappa_5 = 1, \quad \kappa_6 = 2.$$

We recall that homogeneous Dirichlet boundary conditions are imposed on the boundaries $x = 0, 1$. So, using the initial conditions:

$$\theta^0(x) = \zeta^0(x) = \xi^0(x) = T^0 = e^0 = w^0 = x(x-1) \quad \forall x \in (0,1),$$

and adding the following supply terms, for all $(x,t) \in (0,1) \times (0,1)$,

$$F_1(x,t) = e^t \left(\frac{5}{2} x(x-1) - 4x - 6 \right),$$
$$F_2(x,t) = e^t \left(\frac{9}{2} x(x-1) + 4x - 18 \right),$$

the exact solution to Problem P can be easily calculated and it has the form, for $(x,t) \in [0,1] \times [0,1]$:

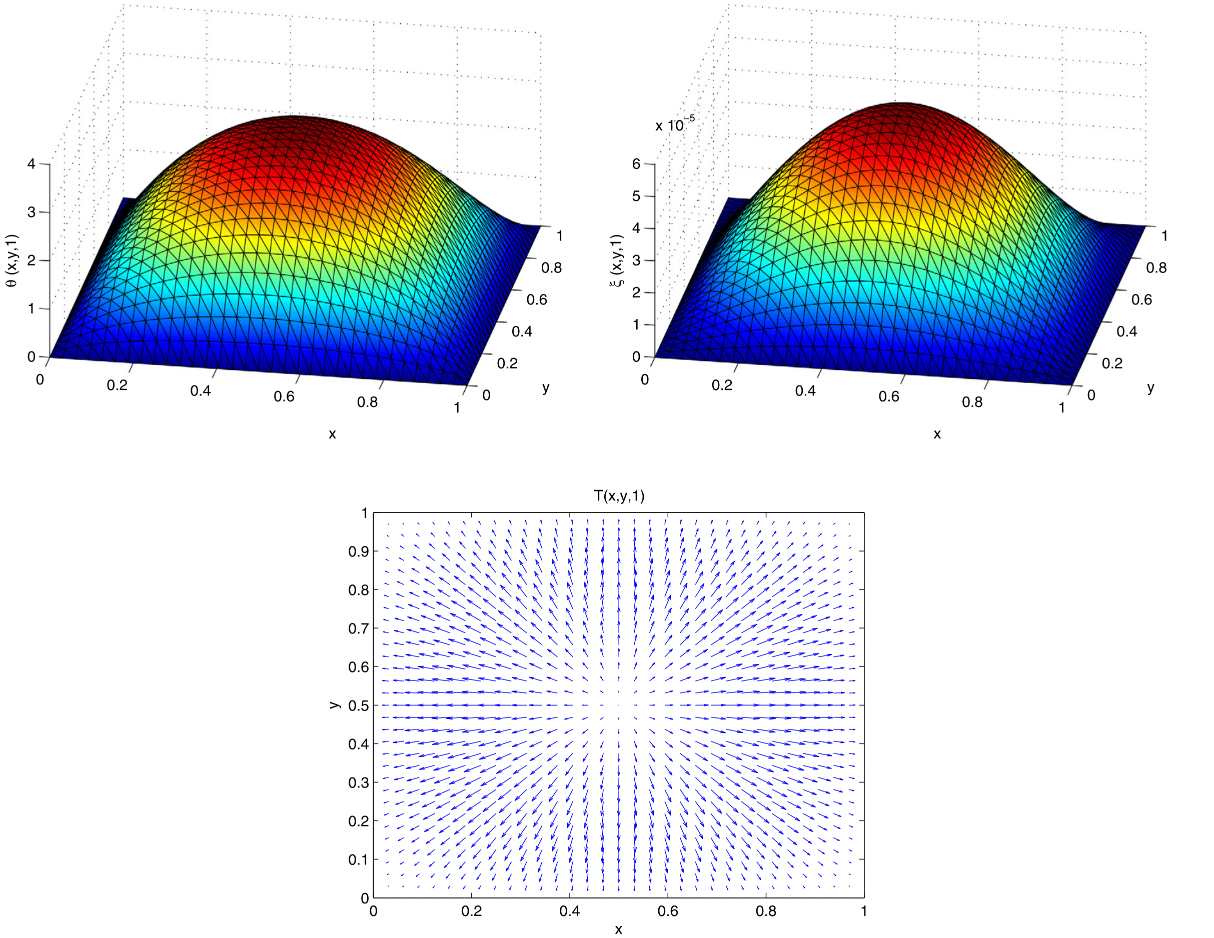

Fig. 8. Example 3: Temperature, temperature acceleration and microtemperatures at final time.

$$\theta(x,t) = T(x,t) = e^t x(x-1).$$

Thus, the approximation errors estimated by

$$\max_{0\le n\le N} \left\{ \|\xi_n - \xi_n^{hk}\|_Y + \|\zeta_n - \zeta_n^{hk}\|_E + \|\theta_n - \theta_n^{hk}\|_E + \|w_n - w_n^{hk}\|_H + \|e_n - e_n^{hk}\|_V + \|T_n - T_n^{hk}\|_V \right\}$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 3.4, is achieved.

If we assume now that there are not supply terms, and we use the final time $T = 25$, the following data

$$\Omega = (0,1), \quad a = 5, \quad b = 1, \quad \tau_1 = 1, \quad \tau_2 = 1, \quad \kappa = 0.1, \quad \kappa_1 = 1, \quad \kappa_2 = 20,$$
$$\kappa_3 = 1, \quad \kappa_4 = 1, \quad \kappa_5 = 1, \quad \kappa_6 = 2,$$

and the initial conditions, for all $x \in (0,1)$,

$$\theta^0(x) = \zeta^0(x) = \xi^0(x) = x(x-1), \quad T^0(x) = e^0(x) = w^0(x) = 0,$$

taking the discretization parameters $h = 10^{-4}$ and $k = 10^{-4}$, the evolution in time of the discrete energy E_n^{hk} is plotted in Fig. 2 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

4.2. Two-dimensional numerical simulations

As a second example, we provide numerical results in a two-dimensional setting but with different initial and boundary conditions. In all the simulations, we considered the domain $\Omega = (0,1) \times (0,1)$. First, we use the following data:

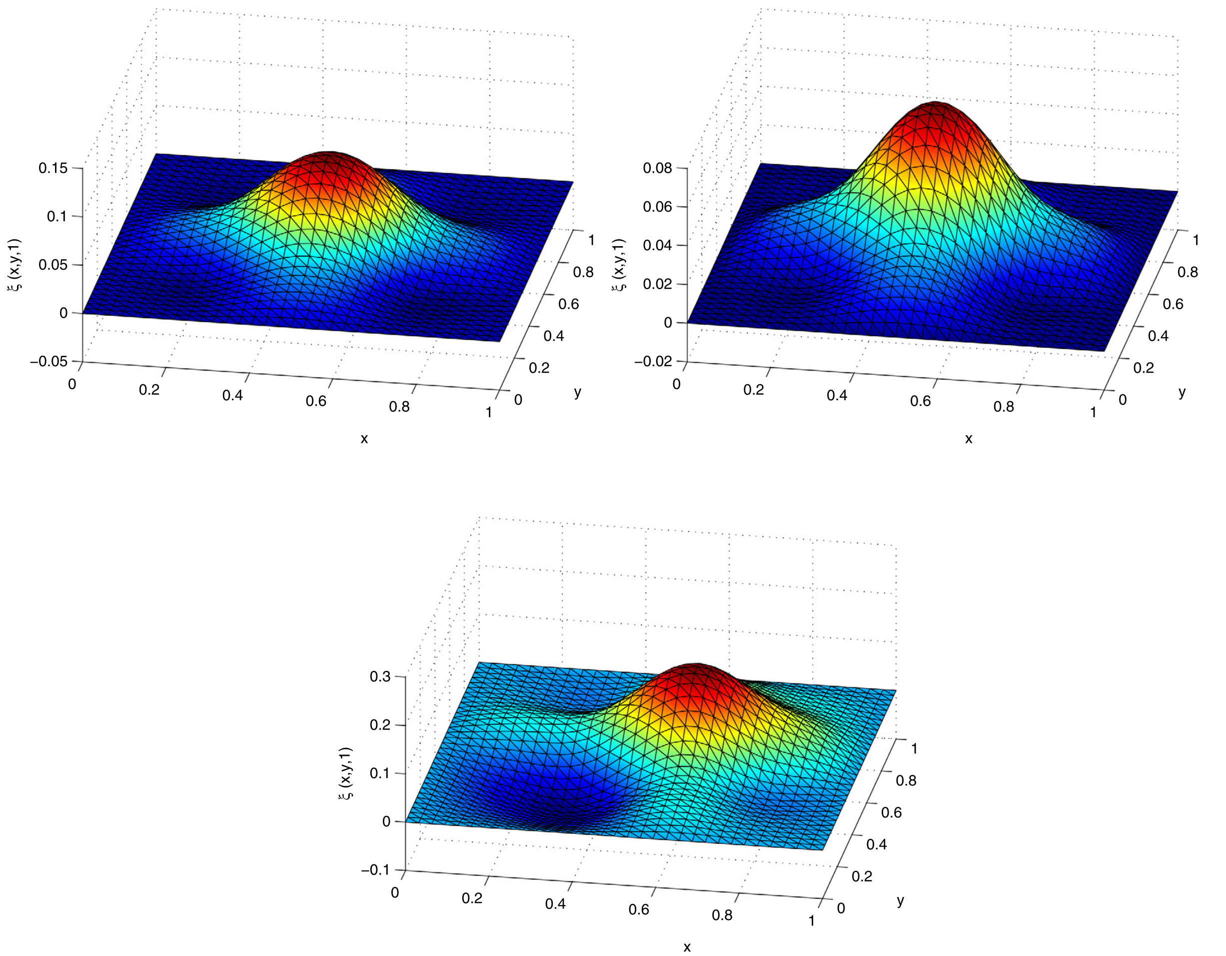

Fig. 9. Example 4: Temperature acceleration at final time for coefficients $\kappa_1 = 0$, $\kappa_3 = 10$ (left upper part), $\kappa_1 = 3$, $\kappa_3 = 4$ (right upper part) and $\kappa_1 = 100$, $\kappa_3 = 0$ (lower part).

$$T = 1, \quad a = 1, \quad b = 1, \quad \tau_1 = 2, \quad \tau_2 = 1, \quad \kappa = 4, \quad \kappa_1 = 1, \quad \kappa_2 = 1,$$
$$\kappa_3 = 2, \quad \kappa_4 = 1, \quad \kappa_5 = 3, \quad \kappa_6 = 2,$$

and the initial conditions, for all $(x, y) \in (0, 1) \times (0, 1)$,

$$\theta^0(x, y) = \zeta^0(x, y) = 0, \quad \xi^0(x, y) = x(x-1)y(y-1),$$
$$\boldsymbol{T}^0(x, y) = \boldsymbol{e}^0(x, y) = \boldsymbol{w}^0(x, y) = (x(x-1)y(y-1), x(x-1)y(y-1)).$$

Taking the discretization parameter $k = 0.01$ and the finite element mesh shown in Fig. 3 (obtained dividing each interval [0, 1] into 32 parts), in Fig. 4 we plot the evolution in time of the temperature, the temperature speed and the temperature acceleration at the middle point $\boldsymbol{x} = (0.5, 0.5)$. We can observe a quadratic behavior for the three functions. Moreover, the evolution in time of the first component of the microtemperatures, the microtemperatures speed and the microtemperatures acceleration (the second one is similar) is plotted in Fig. 5. Now, some oscillations are produced maybe due to the hyperbolic law.

The temperature, the temperature speed and the temperature acceleration are shown at final time in Fig. 6. We can observe that the three functions are quadratic although the temperature has a different curvature.

The microtemperatures, the microtemperatures speed and the microtemperatures acceleration are plotted in Fig. 7 at final time using arrows. As can be seen, there is a oblique direction.

As a second example, we use the following data:

$$T = 1, \quad a = 3, \quad b = 5, \quad \tau_1 = 0.03, \quad \tau_2 = 0.05, \quad \kappa = 2, \quad \kappa_1 = 1, \quad \kappa_2 = 5,$$
$$\kappa_3 = 2, \quad \kappa_4 = 1, \quad \kappa_5 = 3, \quad \kappa_6 = 2,$$

the initial conditions, for all $(x, y) \in (0, 1) \times (0, 1)$,

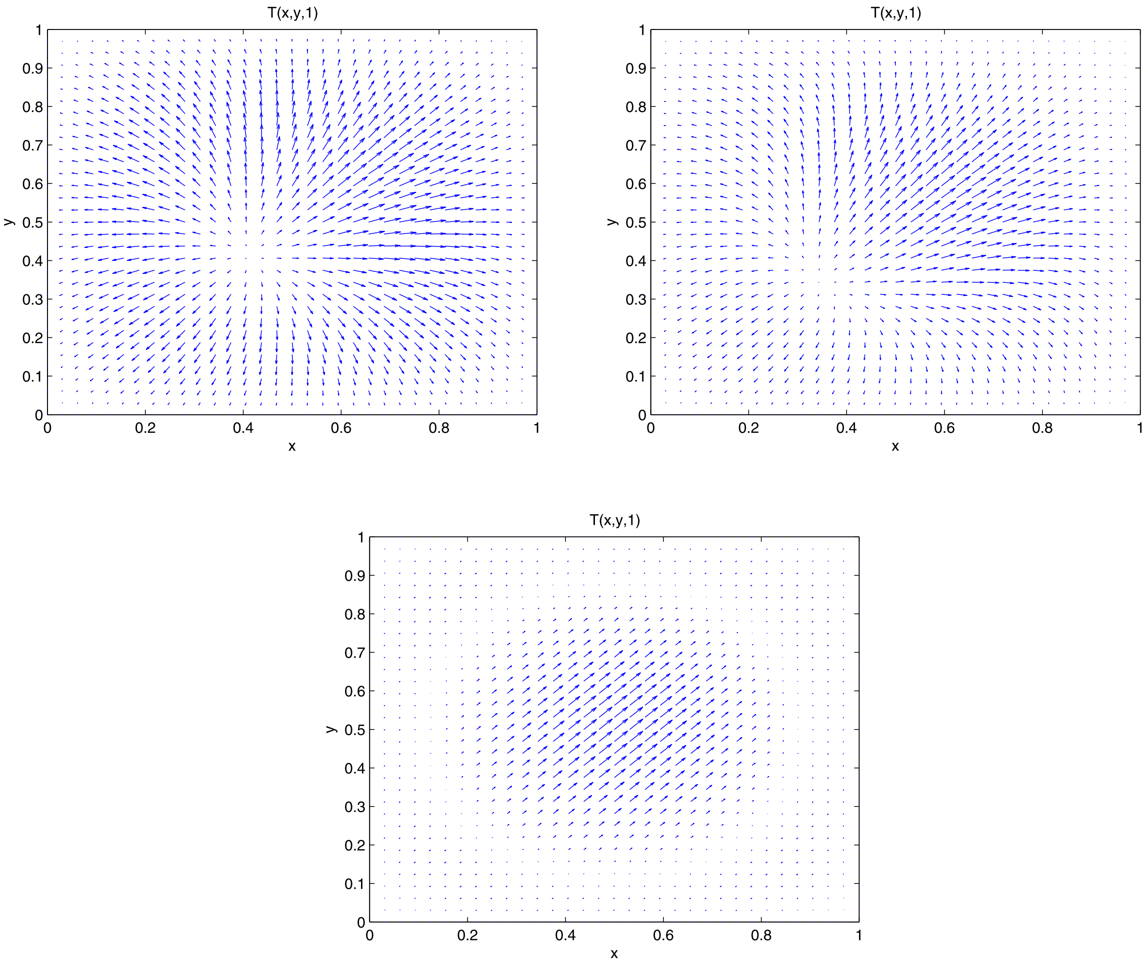

Fig. 10. Example 4: Microtemperatures at final time for coefficients $\kappa_1 = 0$, $\kappa_3 = 10$ (left upper part), $\kappa_1 = 3$, $\kappa_3 = 4$ (right upper part) and $\kappa_1 = 100$, $\kappa_3 = 0$ (lower part).

$$\theta^0(x, y) = \zeta^0(x, y) = 0, \quad \xi^0(x, y) = 0,$$
$$\boldsymbol{T}^0(x, y) = \boldsymbol{e}^0(x, y) = \boldsymbol{w}^0(x, y) = \boldsymbol{0},$$

and a supply term $f(t) = 100t$ in the dual-phase-lag equation. Taking the discretization parameter $k = 0.01$ and the finite element mesh shown in Fig. 3, in Fig. 8 we plot the temperature, the temperature acceleration and microtemperatures (arrows) at final time. As expected, the temperature, temperature speed and temperature acceleration have a quadratic behavior due to the clamping conditions, and the microtemperatures start from the middle point to the boundary in an increasing form. This numerical example compares directly with Example 4.5 of [4]. We note that the temperature and temperature speed are rather similar but the temperature acceleration has a different curvature, being now opposite, due to the microtemperatures effect.

Finally, we analyze the dependence on the coupling coefficients κ_1 and κ_3. Thus, we use the following data:

$$T = 1, \quad a = 3, \quad b = 5, \quad \tau_1 = 2, \quad \tau_2 = 1, \quad \kappa = 3, \quad \kappa_2 = 5,$$
$$\kappa_4 = 1, \quad \kappa_5 = 3, \quad \kappa_6 = 2,$$

and the initial conditions, for all $(x, y) \in (0, 1) \times (0, 1)$,

$$\theta^0(x, y) = \zeta^0(x, y) = 0, \quad \xi^0(x, y) = x(x-1)y(y-1),$$
$$\boldsymbol{T}^0(x, y) = \boldsymbol{e}^0(x, y) = \boldsymbol{w}^0(x, y) = (x(x-1)y(y-1), x(x-1)y(y-1)).$$

Taking the discretization parameter $k = 0.01$ and the finite element mesh shown in Fig. 3, the temperature acceleration are plotted in Fig. 9 at final time for coefficients $\kappa_1 = 0$, $\kappa_3 = 10$, $\kappa_1 = 3$, $\kappa_3 = 4$ and $\kappa_1 = 100$, $\kappa_3 = 0$. In this case, we note that the temperature and the temperature speed are rather similar (we do not show these figures in order to reduce the length of the paper), but we observe clear differences among the temperature accelerations, although they are all quadratic.

Finally, the microtemperatures are shown in Fig. 10 at final time. As we can see, the orientation changes a lot depending on the value of these coupling coefficients.

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