Cooperation on capacitated inventory situations with fixed holding costs

M.G. Fiestras-Janeiro[1] I. García-Jurado[2] A. Meca[3] M.A. Mosquera[1]

Abstract

In this paper we analyze a situation in which several firms deal with inventory problems concerning the same type of product. We consider that each firm uses its limited capacity warehouse for storing purposes and that it faces an economic order quantity model where storage costs are irrelevant (and assumed to be zero) and shortages are allowed. In this setting, we show that firms can save costs by placing joint orders and obtain an optimal order policy for the firms. Besides, we identify an associated class of costs games which we show to be concave. Finally, we introduce and study a rule to share the costs among the firms which provides core allocations and can be easily computed.

1 Introduction

The analysis of centralized inventory models is a flourishing research field in the frontier between game theory and operations research. In a centralized inventory model several agents facing individual inventory problems cooperate by coordinating their orders for the purpose of reducing costs. In the analysis of one of these models two main issues are usually addressed: first, what is the optimal order policy of the group of cooperating agents; second, how the order costs should be shared among the agents. Nagarajan and Sošić (2008), Dror and Hartman (2011) and Fiestras-Janeiro et al. (2012) are recent surveys of centralized inventory models; Fiestras-Janeiro et al. (2011) reviews the applications of cooperative game theory for sharing cost problems.

In most inventory models a positive storage cost per item and time unit is assumed to exist. However, in some situations storage costs are fixed (i.e. independent of the size of the stock) and therefore can be disregarded in the optimization problem. This can be the case, for instance, when the storage costs are only due to the maintenance of the warehouse. Notice that when

[1] *Departamento de Estatística e Investigación Operativa, Universidade de Vigo, Vigo, Spain.*
[2] *Departamento de Matemáticas, Universidade da Coruña, A Coruña, Spain.*
[3] *Centro de Investigación Operativa, Universidad Miguel Hernández de Elche, Elche, Spain.*

storage costs are irrelevant and fixed order costs are positive, in a continuous review setting, the orders should be as large as possible and, thus, the capacities of the warehouses become significant; moreover, the corresponding optimization problem will be trivial, unless shortages are allowed.

In this paper we analyze a situation in which several firms deal with inventory problems concerning the same type of product. We consider that each firm uses its limited capacity warehouse for storing purposes and that it faces an economic order quantity model where storage costs are irrelevant (and assumed to be zero) and shortages are allowed. We first deal with the one decision maker case, and then we study the case with n firms. We show that firms can save costs by placing joint orders and obtain an optimal order policy for the firms; then we obtain some results that can be helpful for allocating the joint costs among them.

An antecedent of this paper is Fiestras-Janeiro et al. (2013), which deals with an inventory problem arising in a farming community in the Northwest of Spain. It considers a collection of stockbreeders (each one owning a relatively small livestock farm) that need livestock feed and place orders to an external supplier. Each farm has its own *silo* (warehouse), with limited capacity, for keeping the feed. The only costs associated with the silos are their building costs since their maintenance costs are irrelevant; thus, the storage cost of each stockbreeder is in fact zero. Fiestras-Janeiro et al. (2013) analyzes then a model with n decision makers, all them facing continuous review inventory problems without holding costs, with limited capacity warehouses and without shortages. The fact that shortages are not allow simplifies strongly the search for optimal policies, as we remarked above. However, the case with shortages can be also used to analyze analogous problems to the ones dealt with in that paper. In fact, the stockbreeders can incur in little shortages since they can get extra feed from their own. Nevertheless, this extra feed is usually of lower quality and, thus, there are losses associated with the shortages.

There are many papers dealing with limited capacity inventory models. In fact, most of the classical and modern books on inventory management include the basic ideas on capacitated inventory; see, for instance, Tersine (1994) and Zipkin (2000). A survey on capacitated lot sizing can be found in Karimi et al. (2003). More recently, Ng et al. (2009) study an economic order quantity model where the warehouse capacity is limited and is, moreover, a decision variable of the model. Parker and Kapucinski (2011) consider the non-cooperative interaction between a retailer and a supplier in a two-stage, periodic review, limited capacity inventory model; they find a Markov equilibrium policy in their model.

On the contrary, as far as we know, apart from Fiestras-Janeiro et al. (2013), the literature has not treated centralized inventory models with limited capacity and fixed storage costs. However, there are a variety of real situations which may be modeled in this way, like the example treated in Fiestras-Janeiro et al. (2013) and in this paper. This example is based on a situation that we have encountered while collaborating with an agricultural cooperative in the Northwest of Spain.

2 The model

An *EOQ (Economic Order Quantity) system without holding costs* is a multiple agent situation where each agent faces a continuous review inventory problem with no holding costs, with shortages and with a limited capacity warehouse. N denotes the finite set of agents. The parameters associated to every $i \in N$ in one of these systems are:

- $a > 0$, the fixed cost per order,
- $b_i > 0$, the shortage cost per item and per time unit,
- $d_i > 0$, the deterministic demand per time unit,
- $K_i > 0$, the capacity of i's warehouse.

This model is in fact a generalization of one introduced in Fiestras-Janeiro et al. (2013): the basic EOQ system without holding costs. These basic systems do not allow for shortages and, then, the analysis of the model we introduce in this paper is fully different. In an EOQ system without holding costs every agent $i \in N$ has to make a decision on his maximum shortage level β_i. In the system operation, every time that i's maximum shortage level is reached, i places an order of size $K_i + \beta_i$ (since the storage cost is zero, i's warehouse should be complete after each order). Then, agent i's average cost per cycle is given by

$$a + b_i \frac{\max\{\beta_i, 0\}}{2} \frac{\max\{\beta_i, 0\}}{d_i}$$

and agent i's average cost per time unit is given by

$$C^i(\beta_i) = \frac{a + \frac{b_i \max^2\{\beta_i, 0\}}{2d_i}}{\frac{K_i + \beta_i}{d_i}} = \frac{ad_i}{K_i + \beta_i} + \frac{b_i \max^2\{\beta_i, 0\}}{2(K_i + \beta_i)},$$

where $\beta_i > -K_i$ in order to guarantee a positive length cycle.[1] We rewrite the agent i's cost function as

$$C^i(\beta_i) = \begin{cases} \frac{ad_i}{K_i+\beta_i} & \text{if } -K_i < \beta_i \leq 0 \\ \frac{ad_i}{K_i+\beta_i} + \frac{b_i\beta_i^2}{2(K_i+\beta_i)} & \text{if } 0 \leq \beta_i. \end{cases}$$

For simplicity we take the number of orders per time unit as the decision variable, that is

$$x_i := \frac{d_i}{K_i + \beta_i}, \tag{1}$$

which implies that

$$\beta_i = \frac{d_i - K_i x_i}{x_i}.$$

[1] In principle, each β_i is non-negative. However, when a group of agents makes joint orders it may be optimal that the maximum shortage level of some agents is negative; notice that in our context storage costs are irrelevant.

Then agent i's cost function can be written as

$$C^i(x_i) = \begin{cases} ax_i & \text{if } x_i \geq \frac{d_i}{K_i} \\ ax_i + \frac{b_i(d_i - K_i x_i)^2}{2x_i d_i} & \text{if } 0 < x_i \leq \frac{d_i}{K_i}. \end{cases}$$

Observe that the ratios demand/capacity (d_i/K_i) play an important role in the cost functions of the agents. They will also play a relevant role in other issues regarding this model as we will see later on, especially in Section 5.

In this paper we explore the possibilities of cooperation in an EOQ system without holding costs. When we look at this model from a cooperative point of view, we consider that a non-empty coalition $S \subset N$ has formed and that all its members place joint orders. It means that the length cycle will be the same for every agent in S, i.e.

$$\frac{1}{x_i} = \frac{K_i + \beta_i}{d_i} = \frac{K_j + \beta_j}{d_j} = \frac{1}{x_j}, \tag{2}$$

for every $i, j \in S$. Equivalently, $x_i = x_j$ for every $i, j \in S$, i.e., the number of orders per time unit will be the same for every agent in S. For simplicity we denote $x = x_i$ for every $i \in S$. Now, the average cost per cycle that coalition S faces is given by

$$a + \sum_{i \in S} b_i \frac{\max\{\beta_i, 0\}}{2} \frac{\max\{\beta_i, 0\}}{d_i}$$

and the average cost per time unit is given by

$$\frac{a + \sum_{i \in S} \frac{b_i \max^2\{\beta_i, 0\}}{2d_i}}{\frac{K_j + \beta_j}{d_j}} = \frac{ad_j}{K_j + \beta_j} + \frac{d_j}{K_j + \beta_j} \sum_{i \in S} \frac{b_i \max^2\{\beta_i, 0\}}{2d_i}.$$

Using the condition of equal length cycle (2), we have that $\beta_i = -K_i + \frac{d_i}{x}$ for all $i \in S$. Thus, for every $x > 0$,

$$\begin{aligned} C^S(x) &= ax + x \sum_{i \in S} \frac{b_i}{2d_i} \max^2\{-K_i + \frac{d_i}{x}, 0\} \\ &= ax + \frac{1}{x} \sum_{i \in S} \frac{b_i}{2d_i} \max^2\{-K_i x + d_i, 0\}. \end{aligned} \tag{3}$$

3 Individual optimal order policies

Now we obtain the optimal order policy and the minimum average cost per time unit of each agent i when ordering alone. Note that C^i is a continuous function for every $x_i > 0$. Besides, it is strictly increasing for every $x_i \geq \frac{d_i}{K_i}$. Then,

$$\min\left\{C^i(x_i) \ : \ x_i \geq \frac{d_i}{K_i}\right\} = \frac{ad_i}{K_i}. \tag{4}$$

If $0 < x_i < \frac{d_i}{K_i}$, then $C^i(x_i)$ can be written as

$$\left(2a + b_i\frac{K_i^2}{d_i}\right)\frac{x_i}{2} + \frac{b_i d_i}{2}\frac{1}{x_i} - b_i K_i. \tag{5}$$

It is a differentiable function and attains a local extreme at x_i if its derivative in x_i equals zero, i.e. if

$$\frac{2a + b_i\frac{K_i^2}{d_i}}{2} - \frac{b_i d_i}{2x_i^2} = 0. \tag{6}$$

The unique value in $(0, \frac{d_i}{K_i})$ satisfying (6) is

$$x_i^* = \sqrt{\frac{b_i d_i}{2a + b_i\frac{K_i^2}{d_i}}}; \tag{7}$$

notice that $x_i^* < \frac{d_i}{K_i}$ because

$$\frac{b_i d_i}{2a + b_i\frac{K_i^2}{d_i}} = \frac{b_i d_i^2}{2ad_i + b_i K_i^2} = \frac{b_i}{\frac{2ad_i}{K_i^2} + b_i}\frac{d_i^2}{K_i^2} < \frac{d_i^2}{K_i^2}. \tag{8}$$

For every $0 < x_i < \frac{d_i}{K_i}$, the second derivative of C^i is

$$\frac{b_i d_i}{x_i^3} > 0.$$

Then, C^i is strictly convex in $(0, \frac{d_i}{K_i})$ and, moreover, x_i^* is the unique minimum of C^i in $(0, \frac{d_i}{K_i})$. Now, the continuity of C^i in $(0, \infty)$, (8), and the fact that C^i is strictly increasing in $[\frac{d_i}{K_i}, \infty)$ imply that x_i^* is in fact the unique minimum of C^i in $(0, \infty)$. Using (5) and (7) it can be easily checked that the minimum average cost per time unit of each agent i when ordering alone $C^i(x_i^*)$ is given by

$$C^i(x_i^*) = \sqrt{b_i d_i(2a + b_i\frac{K_i^2}{d_i})} - b_i K_i. \tag{9}$$

Notice that $C^i(x_i^*)$ can also be written as $C^i(x_i^*) = b_i\beta_i^*$ where

$$\beta_i^* = d_i\sqrt{\frac{2a + b_i\frac{K_i^2}{d_i}}{b_i d_i}} - K_i = \frac{d_i}{x_i^*} - K_i.$$

Clearly, in view of (1), β_i^* is the optimal maximum shortage level for agent i in this context.

4 Coalitional optimal order policies

In this section we obtain the optimal order policy and the minimum average cost per time unit of a non-empty coalition $S \subset N$ when all its members cooperate by placing joint orders. For every such $S \subset N$ and every $x \in (0, +\infty)$ denote by S_x the set $\{i \in S \ : \ x < \frac{d_i}{K_i}\}$. In view of the expression of C^S given in (3), we can write

$$C^S(x) = ax + \frac{1}{x} \sum_{i \in S_x} \frac{b_i}{2d_i}(-K_i x + d_i)^2. \tag{10}$$

It is easy to check that this function is continuous. Moreover, if A_S denotes the set $\{\frac{d_i}{K_i} \ : \ i \in S\}$, it is clear that C^S is differentiable in every $x \in (0, +\infty) \setminus A_S$. It is moreover easy to check that the right and left derivatives of C^S coincide for every $x \in A_S$, so it is in fact differentiable in every $x \in (0, +\infty)$. Its first derivative is given by

$$\frac{d}{dx}C^S(x) = a + \sum_{i \in S_x} \frac{b_i}{2} \frac{K_i^2}{d_i} - \frac{1}{x^2} \sum_{i \in S_x} \frac{b_i d_i}{2}.$$

Again, it is clear that $\frac{d}{dx}C^S$ is differentiable in every $x \in (0, +\infty) \setminus A_S$. Looking at the sign of its derivative we obtain that $\frac{d}{dx}C^S$ is increasing in every $x \in (0, +\infty) \setminus A_S$ and that it is strictly increasing in every $x \in (0, \max_{i \in S} \frac{d_i}{K_i}) \setminus A_S$. Then, taking into account that $\frac{d}{dx}C^S$ is continuous, it is clear that it is increasing in every $x \in (0, +\infty)$ and strictly increasing in every $x \in (0, \max_{i \in S} \frac{d_i}{K_i})$. Thus C^S is a convex function in $(0, +\infty)$ and strictly convex in $(0, \max_{i \in S} \frac{d_i}{K_i})$. Therefore, since

$$\lim_{x \to 0} C^S(x) = \lim_{x \to +\infty} C^S(x) = +\infty$$

and C^S is strictly increasing in $(\max_{i \in S} \frac{d_i}{K_i}, +\infty)$, there exists a unique extreme of C^S in $(0, +\infty)$, which is a minimum. Now, C^S is continuous and differentiable in $(0, +\infty)$ and it has a unique minimum in $(0, +\infty)$ implies that this minimum is attained at the unique point x_S^* in which its first derivative is zero. Thus, x_S^* is the unique solution of the following equation:

$$x_S^* = \sqrt{\frac{\sum_{i \in S_{x_S^*}} b_i d_i}{2a + \sum_{i \in S_{x_S^*}} b_i \frac{K_i^2}{d_i}}}. \tag{11}$$

In order to avoid a cumbersome notation from now on we denote $i(S) := S_{x_S^*}$. In view of (10) and (11), we have that

$$\begin{aligned} C^S(x_S^*) &= \left(2a + \sum_{i \in i(S)} b_i \frac{K_i^2}{d_i}\right) \frac{x_S^*}{2} + \sum_{i \in i(S)} \frac{b_i d_i}{2} \frac{1}{x_S^*} - \sum_{i \in i(S)} b_i K_i \\ &= \sum_{i \in i(S)} b_i d_i \sqrt{\frac{2a + \sum_{i \in i(S)} b_i \frac{K_i^2}{d_i}}{\sum_{i \in i(S)} b_i d_i}} - \sum_{i \in i(S)} b_i K_i. \end{aligned} \tag{12}$$

Notice that, as it should be, (12) reduces to (9) when $S = \{i\}$ (for any $i \in N$). Besides, like in the individual setting treated in the former section, we can express $C^S(x_S^*)$ in terms of the maximum shortage levels of the agents in S when x_S^* is adopted. Indeed, in view of (12),

$$C^S(x_S^*) = \sum_{i \in i(S)} b_i d_i \frac{1}{x_S^*} - \sum_{i \in i(S)} b_i K_i = \sum_{i \in i(S)} b_i \left(\frac{d_i}{x_S^*} - K_i \right) = \sum_{i \in i(S)} b_i \beta_i^{*S},$$

where β_i^{*S} is the maximum shortage level of agent $i \in S$ when coalition S places joint orders and uses an optimal order policy.

One may wonder now how to solve equation (11). Next we describe an iterative algorithm to solve it easily and to compute x_S^* and $i(S)$ for any non-empty $S \subset N$. Denote $s = |S|$.

Algorithm 4.1.

1. *Let $S = \{i_1, i_2, \ldots, i_s\}$ be the agents in S arranged in non-decreasing order of the ratios demand/capacity. Thus $\frac{d_{i_1}}{K_{i_1}} \leq \frac{d_{i_2}}{K_{i_2}} \leq \cdots \leq \frac{d_{i_s}}{K_{i_s}}$.*
2. *Initialize $k = s+1$, $T = \emptyset$, $x_T = 0$, and $S_{x_T} = S$.*
3. *Do while $S_{x_T} \neq T$:*
   Set $k = k-1$, $T = T \cup \{i_k\}$, and compute

$$x_T = \sqrt{\frac{\sum_{i_l \in T} b_{i_l} d_{i_l}}{2a + \sum_{i_l \in T} b_{i_l} \frac{K_{i_l}^2}{d_{i_l}}}} \text{ and } S_{x_T} = \{i_l \in S \ : \ x_T < \frac{d_{i_l}}{K_{i_l}}\}.$$

4. *Let $i(S) = T$ and $x_S^* = x_T$. STOP.*

Let us note that the above algorithm finishes after a finite number of steps (smaller than or equal to s) since equation (11) has a unique solution.

The following result shows a kind of monotonicity of the optimal number of orders of a non-empty coalition. It is an attractive property; moreover we use it later on in this paper.

Theorem 4.1. *Let (N, a, b, d, K) be an EOQ system without holding costs and take a pair of non-empty coalitions $P, S \subset N$ with $P \subset S$. Then $x_P^* \leq x_S^*$.*

Proof. See Appendix. □

To finish this section we present an example that we have encountered while collaborating with an agricultural cooperative in the Northwest of Spain. We use this example to illustrate the concepts introduced up to now, as well as Algorithm 4.1. This example has been also considered in Fiestras-Janeiro et al. (2013), but now we consider that shortages are allowed.

Example 4.1. *This example is based on feedback obtained from dairy farmers in northwestern Spain, although the data considered here are fictitious. A standard dairy farm in northwestern Spain has between 40 and 150 dairy cows. The cow feeding is varied and the feeding ration must have the necessary nutrients to maintaining a high daily production of milk (between 25 and 35 liters). The feeding ration* can be decomposed into two parts. On one hand, a part that has to be stored at the farm in warehouses, called silos. On the other hand, a part that must be daily obtained and that cannot be stored. We are interested in the management of the former part, the one that is stored. From now on, we refer to this part of the feeding ration as the dry feed. The silos, where the dry feed is stored, have a constant maintenance cost. Indeed, this cost is negligible and can be considered to be zero. The dry feed is ordered to an external supplier. There is a fixed cost of a euros each time that an order is made; this fixed cost is mainly due to transportation. Each cow consumes about 10 kg of dry feed for producing about 30 liters of milk per day. When there is a shortage of dry feed, the feeding ration has to be changed. The daily production of milk can be maintained but its quality decreases. So, although the cost of the feeding ration does not change significantly, there is a cost due to the economic impact of the decrease of the quality; this cost is b euros per ton and day.

For simplicity we consider an example with four dairy farms $N = \{1, 2, 3, 4\}$. The dairy cattle is formed by 45, 95, 105 and 120 cows, respectively. The fixed cost per order is $a = 180$ (in euros) and the demand (in tons per day), the shortage costs (in euros per ton and day) and the capacity of silos (in tons) for each dairy farm are given in the next table, whose last column depicts the ratios demand/capacity.

i	d_i	b_i	K_i	$\frac{d_i}{K_i}$
1	0.45	15	5	0.090
2	0.95	15	7.5	0.127
3	1.05	10	8	0.131
4	1.20	12	9	0.133

Assume that the dairy farms 1, 2 and 4 decide to cooperate by ordering together, so $S = \{1, 2, 4\}$. Let us compute $c(S) = C^S(x_S^*)$. First we calculate $i(S)$ and x_S^*. We proceed iteratively using the non-increasing arrangement of the dairy farms' ratios demand/capacity. Take $T = \{4\}$ and compute

$$x_T = \sqrt{\frac{\sum_{i\in T} b_i d_i}{2a + \sum_{i\in T} b_i \frac{K_i^2}{d_i}}} = \sqrt{\frac{b_4 d_4}{2a + b_4 \frac{K_4^2}{d_4}}} = \sqrt{\frac{14.4}{360 + 810}} = 0.11094.$$

Let us note that $S_{x_T} = \{i \in S \,:\, x_T < \frac{d_i}{K_i}\} = \{2, 4\}$ and $S_{x_T} \neq T$. Then, x_T does not satisfy (11) and, consequently, $x_S^* \neq x_T$ and $i(S) \neq T$. Take now $T = \{2, 4\}$ and compute

$$x_T = \sqrt{\frac{\sum_{i\in T} b_i d_i}{2a + \sum_{i\in T} b_i \frac{K_i^2}{d_i}}} = \sqrt{\frac{b_2 d_2 + b_4 d_4}{2a + b_2 \frac{K_2^2}{d_2} b_4 \frac{K_4^2}{d_4}}} = \sqrt{\frac{14.25 + 14.4}{360 + 888.16 + 810}} = 0.117983.$$

Now, since $S_{x_T} = \{2, 4\} = T$, x_T satisfies (11) and, consequently, $i(S) = T = \{2, 4\}$ and $x_S^* = x_T = 0.117983$. Finally, using (12), we have that

$$C^S(x_S^*) = \frac{b_2 d_2 + b_4 d_4}{x_S^*} - b_2 K_2 - b_4 K_4 = \frac{14.25 + 14.4}{0.117983} - 112.5 - 108 = 22.33$$

Following similar operations, one can obtain $C^S(x_S^*)$ for every non-empty $S \subset N$.

S	1	2	3	4	12	13	14
$C^S(x_S^*)$	14.75	20.87	20.90	21.80	20.87	20.90	21.80

S	23	24	34	123	124	134	234	N
$C^S(x_S^*)$	21.92	22.33	22.50	21.92	22.33	22.50	22.67	22.67

5 Profitability of the grand coalition and cost allocation procedures

In the last section we obtained an expression for the minimum cost associated with each non-empty coalition $S \subset N$ when its members place joint orders for a given EOQ system without holding costs (N, a, b, d, K). In terms of cooperative game theory, we have obtained the *cost game* c associated to the system (N, a, b, d, K), c being a map which assigns to every non-empty $S \subset N$ the real number $C^S(x_S^*)$. So, from now on, we write[2]

$$c(S) = \sum_{i \in i(S)} b_i d_i \sqrt{\frac{2a + \sum_{i \in i(S)} b_i \frac{K_i^2}{d_i}}{\sum_{i \in i(S)} b_i d_i}} - \sum_{i \in i(S)} b_i K_i,$$

for every non-empty $S \subset N$. For the results we prove in this section, assume that a system (N, a, b, d, K) is given and that c is its associated cost game.

We consider now the following issue. Is it profitable for the agents in N to form the grand coalition to place joint orders? In this section we prove that the answer to this question is positive because c is a subadditive game, in the sense that

$$c(S \cup T) \leq c(S) + c(T),$$

for all $S, T \in N$ with $S \cup T = \emptyset$. Notice that the superadditivity condition implies that if N is partitioned into disjoint ordering coalitions (whose integrants place joint orders) the corresponding cost will not decrease.

In fact we prove that c is not only subadditive but also concave, in the sense that

$$c(T \cup j) - c(T) \leq c(S \cup j) - c(S) \tag{13}$$

for all $j \in N$ and all $S, T \subset N$ with $S \subsetneq T \subset N \setminus j$. It is a well known result in cooperative game theory that every concave game is subadditive. Moreover, the concavity property provides us with additional information about the game: the marginal contribution of an agent diminishes as a coalition grows (according to (13)).

Theorem 5.1. *Let (N, a, b, d, K) be an EOQ system without holding costs with associated cost game c. Then c is a concave game.*

Proof. See Appendix. □

[2]By convention, $c(\emptyset) = 0$.

So we proved that in an EOQ system without holding costs (N, a, b, d, K) it is efficient that all players place joint orders. In that case, the optimal average cost per time unit is given by

$$c(N) = \sum_{i \in i(N)} b_i d_i \sqrt{\frac{2a + \sum_{i \in i(N)} b_i \frac{K_i^2}{d_i}}{\sum_{i \in i(N)} b_i d_i}} - \sum_{i \in i(N)} b_i K_i.$$

An allocation rule for EOQ systems without holding costs is a map ϕ which assigns a vector $\phi(c) \in \mathbb{R}^N$ to every EOQ system without holding costs (N, a, b, d, K) with associated cost game c, satisfying that $\sum_{i \in N} \phi_i(c) = c(N)$. Each component $\phi_i(c)$ indicates the cost allocated to i, so an allocation rule for EOQ systems without holding costs is a procedure to allocate the optimal cost among the agents in N when they cooperate. An allocation rule should have good properties from the following points of view.

1. The proposal of the rule for a particular system should be computable in a reasonable CPU time, even when the number of agents is large.

2. It is very convenient that the rule proposes for every system an allocation which belongs to the core of the associated cost game (see, for instance, González-Díaz et al. (2010) for details on the core of a cooperative game). This means that, for every EOQ system without holding costs (N, a, b, d, K) with associated cost game c, ϕ should satisfy the following:
$$\sum_{i \in S} \phi_i(c) \leq c(S), \text{for every } S \subset N.$$
Notice that this condition assures that no group S is disappointed with the proposal of the rule, because the cost allocated to it is less than or equal to the cost it would support if its members formed a coalition to place joint orders independently of the agents in $N \setminus S$.

3. The proposal of the rule must be understandable and acceptable by the agents, in the sense that no one of them should feel unfairly treated.

Since the cost games associated to EOQ systems without holding costs are concave, cooperative game theory provides allocation rules for EOQ systems without holding costs with good properties at least with respect to items 2 and 3. We highlight the Shapley value and the nucleolus, which always provide core allocations in this context (see González-Díaz et al. (2010) for details on them). However, both allocations are hard to compute when the number of agents increases.

Next we define an allocation rule for EOQ systems without holding costs and discuss its qualification with respect to the three items enumerated above. In fact, the interest of this rule is that it selects in a very natural way a point in the core. It has excellent properties with respect to items 1 and 2, but its interest from the point of view of item 3 is not so clear.

Definition 5.1. *The rule R we propose assigns to every EOQ system without holding costs* (N, a, b, d, K)

with associated cost game c the allocation vector $R(c) \in \mathbb{R}^N$ given by:

$$R_i(c) = \begin{cases} b_i d_i \sqrt{\dfrac{2a + \sum_{l \in i(N)} b_l \frac{K_l^2}{d_l}}{\sum_{l \in i(N)} b_l d_l}} - b_i K_i & \text{if } i \in i(N) \\ 0 & \text{otherwise.} \end{cases}$$

This rule can be computed easily. Moreover, its complexity increases polynomially on the number of agents. So, it is clear that R is a good rule from the point of view of computability.

With respect to the second item, the following theorem shows that R proposes for every system an allocation which belongs to the core of the associated cost game.[3]

Theorem 5.2. *Let (N, a, b, d, K) be an EOQ system without holding costs with associated cost game c. Then, for every $S \subset N$,*

$$\sum_{i \in S} R_i(c) \leq c(S).$$

Proof. See Appendix. □

Now we make some comments on our rule R which have to do with the third item. R can be explained in the following way.

- Only agents having a *large* ratio demand/capacity will have to contribute to the payment of the ordering costs. With *large* ratio we mean that it is larger than the optimal number of orders per time unit.

- Each agent having a large ratio will pay his own shortage cost plus a part of the fixed cost which depends on his ratio. This part is computed in the following way.

  Agent i's shortage cost per time unit when the number of orders per time unit is x_N^* and $i \in i(N)$ is given by

  $$\frac{1}{x_N^*} \frac{b_i}{2d_i} (-K_i x_N^* + d_i)^2 = \frac{1}{x_N^*} \frac{b_i}{2d_i} (d_i^2 + K_i^2 x_N^{*2}) - b_i K_i.$$

  Assuming that agent $i \in i(N)$ pays his own shortage cost per time unit, then the part of the fixed cost per time unit that he pays according to R is given by

  $$b_i d_i \frac{1}{x_N^*} - b_i K_i - \frac{1}{x_N^*} \frac{b_i}{2d_i} (d_i^2 + K_i^2 x_N^{*2}) + b_i K_i = \frac{1}{x_N^*} (\frac{b_i d_i}{2} - \frac{b_i K_i^2}{2d_i} x_N^{*2}).$$

  Then, the part of the fixed cost a that agent $i \in i(N)$ pays each time that an order is made

[3] Notice that R provides in fact a PMAS of c in the sense of Sprumont (1990); it easily follows from Theorem 5.2 and the definitions of R and c.

is

$$\frac{1}{x_N^{*2}}\left(\frac{b_i d_i}{2}-\frac{b_i K_i^2}{2d_i}x_N^{*2}\right)=\frac{2a+\sum\limits_{l\in i(N)} b_l\frac{K_l^2}{d_l}}{\sum\limits_{l\in i(N)} b_l d_l}\frac{b_i d_i}{2}-\frac{b_i K_i^2}{2d_i}=\left(a+\sum_{l\in i(N)}\frac{b_l K_l^2}{2d_l}\right)\frac{b_i d_i}{\sum\limits_{l\in i(N)} b_l d_l}-\frac{b_i K_i^2}{2d_i},$$

which can be written as

$$\frac{b_i d_i}{\sum_{l\in i(N)} b_l d_l}a+\frac{b_i d_i}{\sum_{l\in i(N)} b_l d_l}\sum_{l\in i(N)}\frac{b_l K_l^2}{2d_l}-\frac{b_i K_i^2}{2d_i}.$$

Notice that R does not divide a in a proportional way among the agents in $i(N)$. The first term is, in fact, a proportional splitting of a. The second term is a proportional division of the total average cost per cycle when each agent's deficit level equals his warehouse size. Finally, the third term is agent i's average cost per cycle when his deficit level is his warehouse size.

- Agents having a large ratio should probably re-dimension the capacity of their warehouses (in view of their demands). That is the reason why it does not seem unfair that they are forced to support the ordering costs.

We finish this section computing the proposal of R in Example 4.1 and comparing it with the proposal of other rules.

Example 5.1. *Consider again the EOQ system without holding costs of Example 4.1 and its corresponding cost game. It can be easily proven that $i(N)=\{2,3,4\}$. Then $R_1(c)=0$. To obtain $R_i(c)$ for $i\in\{2,3,4\}$, we compute*

$$\sqrt{\frac{2a+\sum_{i\in i(N)} b_i\frac{K_i^2}{d_i}}{\sum_{i\in i(N)} b_i d_i}}=\sqrt{\frac{2a+b_2\frac{K_2^2}{d_2}+b_3\frac{K_3^2}{d_3}+b_4\frac{K_4^2}{d_4}}{b_2d_2+b_3d_3+b_4d_4}}=8.2547.$$

Then

$$R_2(c)=8.2547\, b_2d_2-b_2K_2=5.13,$$

$$R_3(c)=8.2547\, b_3d_3-b_3K_3=6.67,$$

$$R_4(c)=8.2547\, b_4d_4-b_4K_4=10.87.$$

Finally, we compute the proposal for this example of other two well-known solution concepts, the Shapley value and the nucleolus. The proposal of the three rules are displayed in the next table.

i	$R_i(c)$	$Sh_i(c)$	$Nu_i(c)$
1	0	3.69	5.35
2	5.13	6.04	5.52
3	6.67	6.14	5.70
4	10.87	6.80	6.10

Observe that in this example the three rules consider that the bigger the ratio demand/capacity of an agent is the more that this agent will have to pay (this is not true in general). With this principle in mind the nucleolus tends to equalize the costs supported by the agents whereas our rule tends to take more account of the differences; the Shapley value plays a more moderate middle.

6 Conclusions

In this paper we analyze multiple agent situations where each agent faces a continuous review inventory problem without holding costs, with shortages and with a limited capacity warehouse. We find a collective optimal policy when a group of agents agrees to cooperate placing joint orders. In this context we show that the formation of the largest possible coalition (the grand coalition) is profitable. Moreover we indicate how cooperative game theory can be used to allocate the cost among the agents and we identify a natural allocation for each problem which satisfies attractive properties from the points of view of computability and stability. We illustrate our results with an example that we have encountered while collaborating with an agricultural cooperative in the Northwest of Spain.

Acknowledgements

Authors acknowledge the financial support of *Ministerio de Ciencia e Innovación* through projects MTM2008-06778-C02-01, ECO2008-03484-C02-02, MTM2011-23205, MTM2011-27731-C03, of *Xunta de Galicia* through project INCITE09-207-064-PR, and of *Generalitat Valenciana* through project ACOMP/2013/092.

Appendix

Proof of Theorem 4.1.

Proof. We distinguish three cases.

1. $P \cap i(S) = \emptyset$**.** Then $i(P) \cap i(S) = \emptyset$. Consequently, for all $i \in i(P)$, $x_S^* \geq \frac{d_i}{K_i}$. Besides $x_P^* < \frac{d_i}{K_i}$ for all $i \in i(P)$. Then, $x_P^* < x_S^*$.

2. $P \cap i(S) = i(S)$**.** In view of Algorithm 4.1 it is clear that in this case $x_P^* = x_S^*$.

3. $\emptyset \neq P \cap i(S) \subsetneq i(S)$**.** Notice that (11) implies that

$$\sqrt{\frac{2a + \sum_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l \in i(S)} b_l d_l}} > \frac{K_j}{d_j}, \quad \text{for every } j \in i(S). \tag{14}$$

By square both sides of (14), multiplying by $b_j d_j$, for every $j \in i(S) \setminus P$, adding up, and dividing by $\sum_{j\in i(S)\setminus P} b_j d_j$, we obtain

$$\frac{2a+\sum_{l\in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l\in i(S)} b_l d_l} > \frac{\sum_{j\in i(S)\setminus P} b_j \frac{K_j^2}{d_j}}{\sum_{j\in i(S)\setminus P} b_j d_j} \tag{15}$$

(note that $\sum_{j\in i(S)\setminus P} b_j d_j \neq 0$ because $P\cap i(S) \subsetneq i(S)$). Expression (15) is equivalent to

$$\sum_{j\in i(S)\setminus P} b_j d_j (2a + \sum_{l\in i(S)} b_l \frac{K_l^2}{d_l}) > \sum_{l\in i(S)} b_l d_l \sum_{j\in i(S)\setminus P} b_j \frac{K_j^2}{d_j}. \tag{16}$$

Substracting $\sum_{l\in i(S)\setminus P} b_l d_l \sum_{l\in i(S)\setminus P} b_l \frac{K_l^2}{d_l}$ and adding $2a\sum_{l\in i(S)\cap P} b_l d_l$ to both sides in (16),

$$2a\sum_{l\in i(S)} b_l d_l + \sum_{l\in i(S)\setminus P} b_l d_l \sum_{l\in i(S)\cap P} b_l \frac{K_l^2}{d_l} > 2a \sum_{l\in i(S)\cap P} b_l d_l + \sum_{l\in i(S)\cap P} b_l d_l \sum_{l\in i(S)\setminus P} b_l \frac{K_l^2}{d_l}, \tag{17}$$

adding $\sum_{l\in i(S)\cap P} b_l d_l \sum_{l\in i(S)\cap P} b_l \frac{K_l^2}{d_l}$ to both sides in (17), we obtain

$$2a\sum_{l\in i(S)} b_l d_l + \sum_{l\in i(S)} b_l d_l \sum_{l\in i(S)\cap P} b_l \frac{K_l^2}{d_l} > 2a \sum_{l\in i(S)\cap P} b_l d_l + \sum_{l\in i(S)\cap P} b_l d_l \sum_{l\in i(S)} b_l \frac{K_l^2}{d_l}. \tag{18}$$

Finally, dividing both sides for $\sum_{l\in i(S)} b_l d_l \sum_{l\in i(S)\cap P} b_l d_l$ in (18), we obtain

$$\frac{2a+\sum_{l\in i(S)\cap P} b_l \frac{K_l^2}{d_l}}{\sum_{l\in i(S)\cap P} b_l d_l} > \frac{2a+\sum_{l\in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l\in i(S)} b_l d_l} \tag{19}$$

(note that $\sum_{l\in i(S)} b_l d_l \sum_{l\in i(S)\cap P} b_l d_l \neq 0$ because $\emptyset \neq P\cap i(S)$). Rewriting (19) and combining it with (11), we have

$$\frac{\sum_{l\in i(S)\cap P} b_l d_l}{2a+\sum_{l\in i(S)\cap P} b_l \frac{K_l^2}{d_l}} < \frac{\sum_{l\in i(S)} b_l d_l}{2a+\sum_{l\in i(S)} b_l \frac{K_l^2}{d_l}} < \frac{d_j^2}{K_j^2}, \text{ for every } j\in i(S).$$

In particular,

$$\frac{\sum_{l\in i(S)\cap P} b_l d_l}{2a+\sum_{l\in i(S)\cap P} b_l \frac{K_l^2}{d_l}} < \frac{\sum_{l\in i(S)} b_l d_l}{2a+\sum_{l\in i(S)} b_l \frac{K_l^2}{d_l}} < \frac{d_j^2}{K_j^2}, \text{ for every } j\in P\cap i(S).$$

The definition of $i(P)$ and this last inequality imply that $P\cap i(S) \subset i(P)$; then $P\cap i(S) \subset i(P)\cap i(S)$ and thus $P\cap i(S) = i(P)\cap i(S)$. Now we check that

$$\frac{2a+\sum_{l\in i(P)} b_l \frac{K_l^2}{d_l}}{\sum_{l\in i(P)} b_l d_l} > \frac{2a+\sum_{l\in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l\in i(S)} b_l d_l}. \tag{20}$$

If $i(P) = P \cap i(S)$, (20) is in fact (19). If $P \cap i(S) \neq i(P)$ then $i(P) \setminus i(S) \neq \emptyset$ and, by the definition of $i(S)$, we have

$$\frac{K_j^2}{d_j^2} \geq \frac{2a + \sum_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l \in i(S)} b_l d_l}, \text{ for every } j \in i(P) \setminus i(S) \tag{21}$$

and then, multiplying by $b_j d_j$ in both sides of (21), adding up for $j \in i(P) \setminus i(S)$, and dividing by $\sum_{j \in i(P) \setminus i(S)} b_j d_j$, we obtain

$$\frac{\sum_{l \in i(P) \setminus i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{j \in i(P) \setminus i(S)} b_j d_j} \geq \frac{2a + \sum_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l \in i(S)} b_l d_l} \tag{22}$$

(note that $\sum_{j \in i(P) \setminus i(S)} b_j d_j \neq 0$ because $i(P) \setminus i(S) \neq \emptyset$). Besides, using (22) and taking into account that $i(P) = (i(P) \cap i(S)) \cup (i(P) \setminus i(S))$, we have

$$2a + \sum_{l \in i(P)} b_l \frac{K_l^2}{d_l} \geq 2a + \sum_{l \in i(P) \cap i(S)} b_l \frac{K_l^2}{d_l} + \sum_{j \in i(P) \setminus i(S)} b_j d_j \frac{2a + \sum_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l \in i(S)} b_l d_l}.$$

Using (19) and $i(P) \cap i(S) = P \cap i(S)$, we obtain

$$2a + \sum_{l \in i(P)} b_l \frac{K_l^2}{d_l} > \sum_{j \in i(P) \cap i(S)} b_j d_j \frac{2a + \sum_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l \in i(S)} b_l d_l} + \sum_{j \in i(P) \setminus i(S)} b_j d_j \frac{2a + \sum_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum_{l \in i(S)} b_l d_l}$$

and then (20) holds. □

Lemma A.1. *Let (N, a, b, d, K) be an EOQ system without holding costs and take a pair of non-empty coalitions $S, T \subset N$ with $S \subsetneq T$. Then $C^T(x) - C^S(x) = C^{T \setminus S}(x) - ax$ for all $x \in (0, +\infty)$. Moreover, $C^P(x) - ax$ is a non-increasing function in $(0, +\infty)$ for all non-empty $P \subset N$.*

Proof. Clearly

$$\begin{aligned} C^T(x) - C^S(x) &= \tfrac{1}{x} \textstyle\sum_{j \in T_x} \tfrac{b_j}{2d_j} (-K_j x + d_j)^2 - \tfrac{1}{x} \textstyle\sum_{j \in S_x} \tfrac{b_j}{2d_j} (-K_j x + d_j)^2 \\ &= \tfrac{1}{x} \textstyle\sum_{j \in (T \setminus S)_x} \tfrac{b_j}{2d_j} (-K_j x + d_j)^2 = C^{T \setminus S}(x) - ax. \end{aligned}$$

In Section 4 we proved that $C^P(x)$ is differentiable in $(0, +\infty)$ for all non-empty $P \subset N$. Thus $C^P(x) - ax$ is differentiable in $(0, +\infty)$. Its first derivative is given by

$$-\frac{1}{x^2} \sum_{j \in P_x} \frac{b_j}{2d_j} (-K_j x + d_j)^2 - \frac{1}{x} \sum_{j \in P_x} \frac{K_j b_j}{d_j} (-K_j x + d_j). \tag{23}$$

Notice that (23) is smaller than or equal to zero because $x \leq \frac{d_j}{K_j}$ for every $j \in P_x$, so the proof is finished. □

Proof of Theorem 5.1.

Proof. Take $j \in N$ and $S \subsetneq T \subset N \setminus j$. We will prove that $c(T \cup j) - c(T) \leq c(S \cup j) - c(S)$. We distinguish two cases.

- If $x^*_T \geq x^*_{S \cup j}$, then

$$\begin{aligned} c(T \cup j) - c(T) &= C^{T \cup j}(x^*_{T \cup j}) - C^T(x^*_T) \leq C^{T \cup j}(x^*_T) - C^T(x^*_T) \\ &= C^j(x^*_T) - a x^*_T \end{aligned}$$

  where the first inequality follows from the fact that $x^*_{T \cup j}$ gives the minimum value of $C^{T \cup j}$ and the second line follows from Lemma A.1. If $S \neq \emptyset$, by Lemma A.1 and the fact that x^*_S gives the minimum value of C^S,

$$C^j(x^*_T) - a x^*_T \leq C^j(x^*_{S \cup j}) - a x^*_{S \cup j} = C^{S \cup j}(x^*_{S \cup j}) - C^S(x^*_{S \cup j}) \leq c(S \cup j) - C^S(x^*_S) = c(S \cup j) - c(S).$$

  If $S = \emptyset$ then, by Lemma A.1

$$C^j(x^*_T) - a x^*_T \leq C^j(x^*_{S \cup j}) - a x^*_{S \cup j} = C^j(x^*_j) - a x^*_j \leq C^j(x^*_j) = c(j) - c(\emptyset).$$

- If $x^*_T < x^*_{S \cup j}$ then following a similar reasoning as above we have

$$\begin{aligned} c(T \cup j) - c(S \cup j) &= C^{T \cup j}(x^*_{T \cup j}) - C^{S \cup j}(x^*_{S \cup j}) \leq C^{T \cup j}(x^*_{S \cup j}) - C^{S \cup j}(x^*_{S \cup j}) \\ &= C^{T \setminus S}(x^*_{S \cup j}) - a x^*_{S \cup j}. \end{aligned}$$

  If $S \neq \emptyset$, by Lemma A.1 and the fact that x^*_S gives the minimum value of C^S,

$$C^{T \setminus S}(x^*_{S \cup j}) - a x^*_{S \cup j} = C^T(x^*_{S \cup j}) - C^S(x^*_{S \cup j}) \leq C^T(x^*_T) - C^S(x^*_T) \leq c(T) - C^S(x^*_S) = c(T) - c(S).$$

  If $S = \emptyset$ then $x^*_T < x^*_{S \cup j}$ becomes $x^*_T < x^*_j$ and, by Lemma A.1

$$C^{T \setminus S}(x^*_{S \cup j}) - a x^*_{S \cup j} = C^T(x^*_j) - a x^*_j \leq C^T(x^*_T) - a x^*_T \leq c(T) = c(T) - c(\emptyset).$$

□

Proof of Theorem 5.2.

Proof. By the definition of the allocation rule R, it is clear that $\sum_{i \in N} R_i(c) = c(N)$. Take $S \subset N$. If $S \cap i(N) = \emptyset$, then

$$\sum_{i \in S} R_i(c) \leq c(S).$$

If $S \cap i(N) \neq \emptyset$, Theorem 4.1 implies that

$$\sqrt{\frac{2a + \sum\limits_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum\limits_{l \in i(S)} b_l d_l}} \geq \sqrt{\frac{2a + \sum\limits_{l \in i(N)} b_l \frac{K_l^2}{d_l}}{\sum\limits_{l \in i(N)} b_l d_l}}. \tag{24}$$

Moreover, it is clear that $i(S) \cap i(N) \subset S \cap i(N)$. In view of Algorithm 4.1 it is easy to check that $S \cap i(N) \subset i(S) \cap i(N)$. Now, using (11), (24) and the definition of $i(S)$, we have

$$\begin{aligned}
\textstyle\sum_{i \in S} R_i(c) - c(S) &= \textstyle\sum_{i \in S \cap i(N)} R_i(c) + \sum_{i \in i(S)} b_i K_i - \sum_{i \in i(S)} b_i d_i \sqrt{\frac{2a + \sum\limits_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum\limits_{l \in i(S)} b_l d_l}} \\
&= \textstyle\sum_{i \in i(S) \setminus i(N)} [b_i d_i \frac{K_i}{d_i} - b_i d_i \sqrt{\frac{2a + \sum\limits_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum\limits_{l \in i(S)} b_l d_l}}] \\
&+ \textstyle\sum_{i \in i(S) \cap i(N)} b_i d_i [\sqrt{\frac{2a + \sum\limits_{l \in i(N)} b_l \frac{K_l^2}{d_l}}{\sum\limits_{l \in i(N)} b_l d_l}} - \sqrt{\frac{2a + \sum\limits_{l \in i(S)} b_l \frac{K_l^2}{d_l}}{\sum\limits_{l \in i(S)} b_l d_l}}] \leq 0.
\end{aligned}$$

Then, $\sum_{i \in S} R_i(c) \leq c(S)$. □

References

Dror M, Hartman BC (2011) Survey of cooperative inventory games and extensions. Journal of the Operational Research Society 62, 565-580

Fiestras-Janeiro MG, García-Jurado I, Meca A, Mosquera MA (2012) Cooperative game theory and inventory management. European Journal of Operational Research 210, 459-466

Fiestras-Janeiro MG, García-Jurado I, Meca A, Mosquera MA (2013) Centralized inventory in a farming community. Submitted for publication. Available at SSRN: http://ssrn.com/abstract=2302588

Fiestras-Janeiro MG, García-Jurado I, Mosquera MA (2011) Cooperative games and cost allocation problems. Top 19, 1-22

González-Díaz J, García-Jurado I, Fiestras-Janeiro MG (2010) An introductory course on mathematical game theory. Graduate Studies in Mathematics, vol. 115. American Mathematical Society

Karimi B, Fatemi Ghomi SMT, Wilson JM (2003) The capacitated lot sizing problem: a review of models and algorithms. Omega 31, 365-378

Nagarajan M, Sošić G (2008) Game-theoretic analysis of cooperation among supply chain agents: review and extensions. European Journal of Operational Research 187, 719-745

Ng CT, Cheng TCE, Kotov V, Kovalyov MY (2009) The EOQ problem with decidable warehouse capacity: Analysis, solution approaches and applications. Discrete Applied Mathematics 157, 1806-1824

Parker RP, Kapuscinski R (2011) Managing a noncooperative supply chain with limited capacity. Operations Research 59, 866-881

Sprumont Y (1990) Population monotonic allocation schemes for cooperative games with transferable utility. Games and Economic Behavior 2, 378-394

Tersine RJ (1994) Principles of Inventory and Material Management. Prentice Hall

Zipkin PH (2000) Foundations of Inventory Management. McGraw-Hill