Computing Banzhaf-Coleman and Shapley-Shubik power indices with incompatible players

J. M. Alonso-Meijide[1], B. Casas-Méndez[2], M. G. Fiestras-Janeiro[3]

Abstract

In this paper, we present methods to compute Banzhaf-Coleman and Shapley-Shubik power indices for weighted majority games when some players are incompatible. We use the so-called generating functions as a tool.

Keywords: weighted majority games, incompatible players, Banzhaf-Coleman power index, Shapley-Shubik power index, generating functions

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[1]josemaria.alonso@usc.es, MODESTYA Research Group, Departamento de Estatística e Investigación Operativa, and Facultade de Ciencias, Universidade de Santiago de Compostela, Spain

[2]balbina.casas.mendez@usc.es, MODESTYA Research Group, Departamento de Estatística e Investigación Operativa, and Facultade de Matemáticas, Universidade de Santiago de Compostela, Spain

[3]fiestras@uvigo.es, SIDOR Research Group, Departamento de Estatística e Investigación Operativa, and Facultade de Económicas, Universidade de Vigo, Spain

1 Introduction

An important family of the so-called cooperative games with transferable utility is the formed by simple games, which have numerous applications, especially in the field of political science. Instead of speaking of value as in general cooperative games, when working with simple games, we usually use the term power index because simple games are normally used as model of organisms in which decisions are determined by voting agreements. The interest of these games usually focus on knowing the power or influence a player has on the final outcome. The well-known Shapley value (Shapley, 1953) and Banzhaf value (Banzhaf, 1965) are called in the context of simple games Shapley-Shubik power index (Shapley and Shubik, 1954) and Banzhaf-Coleman power index (Banzhaf, 1965, and Coleman, 1971), respectively. For the interested reader, there are some applications and specific studies about simple games in Ishikawa (2009) and Kojima and Inohar (2010), among others.

Shapley-Shubik and Banzhaf-Coleman power indices can be obtained using different tools. Two of the most commonly used are the multilinear extension and the generating function. The latter, mainly used in the case of so-called weighted majority games, are based on the use of a combinatorial analysis technique. Roughly speaking, a generating function is a polynomial that allows listing all the possible coalitions, tracked by their respective weights. This is very useful because we can get the exact value of the indices even in games with many players. This technique was used by David G. Cantor (1962) (it appears in Lucas, 1983) to calculate the Shapley-Shubik power index and by Brams and Afuso (1976) for the Banzhaf-Coleman power index. Bilbao et al. (2000) compute and apply the Shapley-Shubik and the Banzhaf-Coleman power indices for weighted majority games and study the time complexity of the corresponding algorithms. Alonso-Meijide et al. (2012) propose methods based in generating functions to compute the Deegan-Packel (Deegan and Packel, 1979), the Public Good (Holler, 1982), and the Shift (Alonso-Meijide and Freixas, 2010) power indices. Chessa (2014) provides a new method also based in generating functions to compute the Public Good power index.

In the classical theory of cooperative games, it is assumed that all players can communicate freely. In other situations, the players act non-cooperatively, but intermediate situations are also possible, where communication is restricted. Myerson (1977), Owen (1986), and Fernández et al. (2002) use graphs to model and study situations in which there are affinities between players.

In this paper, we consider situations where some players are incompatible, that is, some players cannot cooperate among them by ideological or economical reasons. The incompatibilities among players are modelled by a graph in such a way that if two players are linked by an arc of the graph, these two players are incompatible. These situations are studied in Carreras (1991), Carreras and Owen (1996), Yakuba (2008), Bergantiños et al. (1993), and Alonso-Meijide et al. (2009), among others. In the last two papers, modifications of the Shapley and Banzhaf values, respectively, for these situations are defined and characterized. At this stage, in the case of weighted majority games, we face the task of obtaining the Shapley-Shubik and Banzhaf-Coleman power indices when some players are incompatible, using generating functions. Besides, we study the complexity of the proposed methods. The first part of both algorithms requires to find the maximal clique of a graph (cf. Akkoyunlu, 1973, Bron and Kerbosch, 1973, Douglas, 2001, and Bahls et al., 2013). Finally, we apply both power indices to an example taken from the real world.

2 Preliminaries

2.1 Simple games

A cooperative game with transferable utility (or TU game) is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and v, the characteristic function, is a real function on $2^N = \{S \mid S \subseteq N\}$ with $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a coalition.

A simple game is a pair (N, W) where W is a family of coalitions of N satisfying $N \in W$, $\emptyset \notin W$, and the monotonicity property, *i.e.*, $S \subseteq T \subseteq N$ and $S \in W$ implies $T \in W$. (N, W) is proper if for every $S, T \in W$, $S \cap T \neq \emptyset$. Intuitively, N is the set of members of a committee and W is the set of coalitions that fully control the involved decision problem.

In a simple game (N, W), a coalition $S \subseteq N$ is winning if $S \in W$ and it is losing if $S \notin W$. We denote by $SI(N)$ the set of proper simple games with player set N.

A winning coalition $S \in W$ is a minimal winning coalition (MWC) if every proper subset of S is a losing coalition, that is, S is an MWC in (N, W) if $S \in W$ and $T \notin W$ for any $T \subset S$. We denote by $M(W)$ the set of MWC of the simple game (N, W).

Given a simple game (N, W), a swing for a player $i \in N$ is a coalition $S \subseteq N \backslash i$ such that $S \notin W$, and $S \cup i \in W$. We denote by $S(i)$ the set of swings for player $i \in N$ and by $\eta_i(W)$ the number of swings for player $i \in N$.

It has become common practice to associate a TU game (N, v) to every simple game (N, W). In the following, we identify every simple game with its associated TU game. A simple game could be defined as a TU game (N, v) such that $v(S) = 1$ if $S \in W$ and $v(S) = 0$ otherwise, for every $S \subseteq N$.

A game $(N, W) \in SI(N)$ is said to be a weighted majority game if there exists a set of weights $w_1, \ldots, w_n$ for the players, with $w_i \in \mathbb{N} \cup \{0\}$, $1 \leq i \leq n$, and a quota $q \in \mathbb{N}$ $(q > 0)$ such that $S \in W$ if and only if $w(S) = \sum_{i \in S} w_i \geq q$. The amount q is called majority of the game. Typically, a weighted majority game is represented by $[q; w_1, \ldots, w_n]$. A parliament can be seen as a weighted majority game, in which the players are the political parties, the weights are the number of seats available to each party, and the majority of the game coincides with the minimum number of votes needed to win a voting. If the criterion is the most simple, then $q = E(t) + 1$, where t is the half of the total number of seats in the parliament[1]. In general, $q \geq E(t) + 1$ and this implies that the resulting simple game is always proper. Other examples of weighted majority games are the Security Council of the United Nations or the Council of the European Union.

[1] If $x \in \mathbb{R}$, $E(x)$ denotes the integer part of x.

2.2 Power indices

Given a family of games $H \subseteq SI(N)$, a power index on H is a function f, which assigns to a simple game $(N, W) \in H$ a vector

$$(f_1(N, W), \ldots, f_n(N, W)) \in \mathbb{R}^n,$$

where the real number $f_i(N, W)$ is the "power" of the player i in the game (N, W) according to f. The power index of a simple game can be interpreted as a measure of the ability of the different players to turn a losing coalition into a winning one.

Some important power indices for $SI(N)$ that we can find in the literature are the Shapley-Shubik (Shapley and Shubik, 1954) and the Banzhaf-Coleman (Banzhaf, 1965 and Coleman, 1971) indices. The Shapley-Shubik index assigns to each player $i \in N$ the real number:

$$\varphi_i(N, W) = \sum_{S \in S(i)} \frac{s!\,(n-s-1)!}{n!},$$

where $s = |S|$. The Banzhaf-Coleman index assigns to each player $i \in N$ the real number:

$$\beta_i(N, W) = \frac{\eta_i(W)}{2^{n-1}}.$$

2.3 Generating functions

Each sequence of real numbers $a = \{a_0, a_1, a_2, \ldots\}$ could be held to correspond with the power series

$$f_a(t) = \sum_{j=0}^{\infty} a_j t^j.$$

This series, $f_a(t)$, is called the generating function of the sequence a, and may be finite or infinite. Note that in this series, the variable t has no proper meaning and it only serves to identify a_j as the coefficient corresponding to t^j in developing $f_a(t)$. For example, consider the finite product or linear binomials $\prod_{r=1}^{n}(1+\alpha_r t) = \sum_{r=0}^{n} a_r t^r$, with $\alpha_r \in \mathbb{R}$, where $a_0 = 1$ and for $r > 0$, a_r is given by

$$a_r = \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_r}.$$

If all values α_r are equal to 1, we will have that $(1+t)^n = \sum_{r=0}^{n} \binom{n}{r} t^r$. Therefore, the function $f(t) = (1+t)^n$ is the generating function of the sequence $a = \{\binom{n}{r} \,|\, r = 0, 1, \ldots, n\}$. The generating functions provide a method for counting the number of elements $c(r)$ from a finite set, when these elements have a configuration depending on a property r. Levitin (2002 and 2003) apply the universal generating function in weighted voting systems. Levitin (2005) gives a comprehensive description of generating functions including applications in several fields.

The following proposition determines the number of swings of player i in a weighted majority game.

Proposition 1 *Let (N, W) be a weighted majority game given by $[q; w_1, \ldots, w_n]$. Then, the number of swings of player $i \in N$ is equal to*

$$\eta_i(W) = \sum_{k=q-w_i}^{q-1} b_k^i,$$

being b_k^i the number of coalitions $S \subseteq N \backslash i$ such that $w(S) = \sum_{i \in S} w_i = k$.

The following result due to Brams and Affuso (1976) provides a generating function that allows the calculation of the numbers $\{b_k^i\}_{k \geq 0}$ and, then, the calculation of the Banzhaf-Coleman index.

Proposition 2 *Let (N, W) be a weighted majority game given by $[q; w_1, \ldots, w_n]$. For a player $i \in N$, the generating function of the numbers $\{b_k^i\}_{k \geq 0}$ defined above, is given by*

$$\prod_{j=1, j \neq i}^{n} (1 + x^{w_j}).$$

The allocation of the Shapley-Shubik index of each player $i \in N$ in a weighted majority game can be expressed as:

$$\varphi_i(N, W) = \sum_{s=0}^{n-1} \frac{s!\,(n-s-1)!}{n!} d_s^i(W),$$

where $d_s^i(W)$ represents the number of swings for the player i in coalitions of size s. We have that for any value of s between 0 and $n-1$,

$$d_s^i(W) = \sum_{k=q-w_i}^{q-1} a_{ks}^i,$$

where a_{ks}^i is the number of coalitions $S \subseteq N$ formed by s players such that $i \notin S$ and $w(S) = k$. The following result (Lucas, 1983) provides a generating function that allows the calculation of the numbers $\{a_{ks}^i\}_{k \geq 0, s \geq 0}$ and, then, the calculation of the Shapley-Shubik index.

Proposition 3 *Let (N, W) be a weighted majority game given by $[q; w_1, \ldots, w_n]$. For a player $i \in N$, the generating function of the numbers $\{a_{ks}^i\}_{k \geq 0, s \geq 0}$ defined above, is given by*

$$\prod_{j=1, j \neq i}^{n} (1 + x^{w_j} z).$$

3 Situations with incompatibilities

Let $N = \{1, \ldots, n\}$ be a finite set. An undirected graph without loops on N is a set of unordered pairs, denoted $(i : j)$, of different elements. These pairs are called arcs. (Note that $(i : j) = (j : i)$).

We denote by g^N the complete graph on N and by $GR(N)$ the set of undirected graphs on N, that is:

$$g^N = \{(i : j) \mid i \in N, j \in N, i \neq j\} \text{ and } GR(N) = \{g \mid g \subseteq g^N\}.$$

Let us take $g \in GR(N)$. Given a pair of players $i, j \in N$, i and j are incompatible if $(i : j) \in g$. Two players i and j are incompatible if they cannot cooperate at all between them. A coalition S is a subset of compatible players if and only if $(i : j) \notin g$ for all $i, j \in S$. The case $g = \emptyset$, represents the situation in which no incompatible players exist. We denote by $\mathcal{C}$ the family of all maximal subsets of compatible players. For each $i \in N$, we define the set of player i's compatible players, $C(i)$, as follows: $C(i) = \{j \in N \mid (i : j) \notin g\}$. For any coalition $S \subseteq N$, we denote by $P(S, g)$ the set of all partitions of S whose classes are subsets of compatible players.

Proposition 4 *If $g \in GR(N)$ is a graph of incompatibilities, the family of all maximal sets of compatible players, $\mathcal{C}$, is given by*

$$S \in \mathcal{C} \text{ if and only if } \cap_{j \in S} C(j) = S.$$

Proof. Let $S \in \mathcal{C}$ be, then we have that

$$i \in S \Leftrightarrow i \in C(j), \text{ for all } j \in S \Leftrightarrow i \in \cap_{j \in S} C(j)$$

and the result follows. □

A situation with incompatibilities is a triplet (N, v, g) where (N, v) is a TU game and $g \in GR(N)$. In Bergantiños et al. (1993), a rule is introduced, which selects a payoff for every possible situation with incompatibilities, and they prove that it is uniquely determined. They define the g-restriction of a game in a situation with incompatibilities as follows.

Definition 5 *Given a situation with incompatibilities (N, v, g), the g-restriction of v is the TU-game (N, v^g) given by:*

$$v^g(S) = \max_{P \in P(S,g)} \sum_{U \in P} v(U) \text{ for every } S \subseteq N.$$

We find worth mentioning that in this setup the term incompatible is understood as incompatibility for making collective proposals to pass but not incompatibility for making collective proposals to fail. This is different to the total incompatibility, when it is understood as incompatibility in forming a coalition for passing proposals but also incompatibility for making proposals to fail.

Bergantiños et al. (1993) extend the Shapley value to situations with incompatibilities. In a similar way, Alonso-Meijide et al. (2009) extend the Banzhaf value to situations with incompatibilities. The Shapley value for situations with incompatibilities assigns to each situation (N, v, g) the Shapley value of (N, v^g), the g-restriction of v. The Banzhaf value for situations with incompatibilities assigns to each situation (N, v, g) the Banzhaf value of (N, v^g).

The following proposition explores the structure of the g-restriction of a proper simple game.

Proposition 6 *Let (N, v, g) be a situation with incompatibilities where (N, v) is a proper simple game. Let W be its family of winning coalitions and $M(W)$ its family of minimal winning coalitions. If for every $S \in M(W)$ there are some $i_S, j_S \in S$ such that $(i_S : j_S) \in g$, then the $g-$restriction game, (N, v^g), is the null game*[2]*. Otherwise, the g-restriction game (N, v^g) is a simple game with the minimal winning coalition family given by* $M(W^g) = \{S \in M(W) | \ (i : j) \notin g, \text{ for all } i, j \in S\}$.

Proof. First, assume that for every $S \in M(W)$ there are $i_S, j_S \in S$ such that $(i_S : j_S) \in g$. Take $R \subseteq N$. Then, for every $P \in P(R, g)$ and $T \in P$ there is no $S \in M(W)$ such that $S \subseteq T$. Then, $v(T) = 0$ and $v^g(R) = 0$.

Second, assume that the set $A = \{S \in M(W) | \ (i : j) \notin g, \ \forall \, i, j \in S\}$ is nonempty. It is clear that for every $S \in A$, we have $v^g(S) = 1$. Take $R \subseteq N$. Since the simple game (N, v) is proper, for every $S_1, S_2 \in A$, $S_1 \cap S_2 \neq \emptyset$ holds. Thus, for every $P \in P(R, g)$ there is at most a coalition $T \in P$ with $v(T) = 1$. Then, in case there is some $S \in A$ with $S \subseteq R$, by the definition of v^g and the properness property of (N, v), we have $v^g(R) = 1$; otherwise, $v^g(R) = 0$. Now we check that $v^g(R_1) \leq v^g(R_2)$ for every $R_1, \ R_2 \subseteq N$ with $R_1 \subseteq R_2$. If $v^g(R_1) = 0$, it is clear that $v^g(R_1) = 0 \leq v^g(R_2) \in \{0, 1\}$. If $v^g(R_1) = 1$, then there is some $S \in A$ with $S \subseteq R_1 \subseteq R_2$ and we prove that $v^g(R_2) = 1$. As a consequence of all this, we prove that (N, v^g) is a simple game.

Now we characterize the family of minimal winning coalitions of (N, v^g). We denote by W^g the family of winning coalitions of (N, v^g). We have $A \subseteq W^g$. Take $S \in A$. For every $i \in S$, we have $v(S \setminus i) = v^g(S \setminus i) = 0$ and then $A \subseteq M(W^g)$. Take $R \in M(W^g)$. Then, $v^g(R) = 1$ and for all $S \subset R$ we have $v^g(S) = 0$. Then, there is some $P \in P(R, g)$ and $T \in P$ with $v(T) = 1$. Notice that $T \in W$, $T \subseteq R$ and there are no incompatible players in T. Then, there is some $S \in A$ such that $S \subseteq T \subseteq R$. Finally,

[2]Note that in this case the game (N, v^g) is not a simple game.

$S = R$ because $S, R \in M(W^g)$. $\square$

Next examples illustrate the result above.

Example 7 *Take the weigthed majority game* (N, W) *given by* $[5; 3, 2, 2, 1, 1]$. *Consider the graph* $g = \{(1 : 2), (3 : 4)\}$ *as the one which describes the incompatibilities among the players in* N. *The set of minimal winning coalitions is given by*

$$M(W) = \{\{1, 2\},\ \{1, 3\},\ \{1, 4, 5\},\ \{2, 3, 4\},\ \{2, 3, 5\}\}.$$

Taking into account the graph of incompatibilities, the g*-restriction game is the simple game* (N, W^g) *with the following minimal winning coalition set*

$$M(W^g) = \{\{1, 3\},\ \{1, 4, 5\},\ \{2, 3, 5\}\},$$

as it follows from Proposition 6. Notice that this game is not a weigthed majority game. For any system of weights, we must have $w_1 > w_3$ *because coalition* $\{1, 4, 5\}$ *is winning but coalition* $\{3, 4, 5\}$ *is losing. Then, coalition* $\{1, 2, 5\}$ *should be also winning, but this is not the case.*

Example 8 *Take the weigthed majority game* (N, W) *given in Example 7. Consider the graph* $\tilde{g} = \{(1 : 2), (3 : 4), (1 : 3), (1 : 4), (2 : 4), (3 : 5)\}$ *as the one which describes the incompatibilities among the players in* N. *Since the set of minimal winning coalitions in* (M, W) *is given by*

$$M(W) = \{\{1, 2\},\ \{1, 3\},\ \{1, 4, 5\},\ \{2, 3, 4\},\ \{2, 3, 5\}\},$$

if we take into account the graph of incompatibilities $\tilde{g}$, *the* $\tilde{g}$*-restriction game is the null game.*

As we can see in Example 7, the $g-$restriction of a weighted majority game (N, W), despite being a simple game, can not be a weighted majority game. In such situation we can not apply the procedure of generating functions directly in order to compute its Shapley-Shubik and Banzhaf-Coleman power indices. We therefore propose a new method, also based on generating functions, which solves this difficulty.

In what follows, we assume that if (N, v, g) is a situation with incompatibilities then the g-restriction game (N, v^g) is a simple game.

4 Generating functions and situations with incompatibilities

In this section, we consider weighted majority games with incompatibilities and we use generating functions to compute the Banzhaf-Coleman and Shapley-Shubik indices. We also provide some results concerning the complexity and an illustrative numerical example.

4.1 The Banzhaf-Coleman index with incompatibilities

Proposition 9 *Let (N,W) be the weigthed majority game given by $[q;w_1,\dots,w_n]$ and $g\in GR(N)$ the graph describing the incompatibilities among the players. Let (N,W^g) be the g-restriction of (N,W) and $\mathcal{C}=\{C_1,\dots,C_k\}$ the family of maximal subsets of compatible players. Then, for each $i\in N$,*

i) the number of swings of player i is given by

$$\eta_i(N,W^g)=$$
$$\sum_{l\in K_i}\sum_{r_l=q-w_i}^{q-1}\sum_{j\in K\setminus\{K_i^-(l)\cup\{l\}\}}\sum_{r_j=0}^{q-1}\sum_{m\in K_i^-(l)}\sum_{r_m=0}^{q-w_i-1}A^i(r_1,\dots,r_k),$$

where $K=\{1,\dots,k\}$, $K_i=\{l\in K\mid i\in C_l\}$, $K_i^-(l)=\{m\in K_i\mid m<l\}$, for every $l\in K_i$, and $A^i(r_1,\dots,r_k)$ is the number of coalitions $S\subseteq N\setminus i$ such that $w(S\cap C_l)=\sum_{j\in S\cap C_l}w_j=r_l$, for every $l\in K$.

ii) The generating functions for the numbers $\{A^i(r_1,\dots,r_k)\}_{r_1,\dots,r_k\geq 0}$ are given by

$$BC_i(t_1,\dots,t_k)=\Pi_{j=1,j\neq i}^n(1+\Pi_{l\in K_j}t_l^{w_j}).$$

Proof.

i) Take $i\in N$ and $S\subseteq N\setminus i$ such that $S\notin W^g$. This implies that $w(S\cap C_j)\leq q-1$, for every $j\in K$. If S is a swing for player i, we have $S\cup i\in W^g$. Then, there is $l\in K_i$ with $w((S\cup i)\cap C_l)\geq q$. Then, $q-w_i\leq w(S\cap C_l)\leq q-1$. Take $l\in K_i$ with $l=\min\{r\in K_i\,|\,q-w_i\leq w(S\cap C_r)\leq q-1\}$. Thus, $w(S\cap C_m)\leq q-w_i-1$, for every $m\in K_i^-(l)$. Then, the result directly follows.

ii) Consider the function

$$\begin{aligned}BC(t_1,\dots,t_k)&=\prod_{j=1}^n(1+\prod_{l\in K_j}t_l^{w_j})\\&=1+\sum_{\emptyset\neq S\subseteq N}\prod_{j\in S}\prod_{l\in K_j}t_l^{w_j}=\sum_{S\subseteq N}\prod_{l\in K}t_l^{w(S\cap C_l)}\\&=\sum_{r_1=0}^{w(C_1)}\dots\sum_{r_k=0}^{w(C_k)}A(r_1,\dots,r_k)t_1^{r_1}\dots t_k^{r_k}\\&=\sum_{1\leq l\leq k}\sum_{r_l=0}^{w(C_l)}A(r_1,\dots,r_k)t_1^{r_1}\dots t_k^{r_k}.\end{aligned}$$

The function $BC(t_1,\dots,t_k)$ is the generating function of the numbers $A(r_1,\dots,r_k)$ where each $A(r_1,\dots,r_k)$ is the number of coalitions $S\subseteq N$ with $w(S\cap C_l)=r_l$, for every $l\in K$. It is clear that the numbers $\{A^i(r_1,\ldots,r_k)\}_{r_1,\ldots,r_k\geq 0}$ are given by deletion of the factor $(1+\prod_{l\in K_i} t_l^{w_i})$ in the function $BC(t_1,\ldots,t_k)$. $\square$

Proposition 10 *Take the weigthed majority game given by $[q; w_1,\ldots,w_n]$. Take the graph g as the one describing the incompatibilities among the players in N. Then*

1. *The number c of terms of:*

$$BC(t_1,\ldots,t_k)=\Pi_{j=1}^{n}(1+\Pi_{l\in K_j} t_l^{w_j})$$

   satisfies that $n+1\leq c\leq \min\{2^n, \prod_{l=1}^{k}(w(C_l)+1)\}$.

2. *The number of terms of $BC_i(t_1,\ldots,t_k)$, for every $i\in N$, is bounded by c.*

Proof.

1. A lower bound of c is obtained in the case in which the weights of all players are equal and there are no incompatibilities, that is $w_1=\ldots=w_n$ and $g=\emptyset$. In this case, we have to consider the number of terms of $(1+t^{w_j})^n$, that is, $n+1$. On the other hand, we have that:

$$BC(t_1,\ldots,t_k)=\sum_{r_1=0}^{w(C_1)}\cdots\sum_{r_k=0}^{w(C_k)} A(r_1,\ldots,r_k)t_1^{r_1}\ldots t_k^{r_k}$$

   is a polynomial of degree $w(C_l)$ in t_l, $1\leq l\leq k$. Therefore, $c\leq \prod_{l=1}^{k}(w(C_l)+1)$. Moreover, at worst, in case all exponents of the terms of $BC(t_1,\ldots,t_k)$ are different, the number c coincides with the number of subsets of N, 2^n.

2. It is straightforward by part 1. $\square$

The time complexity function $f:\mathbb{N}\to\mathbb{N}$ of an algorithm give us the maximum time $f(n)$ needed to solve any problem instance of encoding length at most $n\in\mathbb{N}$. A function $f(n)$ is $O(g(n))$ if there is a constant k such that $|f(n)|\leq k|g(n)|$ for all integers $n\in\mathbb{N}$. We analyze our algorithms in the arithmetic model, that is, we count elementary arithmetic operations and assignments. For instance, the algorithm for computing the product of two $n\times n$ matrices is $O(n^3)$. We left the proofs to the readers.

Proposition 11 *Take the weigthed majority game given by $[q; w_1,\ldots,w_n]$. Take the graph g which describes the incompatibilities among the players in N. Then*

1. *To expand the polynomial* $BC(t_1,\dots,t_k)$*, a time* $O(nC)$ *is required where* $C=\min\{2^n,\prod_{l=1}^{k}(w(C_l)+1)\}$.

2. *For each* $i\in N$*, to expand the polynomial* $BC_i(t_1,\dots,t_k)$*, a time* $O(nC)$ *is required.*

4.2 The Shapley-Shubik index with incompatibilities

Proposition 12 *Let* (N,W) *be the weigthed majority game given by* $[q;w_1,\dots,w_n]$ *and* $g\in GR(N)$ *a graph describing the incompatibilities among the players. Let* (N,W^g) *be the g-restriction of* (N,W) *and* $\mathcal{C}=\{C_1,\dots,C_k\}$ *the family of maximal subsets of compatible players. Then, for each* $i\in N$*,*

i) the number of swings of player i in coalitions of size s is given by

$$d_s^i(W^g)=$$
$$\sum_{l\in K_i}\sum_{r_l=q-w_i}^{q-1}\sum_{j\in K\setminus\{K_i^-(l)\cup\{l\}\}}\sum_{r_j=0}^{q-1}\sum_{m\in K_i^-(l)}\sum_{r_m=0}^{q-w_i-1}A_s^i(r_1,\dots,r_k),$$

where $A_s^i(r_1,\dots,r_k)$ *is the number of coalitions* $S\subseteq N\backslash i$*, with cardinal* s*, such that* $w(S\cap C_l)=r_l$*, for every* $l\in K$*.*

ii) The generating functions for the numbers $\{A_s^i(r_1,\dots,r_k)\}_{r_1,\dots,r_k,s\geq 0}$ *are given by*

$$SS_i(t_1,\dots,t_k,z)=\Pi_{j=1,j\neq i}^{n}(1+z\Pi_{l\in K_j}t_l^{w_j}).$$

Proof.

i) This proof follows similarly as in Proposition 9.

ii) Consider the function

$$\begin{aligned}SS(t_1,\dots,t_k,z)&=\prod_{j=1}^{n}(1+z\prod_{l\in K_j}t_l^{w_j})\\&=1+\sum_{\emptyset\neq S\subseteq N}z^{|S|}\prod_{j\in S}\prod_{l\in K_j}t_l^{w_j}=\sum_{S\subseteq N}z^{|S|}\prod_{l\in K}t_l^{w(S\cap C_l)}\\&=\sum_{r_1=0}^{w(C_1)}\dots\sum_{r_k=0}^{w(C_k)}\sum_{s=0}^{n}A_s(r_1,\dots,r_k)z^st_1^{r_1}\dots t_k^{r_k}\\&=\sum_{1\leq l\leq k}\sum_{r_l=0}^{w(C_l)}\sum_{s=0}^{n}A_s(r_1,\dots,r_k)z^st_1^{r_1}\dots t_k^{r_k}.\end{aligned}$$

The function $SS(t_1,\dots,t_k,z)$ is the generating function of the numbers $A_s(r_1,\dots,r_k)$ where each $A_s(r_1,\dots,r_k)$ is the number of coalitions $S\subseteq N$ with $|S|=s$ and $w(S\cap C_l)=r_l$, for every $l\in K$. It is clear that the numbers $\{A_s^i(r_1,\dots,r_k)\}_{r_1,\dots,r_k,s\geq 0}$ are given by deletion of the factor $(1+z\prod_{l\in K_i}t_l^{w_i})$ in the function $SS(t_1,\dots,t_k,z)$. $\square$

Proposition 13 *Take the weigthed majority game given by* $[q; w_1, \ldots, w_n]$. *Take the graph* g *that describes the incompatibilities among the players in* N. *Then*

1. *The number* c *of terms of:*

$$SS(t_1, \ldots, t_k, z) = \Pi_{j=1}^{n}(1 + z\Pi_{l\in K_j} t_l^{w_j})$$

satisfies that $n + 1 \leq c \leq \min\{2^n, (n + 1)\prod_{l=1}^{k}(w(C_l) + 1)\}$.

2. *The number of terms of* $SS_i(t_1, \ldots, t_k, z)$, *for every* $i \in N$, *is bounded by* c.

Proof.

1. A lower bound of c is obtained in the case in which the weights of all players are equal, that is $w_1 = \ldots = w_n$, and there are no incompatibilities. In this case, we have to consider the number of terms of $(1 + zt^{w_j})^n$, that is, $n + 1$. On the other hand, we have that:

$$SS(t_1, \ldots, t_k, z) = \sum_{r_1=0}^{w(C_1)} \cdots \sum_{r_k=0}^{w(C_k)} \sum_{s=0}^{n} A_s(r_1, \ldots, r_k) z^s t_1^{r_1} \ldots t_k^{r_k}$$

is a polynomial of degree $w(C_l)$ in t_l, $1 \leq l \leq k$, and degree n in z. Therefore, $c \leq (n + 1)\prod_{l=1}^{k}(w(C_l) + 1)$. Moreover, at worst, in case all exponents of the terms of $SS(t_1, \ldots, t_k, z)$ are different, the number c coincides with the number of subsets of N, 2^n.

2. It is straightforward by part 1. □

Proposition 14 *Take the weigthed majority game given by* $[q; w_1, \ldots, w_n]$. *Take the graph* g *that describes the incompatibilities among the players in* N. *Then*

1. *To expand the polynomial* $SS(t_1, \ldots, t_k, z)$, *a time* $O(nC)$ *is required where*

$$C = \min\{2^n, (n + 1)\prod_{l=1}^{k}(w(C_l) + 1)\}.$$

2. *For each* $i \in N$, *to expand the polynomial* $SS_i(t_1, \ldots, t_k, z)$, *a time* $O(nC)$ *is required.*

Remark 15 *In Proposition 9 and Proposition 12, we need $\mathcal{C}$, the family of maximal subsets of compatible players related with the weighted majority game with incompatibilities (N, W, g). That involves to finding the maximal cliques of the so-called dual graph of the graph g. In the literature, there are efficient algorithms to do this (cf. Akkoyunlu, 1973, Bron and Kerbosch, 1973, and Tomita et al., 2011).*

4.3 The numerical example

Example 16 *Take again the weigthed majority game (N, W) and the graph g analyzed in Example 7. We will use the results of Proposition 9 and Proposition 12 to obtain the Banzhaf-Coleman and Shapley-Shubik power indices of the situation with incompatibilities (N, W, g).*

In Table 1 we show, for each player, his set of compatible players.

Player	$C(i)$
1	$\{1,3,4,5\}$
2	$\{2,3,4,5\}$
3	$\{1,2,3,5\}$
4	$\{1,2,4,5\}$
5	$\{1,2,3,4,5\}$

Table 1: Compatible players.

The family of maximal subsets of compatible players is given by

$$\mathcal{C} = \{\{1,3,5\},\ \{1,4,5\},\ \{2,3,5\},\ \{2,4,5\}\}.$$

We take player 5 to illustrate the computation of the number of swings. Once we associate the variable t_r to the coalition $C_r \in \mathcal{C}$, for every $r = 1, \ldots, 4$, we compute the function $BC_5(t_1, t_2, t_3, t_4)$:

$$\begin{aligned}
&(1 + t_1^{w_1} t_2^{w_1})(1 + t_3^{w_2} t_4^{w_2})(1 + t_1^{w_3} t_3^{w_3})(1 + t_2^{w_4} t_4^{w_4}) &= \\
&1 + t_2^{w_4} t_4^{w_4} + t_1^{w_3} t_3^{w_3} + t_1^{w_3} t_2^{w_4} t_3^{w_3} t_4^{w_4} + t_3^{w_2} t_4^{w_2} + t_2^{w_4} t_3^{w_2} t_4^{w_2+w_4} &+ \\
&t_1^{w_3} t_3^{w_2+w_3} t_4^{w_2} + t_1^{w_3} t_2^{w_4} t_3^{w_2+w_3} t_4^{w_2+w_4} + t_1^{w_1} t_2^{w_1} + t_1^{w_1} t_2^{w_1+w_4} t_4^{w_4} &+ \\
&t_1^{w_1+w_3} t_2^{w_1} t_3^{w_3} + t_1^{w_1+w_3} t_2^{w_1+w_4} t_3^{w_3} t_4^{w_4} + t_1^{w_1} t_2^{w_1} t_3^{w_2} t_4^{w_2} &+ \\
&t_1^{w_1} t_2^{w_1+w_4} t_3^{w_2} t_4^{w_2+w_4} + t_1^{w_1+w_3} t_2^{w_1} t_3^{w_2+w_3} t_4^{w_2} + t_1^{w_1+w_3} t_2^{w_2+w_4} t_3^{w_2+w_3} t_4^{w_2+w_4}. &
\end{aligned}$$

In the simple game (N, W^g), player 5 is pivot for coalitions

$$\{2,3\},\ \{2,3,4\},\ \{1,4\},\ \{1,2,4\}^3.$$

[3]Note that the g-restriction of the game associated with the graph of incompatibilities is different to the cooperation game associated with the so-called dual graph according to Myerson (1977).

Each one of these coalitions corresponds with one of the following terms

$$t_1^{w_3} t_3^{w_2+w_3} t_4^{w_2}, \; t_1^{w_3} t_2^{w_4} t_3^{w_2+w_3} t_4^{w_2+w_4}, \; t_1^{w_1} t_2^{w_1+w_4} t_4^{w_4}, \; t_1^{w_1} t_2^{w_1+w_4} t_3^{w_2} t_4^{w_2+w_4}.$$

Each one of these terms has one of the exponents (note that player 5 *belongs to all the maximal subsets of compatible players) with value* 4 $(q-1 = q-w_5 = 4)$ *and the others are* 0, 1, 2, or 3 $(q-1=4)$.

In Table 2, we depict the Banzhaf-Coleman and Shapley-Shubik power indices of the simple game (N, W) *and the Banzhaf-Coleman and Shapley-Shubik power indices of the situation with incompatibilities* (N, W, g).

Player	Banzhaf–Coleman	Shapley–Shubik	Banzhaf – Coleman & incompatibility	Shapley – Shubik & incompatibility
1	0.625	0.4000	0.500	0.3333
2	0.375	0.2333	0.125	0.0833
3	0.375	0.2333	0.500	0.3333
4	0.125	0.0667	0.125	0.0833
5	0.125	0.0667	0.250	0.1667

Table 2: Some power indices in the numerical example.

5 An example taken from the real world

In this section, we compute the Banzhaf-Coleman and Shapley-Shubik power indices to analyze the Parliament of Catalonia, an autonomous region in eastern Spain, which has been arisen from the election held on November 25^{th}, 2012. Following these elections, the Parliament was composed of:

1. 50 members of CIU, *Convergència i Unió*,
2. 21 members of ERC, *Esquerra Republicana de Catalunya*,
3. 20 members of PSC, *Partit dels Socialistes de Catalunya*,
4. 19 members of PPC, *Partit Popular*,
5. 13 members of $ICV-EUiA$, *Iniciativa por Catalunya-Verds-Esquerra Unida i Alternativa*,
6. 9 members of $C's$, *Ciudadanos-Partidos de la Ciudadanía*, and
7. 3 members of CUP, *Candidatura d'Unitat Popular-Alternativa d'Esquerres.*

This Parliament can be represented as the following weighted majority game:

$$[68; 50, 21, 20, 19, 13, 9, 3].$$

For the sake of clarity, we identify CIU as player 1, ERC as player 2, PSC as player 3, PPC as player 4, $ICV-EUiA$ as player 5, $C's$ as player 6, and CUP as player 7. Then, taking $N = \{1,2,3,4,5,6,7\}$, the corresponding minimal winning coalitions are:

$$M(W) = \{\{1,2\},\{1,3\},\{1,4\},\{1,5,6\},\{2,3,4,5\},\{2,3,4,6\}\}.$$

We see that CUP party is a null player, ERC, PSC, and PPC are symmetric players, and $ICV-EUiA$, and $C's$ are symmetric players, too.

Further, in order to provide a more accurate representation of the parliament, we introduce an incompatibility graph in the model. Taking into account the behavior of the parties in previous legislatures, we consider the next incompatibility graph:

$$g = \{(1:6),(1:7),(2:4),(2:6),(3:7),(4:5),(4:7),(5:6),(6:7)\}.$$

The associated g-restriction game is the simple game with the following minimal winning coalition set

$$M(W^g) = \{\{1,2\},\ \{1,3\},\ \{1,4\}\}.$$

In Table 3 we show, for each player, his set of compatible players.

Player	$C(i)$
1	$\{1,2,3,4,5\}$
2	$\{1,2,3,5,7\}$
3	$\{1,2,3,4,5,6\}$
4	$\{1,3,4,6\}$
5	$\{1,2,3,5,7\}$
6	$\{3,4,6\}$
7	$\{2,5,7\}$

Table 3: Compatible players.

The family of maximal subsets of compatible players is given by

$$\mathcal{C} = \{\{1,2,3,5\},\ \{1,3,4\},\ \{2,5,7\},\ \{3,4,6\}\}.$$

We take player 2 to illustrate the computation of the number of swings. Once we associate the variable t_r to the coalition $C_r \in \mathcal{C}$, for every $r = 1,\ldots,4$, we compute the function $BC_2(t_1,t_2,t_3,t_4)$:

$$(1+t_1^{w_1}t_2^{w_1})(1+t_1^{w_3}t_2^{w_3}t_4^{w_3})(1+t_2^{w_4}t_4^{w_4})(1+t_1^{w_5}t_3^{w_5})(1+t_4^{w_6})(1+t_3^{w_7}).$$

In the g-restriction game, player 2 is pivot for coalitions

$$\{1\},\ \{1,5\},\ \{1,6\},\ \{1,7\},\ \{1,5,6\},\ \{1,5,7\},\ \{1,6,7\},\ \{1,5,6,7\}.$$

Coalition $\{1\}$ corresponds with the term $t_1^{w_1} t_2^{w_1}$ because intersecting $\{1\}$ with the coalitions of $\mathcal{C}$ we obtain $\{1\}$, $\{1\}$, $\{\emptyset\}$, $\{\emptyset\}$, respectively. By reasoning in a similar way, we obtain the remaining swings by considering the following terms

$$t_1^{w_1+w_5} t_2^{w_1} t_3^{w_5},\ t_1^{w_1} t_2^{w_1} t_4^{w_6},\ t_1^{w_1} t_2^{w_1} t_3^{w_7},\ t_1^{w_1+w_5} t_2^{w_1} t_3^{w_5} t_4^{w_6},\ t_1^{w_1+w_5} t_2^{w_1} t_3^{w_5+w_7},$$

$$t_1^{w_1} t_2^{w_1} t_3^{w_7} t_4^{w_6},\ t_1^{w_1+w_5} t_2^{w_1} t_3^{w_5+w_7} t_4^{w_6}.$$

Hence, the Banzhaf-Coleman power index with incompatible players allocates to player 2 a quantity of $\frac{8}{2^{7-1}} = 0.125$.

In Table 4, we present the Banzhaf-Coleman and Shapley-Shubik power indices and the Banzhaf-Coleman and Shapley-Shubik power indices taking into account the incompatibility graph.

Party	Shares of seats	Banzhaf-Coleman	Shapley-Shubik	Banzhaf-Coleman & incompatibility	Shapley-Shubik & incompatiblity
CIU	0.3703	0.8125	0.5333	0.875	0.7500
ERC	0.1556	0.1875	0.1333	0.125	0.0833
PSC	0.1481	0.1875	0.1333	0.125	0.0833
PPC	0.1407	0.1875	0.1333	0.125	0.0833
ICV-EUiA	0.0963	0.0625	0.0333	0.000	0.0000
C's	0.0667	0.0625	0.0333	0.000	0.0000
CUP	0.0222	0.0000	0.0000	0.000	0.0000

Table 4: Some power indices in the Catalonian Parliament.

When we take into account the incompatibility graph, $ICV - EUiA$ becomes a null player because his rejection to PPC and $C's$, and $C's$ becomes a null player too, because his rejection to CIU and ERC. Although the symmetry of the three intermediate players maintains when the incompatibility graph is introduced, these parties lose power for the benefit of the more powerful party. The main reason of this fact is the incompatibility between ERC and PPC.

6 Concluding remarks

Generating functions is a suitable tool to compute power indices for so-called weighted majority games. The method has been applied in the literature for computing several power indices. In this work, we consider the case of incompatible players in weighted majority games. Two players are incompatible in a simple game if never will form a joint coalition for passing an amendment. Although it could happen that the emergent game with incompatibilities is not a weighted majority game, we propose a method to compute the two more highlighted power indices, Banzhaf-Coleman and Shapley-Shubik power indices, by applying generating functions to the weighted majority representation of the original weighted majority game with incompatibilities.

With regard to the generality of the proposed model, we point out that the proposed method applies when the original game (N, W) is weighted majority independently if (N, W^g) is also weighted majority. However, if (N, W^g) is also weighted majority, it is then possible to apply generating functions directly to the game (N, W^g). Even more, assume that the original game (N, W) is not weighted majority but (N, W^g) is. Then, it is possible to compute Banzhaf-Coleman and Shapley-Shubik power indices of (N, W^g) by using generating functions directly.

Moreover, with respect to the apparent limitation of the approach considered, it seems that the condition of proper simple game (see Proposition 6) becomes essential. The limitation is because of the definition of g-restriction of v given in Definition 5, which is appropriate for general cooperative games, but possibly is not the best choice for simple games. In this definition, we can replace the sum by the maximum.Thus, with this different definition, the game v^g over coalitions only takes values 0 and 1, and therefore the model would be more general without the need of demanding properness to the initial game (N, v). However, we have chosen to write the article using the original definition of g-restriction of v and, then, maintained the condition of proper simple game.

Finally, we think that this work supports future extensions when considering the case of incompatible players in weighted majority games and alternative power indices to Shapley-Shubik and Banzhaf-Coleman. For example, Alonso-Meijide and Casas-Méndez (2007) introduce the Public Good Index when some voters are incompatible. It is worth trying to obtain this index following the method introduced in Chessa (2014) to calculate the Public Good Index by generating functions, adapted by the ideas of current work.

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