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Representation of (p,q)-Bernstein polynomials in terms of (p,q)-Jacobi polynomials

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Abstract

A representation of (p,q)-Bernstein polynomials in terms of (p,q)-Jacobi polynomials is obtained.

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Keywords: (p,q)-Bernstein polynoimals; (p,q)-Pearson difference equation; (p,q)-orthogonal solutions; (p,q)-difference operator

1 Introduction

Classical univariate Bernstein polynomials were introduced by Bernstein in a constructive proof for the Stone-Weierstrass approximation theorem [1], and they are defined as [2]

$$b_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0,1,\ldots,n.$$

They form a basis of polynomials and satisfy a number of important properties as non-negativity ($b_i^n(x) \geq 0$ for $0 \leq x \leq 1$), partition of unity ($\sum_{i=0}^n b_i^n(x) = 1$) or symmetry ($b_i^n(x) = b_{n-i}^n(1-x)$).

For a given real-valued defined and bounded function f on the interval $[0,1]$, the nth Bernstein polynomial for f is

$$B_n(f)(x) = \sum_{k=0}^{n} b_k^n(x) f\left(\frac{k}{n}\right).$$

Then, for each point x of continuity of f, we have $B_n(f)(x) \to f(x)$ as $n \to \infty$. Moreover, if f is continuous on $[0,1]$ then $B_n(f)$ converges uniformly to f as $n \to \infty$. Also, for each point x of differentiability of f, we have $B_n'(f)(x) \to f'(x)$ as $n \to \infty$ and if f is continuously differentiable on $[0,1]$ then $B_n'(f)$ converges to f' uniformly as $n \to \infty$.

Bernstein polynomials have been generalized in the framework of q-calculus. More precisely, Lupaş [3] initiated the application of q-calculus in area of the approximation theory, and introduced the q-Bernstein polynomials. Later on, Philips [4] proposed and studied other q-Bernstein polynomials. In both the classical case and in its q-analogs, expansions of Bernstein polynomials have been obtained in terms of appropriate orthogonal bases [5, 6].

Mursaleen *et al.* [7] recently introduced first the concept of (p,q)-calculus in approximation theory and studied the (p,q)-analog of Bernstein operators. The approximation properties for these operators based on Korovkin's theorem and some direct theorems were considered [8]. Also, many well-known approximation operators have been introduced using these techniques, such as Bleimann-Butzer-Hahn operators [9] and Szász-Mirakyan operators [10]. Very recently Milovanović *et al.* [11] considered a (p,q)-analog of the beta operators and using it proposed an integral modification of the generalized Bernstein polynomials. (p,q)-analogs of classical orthogonal polynomials have been characterized in [12].

The main aim of this work is to obtain a representation of (p,q)-Bernstein polynomials in terms of suitable (p,q)-orthogonal polynomials, where the connection coefficients are proved to satisfy a three-term recurrence relation. For this purpose, we have divided the work in two sections. First, we present the basic definitions and notations. Later, in Section 3 we obtain the main results of this work relating (p,q)-Bernstein polynomials and (p,q)-Jacobi orthogonal polynomials.

2 Basic definitions and notations

Next, we summarize the basic definitions and results which can be found in [13–18] and the references therein.

The (p,q)-power is defined as

$$\big((a,b);(p,q)\big)_k = \prod_{j=0}^{k-1}\big(ap^j - bq^j\big) \quad \text{with } \big((a,b);(p,q)\big)_0 = 1. \tag{1}$$

The (p,q)-hypergeometric series is defined as

$$\begin{aligned} &{}_r\Phi_s\left(\begin{matrix}(a_{1p},a_{1q}),\dots,(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\middle|(p,q);z\right)\\ &\quad=\sum_{j=0}^{\infty}\frac{((a_{1p},a_{1q}),\dots,(a_{rp},a_{rq});(p,q))_j}{((b_{1p},b_{1q}),\dots,(b_{sp},b_{sq});(p,q))_j}\frac{z^j}{((p,q);(p,q))_j}\big((-1)^j(q/p)^{\frac{j(j-1)}{2}}\big)^{1+s-r}, \end{aligned} \tag{2}$$

where

$$\big((a_{1p},a_{1q}),\dots,(a_{rp},a_{rq});(p,q)\big)_j = \prod_{s=1}^{r}\big((a_{sp},a_{sq});(p,q)\big)_j,$$

and $r,s\in\mathbb{Z}_+$ and $a_{1p},a_{1q},\dots,a_{rp},a_{rq},b_{1p},b_{1q},\dots,b_{sp},b_{sq},z\in\mathbb{C}$.

The (p,q)-difference operator is defined as (see *e.g.* [14])

$$(\mathcal{D}_{p,q}f)(x) = \frac{\mathcal{L}_pf(x)-\mathcal{L}_qf(x)}{(p-q)x}, \quad x\neq 0, \tag{3}$$

where the shift operator is defined by

$$\mathcal{L}_ah(x) = h(ax), \tag{4}$$

and $(\mathcal{D}_{p,q}f)(0) = f'(0)$, provided that f is differentiable at 0.

The (p,q)-Bernstein polynomials are defined as

$$b_i^n(x;p,q) = p^{n(1-n)/2} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q} p^{i(i-1)/2} x^i \bigl((1,x);(p,q)\bigr)_{n-i}, \tag{5}$$

and can be expanded in the basis $\{x^k\}_{k\geq 0}$ as

$$b_i^n(x;p,q) = \sum_{k=i}^{n} (-1)^{k-i} q^{(k-i)(k-i-1)/2} p^{\frac{1}{2}((i-1)i+k(k-2n+1))} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \begin{bmatrix} k \\ i \end{bmatrix}_{p,q} x^k. \tag{6}$$

From the definition of (p,q)-Bernstein polynomials it is possible to derive the basic properties of (p,q)-Bernstein polynomials.

(1) Partition of unity

$$\sum_{i=0}^{n} b_i^n(x;p,q) = 1.$$

(2) End-point properties

$$b_i^n(0;p,q) = \begin{cases} 1, & i = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad b_i^n(1;p,q) = \begin{cases} 1, & i = n, \\ 0, & \text{otherwise}. \end{cases}$$

The (p,q)-Jacobi polynomials are defined by

$$P_n(x;\alpha,\beta;p,q) = {}_2\Phi_1 \left(\begin{matrix} (p^{-n},q^{-n}),(p^{\alpha+\beta+n+1},q^{\alpha+\beta+n+1}) \\ (p^{\beta+1},q^{\beta+1}) \end{matrix} \middle| (p,q); \frac{xq^{-\alpha}}{p} \right), \tag{7}$$

and they satisfy the second order (p,q)-difference equation

$$\begin{aligned} &\frac{qx(qx-p)}{p^2} \bigl(\mathcal{D}_{p,q}^2 y\bigr)(x) + \left(\frac{x(p^{\alpha+\beta+2}q^{-\alpha-\beta} - q^2) - p^{\beta+2}q^{-\beta} + pq}{p^2(p-q)} \right) \mathcal{L}_p\bigl((\mathcal{D}_{p,q}y)(x)\bigr) \\ &\quad + [n]_{p,q} \left(\frac{qp^{-n-2} - p^{\alpha+\beta-1}q^{-\alpha-\beta-n}}{p-q} \right) \mathcal{L}_{pq} y(x) = 0. \end{aligned} \tag{8}$$

The (p,q)-Jacobi polynomials satisfy the three-term recurrence relation

$$\begin{aligned} &P_0(x;\alpha,\beta;p,q) = 1, \qquad P_1(x;\alpha,\beta;p,q) = x - B_0(\alpha,\beta;p,q), \\ &P_{n+1}(x;\alpha,\beta;p,q) = \bigl(x - B_n(\alpha,\beta;p,q)\bigr) P_n(x;\alpha,\beta;p,q) - C_n(\alpha,\beta;p,q) P_{n-1}(x;\alpha,\beta;p,q), \end{aligned}$$

where

$$\begin{aligned} B_n(\alpha,\beta;p,q) &= \frac{p^{n+2}q^{\alpha+n+1}}{(p-q)^2[\alpha+\beta+2n]_{p,q}[\alpha+\beta+2n+2]_{p,q}} \\ &\quad \times \bigl((p^\beta + q^\beta) q^{\alpha+\beta+2n+1} - (p+q)(p^\alpha + q^\alpha) p^{\beta+n} q^{\beta+n} \\ &\quad + (p^\beta + q^\beta) p^{\alpha+\beta+2n+1} \bigr) \end{aligned} \tag{9}$$

and

$$C_n(\alpha,\beta;p,q)=\frac{p^{\beta+2n+3}q^{2\alpha+\beta+2n+1}[n]_{p,q}[\alpha+n]_{p,q}[\beta+n]_{p,q}[\alpha+\beta+n]_{p,q}}{[\alpha+\beta+2n-1]_{p,q}([\alpha+\beta+2n]_{p,q})^2[\alpha+\beta+2n+1]_{p,q}}. \tag{10}$$

3 Representation of (p,q)-Bernstein polynomials in terms of (p,q)-Jacobi polynomials

Lemma 3.1 *The (p,q)-Bernstein polynomials satisfy the following first order (p,q)-difference equation*:

$$(px-1)x\big(D_{p,q}b_i^n\big)(x;p,q)+\big(-p^{1-n}[n]_{p,q}x+p^{-i}[i]_{p,q}\big)b_i^n(px;p,q)=0. \tag{11}$$

Proof The result can be obtained by equating the coefficients in x^j. □

If we introduce the first order (p,q)-difference operator

$$L_{i,n}=(px-1)xD_{p,q}+\big(-p^{1-n}[n]_{p,q}x+p^{-i}[i]_{p,q}\big)\mathcal{L}_p, \tag{12}$$

then

$$L_{i,n}b_i^n(x;p,q)=0.$$

Lemma 3.2 *The (p,q)-Jacobi polynomials satisfy the following structure relation*:

$$\begin{aligned}
&x(px-1)D_{p,q}\big(P_n\big(p^2x;\alpha,\beta;p,q\big)\big)\\
&\quad=[n]_{p,q}p^{-n-2}P_{n+1}\big(p^3x;\alpha,\beta;p,q\big)+\varpi_1(n)P_n\big(p^3x;\alpha,\beta;p,q\big)\\
&\qquad+\varpi_2(n)P_{n-1}\big(p^3x;\alpha,\beta;p,q\big),
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\varpi_1(n)&=-\frac{[n]_{p,q}(-(p+q)q^{\alpha+n}-p^{\beta+n}+p^{\alpha+\beta+2n+1}+q^{\alpha+\beta+2n+1})[\alpha+\beta+n+1]_{p,q}}{(p-q)[\alpha+\beta+2n]_{p,q}[\alpha+\beta+2n+2]_{p,q}},\\
\varpi_2(n)&=\frac{q^{\alpha+n}p^{\beta+2n+1}[n]_{p,q}[\alpha+n]_{p,q}[\beta+n]_{p,q}[\alpha+\beta+n]_{p,q}[\alpha+\beta+n+1]_{p,q}}{[\alpha+\beta+2n-1]_{p,q}([\alpha+\beta+2n]_{p,q})^2[\alpha+\beta+2n+1]_{p,q}}.
\end{aligned}$$

Proof The result follows from (7) by equating the coefficients in x^j. □

Theorem 3.1 *The (p,q)-Bernstein polynomials defined in* (5) *have the following representation in terms of (p,q)-Jacobi polynomials defined in* (7):

$$b_i^n(x;p,q)=\sum_{k=0}^{n}H_k(i,n;\alpha,\beta;p,q)P_k\big(p^2x;\alpha,\beta;p,q\big), \tag{14}$$

where the connection coefficients $H_k(i,n;\alpha,\beta;p,q)$ satisfy the following three-term recurrence relation:

$$\begin{aligned}
&H_{k-1}(i,n;\alpha,\beta;p,q)\Lambda_1(k-1,i,n;\alpha,\beta;p,q)+H_k(i,n;\alpha,\beta;p,q)\Lambda_2(k,i,n;\alpha,\beta;p,q)\\
&\quad+H_{k+1}(i,n;\alpha,\beta;p,q)\Lambda_3(k+1,i,n;\alpha,\beta;p,q)=0,
\end{aligned} \tag{15}$$

valid for $1 \leq k \leq n-1$ *with initial conditions*

$$H_{n+1}(i,n;\alpha,\beta;p,q) = 0, \tag{16}$$

$$H_n(i,n;\alpha,\beta;p,q) = (-1)^{n+1} q^{-\frac{1}{2}(1-n)n} p^{-n(n+3)/2+k(k+1)/2} \begin{bmatrix} n \\ i \end{bmatrix}_{p,q}, \tag{17}$$

and

$$\begin{cases} \Lambda_1(k,i,n;\alpha,\beta;p,q) = p^{-k-2}[k]_{p,q} - p^{-n-2}[n]_{p,q}, \\ \Lambda_2(k,i,n;\alpha,\beta;p,q) = p^{-i}[i]_{p,q} - p^{-2-n}[n]_{p,q} B_k(\alpha,\beta;p,q) + \varpi_1(k), \\ \Lambda_3(k,i,n;\alpha,\beta;p,q) = -p^{-n-2}[n]_{p,q} C_k(\alpha,\beta;p,q) + \varpi_2(k). \end{cases} \tag{18}$$

Proof In order to obtain the result we shall apply the so-called Navima algorithm (see *e.g.* [19, 20] and the references therein) for solving connection problems. If we apply the first order linear operator $L_{i,n}$ defined in (12) to both sides of (14) we have

$$\begin{aligned} 0 &= \sum_{k=0}^{n} H_k(i,n;\alpha,\beta;p,q) L_{i,n} P_k\bigl(p^2x;\alpha,\beta;p,q\bigr) \\ &= \sum_{k=0}^{n} H_k(i,n;\alpha,\beta;p,q)\bigl((px-1)xD_{p,q}\bigl(P_k\bigl(p^2x;\alpha,\beta;p,q\bigr)\bigr) \\ &\quad + \bigl(-p^{1-n}[n]_{p,q}x + p^{-i}[i]_{p,q}\bigr)P_k\bigl(p^3x;\alpha,\beta;p,q\bigr)\bigr). \end{aligned}$$

From the three-term recurrence relation for (p,q)-Jacobi polynomials it yields

$$\begin{aligned} &\bigl(-p^{1-n}[n]_{p,q}x + p^{-i}[i]_{p,q}\bigr)P_k\bigl(p^3x;\alpha,\beta;p,q\bigr) \\ &\quad = -p^{-n-2}[n]_{p,q}P_{k+1}\bigl(p^3x;\alpha,\beta;p,q\bigr) \\ &\qquad + p^{-2-n-i}\bigl(-p^{n+2}[i]_{p,q} + p^{i}[n]_{p,q}B_k(\alpha,\beta;p,q)\bigr)P_k\bigl(p^3x;\alpha,\beta;p,q\bigr) \\ &\qquad - p^{-n-2}[n]_{p,q}C_k(\alpha,\beta;p,q)P_{k-1}\bigl(p^3x;\alpha,\beta;p,q\bigr). \end{aligned}$$

Therefore, by using the structure relation for (p,q)-Jacobi polynomials (13) we have

$$\begin{aligned} &(px-1)xD_{p,q}\bigl(P_k\bigl(p^2x;\alpha,\beta;p,q\bigr)\bigr) + \bigl(-p^{1-n}[n]_{p,q}x + p^{-i}[i]_{p,q}\bigr)P_k\bigl(p^3x;\alpha,\beta;p,q\bigr) \\ &\quad = \Lambda_1(k,i,n;\alpha,\beta;p,q)P_{k+1}\bigl(p^3x;\alpha,\beta;p,q\bigr) + \Lambda_2(k,i,n;\alpha,\beta;p,q)P_k\bigl(p^3x;\alpha,\beta;p,q\bigr) \\ &\qquad + \Lambda_3(k,i,n;\alpha,\beta;p,q)P_{k-1}\bigl(p^3x;\alpha,\beta;p,q\bigr), \end{aligned}$$

where $\Lambda_i(k,i,n;\alpha,\beta;p,q)$ are given in (18).

As a consequence,

$$\begin{aligned} 0 &= \sum_{k=0}^{n} H_k(i,n;\alpha,\beta;p,q)\bigl(\Lambda_1(k,i,n;\alpha,\beta;p,q)P_{k+1}\bigl(p^3x;\alpha,\beta;p,q\bigr) \\ &\quad + \Lambda_2(k,i,n;\alpha,\beta;p,q)P_k\bigl(p^3x;\alpha,\beta;p,q\bigr) + \Lambda_3(k,i,n;\alpha,\beta;p,q)P_{k-1}\bigl(p^3x;\alpha,\beta;p,q\bigr)\bigr). \end{aligned}$$

By using the linear independence of $\{P_k(p^3x;\alpha,\beta;p,q)\}$ we obtain the three-term recurrence relation (15) for the connection coefficients $H_k(i,n;\alpha,\beta;p,q)$, where the initial conditions are obtained by equating the highest power in x^k. □

4 Conclusions

In this work we have obtained a three-term recurrence relation for the coefficients in the expansion of (p,q)-Bernstein polynomials in terms of (p,q)-Jacobi polynomials. For our purposes some auxiliary results both for (p,q)-Bernstein polynomials and (p,q)-Jacobi polynomials have been derived.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, FS, IA, MMJ, and JJN contributed to each part of this study equally and read and approved the final version of the manuscript.

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