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Numerical approximation of some poro-elastic problems with MGT-type dissipation mechanisms

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A B S T R A C T

In this work, we numerically analyze a porous elastic problem including several dissipation mechanisms of MGT type. The resulting variational problem is written in terms of the acceleration and the porosity speed. An existence and uniqueness result is recalled. Then, fully discrete approximations are introduced by using the classical finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are proved from which the linear convergence of the approximation is derived. Finally, some numerical simulations are presented to show the accuracy of the approximation, the discrete energy decay and the behavior of the solution.

1. Introduction

Porous elastic materials are widely used in common life. For this reason, several theories have been considered to describe them. In this paper, we want to consider one of these theories which is capable of describing certain motions in porous media. This is the theory containing voids which was proposed by Nunziato and Cowin [8,9,25]. The potential applications of this theory are huge. A brief description of the possibilities of application can be found at the pages 307-308 of the book of B. Straughan [27]. Therefore, we can say that they go from the biomedicine to the building industries.

The theory of Nunziato and Cowin extends the classical theory of elasticity describing the behavior of elastic solids with voids. It is assumed that the materials have a matrix material (or skeleton) which is elastic and the interstices are voids of the material. It is suitable to recall that a great amount of papers have been published concerning this theory. We can cite [11–13,18,22–24] among others. Moreover, extensions of this theory to double porosity [10,20,21] and strain gradient porous-elasticity [19] have been recently considered. The theory proposed by Ieşan is conservative. That is, the energy of the system does not dissipate by means of structural mechanisms but it can be obtained introducing dissipative structural mechanisms. This should be reflected by means of a good selection of the independent variables and the constitutive equations must be compatible with the second law of thermodynamics. Structural viscous mechanisms can bring the systems to the equilibrium position. Perhaps, the most known dissipative mechanism of mechanical type is the one associated with Kelvin-Voigt viscosity. A similar mechanism can be introduced in the porous structure. However, it allows the instantaneous

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propagation of the porous-elastic waves which is not compatible with the causality principle. Other viscosity mechanisms as memory effects overcome this drawback.

In the recent years, a big deal has been developed to study the Moore-Gibson-Thompson (MGT) equation. This was proposed from the mechanics of fluids, but very recently we have seen it as a heat equation when we propose to introduce a relaxation parameter to the type III Green-Naghdi theory [26]. Indeed, these ideas have been extended from several points of view and it is also possible to consider several dissipation mechanisms associated to different theories of mechanics of solids [5–7,14,15]. We here want to consider a strain-gradient porous elastic theory when the dissipation mechanism corresponds to a Moore-Gibson-Thompson type of dissipation. It is important to say that this kind of materials can be described by high-order in time partial differential equations and they can be obtained by choosing suitable kernels in the theory of materials with memory [17].

It is worth remarking that the MGT-dissipation allow us to obtain mechanical dissipation mechanisms that can be expressed by means of partial differential equations (not integro-differential equations). They are compatible with the causality principle and so, they propose an interesting field of problems to be analyzed.

The paper is structured as follows. The mechanical problem is described in Section 2, recalling an existence and uniqueness result proved in [16]. Then, in Section 3 a fully discrete approximation is introduced, based on the finite element method to approximate the spatial domain and the backward Euler scheme to discretize the time derivatives. A discrete stability property and a priori error estimates are obtained, from which, under suitable additional regularity conditions, the linear convergence of the algorithm is deduced. Finally, some numerical simulations are presented in Section 4.

2. The poro-elastic problem

In this section, we briefly describe the model and its mechanical form, and we state an existence and uniqueness result (we refer the readers to the companion paper [16] for further details).

Let $(0,\ell)$ be the poro-elastic rod and denote by $[0,T]$, $T>0$, the time interval of interest. We restrict ourselves to the one-dimensional setting for the sake of simplicity but we note that the numerical analysis provided in the next section could be extended, with some minor modifications, to the multidimensional setting. Moreover, let $x\in(0,\ell)$ and $t\in[0,T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on x and t. The time derivatives are represented as a dot for the first order, two dots for the second order and three dots for the third order, over each variable.

In this case, following [16] we include some MGT-type dissipation mechanisms. Therefore, if we denote by u the displacements and φ the volume fraction or porosity, and we assume that the dissipation is produced in the second-order viscous term, the system of equations is written as:

$$\rho(\ddot{u}+\tau\dddot{u}) = au_{xx}+a^*\dot{u}_{xx}+b\varphi_x-\eta\varphi_{xxx}-k_1(u_{xxxx}+\tau\dot{u}_{xxxx}), \tag{1}$$

$$J\ddot{\varphi} = \eta(u_{xxx}+\tau\dot{u}_{xxx})-b(u_x+\tau\dot{u}_x)+\delta^*\varphi_{xx}-\xi\varphi-k_2\varphi_{xxxx}, \tag{2}$$

where we assume that parameters ρ, a, k_1, a^*, J, δ^*, ξ and k_2 are positive as well as we impose that $a\xi>b^2$, $k_1\delta^*>\eta^2$ and $a^*>\tau a$. We also suppose that $\eta\neq 0$, which implies a strong coupling.

The initial conditions we will consider in this case are, for a.e. $x\in(0,\ell)$,

$$\begin{aligned} &u(x,0)=u^0(x), \quad \dot{u}(x,0)=v^0(x), \quad \ddot{u}(x,0)=c^0(x),\\ &\varphi(x,0)=\varphi^0(x), \quad \dot{\varphi}(x,0)=\psi^0(x). \end{aligned} \tag{3}$$

In this section, we are going to assume the boundary conditions, for a.e. $t\in(0,T)$,

$$\begin{aligned} &u(0,t)=u(\ell,t)=u_{xx}(0,t)=u_{xx}(\ell,t)=0,\\ &\varphi_x(0,t)=\varphi_x(\ell,t)=\varphi_{xxx}(0,t)=\varphi_{xxx}(\ell,t)=0. \end{aligned} \tag{4}$$

Remark 1. We note that in [16] two other dissipation mechanisms were considered, including the MGT-type dissipation into the fourth-order term of the displacements (hyperviscosity) and the zeroth-order term of the porosity (weak viscoporosity). In these cases, the resulting systems are the following:

$$\begin{aligned} &\rho(\ddot{u}+\tau\dddot{u}) = a(u_{xx}+\tau\dot{u}_{xx})+b\varphi_x-\eta\varphi_{xxx}-k_1u_{xxxx}-k_1^*\dot{u}_{xxxx},\\ &J\ddot{\varphi} = \eta(u_{xxx}+\tau\dot{u}_{xxx})-b(u_x+\tau\dot{u}_x)+\delta^*\varphi_{xx}-\xi\varphi-k_2\varphi_{xxxx}, \end{aligned}$$

for the hyperviscosity, or

$$\begin{aligned} &\rho\ddot{u} = au_{xx}+b(\varphi_x+\tau\dot{\varphi}_x)-\eta(\varphi_{xxx}+\tau\dot{\varphi}_{xxx})-k_1u_{xxxx},\\ &J(\ddot{\varphi}+\tau\dddot{\varphi}) = \eta u_{xxx}-bu_x+\delta^*(\varphi_{xx}+\tau\dot{\varphi}_{xx})-\xi\varphi-\xi^*\dot{\varphi}-k_2(\varphi_{xxxx}+\tau\dot{\varphi}_{xxxx}) \end{aligned}$$

for the weak viscoporosity.

The boundary conditions (4) can be imposed for both problems but adequate modified initial conditions are needed for the weak viscoporosity case because now the problem is second-order in time for the displacements and third-order in time for the porosity. However, the modifications required to obtain their numerical analysis, as in the next section, are minor and we do not give details for the sake of simplicity.

The following existence and uniqueness result was proved in [16].

Theorem 2. The differential operator associated to system (1)-(2) generates a C^0-semigroup of contractions. Therefore, if the initial conditions have the regularity:

$$u^0 \in H^2(0,\ell), \quad v^0 \in H^2(0,\ell), \quad c^0 \in L^2(0,\ell),$$
$$\varphi^0 \in H^2(0,\ell), \quad \psi^0 \in H^2(0,\ell),$$

there exists a unique solution with the following regularity:

$$u \in C^3([0,T]; L^2(0,\ell)) \cap C^1([0,T]; H^2(0,\ell)),$$
$$\varphi \in C^2([0,T]; L^2(0,\ell)) \cap C([0,T]; H^2(0,\ell))$$

to our problem. Moreover, the energy decay is polynomial with order 1/2.

To be precise, we recall that in the reference [16] there were used the following aspects. First, the problem was transformed into a Cauchy problem on a suitable Hilbert space by means of the definition of an operator. Second, it was used the Lumer-Phillips corollary to the Hille-Yosida theorem in order to prove that the operator generates a contractive semigroup. In fact, it was shown that the operator is defined in a dense subspace of the Hilbert space, that zero belongs to the resolvent of the operator and that the product of the image of a vector by the vector has always a negative real part. Then, it was used the results of Borichev and Tomilov [1], which characterize the polynomial decay of the contractive semigroup, and we proved that the imaginary axis is contained in the resolvent of the operator and that the asymptotic condition

$$\overline{\lim_{|\lambda|\to\infty}} \lambda^{-2} \|(\lambda I - \mathcal{A})^{-1}\| < \infty$$

holds. Here, $\mathcal{A}$ denotes the differential operator associated to the continuous problem.

3. Fully discrete approximations: an a priori error analysis

In order to obtain the numerical approximation of the above mechanical problem, let us denote $Y = L^2(0,\ell)$ and $V = H_0^2(0,\ell)$. Moreover, let $(\cdot,\cdot)$ and $\|\cdot\|$ be the inner product and the norm defined on $L^2(0,\ell)$, respectively, and denote by $\|\cdot\|_V$ the norm in the variational space V. We recall that the norm in this space can be defined as

$$\|v\|_V^2 = \|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 \quad \forall v \in V.$$

As usual in this type of numerical analysis, we replace boundary conditions (4) by the following ones:

$$u(0,t) = u(\ell,t) = u_x(0,t) = u_x(\ell,t) = 0,$$
$$\varphi(0,t) = \varphi(\ell,t) = \varphi_x(0,t) = \varphi_x(\ell,t) = 0. \tag{5}$$

Integrating by parts equations (1)-(2) and using initial conditions (3) and boundary conditions (5), we obtain the following weak formulation of problem (1), (2), (3) and (5) written using the velocity $v = \dot{u}$, the acceleration $c = \ddot{u}$ and the porosity speed $\psi = \dot{\varphi}$.

Problem VP. Find the acceleration $c : [0,T] \to V$ and the porosity speed $\psi : [0,T] \to V$ such that $c(0) = c^0$, $\psi(0) = \psi^0$ and, for a.e. $t \in (0,T)$ and for all $w, r \in V$,

$$\rho(c(t) + \tau\dot{c}(t), w) + k_1(u_{xx}(t) + \tau v_{xx}(t), w_{xx}) + (au_x(t) + a^* v_x(t), w_x)$$
$$= -b(\varphi(t), w_x) - \eta(\varphi_x(t), w_{xx}), \tag{6}$$

$$J(\dot{\psi}(t), r) + k_2(\varphi_{xx}(t), r_{xx}) + \delta^*(\varphi_x(t), r_x) + \xi(\varphi(t), r)$$
$$= -b(u_x(t) + \tau v_x(t), r) - \eta(u_{xx}(t) + \tau v_{xx}(t), r_x), \tag{7}$$

where the displacements, the velocity and the porosity are then recovered from the relations:

$$u(t) = \int_0^t v(s)\,ds + u^0, \quad v(t) = \int_0^t c(s)\,ds + v^0,$$
$$\varphi(t) = \int_0^t \psi(s)\,ds + \varphi^0. \tag{8}$$

In the rest of this section, we study a fully discrete approximation of the above variational problem. First, we assume that the interval $[0,\ell]$ is divided into M subintervals $a_0=0<a_1<\ldots<a_M=\ell$ of length $h=a_{i+1}-a_i=\ell/M$ and so, to approximate the variational space V, we define the finite dimensional space $V^h\subset V$ given by

$$V^h=\{w^h\in C^1([0,\ell])\cap H^2(0,\ell)\,;\ w^h_{|[a_i,a_{i+1}]}\in P_3([a_i,a_{i+1}])\ \ i=0,\ldots,M-1,\\ w^h_x(0)=w^h_x(\ell)=w^h(0)=w^h(\ell)=0\},\tag{9}$$

where $P_3([a_i,a_{i+1}])$ represents the space of polynomials of degree less or equal to 3 in the subinterval $[a_i,a_{i+1}]$; i.e. the finite element space V^h is made C^1 and piecewise cubic functions. Here, $h>0$ denotes the spatial discretization parameter. Furthermore, let the discrete initial conditions u^{0h}, v^{0h}, c^{0h}, φ^{0h} and ψ^{0h} be defined as

$$\begin{aligned}&u^{0h}=\mathcal{P}^h u^0,\quad v^{0h}=\mathcal{P}^h v^0,\quad c^{0h}=\mathcal{P}^h c^0,\\&\varphi^{0h}=\mathcal{P}^h\varphi^0,\quad \psi^{0h}=\mathcal{P}^h\psi^0,\end{aligned}\tag{10}$$

where $\mathcal{P}^h$ represents the projection operator over the finite element space V^h (see [4] for details).

In order to discretize the time derivatives, we consider a uniform partition of the time interval $[0,T]$, denoted by $0=t_0<t_1<\ldots<t_N=T$, with step size $k=T/N$ and nodes $t_n=nk$ for $n=0,1,\ldots,N$. For a continuous function $z(t)$ let $z^n=z(t_n)$ and, given a sequence $\{z^n\}_{n=0}^N$, we denote by $\delta z^n=(z^n-z^{n-1})/k$ its divided differences.

Therefore, using the implicit Euler scheme, the fully discrete approximations of Problem VP are the following.

Problem VP^{hk}. Find the discrete acceleration $c^{hk}=\{c^{hk,n}\}_{n=0}^N\subset V^h$ and the discrete porosity speed $\psi^{hk}=\{\psi^{hk,n}\}_{n=0}^N\subset V^h$ such that $c^{hk,0}=c^{0h}$, $\psi^{hk,0}=\psi^{0h}$, and, for all w^h, $r^h\in V^h$ and $n=1,\ldots,N$,

$$\begin{aligned}&\rho(c^{hk,n}+\tau\delta c^{hk,n},w^h)+k_1(u^{hk,n}_{xx}+\tau v^{hk,n}_{xx},w^h_{xx})+(au^{hk,n}_x+a^*v^{hk,n}_x,w^h_x)\\&\qquad=-b(\varphi^{hk,n},w^h_x)-\eta(\varphi^{hk,n}_x,w^h_{xx}),\end{aligned}\tag{11}$$

$$\begin{aligned}&J(\delta\psi^{hk}_n,r^h)+k_2(\varphi^{hk,n}_{xx},r^h_{xx})+\delta^*(\varphi^{hk,n}_x,r^h_x)+\xi(\varphi^{hk,n},r^h)\\&\qquad=-b(u^{hk,n}_x+\tau v^{hk,n}_x,r^h)-\eta(u^{hk,n}_{xx}+\tau v^{hk,n}_{xx},r^h_x),\end{aligned}\tag{12}$$

where the discrete displacements $u^{hk,n}$, the discrete velocity $v^{hk,n}$ and the discrete porosity $\varphi^{hk,n}$ are now recovered from the relations:

$$\begin{aligned}&u^{hk,n}=k\sum_{j=1}^n v^{hk,j}+u^{0h},\quad v^{hk,n}=k\sum_{j=1}^n c^{hk,j}+v^{0h},\\&\varphi^{hk,n}=k\sum_{j=1}^n\psi^{hk,j}+\varphi^{0h}.\end{aligned}\tag{13}$$

It is straightforward to show that Problem VP^{hk} admits a unique solution applying the well-known Lax Milgram lemma and the assumptions on the constitutive coefficients provided in the previous section.

The aim of this section is to obtain a priori error estimates on the numerical errors. First, we prove a discrete stability result.

We have the following.

Lemma 3. Let the assumptions of Theorem 2 hold. Then, it follows that the sequences $\{u^{hk},v^{hk},c^{hk},\varphi^{hk},\psi^{hk}\}$ generated by Problem VP^{hk} satisfy the stability estimate:

$$\begin{aligned}&\|c^{hk,n}\|^2+\|v^{hk,n}\|_V^2+\|u^{hk,n}\|_V^2+\|\psi^{hk,n}\|^2+\|\varphi^{hk,n}\|_V^2\\&\qquad\le C\Big(\|c^{0h}\|^2+\|v^{0h}\|_V^2+\|u^{0h}\|_V^2+\|\psi^{0h}\|^2+\|\varphi^{0h}\|_V^2\Big),\end{aligned}$$

where C is a positive constant which is independent of the discretization parameters h and k.

Proof. For the sake of simplicity in the calculations, we assume that $\tau=1$.

First, taking as a discrete function $w^h=c^{hk,n}$ in variational equation (11) we have

$$\begin{aligned}&\rho(c^{hk,n}+\delta c^{hk,n},c^{hk,n})+k_1(u^{hk,n}_{xx}+v^{hk,n}_{xx},c^{hk,n}_{xx})+(au^{hk,n}_x+a^*v^{hk,n}_x,c^{hk,n}_x)\\&\qquad=-b(\varphi^{hk,n},c^{hk,n}_x)-\eta(\varphi^{hk,n}_x,c^{hk,n}_{xx}).\end{aligned}$$

Keeping in mind that

$$
\begin{aligned}
&(\delta c^{hk,n}, c^{hk,n}) \geq \frac{1}{2k}\Big[\|c^{hk,n}\|^2 - \|c^{hk,n-1}\|^2\Big],\\
&(a^* v_x^{hk,n}, c_x^{hk,n}) = \frac{a^*}{2k}\Big[\|v_x^{hk,n}\|^2 - \|v_x^{hk,n-1}\|^2 + \|(v^{hk,n} - v^{hk,n-1})_x\|^2\Big],\\
&(k_1 v_{xx}^{hk,n}, c_{xx}^{hk,n}) = \frac{k_1}{2k}\Big[\|v_{xx}^{hk,n}\|^2 - \|v_{xx}^{hk,n-1}\|^2 + \|(v^{hk,n} - v^{hk,n-1})_{xx}\|^2\Big],
\end{aligned}
$$

where we have used several times Cauchy's inequality

$$
ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b, \epsilon \in \mathbb{R},\ \epsilon > 0, \tag{14}
$$

we obtain

$$
\begin{aligned}
&\frac{\rho}{2k}\Big[\|c^{hk,n}\|^2 - \|c^{hk,n-1}\|^2\Big] + \frac{a^*}{2k}\Big[\|v_x^{hk,n}\|^2 - \|v_x^{hk,n-1}\|^2 + \|(v^{hk,n} - v^{hk,n-1})_x\|^2\Big]\\
&\quad + \frac{k_1}{2k}\Big[\|v_{xx}^{hk,n}\|^2 - \|v_{xx}^{hk,n-1}\|^2 + \|(v^{hk,n} - v^{hk,n-1})_{xx}\|^2\Big] + a(u_x^{hk,n}, c_x^{hk,n})\\
&\quad + k_1(u_{xx}^{hk,n}, c_{xx}^{hk,n}) + \eta(\varphi_x^{hk,n}, c_{xx}^{hk,n}) + b(\varphi^{hk,n}, c_x^{hk,n}) \leq 0.
\end{aligned} \tag{15}
$$

Now, taking a discrete function $r^h = \psi^{hk,n}$ in variational equation (12) we find that

$$
\begin{aligned}
&J(\delta\psi_n^{hk}, \psi^{hk,n}) + k_2(\varphi_{xx}^{hk,n}, \psi_{xx}^{hk,n}) + \delta^*(\varphi_x^{hk,n}, \psi_x^{hk,n}) + \xi(\varphi^{hk,n}, \psi^{hk,n})\\
&\qquad = -b(u_x^{hk,n} + v_x^{hk,n}, \psi^{hk,n}) - \eta(u_{xx}^{hk,n} + v_{xx}^{hk,n}, \psi_x^{hk,n}).
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
&J(\delta\psi^{hk,n}, \psi^{hk,n}) \geq \frac{J}{2k}\Big[\|\psi^{hk,n}\|^2 - \|\psi^{hk,n-1}\|^2\Big],\\
&\delta^*(\varphi_x^{hk,n}, \psi_x^{hk,n}) = \frac{\delta^*}{2k}\Big[\|\varphi_x^{hk,n}\|^2 - \|\varphi_x^{hk,n-1}\|^2 + \|(\varphi^{hk,n} - \varphi^{hk,n-1})_x\|^2\Big],\\
&k_2(\varphi_{xx}^{hk,n}, \psi_{xx}^{hk,n}) \geq \frac{k_2}{2k}\Big[\|\varphi_{xx}^{hk,n}\|^2 - \|\varphi_{xx}^{hk,n-1}\|^2\Big],\\
&\xi(\varphi^{hk,n}, \psi^{hk,n}) = \frac{\xi}{2k}\Big[\|\varphi^{hk,n}\|^2 - \|\varphi^{hk,n-1}\|^2 + \|\varphi^{hk,n} - \varphi^{hk,n-1}\|^2\Big],
\end{aligned}
$$

using again inequality (14) we have

$$
\begin{aligned}
&\frac{J}{2k}\Big[\|\psi^{hk,n}\|^2 - \|\psi^{hk,n-1}\|^2\Big] + \frac{\delta^*}{2k}\Big[\|\varphi_x^{hk,n}\|^2 - \|\varphi_x^{hk,n-1}\|^2 + \|(\varphi^{hk,n} - \varphi^{hk,n-1})_x\|^2\Big]\\
&\quad + \frac{k_2}{2k}\Big[\|\varphi_{xx}^{hk,n}\|^2 - \|\varphi_{xx}^{hk,n-1}\|^2\Big] + \frac{\xi}{2k}\Big[\|\varphi^{hk,n}\|^2 - \|\varphi^{hk,n-1}\|^2 + \|\varphi^{hk,n} - \varphi^{hk,n-1}\|^2\Big]\\
&\quad + \eta(u_{xx}^{hk,n}, \psi_x^{hk,n}) + b(u_x^{hk,n}, \psi^{hk,n}) + \eta(v_{xx}^{hk,n}, \psi_x^{hk,n}) + b(v_x^{hk,n}, \psi^{hk,n}) \leq 0.
\end{aligned} \tag{16}
$$

We observe that

$$
\begin{aligned}
&b(v_x^{hk,n}, \psi^{hk,n}) + b(c_x^{hk,n}, \varphi^{hk,n}) = \frac{1}{k}\Big[b(v_x^{hk,n}, \varphi^{hk,n}) - b(v_x^{hk,n-1}, \varphi^{hk,n-1})\\
&\qquad + b(v_x^{hk,n} - v_x^{hk,n-1}, \varphi^{hk,n} - \varphi^{hk,n-1})\Big],\\
&\eta(v_{xx}^{hk,n}, \psi_x^{hk,n}) + \eta(c_{xx}^{hk,n}, \varphi_x^{hk,n}) = \frac{1}{k}\Big[\eta(v_{xx}^{hk,n}, \varphi_x^{hk,n}) - \eta(v_{xx}^{hk,n-1}, \varphi_x^{hk,n-1})\\
&\qquad + \eta(v_{xx}^{hk,n} - v_{xx}^{hk,n-1}, \varphi_x^{hk,n} - \varphi_x^{hk,n-1})\Big].
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
&a^*\|v_x^{hk,n} - v_x^{hk,n-1}\|^2 + \xi\|\varphi^{hk,n} - \varphi^{hk,n-1}\|^2 + 2b(v_x^{hk,n} - v_x^{hk,n-1}, \varphi^{hk,n} - \varphi^{hk,n-1}) \geq 0,\\
&k_1\|v_{xx}^{hk,n} - v_{xx}^{hk,n-1}\|^2 + \delta^*\|\varphi_x^{hk,n} - \varphi_x^{hk,n-1}\|^2 + 2\eta(v_{xx}^{hk,n} - v_{xx}^{hk,n-1}, \varphi_x^{hk,n} - \varphi_x^{hk,n-1}) \geq 0,
\end{aligned}
$$

thanks to the required assumptions on the constitutive coefficients, combining estimates (15) and (16), multiplying the resulting estimates by k and summing up to n, it follows that

$$
\begin{aligned}
&\rho\|c^{hk,n}\|^2 + a^*\|v_x^{hk,n}\|^2 + k_1\|v_{xx}^{hk,n}\|^2 + J\|\psi^{hk,n}\|^2 + \delta^*\|\varphi_x^{hk,n}\|^2 + k_2\|\varphi_{xx}^{hk,n}\|^2 + \xi\|\varphi^{hk,n}\|^2\\
&\quad +2b(\varphi^{hk,n}, v_x^{hk,n}) + 2\eta(\varphi_x^{hk,n}, v_{xx}^{hk,n}) + 2ak\sum_{j=1}^n (u_x^{hk,j}, c_x^{hk,j}) + 2k_1 k\sum_{j=1}^n (u_{xx}^{hk,j}, c_{xx}^{hk,j})\\
&\quad +2\eta k\sum_{j=1}^n (u_{xx}^{hk,j}, \psi_x^{hk,j}) + 2bk\sum_{j=1}^n (u_x^{hk,j}, \psi^{hk,j})\\
&\le C\Big(\|c^{0h}\|^2 + \|v^{0h}\|_V^2 + \|u^{0h}\|_V^2 + \|\psi^{0h}\|^2 + \|\varphi^{0h}\|_V^2\Big).
\end{aligned}
$$

Finally, keeping in mind that

$$
\begin{aligned}
&a^*\|v_x^{hk,n}\|^2 + \xi\|\varphi^{hk,n}\|^2 + 2b(\varphi^{hk,n}, v_x^{hk,n}) \ge C\Big(\|v_x^{hk,n}\|^2 + \|\varphi^{hk,n}\|^2\Big),\\
&k_1\|v_{xx}^{hk,n}\|^2 + \delta^*\|\varphi_x^{hk,n}\|^2 + 2\eta(\varphi_x^{hk,n}, v_{xx}^{hk,n}) \ge C\Big(\|v_{xx}^{hk,n}\|^2 + \|\varphi_x^{hk,n}\|^2\Big),\\
&k\sum_{j=1}^n (u_x^{hk,j}, c_x^{hk,j}) = (u_x^{hk,n}, v_x^{hk,n}) - k\sum_{j=1}^n (v_x^{hk,n}, v_x^{hk,n}) + (u_x^{0h}, v_x^{0h}),\\
&k\sum_{j=1}^n (u_{xx}^{hk,j}, c_{xx}^{hk,j}) = (u_{xx}^{hk,n}, v_{xx}^{hk,n}) - k\sum_{j=1}^n (v_{xx}^{hk,n}, v_{xx}^{hk,n}) + (u_{xx}^{0h}, v_{xx}^{0h}),\\
&k\sum_{j=1}^n (u_{xx}^{hk,j}, \psi_x^{hk,j}) = (u_{xx}^{hk,n}, \varphi_x^{hk,n}) - k\sum_{j=1}^n (v_{xx}^{hk,n}, \varphi_x^{hk,n}) + (u_{xx}^{0h}, \varphi_x^{0h}),\\
&k\sum_{j=1}^n (u_x^{hk,j}, \psi^{hk,j}) = (u_x^{hk,n}, \varphi^{hk,n}) - k\sum_{j=1}^n (v_x^{hk,n}, \varphi^{hk,n}) + (u_x^{0h}, \varphi^{0h}),\\
&\|u_x^{hk,n}\|^2 \le C\Big(\|u_x^{0h}\|^2 + k\sum_{j=1}^n \|v_x^{hk,n}\|^2\Big),\\
&\|u_{xx}^{hk,n}\|^2 \le C\Big(\|u_{xx}^{0h}\|^2 + k\sum_{j=1}^n \|v_{xx}^{hk,n}\|^2\Big),
\end{aligned}
$$

applying a discrete version of Gronwall's inequality (see, for instance, [2]) we conclude the desired discrete stability.

Now, we prove a main error estimates result.

Theorem 4. Let the assumptions of Theorem 2 still hold. If we denote by (u, v, c, φ, ψ) the solution to problem (6)-(8) and by $(u^{hk}, v^{hk}, c^{hk}, \varphi^{hk}, \psi^{hk})$ the solution to problem (11)-(13), then we have the following a priori error estimates, for all $w^h = \{w^{h,j}\}_{j=0}^N,\ r^h = \{r^{h,j}\}_{j=0}^N \subset V^h$,

$$
\begin{aligned}
&\max_{0\le n\le N}\Big\{\|c^n - c^{hk,n}\|^2 + \|v^n - v^{hk,n}\|_V^2 + \|u^n - u^{hk,n}\|_V^2 + \|\psi^n - \psi^{hk,n}\|^2\\
&\quad +\|\varphi^n - \varphi^{hk,n}\|_V^2\Big\}\\
&\le Ck\sum_{j=1}^N\Big(\|\dot c^j - \delta c^j\|^2 + \|\dot v^j - \delta v^j\|_V^2 + \|c^j - w^{h,j}\|_V^2 + \|\dot\psi^j - \delta\psi^j\|^2\\
&\quad +\|\dot\varphi^j - \delta\varphi^j\|_V^2 + \|\psi^j - r^{h,j}\|_V^2 + I_j\Big) + C\max_{0\le n\le N}\Big\{\|c^n - w^{h,n}\|^2 + \|\psi^n - r^{h,n}\|^2\Big\}\\
&\quad +\frac{C}{k}\sum_{j=1}^{N-1}\Big(\|c^j - w^{h,j} - (c^{j+1} - w^{h,j+1})\|^2 + \|\psi^j - r^{h,j} - (\psi^{j+1} - r^{h,j+1})\|^2\Big)\\
&\quad +C\Big(\|c^0 - c^{0h}\|^2 + \|v^0 - v^{0h}\|_V^2 + \|u^0 - u^{0h}\|_V^2 + \|\psi^0 - \psi^{0h}\|^2 + \|\varphi^0 - \varphi^{0h}\|_V^2\Big),
\end{aligned}
$$

where C is a positive constant which does not depend on parameters h and k, and I_j is the integration error given by

$$
I_j = \left\|\int_0^{t_j} v(s)\,ds - k\sum_{l=1}^j v_l\right\|_V^2.
$$

Proof. Again, to simplify the calculations we assume that $\tau = 1$. It is easy to extend the calculations presented below to a more general situation.

First, we obtain the error estimates for the acceleration. Therefore, we subtract variational equation (6) at time $t = t_n$ for a test function $w = w^h \in V^h \subset V$ and discrete variational equation (11) to obtain

$$\begin{aligned}
&\rho(c^n - c^{hk,n} + \dot{c}^n - \delta c^{hk,n}, w^h) + k_1((u^n - u^{hk,n})_{xx} + (v^n - v^{hk,n})_{xx}, w^h_{xx})\\
&\quad +(a(u^n - u^{hk,n})_x + a^*(v^n - v^{hk,n})_x, w^h_x)\\
&\quad +b(\varphi^n - \varphi^{hk,n}, w^h_x) + \eta((\varphi^n - \varphi^{hk,n})_x, w^h_{xx}) = 0 \quad \forall w^h \in V^h,
\end{aligned}$$

and so, we have

$$\begin{aligned}
&\rho(c^n - c^{hk,n} + \dot{c}^n - \delta c^{hk,n}, c^n - c^{hk,n}) + k_1((u^n - u^{hk,n})_{xx} + (v^n - v^{hk,n})_{xx}, (c^n - c^{hk,n})_{xx})\\
&\quad +(a(u^n - u^{hk,n})_x + a^*(v^n - v^{hk,n})_x, (c^n - c^{hk,n})_x)\\
&\quad +b(\varphi^n - \varphi^{hk,n}, (c^n - c^{hk,n})_x) + \eta((\varphi^n - \varphi^{hk,n})_x, (c^n - c^{hk,n})_{xx})\\
&= \rho(c^n - c^{hk,n} + \dot{c}^n - \delta c^{hk,n}, c^n - w^h) + k_1((u^n - u^{hk,n})_{xx} + (v^n - v^{hk,n})_{xx}, (c^n - w^h)_{xx})\\
&\quad +(a(u^n - u^{hk,n})_x + a^*(v^n - v^{hk,n})_x, (c^n - w^h)_x)\\
&\quad +b(\varphi^n - \varphi^{hk,n}, (c^n - w^h)_x) + \eta((\varphi^n - \varphi^{hk,n})_x, (c^n - w^h)_{xx}) \quad \forall w^h \in V^h.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&(\dot{c}^n - \delta c^{hk,n}, c^n - c^{hk,n}) = (\dot{c}^n - \delta c^n, c^n - c^{hk,n})\\
&\quad +(\delta c^n - \delta c^{hk,n}, c^n - c^{hk,n}),\\
&\Big(\delta c^n - \delta c^{hk,n}, c^n - c^{hk,n}\Big) \geq \frac{1}{2k}\Big\{\|c^n -^{hk,n}\|^2 - \|c^{n-1} - c^{hk,n-1}\|^2\Big\},\\
&((v^n - v^{hk,n})_x, (c^n - c^{hk,n})_x) = ((v^n - v^{hk,n})_x, (\dot{v}^n - \delta v^n)_x)\\
&\quad +\frac{1}{2k}\Big\{\|(v^n - v^{hk,n})_x\|^2 - \|(v^{n-1} - v^{hk,n-1})_x\|^2\\
&\quad +\|(v^n - v^{hk,n} - (v^{n-1} - v^{hk,n-1}))_x\|^2\Big\},\\
&((v^n - v^{hk,n})_{xx}, (c^n - c^{hk,n})_{xx}) = ((v^n - v^{hk,n})_{xx}, (\dot{v}^n - \delta v^n)_{xx})\\
&\quad +\frac{1}{2k}\Big\{\|(v^n - v^{hk,n})_{xx}\|^2 - \|(v^{n-1} - v^{hk,n-1})_{xx}\|^2\\
&\quad +\|(v^n - v^{hk,n} - (v^{n-1} - v^{hk,n-1}))_{xx}\|^2\Big\},
\end{aligned}$$

and using several times Cauchy's inequality (14) it follows that

$$\begin{aligned}
&\frac{\rho}{2k}\Big\{\|c^n - c^{hk,n}\|^2 - \|c^{n-1} - c^{hk,n-1}\|^2\Big\} + \eta((\varphi^n - \varphi^{hk,n})_x, (c^n - c^{hk,n})_{xx})\\
&\quad +\frac{a^*}{2k}\Big\{\|(v^n - v^{hk,n})_x\|^2 - \|(v^{n-1} - v^{hk,n-1})_x\|^2\\
&\quad +\|(v^n - v^{hk,n} - (v^{n-1} - v^{hk,n-1}))_x\|^2\Big\}\\
&\quad +\frac{k_1}{k}\Big\{\|(v^n - v^{hk,n})_{xx}\|^2 - \|(v^{n-1} - v^{hk,n-1})_{xx}\|^2\\
&\quad +\|(v^n - v^{hk,n} - (v^{n-1} - v^{hk,n-1}))_{xx}\|^2\Big\} + b(\varphi^n - \varphi^{hk,n}, (c^n - c^{hk,n})_x)\\
&\quad +k_1((u^n - u^{hk,n})_{xx}, (c^n - c^{hk,n})_{xx}) + (a(u^n - u^{hk,n})_x, (c^n - c^{hk,n})_x)\\
&\leq C\Big(\|\dot{c}^n - \delta c^n\|^2 + \|\dot{v}^n - \delta v^n\|^2_V + \|c^n - w^h\|^2_V + \|u^n - u^{hk,n}\|^2_V\\
&\quad +\|(v^n - v^{hk,n})_x\|^2 + \|c^n - c^{hk,n}\|^2 + \|(v^n - v^{hk,n})_{xx}\|^2 + \|(\varphi^n - \varphi^{hk,n})_x\|^2\\
&\quad +\|\varphi^n - \varphi^{hk,n}\|^2 + (\delta c^n - \delta c^{hk,n}, c^n - w^h)\Big) \quad \forall w^h \in V^h.
\end{aligned} \tag{17}$$

Proceeding in a similar form for the porosity speed, we obtain the following estimates, for all $r^h \in V^h$,

$$
\begin{aligned}
&\frac{J}{2k}\left\{\|\psi^n-\psi^{hk,n}\|^2-\|\psi^{n-1}-\psi^{hk,n-1}\|^2\right\}+\eta((\psi^n-\psi^{hk,n})_x,(u^n-u^{hk,n}+v^n-v^{hk,n})_{xx})\\
&\quad+\frac{\delta^*}{2k}\left\{\|(\varphi^n-\varphi^{hk,n})_x\|^2-\|(\varphi^{n-1}-\varphi^{hk,n-1})_x\|^2\right.\\
&\quad\left.+\|(\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1}))_x\|^2\right\}+b(\psi^n-\psi^{hk,n},(u^n-u^{hk,n}+v^n-v^{hk,n})_x)\\
&\quad+\frac{k_2}{k}\left\{\|(\varphi^n-\varphi^{hk,n})_{xx}\|^2-\|(\varphi^{n-1}-\varphi^{hk,n-1})_{xx}\|^2\right\}\\
&\quad+\frac{\xi}{k}\left\{\|\varphi^n-\varphi^{hk,n}\|^2-\|\varphi^{n-1}-\varphi^{hk,n-1}\|^2+\|\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1})\|^2\right\}\\
&\le C\Big(\|\dot\psi^n-\delta\psi^n\|^2+\|\dot\varphi^n-\delta\varphi^n\|_V^2+\|\psi^n-r^h\|_V^2+\|\psi^n-\psi^{hk,n}\|^2\\
&\quad+\|(\varphi^n-\varphi^{hk,n})_x\|^2+\|(\varphi^n-\varphi^{hk,n})_{xx}\|^2+\|\varphi^n-\varphi^{hk,n}\|^2+\|u^n-u^{hk,n}\|_V^2\\
&\quad+\|(v^n-v^{hk,n})_x\|^2+\|(v^n-v^{hk,n})_{xx}\|^2+(\delta\psi^n-\delta\psi^{hk,n},\psi^n-r^h)\Big).
\end{aligned}\tag{18}
$$

Combining estimates (17) and (18) it follows that, for all $w^h,\,r^h\in V^h$,

$$
\begin{aligned}
&\frac{\rho}{2k}\left\{\|c^n-c^{hk,n}\|^2-\|c^{n-1}-c^{hk,n-1}\|^2\right\}+\eta((\varphi^n-\varphi^{hk,n})_x,(c^n-c^{hk,n})_{xx})\\
&\quad+\frac{a^*}{2k}\left\{\|(v^n-v^{hk,n})_x\|^2-\|(v^{n-1}-v^{hk,n-1})_x\|^2\right.\\
&\quad\left.+\|(v^n-v^{hk,n}-(v^{n-1}-v^{hk,n-1}))_x\|^2\right\}+\eta((\psi^n-\psi^{hk,n})_x,(u^n-u^{hk,n}+v^n-v^{hk,n})_{xx})\\
&\quad+\frac{k_1}{k}\left\{\|(v^n-v^{hk,n})_{xx}\|^2-\|(v^{n-1}-v^{hk,n-1})_{xx}\|^2\right.\\
&\quad\left.+\|(v^n-v^{hk,n}-(u^{n-1}-u^{hk,n-1}))_{xx}\|^2\right\}+b(\varphi^n-\varphi^{hk,n},(c^n-c^{hk,n})_x)\\
&\quad+k_1((u^n-u^{hk,n})_{xx},(c^n-c^{hk,n})_{xx})+a((u^n-u^{hk,n})_x,(c^n-c^{hk,n})_x)\\
&\quad+\frac{J}{2k}\left\{\|\psi^n-\psi^{hk,n}\|^2-\|\psi^{n-1}-\psi^{hk,n-1}\|^2\right\}\\
&\quad+\frac{\delta^*}{2k}\left\{\|(\varphi^n-\varphi^{hk,n})_x\|^2-\|(\varphi^{n-1}-\varphi^{hk,n-1})_x\|^2\right.\\
&\quad\left.+\|(\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1}))_x\|^2\right\}+b(\psi^n-\psi^{hk,n},(u^n-u^{hk,n}+v^n-v^{hk,n})_x)\\
&\quad+\frac{k_2}{k}\left\{\|(\varphi^n-\varphi^{hk,n})_{xx}\|^2-\|(\varphi^{n-1}-\varphi^{hk,n-1})_{xx}\|^2\right\}\\
&\quad+\frac{\xi}{k}\left\{\|\varphi^n-\varphi^{hk,n}\|^2-\|\varphi^{n-1}-\varphi^{hk,n-1}\|^2+\|\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1})\|^2\right\}\\
&\le C\Big(\|\dot c^n-\delta c^n\|^2+\|\dot v^n-\delta v^n\|_V^2+\|c^n-w^h\|_V^2+\|u^n-u^{hk,n}\|_V^2\\
&\quad+\|(v^n-v^{hk,n})_x\|^2+\|c^n-c^{hk,n}\|^2+\|(v^n-v^{hk,n})_{xx}\|^2+\|(\varphi^n-\varphi^{hk,n})_x\|^2\\
&\quad+\|\varphi^n-\varphi^{hk,n}\|^2+(\delta c^n-\delta c^{hk,n},c^n-w^h)+\|\dot\psi^n-\delta\psi^n\|^2+\|\dot\varphi^n-\delta\varphi^n\|_V^2\\
&\quad+\|\psi^n-r^h\|_V^2+\|\psi^n-\psi^{hk,n}\|^2+\|(\varphi^n-\varphi^{hk,n})_{xx}\|^2+(\delta\psi^n-\delta\psi^{hk,n},\psi^n-r^h)\Big).
\end{aligned}
$$

Now, thanks again to the assumptions on the constitutive coefficients we observe that

$$
\begin{aligned}
&\eta((\varphi^n-\varphi^{hk,n})_x,(\delta v^n-\delta v^{hk,n})_{xx})+\eta((\delta\varphi^n-\delta\varphi^{hk,n})_x,(v^n-v^{hk,n})_{xx})\\
&\quad=\frac{\eta}{k}\left\{((\varphi^n-\varphi^{hk,n})_x,(v^n-v^{hk,n})_{xx})-((\varphi^{n-1}-\varphi^{hk,n-1})_x,(v^{n-1}-v^{hk,n-1})_{xx})\right.\\
&\qquad\left.+((\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1}))_x,(v^n-v^{hk,n}-(v^{n-1}-v^{hk,n-1}))_{xx})\right\},\\
&b(\varphi^n-\varphi^{hk,n},(\delta v^n-\delta v^{hk,n})_x)+b(\delta\varphi^n-\delta\varphi^{hk,n},(v^n-v^{hk,n})_x)\\
&\quad=\frac{b}{k}\left\{(\varphi^n-\varphi^{hk,n},(v^n-v^{hk,n})_x)-(\varphi^{n-1}-\varphi^{hk,n-1},(v^{n-1}-v^{hk,n-1})_x)\right.\\
&\qquad\left.+(\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1}),(v^n-v^{hk,n}-(v^{n-1}-v^{hk,n-1}))_x)\right\},\\
&k_1\|(v^n-v^{hk,n}-(v^{n-1}-v^{hk,n-1}))_{xx}\|^2+\delta^*\|(\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1}))_x\|^2\\
&\qquad+2\eta((\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1}))_x,(v^n-v^{hk,n}-(v^{n-1}-v^{hk,n-1}))_{xx})\ge 0,
\end{aligned}
$$

$$a^*\|(v^n-v^{hk,n}-(v^{n-1}-v^{hk,n-1}))_x\|^2+\xi\|\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1})\|^2$$
$$+2b(\varphi^n-\varphi^{hk,n}-(\varphi^{n-1}-\varphi^{hk,n-1}),(v^n-v^{hk,n}-(v^{n-1}-v^{hk,n-1}))_x)\geq 0.$$

Multiplying the above estimates by k and summing up to n we find that, for all $w^h, r^h\in V^h$,

$$\begin{aligned}
&\rho\|c^n-c^{hk,n}\|^2+2\eta((\varphi^n-\varphi^{hk,n})_x,(v^n-v^{hk,n})_{xx})+a^*\|(v^n-v^{hk,n})_x\|^2\\
&\quad+k_1\|(v^n-v^{hk,n})_{xx}\|^2+2b(\varphi^n-\varphi^{hk,n},(v^n-v^{hk,n})_x)\\
&\quad+k\sum_{j=1}^n\Big\{k_1((u^j-u^{hk,j})_{xx},(\delta v^j-\delta v^{hk,j})_{xx})+a((u^j-u^{hk,j})_x,(\delta v^j-\delta v^{hk,j})_x)\Big\}\\
&\quad+J\|\psi^n-\psi^{hk,n}\|^2+\delta^*\|(\varphi^n-\varphi^{hk,n})_x\|^2+k_2\|(\varphi^n-\varphi^{hk,n})_{xx}\|^2+\xi\|\varphi^n-\varphi^{hk,n}\|^2\\
&\leq Ck\sum_{j=1}^n\Big(\|\dot c^j-\delta c^j\|^2+\|\dot v^j-\delta v^j\|_V^2+\|c^j-w^{h,j}\|_V^2+\|u^j-u^{hk,j}\|_V^2\\
&\quad+\|(v^j-v^{hk,j})_x\|^2+\|c^j-c^{hk,j}\|^2+\|(v^j-v^{hk,j})_{xx}\|^2+\|(\varphi^j-\varphi^{hk,j})_x\|^2\\
&\quad+\|\varphi^j-\varphi^{hk,j}\|^2+(\delta c^j-\delta c^{hk,j},c^j-w^{h,j})+\|\dot\psi^j-\delta\psi^j\|^2+\|\dot\varphi^j-\delta\varphi^j\|_V^2\\
&\quad+\|\psi^j-r^{h,j}\|_V^2+\|\psi^j-\psi^{hk,j}\|^2+\|(\varphi^j-\varphi^{hk,j})_{xx}\|^2+(\delta\psi^j-\delta\psi^{hk,j},\psi^j-r^{h,j})\Big)\\
&\quad+C\Big(\|c^0-c^{0h}\|^2+\|v^0-v^{0h}\|_V^2+\|\psi^0-\psi^{0h}\|^2+\|\varphi^0-\varphi^{0h}\|_V^2\Big).
\end{aligned}$$

By using again the conditions on the constitutive coefficients it follows that

$$k_1\|(v^n-v^{hk,n})_{xx}\|^2+\delta^*\|(\varphi^n-\varphi^{hk,n})_x\|^2+2\eta((\varphi^n-\varphi^{hk,n})_x,(v^n-v^{hk,n})_{xx})\geq 0,$$
$$a^*\|(v^n-v^{hk,n})_x\|^2+\xi\|\varphi^n-\varphi^{hk,n}\|^2+2b(\varphi^n-\varphi^{hk,n},(v^n-v^{hk,n})_x)\geq 0.$$

Finally, taking into account that

$$\begin{aligned}
&\sum_{j=1}^n(c^j-c^{hk,j}-(c^{j-1}-c^{hk,j-1}),c^j-w^{h,j})=(c^n-c^{hk,n},c^n-w^{h,n})\\
&\quad+(c^{0h}-c^0,c^1-w^{h,1})+\sum_{j=1}^{n-1}(c^j-c^{hk,j},c^j-w^{h,j}-(c^{j+1}-w^{h,j+1})),\\
&\sum_{j=1}^n(\psi^j-\psi^{hk,j}-(\psi^{j-1}-\psi^{hk,j-1}),\psi^j-r^{h,j})=(\psi^n-\psi^{hk,n},\psi^n-r^{h,n})\\
&\quad+(\psi^{0h}-\psi^0,\psi^1-r^{h,1})+\sum_{j=1}^{n-1}(\psi^j-\psi^{hk,j},\psi^j-r^{h,j}-(\psi^{j+1}-r^{h,j+1})),\\
&k\sum_{j=1}^n((u^j-u^{hk,j})_{xx},(\delta v^j-\delta v^{hk,j})_{xx})=((u^n-u^{hk,n})_{xx},(v^n-v^{hk,n})_{xx})\\
&\quad+((u^{0h}-\psi^0)_{xx},(v^0-v^{0h})_{xx})-k\sum_{j=1}^{n-1}((v^j-v^{hk,j})_{xx},(v^j-v^{hk,j})_{xx}),\\
&k\sum_{j=1}^n((u^j-u^{hk,j})_x,(\delta v^j-\delta v^{hk,j})_x)=((u^n-u^{hk,n})_x,(v^n-v^{hk,n})_x)\\
&\quad+((u^{0h}-\psi^0)_x,(v^0-v^{0h})_x)-k\sum_{j=1}^{n-1}((v^j-v^{hk,j})_x,(v^j-v^{hk,j})_x),\\
&\|u^n-u^{hk,n}\|_V^2\leq C\Big(\|u^0-u^{0h}\|_V^2+I_n+k\sum_{j=1}^n\|v^j-v^{hk,j}\|_V^2\Big),
\end{aligned}$$

using again a discrete version of Gronwall's inequality (see [2]) we obtain the desired a priori error estimates.

The estimates provided in the above theorem can be used to obtain the convergence order of the approximations given by discrete problem (11)-(13). Hence, as an example, if we assume the additional regularity:

$$u \in H^4(0,T;Y)\cap H^3(0,T;V)\cap C^2([0,T];H^3(0,\ell)),$$
$$\varphi \in H^3(0,T;Y)\cap H^2(0,T;V)\cap C^1([0,T];H^3(0,\ell)), \tag{19}$$

we obtain the linear convergence of the algorithm applying some results on the approximation by finite elements (see [3]) and previous estimates already derived in [2]. We have the following.

Corollary 5. Under the assumptions of Theorem 4, it follows that there exists a positive constant $C > 0$, independent of the discretization parameters h and k, such that

$$\max_{0\le n\le N}\Big\{\|c^n - c^{hk,n}\| + \|v^n - v^{hk,n}\|_V + \|u^n - u^{hk,n}\|_V + \|\psi^n - \psi^{hk,n}\| \\ +\|\varphi^n - \varphi^{hk,n}\|_V\Big\} \le C(h+k).$$

4. Numerical results

In this final section, we describe the numerical scheme implemented in MATLAB for solving Problem VP^{hk}, and we show some numerical examples to demonstrate the accuracy of the approximations and the behavior of the solution

4.1. Numerical scheme

As a first step, given the solution $u^{hk,n-1}$, $v^{hk,n-1}$ and $\varphi^{hk,n-1}$ at time t_{n-1}, variables $c^{hk,n}$ and $\psi^{hk,n}$ are obtained by solving the discrete linear system, for all w^h, $r^h \in V^h$.

$$\begin{aligned}
&\rho(c^{hk,n} + \frac{\tau}{k}c^{hk,n}, w^h) + k_1(k^2 c^{hk,n}_{xx} + \tau k c^{hk,n}_{xx}, w^h_{xx}) + (ak^2 c^{hk,n}_x + a^* k c^{hk,n}_x, w^h_x)\\
&\quad = \rho(\frac{\tau}{k}c^{hk,n-1}, w^h) - b(\varphi^{hk,n}, w^h_x) - \eta(\varphi^{hk,n}_x, w^h_{xx})\\
&\qquad -k_1(u^{hk,n-1}_{xx} + k v^{hk,n-1}_{xx} + \tau v^{hk,n-1}_{xx}, w^h_{xx}))\\
&\qquad -(au^{hk,n-1}_x + akv^{hk,n-1}_x + a^* v^{hk,n-1}_x, w^h_x),\\
&\frac{J}{k}(\psi^{hk,n}, r^h) + k_2(k\psi^{hk,n}_{xx}, r^h_{xx}) + \delta^*(k\psi^{hk,n}_x, r^h_x) + \xi(k\psi^{hk,n}, r^h)\\
&\quad = \frac{J}{k}(\psi^{hk,n-1}, r^h) - b(u^{hk,n}_x + \tau v^{hk,n}_x, r^h) - \eta(u^{hk,n}_{xx} + \tau v^{hk,n}_{xx}, r^h_x)\\
&\qquad -k_2(\varphi^{hk,n-1}_{xx}, r^h_{xx}) - \delta^*(\varphi^{hk,n-1}_x, r^h_x) - \xi(\varphi^{hk,n-1}, r^h).
\end{aligned}$$

We note that the numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical run ($h = k = 0.001$) took about 6.05 seconds of CPU time.

4.2. First example: numerical convergence

As an academical example, in order to show the accuracy of the approximations we consider problem (1)-(4) with the data:

$$\ell = 1,\quad T = 1,\quad \rho = 1,\quad a = 2,\quad a^* = 3,\quad b = 1,\quad \eta = 2,\quad k_1 = 2,$$
$$k_2 = 1,\quad \tau = 1,\quad J = 1,\quad \delta^* = 3,\quad \xi = 1.$$

By using the following initial conditions, for all $x \in (0,1)$,

$$u^0(x) = v^0(x) = c^0 = \varphi^0(x) = \psi^0(x) = x^3(x-1)^3,$$

and considering the (artificial) supply terms, for all $(x,t) \in (0,1)\times(0,1)$,

$$F_1(x,t) = e^t(2x^6 - 12x^5 - 129x^4 + 526x^3 + 903x^2 - 1266x + 276),$$
$$F_2(x,t) = e^t(2x^6 + 6x^5 - 114x^4 - 278x^3 + 966x^2 - 630x + 96),$$

the exact solution to the above problem can be easily calculated and it has the form, for $(x,t) \in [0,1]\times[0,1]$:

$$u(x,t) = \varphi(x,t) = e^t x^3(x-1)^3.$$

Table 1
Example 1: Numerical errors for some h and k.

$h\downarrow k\rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	1.127922	1.128313	1.128495	1.128350	1.128021	1.127642	1.127487
$1/2^4$	0.613492	0.613761	0.613874	0.613698	0.613336	0.612917	0.612738
$1/2^5$	0.319827	0.320014	0.320081	0.319878	0.319485	0.319032	0.318837
$1/2^6$	0.164731	0.164864	0.164901	0.164681	0.164266	0.163794	0.163594
$1/2^7$	0.085343	0.085424	0.085439	0.085203	0.084768	0.084283	0.084086
$1/2^8$	0.045402	0.045412	0.045401	0.045150	0.044689	0.044200	0.044019
$1/2^9$	0.025669	0.025557	0.025508	0.025237	0.024751	0.024273	0.024104
$1/2^{10}$	0.016254	0.015921	0.015807	0.015509	0.014999	0.014526	0.014335
$1/2^{11}$	0.012159	0.011418	0.011250	0.010903	0.010416	0.009857	0.009710

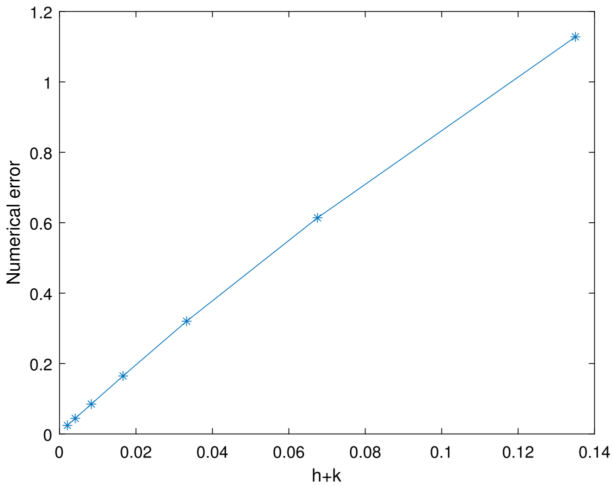

Fig. 1. Example 1: Asymptotic constant error.

Thus, the approximation errors estimated by

$$\max_{0\le n\le N}\Big\{\|c^n-c^{hk,n}\|+\|v^n-v^{hk,n}\|_V+\|u^n-u^{hk,n}\|_V+\|\psi^n-\psi^{hk,n}\|$$
$$+\|\varphi^n-\varphi^{hk,n}\|_V\Big\}$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 5, is achieved.

4.3. Second example: dependence of the energy decay on some viscous parameters

In this second example, we will investigate the dependence on parameters a^*, k_1^* and ξ^* of the energy for the poroelastic problem assuming the three dissipation mechanisms introduced in Section 2.

First, we consider the viscoelastic case. Then, we assume that there are not supply terms, and we use the final time $T=10$, the data

$$\ell=1,\quad \rho=1,\quad a=2,\quad b=1,\quad \eta=2,\quad k_1=2,$$
$$k_2=1,\quad \tau=1,\quad J=1,\quad \delta^*=3,\quad \xi=1,$$

and the initial conditions, for all $x\in(0,1)$,

$$u^0(x)=v^0(x)=c^0(x)=\varphi^0(x)=\psi^0(x)=x^3(x-1)^3.$$

Taking the discretization parameters $h=0.001$ and $k=0.0001$, the evolution in time of the discrete energy is plotted in Fig. 2 for some values of the viscous parameter a^* (in both natural and semi-log scales). As we can see, the exponential decay seems to be achieved for every value of parameter a^* although the energy decreases faster when it increases.

Secondly, we simulate the hyperviscosity case and we use the data and the initial conditions given in the previous viscosity case. Taking the discretization parameters $h=0.001$ and $k=0.0001$, the evolution in time of the discrete energy

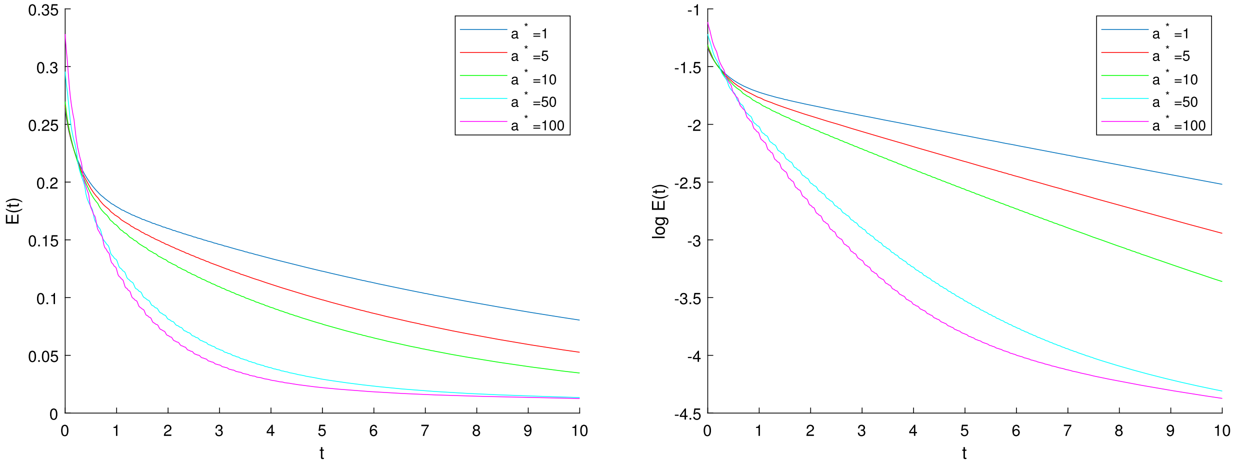

Fig. 2. Example 2: Evolution in time of the discrete energy in the viscoelastic case for different values of a^* (natural and semi-log scales).

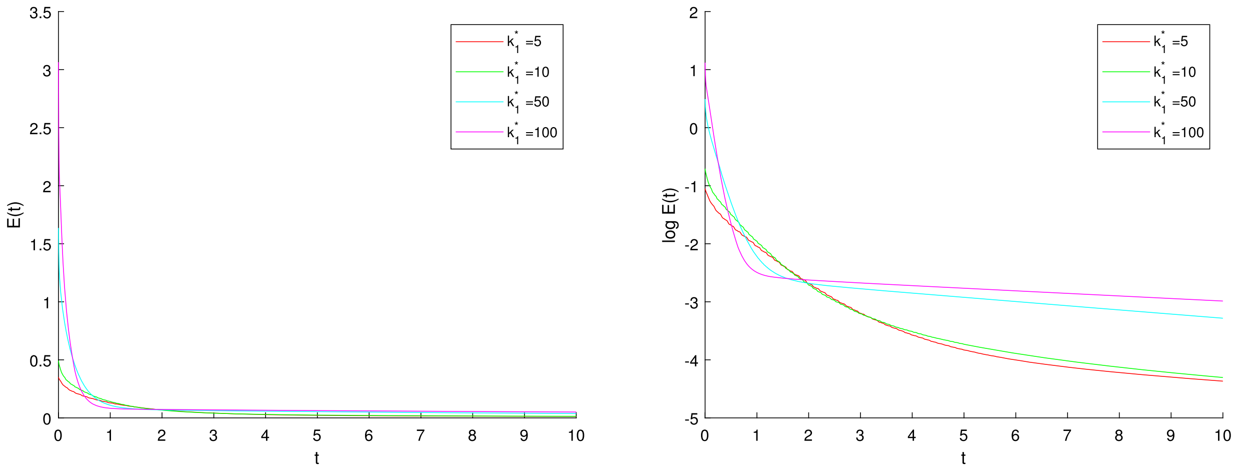

Fig. 3. Example 2: Evolution in time of the discrete energy in the hyperviscosity case for different values of k_1^* (natural and semi-log scales).

is shown in Fig. 3 for some values of the viscous parameter k_1^* (in both natural and semi-log scales). An exponential energy decay seems to be achieved for all the values of the parameter although now the fastest convergence is found for the minimum value of the chosen parameter.

Finally, we consider the weak viscoporosity case. We use the same data as in the two previous cases, but now, since the constitutive equations are different (see Section 2), we use the following initial conditions, for all $x \in (0, 1)$,

$$u^0(x) = v^0(x) = \varphi^0(x) = \psi^0(x) = \xi^0(x) = x^3(x-1)^3,$$

where ξ denotes the porosity acceleration and ξ^0 is a new initial condition for this variable.

Taking the discretization parameters $h = 0.001$ and $k = 0.0001$, the evolution in time of the discrete energy is plotted in Fig. 4 for some values of the viscoporous parameter ξ^* (in both natural and semi-log scales). We can see again that an exponential energy decay seems to be achieved, being the convergence faster for the minimum value of the parameter.

4.4. Third example: comparison of the dissipation mechanisms

In this example we are going to compare the energy decay of the problem by using the three dissipation mechanisms: viscosity (second-order term in the elasticity equation), hyperviscosity (fourth-order term in the elasticity equation) and weak viscoporosity (zeroth-order term in the porosity equation).

In all the cases, we assume now that there are not supply terms, and we use the final time $T = 10$, the data

$$\ell = 1, \quad \rho = 1, \quad a = 10, \quad b = 1, \quad \eta = 2, \quad k_1 = 10,$$

$$k_2 = 10, \quad \tau = 0.1, \quad J = 1, \quad \delta^* = 3, \quad \xi = 5,$$

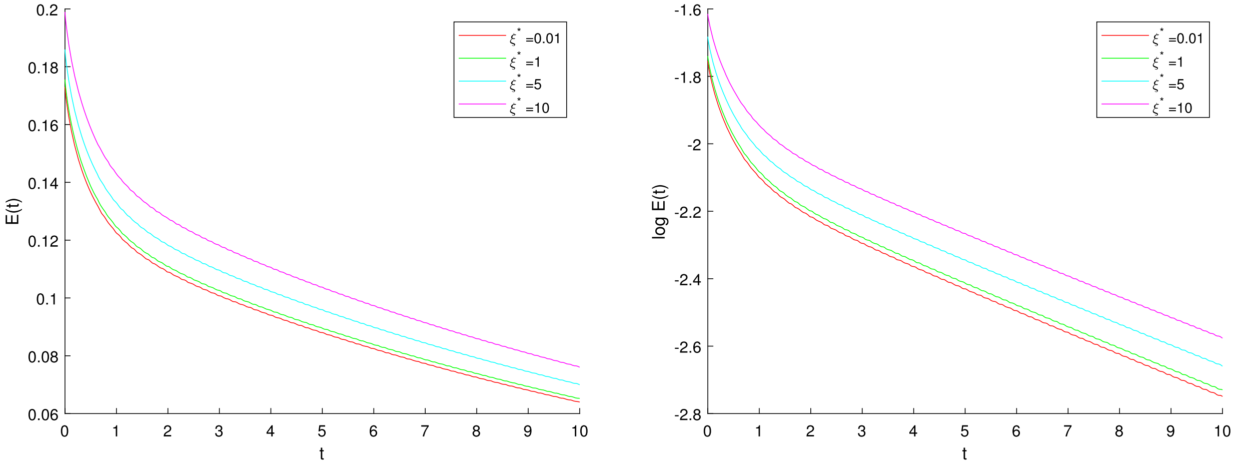

Fig. 4. Example 2: Evolution in time of the discrete energy in the weak viscoporosity case for different values of ξ^* (natural and semi-log scales).

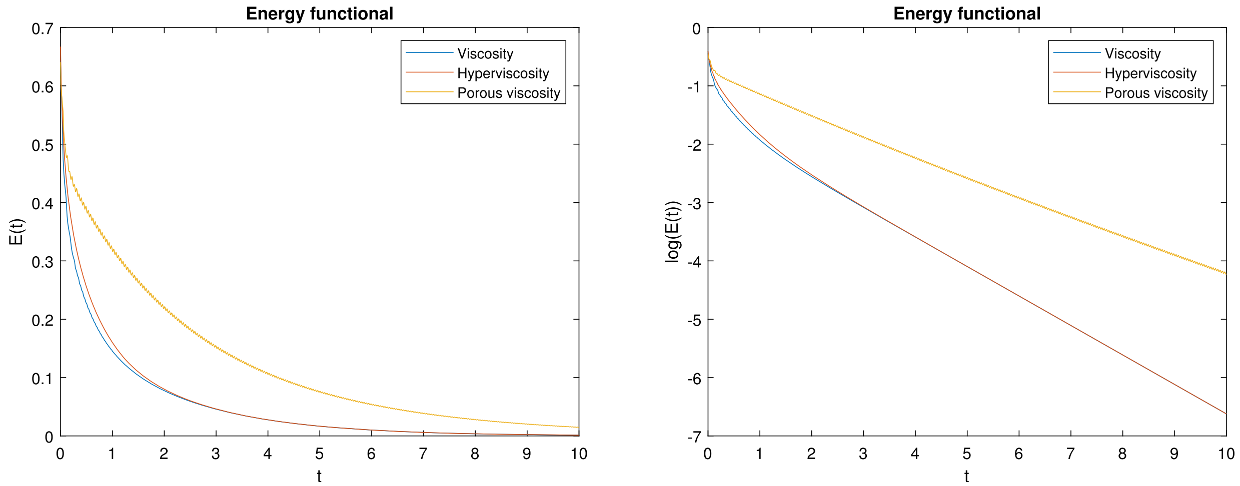

Fig. 5. Example 3: Evolution in time of the discrete energy for different dissipation mechanisms (natural and semi-log scales).

and the initial conditions, for all $x \in (0, 1)$,

$$u^0(x) = v^0(x) = c^0(x) = \varphi^0(x) = \psi^0(x) = \xi^0(x) = x^3(x-1)^3.$$

Regarding the three simulated cases, we have used coefficients $a^* = 10$ and $\xi^* = k_1^* = 0$ for the viscosity case, coefficients $k_1^* = 10$ and $a^* = \xi^* = 0$ for the hyperviscosity case, and coefficients $\xi^* = 10$ and $a^* = k_1^* = 0$ for the weak viscoporosity case. Moreover, we have taken the discretization parameters $h = k = 10^{-4}$ in all the cases.

Therefore, the evolution in time of the three discrete energies is plotted in Fig. 5 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved by using the three dissipation mechanisms. From these results, we may conclude that the best dissipation mechanism is the viscosity case, with a similar behavior for the hyperviscosity one, being clearly the worst dissipation mechanism that one corresponding to the weak viscoporosity case.

5. Conclusions

In this work, we analyzed, from the numerical point of view, a poro-elastic problem assuming three possible dissipation mechanisms of MGT-type: viscosity dissipation (second-order term), hyperviscosity dissipation (fourth-order term) and weak viscoporosity dissipation (zeroth-order term). A fully discrete approximation was introduced by using the finite element method for the spatial variable and the implicit Euler scheme to discretize the time derivatives. A discrete stability property and a priori error estimates were proved, from which the linear convergence of the algorithm was derived under adequate additional regularity conditions. Finally, we performed some numerical simulations to analyze this theoretical behavior. First, a simple one-dimensional example was chosen to demonstrate the numerical convergence and the linear convergence with respect to parameter $k+h$. Secondly, the discrete energy decay was shown for each dissipation mechanism and different constitutive parameters. Thirdly, we compared the three cases in a similar example, concluding that the best dissipation mechanism was the viscosity one.

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References

[1] A. Borichev, Y. Tomilov, Optimal polynomial decay of function semigroups, Math. Ann. 347 (2009) 455–478.
[2] M. Campo, J.R. Fernández, K. Kuttler, M. Shillor, J. Viaño, Numerical analysis and simulations of a dynamic frictionless contact problem with damage, Comput. Methods Appl. Mech. Eng. 196 (1–3) (2006) 476–488.
[3] P.G. Ciarlet, The finite element method for elliptic problems, in: Handbook of Numerical Analysis. Vol. II, Handb. Numer. Anal., North-Holland, 1991, pp. 17–352.
[4] P. Clement, Approximation by finite element functions using local regularization, RAIRO Math. Model. Numer. Anal. 9 (1975) 77–84.
[5] M. Conti, V. Pata, M. Pellicer, R. Quintanilla, On the analiticity of the MGT-viscoelastic plate with heat conduction, J. Differ. Equ. 269 (2020) 7862–7880.
[6] M. Conti, V. Pata, R. Quintanilla, Thermoelasticity of Moore-Gibson-Thompson type with history dependence in the temperature, Asymptot. Anal. 120 (2020) 1–21.
[7] M. Conti, V. Pata, M. Pellicer, R. Quintanilla, A new approach to MGT thermoviscoelasticity, Discrete Contin. Dyn. Syst. 41 (2021) 4645–4666.
[8] S. Cowin, The viscoelastic behavior of linear elastic materials with voids, J. Elast. 15 (1985) 185–191.
[9] S. Cowin, J. Nunziato, Linear elastic materials with voids, J. Elast. 13 (2) (1983) 125–147, https://doi.org/10.1007/BF00041230.
[10] S. De Cicco, D. Ieşan, On the theory of thermoelastic materials with a double porosity structure, J. Therm. Stresses (2021).
[11] B. Feng, T. Apalara, Optimal decay for a porous elasticity system with memory, J. Math. Anal. Appl. 470 (2019) 1108–1128.
[12] B. Feng, M. Yin, Decay of solutions for one-dimensional porous elasticity system with memory: the case of nonequal waves speed, Math. Mech. Solids 24 (2019) 2361–2373.
[13] B. Feng, L. Yang, D. Almeida Junior, Stabilization for an inhomogeneous porous-elastic system with temperatures and microtemperatures, Z. Angew. Math. Mech. 101 (2021) e202000058.
[14] J.R. Fernández, R. Quintanilla, Moore-Gibson-Thompson theory for thermoelastic dielectrics, Appl. Math. Mech. 42 (2021) 309–316.
[15] J.R. Fernández, R. Quintanilla, On a mixture of an MGT viscous material and an elastic solid, Acta Mech. 233 (2022) 291–297, https://doi.org/10.1007/s00707-021-03124-z.
[16] J.R. Fernández, A. Magaña, R. Quintanilla, Decay of waves in strain gradient porous elasticity with MGT dissipation, Philos. Trans. R. Soc. A-Math. Phys. Eng. Sci. (2022), submitted for publication.
[17] M.E. Gurtin, Time-reversal and symmetry in the thermodynamics of materials with memory, Arch. Ration. Mech. Anal. 44 (1972) 387–399.
[18] D. Ieşan, A theory of thermoelastic materials with voids, Acta Mech. 60 (1–2) (1986) 67–89, https://doi.org/10.1007/BF01302942.
[19] D. Ieşan, A gradient theory of porous elastic solids, Z. Angew. Math. Mech. 100 (2020) e201900241.
[20] D. Ieşan, R. Quintanilla, On a theory of thermoelastic materials with a double porosity structure, J. Therm. Stresses 37 (2014) 1017–1036.
[21] D. Ieşan, R. Quintanilla, Viscoelastic materials with a double porosity structure, C. R. Mecanique 347 (2019) 124–130.
[22] Z. Liu, R. Quintanilla, Dual-phase-lag one-dimensional thermo-porous-elasticity with microtemperatures, Comput. Appl. Math. 40 (2021) 231.
[23] A. Miranville, R. Quintanilla, Exponential stability in type III thermoelasticity with voids, Appl. Math. Lett. 94 (2019) 30–37.
[24] A. Miranville, R. Quintanilla, Exponential decay in one-dimensional type II thermoviscoelasticity with voids, J. Comput. Appl. Math. 368 (2020) 112573.
[25] J.W. Nunziato, S.C. Cowin, A nonlinear theory of elastic materials with voids, Arch. Ration. Mech. Anal. 72 (2) (1979) 175–201, https://doi.org/10.1007/BF00249363.
[26] R. Quintanilla, Moore-Gibson-Thompson thermoelasticity, Math. Mech. Solids 24 (2019) 4020–4031.
[27] B. Straughan, Stability and Wave Motion in Porous Media, Applied Mathematical Sciences, vol. 165, Springer-Verlag, Berlin, 2009.