Cost allocation in inventory transportation systems*

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Abstract

In this paper we deal with the cost allocation problem arising in an inventory transportation system with a single item and multiple agents that place joint orders using an EOQ policy. In our problem, the fixed order cost of each agent is the sum of a first component (common to all agents) plus a second component which depends on the distance from the agent to the supplier. We assume that agents are located on a line route, in the sense that if any subgroup of agents places a joint order, its fixed cost is the sum of the first component plus the second component of the agent in the group at maximal distance from the supplier. For these inventory transportation systems we introduce and characterize a rule which allows us to allocate the costs generated by the joint order. This rule has the same flavor as the Shapley value, but requires less computational effort. We show that our rule has good properties from the point of view of stability.

Key words: inventory transportation systems, cooperative games, core, cost allocation rule.

1 Introduction

In the context of modern commerce, franchising is the process of expanding a business whereby a company (franchisor) grants a license to an independent business owner (franchisee) to sell its products or render its services. Each party of a franchise agreement gives up some legal

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*Authors acknowledge the valuable comments of Fernando Bernstein and the financial support of Ministerio de Ciencia e Innovación through projects CSD2006-00032, MTM2008-06778-C02-01 and ECO2008-03484-C02-02/ECO, of *Generalitat Valenciana* through project ACOMP/2011/129, and of *Xunta de Galicia* through project INCITE09-207-064-PR.

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rights to gain others and to obtain benefits. The franchisor increases its number of outlets and receives additional income. The franchisee opens an established business with strong potential for success. Franchising offers economic agents a chance to manage and direct their own firms without having to take all the associated risks and, at the same time, has proven to be a powerful and efficient means of building up a business and of creating employment and wealth both at local and international level.

An important issue in franchising is the distribution of the products from the franchisor to the franchisees. In this context, the centralized inventory models are especially interesting. Consider a finite set of franchisees that distribute a single product manufactured by the franchisor. The franchisees place their orders according to Economic Order Quantity models (abbreviated to EOQ models; see for instance Zipkin, 2000 for details of these models). In EOQ models the firms (the franchisees in this case) face two main types of costs: fixed costs per order and holding costs. When there are two or more firms, and their fixed costs can be written as the sum of two components, one due to common setup costs and other due to firm-dependent transportation costs, we have what we call an inventory transportation system. In order to study one of those systems and to propose optimal policies for the franchisees we should use an inventory model. If we allow the franchisees to cooperate and to place orders jointly in order to reduce the total fixed costs, then we should use a centralized inventory model. In this case, we should in addition propose a rule to allocate the costs among the cooperating franchisees. In this paper we introduce and analyze a centralized inventory model and an allocation rule to tackle inventory transportation systems. Obviously, although we base our problem on the framework of franchising, its potential applications go much further and can be extended to many other areas.

To be more precise, in this paper we deal with the cost allocation problem in an inventory transportation system with a single item and multiple agents that place joint orders using an EOQ policy. In the case of our problem, the fixed order cost of each agent is the sum of a first component common to all agents, which reflects the setup cost, plus a second component which depends on the distance of the agent to the supplier. We assume that agents are located on a line route. By this we mean that if a group of agents places a joint order, the fixed cost incurred by this group of agents is the sum of the first component plus the second component of an agent whose distance from the supplier is maximal within the agents in the group. Notice that this kind of problem often arises in transportation management when firms use shipment consolidation strategies and other pooling strategies (see for instance Ülkü, 2009, for details on shipment consolidation).

This problem is related to other problems studied in the literature. For instance, it extends the problem studied in Meca et al. (2004). In both settings several agents need a certain product, which is sold by a single supplier, and they agree to place the orders jointly by means of the classical EOQ policy. The difference between this paper and Meca et al. (2004) is in the structure of the fixed costs. Meca et al. (2004) considers an identical fixed cost for all agents, which is shared by them if they cooperate; on the contrary we consider that a part of the fixed cost depends on each agent and that it is proportional to the distance between the agent and the supplier. Other variations of Meca et al. (2004) can be found in Anily and Haviv (2007), in Zhang (2009) and in Dror and Hartman (2007), but in all these papers authors consider problems which are different from that dealt with here. In the two first papers it is considered that agents use POT (Power of Two) policies instead of EOQ policies, while in the third paper it is considered that, if a group of agents place a joint order, its fixed cost is the sum of the first component plus the sum of the second component of the agents in the group.

In order to analyze our cost allocation problem we use cooperative game theory. In particular, we model our problem as a cost game and study when cooperation is profitable and when the core of the game is non-empty. Informally, the core of a cost game is the set of allocations according to which no sub-coalition pays more than it would pay if separated from the grand coalition. We also propose a cost allocation rule that always provides allocations in the core of the cost game. We introduce in the paper the game theoretical concepts and results that we use, but for more details on cooperative game theory the readers may consult, for instance, González-Díaz et al. (2010).

The paper is organized as follows. Section 2 is devoted to introducing the inventory transportation systems and to studying conditions under which cooperation among agents in those systems is profitable. Then, we analyze the corresponding class of cooperative cost games and give a sufficient condition for the non-emptiness of the cores of the games in this class. Section 3 is devoted to studying and characterizing a cost allocation rule in this framework. We complete the paper with a section of conclusions and an Appendix containing the proofs of the main results.

2 Inventory transportation systems

An inventory transportation system is a multiple agent situation where each agent is faced with a basic EOQ problem and where the fixed order cost of each agent is the sum of a first component (common to all agents) plus a second component which is proportional to the distance of the agent to the supplier. N denotes the finite set of agents. The parameters associated to every $i \in N$ in one of those systems are:

- $a > 0$, the first component (common to all agents) of the fixed cost per order,
- $a_i > 0$, the second component of the fixed cost per order, which is i's distance to the supplier (or i's distance multiplied by a constant common to all agents),
- $d_i > 0$, the deterministic demand per time unit,
- $h_i > 0$, the holding cost per item and per time unit.

Each agent $i \in N$ has to meet the demand in time. To attain this, i keeps stock in hand by placing orders of size $Q_i > 0$. It is well-known that the optimal size of the order and the minimum cost for agent i are

$$Q_i^* = \sqrt{\frac{2(a+a_i)d_i}{h_i}} \qquad \text{and} \qquad C^i(Q_i^*) = \sqrt{2(a+a_i)d_i h_i}.$$

However, in a multiple agent inventory situation such as this one, the agents in every coalition $S \subset N$ can cooperate by placing joint orders (forming what we call an order coalition). At this point we make two assumptions.

1. All the agents are located on the same line route. By this we mean that if a group of agents S places a joint order, its fixed cost is the sum of the first component a plus the second component of an agent in S whose distance from the supplier is maximal (that we denote by a_S; i.e., $a_S = max\{a_i \mid i \in S\}$).
2. The supplier accepts and even encourages agents to form order coalitions at the beginning of each term. But, because of organizational reasons, once an order coalition S has been formed, the fixed cost that the supplier charges to this coalition, for each order throughout the term, is $a + a_S$. This means that, if in a particular order an agent $i \in S$ does not buy units of the product, then the supplier even charges $a + a_S$ to S.

The questions we want to answer are: (a) under what conditions it **can be** reasonable that the agents in N form an order coalition, and (b) in the case that the order coalition N has been formed, how the total cost should be allocated to the members of this coalition.

Take now an inventory transportation system $(N, \mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i\in N})$ and let us compute the optimal total average cost per time unit for every $S \subset N$, in the case that the order coalition S forms. As a consequence of Assumption 2 it is clear that, if S forms, all its members must coordinate their cycles, i.e.

$$\frac{Q_i}{d_i} = \frac{Q_j}{d_j} \qquad \forall i, j \in S. \tag{1}$$

The total average cost per time unit for S and order size[1] Q_i is given by

$$\begin{aligned} C(S, Q_i) &= \frac{(a + a_S)d_i}{Q_i} + \sum_{j\in S} h_j \frac{Q_j}{2} \\ &= \frac{(a + a_S)d_i}{Q_i} + \frac{Q_i}{2d_i} \sum_{j\in S} d_j h_j \end{aligned}$$

(the second equality follows from the equality of the cycle lengths).

After some algebra one obtains that, if S forms, the optimal size of the order for agent $i \in S$ is

$$\hat{Q}_i = \sqrt{\frac{2(a + a_S)d_i^2}{\sum_{j\in S} d_j h_j}},$$

the optimal number of orders per time unit is

$$\hat{m}_S = \frac{d_i}{\hat{Q}_i} = \sqrt{\frac{\sum_{j\in S} d_j h_j}{2(a + a_S)}},$$

and the optimal total average cost per time unit is

$$C(S, \hat{Q}_i) = \sqrt{2(a + a_S) \sum_{j\in S} d_j h_j} = 2(a + a_S)\hat{m}_S.$$

These computations allow us to associate a cost game to every inventory transportation system. A cost game is a pair (N, c), where N is the finite set of agents and $c : 2^N \longrightarrow \mathbb{R}$ is the so-called characteristic function of the game, which assigns to each subset $S \subset N$ the cost $c(S)$ that has to be paid if agents in S cooperate. By convention, $c(\emptyset) = 0$. Cost games

[1]Notice that, once the order size Q_i is fixed all the order sizes Q_j for all $j \in S \setminus \{i\}$ are fixed by (1).

are commonly used to model cost allocation situations. In one of those situations a group of agents N cooperate to develop a joint project. Then, for every S, $c(S)$ is the minimal cost of the project with the specifications needed by the agents in S. For an inventory transportation system $(N,\mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i\in N})$, one can naturally build the cost game (N, c) given by:

$$c(S) := C(S, \hat{Q}_i) = \sqrt{2(a + a_S)\sum_{j\in S} d_j h_j} = 2(a + a_S)\hat{m}_S. \tag{2}$$

Then, a cost game (N, c) is said to be an inventory transportation game if there exists an inventory transportation system $(N,\mathcal{I})$ whose associated cost game is (N, c). Meca et al. (2003), Meca et al. (2004) and Mosquera et al. (2008), among others, study intensively the class of the so-called inventory games. An inventory game is any cost game (N, c) satisfying that, for all $S \subset N$, $c(S) \geq 0$ and, moreover, $c(S)^2 = \sum_{i\in S} c(i)^2$. Note that, in general, inventory transportation games are not inventory games (see Expression (2)).

A cost game is said to be subadditive if it is never beneficial for a coalition to split into several smaller disjointed coalitions. Formally, for each $S, T \subset N$ such that $S \cap T = \emptyset$, it holds that $c(S) + c(T) \geq c(S \cup T)$. This concept will help us to answer the first question posed above. In an inventory transportation system, under what conditions can it be reasonable for the agents in N to form an order coalition? When the corresponding cost game is subadditive it can be reasonable that the grand coalition N forms.

The next example shows that the cooperation by placing joint orders in the model we are dealing with is not always profitable. In other words, it shows an inventory transportation system whose associated cost game is not subadditive.

Example 2.1. *Consider the inventory transportation system with* $N = \{1, 2\}$, $a = 20$, $a_1 = 20$, $a_2 = 90$, $d = (800, 300)$ *and* $h = (0.1, 0.06)$. *In this situation, the cooperation is not profitable since:*

$$\begin{aligned} c(1) + c(2) = C(\{1\}, Q_1^*) + C(\{2\}, Q_2^*) &= \sqrt{6400} + \sqrt{3960} = 80 + 62.93 \\ &< c(\{1,2\}) = C(\{1,2\}, \hat{Q}_1) = \sqrt{21560} = 146.83. \end{aligned}$$

The following theorem provides a necessary and sufficient condition for the subadditivity of an inventory transportation game. It says that an inventory transportation game is subadditive if and only if, for all disjoint pairs $S, T \subset N$ such that $a_S \leq a_T$, the optimal number of orders for coalition T cannot be too small in relation with the optimal number of orders for coalition S. Roughly speaking, this condition means that cooperation is profitable when the remote agents are not "rare clients" in the sense that their individual optimal numbers of orders are not too small in comparison with the others' individual optimal numbers of orders.

Theorem 2.1. *Consider an inventory transportation game (N, c) associated to an inventory transportation system $(N, \mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i \in N})$. (N, c) is subadditive if and only if*

$$\hat{m}_T \geq \frac{1}{2} \frac{a_T - a_S}{a + a_T} \hat{m}_S$$

for all $S, T \subset N$ such that $S \cap T = \emptyset$ and $a_S \leq a_T$.

Proof. See Appendix. □

In the next section we answer our second question: if we have an inventory transportation system whose associated game (N, c) is subadditive, i.e., an inventory transportation system in which it is reasonable for the agents in N to form an order coalition, how should $c(N)$ be allocated to the members of N?

3 A cost allocation rule for inventory transportation systems

We start this section with a result on the core of an inventory transportation game. Take an inventory transportation game (N, c) and assume that the agents in N form an order coalition. In this case, if we want to allocate $c(N)$ to the members of N, it would be very convenient for our allocation to belong to the core of (N, c) given by the set

$$\mathcal{C}(N, c) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for each } S \subset N \right\}.$$

Notice that the allocations in the core are those for which no group of agents is disappointed, in the following sense. If x belongs to the core and a group S separates from N and forms an order coalition, the cost $c(S)$ it will pay is greater than or equal to the cost $\sum_{i \in S} x_i$ allocated by x to the members of S.

Although the core is an appealing concept in this context, it is easy to check that the core of an inventory transportation game may be empty. For instance, this is obviously the case in the game in Example 2.1. However, the following result shows that subadditive inventory transportation games always have a non-empty core.

Theorem 3.1. *Consider a subadditive inventory transportation game* (N, c). *Then,* $\mathcal{C}(N, c)$ *is non-empty.*

We leave the formal proof of this theorem for the Appendix. However, we now give some definitions and an outline of the proof which will be useful for the rest of this section.

Take a subadditive inventory transportation game (N, c) associated to an inventory transportation system $(N, \mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i \in N})$. We say that $i \in N$ is an extreme agent of $(N, \mathcal{I})$ if $a_i = a_N$, i.e. if its distance to the supplier is greater than or equal to the distance to the supplier of all the other agents. We denote by $E_{(N,\mathcal{I})}$ the set of extreme agents of $(N, \mathcal{I})$. We will soon see that extreme agents play an important role in the proof of this result.

Let us introduce now some concepts and notations in relation with (N, c). First we denote by $\Pi(N)$ the set of all orderings in N. Formally, every $\sigma \in \Pi(N)$ is a one-to-one map which associates to every element of N a natural number in $\{1, 2, \ldots, n\}$ (n denotes the number of elements of N). $\sigma(i) = j$ means that i has the j-th position in the ordering given by σ. Denote by σ^{-1} the inverse of map σ. For every $i \in N$, the set of predecessors of i with respect to $\sigma \in \Pi(N)$ is $P_i^\sigma = \{j \in N \mid \sigma(j) < \sigma(i)\}$. Now take $\sigma \in \Pi(N)$; the marginal vector associated with σ is defined as $m^\sigma(N, c) = (m_i^\sigma(N, c))_{i \in N}$, where $m_i^\sigma(N, c) = c(P_i^\sigma \cup \{i\}) - c(P_i^\sigma)$ for each $i \in N$. Notice that for every marginal vector m^σ, it holds that $\sum_{i \in N} m_i^\sigma(N, c) = c(N)$. Hence, a marginal vector of (N, c) is an allocation of $c(N)$ which allocates to every i its contribution to its predecessors according to a particular ordering.

Now the proof of Theorem 3.1 simply consists of demonstrating that if $m^\sigma(N, c)$ is a marginal vector and its ordering σ satisfies that $\sigma^{-1}(1)$ is an extreme agent of $(N, \mathcal{I})$, then $m^\sigma(N, c)$ belongs to the core of (N, c). This proves the theorem, but provides a second output: we know that all those marginal vectors are allocations in the core. Below we use this feature to define our allocation rule.

First let us say what we actually mean by allocation rule. Our target is to propose for every inventory transportation system whose associated game (N, c) is subadditive an allocation of $c(N)$ among the agents in N. An allocation rule is a mechanism with which to do this. Formally, an allocation rule for inventory transportation systems is a map ϕ which associates to every inventory transportation system $(N, \mathcal{I})$, with associated cost game (N, c), a vector $\phi(N, \mathcal{I}) = (\phi_i(N, \mathcal{I}))_{i \in N}$ satisfying that $\sum_{i \in N} \phi_i(N, \mathcal{I}) = c(N)$.

Now we define an allocation rule which always proposes allocations in the core of the corresponding inventory transportation game whenever it is subadditive. Take an inventory transportation system $(N, \mathcal{I})$ and consider all the orderings $\sigma \in \Pi(N)$ which invert the ordering given by the distances from the agents to the supplier, i.e., all the orderings $\sigma \in \Pi(N)$ which satisfy that $\sigma(i) \leq \sigma(j)$ implies that a_i (distance of i to the supplier) is greater than or equal to a_j (distance of j to the supplier), for all $i, j \in N$. We denote by $\Pi(N, \mathcal{I})$ the set of those orderings in $(N, \mathcal{I})$. We define our allocation rule as the rule that proposes for every inventory transportation system $(N, \mathcal{I})$ the average of the marginal vectors associated to orderings in $\Pi(N, \mathcal{I})$. We give now the formal definition of our rule, that we call the line rule.

Definition 3.1. *The line rule is the allocation rule which associates to every inventory transportation system $(N, \mathcal{I})$, with associated cost game (N, c), the allocation $L(N, \mathcal{I}) = (L_i(N, \mathcal{I}))_{i \in N}$ given by:*

$$L_i(N, \mathcal{I}) = \frac{1}{|\Pi(N, \mathcal{I})|} \sum_{\sigma \in \Pi(N, \mathcal{I})} m_i^{\sigma}(N, c)$$

for all $i \in N$.

Notice that all orderings $\sigma \in \Pi(N, \mathcal{I})$ satisfy that $\sigma^{-1}(1)$ is an extreme agent of $(N, \mathcal{I})$, **and** then $m^{\sigma}(N, c) \in \mathcal{C}(N, c)$ when (N, c) is a subadditive game. Since $\mathcal{C}(N, c)$ is a convex set, then $L(N, \mathcal{I}) \in \mathcal{C}(N, c)$ whenever (N, c) is subadditive. We formally state this in the following result.

Theorem 3.2. *Consider an inventory transportation system $(N, \mathcal{I})$ having a subadditive associated cost game (N, c). Then, $L(N, \mathcal{I}) \in \mathcal{C}(N, c)$.*

Now we include an example in which we compute the line rule of an inventory transportation system. We also compute the Shapley value for the system in the example. The Shapley value is a very well-known solution concept for cooperative games. For details on the Shapley value, for instance, the survey in Moretti and Patrone (2008) can be consulted. With the notation of this paper, the Shapley value of an inventory transportation system $(N, \mathcal{I})$ with cost game (N, c) is the vector $\Phi(N, \mathcal{I}) = (\Phi_i(N, \mathcal{I}))_{i \in N}$ given by

$$\Phi_i(N, \mathcal{I}) = \frac{1}{|\Pi(N)|} \sum_{\sigma \in \Pi(N)} m_i^{\sigma}(N, c)$$

for all $i \in N$.

Example 3.1. *Consider the inventory transportation system $(N, \mathcal{I})$ with $N = \{1, 2, 3\}$, $a = 200$ and*

i	a_i	d_i	h_i
1	300	90	0.06
2	300	80	0.06
3	900	20	0.1

The associated inventory transportation game is

S	$\emptyset$	1	2	3	12	13	23	N
$c(S)$	0	73.48	69.28	66.33	101	127.59	122.31	163.83

It is easily checked that (N, c) is subadditive. Moreover $E_{(N,\mathcal{I})} = \{3\}$ and $\Pi(N, \mathcal{I}) = \{(3\,1\,2), (3\,2\,1)\}$. So,

$$\{m^{\sigma}(N, \mathcal{I}) \mid \sigma \in \Pi(N, \mathcal{I})\} = \{(61.26, 36.24, 66.33), (41.52, 55.98, 66.33)\}.$$

Then, the line rule for this system is

$$L(N, \mathcal{I}) = (51.39, 46.11, 66.33) \in \mathcal{C}(N, c),$$

The Shapley value requires more computational effort and, moreover, may lie outside the core. In this example

$$\Phi(N, \mathcal{I}) = (53.83, 49.09, 60.91) \notin \mathcal{C}(N, c),$$

since the core condition fails for coalition $S = \{1, 2\}$. ◇

We see that the line rule is an allocation rule for inventory transportation systems which has the same flavor as the Shapley value, but requires less computational effort and relates better with the core. Let us now see two properties which are intimately connected with the line rule, in the sense that they characterize it within the set of possible rules for inventory transportation systems.

The two properties are concerned with the distances of the agents to the supplier. These distances are an important feature in this model, the feature which distinguishes it from other centralized inventory models. The first property is a fairness property that states that if two agents are equally distant from the supplier, they must be treated in a balanced way by the rule.

Balanced Treatment for Equally Distant Agents (BT). An allocation rule ϕ for inventory transportation systems satisfies BT if the following is fulfilled. Take an inventory transportation system $(N, \mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i \in N})$ and take $j, k \in N$ with $a_j = a_k$. For every $l \in N$

denote by $(N \setminus \{l\}, \mathcal{I})$ the inventory transportation system $(N \setminus \{l\}, a, \{a_i, d_i, h_i\}_{i \in N \setminus \{l\}})$. Then,

$$\phi_j(N, \mathcal{I}) - \phi_j(N \setminus \{k\}, \mathcal{I}) = \phi_k(N, \mathcal{I}) - \phi_k(N \setminus \{j\}, \mathcal{I}).$$

The second property, Free Participation of Costless Agents (FP), states that if an order group has formed and a set of new agents joins the group, this incorporation will not affect the allocation to the agents in the original group if the new agents are closer to the supplier than all the other agents. The idea under this property is that these new agents can be considered as costless agents, because they do not perturb either the original group or any subgroup of it, in the sense that the transportation route can be maintained without any variations after their incorporation (apart maybe from some additional stops). Then, costless agents may be incorporated to the order group provided that they do not modify the allocation for the agents in the original group. FP is a desirable property in many situations. First, it is important to notice that it is a clear, understandable for everyone, and reasonable property. Moreover, if the supplier wants to form a large order coalition, FP seems to be a good property. To simplify our argument assume that the agents lie in towns located on a line route; we identify the agents in a town with the town itself. Then, for every two towns, both are not against merging: the furthest town from the supplier is in fact indifferent, and the closest town will benefit from the merging (if the game is subadditive).

Free Participation of Costless Agents (FP). An allocation rule ϕ for inventory transportation systems satisfies FP if the following condition is fulfilled. Take an inventory transportation system $(N \cup N', \mathcal{I}_{N \cup N'}) = (N \cup N', a, \{a_i, d_i, h_i\}_{i \in N \cup N'})$ such that $N \cap N' = \emptyset$ and $a_{N'} < a_i$ for all $i \in N$. Denote by $(N, \mathcal{I})$ the inventory transportation system $(N, a, \{a_i, d_i, h_i\}_{i \in N})$. Then,

$$\phi_j(N \cup N', \mathcal{I}_{N \cup N'}) = \phi_j(N, \mathcal{I})$$

for all $j \in N$.

These two properties characterize the line rule. We formally state this in the following theorem.

Theorem 3.3. *The line rule is the unique rule for inventory transportation systems satisfying BT and FP.*

Proof. See Appendix. □

4 Conclusions

This paper provides a new contribution to centralized inventory problems. It examines a cost allocation problem in an inventory transportation system with a single item, a single supplier and multiple retailers that place joint orders using an EOQ policy with a specific cost structure: the joint order cost is the addition of a fixed part plus a transportation cost which is the maximum of the individual transportation costs. This problem corresponds to a situation in which the retailers are located on the same line route, in the sense that, if a retailer is served, all retailers which are closer to the supplier are served without any additional transportation cost. For these inventory transportation systems cooperation is not always reasonable, but if we impose a simple condition that compares the optimal numbers of orders for every disjoint pair of coalitions, we can ensure that cooperation is reasonable. Moreover, when cooperation is reasonable, we prove that we can always find stable allocations (allocations belonging to the core of the corresponding cost game). Finally we introduce the line rule, an allocation rule which provides stable allocations when cooperating is reasonable. We also provide two properties that characterize this rule. All these results provide satisfactory answers to the questions initially addressed. From a practical point of view, we can apply our results to franchises operations.

5 Appendix

Here the reader can find the proofs of the theorems stated in this paper.

Proof of Theorem 2.1. Let $S, T \subset N$ be such that $S \cap T = \emptyset$ and $a_S \leq a_T$. It has to be proven that $c(S) + c(T) \geq c(S \cup T) \Leftrightarrow \hat{m}_T \geq \frac{1}{2}\frac{a_T - a_S}{a + a_T}\hat{m}_S$. Since $c(S) \geq 0$,

$$\begin{aligned} c(S) + c(T) \geq c(S \cup T) \quad &\Leftrightarrow \quad c(S)^2 + c(T)^2 + 2c(S)c(T) \geq c(S \cup T)^2 \\ &\Leftrightarrow \quad 2c(S)c(T) \geq c(S \cup T)^2 - c(S)^2 - c(T)^2. \end{aligned}$$

But

$$\begin{aligned} c(S \cup T)^2 - c(S)^2 - c(T)^2 \quad &= \quad 2(a + a_{S \cup T}) \sum_{i \in S \cup T} h_i d_i - 2(a + a_S) \sum_{i \in S} h_i d_i - 2(a + a_T) \sum_{i \in T} h_i d_i \\ &= \quad 2(a + a_T) \sum_{i \in S} h_i d_i - 2(a + a_S) \sum_{i \in S} h_i d_i \end{aligned}$$

$$= \quad 2(a_T - a_S)\sum_{i\in S} h_i d_i.$$

Then,

$$c(S) + c(T) \geq c(S \cup T) \quad \Leftrightarrow \quad c(S)c(T) \geq (a_T - a_S)\sum_{i\in S} h_i d_i.$$

Moreover, the right side of the equivalence can be written as

$$4(a+a_S)(a+a_T)\hat{m}_S\hat{m}_T \geq (a_T - a_S)2(a+a_S)\hat{m}_S^2 \quad \Leftrightarrow \quad \hat{m}_T \geq \frac{1}{2}\frac{a_T - a_S}{a + a_T}\hat{m}_S. \qquad \square$$

Proof of Theorem 3.1. In this proof we use the notation and concepts introduced in the informal proof of this result provided in Section 3. Take a subadditive inventory transportation game (N, c) associated to an inventory transportation system $(N, \mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i\in N})$. Take now a marginal vector $m^\sigma(N, c)$ such that σ satisfies that $\sigma^{-1}(1)$ is an extreme agent of $(N, \mathcal{I})$. We have to prove that $m^\sigma(N, c)$ belongs to the core of (N, c). To do this, it suffices to show that for every non-empty coalition $S \subset N$, it holds that $\sum_{i\in S} m_i^\sigma(N, c) \leq c(S)$. We distinguish two cases.

a) S contains the extreme agent $\sigma^{-1}(1)$. Then,

$$\begin{aligned}
\sum_{i\in S} m_i^\sigma(N,c) &= c(\sigma^{-1}(1)) + \sum_{j\in S\setminus\{\sigma^{-1}(1)\}} \Big(c(P_j^\sigma \cup \{j\}) - c(P_j^\sigma)\Big) \\
&= c(\sigma^{-1}(1)) + \sum_{j\in S\setminus\{\sigma^{-1}(1)\}} \left(\sqrt{2(a+a_N)\sum_{i\in P_j^\sigma\cup\{j\}} h_i d_i} - \sqrt{2(a+a_N)\sum_{i\in P_j^\sigma} h_i d_i}\right) \\
&= \sum_{j\in S}\sqrt{2(a+a_N)}\left(\sqrt{\sum_{i\in P_j^\sigma\cup\{j\}} h_i d_i} - \sqrt{\sum_{i\in P_j^\sigma} h_i d_i}\right) \\
&\leq \sum_{j\in S}\sqrt{2(a+a_N)}\left(\sqrt{\sum_{i\in (P_j^\sigma\cup\{j\})\cap S} h_i d_i} - \sqrt{\sum_{i\in P_j^\sigma\cap S} h_i d_i}\right) \\
&= c(S)
\end{aligned}$$

where the inequality follows from the fact that the function $\sqrt{x+y} - \sqrt{x}$ is decreasing in x for all $y \in [0, \infty)$.

b) S does not contain the extreme agent $\sigma^{-1}(1)$. In this case denote $\bar{S} = S \cup \{\sigma^{-1}(1)\}$. Using the same proof above we conclude that

$$\sum_{i \in \bar{S}} m_i^\sigma(N, c) \leq c(\bar{S}).$$

Now, taking into account that $m^\sigma_{\sigma^{-1}(1)}(N, c) = c(\sigma^{-1}(1))$ and that c is subadditive, it holds that

$$c(\sigma^{-1}(1)) + \sum_{i \in S} m_i^\sigma(N, c) = \sum_{i \in \bar{S}} m_i^\sigma(N, c) \leq c(\bar{S}) \leq c(\sigma^{-1}(1)) + c(S),$$

which implies that $\sum_{i \in S} m_i^\sigma(N, c) \leq c(S)$. □

Proof of Theorem 3.3. It is clear that the line rule satisfies FP. Namely, take an inventory transportation system $(N \cup N', \mathcal{I}_{N \cup N'}) = (N \cup N', a, \{a_i, d_i, h_i\}_{i \in N \cup N'})$ such that $N \cap N' = \emptyset$ and $a_{N'} < a_i$ for every $i \in N$, and its associated inventory transportation game $(N \cup N', c)$. Then, it is easy to check that $|\Pi(N \cup N', \mathcal{I}_{N \cup N'})| = |\Pi(N, \mathcal{I}_N)||\Pi(N', \mathcal{I}_{N'})|$. Moreover, for every $i \in N$ and $\sigma \in \Pi(N \cup N', \mathcal{I}_{N \cup N'})$,

$$\{j \in N \cup N' \mid \sigma(j) < \sigma(i)\} = \{j \in N \mid \sigma(j) < \sigma(i)\} = \{j \in N \mid \sigma_{|N}(j) < \sigma_{|N}(i)\}.$$

Therefore, for every $i \in N$,

$$\begin{aligned} L_i(N \cup N', \mathcal{I}_{N \cup N'}) &= \frac{1}{|\Pi(N \cup N', \mathcal{I}_{N \cup N'})|} \sum_{\sigma \in \Pi(N \cup N', \mathcal{I}_{N \cup N'})} m_i^\sigma(N \cup N', c) \\ &= \frac{1}{|\Pi(N, \mathcal{I}_N)|} \sum_{\sigma \in \Pi(N, \mathcal{I}_N)} m_i^\sigma(N, c) = L_i(N, \mathcal{I}_N). \end{aligned}$$

To conclude the proof it must be shown, for every arbitrary inventory transportation system $(N, \mathcal{I}) = (N, a, \{a_i, d_i, h_i\}_{i \in N})$ and its associated inventory transportation game (N, c), that:

Claim 1. For every $i, j \in N$ with $a_i = a_j$, it holds that $L_i(N, \mathcal{I}) - L_i(N \setminus \{j\}, \mathcal{I}) = L_j(N, \mathcal{I}) - L_j(N \setminus \{i\}, \mathcal{I})$.

Claim 2. If ϕ is an allocation rule for inventory transportation systems satisfying BT and FP, then $\phi_i(N, \mathcal{I}) = L_i(N, \mathcal{I})$ for all $i \in N$.

First of all, assume without loss of generality that $N = \{1, \ldots, n\}$ and that $a_1 \geq \ldots \geq a_i \geq \ldots \geq a_n$. Now denote

$$
\begin{aligned}
&N_1 = \{i \in N \mid a_i = a_1\} \text{ and } n_1 = |N_1|, \\
&N_2 = \{i \in N \mid a_i = a_{n_1+1}\} \text{ and } n_2 = |N_2|, \\
&\ldots \\
&N_k = \{i \in N \mid a_i = a_{n_1+\ldots+n_{k-1}+1} = a_n\} \text{ and } n_k = |N_k|.
\end{aligned}
\tag{3}
$$

where $|\cdot|$ denotes the cardinality function. Notice that $N = N_1 \cup \ldots \cup N_k$ and $N_r \cap N_t = \emptyset$ for every $r, t \in \{1, \ldots, k\}$, $r \neq t$.

Let us prove Claim 1. Assume that $i, j \in N_l$ with $1 \leq l \leq k$. Define the cost game (N_l, c_l) by

$$c_l(S) := c(\cup_{r=1}^{l-1} N_r \cup S) - c(\cup_{r=1}^{l-1} N_r)$$

for all $S \subset N_l$. The Shapley value of a cost game (N, c) is $\Phi_i(N, c) = \frac{1}{|\Pi(N)|} \sum_{\sigma \in \Pi(N)} m^{\sigma}(N, c)$ (see Moretti and Patrone, 2008), and then

$$\Phi_i(N_l, c_l) = \frac{1}{|\Pi(N_l)|} \sum_{\sigma \in \Pi(N_l)} m_i^{\sigma}(N_l, c_l) = L_i(N, \mathcal{I}),$$

where the last equality follows from $\Pi(N_l, \mathcal{I}_{N_l}) = \Pi(N_l)$.

In fact what we want to prove is that

$$\Phi_i(N_l, c_l) - \Phi_i(N_l \setminus \{j\}, c_l) = \Phi_j(N_l, c_l) - \Phi_j(N_l \setminus \{i\}, c_l)$$

(in the cost game $(N_l \setminus \{i\}, c_l)$ c_l means the restriction of c_l to $N_l \setminus \{i\}$). But this is immediately obtained from the so-called balanced contributions property of the Shapley value (proved in Hart and Mas-Colell, 1989) which states that, for every cost game (N, c) and every $i, j \in N$,

$$\Phi_i(N, c) - \Phi_i(N \setminus \{j\}, c) = \Phi_j(N, c) - \Phi_j(N \setminus \{i\}, c).$$

Let us prove now Claim 2.

a) We first prove that $\sum_{i \in N_j} \phi_i(N, \mathcal{I}) = \sum_{i \in N_j} L_i(N, \mathcal{I})$ for all $j \in \{1, \ldots, k\}$. We do this by induction on j. For $j = 1$, since ϕ and L satisfy FP and $N = N_1 \cup (N \setminus N_1)$, then

$\phi_i(N_1, \mathcal{I}_{N_1}) = \phi_i(N, \mathcal{I})$ and $L_i(N_1, \mathcal{I}_{N_1}) = L_i(N, \mathcal{I})$ for all $i \in N_1$.

Therefore, by definition of allocation rule

$$\begin{aligned}\sum_{i \in N_1} \phi_i(N, \mathcal{I}) &= \sum_{i \in N_1} \phi_i(N_1, \mathcal{I}_{N_1}) = c(N_1) \\ &= \sum_{i \in N_1} L_i(N_1, \mathcal{I}_{N_1}) = \sum_{i \in N_1} L_i(N, \mathcal{I}).\end{aligned}$$

Take $j > 1$. Assume that the statement is true for all $l \leq j-1$. Now, ϕ and L satisfying FP implies that

$$\sum_{i \in N_1 \cup \ldots \cup N_j} \phi_i(N, \mathcal{I}) = c(N_1 \cup \ldots \cup N_j) = \sum_{i \in N_1 \cup \ldots \cup N_j} L_i(N, \mathcal{I}),$$

and the induction hypothesis implies that $\sum_{i \in N_j} \phi_i(N, \mathcal{I}) = \sum_{i \in N_j} L_i(N, \mathcal{I})$.

b) We now prove that $\phi_i(N, \mathcal{I}) = L_i(N, \mathcal{I})$ for all $i \in N$. Take $i \in N$ and $l \in \{1, \ldots, k\}$ such that $i \in N_l$ (remember that $\{N_1, \ldots, N_k\}$ defined as in (3) is a partition of N). We prove the statement by induction on $n_l = |N_l|$. If $n_l = 1$, a) obviously implies it. Take $n_l > 1$. Assume now that the statement is true for $n_l - 1$. Take $j \in N_l$ different from i. Considering that ϕ and L satisfy BT we have that

$$\phi_i(N, \mathcal{I}) - \phi_j(N, \mathcal{I}) = \phi_i(N \setminus \{j\}, \mathcal{I}) - \phi_j(N \setminus \{i\}, \mathcal{I}),$$

$$L_i(N, \mathcal{I}) - L_j(N, \mathcal{I}) = L_i(N \setminus \{j\}, \mathcal{I}) - L_j(N \setminus \{i\}, \mathcal{I}).$$

The induction hypothesis implies that $L_i(N \setminus \{j\}, \mathcal{I}) = \phi_i(N \setminus \{j\}, \mathcal{I})$ and $L_j(N \setminus \{i\}, \mathcal{I}) = \phi_j(N \setminus \{i\}, \mathcal{I})$.

Therefore,

$$\phi_i(N, \mathcal{I}) - \phi_j(N, \mathcal{I}) = L_i(N, \mathcal{I}) - L_j(N, \mathcal{I}).$$

Thus $\phi_j(N, \mathcal{I}) - L_j(N, \mathcal{I})$ is constant for all $j \in N_l$. Now a) implies that this constant has to be zero and hence $\phi_i(N, \mathcal{I}) = L_i(N, \mathcal{I})$. $\square$

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