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Existence results for damped regular equations under periodic or Neumann boundary conditions ☆

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ABSTRACT

We present a version of the averaging and a method to construct lower and upper solutions in the reversed order that are suitable for both the periodic and the Neumann boundary conditions. As an application, we obtain new results for a periodic problem related to the Liebau phenomenon and for a Neumann boundary value problem arising in fluid dynamics.

1. Introduction

We study the existence of positive solutions of the damped regular differential equation

$$\ddot{x} + c\dot{x} = g(t)x^{\lambda} - h(t)x^{\mu} \tag{1.1}$$

under periodic or Neumann boundary conditions, where $c \in \mathbb{R}$, $g, h \in C([0,T])$ and λ, $\mu > 0$.

Throughout the paper, by speaking about a T-periodic function x, we mean that both x and $\dot{x}$ are periodic functions, i.e.,

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$$x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \tag{1.2}$$

and we will identify any function defined on $[0, T)$ with its T-periodic extension to the whole of $\mathbb{R}$. We also deal with the existence of positive solutions for equation (1.1) subject to the Neumann boundary conditions

$$\dot{x}(0) = \dot{x}(T) = 0. \tag{1.3}$$

Due to common features shared by the periodic and the Neumann boundary conditions, some methods can be applied to both the problems: it is the case of a recent version of the averaging [6] and the lower and upper solutions technique [9,15].

The models for (1.1) we have in mind are the periodic problem for the equation

$$\ddot{x} + c\dot{x} = (b+1)e(t)x^{\frac{b-1}{b+1}} - a(b+1)x^{\frac{b}{b+1}}, \qquad a > 0, \quad b > 1,$$

related to the Liebau phenomenon and the Neumann boundary value problem for the equation describing fluid dynamics

$$\ddot{x} + c\dot{x} = q^2 x - \left(1 + \delta \sin\left(\frac{\pi t}{T}\right)\right) x^2.$$

More details about the models and the related bibliography are given in the corresponding sections.

The following notation is used throughout the paper: $AC^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ is the set of all T-periodic functions $x\colon \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ such that x and $\dot{x}$ are absolutely continuous and for a given continuous function x on $[0, T]$ we denote

$$M_x = \max_{t\in[0,T]} x(t), \quad m_x = \min_{t\in[0,T]} x(t),$$
$$Q_{\mathcal{P}}(x) = \frac{1}{T}\int_0^T x(s)ds$$

and

$$Q_{\mathcal{N},c}(x) = \begin{cases} \dfrac{1}{T}\displaystyle\int_0^T x(s)ds & \text{if } c = 0, \\ \dfrac{c}{e^{cT}-1}\displaystyle\int_0^T e^{cs}x(s)ds & \text{if } c \neq 0. \end{cases}$$

The rest part of this paper is organized as follows. In Section 2, the asymptotic behavior of solutions of general second order differential equations with small parameters is given, which is used to deal with the regular damped differential equations; finally, it is applied to the fluid dynamics equation. In Section 3, we provide some results based on lower and upper solutions in the reversed order, then the results are applied to the Liebau-type differential equation improving and complementing some previous results in the literature.

2. The averaging method

In this section we deal with the equation

$$\ddot{x} + c\dot{x} = \nu g(t)x^{\lambda} - h(t)x^{\mu} \tag{2.1}$$

in the presence of a parameter ν.

We are going to apply an averaging technique recently developed in [6] that allows us to deal simultaneously with both the periodic and the Neumann boundary conditions. Let $I \subset \mathbb{R}$ be an open interval and let

$$f\colon [0,T]\times I\times\mathbb{R}\times(-\varepsilon_1,\varepsilon_1)\to\mathbb{R},\qquad (t,u_0,u_1,\varepsilon)\mapsto f(t,u_0,u_1,\varepsilon)$$

be continuous and such that $\dfrac{\partial f}{\partial u_0}, \dfrac{\partial f}{\partial u_1}$ exist and are continuous. Let us consider the problem

$$\ddot{x}+c\dot{x}=\varepsilon f(t,x,\dot{x},\varepsilon) \tag{2.2}$$

and define the *periodic averaged function*

$$F_{\mathcal{P}}\colon I\to\mathbb{R},\quad \kappa\mapsto F_{\mathcal{P}}(\kappa):=Q_{\mathcal{P}}(f(\cdot,\kappa,0,0))$$

and the *Neumann averaged function*

$$F_{\mathcal{N},c}\colon I\to\mathbb{R},\quad \kappa\mapsto F_{\mathcal{N},c}(\kappa):=Q_{\mathcal{N},c}(f(\cdot,\kappa,0,0)).$$

Theorem 2.1. *If there exists $\kappa_0\in I$ such that*

$$F_{\mathcal{P}}(\kappa_0)=0 \quad \textit{and} \quad F'_{\mathcal{P}}(\kappa_0)\neq 0 \quad (\textit{resp. } F_{\mathcal{N},c}(\kappa_0)=0 \quad \textit{and} \quad F'_{\mathcal{N},c}(\kappa_0)\neq 0),$$

then there exists $\varepsilon_0\in(0,\varepsilon_1)$ such that for $0<|\varepsilon|<\varepsilon_0$, the periodic problem (2.2), (1.2) (resp. the Neumann problem (2.2), (1.3)) has a unique solution $x(t,\varepsilon)$ in a neighborhood of κ_0 and $\lim_{\varepsilon\to 0^+} x(t,\varepsilon)=\kappa_0$ uniformly in $t\in[0,T]$.

Proof. The result for the periodic problem (2.2), (1.2) is just a particular case of [6, Theorem 4]. The proof for the Neumann problem (2.2), (1.3) is similar by taking into account that for $z\in C([0,T])$ the Neumann problem

$$\ddot{x}+c\dot{x}=z(t),\quad \dot{x}(0)=0=\dot{x}(T)$$

is solvable if and only if $Q_{\mathcal{N},c}(z)=0$. Then the rest of the proof goes through as in [6, Remark 5]. □

Therefore, as an application of Theorem 2.1 to equation (2.1) with the periodic or the Neumann boundary conditions, we obtain the following corollary. In the periodic setting it is a complementary result to [6, Theorem 7].

Corollary 2.2. *Assume that $c\in\mathbb{R}$, $\mu\neq 1$, $\lambda\neq\mu$ and g and h are continuous functions with $Q(g)\cdot Q(h)>0$, where Q stands for $Q_{\mathcal{P}}$ or $Q_{\mathcal{N},c}$.*

If $Q=Q_{\mathcal{P}}$ (resp. $Q=Q_{\mathcal{N},c}$), then the periodic problem (2.1), (1.2) (resp. the Neumann problem (2.1), (1.3)) has a solution $x(t,\nu)$ provided that either one of the following conditions holds:

(i) $\dfrac{\mu-1}{\mu-\lambda}>0$ *and $\nu>0$ is small enough,*
(ii) $\dfrac{\mu-1}{\mu-\lambda}<0$ *and $\nu>0$ is large enough.*

Moreover, in case (i) the following asymptotic behavior holds

$$\lim_{\nu\to 0^+} \nu^{\frac{1}{\lambda-\mu}} x(t,\nu) = \left(\frac{Q(g)}{Q(h)}\right)^{\frac{1}{\mu-\lambda}} \quad \text{uniformly in } t \in [0,T],$$

while in case (ii)

$$\lim_{\nu\to +\infty} \nu^{\frac{1}{\lambda-\mu}} x(t,\nu) = \left(\frac{Q(g)}{Q(h)}\right)^{\frac{1}{\mu-\lambda}} \quad \text{uniformly in } t \in [0,T].$$

Proof. Setting $x = \nu^{\frac{1}{\mu-\lambda}} y$, equation (2.1) becomes, after simplification by $\nu^{\frac{1}{\mu-\lambda}}$,

$$\ddot{y} + c\dot{y} = \nu^{\frac{\mu-1}{\mu-\lambda}} (g(t)y^{\lambda} - h(t)y^{\mu}),$$

and hence, choosing $\varepsilon = \nu^{\frac{\mu-1}{\mu-\lambda}}$, we have

$$\ddot{y} + c\dot{y} = \varepsilon(g(t)y^{\lambda} - h(t)y^{\mu}). \tag{2.3}$$

Now, it is easy to show that equation (2.1) satisfies the conditions of Theorem 2.1. To see this, we prove the corresponding result for $Q = Q_{\mathcal{P}}$. Indeed, for any $\kappa > 0$ we have

$$F_{\mathcal{P}}(\kappa) = Q_{\mathcal{P}}(g)\kappa^{\lambda} - Q_{\mathcal{P}}(h)\kappa^{\mu} \quad \text{and} \quad F'_{\mathcal{P}}(\kappa) = \lambda Q_{\mathcal{P}}(g)\kappa^{\lambda-1} - \mu Q_{\mathcal{P}}(h)\kappa^{\mu-1},$$

and then for $\kappa_0 = \left(\frac{Q_{\mathcal{P}}(g)}{Q_{\mathcal{P}}(h)}\right)^{\frac{1}{\mu-\lambda}} > 0$ it holds

$$F_{\mathcal{P}}(\kappa_0) = 0 \quad \text{and} \quad F'_{\mathcal{P}}(\kappa_0) = (\lambda-\mu)\frac{Q_{\mathcal{P}}(g)^{\frac{\mu-1}{\mu-\lambda}}}{Q_{\mathcal{P}}(h)^{\frac{\lambda-1}{\mu-\lambda}}} \neq 0.$$

A similar argument holds for the case $Q = Q_{\mathcal{N},c}$. Namely, it is enough to replace $F_{\mathcal{P}}$ by $F_{\mathcal{N},c}$. □

2.1. Application to a fluid dynamics equation

The following Neumann problem, related to an equation arising in fluid dynamics, has been studied for example in [1,3,11,12,17]

$$\begin{cases} \ddot{x} = q^2 x - \left(1 + \delta \sin\left(\frac{\pi t}{T}\right)\right) x^2, & t \in (0,T),\ q > 0,\ \delta \geq 0, \\ \dot{x}(0) = \dot{x}(T) = 0. \end{cases} \tag{2.4}$$

In fact, we are dealing with a more general version of the problem, namely,

$$\begin{cases} \ddot{x} + c\dot{x} = q^2 x - h(t)x^p, & t \in (0,T), \\ \dot{x}(0) = \dot{x}(T) = 0, \end{cases} \tag{2.5}$$

and we provide an existence result that complements [1, Theorem 3].

Theorem 2.3. *Suppose that $c \in \mathbb{R}$, $q > 0$, $0 < p \neq 1$ and h is a continuous function in $[0,T]$ such that $Q_{\mathcal{N},c}(h) > 0$.*

Then, there exists $q_0 \in (0, \infty)$ such that, for $0 < q < q_0$, problem (2.5) has a solution $x(t, q)$ such that

$$\lim_{q\to 0^+} q^{\frac{2}{1-p}} x(t,q) = \left(\frac{1}{Q_{\mathcal{N},c}(h)}\right)^{\frac{1}{p-1}} \quad \textit{uniformly in } t \in [0,T].$$

Proof. It is a particular case of Corollary 2.2 with $\nu = q^2$, $g(t) \equiv 1$, $\lambda = 1$, $\mu = p$ and $Q = Q_{\mathcal{N},c}$. Take into account that $Q_{\mathcal{N},c}(1) = 1$ for each $c \in \mathbb{R}$. □

As an application of the previous theorem to problem (2.4), we obtain the following result.

Corollary 2.4. *There exists $q_0 \in (0, \infty)$ such that, for $0 < q < q_0$, problem (2.4) has a solution $x(t, q)$ such that $\lim_{q\to 0^+} \frac{1}{q^2} x(t,q) = \frac{\pi}{\pi + 2\delta}$ uniformly in $t \in [0, T]$.*

Proof. It is a particular case of Theorem 2.3 with $c = 0$, $h(t) = 1 + \delta \sin\left(\frac{\pi t}{T}\right)$ and $p = 2$. Notice that in this case $Q_{\mathcal{N},0}(h) = \frac{\pi + 2\delta}{\pi} > 0$. □

Remark 2.5. The existence of a positive solution for problem (2.4) was established by Torres in [17] for each $q > 0$ by means of the Krasnosels'kii fixed point theorem. The uniqueness of such positive solution was added in [1] for $q \in (0, 0.354446\ldots)$. The asymptotic information provided by Corollary 2.4 for the unique positive solution of problem (2.4) as $q \to 0^+$ seems to be new.

3. Lower and upper solutions in the reversed order

We present sufficient conditions such that problem (1.1), (1.2) (resp. (1.1), (1.3)) has a lower solution α and an upper solution β in the reversed order, that is $\beta \leq \alpha$. The construction of the lower and upper solutions follows the approach initiated in [10]. We point out that in the reversed order case the monotone method can be used to approximate the extremal solutions between β and α for any boundary value problem such that a uniform anti-maximum principle holds, as is the case for the periodic or the Neumann problems [2,9,13]. Let us also mention that the method of lower and upper solutions in the reversed order case requires a growth condition imposed on the nonlinear part of equation (1.1). Roughly speaking, the partial derivative ρ_x of the function

$$\rho(t, x) = h(t)x^\mu - g(t)x^\lambda$$

has to be bounded from above by an appropriate constant for $x \in [\beta(t), \alpha(t)]$ (see [13,16]).

Theorem 3.1. *Let $\lambda, \mu > 0$, $c_2 > 0$, $c_1, d_1, d_2 \in [0, +\infty)$, and the functions $\omega, \sigma \in C^2([0,T], \mathbb{R})$ be the solutions of the differential equations*

$$\ddot{\omega} + c\dot{\omega} = c_1 h(t) - c_2 g(t),\ t \in [0,T] \tag{3.1}$$

and

$$\ddot{\sigma} + c\dot{\sigma} = d_1 h(t) - d_2 g(t),\ t \in [0,T] \tag{3.2}$$

under the periodic boundary conditions (1.2) (resp. under the Neumann boundary conditions (1.3)).

Suppose there exists $x_1 \in (0, +\infty)$ such that

$$x_1(\omega(t) - m_\omega) + \sigma(t) - m_\sigma \leq (c_1 x_1 + d_1)^{1/\mu} - (c_2 x_1 + d_2)^{1/\lambda} \text{ for } t \in [0,T] \tag{3.3}$$

holds and one of the following conditions is satisfied:

i) $0 < \lambda < \mu < 1$,
ii) $0 < \lambda < 1$ *and* $c_1 = 0$,
iii) $1 = \lambda < \mu$ *and* $c_2 > M_\omega - m_\omega$,
iv) $\lambda = 1$, $c_1 = 0$ *and* $c_2 > M_\omega - m_\omega$.

Then, there exist an upper solution β *and a lower solution* α *of the periodic problem* (1.1), (1.2) *(resp. of the Neumann boundary value problem* (1.1), (1.3)*) such that*

$$0 < \beta \le \alpha.$$

Proof. By condition (3.3) the function defined by

$$\beta(t) = (c_1 x_1 + d_1)^{1/\mu} - [x_1(\omega(t) - m_\omega) + \sigma(t) - m_\sigma] \text{ for } t \in [0, T] \tag{3.4}$$

satisfies

$$(c_2 x_1 + d_2)^{1/\lambda} \le \beta(t) \le (c_1 x_1 + d_1)^{1/\mu} \text{ for } t \in [0, T]. \tag{3.5}$$

Moreover, in view of (3.1) and (3.2) we have

$$\ddot{\beta} + c\dot{\beta} = (c_2 x_1 + d_2) g(t) - (c_1 x_1 + d_1) h(t) \text{ for } t \in [0, T]. \tag{3.6}$$

So, (3.5) and (3.6) imply

$$\ddot{\beta} + c\dot{\beta} \le g(t)\beta^\lambda - h(t)\beta^\mu \text{ for } t \in [0, T],$$

and then $0 < \beta$ is an upper solution of problem (1.1), (1.2) (resp. (1.1), (1.3)).

On the other hand, note that each of the conditions i), ii), iii) or iv) implies that

$$\lim_{x \to +\infty} [(c_2 x + d_2)^{\frac{1}{\lambda}} - (c_1 x + d_1)^{\frac{1}{\mu}} - x(M_\omega - m_\omega) - (M_\sigma - m_\sigma)] = +\infty.$$

Therefore, we can choose $x_0 > x_1$ such that

$$x_0(\omega(t) - m_\omega) + \sigma(t) - m_\sigma \le (c_2 x_0 + d_2)^{1/\lambda} - (c_1 x_0 + d_1)^{1/\mu} \text{ for } t \in [0, T]. \tag{3.7}$$

Then the function

$$\alpha(t) := (c_2 x_0 + d_2)^{1/\lambda} - [x_0(\omega(t) - m_\omega) + \sigma(t) - m_\sigma] \text{ for } t \in [0, T] \tag{3.8}$$

satisfies

$$(c_1 x_0 + d_1)^{1/\mu} \le \alpha(t) \le (c_2 x_0 + d_2)^{1/\lambda} \text{ for } t \in [0, T]. \tag{3.9}$$

In a similar way as before it can be checked that α is a lower solution of problem (1.1), (1.2) (resp. (1.1), (1.3)). Finally, from (3.5), (3.9) and $x_1 < x_0$ we obtain that for all $t \in [0, T]$

$$(c_2 x_1 + d_2)^{1/\lambda} \le \beta(t) \le (c_1 x_1 + d_1)^{1/\mu} \le (c_1 x_0 + d_1)^{1/\mu} \le \alpha(t) \le (c_2 x_0 + d_2)^{1/\lambda}. \quad \square$$

3.1. Applications to a periodic problem

Consider the Liebau-type equation

$$\ddot{u} + c\dot{u} = \frac{1}{u}(e(t) - b\dot{u}^2) - a, \tag{3.10}$$

where $a > 0, b > 1, c \geq 0$ are constants and $e \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$. The equation (3.10) appeared in [14] as a simple model for the "valveless pumping" phenomenon firstly noticed by the German cardiologist G. Liebau: a periodic external forcing acting in an asymmetric configuration can produce a directed flow without the use of valves (see also [18, Chapter 8] for a discussion of this singular model).

It was pointed out in [7] that $\bar{e} := \frac{1}{T}\int_0^T e(s)ds > 0$ is a necessary condition in order that (3.10) has a positive T-periodic solution but it is still an open problem to know whether or not this condition is also sufficient. Therefore, any new existence criteria are welcome.

Although (3.10) is singular, it was observed in [7] and later exploited in [4,5,8,19] that the change of variable $u = x^{\frac{1}{b+1}}$ leads to the regular equation

$$\ddot{x} + c\dot{x} = (b+1)e(t)x^{\frac{b-1}{b+1}} - a(b+1)x^{\frac{b}{b+1}}, \quad a > 0, \quad b > 1, \tag{3.11}$$

which is a particular case of (1.1) by setting

$$g(t) = (b+1)e(t), \quad h(t) = a(b+1), \quad \lambda = \frac{b-1}{b+1} \quad \text{and} \quad \mu = \frac{b}{b+1}.$$

Theorem 3.2. *(Existence) Let $a > 0, b > 1, c \geq 0$ and $e \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\bar{e} > 0$. Suppose that $\omega \in AC^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ is a periodic solution of the equation*

$$\ddot{\omega} + c\dot{\omega} = (b+1)(\bar{e} - e(t)) \tag{3.12}$$

that satisfies

$$M_\omega - m_\omega \leq \widehat{u}(x_1), \tag{3.13}$$

where

$$\widehat{u}(x) = \left(\frac{\overline{e}}{a}\right)^{\frac{b+1}{b}} x^{\frac{1}{b}} - x^{\frac{2}{b-1}}$$

and

$$x_1 = \left[\frac{\overline{e}}{a}\left(\frac{b-1}{2b}\right)^{\frac{b}{b+1}}\right]^{b-1}.$$

Moreover, let us define

$$\beta(t) = \left(\frac{\overline{e}}{a}x_1\right)^{\frac{b+1}{b}} - x_1(\omega(t) - m_\omega) \text{ for } t \in [0, T] \tag{3.14}$$

and assume that $\left\|\frac{ab}{\beta^{\frac{1}{b+1}}}\right\|_\infty \leq \frac{\pi^2}{T^2} + \frac{c^2}{4}$.

Then equation (3.11) has a positive T-periodic solution x such that

$$0 < \beta(t) \le x(t) \le \alpha(t),$$

where

$$\alpha(t) = x_0^{\frac{b+1}{b-1}} - x_0(\omega(t) - m_\omega) \tag{3.15}$$

with x_0 satisfying (3.7) and $x_0 > x_1$.
(Stability) Moreover, if $c > 0$, the functions α and β are strict lower and upper solutions of the periodic problem (3.11), (1.2) and

$$m_\beta > \left(\frac{(b-1)M_e}{ab}\right)^{b+1}, \tag{3.16}$$

then equation (3.11) has a unique asymptotically stable T-periodic solution between β and α.

Proof. Firstly, we are going to apply Theorem 3.1 with $c_2 = 1$, $c_1 = \frac{\bar{e}}{a}$ and $d_1 = d_2 = 0$. So, take ω as a T-periodic solution of (3.12) and $\sigma \equiv 0$. Then assumption (3.13) implies (3.3) and condition i) in Theorem 3.1 is also satisfied. Hence β and α given by (3.14) and (3.15) are, respectively, upper and lower solutions of the periodic problem (3.11), (1.2) such that

$$0 < \beta \le \alpha.$$

Define now

$$\rho(t, x) = h(t)x^\mu - g(t)x^\lambda = a(b+1)x^{\frac{b}{b+1}} - (b+1)e(t)x^{\frac{b-1}{b+1}}.$$

For $t \in \mathbb{R}$ and $0 < \beta(t) < x < \alpha(t)$ we have

$$\begin{aligned}\rho_x(t, x) &= \frac{b}{b+1}a(b+1)x^{\frac{-1}{b+1}} - \frac{b-1}{b+1}(b+1)e(t)x^{\frac{-2}{b+1}} \\ &\le \frac{ab}{\beta^{\frac{1}{b+1}}} \le \frac{\pi^2}{T^2} + \frac{c^2}{4},\end{aligned}$$

then the existence part follows from the lower/upper solution technique for the periodic problem in the reversed case, see for instance [16, Theorem 4.2].

As for the stability part, for $t \in \mathbb{R}$ and $x > 0$ we have

$$\begin{aligned}\rho_x(t, x) &= \frac{b}{b+1}a(b+1)x^{\frac{-1}{b+1}} - \frac{b-1}{b+1}(b+1)e(t)x^{\frac{-2}{b+1}} \\ &= x^{\frac{-2}{b+1}}[abx^{\frac{1}{b+1}} - (b-1)e(t)] \\ &\ge x^{\frac{-2}{b+1}}[abx^{\frac{1}{b+1}} - (b-1)M_e].\end{aligned}$$

So, for $m_2 > m_1 > \left(\frac{(b-1)M_e}{ab}\right)^{b+1}$ there exists $k(m_1, m_2) > 0$ such that for all $t \in \mathbb{R}$ and $m_1 \le x \le m_2$ we have

$$\rho_x(t, x) \ge k(m_1, m_2) > 0.$$

Then, by (3.16) we can take $m_1 := m_\beta$, $m_2 := M_\alpha$ and we obtain that

$$\rho_x(t,x) \geq k(m_1,m_2) > 0 \quad \text{for all } t \in \mathbb{R} \text{ and } \beta(t) \leq x \leq \alpha(t).$$

Thus, the existence of a unique asymptotically stable T-periodic solution between β and α follows now from [13, Remark 3.2]. □

We conclude this section with an example which illustrates Theorem 3.2. The graphs and some computations were made with the help of the software system *Mathematica*.

Example 3.3. Consider equation (3.11) with $a = 0.5$, $b = 2$, $c = 1$, and $e(t) = \sin 2\pi t + 2$, that is,

$$\ddot{x} + \dot{x} = 3(\sin 2\pi t + 2)x^{\frac{1}{3}} - \tfrac{3}{2}x^{\frac{2}{3}}. \tag{3.17}$$

It is clear that e is continuous and T-periodic with $T = 1$. Moreover, $\overline{e} = 2$, $m_e = 1$ and $M_e = 3$. Then equation (3.12) reads

$$\ddot{\omega} + \dot{\omega} = -3\sin 2\pi t. \tag{3.18}$$

By standard calculations we have that

$$\omega(t) = \frac{3}{2\pi(1+4\pi^2)}\cos 2\pi t + \frac{3}{1+4\pi^2}\sin 2\pi t$$

is a 1-periodic solution of (3.18). We also have

$$x_1 = 2^{\frac{2}{3}} \approx 1.5874 \;\text{ and }\; \widehat{u}(x_1) = 6\cdot 2^{\frac{1}{3}} \approx 7.5595.$$

Since $M_\omega - m_\omega = \dfrac{3}{\pi\sqrt{1+4\pi^2}} \leq 0.16$, the inequality $M_\omega - m_\omega \leq \widehat{u}(x_1)$ holds. Next, we obtain

$$\beta(t) = 8x_1^{\frac{3}{2}} - x_1(\omega(t) - m_\omega),$$
$$m_\beta = 8x_1^{\frac{3}{2}} - x_1(M_\omega - m_\omega) = 16 - 3\frac{2^{2/3}}{\pi\sqrt{1+4\pi^2}} \approx 15.7617,$$

and

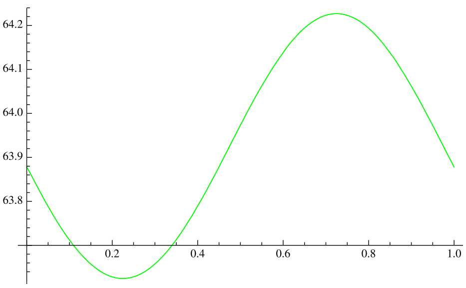
$$\left\|\frac{1}{\sqrt[3]{\beta}}\right\|_\infty = \frac{1}{\sqrt[3]{m_\beta}} \approx 0.3988 < \frac{\pi^2}{T^2} + \frac{c^2}{4} = \pi^2 + \frac{1}{4}.$$

Moreover, for $x_0 = 4.1 > x_1$ inequality (3.7) is satisfied. Defining

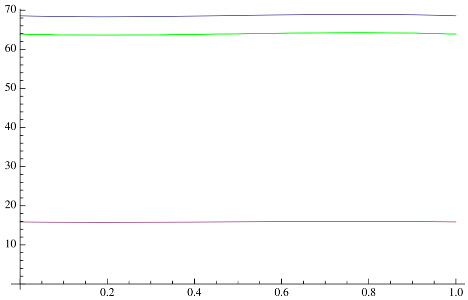
$$\alpha(t) = x_0^3 - x_0(\omega(t) - m_\omega),$$

by Theorem 3.2 we have that equation (3.17) has a positive periodic solution x such that

$$\beta(t) \leq x(t) \leq \alpha(t).$$



Numerical 1-periodic solution of equation (3.17).



From down to up: β, the solution and α.

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