Journal of Nonlinear Mathematical Physics Volume 15, Supplement 2 (2008), 50–57 ARTICLE

On the Recovery of the Free Surface from the Pressure within Periodic Traveling Water Waves

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Received on April 22, 2008

Abstract

We present a consistent derivation of the pressure transfer function for small amplitude waves within the framework of linear wave theory and discuss some nonlinear aspects.

1 Introduction

Wave measurements are extremely difficult [12, 13, 16] and a widely used approach to gather information about waves is by recording pressure data [22]. Among the reasons for relying on underwater pressure transducers for wave measurements are the fact that they are simple and less expensive than other devices, and their deployment under water minimizes their exposure to damage by sea traffic and fishing activities. The main thrust of this approach relies on the ability to convert the measured pressure data to surface wave information [1, 2, 3, 18, 22]. Linear wave theory provides a realistic framework for small amplitude waves and in this case a simple formula, involving the so-called "pressure transfer function" relates the exact shape of the free surface to the exact formula for the dynamic pressure. Various ways to obtain this pressure transfer function are available (see [22] and the comments therein) so that a consistent derivation starting with the governing equations for water waves seems timely. A considerable difficulty in dealing with the governing equations for the water wave problem lies in the fact that the free surface is not known *a priori*. A detailed analysis of the governing equations can be pursued in the case of waves that travel at constant speed and without change of shape at the surface of water with a flat bed (see [4, 6, 11, 21]) but for our specific purpose, which requires an understanding of the coupling between the free surface and the pressure within the fluid, the governing equations for water waves are too complicated and one has to simplify them by passing to certain regimes where certain approximations are possible. After presenting the governing equations for the water wave problem in Section 2, we pursue a detailed analysis of waves of small amplitude (within the linear approximation framework) in Section 3 and we conclude in Section 4 with some considerations about the nonlinear problem.

2 Preliminaries

In our analysis the water waves propagating at sea are two-dimensional i.e. parallel the crest of the wave the motion is identical. So using Caresian coordinates (x, y) we will look at a cross section of the wave form perpendicular to the crest line with the y-axis pointing vertically upward, the x-axis pointing in the direction of motion of the wave and the origin at the sea bed.

Let $(u(t, x, y), v(t, x, y))$ be the velocity field of the water flow, take the sea bed to be flat (and positioned at $y = 0$) and let $y = h_0 + \eta(t, x)$ be the water's free surface, where $h_0 > 0$ is the average height of the water above the flat bed. Denoting by P the pressure within the fluid and letting g to be the constant acceleration of gravity, the governing equations for water waves are [17]

1. Euler's equation:
$$\begin{cases} u_t + uu_x + vu_y &= -P_x, \\ v_t + uv_x + vv_y &= -P_y - g. \end{cases}$$

2. Equation for mass conservation:
$$u_x + v_y = 0.$$

3. Dynamic boundary condition:
$$P = P_a \quad \text{on} \quad y = h_0 + \eta(t, x),$$
where P_a is the (constant) atmospheric pressure at the water's free surface.

4. Kinematic boundary condition at the free surface:
$$v = \eta_t + u\eta_x \quad \text{on} \quad y = h_0 + \eta(t, x)$$

5. Kinematic boundary boundary condition on the flat bed:
$$v = 0 \quad \text{on} \quad y = 0.$$

In our treatment of water waves we will assume that at $t = 0$ the flat surface of still water is disturbed and we set out to analyze the result of this disturbance over time. We assume the flow has zero vorticity $\omega = 0$. The vorticity of a flow is defined at the measurement of local spin of a fluid element and is given by $\omega = u_y - v_x$, so zero vorticity implies

$$u_y = v_x. \tag{2.1}$$

This assumption is physically correct as there is experimental evidence to show that waves entering a region of still water can be taken to have zero vorticity. Also water waves which are initially irrotational will remain irrotational at later times [17, 19].

3 Linear theory

The governing equations can be nondimensionalised (see [17]) using the following transformations involving the typical wave amplitude a and the typical wavelength λ,

$$x \to \lambda x, \quad y \to h_0 y, \quad t \to \frac{\lambda}{\sqrt{gh_0}} t, \quad u \mapsto u\,\sqrt{gh_0}, \quad v \mapsto v\,\frac{h_0\,\sqrt{gh_0}}{\lambda}, \quad \eta \to a\eta, \tag{3.1}$$

in the sense that x is replaced by $\lambda\, x$ so that the new symbol x stands for a nondimensional variable. Introducing the nondimensional pressure variable p as a measure of the deviation of the pressure within the fluid from the hydrostatic pressure distribution,

$$P = P_a + gh_0(1-y) + gh_0 p, \tag{3.2}$$

and applying these transformations to the governing equations for irrotational flow, we get the equivalent system

$$\begin{cases} u_t + uu_x + vu_y = -p_x \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1 + \epsilon\,\eta(t,x), \\ \delta^2(v_t + uv_x + vv_y) = -p_y \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1 + \epsilon\,\eta(t,x), \\ u_x + v_y = 0 \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1 + \epsilon\,\eta(t,x), \\ u_y - \delta^2\, v_x = 0 \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1 + \epsilon\,\eta(t,x), \\ p = \epsilon\,\eta \quad \text{on} \quad y = 1 + \epsilon\,\eta(t,x) \quad \text{with} \quad x \in \mathbb{R}, \\ v = \epsilon\,(\eta_t + u\eta_x) \quad \text{on} \quad y = 1 + \epsilon\,\eta(t,x) \quad \text{with} \quad x \in \mathbb{R}, \\ v = 0 \quad \text{on} \quad y = 0 \quad \text{with} \quad x \in \mathbb{R}, \end{cases} \tag{3.3}$$

where

$$\epsilon = \frac{a}{h_0} \tag{3.4}$$

is the amplitude parameter and

$$\delta = \frac{h_0}{\lambda} \tag{3.5}$$

is the shallowness parameter. The fifth and sixth equation in (3.3) show that the evaluations of both v and p to the free surface are proportional to ϵ. This is consistent with the fact that as $\epsilon \to 0$ we expect $v = p = 0$ on the free surface $z = 1$ as in this limit no disturbance means still water with a flat free surface. This leads us to the scaling

$$(u,v) \mapsto \epsilon\,(u,v), \qquad p \mapsto \epsilon\, p. \tag{3.6}$$

The problem (3.3) is transformed after performing the non-dimensionalisation (3.1) and the scaling (3.6) to the equivalent problem

$$\begin{cases} u_t + \epsilon\,(uu_x + vu_y) = -p_x \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1 + \epsilon\,\eta(t,x), \\ \delta^2(v_t + \epsilon\,(uv_x + vv_y)) = -p_y \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1 + \epsilon\,\eta(t,x), \\ u_x + v_y = 0 \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1 + \epsilon\,\eta(t,x), \\ u_y - \delta^2\, v_x = 0 \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1 + \epsilon\,\eta(t,x), \\ p = \eta \quad \text{on} \quad y = 1 + \epsilon\,\eta(t,x) \quad \text{with} \quad x \in \mathbb{R}, \\ v = \eta_t + \epsilon\, u\eta_x \quad \text{on} \quad y = 1 + \epsilon\,\eta(t,x) \quad \text{with} \quad x \in \mathbb{R}, \\ v = 0 \quad \text{on} \quad y = 0 \quad \text{with} \quad x \in \mathbb{R}. \end{cases} \tag{3.7}$$

The linear approximation, suitable for waves of small amplitude, is obtained by letting $\epsilon \to 0$ in (3.7):

$$\begin{cases} u_t = -p_x \quad \text{and} \quad \delta^2\, v_t = -p_y \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1, \\ u_x + v_y = 0 \quad \text{and} \quad u_y - \delta^2\, v_x = 0 \quad \text{for} \quad x \in \mathbb{R},\ 0 < y < 1, \\ p = \eta \quad \text{and} \quad v = \eta_t \quad \text{on} \quad y = 1 \quad \text{with} \quad x \in \mathbb{R}, \\ v = 0 \quad \text{on} \quad y = 0 \quad \text{with} \quad x \in \mathbb{R}. \end{cases} \tag{3.8}$$

The fundamental Fourier mode Ansatz

$$\eta(t,x) = \cos\left[2\pi(x - c_0 t)\right]$$

arises in the context of looking for solutions of (3.8) representing periodic waves (of unit period) traveling without change of shape at constant speed c_0, setting in which all functions u, v, p, η exhibit an (x,t)-dependence as a periodic function of $x - c_0 t$. For this specific η the linearised problem (3.8) has the explicit solution

$$\begin{cases} u(t,x,y) = 2\pi c_0 \delta\, \dfrac{\cosh(2\pi\delta y)}{\sinh(2\pi\delta)} \cos\left[2\pi(x - c_0 t)\right], \\ v(t,x,y) = 2\pi c_0\, \dfrac{\sinh(2\pi\delta y)}{\sinh(2\pi\delta)} \sin\left[2\pi(x - c_0 t)\right], \\ p(t,x,y) = \dfrac{\cosh(2\pi\delta y)}{\cosh(2\pi\delta)} \cos\left[2\pi(x - c_0 t)\right], \end{cases} \tag{3.9}$$

provided

$$c_0^2 = \frac{\tanh(2\pi\delta)}{2\pi\delta}. \tag{3.10}$$

In the original physical variables the solution (3.9) takes the form

$$\begin{cases} \eta(t,x) = \epsilon h_0 \cos(kx - \omega t), \\ u(t,x,y) = \epsilon\omega h_0\, \dfrac{\cosh(ky)}{\sinh(kh_0)} \cos(kx - \omega t), \\ v(t,x,y) = \epsilon\omega h_0\, \dfrac{\sinh(ky)}{\sinh(kh_0)} \sin(kx - \omega t), \\ P(t,x,y) = P_a + g(h_0 - y) + \epsilon g h_0\, \dfrac{\cosh(ky)}{\cosh(kh_0)} \cos(kx - \omega t), \end{cases} \tag{3.11}$$

where

$$k = \frac{2\pi}{\lambda}, \quad \omega = \sqrt{gk\, \tanh(kh_0)}, \tag{3.12}$$

are the wave number, respectively the frequency of the linear wave propagating at speed

$$c = \frac{\omega}{k} = \sqrt{g\, \frac{\tanh(kh_0)}{k}} \tag{3.13}$$

and of period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{gk\,\tanh(kh_0)}} \tag{3.14}$$

over the flat bed $y = 0$, with mean water level h_0 and wavelength $\lambda > 0$.

Defining the **dynamic pressure** P_d as the difference between the pressure P and the hydrostatic pressure

$$P_h = P_a + g(h_0 - y),$$

from the first and last expressions in (3.11) we see that

$$P_d(x, y, t) = g\,\frac{\cosh(ky)}{\cosh(kh_0)}\,\eta(t, x). \tag{3.15}$$

so that in linear theory the pressure transfer function is given by

$$K(y) = g\,\frac{\cosh(ky)}{\cosh(kh_0)}. \tag{3.16}$$

This function allows the reconstruction of the free surface from measurements of the pressure along the flat bed by means of the simple formula (3.15).

4 Towards a nonlinear theory

The lack of explicit general solutions for the governing equations makes it impossible to expect a formula of type (3.15) to be valid for waves of large amplitude - a setting where the linear theory presented in the previous section is not appropriate. Therefore one has to expect incremental results relating various properties of the pressure in the fluid with features of the shape of the free surface. The reconstruction of the free surface from pressure measurements on the bed is also relevant in relation to tsunami waves (see [9, 10] for recent discussions of this type of water waves) and should also be pursued in the context of solitary waves.

At present very little is known about qualitative properties of the pressure below a periodic wave of permanent form traveling at constant speed at the free surface of water with a flat bed. For irrotational flows (Stokes waves), however, some basic results about the shape of the free surface are available: the profile is strictly monotonic between crests and troughs [21] and has to be symmetric about the wave crest, with the symmetry persisting even in the presence of vorticity [7, 8, 5]. As for the pressure, recall [4, 11] that in a frame moving at the constant wave speed $c > 0$, the governing equations for periodic waves of permanent form traveling at constant speed in irrotational flow can be expressed in terms of the stream function ψ defined up to a constant by

$$\psi_x = -v, \qquad \psi_y = u - c, \tag{4.1}$$

in the form of the system

$$\begin{cases} \Delta\psi = 0 \quad \text{in} \quad 0 < y < h_0 + \eta(x), \\ \dfrac{|\nabla\psi|^2}{2} + gy = Q \quad \text{on} \quad y = h_0 + \eta(x), \\ \psi = 0 \quad \text{on} \quad y = h_0 + \eta(x), \\ \psi = m \quad \text{on} \quad y = 0. \end{cases} \tag{4.2}$$

Here $m > 0$ and $Q > 0$ are physical constants (relative mass flux, hydraulic head). We recover the pressure P via Bernoulli's theorem which ensures that

$$\frac{|\nabla\psi|^2}{2} + gy + P = Q + P_a$$

throughout the fluid so that

$$P = Q + P_a - \frac{|\nabla\psi|^2}{2} - gy. \tag{4.3}$$

A direct calculation now shows that the function P is superharmonic in the fluid domain as

$$\Delta P = -\psi_{xx}^2 - 2\psi_{xy}^2 - \psi_{yy}^2 \leq 0.$$

Since $P_y = -g < 0$ on $y = 0$, taking into account both the second component of Euler's equation in the form

$$(u - c)v_x + vv_y = -P_y - g,$$

and the kinematic boundary condition $v = 0$ on the flat bed $y = 0$, Hopf's maximum principle [15] yields that the minimum of P is attained only along the free surface $\{y = h_0 + \eta(t, x)\}$, where $P = P_a$ in view of the dynamic boundary condition. As for the behaviour of the pressure P on the flat bed, notice that the first component of Euler's equation,

$$(u - c)u_x + vu_y = -P_x,$$

coupled with the kinematic boundary condition $v = 0$ on the flat bed $y = 0$, yields

$$P_x = (c - u)u_x \quad \text{along the flat bed} \quad y = 0.$$

But for Stokes waves we know (see [21]) that $u < c$ except perhaps at the wave crest where equality holds for the extreme waves (the "waves of greatest height"). Also, assuming that the point $(0, 0)$ lies on the bed straight below the wave crest (in the moving frame), while $(\pm\pi, 0)$ is straight below the wave troughs, we know (see [4] that the function $u(x, 0)$ is even in the x-variable with $u_x(x, 0) < 0$ for $x \in (0, \pi)$. This means that along the flat bed the pressure is maximal below the wave crest and minimal below the wave trough (as well as strictly monotone in-between). These properties parallel those established for small amplitude waves within the framework of linear theory and allow us to predict from measurements of the pressure on the flat bed the passage at the free surface of a wave crest or of a wave trough.

The above considerations show that it is possible to obtain relevant information about the pressure within the fluid. Advances of the state-of-the-art depend on a fruitful interaction between theoretical studies (building upon recent insights obtained in [4, 6, 11]), numerical simulations (as in [14]) and experimental evidence gathered in the laboratory (see e.g. [20]) or by collection of field data. This is work in progress.

Acknowledgments. *The authors are grateful to Adrian Constantin for several interesting and stimulating discussions.*

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