Upper density problems in infinite Ramsey theory

Dissertation zur Erlangung des Grades
eines Doktors der Naturwissenschaften (Dr. rer. nat.)
am Fachbereich Mathematik und Informatik
der Freien Universität Berlin

von

Ander Lamaison Vidarte

Berlin
September 2020

Defense date:
26th February 2021

Supervisor and first reviewer:
Prof. Tibor Szabó

Institut für Mathematik
Freie Universität Berlin

Second reviewer:
Prof. Daniel Král'

Faculty of Informatics
Masaryk University
Brno, Czech Republic

Summary

We consider the following question in infinite Ramsey theory, introduced by Erdős and Galvin [EG93] in a particular case and by DeBiasio and McKenney [DM19] in a more general setting. Let H be a countably infinite graph. If the edges of the complete graph on $\mathbb{N}$ are colored red or blue, what is the maximum value of λ such that we are guaranteed to find a monochromatic copy of H whose vertex set has upper density at least λ? We call this value the Ramsey density of H.

The problem of determining the Ramsey density of the infinite path was first studied by Erdős and Galvin, and was recently solved by Corsten, DeBiasio, Lang and the author [CDLL19]. In this thesis we study the problem of determining the Ramsey density of arbitrary graphs H. On an intuitive level, we show that three properties of a graph H have an effect on the Ramsey density: the chromatic number, the number of components, and the expansion of its independent sets. We deduce the exact value of the Ramsey density for a wide variety of graphs, including all locally finite forests, bipartite factors, clique factors and odd cycle factors. We also determine the value of the Ramsey density of all locally finite graphs, up to a factor of 2.

We also study a list coloring variant of the same problem. We show that there exists a way of assigning a list of size two to every edge in the complete graph on $\mathbb{N}$ such that, in every list coloring, there are monochromatic paths with density arbitrarily close to 1.

Zusammenfassung

Wir betrachten die folgende Fragestellung aus der Ramsey-Theorie, welche von Erdős und Galvin [EG93] in einem Spezialfall sowie von DeBiasio und McKenney [DM19] in einem allgemeineren Kontext formuliert wurde: Es sei H ein abzählbar unendlicher Graph. Welches ist der größtmögliche Wert λ, sodass wir, wenn die Kanten des vollständigen Graphen mit Knotenmenge $\mathbb{N}$ jeweils entweder rot oder blau gefärbt sind, stets eine einfarbige Kopie von H, dessen Knotenmenge eine obere asymptotische Dichte von mindestens λ besitzt, finden können? Wir nennen diesen Wert die *Ramsey-Dichte* von H.

Das Problem, die Ramsey-Dichte des unendlichen Pfades zu bestimmen wurde erstmals von Erdős und Galvin untersucht und wurde vor kurzem von Corsten, DeBiasio, Lang und dem Autor [CDLL19] gelöst. Gegenstand der vorliegenden Dissertation ist die Bestimmung der Ramsey-Dichten von Graphen. Auf einer intuitiven Ebene zeigen wir, dass drei Parameter eines Graphen die Ramsey-Dichte beeinflussen: die chromatische Zahl, die Anzahl der Zusammenhangskomponenten sowie die Expansion seiner unabhängigen Mengen. Wir ermitteln die exakten Werte der Ramsey-Dichte für eine Vielzahl von Graphen, darunter alle lokal endlichen Wälder, bipartite Faktoren, K_r-Faktoren sowie C_k-Faktoren für ungerade k. Ferner bestimmen wir den Wert der Ramsey-Dichte aller lokal endlichen Graphen bis auf einen Faktor 2.

Darüber hinaus untersuchen wir eine Variante des oben beschriebenen Problems für Listenfärbungen. Wir zeigen, dass es möglich ist, jeder Kante des vollständigen Graphen mit Knotenmenge $\mathbb{N}$ eine Liste der Größe Zwei zuzuweisen, sodass in jeder zugehörigen Listenfärbung monochromatische Pfade mit beliebig nah an 1 liegender Dichte existieren.

Acknowledgements

To my advisor, Tibor Szabó, for always providing guidance both in my research and in my academic life, and to everyone in the combinatorics group at FU for creating an active and positive work environment.

To the Berlin Mathematical School, for awarding me a scholarship both in Phase I and Phase II of my PhD. This scholarship, in turn, was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689). Special thanks for additionally funding my visits to IMPA, University of Illinois at Urbana-Champaign and Princeton University, where much of the research contained in this thesis was conducted.

To all my coauthors and collaborators. Special thanks to Louis DeBiasio, for suggesting the problem of the Ramsey density of the path during a workshop at IMPA. Also special thanks to Noga Alon, József Balogh and Dan Král, for welcoming me in research visits. Additional thanks to József Balogh, for working with me on what would become Chapter 4 in this thesis.

And to my family, for supporting me all the way.

Contents

Summary v

Zusammenfassung vii

Acknowledgements ix

1 Introduction 1
- 1.1 Definitions . . . 5
- 1.2 Results in this thesis . . . 6
  - 1.2.1 Results on locally finite graphs . . . 7
  - 1.2.2 Results for infinite factors . . . 9
  - 1.2.3 Further results for infinite factors . . . 10
  - 1.2.4 Results on list coloring . . . 11
  - 1.2.5 Structure of the thesis . . . 13

2 Ramsey density of infinite paths 15
- 2.1 Proof of Theorem 1.2 . . . 15
- 2.2 Proof of Theorem 1.3 . . . 18
  - 2.2.1 From path forests to paths . . . 19
  - 2.2.2 From simple forests to path forests . . . 20
  - 2.2.3 Upper density of simple forests . . . 23
  - 2.2.4 Sequences and oscillation . . . 27

3 Ramsey density of locally finite graphs 31
- 3.1 General upper bounds . . . 31
- 3.2 General lower bounds . . . 35
- 3.3 Bounds for particular families of graphs . . . 48

4 Ramsey density of infinite factors 53
- 4.1 Proof of Theorem 1.17 . . . 53
- 4.2 Proof of Theorem 1.19 . . . 60

5 Ramsey density and list coloring 65
- 5.1 Proof of Theorem 1.20 and Theorem 1.21 . . . 65
- 5.2 Proof of Theorem 1.22 . . . 68

6 Open problems 71

A Proof of Lemma 3.10 75

B Properties of $f(\lambda)$ **79**

Declaration of Authorship **85**

Bibliography **87**

Chapter 1

Introduction

The field of Ramsey theory studies the partitioning of discrete structures, and the patterns that necessarily appear in such partitions. A typical Ramsey theory question can be phrased in the following form: if the elements of [discrete structure] are partitioned into a fixed number of classes in any way we want, do we always find a [substructure] in which every element belongs to the same class?

For example, a classic result of Erdős and Szekeres [ES35] can be stated as follows: if every edge in K_n with $n = \binom{2k-2}{k-1}$ is colored red or blue, then the resulting graph always contains a monochromatic copy of K_k. Here, the structure that we partition is $E(K_n)$, and the substructure that we look for is the set of edges in a k-clique. Observe that we describe the partition into two classes (the class of red edges and the class of blue edges) as a color assignment. Indeed, it is common in Ramsey theory to talk about colorings instead of partitions, since for our purposes it is an equivalent concept. When the partition has two classes, they are usually assigned the colors red and blue.

Although there had been some earlier results that fit into this framework, a 1929 paper by Ramsey [Ram30] is considered a turning point in the namesake theory. While graphs are one of the most widely studied structures in connection to Ramsey theory, over the decades the theory has been applied to colorings of hypergraphs, arithmetic structures, geometric spaces, function spaces or partially ordered sets, to name a few [Gra07].

When the structures studied are finite, much emphasis is often placed in quantitative results. Consider the theorem of Erdős and Szekeres above. What is the smallest function $R(k)$ which can replace the n in this statement? The number $R(k)$ is called the k-th diagonal Ramsey number, and determining the asymptotics of these numbers is considered one of the most important open questions in Ramsey theory, if not the most important.

Two popular quantitative problems in Ramsey theory are the following: let F be a finite graph. We say that a graph G has the Ramsey property for F if in every two-coloring of $E(G)$ there exists a monochromatic copy of F. What is the smallest number of vertices in a graph G that is Ramsey for F? And the smallest number of edges? These numbers are called, respectively, the Ramsey number and the size Ramsey number of F.

On the other hand, there have also been Ramsey theoretic results involving infinite structures. Indeed, Ramsey himself proved in [Ram30] that, when the edges of an infinite complete graph are colored using finitely many colors, there is an infinite subset of the vertices which induces a monochromatic clique. A generalization, called the canonical Ramsey theorem [ER50], determines the colored patterns in infinite cliques that must necessarily appear if the family of available colors is not necessarily finite.

The analysis of the Ramsey theoretic properties of infinite sets of large cardinalities has also attracted some attention [EH62]. But for the most part, results in infinite Ramsey theory have been existential, rather than quantitative.

One relevant result in infinite Ramsey theory is a 1978 theorem of Rado [Rad78]. Let $K_{\mathbb{N}}$ denote the complete graph on the natural numbers. The theorem states that, whenever the edges of $K_{\mathbb{N}}$ are colored using one of k colors, there is a family $Z_1, Z_2, \ldots, Z_k$ of paths, one in each color, such that every vertex is contained in exactly one path.

We note at this point that, in contrast to finite paths, there are two different ways in which an infinite path can be defined. One can define the so-called one-way infinite path, which is the graph on $\mathbb{N}$ where every pair of vertices at distance 1 is joined by an edge, or the two-way infinite path, defined similarly over $\mathbb{Z}$. In this thesis, by 'infinite path' we will always mean one-way infinite, and we will denote it by P_∞. Nevertheless, our results would hold equally for two-way infinite paths.

Intuitively, if the vertex set of $K_{\mathbb{N}}$ is partitioned into k monochromatic paths, then one of them must contain at least "a $1/k$ fraction of the vertices". But in infinite sets, what "proportion of the vertices" a certain subset represents can be defined in different ways, and these definitions will not be equivalent for our purposes.

A natural measure of this proportion is the following: for a set $S \subseteq \mathbb{N}$, if $\lim\limits_{n\to\infty} \frac{|S\cap[n]|}{n}$ exists, the value of this limit is called the natural density of S. This notion has the drawback of not being defined for all subsets $S \subseteq \mathbb{N}$. There are two extensions of this measure that match the natural density whenever it is defined: upper density and lower density.

In this thesis we will use upper density as a measure of subsets of $\mathbb{N}$:

Definition 1.1. *Given a subset $S \subseteq \mathbb{N}$, we define its (upper) density as*

$$\bar{d}(S) := \limsup_{n\to\infty} \frac{|S \cap [n]|}{n}.$$

For a subgraph $H \subseteq K_{\mathbb{N}}$, we define $\bar{d}(H) := \bar{d}(V(H))$.

Because we will not use any other definition of density, we will omit the word 'upper'. Lower density replaces the lim sup in the definition by a lim inf. In our problem, in the particular case of the path, choosing lower density as the measure of subgraphs results in a rather easy problem, as Erdős and Galvin [EG93] constructed a coloring in which every monochromatic path has lower density 0. The same coloring has maximum lower density 0 for most of the graphs that we will see here, which is why we will not consider this problem. For a study of similar problems involving lower density, in which the target graph is something other than the path, see [DM19, CDM20].

Our density satisfies subadditivity: for any two subsets $A, B \subseteq \mathbb{N}$, we have $\bar{d}(A \cup B) \leq \bar{d}(A) + \bar{d}(B)$. With this definition of density (in contrast to lower density), Rado's theorem does imply, by subadditivity, that every two-coloring of $E(K_{\mathbb{N}})$ contains a monochromatic path of density at least $1/2$.

Erdős and Galvin [EG93] were the first to study whether one can always find an infinite path with density higher than $1/2$, improving the bound that can be deduced from Rado's theorem. They showed that, even more, every two-coloring of $E(K_{\mathbb{N}})$ contains a monochromatic infinite path of density at least $2/3$. The bound here is intriguing, because of a similar result for finite paths: Gerencsér and Gyárfás [GG67] proved that in every two-coloring of $E(K_n)$ there is a path on $\lceil 2n/3 + 1\rceil$ vertices, and this is best possible (both results use similar techniques, which is why they produce similar asymptotics). Erdős and Galvin also gave a construction of a coloring without monochromatic paths of density higher than $8/9$.

This raises the question: what is the largest value of M such that one can always find a monochromatic infinite path with density at least M? Clearly $2/3 \leq M \leq 8/9$. DeBiasio and McKenney [DM19] improved the lower bound to $M \geq 3/4$, and conjectured that $M = 8/9$. Lo, Sanhueza-Matamala and Wang [LSMW18] further showed $M \geq (9 + \sqrt{17})/16 = 0.82019\ldots$ Finally, Corsten, DeBiasio, Lang and the author determined in [CDLL19] that neither this lower bound nor Erdős and Galvin's upper bound is tight. They established the exact value of M, which disproves the conjecture of DeBiasio and McKenney:

Theorem 1.2. *There exists a 2-coloring of the edges of $K_{\mathbb{N}}$ such that every monochromatic path has upper density at most $(12 + \sqrt{8})/17 = 0.87226\ldots$.*

Theorem 1.3. *In every 2-coloring of the edges of $K_{\mathbb{N}}$, there exists a monochromatic path of upper density at least* $(12+\sqrt{8})/17 = 0.87226\ldots$.

In this thesis we will deal with this question: what if, instead of a monochromatic infinite path, we are looking for a monochromatic copy of some other infinite graph H? Clearly, by Ramsey's theorem, every two-coloring of $E(K_{\mathbb{N}})$ contains a monochromatic infinite clique, of which H is a subgraph. But does there exist a *dense* monochromatic copy of H? And if so, how dense? The definition below formalizes this question.

Definition 1.4. *Let H be a countably infinite graph. The Ramsey (upper) density of H, denoted by $\rho(H)$, is the supremum of the values λ satisfying the following property: in every two-coloring of $E(K_{\mathbb{N}})$ there exists a monochromatic subgraph H' isomorphic to H such that $\bar{d}(H') \geq \lambda$.*

Theorems 1.2 and 1.3 [CDLL19] can then be combined[1] as as $\rho(P_\infty) = (12+\sqrt{8})/17$.

The first paper discussing this parameter for graphs other than P_∞ was that of DeBiasio and McKenney [DM19]. Their starting point was a theorem of Elekes, Soukup, Soukup and Szentmiklóssy [ESSS17] which, disregarding some color considerations, is an extension of Rado's theorem. The k-th power of the infinite path, denoted by P_∞^k, is the graph on $\mathbb{N}$ where two vertices ij form an edge if and only if $|i-j| \leq k$. Elekes et al. proved that, in every two-coloring of $E(K_{\mathbb{N}})$, the vertex set can be partitioned into at most 2^{2k-1} monochromatic copies of P_∞^k, plus a finite set. In the particular case $k=2$, the number of monochromatic squares of paths can be reduced to 4. DeBiasio and McKenney noted that this implies that $\rho(P_\infty^k) \geq 2^{-(2k-1)}$ and $\rho(P_\infty^2) = 1/4$, just like Rado's theorem implies $\rho(P_\infty) \geq 1/2$.

In the same paper, DeBiasio and McKenney state a conjecture that, using our notation, can be phrased as follows:

Conjecture 1.5 ([DM19])**.** *For every $\Delta \in \mathbb{N}$ there exists a constant $c = c(\Delta) > 0$ such that, for every infinite graph H with maximum degree at most Δ, we have $\rho(H) \geq c$.*

In this thesis we will prove a number of results regarding the Ramsey density of graphs. We will state them in Section 1.2. But before that, we will need to define some concepts and some notation that we will use in these statements.

[1] Actually the theorems are slightly stronger, since $\rho(P_\infty) = (12+\sqrt{8})/17$ only allows one to get arbitrarily close to $(12+\sqrt{8})/17$. This distinction is important: we will see that there exist graphs H with $\rho(H) = 0$ where every two-coloring of $E(K_{\mathbb{N}})$ contains a monochromatic copy of H with positive density.

1.1 Definitions

We say that an infinite graph is *locally finite* if every vertex has finite degree.

If F is a finite graph, we denote by $\omega \cdot F$ the graph obtained by taking the disjoint union of a countably infinite number of copies of F. We will call this the infinite F-factor.

Given $S \subseteq V(H)$, we will denote by $N(S) = (\cup_{v \in S} N(v)) \setminus S$ the set of vertices outside S with a neighbor in S. We let $\mu(H, n)$ be the minimum value of $|N(I)|$, where $I \subseteq V(H)$ is an independent set in H of size n. Note that, if an infinite graph H is locally finite, then it contains arbitrarily large finite independent sets, each with a finite neighborhood, and in particular $\mu(H, n)$ is well-defined and finite. We say that a set $I \subseteq V(H)$ is *doubly independent* if both I and $N(I)$ are independent.

Finally, we define a function $f(x)$ which will be crucial in relating the values of $\rho(H)$ and $|N(I)|/|I|$, where I is an independent set of H. Unfortunately, there is no satisfying intuition (at least not before seeing the proof of Theorem 1.9) for why this particular choice of $f(x)$, and not another, is behind the relation between these two parameters. One can reverse engineer the definition of the function $f(x)$ from the proof of Theorem 1.9, by checking which choice of parameters leads to the best upper bound on the Ramsey density. The surprising part is that the same function $f(x)$ also comes up in the analysis of lower bounds.

Since the definition of $f(x)$ is quite complicated and its comprehension is not essential to the appreciation of our results, we encourage the reader to skip it for now. For the reading of this chapter, knowing that such a function exists is sufficient. Of course, for the reading of the proofs, the precise definition becomes necessary. Its precise definition is used only in the proof of Theorem 1.9 (where it acts as an upper bound), in Appendix A (where it acts as a lower bound), and in Appendix B (for properties of $f(x)$ itself). All other uses of $f(x)$ treat the aforementioned as black boxes.

Definition 1.6. *Let $\gamma \in (-1, 1)$. For a continuous function $g(x) : [0, +\infty) \to \mathbb{R}$, and $t > 0$, define*

$$\Gamma_\gamma^+(g, t) = \min\{x : \gamma x + g(x) \geq t\} \qquad \Gamma_\gamma^-(g, t) = \min\{x : \gamma x - g(x) \geq t\},$$

where we take the minimum of the empty set to be $+\infty$. We define $h(\gamma)$ to be the infimum, over all 1-Lipschitz[2] functions g with $g(0) = 0$, of

$$h(\gamma) = \inf_g \limsup_{t \to \infty} \frac{\Gamma_\gamma^+(g, t) + \Gamma_\gamma^-(g, t)}{t}. \tag{1.1}$$

[2]A 1-Lipschitz function is a function satisfying $|g(x) - g(y)| \leq |x - y|$ for every x, y in the domain.

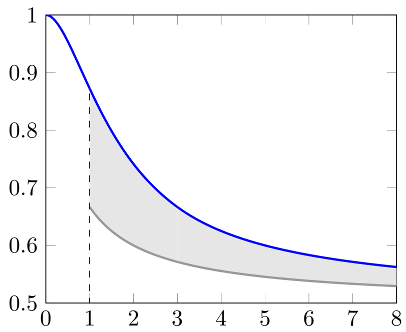

FIGURE 1.1: Plot of the function $f(x)$ on the interval $[0,1]$, and the upper and lower bounds elsewhere. The conjectured value is given in blue.

Define $f : (0,+\infty) \to \mathbb{R}$ *as*

$$f(\lambda) = 1 - \frac{1}{\frac{2\lambda}{(1+\lambda)^2} h\left(\frac{\lambda-1}{\lambda+1}\right) + \frac{2\lambda}{1+\lambda}}.$$

We define $f(0) = 1$ *and* $f(+\infty) = 1/2$ *(by* (1.2) *below, we have* $\lim_{t\to 0} f(t) = 1$ *and* $\lim_{t\to+\infty} f(t) = 1/2$*.)*

In Appendix B we prove some properties of $f(x)$, including the following bounds:

$$\frac{x+1}{2x+1} \leq f(x) \leq \begin{cases} \frac{2x^2+3x+7+2\sqrt{x+1}}{4x^2+4x+9} & \text{for} \quad 0 \leq x < 3, \\ \\ \frac{x+1}{2x} & \text{for} \quad x \geq 3. \end{cases} \tag{1.2}$$

The upper bound is tight for $x \in [0,1]$, and we conjecture[3] that it is tight everywhere. Observe that $f(1) = (12+\sqrt{8})/17 = \rho(P_\infty)$.

1.2 Results in this thesis

The first results proved in this thesis are Theorem 1.2 and Theorem 1.3. These proofs, reproductions of the ones in [CDLL19], will make up Chapter 2. The two theorems are implied by some of the other results in the thesis (in particular Theorem 1.11), but nevertheless we include the proofs in order to introduce the methods that will be used later in the thesis in a more involved context. However, this thesis is organized in such a way that a reader who is only interested in the stronger results can skip Chapter 2 entirely.

[3] An extended abstract for the paper [Lam20] (on which Chapter 3 is based), published in *Acta Math. Univ. Comenianae* for EUROCOMB 2019, stated this conjecture as proved. Since then, a mistake in the proof has been found.

In Chapter 3 and Chapter 4 we prove a number of results on the value of $\rho(H)$ for different types of graphs. Our goal is to compute the exact value of $\rho(H)$ for a family of graphs H as wide as possible, but in the process we will also build some intuition into how $\rho(H)$ behaves in general. One of the main takeaways from these results is the intuition that the following three parameters have a considerable effect on the value of $\rho(H)$: the ratio $\liminf_{n\to\infty} \frac{\mu(H,n)}{n}$ (which we will refer to as the *expansion ratio of the independent sets* of H), the chromatic number of H, and the number of components of H. The relation between these parameters and $\rho(H)$ can be found, for example, in Theorems 1.7, 1.9, 1.11, 3.2 and Corollary 1.18.

As a sample of these results, we will state here the main theorems from these chapters, and the most important corollaries that can be derived from those. Some of these results follow from even more general bounds, which we will not state here because their statement is very involved (as an example, Theorem 3.2 is arguably the most powerful lower bound on $\rho(H)$ proved in this thesis, but due to the many technical details in the statement, we will not present it here, and instead we will present consequences of this theorem that are more ready for immediate application and easier to understand).

After that, we present the results from Chapter 5, which deals with a related problem which combines Ramsey density and list coloring.

1.2.1 Results on locally finite graphs

The first theorem that we present gives the value of $\rho(H)$ for every locally finite graph H, up to a factor of 2:

Theorem 1.7. *Let H be a locally finite graph.*

(i) If H has infinitely many components, then $\rho(H) \geq 1/2$.

(ii) If H has finitely many components:

(a) If H has infinite chromatic number, then $\rho(H) = 0$.

(b) If H has finite chromatic number, then

$$\min\left\{\frac{b}{2(\chi(H)-1)}, \frac{1}{2}\right\} \leq \rho(H) \leq \min\left\{\frac{b}{\chi(H)-1}, 1\right\},$$

where b is the number of infinite components of H.

Theorem 1.7 implies Conjecture 1.5, because every graph satisfies $\chi(H) \leq \Delta(H) + 1$:

Corollary 1.8. *For every $\Delta \in \mathbb{N}$, every infinite graph H with maximum degree at most Δ has $\rho(H) \geq 1/(2\Delta)$.*

Theorem 1.7 also improves the lower bound on the Ramsey density of powers of paths given by DeBiasio and McKenney. Where for $k \geq 3$ they observed that $\rho(P_\infty^k) \geq 2^{-(2k-1)}$, Theorem 1.7 gives $\rho(P_\infty^k) \geq 1/(2k)$, since $\chi(P_\infty^k) = k+1$. For $k = 2$, we match the lower bound $\rho(P_\infty^2) \geq 1/4$.

The case (ii)a in Theorem 1.7 connects to a result of Corsten, DeBiasio and McKenney [CDM20]. While we show that every locally finite graph H with finitely many components and infinite chromatic number has $\rho(H) = 0$, they show that these graphs are "2-Ramsey-dense", as they call it (see Corollary 1.7 in their paper). This property means that, in every two-coloring of $E(K_\mathbb{N})$, there exists $\epsilon > 0$ and a monochromatic copy of H with density at least ϵ. Of course this is not a contradiction, because there does not exist a choice of ϵ that is valid for every coloring.

While no graph H is known for which the lower bound in (ii)b is tight and not equal to 1/2, the upper bound is tight, for example, in the graph $b \cdot T + K_a$ described in Section 3.2. It is worth pointing out that, unlike the other graphs H for which we obtain the exact value of $\rho(H)$ in this thesis, the value here does not depend on the function $f(x)$: for all $a > b \geq 1$ we have $\rho(b \cdot T + K_a) = \frac{b}{a-1}$.

The second theorem is a general upper bound on $\rho(H)$ that is valid for all locally finite graphs. The importance of this theorem lies in the fact that it is sharp for a variety of choices of H, including forests (Theorem 1.11), bipartite infinite factors (Corollary 1.14), clique factors and odd cycle factors (Corollary 1.18).

Theorem 1.9. *Let H be a locally finite graph. Then*

$$\rho(H) \leq f\left(\liminf_{n\to\infty} \frac{\mu(H,n)}{n}\right).$$

For bipartite graphs, there is a relatively simple condition which implies that this theorem is tight:

Theorem 1.10. *Let H be a locally finite bipartite graph, and let $\lambda = \liminf_{n\to\infty} \frac{\mu(H,n)}{n}$. Suppose that for every $\lambda' > \lambda$ there exist infinitely many pairwise disjoint independent sets $I_1, I_2, \ldots$, all of the same size, with $\frac{|N(I_i)|}{|I_i|} \leq \lambda'$. Then $\rho(H) = f(\lambda)$.*

Intuitively, the difference between Theorem 1.9 and Theorem 1.10 is that, for each size n, in order to have $\mu(H,n) \leq k$ it is enough if we only have one independent size with size n and neighborhood of size k, whereas in Theorem 1.10 we require that we have infinitely many independent sets with the same size. The following theorem, and the three corollaries that follow, capture some cases in which Theorem 1.10 applies:

Theorem 1.11. *Let H be a locally finite forest, or a locally finite bipartite graph in which every orbit of the automorphism group acting on $V(H)$ has infinite size. Then*

$$\rho(H) = f\left(\liminf_{n\to\infty} \frac{\mu(H,n)}{n}\right).$$

Corollary 1.12. *Let T_k be the infinite k-ary tree, that is, the rooted tree in which every vertex has k children. Then $\rho(T_k) = f(k)$.*

Corollary 1.13. *For any $d \geq 1$, let Grid_d be the infinite d-dimensional grid, that is, the graph on $\mathbb{Z}^d$ where two vertices are connected if they are at Euclidean distance 1. Then $\rho(\mathrm{Grid}_d) = f(1) = (12+\sqrt{8})/17 = 0.87226\ldots$*

Corollary 1.14. *Let F be a finite bipartite graph. Then*

$$\rho(\omega \cdot F) = f\left(\min_{\substack{I \text{ indep. in } F\\ I\neq\emptyset}} \frac{|N(I)|}{|I|}\right).$$

In particular, we have $\rho(\omega \cdot C_{2k}) = f(1)$ for every $k \geq 2$, and for every $1 \leq a \leq b$ we have

$$\rho(\omega \cdot K_{a,b}) = f\left(\frac{a}{b}\right) = \frac{2\left(\frac{a}{b}\right)^2 + 3\left(\frac{a}{b}\right) + 7 + 2\sqrt{\frac{a}{b}+1}}{4\left(\frac{a}{b}\right)^2 + 4\left(\frac{a}{b}\right) + 9}.$$

In a finite bipartite graph F, there is always an independent set satisfyng $|N(I)| \leq |I|$ (one of the two partition classes has this), so the value of $\rho(\omega \cdot F)$ always falls on the range in which $f(x)$ is known explicitly.

1.2.2 Results for infinite factors

We now present two results specific to infinite factors, which can be applied to non-bipartite graphs. The first one is again related to the expansion of independent sets:

Theorem 1.15. *Let F be a finite connected graph, and let $I \subseteq V(F)$ be a non-empty doubly independent set. Then $\rho(\omega \cdot F) \geq f\left(\frac{|N(I)|}{|I|}\right)$.*

If one of the independent sets $I \subseteq V(F)$ that minimize $|N(I)|/|I|$ is doubly independent, then Theorem 1.9 and Theorem 1.15 together give the exact value for $\rho(\omega \cdot F)$. This is always true in bipartite graphs, which we stated as Corollary 1.14 (this result can be proved either from Theorem 1.11 or from Theorem 1.15). Figure 1.2 shows four non-bipartite graphs F for which the same holds.

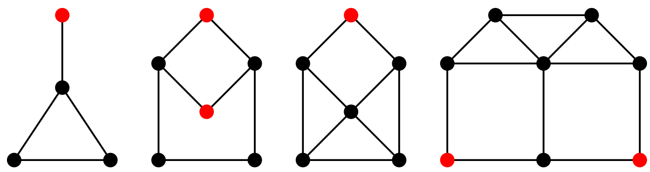

FIGURE 1.2: Four non-bipartite graphs F for which $\rho(\omega \cdot F)$ equals $f(1)$, $f(1)$, $f(2)$ and $f(3/2)$ respectively, with their doubly independent sets indicated.

If the graph F does not contain any non-empty doubly independent sets (for example $F = K_3$), then Theorem 1.15 does not give any lower bound at all. Instead, in these cases one can obtain the following lower bound on $\rho(\omega \cdot F)$ using finite Ramsey theory:

Theorem 1.16. *For every finite graph F, we have*

$$\rho(\omega \cdot F) \geq \frac{|V(F)|}{2|V(F)| - \alpha(F)}.$$

In the particular case $F = K_3$, combining Theorem 1.16 with Theorem 1.9 and (1.2) we obtain

$$3/5 \leq \rho(\omega \cdot K_3) \leq f(2) \leq \frac{21 + \sqrt{12}}{33} \approx 0.74133. \tag{1.3}$$

1.2.3 Further results for infinite factors

The results in Chapter 4, which can be found in [BL20], represent joint work with József Balogh.

In Chapter 4 we continue looking at the Ramsey density of infinite factors $\omega \cdot F$. This is because, even though they are graphs with a relatively simple structure, there are still plenty of choices of F for which the bounds from the previous sections are not enough to determine $\rho(\omega \cdot F)$ exactly. Notably, this is the case for cliques and odd cycles:

- In the case of cliques, we have $\rho(\omega \cdot K_k) \leq f(k-1) \leq \frac{k}{2k-2}$ from Theorem 1.9 and $\rho(\omega \cdot K_k) \geq \frac{k}{2k-1}$ from Theorem 1.16. We cannot obtain any lower bound from Theorem 1.15 because there is no non-empty independent set I such that $N(I)$ is independent, if $k \geq 3$.
- In the case of odd cycles, we have $\rho(\omega \cdot C_{2k+1}) \leq f\left(\frac{k+1}{k}\right)$ from Theorem 1.9, $f(\omega \cdot C_{2k+1}) \geq f\left(\frac{k}{k-1}\right)$ from Theorem 1.15 (if $k \geq 2$), and $\rho(\omega \cdot C_{2k+1}) \geq \frac{2k+1}{2(2k+1)-k} = \frac{2k+1}{3k+2}$ from Theorem 1.16. For k large enough, the lower bound from Theorem 1.15 is better than that from Theorem 1.16, although the latter has the advantage of being easier to compute.

We will prove a new lower bound on $\rho(\omega \cdot F)$ for all graphs F, that will solve these and other cases:

Theorem 1.17. *Let F be a finite graph. Then $\rho(\omega \cdot F) \geq f\left(\frac{|V(F)|}{\alpha(F)} - 1\right)$.*

This bound is not weaker[4] than Theorem 1.16. That result can be expressed as $\rho(\omega \cdot F) \geq \bar{f}\left(\frac{|V(F)|}{\alpha(F)} - 1\right)$, for $\bar{f}(x) = \frac{x+1}{2x+1}$. By Proposition B.5, we have $f(x) \geq \bar{f}(x)$.

We can compare this lower bound to the upper bound in Theorem 1.9 to find a new family of graphs for which we can obtain an exact result:

Corollary 1.18. *Let F be a finite graph. Suppose that, among the non-empty independent sets I that minimize $\frac{|N(I)|}{|I|}$, there is at least one with size $\alpha(F)$. Then $\rho(\omega \cdot F) = f\left(\frac{|V(F)|}{\alpha(F)} - 1\right)$. In particular:*

- *$\rho(\omega \cdot K_k) = f(k-1)$ for all $k \geq 2$.*
- *$\rho(\omega \cdot C_{2k+1}) = f\left(\frac{k+1}{k}\right)$ for all $k \geq 1$.*

For the triangle, which is both a clique and an odd cycle, Corollary 1.18 shows that $\rho(\omega \cdot K_3) = f(2)$. Note however that, since we do not know the actual value of $f(2)$, the explicit bounds on (1.3) have not improved: all we know is that $\frac{3}{5} \leq \rho(\omega \cdot K_3) \leq (21 + \sqrt{12})/33 = 0.74133\ldots$ (we suspect that the upper bound is sharp). The next result improves upon this lower bound:

Theorem 1.19. $\rho(\omega \cdot K_3) \geq 1 - \frac{1}{\sqrt{7}} = 0.62204\ldots$ *(and therefore $f(2) \geq 1 - \frac{1}{\sqrt{7}}$).*

Unlike Theorem 1.17, whose proof is based on the techniques from [CDLL19] and Theorem 3.2 (which is where the function $f(x)$ comes from), Theorem 1.19 is based on a careful analysis of the technique of Burr, Erdős and Spencer [BES75].

1.2.4 Results on list coloring

Let $\mathcal{C}$ be an infinite set of colors (which we will often identify with $\mathbb{N}$). A list assignment of size k on a set S is a function $L : S \to \binom{\mathcal{C}}{k}$. When not specified, we will assume that the size of a list assignment is two and the set S is $E(K_\mathbb{N})$. Given a list assignment L as above, an L-coloring is a function $\Psi : S \to \mathcal{C}$, where $\Psi(s) \in L(s)$ for every element $s \in S$. Note that, in this context, a red-blue coloring of $E(K_\mathbb{N})$ is an L-coloring, where L is the list assignment that maps every edge to $\{R, B\}$.

[4] However, this does not make Theorem 1.16 redundant. While it is true that Theorem 1.17 gives a bound that is not weaker than Theorem 1.16, *the proof of this fact* (namely that $f(x) \geq \bar{f}(x)$) uses Theorem 1.16 itself.

Most classical Ramsey theory questions admit a list coloring version. Remember that a typical classical Ramsey theory question looks as follows: if the elements of [discrete structure] are colored in any way using k colors, is it always possible to find a monochromatic [substructure]? The corresponding list coloring version would therefore take the following form: does there exist a list assigment L of size k on the elements of [discrete structure], such that every L-coloring contains a monochromatic [substructure]?

This merging of Ramsey theory with list coloring was introduced by Alon, Bucić, Kalvari, Kupperwasser and Szabó [ABK$^+$19]. They proved that the list coloring equivalent of the diagonal Ramsey numbers of hypergraph cliques of uniformity at least 3 are much smaller than their non-list coloring counterparts. In other words, there are lists assignments L on the edges of complete hypergraphs that are better at forcing large monochromatic subcliques than the uniform list assignment.

We want to study what happens in the Ramsey density problem when we consider its list coloring analogue. Let us consider the case in which the graph that we are looking for is P_∞, the infinite path. As we know from [CDLL19], the Ramsey density of this graph is $f(1) = \frac{12+\sqrt{8}}{17} = 0.87226\ldots$, and in a slightly stronger sense[5], every two-coloring of $E(K_\mathbb{N})$ contains a monochromatic path of density at least $f(1)$ and there exists a two coloring in which every monochromatic path has density at most $f(1)$.

Thus the question here would be: does there exist a list assignment L and $\lambda > f(1)$ such that, in every L-coloring, there exists a monochromatic path with density at least λ? And if so, how large can λ be? This question was suggested to the author by Alon. In this chapter we will answer this question in the affirmative, and show that in fact we can make λ be as close to 1 as we want.

Theorem 1.20. *There exists a list assignment L which satisfies that, for every L-coloring and every $\epsilon > 0$, there exists a monochromatic path with density at least $1 - \epsilon$.*

In fact, we will prove something stronger:

Theorem 1.21. *There exists a list assignment L with the following property: in every L-coloring there exists an infinite family $\mathcal{P}$ of infinite monochromatic paths, satisfying that every vertex $v \in V(K_\mathbb{N})$ is missed by at most one of the paths in $\mathcal{P}$.*

This is similar to the result of Rado [Rad78], which states that when the edges of $K_\mathbb{N}$ are k-colored, there exist k monochromatic paths, one in each color, such that every vertex $v \in V(K_\mathbb{N})$ is contained in exactly one path. In fact, Theorem 1.21 can be strengthened

[5] $\rho(P_\infty) = f(1)$ only means that, for every $\epsilon > 0$, every two-coloring of $E(K_\mathbb{N})$ contains a monochromatic path of density at least $f(1) - \epsilon$ and there exists a two coloring in which every monochromatic path has density at most $f(1) + \epsilon$. As we will see in this section, this distinction is important.

to say that each vertex is missed by exactly one path. A proof of this fact is sketched after the proof of Theorem 1.21.

However, we we try to make λ equal to 1, we encounter an interesting phenomenon. There exists a list assignment L which satisfies that, for every L-coloring and every $\epsilon > 0$, there exists a monochromatic path with density at least $1 - \epsilon$. Nonetheless, for every list assignment L there exists an L-coloring in which no single monochromatic path has density exactly 1.

Theorem 1.22. *For every list assignment L there exists an L-coloring in which no single monochromatic path has density 1.*

In a certain way, this phenomenon, in which we can have density arbitrarily close to 1 but not exactly 1, can be seen as analogous of what happens in Theorem 1.7(ii)a. If a locally finite graph H has finitely many components and infinite chromatic number, then for every $\epsilon > 0$ there exists a two-coloring of $E(K_{\mathbb{N}})$ in which every monochromatic copy of H has density at most ϵ (which implies that $\rho(H) = 0$), but every two-coloring of $E(K_{\mathbb{N}})$ contains a monochromatic copy of H of positive density.

1.2.5 Structure of the thesis

This thesis is organized as follows: on Chapter 2, based on [CDLL19], we prove Theorems 1.2 and 1.3. on Chapter 3, based on [Lam20], we prove the results in Section 1.2.1 and Section 1.2.2, related to the Ramsey density of locally finite graphs. Chapter 4, which represents joint work with József Balogh, proves the results in Section 1.2.3, which focuses on infinite factors. Chapter 5 proves the results in Section 1.2.4, relating Ramsey density to list coloring. In Chapter 6 we propose some conjectures and open problems regarding Ramsey density. Appendix A contains the proof of a lemma from Chapter 3 which, due to its length and the fact that consists mostly of analytic manipulations, we decided against featuring in the main body of the thesis. Finally, Appendix B contains the proof of some properties of the function $f(x)$ (again excluded from the main body for the same reasons).

Chapter 2

Ramsey density of infinite paths

This chapter, which is adapted from [CDLL19] (and hence is joint work with Jan Corsten, Louis DeBiasio and Richard Lang), contains the proof of the statement $\rho(P_\infty) = (12+\sqrt{8})/17$. The upper bound on $\rho(P_\infty)$ is implied by Theorem 1.2, while the lower bound is implied by Theorem 1.3. This will act as a warm-up exercise, to introduce the techniques that will be used in the more general results in Chapter 3. In particular, Theorem 1.2 will be generalized as Theorem 1.9, and Theorem 1.3 will be generalized as Theorem 3.2. Some comments will be made in the relevant sections of Chapter 3 about the analogy of these results.

2.1 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Let $q > 1$ be a real number, whose exact value will be chosen later on. We start by defining a coloring of the edges of the infinite complete graph. Let $A_0, A_1, \dots$ be a partition of $\mathbb{N}$, such that every element of A_i precedes every element of A_{i+1} and $|A_i| = \lfloor q^i \rfloor$. We color the edges of $K_\mathbb{N}$ such that every edge uv with $u \in A_i$ and $v \in A_j$ is red if $\min\{i,j\}$ is odd, and blue if it is even. A straightforward calculation shows that for $q = 2$, every monochromatic path P in $K_\mathbb{N}$ satisfies $\bar{d}(P) \leq 8/9$ (see Theorem 1.5 in [EG93]). We will improve this bound by reordering the vertices of $K_\mathbb{N}$ and then optimizing the value of q.

For convenience, we will say that the vertex $v \in A_i$ is red if i is odd and blue if i is even. We also denote by B the set of blue vertices and by R be the set of red vertices. Let b_i and r_i denote the i-th blue vertex and the i-th red vertex, respectively. We define a monochromatic red matching M_r by forming a matching between A_{2i-1} and the first $|A_{2i-1}|$ vertices of A_{2i} for each $i \geq 1$. Similarly, we define a monochromatic blue matching M_b by forming a matching between A_{2i} and the first $|A_{2i}|$ vertices of A_{2i+1} for each $i \geq 0$.

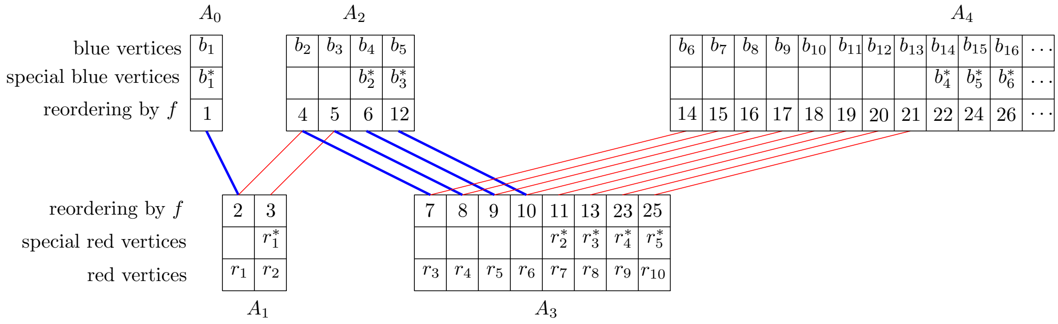

FIGURE 2.1: The coloring for $q = 2$ and the reordering by ϕ.

Next, let us define a bijection $\phi\colon \mathbb{N} \to V(G)$, which will serve as a reordering of G. Let r_t^* denote the t-th red vertex not in M_b, and b_t^* denote the t-th blue vertex not in M_r. The function ϕ is defined as follows. We start enumerating blue vertices, in their order, until we reach b_1^*. Then we enumerate red vertices, in their order, until we reach r_1^*. Then we enumerate blue vertices again until we reach b_2^*. We continue enumerating vertices in this way, changing colors whenever we find an r_t^* or a b_t^*. (See Figure 2.1.) Finally, for every $H \subset G$, we define

$$\bar{d}(H;\phi) = \limsup_{t\to\infty} \frac{|V(H) \cap \phi([t])|}{t}.$$

Note that $\bar{d}(H;\phi)$ is the upper density of H in the reordered graph $\phi^{-1}(G)$.

Claim 2.1. *Let P_r and P_b be infinite monochromatic red and blue paths in G, respectively. Then $\bar{d}(P_r;\phi) \leq \bar{d}(M_r;\phi)$ and $\bar{d}(P_b;\phi) \leq \bar{d}(M_b;\phi)$.*

Claim 2.2. *We have*

$$\bar{d}(M_r;\phi),\ \bar{d}(M_b;\phi) \leq \frac{q^2+2q-1}{q^2+3q-2}.$$

We can easily derive Theorem 1.2 from these two claims. Note that the rational function in Claim 2.2 evaluates to $(12+\sqrt{8})/17$ at $q \coloneqq \sqrt{2}+1$. It then follows from Claim 2.1 and 2.2, that every monochromatic path P in G satisfies $\bar{d}(P;\phi) \leq (12+\sqrt{8})/17$. Thus we can define the desired coloring of $K_{\mathbb{N}}$, by coloring each edge ij with the color of the edge $\phi(i)\phi(j)$ in G.

It remains to prove Claim 2.1 and 2.2. The intuition behind Claim 2.1 is that in every monochromatic red path P_r there is a red matching with the same vertex set, and that M_r has the largest upper density among all red matchings, as it contains every red vertex and has the largest possible upper density of blue vertices. Note that the proof of Claim 2.1 only uses the property that ϕ preserves the order of the vertices inside R and inside B.

Proof of Claim 2.1. We will show $\bar{d}(P_r;\phi) \le \bar{d}(M_r;\phi)$. (The other case is analogous.) We prove that, for every positive integer k, we have $|V(P_r)\cap\phi([k])| \le |V(M_r)\cap\phi([k])|$. Assume, for contradiction, that this is not the case and let k be the minimum positive integer for which the inequality does not hold. Every red vertex is saturated by M_r, so $|V(P_r)\cap\phi([k])\cap B| > |V(M_r)\cap\phi([k])\cap B|$. By the minimality of k, $\phi(k)$ must be in P_r but not in M_r, and in particular it must be blue.

Let $\phi(k)\in A_{2i}$. Since $\phi(k)\notin M_r$, we know that $f(k)$ is not among the first $|A_{2i-1}|$ vertices of A_{2i}. Therefore, since ϕ preserves the order of the vertices inside B, $\phi([k])$ contains the first $|A_{2i-1}|$ blue vertices in A_{2i}, and hence

$$|V(P_r)\cap\phi([k])\cap B| > |V(M_r)\cap\phi([k])\cap B| = \sum_{j=1}^{i}|A_{2j-1}|. \tag{2.1}$$

On the other hand, every edge between two blue vertices is blue, so the successor of every blue vertex in P_r is red, and in particular there is a red matching between $V(P_r)\cap B$ and R saturating $V(P_r)\cap B$. So by (2.1), the number of red neighbors of $V(P_r)\cap\phi([k])\cap B$ is at least $|V(P_r)\cap\phi([k])\cap B| > \sum_{j=1}^{i}|A_{2j-1}|$. Observe that by the definition of ϕ, we have $V(P_r)\cap\phi([k])\cap B\subseteq\bigcup_{j=0}^{i}A_{2j}$. Hence the red neighborhood of $V(P_r)\cap\phi([k])\cap B$ is contained in $\bigcup_{j=1}^{i}A_{2j-1}$, a contradiction. □

Proof of Claim 2.2. Let $\ell_r(t)$ and $\ell_b(t)$ denote the position of r_t^* among the red vertices and of b_t^* among the blue vertices, respectively. In other words, let $\ell_r(t)=i$ where $r_t^*=r_i$ and $\ell_b(t)=j$ where $b_t^*=b_j$ (so for example in Figure 2.1, $\ell_r(4)=9$ and $\ell_b(4)=14$). Note that $\phi(\ell_b(t)+\ell_r(t))=r_t^*$, so for $\ell_b(t-1)+\ell_r(t-1)\le k\le\ell_b(t)+\ell_r(t)-1$, $\phi([k])$ has exactly $t-1$ vertices outside of M_b and at least $t-1$ vertices outside of M_r. As a consequence, we obtain

$$\bar{d}(M_r;\phi),\ \bar{d}(M_b;\phi) \le \limsup_{k\to\infty}(1-\sigma(k)) = \limsup_{t\to\infty}\left(1-\frac{t-1}{\ell_r(t)+\ell_b(t)-1}\right), \tag{2.2}$$

where $\sigma(k)=(t-1)/k$ if $\ell_b(t-1)+\ell_r(t-1)\le k\le\ell_b(t)+\ell_r(t)-1$. It is easy to see that

$$\begin{aligned}
\ell_r(t) &= t+\sum_{j=0}^{i}|A_{2j}| \quad \text{for} \quad \sum_{j=0}^{i-1}(|A_{2j+1}|-|A_{2j}|) < t \le \sum_{j=0}^{i}(|A_{2j+1}|-|A_{2j}|), \text{ and}\\
\ell_b(t) &= t+\sum_{j=1}^{i}|A_{2j-1}| \quad \text{for} \quad \sum_{j=1}^{i-1}(|A_{2j}|-|A_{2j-1}|) < t-|A_0| \le \sum_{j=1}^{i}(|A_{2j}|-|A_{2j-1}|).
\end{aligned}$$

Note that $\ell_r(t)-t$ and $\ell_b(t)-t$ are piecewise constant and non-decreasing. We claim that, in order to compute the right hand side of (2.2), it suffices to consider values of t for which $\ell_r(t)-t>\ell_r(t-1)-(t-1)$ or $\ell_b(t)-t>\ell_b(t-1)-(t-1)$. This is because we can write

$$1 - \frac{t-1}{\ell_r(t)+\ell_b(t)-1} = \frac{1}{2} + \frac{(\ell_r(t)-t)+(\ell_b(t)-t)+1}{2(\ell_r(t)+\ell_b(t)-1)}.$$

In this expression, the second fraction has a positive, piecewise constant numerator and a positive increasing denominator. Therefore, the local maxima are attained precisely at the values for which the numerator increases. We will do the calculations for the case when $\ell_r(t) - t > \ell_r(t-1) - (t-1)$ (the other case is similar), in which we have

$$t = 1 + \sum_{j=0}^{i-1}(|A_{2j+1}| - |A_{2j}|) = 1 + \sum_{j=0}^{i-1}(1+o(1))q^{2j}(q-1) = (1+o(1))\frac{q^{2i}}{q+1},$$

$$\ell_r(t) = t + \sum_{j=0}^{i}|A_{2j}| = (1+o(1))\left(\frac{q^{2i}}{q+1} + \sum_{j=0}^{i} q^{2j}\right) = (1+o(1))\frac{(q^2+q-1)q^{2i}}{q^2-1}, \text{ and}$$

$$\ell_b(t) = t + \sum_{j=1}^{i}|A_{2j-1}| = (1+o(1))\left(\frac{q^{2i}}{q+1} + \sum_{j=1}^{i} q^{2j-1}\right) = (1+o(1))\frac{(2q-1)q^{2i}}{q^2-1}.$$

Plugging this into (2.2) gives the desired result. □

2.2 Proof of Theorem 1.3

This section is dedicated to the proof of Theorem 1.3. A *total coloring* of a graph G is a coloring of the vertices and edges of G. Due to an argument of Erdős and Galvin, the problem of bounding the upper density of monochromatic paths in edge colored graphs can be reduced to the problem of bounding the upper density of monochromatic path forests in totally colored graphs.

Definition 2.1 (Monochromatic path forest)**.** Given a totally colored graph G, a forest $F \subset G$ is said to be a *monochromatic path forest* if $\Delta(F) \leq 2$ and there is a color c such that all leaves, isolated vertices, and edges of F receive color c.

Lemma 2.3. *For every $\gamma > 0$ and $k \in \mathbb{N}$, there is some $n_0 = n_0(k,\gamma)$ so that the following is true for every $n \geq n_0$. For every total 2-coloring of K_n, there is an integer $t \in [k,n]$ and a monochromatic path forest F with $|V(F) \cap [t]| \geq ((12+\sqrt{8})/17 - \gamma)t$.*

Some standard machinery related to Szemerédi's regularity lemma, adapted to the ordered setting, will allow us to reduce the problem of bounding the upper density of monochromatic path forests to the problem of bounding the upper density of monochromatic simple forests.

Definition 2.2 (Monochromatic simple forest)**.** Given a totally colored graph G, a forest $F \subset G$ is said to be a *monochromatic simple forest* if $\Delta(F) \leq 1$ and there is a color c such that all edges and isolated vertices of F receive color c and at least one endpoint of each edge of F receives color c.

Lemma 2.4. *For every $\gamma > 0$, there exists $k_0, N \in \mathbb{N}$ and $\alpha > 0$ such that the following holds for every integer $k \geq k_0$. Let G be a totally 2-colored graph on kN vertices with minimum degree at least $(1-\alpha)kN$. Then there exists an integer $t \in [k/8, kN]$ and a monochromatic simple forest F such that $|V(F) \cap [t]| \geq ((12+\sqrt{8})/17 - \gamma)t$.*

The heart of the proof is Lemma 2.4, which we shall prove in Section 2.2.3. But first, in the next two sections, we show how to deduce Theorem 1.3 from Lemmas 2.3 and 2.4.

2.2.1 From path forests to paths

In this section we use Lemma 2.3 to prove Theorem 1.3. Our exposition follows that of Theorem 1.6 in [DM19].

Proof of Theorem 1.3. Fix a 2-coloring of the edges of $K_{\mathbb{N}}$ in red and blue. We define a 2-coloring of the vertices by coloring $n \in \mathbb{N}$ red if there are infinitely many $m \in \mathbb{N}$ such that the edge nm is red and blue otherwise.

Case 1. Suppose there are vertices x and y of the same color, say red, and a finite set $S \subset \mathbb{N}$ such that there is no red path disjoint from S which connects x to y.

We partition $\mathbb{N} \setminus S$ into sets X, Y, Z, where $x' \in X$ if and only if there is a red path, disjoint from S, which connects x' to x and $y' \in Y$ if and only if there is a red path disjoint from S which connects y to y'. Note that every edge from $X \cup Y$ to Z is blue. Since x and y are colored red, both X and Y are infinite, and by choice of x and y all edges in the bipartite graph between X and $Y \cup Z$ are blue. Hence there is a blue path with vertex set $X \cup Y \cup Z = \mathbb{N} \setminus S$.

Case 2. Suppose that for every pair of vertices x and y of the same color c, and every finite set $S \subset \mathbb{N}$, there is a path from x to y of color c which is disjoint from S.

Let γ_n be a sequence of positive reals tending to zero, and let a_n and k_n be increasing sequences of integers such that

$$a_n \geq n_0(k_n, \gamma_n) \text{ and } k_n/(a_1 + \cdots + a_{n-1} + k_n) \to 1,$$

where $n_0(k, \gamma)$ is as in Lemma 2.3. Let $\mathbb{N} = (A_i)$ be a partition of $\mathbb{N}$ into consecutive intervals with $|A_n| = a_n$. By Lemma 2.3 there are monochromatic path forests F_n with

$V(F_n) \subset A_n$ and initial segments $I_n \subset A_n$ of length at least k_n such that

$$|V(F_n) \cap I_n| \geq \left(\frac{12 + \sqrt{8}}{17} - \gamma_n \right) |I_n|.$$

It follows that for any $G \subset K_{\mathbb{N}}$ containing infinitely many F_n's we have

$$\bar{d}(G) \geq \limsup_{n\to\infty} \frac{|V(F_n) \cap I_n|}{a_1 + \cdots + a_{n-1} + |I_n|} \geq \limsup_{n\to\infty} \frac{12 + \sqrt{8}}{17} - \gamma_n = \frac{12 + \sqrt{8}}{17}.$$

By the pigeonhole principle, there are infinitely many F_n's of the same color, say blue. We will recursively construct a blue path P which contains infinitely many of these F_n's. To see how this is done, suppose we have constructed a finite initial segment p of P. We will assume as an inductive hypothesis that p ends at a blue vertex v. Let n be large enough that $\min(A_n)$ is greater than every vertex in p, and F_n is blue. Let $F_n = \{P_1, \ldots, P_s\}$ for some $s \in \mathbb{N}$ and let w_i, w_i' be the endpoints of the path P_i (note that w_i and w_i' could be equal) for every $i \in [s]$. By the case assumption, there is a blue path q_1 connecting v to w_1, such that q_1 is disjoint from $A_1 \cup \cdots \cup A_n$. Similarly, there is a blue path q_2 connecting w_1' to w_2, such that q_2 is disjoint from $A_1 \cup \cdots \cup A_n \cup \{q_1\}$. Continuing in this fashion, we find disjoint blue paths $q_3, \ldots, q_s$ such that q_i connects w_{i-1}' to w_i. Hence, we can extend p to a path p' which contains all of the vertices of F_n and ends at a blue vertex. $\square$

2.2.2 From simple forests to path forests

In this section we use Lemma 2.4 to prove Lemma 2.3. The proof is based on Szemerédi's regularity lemma. The main difference to standard applications of the regularity lemma is that we have to define an ordering of the reduced graph, which approximately preserves densities. This is done by choosing a suitable initial partition.

Let $G = (V, E)$ be a graph and A and B be non-empty, disjoint subsets of V. We write $e_G(A, B)$ for the number of edges in G with one vertex in A and one in B and define the *density* of the pair (A, B) to be $d_G(A, B) = e_G(A, B)/(|A||B|)$. The pair (A, B) is ε-*regular* (in G) if we have $|d_G(A', B') - d_G(A, B)| \leq \varepsilon$ for all $A' \subseteq A$ with $|A'| \geq \varepsilon|A|$ and $B' \subseteq B$ with $|B'| \geq \varepsilon|B|$. It is well-known (see for instance [Hax97]) that dense regular pairs contain almost spanning paths. We include a proof of this fact for completeness.

Lemma 2.5. *For $0 < \varepsilon < 1/4$ and $d \geq 2\sqrt{\varepsilon} + \varepsilon$, every ε-regular pair (A, B) with density at least d contains a path with both endpoints in A and covering all but at most $2\sqrt{\varepsilon}|A|$ vertices of $A \cup B$.*

Proof. We will construct a path $P_k = (a_1 b_1 \dots a_k)$ for every $k = 1, \dots, \lceil (1-\sqrt{\varepsilon})|A| \rceil$ such that $B_k := N(a_k) \setminus V(P_k)$ has size at least $\varepsilon|B|$. As $d \geq \varepsilon$, this is easy for $k = 1$. Assume now that we have constructed P_k for some $1 \leq k < (1-\sqrt{\varepsilon})|A|$. We will show how to extend P_k to P_{k+1}. By ε-regularity of (A, B), the set $\bigcup_{b \in B_k} N(b)$ has size at least $(1-\varepsilon)|A|$. So $A' := \bigcup_{b \in B_k} N(b) \setminus V(P_k)$ has size at least $(\sqrt{\varepsilon} - \varepsilon)|A| \geq \varepsilon|A|$. Let $B' = B \setminus V(P_k)$ and note that $|B'| \geq \sqrt{\varepsilon}|B|$ as $k < (1-\sqrt{\varepsilon})|A|$ and $|A| = |B|$. By ε-regularity of (A, B), there exists $a_{k+1} \in A'$ with at least $(d-\varepsilon)|B'| \geq 2\varepsilon|B|$ neighbors in B'. Thus we can define $P_{k+1} = (a_1 b_1 \dots a_k b_k a_{k+1})$, where $b_k \in B_k \cap N(a_{k+1})$. □

A family of disjoint subsets $\{V_i\}_{i \in [m]}$ of a set V is said to *refine* a partition $\{W_j\}_{j \in [\ell]}$ of V if, for all $i \in [m]$, there is some $j \in [\ell]$ with $V_i \subset W_j$.

The specific version of the regularity lemma that we will use here comes from [KS96], and its statement can be found later in this thesis as Lemma 3.7.

Before we start with the proof, we will briefly describe the setup and proof strategy of Lemma 2.3. Consider a totally 2-colored complete graph $G = K_n$. Denote the sets of red and blue vertices by R and B, respectively. For $\ell \geq 4$, let $\{W_j\}_{j \in [\ell]}$ be a partition of $[n]$ such that each W_j consists of at most $\lceil n/\ell \rceil$ subsequent vertices. The partition $\{W'_j\}_{j \in [2\ell]}$, with parts of the form $W_i \cap R$ and $W_i \cap B$, refines both $\{W_j\}_{j \in [\ell]}$ and $\{R, B\}$. Suppose that $V_0 \cup \dots \cup V_m$ is a partition obtained from Lemma 3.7 applied to G and $\{W'_j\}_{j \in [2\ell]}$ with parameters ε, m_0, 2ℓ and d. We define the (ε, d)-*reduced graph* G' to be the graph with vertex set $V(G') = [m]$ where ij is an edge of G' if and only if if (V_i, V_j) is an ε-regular pair of density at least d in the red subgraph of H or in the blue subgraph of H. Furthermore, we color ij red if (V_i, V_j) is an ε-regular pair of density at least d in the red subgraph of H, otherwise we color ij blue. As $\{V_i\}_{i \in [m]}$ refines $\{R, B\}$, we can extend this to a total 2-coloring of G' by coloring each vertex i red, if $V_i \subset R$, and blue otherwise. By relabelling the clusters, we can furthermore assume that $i < j$ if and only if $\max\{V_i\} < \max\{V_j\}$. Note that, by choice of $\{W_j\}_{j \in [\ell]}$, any two vertices in V_i differ by at most n/ℓ. Moreover, a simple calculation (see [KO09, Proposition 42]) shows that G' has minimum degree at least $(1 - d - 3\varepsilon)m$.

Given this setup, our strategy to prove Lemma 2.3 goes as follows. First, we apply Lemma 2.4 to obtain $t' \in [m]$ and a, red say, simple forest $F' \subset G'$ with $d(F', t') \approx (12 + \sqrt{8})/17$. Next, we turn F' into a red path forest $F \subset G$. For every isolated vertex $i \in V(F')$, this is straightforward as $V_i \subset R$ by the refinement property. For every edge $ij \in E(F')$ with $i \in R$, we apply Lemma 2.5 to obtain a red path that almost spans (V_i, V_j) and has both ends in V_i. So the union F' of these paths and vertices is indeed a red path forest. Since the vertices in each V_i do not differ too much, it will follow that $d(F, t) \approx (12 + \sqrt{8})/17$ for $t = \max\{V_{t'}\}$.

Proof of Lemma 2.3. Suppose we are given $\gamma > 0$ and $k \in \mathbb{N}$ as input. Let $k_0, N \in \mathbb{N}$ and $\alpha > 0$ be as in Lemma 2.4 with input $\gamma/4$. We choose constants $d, \varepsilon > 0$ and $\ell, m_0 \in \mathbb{N}$ satisfying

$$2\sqrt{\varepsilon} + \varepsilon \leq 1/\ell, d \leq \alpha/8 \text{ and } m_0 \geq 4N/d, 2k_0N.$$

We obtain M from Lemma 3.7 with input ε, m_0 and 2ℓ. Finally, set $n_0 = 16k\ell MN$.

Now let $n \geq n_0$ and suppose that K_n is an ordered complete graph on vertex set $[n]$ and with a total 2-coloring in red and blue. We have to show that there is an integer $t \in [k, n]$ and a monochromatic path forest F such that $|V(F) \cap [t]| \geq ((12 + \sqrt{8})/17 - \gamma)t$.

Denote the red and blue vertices by R and B, respectively. Let $\{W'_j\}_{j \in [\ell]}$ refine $\{R, B\}$ as explained in the above setting. Let $\{V_0, \ldots, V_m\}$ be a partition of $[n]$ with respect to $G = K_n$ and $\{W'_j\}_{j \in [\ell]}$ as detailed in Lemma 3.7 with totally 2-colored (ε, d)-reduced graph G'' of minimum degree $\delta(G'') \geq (1 - 4d)m$. Set $k' = \lfloor m/N \rfloor \geq k_0$ and observe that the subgraph G' induced by G'' in $[k'N]$ satisfies $\delta(G') \geq (1 - 8d)m \geq (1 - \alpha)m$ as $m \geq 4N/d$. Thus we can apply Lemma 2.4 with input G', k', $\gamma/4$ to obtain an integer $t' \in [k'/8, k'N]$ and a monochromatic (say red) simple forest $F' \subset G'$ such that $d(F', t') \geq (12 + \sqrt{8})/17 - \gamma/4$.

Set $t = \max V_{t'}$. We have that $V_{t'} \subset W_j$ for some $j \in [\ell]$. Recall that $i < j$ if and only if $\max\{V_i\} < \max\{V_j\}$ for any $i, j \in [m]$. It follows that $V_i \subset [t]$ for all $i \leq t'$. Hence

$$t \geq t'|V_1| \geq \frac{k'}{8}|V_1| \geq \left\lfloor \frac{m}{N} \right\rfloor \frac{(1-\varepsilon)n}{8m} \geq \frac{n}{16N}. \tag{2.3}$$

This implies $t \geq k$ by choice of n_0. Since $[t]$ is covered by $V_0 \cup W_j \cup \bigcup_{i \in [t']} V_i$, it follows that

$$\begin{aligned} t'|V_1| &\geq t - |V_0| - |W_j| \\ &\geq \left(1 - \varepsilon\frac{n}{t} - \frac{4}{\ell}\frac{n}{t}\right)t \\ &\geq \left(1 - 16\varepsilon N - \frac{64N}{\ell}\right)t \qquad \text{(by (2.3))} \\ &\geq \left(1 - \frac{\gamma}{2}\right)t. \end{aligned} \tag{2.4}$$

For every edge $ij \in E(F')$ with $V_i \subset R$, we apply Lemma 2.5 to choose a path P_{ij} which starts and ends in V_i and covers all but at most $2\sqrt{\varepsilon}|V_1|$ vertices of each V_i and V_j. We denote the isolated vertices of F' by I'. For each $i \in I'$ we have $V_i \subset R$. Hence the red path forest $F := \bigcup_{i \in I'} V_i \cup \bigcup_{ij \in E(F')} P_{ij} \subset K_n$ satisfies

$$
\begin{aligned}
|V(F) \cap [t]| &= \sum_{i \in I'} |V_i \cap [t]| + \sum_{ij \in E(F')} |V(P_{ij}) \cap [t]| \\
&\geq \sum_{i \in I' \cap [t']} |V_i| + \sum_{i \in V(F' - I') \cap [t'],} (|V_i| - 2\sqrt{\varepsilon}|V_1|) \\
&\geq (1 - 2\sqrt{\varepsilon})|V_1||V(F') \cap [t']| \\
&\geq (1 - 2\sqrt{\varepsilon}) \left(\frac{12 + \sqrt{8}}{17} - \frac{\gamma}{4} \right) t'|V_1| \\
&\overset{(2.4)}{\geq} \left(\frac{12 + \sqrt{8}}{17} - \gamma \right) t
\end{aligned}
$$

as desired. □

2.2.3 Upper density of simple forests

In this section we prove Lemma 2.4. For a better overview, we shall define all necessary constants here. Suppose we are given $\gamma' > 0$ as input and set $\gamma = \gamma'/4$. Fix a positive integer $N = N(\gamma)$ and let $0 < \alpha \leq \gamma/(8N)$. The exact value of N will be determined later on. Let $k_0 = \lceil 8/\gamma \rceil$ and fix a positive integer $k \geq k_0$. Consider a totally 2-colored graph G' on $n = kN$ vertices with minimum degree at least $(1 - \alpha)n$.

Denote the sets of red and blue vertices by R and B, respectively. As it turns out, we will not need the edges inside R and B. So let G be the spanning bipartite subgraph, obtained from G' by deleting all edges within R and B. For each red vertex v, let $d_b(v)$ be the number of blue edges incident to v in G. Let $a_1 \leq \cdots \leq a_{|R|}$ denote the degree sequence taken by $d_b(v)$. The whole proof of Lemma 2.4 revolves around analysing this sequence.

Fix an integer $t = t(\gamma, N, k)$ and subset $R' \subset R$, $B' \subset B$. The value of t and nature of R', B' will be determined later. The following two observations explain our interest in the sequence $a_1 \leq \cdots \leq a_{|R|}$.

Claim 2.6. *If $a_j > j - t$ for all $1 \leq j \leq |R'| - 1$, then there is a blue simple forest covering all but at most t vertices of $R' \cup B$.*

Proof. We write $R' = \{v_1, \ldots, v_{|R'|}\}$ such that $d_b(v_i) \leq d_b(v_j)$ for every $1 \leq i \leq j \leq |R'|$. By assumption, we have $d_b(v_j) \geq a_j > j - t$ for all $1 \leq j \leq |R'| - 1$. Thus we can greedily select a blue matching containing $\{v_t, v_{t+1}, \ldots, v_{|R'|-1}\}$, which covers all but t vertices of R'. Together with the rest of B, this forms the desired blue simple forest. □

Claim 2.7. *If $a_i < i+t$ for all $1 \le i \le |B'|-t$, then there is a red simple forest covering all but at most $t + \alpha n$ vertices of $R \cup B'$.*

Proof. Let X' be a minimum vertex cover of the red edges in the subgraph of G induced by $R \cup B'$. If $|X'| \ge |B'| - t - \alpha n$, then by König's theorem there exists a red matching covering at least $|B'| - t - \alpha n$ vertices of B'. This together with the vertices in R yields the desired red simple forest.

Suppose now that $|X'| < |B'| - t - \alpha n$. Since every edge between $R \setminus (X' \cap R)$ and $B' \setminus (X' \cap B')$ is blue, we have for every vertex v in $R \setminus (X' \cap R)$,

$$d_b(v) \ge |B'| - |X' \cap B'| - \alpha n = |X' \cap R| + |B'| - |X'| - \alpha n > |X' \cap R| + t.$$

In particular, this implies $a_i \ge i + t$ for $i = |X' \cap R| + 1$. So $|B'| - t + 1 \le |X' \cap R| + 1$ by the assumption in the statement. Together with

$$|X' \cap R| + 1 \le |X'| + 1 < |B'| - t - \alpha n + 1 < |B'| - t + 1,$$

we reach a contradiction. □

Motivated by this, we introduce the following definitions.

Definition 2.3 (Oscillation, $\ell^+(t)$, $\ell^-(t)$)**.** Let $a_1, \ldots, a_n$ be a non-decreasing sequence of non-negative real numbers. We define its *oscillation* as the maximum value T, for which there exist indices $i, j \in [n]$ with $a_i - i \ge T$ and $j - a_j \ge T$. For all $0 < t \le T$, set

$$\begin{aligned} \ell^+(t) &= \min\{i \in [n] \colon\ a_i \ge i + t\}, \\ \ell^-(t) &= \min\{j \in [n] \colon\ a_j \le j - t\}. \end{aligned}$$

Suppose that the degree sequence $a_1, \ldots, a_{|R|}$ has oscillation T and fix some positive integer $t \le T$. We define ℓ and λ by

$$\ell = \ell^+(t) + \ell^-(t) = \lambda t. \tag{2.5}$$

The next claim combines Claims 2.6 and 2.7 into a density bound of a monochromatic simple forest in terms of the ratio $\ell/t = \lambda$. (Note that, in practice, the term αn will be of negligible size.)

Claim 2.8. *There is a monochromatic simple forest $F \subset G$ with*

$$d(F, \ell + t) \ge \frac{\ell - \alpha n}{\ell + t} = \frac{\lambda t - \alpha n}{(1+\lambda)t}.$$

Proof. Let $R' = R \cap [\ell + t]$ and $B' = B \cap [\ell + t]$ so that $\ell^+(t) + \ell^-(t) = \ell = |R'| + |B'| - t$. Thus we have either $\ell^-(t) \geq |R'|$ or $\ell^+(t) > |B'| - t$. If $\ell^-(t) \geq |R'|$, then $a_j > j - t$ for every $1 \leq j \leq |R'| - 1$. Thus Claim 2.6 provides a blue simple forest F covering all but at most t vertices of $[\ell + t]$. On the other hand, if $\ell^+(t) > |B'| - t$, then $a_i < i + t$ for every $1 \leq i \leq |B'| - t$. In this case Claim 2.7 yields a red simple forest F covering all but at most $t + \alpha n$ vertices of $[\ell + t]$. □

Claim 2.8 essentially reduces the problem of finding a dense linear forest to a problem about bounding the ratio ℓ/t in integer sequences. It is, for instance, not hard to see that we always have $\ell \geq 2t$ (which, together with the methods of the previous two subsections, would imply the bound $\bar{d}(P) \geq 2/3$ of Erdős and Galvin). The following lemma provides an essentially optimal lower bound on $\ell/t = \lambda$. Note that for $\lambda = 4+\sqrt{8}$, we have $\frac{\lambda}{\lambda+1} = (12 + \sqrt{8})/17$.

Lemma 2.9. *For all $\gamma \in \mathbb{R}^+$, there exists $N \in \mathbb{N}$ such that, for all $k \in \mathbb{R}^+$ and all sequences with oscillation at least kN, there exists a real number $t \in [k, kN]$ with*

$$\ell := \ell^+(t) + \ell^-(t) \geq \left(4 + \sqrt{8} - \gamma\right) t.$$

The proof of Lemma 2.9 is deferred to the last section. We now finish the proof of Lemma 2.4. Set $N = N(\gamma)$ to be the integer returned by Lemma 2.9 with input $\gamma = \gamma'/4$. In order to use Lemma 2.9, we have to bound the oscillation of $a_1, \ldots, a_{|R|}$:

Claim 2.10. *The degree sequence $a_1, \ldots, a_{|R|}$ has oscillation $T \geq kN/8$ or there is a monochromatic simple forest $F \subset G$ with $d(F, n) \geq (12 + \sqrt{8})/17 - \gamma$.*

Before we prove Claim 2.10, let us see how this implies Lemma 2.4.

Proof of Lemma 2.4. By Claim 2.10, we may assume that the sequence $a_1, \ldots, a_{|R|}$ has oscillation at least $kN/8$. By Lemma 2.9, there is a real number $t' \in [k/8, kN/8]$ with

$$\ell = \ell^+(t') + \ell^-(t') \geq (4 + \sqrt{8} - \gamma)t'.$$

Let $t = t(\gamma, N, k) = \lceil t' \rceil$. Since the a_i's are all integers, we have $\ell^+(t) = \ell^+(t')$ and $\ell^-(t) = \ell^-(t')$. Let $F \subset G$ be the monochromatic simple forest obtained from Claim 2.8.

As $n = kN$, $\ell \geq t' \geq k/8 \geq 1/\gamma$, $\alpha \leq \gamma/(8N)$, and by (2.5), it follows that

$$\begin{aligned} d(F, \ell + t) &\geq \frac{\ell - \alpha n}{\ell + t} = \frac{1 - \alpha n/\ell}{1 + \frac{t}{\ell}} \geq \frac{1 - 8\alpha N}{1 + \frac{t'}{\ell} + \frac{1}{\ell}} \geq \frac{1}{1 + \frac{t'}{\ell}} - 2\gamma \\ &\geq \frac{1}{1 + \frac{1}{4+\sqrt{8}-\gamma}} - 2\gamma = \frac{4 + \sqrt{8} - \gamma}{5 + \sqrt{8} - \gamma} - 2\gamma \\ &\geq \frac{4 + \sqrt{8}}{5 + \sqrt{8}} - 4\gamma = \frac{12 + \sqrt{8}}{17} - \gamma', \end{aligned}$$

as desired. $\square$

To finish, it remains to show Claim 2.10. The proof uses König's theorem and is similar to the proof of Claim 2.7.

Proof of Claim 2.10. Let X be a minimum vertex cover of the red edges. If $|X| \geq |B| - (1/8 + \alpha)n$, then König's theorem implies that there is a red matching covering all but at most $(1/8 + \alpha)n$ blue vertices. Thus adding the red vertices, we obtain a red simple forest F with $d(F, kN) \geq 7/8 - \alpha \geq (12 + \sqrt{8})/17 - \gamma$. Therefore, we may assume that $|X| < |B| - (1/8 + \alpha)n$. Every edge between $R \setminus (X \cap R)$ and $B \setminus (X \cap B)$ is blue. So there are at least $|R| - |X \cap R|$ red vertices v with

$$d_b(v) \geq |B| - |X \cap B| - \alpha n = |X \cap R| + |B| - |X| - \alpha n > |X \cap R| + n/8.$$

This implies that $a_i \geq i + n/8$ for $i = |X \cap R| + 1$. (See Figure 2.2.)

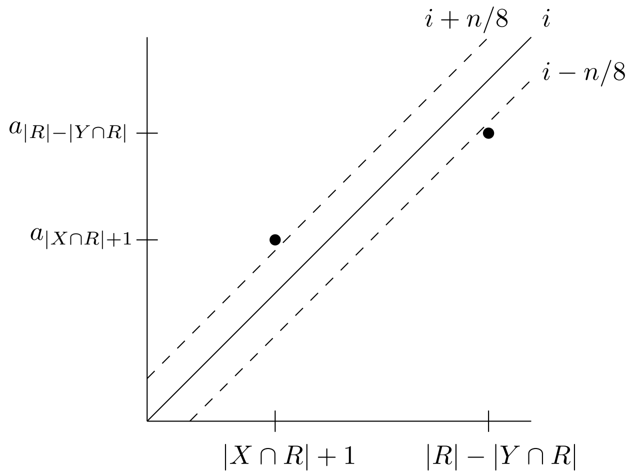

FIGURE 2.2: The sequence $a_1, \ldots, a_{|R|}$ has oscillation at least $kN/8$.

Let Y be a minimum vertex cover of the blue edges. Using König's theorem as above, we can assume that $|Y| \leq |R| - n/8$. Every edge between $R \setminus (Y \cap R)$ and $B \setminus (Y \cap B)$

is red. It follows that there are at least $|R| - |Y \cap R|$ red vertices v with

$$d_b(v) \leq |Y \cap B| = |Y| - |Y \cap R| \leq |R| - |Y \cap R| - \frac{n}{8}.$$

This implies that $a_j \leq j - n/8$ for $j = |R| - |Y \cap R|$. Thus $a_1, \ldots, a_{|R|}$ has oscillation at least $n/8 = kN/8$. □

2.2.4 Sequences and oscillation

We now present the quite technical proof of Lemma 2.9. We will use the following definition and related lemma in order to describe the oscillation from the diagonal.

Definition 2.4 (k-good, $u_\mathrm{o}(k)$, $u_\mathrm{e}(k)$)**.** Let $a_1, \ldots, a_n$ be a sequence of non-negative real numbers and let k be a positive real number. We say that the sequence is *k-good* if there exists an odd i and an even j such that $a_i \geq k$ and $a_j \geq k$. If the sequence is k-good, we define for all $0 < t \leq k$

$$\begin{aligned} u_\mathrm{o}(t) &= a_1 + \cdots + a_{i_o - 1} \quad \text{where } i_o = \min\{i\colon\ a_i \geq t, i \text{ odd}\}, \\ u_\mathrm{e}(t) &= a_1 + \cdots + a_{i_e - 1} \quad \text{where } i_e = \min\{i\colon\ a_i \geq t, i \text{ even}\}. \end{aligned}$$

Lemma 2.11. *For all $\gamma \in \mathbb{R}^+$ there exists $N \in \mathbb{N}$ such that for all $k \in \mathbb{R}^+$ and all (kN)-good sequences, there exists a real number $t \in [k, kN]$ with*

$$u_o(t) + u_e(t) \geq \left(3 + \sqrt{8} - \gamma\right) t.$$

First we use Lemma 2.11 to prove Lemma 2.9.

Proof of Lemma 2.9. Given $\gamma > 0$, let N be obtained from Lemma 2.11. Let $k \in \mathbb{R}^+$ and $a_1, \ldots, a_n$ be a sequence with oscillation at least kN. Suppose first that $a_1 \geq 1$. Partition $[n]$ into a family of non-empty intervals $I_1, \ldots, I_r$ with the following properties:

- For every odd i and every $j \in I_i$, we have $a_j \geq j$.
- For every even i and every $j \in I_i$, we have $a_j < j$.

Define $s_i = \max\{|a_j - j|\colon\ j \in I_i\}$. Intuitively, this is saying that the values in the odd indexed intervals are "above the diagonal" and the values in the even indexed intervals are "below the diagonal" and s_i is the largest gap between sequence values and the "diagonal" in each interval.

Since $a_1, \ldots, a_n$ has oscillation at least kN, the sequence $s_1, \ldots, s_r$ is (kN)-good and thus by Lemma 2.11, there exists $t \in [k, kN]$ such that

$$u_{\mathrm{o}}(t) + u_{\mathrm{e}}(t) \geq \left(3 + \sqrt{8} - \gamma\right) t. \tag{2.6}$$

Since the sequence $a_1, a_2, \ldots, a_n$ is non-decreasing, $a_j - j$ can decrease by at most one in each step and thus we have $|I_i| \geq s_i$ for every $i \in [r-1]$. Moreover, we can find bounds on $\ell^+(t)$ and $\ell^-(t)$ in terms of the s_i:

- $\ell^+(t)$ must lie in the interval I_i with the smallest odd index i_o such that $s_{i_o} \geq t$, therefore $\ell^+(t) \geq s_1 + \cdots + s_{i_o - 1} = u_{\mathrm{o}}(t)$.
- $\ell^-(t)$ must lie in the interval I_j with the smallest even index i_e such that $s_{i_e} \geq t$. Moreover, it must be at least the t-th element in this interval, therefore $\ell^-(t) \geq s_1 + \cdots + s_{i_e - 1} + t = u_{\mathrm{e}}(t) + t$.

Combining the previous two observations with (2.6) gives

$$\ell^+(t) + \ell^-(t) \geq u_{\mathrm{o}}(t) + u_{\mathrm{e}}(t) + t \geq \left(4 + \sqrt{8} - \gamma\right) t,$$

as desired.

If $0 \leq a_1 < 1$, we start by partitioning $[n]$ into a family of non-empty intervals $I_1, \ldots, I_r$ with the following properties:

- For every even i and every $j \in I_i$, we have $a_j \geq j$.
- For every odd i and every $j \in I_i$, we have $a_j < j$.

From this point, the proof is analogous. □

Finally, it remains to prove Lemma 2.11. The proof is by contradiction and the main strategy is to find a subsequence with certain properties which force the sequence to become negative eventually.

Proof of Lemma 2.11. Let $\rho = 3 + \sqrt{8} - \gamma$ and let $m := m(\rho)$ be a positive integer which will be specified later. Suppose that the statement of the lemma is false for $N = 6 \cdot 4^m$ and let $a_1, \ldots, a_n$ be an (Nk)-good sequence without t as in the statement. We first show that a_i has a long strictly increasing subsequence. Set

$$I = \{i\colon\ a_i \geq k, a_i > a_j \text{ for all } j < i\},$$

denote the elements of I by $i_1 \leq i_2 \leq \cdots \leq i_r$ and let $a'_j = a_{i_j}$. Consider any $j \in [r-1]$ and suppose without loss of generality that i_{j+1} is odd. For δ small enough, this implies $u_\mathrm{o}(a'_j + \delta) = a_1 + \cdots + a_{i_{j+1}-1} \geq a'_1 + \cdots + a'_j$, and $u_\mathrm{e}(a'_j + \delta) \geq a_1 + \cdots + a_{i_{j+1}} \geq a'_1 + \cdots + a'_{j+1}$. By assumption we have $u_\mathrm{o}(a'_j + \delta) + u_\mathrm{e}(a'_j + \delta) < \rho(a'_j + \delta)$. Hence, letting $\delta \to 0$ we obtain $2\left(a'_1 + \cdots + a'_j\right) + a'_{j+1} \leq \rho a'_j$, which rearranges to

$$a'_{j+1} \leq (\rho - 2)a'_j - 2\left(a'_1 + \cdots + a'_{j-1}\right). \tag{2.7}$$

In particular, this implies $a'_{j+1} \leq (\rho-2)a'_j < 4a'_j$. Moreover, we have $a'_1 \leq u_\mathrm{o}(k)$ if i_1 is even and $a'_1 \leq u_\mathrm{e}(k)$ if i_1 is odd. Therefore,

$$6k \cdot 4^m = kN \leq a'_r < 4^r \cdot a'_1 \leq 4^r \max\{u_\mathrm{o}(k), u_\mathrm{e}(k)\} \leq 4^r(u_\mathrm{o}(k) + u_\mathrm{e}(k)) < 4^r \cdot \rho k < 6k \cdot 4^r$$

and thus $r \geq m$.

Finally, we show that any sequence of reals satisfying (2.7), will eventually become negative, but since a'_i is non-negative this will be a contradiction.

We start by defining the sequence $b_1, b_2, \ldots$ recursively by $b_1 = 1$ and $b_{i+1} = (\rho-2)b_i - 2(b_1 + \cdots + b_{i-1})$. Note that

$$\begin{aligned}
b_{i+1} &= (\rho-2)b_i - 2(b_1 + \cdots + b_{i-1}) \\
&= (\rho-1)b_i - b_i - 2(b_1 + \cdots + b_{i-1}) \\
&= (\rho-1)b_i - ((\rho-2)b_{i-1} - 2(b_1 + \cdots + b_{i-2})) - 2(b_1 + \cdots + b_{i-1}) \\
&= (\rho-1)b_i - \rho b_{i-1}
\end{aligned}$$

So equivalently the sequence is defined by,

$$b_1 = 1,\ b_2 = \rho - 2,\ \text{and } b_{i+1} = (\rho-1)b_i - \rho b_{i-1} \text{ for } i \geq 2.$$

It is known that a second order linear recurrence relation whose characteristic polynomial has non-real roots will eventually become negative (see [BW81]). Indeed, the characteristic polynomial $x^2 - (\rho-1)x + \rho$ has discriminant $\rho^2 - 6\rho + 1 < 0$ and so its roots $\alpha, \bar{\alpha}$ are non-real. Hence the above recursively defined sequence has the closed form of $b_i = z\alpha^i + \bar{z}\bar{\alpha}^i = 2\mathrm{Re}\left(z\alpha^i\right)$ for some complex number z. By expressing $z\alpha^i$ in polar form we can see that $b_m < 0$ for some positive integer m. Note that the calculation of m only depends on ρ.

Now let $a'_1, \ldots, a'_m$ be a sequence of non-negative reals satisfying (2.7). We will be done if we can show that $a'_j \leq a'_1 b_j$ for all $1 \leq j \leq m$; so suppose $a'_s > a'_1 b_s$ for some s, and such that $\{a'_j\}_{j=1}^m$ and $\{a'_1 b_j\}_{j=1}^m$ coincide on the longest initial subsequence. Let p be the minimum value such that $a'_p \neq a'_1 b_p$. Clearly $p > 1$. Applying (2.7) to $j = p - 1$ we see that

$$
\begin{aligned}
a'_p \leq (\rho - 2)a'_{p-1} - 2(a'_1 + \cdots + a'_{p-2}) &= (\rho - 2)a'_1 b_{p-1} - 2(a'_1 b_1 + \cdots + a'_1 b_{p-2}) \\
&= a'_1((\rho - 2)b_{p-1} - 2(b_1 + \cdots + b_{p-2})) = a'_1 b_p
\end{aligned}
$$

and thus $a'_p < a'_1 b_p$.

Let $\beta = (a'_1 b_p - a'_p)/a'_1 > 0$. Now consider the sequence a''_j where $a''_j = a'_j$ for $j < p$ and $a''_j = a'_j + \beta a'_{j-p+1}$ for $j \geq p$. Then $a''_p = a'_1 b_p = a''_1 b_p$. Clearly, this new sequence satisfies (2.7) for every $j < p$. Furthermore, we have

$$
\begin{aligned}
a''_{p+j} &= a'_{p+j} + \beta a'_{j+1} \\
&\leq (\rho - 2)a'_{p+j-1} - 2\left(a'_1 + \cdots + a'_{p+j-2}\right) + \beta(\rho - 2)a'_j - 2\beta\left(a'_1 + \cdots + a'_{j-1}\right) \\
&= (\rho - 2)a''_{p+j-1} - 2\left(a''_1 + \cdots + a''_{p+j-2}\right)
\end{aligned}
$$

for every $j \geq 0$. Hence, the whole sequence satisfies (2.7). We also have $a''_s \geq a'_s > a'_1 b_s = a''_1 b_s$. This contradicts the fact that a'_j was such a sequence which coincided with $a'_1 b_j$ on the longest initial subsequence. $\square$

Chapter 3

Ramsey density of locally finite graphs

In this Chapter we will prove the results from Section 1.2.1 and Section 1.2.2.

Before jumping into the proofs, we need a definition that will be important when relating the Ramsey density of a graph H and its components. This definition appears in Theorem 3.1 and Theorem 3.2

We say that a family $\{S_1, S_2, \dots\}$ of subsets of $V(H)$ is *concentrated* in at most k components if there are components $C_1, C_2, \dots, C_s$ of H, with $s \leq k$, such that all but finitely many sets S_i intersect some component C_j. We say that $V(H)$ is concentrated in at most k components if $\{\{v\} : v \in V(H)\}$ is concentrated in at most k components.

This chapter is organized as follows: we prove the general upper bounds in Section 3.1, and the general lower bounds in Section 3.2, besides a lemma that is left for Appendix A (the bulk of this proof is a rather long series of calculations). In Section 3.3 we discuss the application of the general bounds to particular families of graphs, and obtain the remaining results from Section 1.2.1 and Section 1.2.2.

3.1 General upper bounds

We will prove two upper bounds in this section: the first one implies the upper bound from items (ii)a and (ii)b in Theorem 1.7, while the second one is Theorem 1.9. In both cases we will construct a coloring of $E(K_{\mathbb{N}})$ in which no dense monochromatic copy of H exists.

Theorem 3.1. *Let H be a locally finite graph with chromatic number at least a, such that $V(H)$ is concentrated in at most b components. There exists a two-coloring of $K_{\mathbb{N}}$ in which every monochromatic copy of H has density at most $b/(a-1)$.*

Proof. Consider the coloring of $K_{\mathbb{N}}$ in which the edge uv is red iff $a-1$ divides $v-u$. The graph formed by the blue edges has chromatic number $a-1$, and thus does not contain H as a subgraph. Every monochromatic copy of H in this coloring is red.

The red graph consists of $a-1$ cliques $C_1, \dots, C_{a-1}$, each with $\bar{d}(C_i) = 1/(a-1)$. The b components that concentrate $V(H')$ must be each contained in a clique C_i. Because modifying finitely many elements does not affect the density of a set, we have $\bar{d}(H') \leq \bar{d}(C_1 \cup \cdots \cup C_b) = b/(a-1)$. We conclude that $\rho(H) \leq b/(a-1)$. □

Next we will prove Theorem 1.9. As before, the goal is to construct a two-coloring of $K_{\mathbb{N}}$ without dense monochromatic copies of H.

The intuition behind the construction to prove Theorem 1.9 is as follows: suppose that we are trying to find a red copy of H in this coloring. If we have a blue clique K which has fewer than k vertices neighboring K through some red edge, and $\mu(H,t) = k$, then we know that fewer than t vertices from K can be in H', because those vertices correspond to an independent set in H. Our goal is to find a construction that maximizes the number of vertices from $[n]$ that can be excluded from a potential red or blue H' using this method.

Compared to the approach of Theorem 1.2, we preserve the idea of a two-step construction: we start with an infinite set of vertices, we first decide the color of the edges between them, and then we choose the element of $\mathbb{N}$ that will correspond to each vertex. One important difference is that, because we were unable to solve explicitly the optimization problem from Definition 1.6, we will write our construction in terms of the optimal solution (in our notation, the function g is an approximation of the solution). This inability to solve explicitly is not due to any difference in the methods used, but is rather subtle: there is a "without loss of generality"-type statement in the proof of Proposition B.3 (the part in which $\frac{1-\gamma}{1+\gamma} > 1$ is assumed) which does lose generality when applying the proof to graphs H with expansion ratio of independent sets greater than 1. However, if this minor technical issue is overcome (and intuition leads us to believe this is possible), then the rest of the proof goes through, and $f(x)$ would be equal to its upper bound in (1.2). If that is the case, then the function g can be defined explicitly, symplifying the analysis.

Proof of Theorem 1.9. Denote $\lambda = \liminf_{n\to\infty} \frac{\mu(H,n)}{n}$. Let $\epsilon > 0$. Let g be a 1-Lipschitz function such that the upper limit in (1.1) is less than $h(\gamma) + \epsilon$, for $\gamma = \frac{\lambda-1}{\lambda+1}$. Take an infinite set of vertices $v_1, v_2, \ldots$, and arrange them from left to right in this order. Color these vertices red and blue in such a way that, for every $n \in \mathbb{N}$, among the n leftmost vertices there are exactly $\lfloor (n+g(n))/2 \rfloor$ red vertices (this is possible because g is 1-Lipschitz). Form a two-colored complete graph by giving each edge the color of its leftmost endpoint.

There must be infinitely many vertices of each color. This is because otherwise one of the non-decreasing functions $\frac{x+g(x)}{2}$ or $\frac{x-g(x)}{2}$ is bounded, giving an absolute upper bound on $x \pm g(x)$. Then $\max\{\Gamma_\gamma^+(g,n), \Gamma_\gamma^-(g,n)\} = +\infty$ for every n large enough.

Let the red vertices be $r_1, r_2, r_3, \ldots$ and the blue vertices be $b_1, b_2, b_3, \ldots$, according to the left-to-right order. Let α_i be the smallest value such that r_{α_i} has at most $\lambda(\alpha_i - i)$ blue vertices to its left, and β_i the smallest value such that b_{β_i} has at most $\lambda(\beta_i - i)$ red vertices to its left. The following discussion will not only prove the existence of α_i and β_i, but also give a bound on them.

Let $w = \frac{2}{1+\lambda}(\lambda i + 2\lambda + 2)$. Let $z^+ = \Gamma_\gamma^+(g,w)$ and $z^- = \Gamma_\gamma^-(g,w)$. By continuity of g and definition of Γ_γ^+ and Γ_γ^-, we have

$$g(z^+) = w - \gamma z^+, \qquad g(z^-) = \gamma z^- - w.$$

One can check that the following identity holds by substution of $g(z^-)$:

$$\frac{z^- + g(z^-)}{2} + 2 = \lambda\left(\frac{z^- - g(z^-)}{2} - i - 2\right).$$

Among the $\lfloor z^- \rfloor$ leftmost vertices there are $\left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor$ red vertices and $\lfloor z^- \rfloor - \left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor$ blue vertices. Observe that

$$\begin{aligned}\left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor \leq & \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \leq \frac{z^- + g(z^-)}{2} < \lambda\left(\frac{z^- - g(z^-)}{2} - i - 2\right) \\ \leq & \lambda\left(\lfloor z^- \rfloor - \left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor - i\right).\end{aligned}$$

If the last blue vertex among those $\lfloor z^- \rfloor$ is b_τ, then the number of red vertices to its left is less than $\lambda(\tau - i)$, meaning that $\beta_i \leq \tau$. Hence

$$\beta_i \le \lfloor z^- \rfloor - \left\lfloor \frac{\lfloor z^- \rfloor + g(\lfloor z^- \rfloor)}{2} \right\rfloor \le \frac{z^- - g(z^-)}{2} + 2 = \frac{1-\gamma}{2} z^- + \frac{w}{2} + 2.$$

Analogously, one has the identity

$$\frac{z^+ - g(z^+)}{2} + 2 = \lambda \left(\frac{z^+ + g(z^+)}{2} - i - 2 \right)$$

and the inequality

$$\lfloor z^+ \rfloor - \left\lfloor \frac{\lfloor z^+ \rfloor + g(\lfloor z^+ \rfloor)}{2} \right\rfloor \le \lambda \left(\left\lfloor \frac{\lfloor z^+ \rfloor + g(\lfloor z^+ \rfloor)}{2} \right\rfloor - i \right).$$

Hence we find

$$\alpha_i \le \left\lfloor \frac{\lfloor z^+ \rfloor + g(\lfloor z^+ \rfloor)}{2} \right\rfloor \le \frac{z^+ + g(z^+)}{2} = \frac{1-\gamma}{2} z^+ + \frac{w}{2}.$$

Adding the two values together, for i large enough we have

$$\alpha_i + \beta_i \le \frac{1-\gamma}{2}(z^+ + z^-) + w + 2 \le \left(\frac{1-\gamma}{2} h(\gamma) + \epsilon + 1 \right) \frac{2\lambda}{1+\lambda} i + o(i).$$

Let $\phi : \mathbb{N} \to \{v_1, v_2, \dots\}$ be an arbitrary bijection that, for every j, maps $[\alpha_j + \beta_j]$ to the set $\{r_1, r_2, \dots, r_{\alpha_j}, b_1, b_2, \dots, b_{\beta_j}\}$. The function ϕ defines a coloring of $K_{\mathbb{N}}$, where the color of the edge ij is the color of the edge $\phi(i)\phi(j)$.

Let R and B be the sets of positive integers i whose image $\phi(i)$ is red or blue, respectively. Let $H' \subseteq K_{\mathbb{N}}$ be a monochromatic copy of H in this coloring. Suppose that H' is red. Let n be a positive integer, and let $B_n = V(H') \cap [n] \cap B$. Because the vertices of B_n form a monochromatic blue clique in our coloring of $K_{\mathbb{N}}$, the set B_n must be independent in H'.

Let j be the minimum value such that $\phi(B \cap [n]) \subseteq \{b_1, b_2, \dots, b_{\beta_j}\}$. We claim first that there are at least $(1 - O(\epsilon))j$ vertices in $[n]$ which do not belong to H'. Indeed, let $B'_n = V(H') \cap \{b_1, b_2, \dots, b_{\beta_{j-1}}\} \subseteq B_n$. From the construction of the coloring, the vertices that are connected to a vertex of $\{b_1, b_2, \dots, b_{\beta_{j-1}}\}$ through a red edge are precisely the red vertices to the left of $b_{\beta_{j-1}}$, of which there are at most $\lambda(\beta_{j-1} - (j-1))$. This means that $\mu(H, |B'_n|) \le \lambda(\beta_{j-1} - (j-1))$. For j large enough, this implies $|B'_n| \le (1 + o(1))(\beta_{j-1} - (j-1))$, and

$$|[n] \setminus V(H')| \ge \beta_{j-1} - |B'_n| \ge (1 + o(1))(j-1) - o(1)\beta_{j-1}.$$

Observe next that we cannot have $\beta_j = \beta_{j+1}$. This is because $b_{\beta_j - 1}$, which is to the left[1] of b_{β_j}, has more than $\lambda((\beta_j - 1) - j) = \lambda(\beta_j - (j+1))$ red vertices to its left. We thus have, by minimality of j, that $b_{\beta_{j+1}} \notin \phi([n])$, and by construction of ϕ we have $\phi([n]) \subset \{r_1, r_2, \ldots, r_{\alpha_{j+1}}, b_1, b_2, \ldots, b_{\beta_{j+1}}\}$. This leads to the desired bound:

$$\frac{|V(H') \cap [n]|}{n} \leq 1 - \frac{(1-o(1))j}{n} \leq 1 - \frac{(1-o(1))j}{\alpha_{j+1} + \beta_{j+1}} \leq 1 - \frac{1-o(1)}{\left(\frac{1-\gamma}{2} h(\gamma) + \epsilon + 1\right) \frac{2\lambda}{1+\lambda}}$$

which for ϵ small enough and n large enough can take values arbitrarily close to $f(\lambda)$.

The case in which H' is monochromatic blue is analogous. Indeed, besides the direction of the rounding, it is equivalent to taking the function $-g(x)$ instead of $g(x)$. $\square$

3.2 General lower bounds

In this section we will prove three lower bounds. One is item (i) from Theorem 1.7, another is the lower bound of item (ii)b in the same theorem, and the final one is the following, which will be used in the proof of Theorem 1.10 and Theorem 1.15:

Theorem 3.2. *Let H be a locally finite graph, a, b, r, s be positive integers with $a > b$, and $\Psi : V(H) \to [a]$ be a proper coloring. Suppose that there exist infinitely many pairwise disjoint doubly independent sets $I_1, I_2, \ldots$ in H, each contained in some component of H and not concentrated in fewer than b components, such that $|I_i| = r$, $|N(I_i)| \leq s$, and $\Psi(N(I_i)) = a$. Then*
$$\rho(H) \geq \frac{b}{a-1} f\left(\frac{s}{r}\right).$$

As an example of a graph whose Ramsey density can be computed from Theorem 3.2, but not from the other lower bounds mentioned above, let T be the tree obtained by taking an infinite path $v_1 v_2 v_3 \ldots$ and adding i leaves to each vertex v_i, for all i. Consider the graph $H = b \cdot T + K_a$, with $a > b$, obtained by taking the disjoint union of b copies of T and one a-clique. We can define a proper coloring $\Psi : V(H) \to [a]$ in which the vertices of K_a all receive different colors, and the trees T are properly two-colored with colors $\{1, a\}$. Then for every $r \in \mathbb{N}$ there exist infinitely many independent pairwise disjoint sets I, in every T-component, where $N(I)$ is a single vertex with color a (just take r leaves of a vertex with label greater than r and color a). Theorem 3.2 then tells us that $\rho(b \cdot T + K_a) \geq \frac{b}{a-1} f(r^{-1})$ for all $r \in \mathbb{N}$, and so $\rho(b \cdot T + K_a) \geq \frac{b}{a-1}$. We have equality here, as this matches the upper bound from Theorem 1.7(ii)b.

[1] We cannot have $\beta_j = 1$ for $j \geq 2$, because then b_1 would have at most $\lambda(1-j) < 0$ red vertices to its left.

Another example is the graph $2 \cdot P_\infty + K_3$, in which the graph can be colored with the colors $\{1, 2, 3\}$ in a way that both paths use only the colors $\{1, 3\}$. Then for every r, each P_∞-component contains infinitely many pairwise disjoint independent sets I with $|I| = r$, $|N(I)| = r+1$ and $N(I)$ being monochromatic in color 3 (just take r consecutive vertices receiving color 1). By Theorem 3.2 we have $\rho(2 \cdot P_\infty + K_3) \geq f\left(\frac{r+1}{r}\right)$ and, by continuity of $f(x)$, we have $\rho(2 \cdot P_\infty + K_3) \geq f(1)$, which matches the upper bound from Theorem 1.9.

However, for every graph for which we know that Theorem 3.2 is tight, we either have $a - 1 = b$ or $r = s$, as in the two examples above.

We start with the proof of Theorem 1.7(i). This result follows easily from the infinite version of Ramsey's theorem:

Proof of Theorem 1.7(i). Let $\chi : E(K_\mathbb{N}) \to \{R, B\}$ be an edge-coloring. Let $\mathcal{F}$ be an inclusion-maximal family of pairwise disjoint monochromatic infinite cliques in χ. Then $\mathbb{N} \setminus V(\mathcal{F})$ is finite, because otherwise by Ramsey's theorem there would be an infinite monochromatic clique in χ restricted to $\mathbb{N} \setminus V(\mathcal{F})$, contradicting the maximality of $\mathcal{F}$. Let $\mathcal{F}_R$ and $\mathcal{F}_B$ be the families of red and blue cliques in $\mathcal{F}$. Since $\bar{d}(V(\mathcal{F}_R) \cup V(\mathcal{F}_B)) = 1$, we have $\max\{\bar{d}(\mathcal{F}_R), \bar{d}(\mathcal{F}_B)\} \geq 1/2$. W. l. o. g. assume $\bar{d}(\mathcal{F}_R) \geq 1/2$. We can suppose that $\mathcal{F}_R$ contains infinitely many cliques, because otherwise we can take one clique $K \in \mathcal{F}_R$ and divide it into infinitely many infinite cliques. Let $K_1, K_2, \ldots,$ be the cliques in $\mathcal{F}_R$. We can partition the vertex set of H into infinitely many parts $S_1, S_2, \ldots,$ each of which is made up of infinitely many components of H. Now take any $\Phi : V(H) \to V(K_\mathbb{N})$ which is a bijection from each S_i to each K_i. The image of H is a monochromatic graph H' and $\bar{d}(H') = \bar{d}(\mathcal{F}_R) \geq 1/2$. □

The proof of Theorem 1.7(ii)b and Theorem 3.2 will both be (partially) algorithmic: given a coloring $\chi : E(K_\mathbb{N}) \to \{R, B\}$, we will define an algorithm that constructs a dense monochromatic copy of H. The algorithms will be similar, so we will first prove Theorem 3.2 and then explain how to adapt the proof to Theorem 1.7(ii)b.

Let H be as in Theorem 3.2, and let $\chi : E(K_\mathbb{N}) \to \{R, B\}$. Our goal is to find a copy of H in $K_\mathbb{N}$ with density at least $b/(a-1)f(s/r)$. In order to find such a copy of H, it will be helpful to also color the vertices of $K_\mathbb{N}$, in a way that encodes information about how the vertices are connected through red or blue edges. The following coloring is a variant of one due to Elekes et al. [ESSS17].

We denote by $N_C(v)$ the set of vertices connected to v through an edge of color C. When C is a color that is either red or blue, we denote the other color by $\bar{C}$.

Definition 3.3. *Let $\chi : E(K_{\mathbb{N}}) \to \{R, B\}$ be a coloring, and let a be a positive integer. An a-good coloring of $V(K_{\mathbb{N}})$ is a partition $\mathbb{N} = \cup_{i=1}^{a}(R_i \cup B_i) \cup X$ into $2a+1$ classes (some of which might be empty), where X is finite, with the following properties:*

- *For every color $C \in \{R, B\}$, every $1 \leq i \leq a-1$ and every nonempty finite subet $S \subseteq C_i$, the set $(\cap_{v\in S} N_C(v)) \cap C_i$ is infinite.*
- *For every color $C \in \{R, B\}$, every $1 \leq i \leq a-1$ and every nonempty finite subet $S \subseteq C_a \cup \left(\cup_{j=i+1}^{a-1} \bar{C}_j\right)$, the set $(\cap_{v\in S} N_C(v)) \cap \bar{C}_i$ is infinite.*

A similar coloring was defined by Corsten, DeBiasio and McKenney [CDM20]. Both their coloring and that by Elekes et al. are defined using ultrafilters. We define ours algorithmically, even though ultrafilters would have worked just as well, in order to make the properties of this coloring more intuitive and, in the process, avoiding an appeal to the axiom of choice.

In Chapter 2, the counterpart is the coloring of the vertices used in Section 2.2.1. This is obviously way simpler than the a-good colorings that we are defining now. There are two main reasons for the difference in complexity. First, P_∞ is a bipartite graph. As we will see, the parameter a in a-good colorings is tied to the chromatic number of the target graph H. Second, the self-similarity of P_∞ was implicitly used in Section 2.2.1 together with this fact: given two vertices u, v in an 2-edge-colored clique, either they are connected by a finite piece of P_∞ formed by red edges or there is a partition of the graph separating u and v where every edge between parts is blue.

We call each class R_i a shade of red and each class B_i a shade of blue. X can be seen as a residual set, which can be removed without affecting the density of the graph. The choice of a is related to the chromatic number of the monochromatic subgraphs that we can find in this graph. Indeed, say that we want to find a red clique of size a containing $v \in R_i$. If $i \leq a-1$, then we can set $v = v_1$, and then greedily select $v_2, v_3, \ldots, v_a \in R_i$, each adjacent to the previous ones through a red edge. If $i = a$, we can set $v = v_a$, and then greedily select $v_{a-1}, v_{a-2}, \ldots, v_1$, with $v_j \in B_j$, each adjacent to the previous ones through a red edge.

We denote by $K_{r,s}^{C}$ a complete bipartite graph in which all edges have color C, all vertices in the part of size s have color C and all vertices in the part of size r have color $\bar{C}$. These subgraphs will be used to embed the sets $I_i \cup N(I_i)$ in our colored graph.

The proof of Theorem 3.2 will have three main steps, which are captured by these lemmas:

Lemma 3.4. *Let $\chi : E(K_{\mathbb{N}}) \to \{R, B\}$ be a coloring, and let a be a positive integer. There exists an a-good coloring in which at least two of $(R_a \cup B_{a-1})$, $(B_a \cup R_{a-1})$ and X are empty.*

Lemma 3.5. *Let $\chi : E(K_{\mathbb{N}}) \cup V(K_{\mathbb{N}})$ be a coloring, and let r, s be positive integers. There exists a color C and a subgraph $W \subseteq K_{\mathbb{N}}$, with $\bar{d}(W) \geq f(s/r)$, in which every component is either an isolated vertex with color C, or a $K^C_{r,s}$. Furthermore, if $V(K_{\mathbb{N}})$ is further subdivided into finitely many shades, then W can be taken in a way that each $K^C_{r,s}$ only uses one shade of each color.*

Lemma 3.6. *Let $\chi : E(K_{\mathbb{N}}) \to \{R, B\}$ be an edge-coloring, let $a \geq a' \geq b$ be positive integers. Let $\mathbb{N} \to \{R_1, \dots, R_a, B_1, \dots, B_a, X\}$ be an a-good coloring in which at most a' shades of each color are non-empty. Let $W \subseteq K_{\mathbb{N}}$ be a subgraph in which every component is either an isolated vertex with color C, or a $K^C_{r,s}$ which uses only one shade of each color. Under the conditions of Theorem 3.2, there exists a monochromatic $H' \subseteq K_{\mathbb{N}}$ of color C, $H' \simeq H$, with $\bar{d}(H') \geq b/a'\bar{d}(W)$.*

Lemma 3.5 is the counterpart of Lemma 2.3 (its proof also includes the counterpart of the proof of Lemma 2.4). Lemma 3.6, which is used to glue some of the components in W, is not needed for P_∞ due to the simple structure of that graph. It is worth noting that part of the difficulty of the generalization of the vertex coloring used comes from finding the right definition of what a "good" coloring should be: weak enough that it is possible to find in each edge-coloring, yet strong enough to allow us to construct a monochromatic copy of H. It is straightforward to combine these three lemmas to deduce Theorem 3.2:

Proof of Theorem 3.2. Let $\chi : E(K_{\mathbb{N}})$ be given. Apply Lemma 3.4 to this edge-coloring to obtain an a-good coloring with at most $a - 1$ shades of each color are non-empty. Assign the color red to the vertices in X. Apply Lemma 3.5 to obtain C and W. Remove from W every component which uses a vertex of X (this does not affect $\bar{d}(W)$ because it only removes finitely many vertices). By Lemma 3.6, we can find a monochromatic $H' \subseteq K_{\mathbb{N}}$ with $\bar{d}(H') \geq b/(a-1)\bar{d}(W) \geq b/(a-1)f(s/r)$. □

Proof of Lemma 3.4. For each vertex v, we will denote by $c(v)$ and $s(v)$ the color and the shade that we assign to it, respectively. The color assigned to a vertex might change while the algorithm is running, but the shade of each vertex is final once assigned and it will match the color that the vertex has at that time.

At some points, the shade assigning algorithm will call the basic coloring algorithm to color an infinite set $V = \{v_1, v_2, \dots\}$ of vertices. We will first describe this algorithm.

Basic coloring algorithm: First, the color $c(v_1)$ is assigned, satisfying that $N_{c(v_1)}(v_1) \cap V$ is infinite. Once the colors of $v_1, \ldots, v_{i-1}$ have been assigned, assuming by induction that $\left(\cap_{i=1}^{n-1} N_{c(v_i)}(v_i)\right) \cap V$ is infinite, the color $c(v_n)$ is chosen so that $\left(\cap_{i=1}^{n} N_{c(v_i)}(v_i)\right) \cap V$ is infinite.

The basic coloring algorithm produces a coloring in which the set $\left(\cap_{i=1}^{n} N_{c(v_i)}(v_i)\right) \cap V$ is infinite for every n. We say that a color C is dominant in this coloring if, for every n, $\left(\cap_{i=1}^{n} N_{c(v_i)}(v_i)\right) \cap V$ contains infinitely many vertices v with $c(v) = C$. Observe that at least one of the colors is dominant.

Now we define the shade assigning algorithm:

1. For every vertex $v \in \mathbb{N}$, start with $c(v)$ and $s(v)$ unassigned.

2. If finitely many vertices v remain with $s(v)$ unassigned, assign $s(v) = X$, and END.

3. Let V be the set of vertices without a shade. Color V with the basic coloring algorithm. Choose a color C that is dominant. Let i be the minimum value such that C_i is empty. For every $v \in V$ with $c(v) = C$, set $s(v) = C_i$.

4. If $i = a-1$, set $s(v) = \bar{C}_a$ for every $v \in V$ with $c(v) = \bar{C}$, and END. If $i \neq a-1$, return to Step 2.

The algorithm runs the loop $2-4$ at most $2a-3$ times before ending. Whenever a set C_i with $i \leq a-1$ is defined, the color C is dominant in the corresponding coloring, meaning that in particular $(\cap_{v \in S} N_C(v)) \cap C_i$ is infinite for every finite non-empty $S \subseteq C_i$, as it is a superset of the color C vertices of $\left(\cap_{i=1}^{n} N_{c(v_i)}(v_i)\right) \cap V$ for n large enough. For the same reason, for any finite subset S of vertices whose shade is not assigned when C_i is defined, we have that $(\cap_{v \in S} N_{\bar{C}}(v)) \cap C_i$ is infinite. If C_a is defined at some point in the algorithm (namely at the end), then $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_{a-1}, C_a$ are defined in this order. This proves that the coloring that we obtained is a-good.

To conclude the proof of Lemma 3.4, simply observe that X is nonempty only if the algorithm terminates at Step 2, the set $(R_a \cup B_{a-1})$ is nonempty only if the algorithm terminates at Step 4 with $C = B$ and $(B_a \cup R_{a-1})$ is nonempty only if the algorithm terminates at Step 4 with $C = R$. $\square$

The proof of Lemma 3.5 divides $K_{\mathbb{N}}$ into infinitely many finite graphs, and then combines the regularity lemma and a max flow/min cut argument, to reduce the problem to an optimization problem equivalent to (1.1). One important feature that is common with the proof of Lemma 2.3 is that our application of the regularity lemma will roughly preserve the ordering of the vertices. We will now state the lemmas that we will need for this:

Lemma 3.7 (Regularity Lemma [KS96]). *For every $\epsilon > 0$ and $m_0, \ell \geq 1$ there exists $M = M(\epsilon, m_0, \ell)$ such that the following holds. Let G be a graph on $n \geq M$ vertices whose edges are colored in red and blue and let $d > 0$. Let $\{W_i\}_{i\in[\ell]}$ be a partition of $V(G)$. Then there exists a partition $\{V_0, \dots, V_m\}$ of $V(G)$ and a subgraph H of G with vertex set $V(G) \setminus V_0$ such that the following holds:*

1. *$m_0 \leq m \leq M$;*
2. *$\{V_i\}_{i\in[m]}$ refines $\{W_i \cap V(H)\}_{i\in[\ell]}$;*
3. *$|V_0| \leq \epsilon n$ and $|V_1| = \dots = |V_m| \leq \lceil \epsilon n \rceil$;*
4. *$\deg_H(v) \geq \deg_G(v) - (d+\epsilon)n$ for each $v \in V(G) \setminus V_0$;*
5. *$H[V_i]$ has no edges for $i \in [m]$;*
6. *all pairs (V_i, V_j) are ϵ-regular and with density either 0 or at least d in each color in H.*

The max flow/min cut result that we will use can be seen as a weighted version of König's Theorem:

Lemma 3.8. *Let G be a finite bipartite graph on $V = (X, Y)$, and let r, s be positive integers. There exists a unique value of D for which both of these exist:*

- *A function $h : E(G) \to \mathbb{N} \cup \{0\}$ such that $\sum_{e\ni v} h(e) \leq r$ if $v \in X$, $\sum_{e\ni v} h(e) \leq s$ if $v \in Y$ and $\sum_{e\in E(G)} h(e) = D$.*
- *A vertex cover Z of G such that $r|Z \cap X| + s|Z \cap Y| = D$.*

Proof. Take an orientation of every edge in G from X to Y, and give it an infinite capacity. Connect every vertex in X to a source σ through an edge with capacity r, and every vertex in y to a sink τ through an edge with capacity s. Let D be the maximum flow in this network. D is the maximum value for which a function h as in the statement exists (by the integrality theorem, there exists a maximum flow in which the flow of every edge is an integer). D is also the minimum value for which a cut (C_1, C_2) with $\sigma \in C_1$ and $\tau \in C_2$ exists. Observe that (C_1, C_2) is a cut with finite capacity iff $(C_2 \cap X) \cup (C_1 \cap Y)$ is a vertex cover of G, in which case the capacity of the cut is $r|C_2 \cap X| + s|C_1 \cap Y|$. Our lemma follows from the Ford-Fulkerson theorem. $\square$

The next lemma that we will introduce requires the definition of two parameters, which up to a change of coordinates are equivalent to Γ^+_γ and Γ^-_γ.

Definition 3.9. *Let $g : [0, +\infty) \to [0, +\infty)$ be a continuous, non-decreasing function. Let λ, t be positive real numbers. We define the following two parameters:*

$$\ell^+_\lambda(g,t) = \min\{x : g(\lambda x) - x \geq t\} \qquad \ell^-_\lambda(g,t) = \min\left\{x : x - \frac{g(x)}{\lambda} \geq t\right\},$$

where we take the minimum of the empty set to be $+\infty$.

Lemma 3.10. *For $\lambda, \epsilon > 0$ there exists $\gamma > 0$ with the following property: for every non-decreasing continuous function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ and every $m > 0$ there exists $t \in [\gamma m, m]$ such that*

$$\frac{\ell^+_\lambda(g,t) + \ell^-_\lambda(g,t)}{t} \geq \frac{f(\lambda)}{1 - f(\lambda)} - \epsilon.$$

The proof of Lemma 3.10 can be found in Appendix A. Combining Lemma 3.8 and Lemma 3.10, we can obtain the following:

Lemma 3.11. *For every $\epsilon, r, s > 0$ there exists $\gamma, \eta > 0$ and N for which the following hold: for every graph G on $[n]$, with $n > N$ and $\delta(G) \geq (1 - \eta)n$, and for every total coloring $\chi : V(G) \cup E(G) \to \{R, B\}$, there exists $t \in [\gamma n, n]$, a color C, and $h : E(G) \to \mathbb{N} \cup \{0\}$, such that the following hold:*

- *For every edge $e = uv$, if $g(e) > 0$ then $\chi(e) = C$ and $\chi(u) \neq \chi(v)$.*
- *$\sum_{e \ni v} h(e) \leq r$ for every v with $\chi(v) = C$ and $\sum_{e \ni v} h(e) \leq s$ for every v with $\chi(v) = \bar{C}$.*
- *$\frac{|C \cap [t]|}{t} + \frac{\sum_{v \in (\bar{C} \cap [t])} \sum_{e \ni v} h(e)}{st} \geq f(s/r) - \epsilon$*

Proof. Let $\lambda = s/r$. Our constants will follow the hierarchy

$$\eta, N^{-1} \ll \gamma \ll \kappa \ll \xi \ll \epsilon, \lambda.$$

That is, after ϵ and λ are given we pick ξ small enough, after fixing ξ we pick κ small enough, and so on.

For every red vertex v, we define its blue degree $d_B(v)$ as the number of blue vertices w such that vw is blue. Let $v_1, v_2, \ldots, v_{|R|}$ be the set of red vertices, sorted from smallest to largest blue degree, and let $d_i = d_B(v_i)$. Define additionally $d_0 = 0$ and $d_k = d_{|R|}$ for $k > |R|$. Let $g : [0, +\infty) \to [0, +\infty)$ be the function that satisfies $g(k) = d_k$ for every integer k and which is linear between every pair of consecutive integers.

By Lemma 3.10 there exists $\tau \in [\gamma n, \kappa n]$ for which $\frac{\ell_\lambda^+(g,\tau)+\ell_\lambda^-(g,\tau)}{\tau} \geq \frac{f(\lambda)}{1-f(\lambda)} - \xi$. Let $t = \left(\frac{1}{1-f(\lambda)} - \xi\right)\tau$. Then either $|R \cap [t]| < \ell_\lambda^-(g,\tau)$ or $|B \cap [t]| \leq \ell_\lambda^+(g,\tau) + \tau$. We consider both cases, in the former we will have $C = B$ and in the latter $C = R$:

Case 1: $|R \cap [t]| < \ell^-(g,\tau)$. Let $R' = R \cap [t]$. Let G' be the graph of blue edges in G between R' and B. Let h, Z and D be as in Lemma 3.8 applied to G', with $X = B$ and $Y = R'$. Suppose that $D \leq s(|R'| - \tau)$. Every vertex $v \in R' \setminus Z$ must have all its blue neighbors in $B \cap Z$, and so $d_B(v) \leq |B \cap Z|$. Therefore

$$d_{|R'|-|Z\cap R'|} \leq |Z \cap B| = \frac{D - s|Z \cap R'|}{r} \leq \frac{s}{r}(|R'| - |Z \cap R'| - \tau).$$

Setting $x = |R'| - |Z \cap R'|$, this expression rearranges to $x - \frac{g(x)}{\lambda} \geq \tau$, so by definition of ℓ_λ^- this means that $x \geq \ell_\lambda^-(g,\tau)$. But this is a contradiction, because $x \leq |R'| < \ell_\lambda^-(g,\tau)$. This means that we have $D > s(|R'| - \tau)$, and

$$\frac{|B \cap [t]|}{t} + \frac{D}{st} \geq \frac{t - |R'|}{t} + \frac{s(|R'| - \tau)}{st} = 1 - \frac{\tau}{t} = 1 - \frac{1}{\frac{1}{1-f(\lambda)} - \xi} \geq f(\lambda) - \epsilon.$$

Case 2: $|B \cap [t]| \leq \ell^+(g,\tau) + \tau$. Let $B' = B \cap [t]$. Let G' be the graph of red edges between R and B'. Let h, Z and D be as in Lemma 3.8 applied to G', with $X = R$ and $Y = B'$. Suppose that $D < s(|B'| - \tau - \eta n - \frac{1}{\lambda})$. Every edge between $R \setminus Z$ and $B' \setminus Z$ is blue. Every vertex v has at most ηn vertices to which it is not connected, and so $d_B(v) \geq |B' \setminus Z| - \eta n$ for all[2] $v \in R \setminus Z$.

$$\begin{aligned} d_{|R\cap Z|+1} \geq& |B'| - |B' \cap Z| - \eta n \geq |B'| - \frac{D - r|R \cap Z|}{s} - \eta n \\ =& \frac{s|B'| - D}{s} + \frac{1}{\lambda}|R \cap Z| - \eta n \geq \tau + \eta n + \frac{1}{\lambda} + \frac{1}{\lambda}|R \cap Z| - \eta n \\ \geq& \tau + \frac{1}{\lambda}(|R \cap Z| + 1). \end{aligned}$$

Setting $x = \frac{1}{\lambda}(|R \cap Z| + 1)$, this expression rearranges to $g(\lambda x) - x \geq \tau$, so by definition of ℓ_λ^+ this means that $x \geq \ell_\lambda^+(g,\tau)$. On the other hand, $x = \frac{|R\cap Z|+1}{\lambda} \leq \frac{D}{s} + \frac{1}{\lambda} < |B'| - \tau - \eta n - \frac{1}{\lambda} + \frac{1}{\lambda} < |B'| - \tau \leq \ell_\lambda^+(g,\tau)$, which is a contradiction. This means that we have $D \geq s(|B'| - \tau - \eta n - \frac{1}{\lambda})$, and

$$\frac{|R \cap [t]|}{t} + \frac{D}{st} \geq \frac{t - |B'|}{t} + \frac{s(|B'| - \tau - \eta n - \frac{1}{\lambda})}{st} \geq 1 - \frac{\tau}{t} - \frac{\eta}{\gamma} - \frac{1}{\lambda\gamma N} \geq f(\lambda) - \epsilon.$$

□

[2]What if $R \setminus Z = \emptyset$ (if this happens we cannot guarantee $d_{|R\cap Z|+1} \geq |B' \setminus Z|$)? Then $r|R| \leq D \leq s|B'| \leq st \leq s\frac{\tau}{1-f(\lambda)} \leq s\frac{\kappa}{1-f(\lambda)}n \Rightarrow |R| \leq (1 - f(\lambda))n \Rightarrow |B| \geq f(\lambda)n$, and thus taking $t' = n$, $h = 0$ and $C = B$ is enough for Lemma 3.11.

To prove Lemma 3.5, we apply the regularity lemma to the graph and use Lemma 3.11. We also use the fact that, by the Kővári-Sós-Turán theorem, every large enough dense bipartite graph contains a large complete bipartite subgraph:

Proof of Lemma 3.5. Let $\lambda = s/r$. We first claim that, for every $\epsilon > 0$, there exists $\gamma(\epsilon) > 0$ and $N(\epsilon)$ such that, for every $n > N$, there exist $t \in [\gamma n, n]$, a color C and a subgraph $\mathcal{F} \subseteq K_{\mathbb{N}}$ contained in $[n]$ in which every component is either an isolated vertex of color C or a $K_{r,s}^C$ using only a shade of each color, with

$$\frac{|V(\mathcal{F}) \cap [t]|}{t} \geq f(\lambda) - \epsilon.$$

Let a be the total number of shades (from both colors). Our constants will follow the hierarchy

$$N^{-1} \ll M^{-1} \ll \rho \ll \delta \ll \zeta \ll \gamma, \eta \ll \epsilon, r^{-1}, s^{-1}, a^{-1}.$$

Let G be the restriction of our coloring to $[n]$. Take a partition of $[n]$ into $\ell = a\lceil \rho^{-1} \rceil$ parts $\{Z_1, \ldots, Z_\ell\}$, such that each Z_i is contained in one shade, and $\max Z_i - \min Z_i < \rho n$. Applying Lemma 3.7 to G with $d = 2\delta$, we find a subgraph $H \subseteq G$ and a partition $[n] = \{V_0, V_1, \ldots, V_m\}$, with $\ell \leq m \leq M$, as in the statement of Lemma 3.7, replacing ϵ with δ.

We suppose that the labeling of the parts is such that $\min V_1 < \min V_2 < \cdots < \min V_m$. We define an auxilliary graph H' as follows: the vertex set is $[m]$. The color of every vertex i is the same as the color of each of its vertices in G. Between any two vertices ij, we draw an edge if the bipartite graph V_iV_j is nonempty in H, and we color it in the most dense color in V_iV_j.

Let $y = |V_1| = \cdots = |V_m|$. Then $\frac{(1-\delta)n}{m} \leq y \leq \frac{n}{m}$. The minimum degree in H' is at least $(1-\eta)m$. Indeed, given i and $v \in V_i$, we have $d_{H'}(i) \geq \frac{d_H(v) - \delta n}{y} \geq \frac{d_G(v) - 4\delta n}{y} \geq (1 - \frac{4\delta}{1-\delta})m > (1-\eta)m$.

Apply Lemma 3.11 to H', with parameters $\epsilon/2, r, s$ to obtain a color C, a function $h : E(H') \to \mathbb{N}$ and a value $\tau \in [\gamma m, m]$ as in the statement of Lemma 3.11, replacing t with τ. Subdivide each V_i with color C into r parts $V_{i,1}, \ldots, V_{i,r}$, each of size at least $\lfloor y/r \rfloor$, and each V_i with color $\bar{C}$ into s parts $V_{i,1}, \ldots, V_{i,s}$, each of size at least $\lfloor y/s \rfloor$. Construct a matching $\mathcal{M}$ of pairs $(V_{i,k}, V_{j,k'})$, where for any fixed values of i and j, the number of pairs $(V_{i,k}, V_{j,k'})$ in $\mathcal{M}$ is $h(ij)$.

Within each pair $(V_{i,k}, V_{j,k'})$, where V_i has color C and V_j has color $\bar{C}$, find a maximum family $\mathcal{F}_{i,k,j,k'}$ of disjoint copies of $K_{r,s}^C$. Since $N \gg M, \delta^{-1}, r, s$, and therefore $\delta y \gg r, s$, then $\min\{|V_{i,k} \setminus V(\mathcal{F}_{i,k,j,k'})|, |V_{j,k'} \setminus V(\mathcal{F}_{i,k,j,k'})|\} < \delta y$. That is because otherwise the bipartite graph between $V_{i,k} \setminus V(\mathcal{F}_{i,k,j,k'})$ and $V_{j,k'} \setminus V(\mathcal{F}_{i,k,j,k'})$ would have density at least δ in the edges of color C, and for δy large enough this implies the existence of a copy of $K^C_{r,s}$, which would contradict the maximality of $\mathcal{F}_{i,k,j,k'}$.

Let $\mathcal{F}$ be the union of all families $\mathcal{F}_{i,k,j,k'}$. Let $t = \min V_\tau$. We will now bound $\frac{|(V(\mathcal{F})\cup C)\cap[t]|}{[t]}$. If $v \geq t + \rho n$, and $v \in V_i$ with $i \neq 0$, then $\min V_i > \max V_i - \rho n \geq v - \rho n \geq t = \min V_\tau$, and thus $i > \tau$. This means that $|(\cup_{i=1}^{\tau} V_i) \setminus [t]| \leq \rho n$, and $t \geq \tau y - \rho n \geq \frac{(1-\delta)\tau n}{m} - \rho n$. On the other hand, if $v \leq t$ then either $v \in V_0$ or $v \in V_i$ with $\min V_i \leq v \leq t = \min V_\tau$, and thus $i \leq \tau$. This implies that $t \leq \sum_{i=0}^{\tau} |V_i| \leq \delta n + \tau y \leq \delta n + \frac{\tau n}{m}$.

Every V_i with color C and $i \in [\tau]$ will trivially be contained in $(V(\mathcal{F}) \cup C) \cap (\cup_{i=1}^{\tau} V_i)$. For any V_i with color $\bar{C}$ and $i \in [\tau]$, there are $\sum_{e \ni i} h(e)$ parts $V_{i,k}$ which are paired up with a different part $V_{j,k'}$. We either have $|V_{i,k} \setminus V(\mathcal{F})| \leq \delta y$ or $|V_{j,k'} \setminus V(\mathcal{F})| \leq \delta y$. In the first case, $|V_{i,k} \cap V(\mathcal{F})| \geq \lfloor y/s \rfloor - \delta y \geq (1/s - 1/y - \delta)y$. In the second case, $|V_{j,k'} \cap V(\mathcal{F})| \geq \lfloor y/r \rfloor - \delta y \geq (1/r - 1/y - \delta)y$. But $\mathcal{F}$ is a family of copies of $K_{r,s}$, so $|V_{i,k} \cap V(\mathcal{F})| = \frac{r}{s}|V_{j,k'} \cap V(\mathcal{F})| \geq (1/s - \lambda^{-1}(1/y + \delta))y$. In either case we have $|V_{i,k} \cap V(\mathcal{F})| \geq (1-\zeta)y/s$.

Putting our bounds together:

$$
\begin{aligned}
\frac{|(V(\mathcal{F}) \cup C_G) \cap [t]|}{t} \geq & \frac{|(V(\mathcal{F}) \cup C_G) \cap (\cup_{i=1}^{\tau} V_i)| - \rho n}{t} \\
\geq & \frac{y|C_{H'} \cap [\tau]|}{t} + (1-\zeta)\,\frac{y}{s}\,\frac{\sum_{v \in (\bar{C}_{H'} \cap [\tau])} \sum_{e \ni v} h(e)}{t} - \frac{\rho n}{t} \\
\geq & (1-\zeta)\,\frac{\tau y}{t}\left(\frac{|C_{H'} \cap [\tau]|}{\tau} + \frac{\sum_{v \in (\bar{C}_{H'} \cap [\tau])} \sum_{e \ni v} h(e)}{s\tau}\right) - \frac{\rho n}{t} \\
\geq & (1-\zeta)\,\frac{\tau y}{t}\left(f(\lambda) - \frac{\epsilon}{2}\right) - \frac{\rho}{\frac{\tau(1-\delta)}{m} - \rho} \\
\geq & (1-\zeta)\,\frac{\tau y}{\delta n + \tau y}\left(f(\lambda) - \frac{\epsilon}{2}\right) - \frac{\rho}{\gamma(1-\delta) - \rho} \\
\geq & (1-\zeta)\,\frac{1}{1 + \delta \frac{n}{my}\frac{m}{\tau}}\left(f(\lambda) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{4} \\
\geq & (1-\zeta)\,\frac{1}{1 + \frac{\delta}{(1-\delta)\gamma}}\left(f(\lambda) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{4} \\
\geq & f(\lambda) - \epsilon
\end{aligned}
$$

To conclude the proof of our initial claim, notice that $t \geq \left(\frac{(1-\delta)\tau}{m} - \rho\right) n \geq ((1-\delta)\gamma - \rho)n \geq \gamma' n$ for a constant $\gamma' > 0$.

We are now ready to construct W. Take a sequence $f(s/r) > \epsilon_1 > \epsilon_2 > \cdots > 0$ with $\epsilon_i \to 0$. Start by applying the claim with $\epsilon = \epsilon_1$ and $n_1 = N(\epsilon)$ to obtain a subgraph $\mathcal{F}_1$ with color C_1 with density at least $f(s/r) - \epsilon_1$ in $[t_1]$. Now proceed by induction, and set $n_i = \max\{N(\epsilon_i/2), 2n_{i-1}(r+s)/(\epsilon_i\gamma(\epsilon_i/2))\}$. Applying the claim with $\epsilon = \epsilon_i/2$ we find a subgraph $\mathcal{F}_i'$ with color C_i contained in $[n_i]$ and with density at least $f(s/r) - \epsilon_i/2$ in $[t_i]$, for some $t_i \in [\gamma(\epsilon)n_i, n_i]$. Remove from $\mathcal{F}_i'$ all components that intersect $[n_{i-1}]$ (this represents at most $n_{i-1}(r+s)$ vertices) to obtain $\mathcal{F}_i$. Then $\mathcal{F}_i$ is disjoint from all previous $\mathcal{F}_j$, and by the choice of n_i, it still has density at least $f(s/r) - \epsilon_i$ in $[t_i]$.

Select a color C such that $C_i = C$ for infinitely many i. Let $W = \cup_{C_i = C}\mathcal{F}_i$. Then by construction $\bar{d}(W) \geq f(s/r)$, since the t_i tend to infinity, and the components of W are isolated vertices of color C or $K_{r,s}^C$. This concludes the proof of Lemma 3.5. □

Finally, we prove Lemma 3.6 by defining an algorithm that constructs a monochromatic H'. This algorithm uses enough components from W (mapping to them either single vertices of H or sets $I_i \cup N(I_i)$) to keep a fraction of its density, and takes advantage of the properties of the a-good coloring to map the remaining vertices of H.

Proof of Lemma 3.6. Without loss of generality, assume that C is red, let S_j denote the vertices in W of shade R_j, plus the blue vertices contained in a copy of $K_{r,s}^R$ in W in which the red side has shade R_j. Removing from W the finite sets S_j does not affect its density, so suppose that each S_j is either empty or infinite. We will show that there exists a set J, of size b, such that $\bar{d}(\cup_{j\in J}S_j) \geq b/a'\bar{d}(W)$.

By definition of density, there exists a sequence $n_1 < n_2 < \ldots$ of positive integers such that $|V(W) \cap [n_i]|/n_i \to \bar{d}(W)$. For each i there exists a subset $J_i \subseteq [a]$ of b indices such that

$$\frac{|(\cup_{j\in J_i}S_j) \cap [n_i]|}{n_i} \geq \frac{b}{a'}\frac{|V(W) \cap [n_i]|}{n_i}.$$

For infinitely many i, the set J_i is the same, which we denote J. By taking an appropriate subsequence of $n_1, n_2, \ldots$, we can suppose without loss of generality that $J_i = J$ for all i and that $n_{i+1}/n_i \to \infty$. Let $\mathcal{F}_j$ is the union of components from W contained in some S_j with $j \in J$, which contain a vertex from $[n_i]$ but no vertex from $[n_{i-1}]$. Let $\mathcal{I} = \{I_i, I_2, \ldots\}$ be the family of doubly independent sets. We can suppose that the elements in $\mathcal{I}$ are such that the sets $I_i \cup N(I_i)$ are pairwise disjoint. Indeed, because H is locally finite, each $I_i \cup N(I_i)$ intersects finitely many sets $I_j \cup N(I_j)$, so we can find an infinite subfamily $\mathcal{I}'$ by including into it only the sets I_i such that $I_i \cup N(I_i)$ does not intersect a set $I_j \cup N(I_j)$ for some $j < i$ with $I_j \in \mathcal{I}'$. This does not change the components in which $\mathcal{I}$ is concentrated.

Let $J' \subseteq J$ be the set of indices in j for which S_j is non-empty. We assign to each component $\mathcal{C} \subseteq H$ a number $\kappa(\mathcal{C}) \in J'$, in such a way that for every $j \in J'$ there are infinitely many sets I_i in components with $\kappa(\mathcal{C}) = j$. Indeed, if finitely many components intersect $\mathcal{I}$, there are at least b components that contain infinitely many elements of $\mathcal{I}$, so give different values of $\kappa(\mathcal{C})$ to $|J'| \leq b$ of them, whereas if there are infinitely many components that intersect $\mathcal{I}$, we can assign each value of J' to infinitely many of them. The purpose of $\kappa(\mathcal{C})$ will be to identify the shade of red to be used in the vertices while embedding $\mathcal{C}$ in the red edges of χ.

We will define an injective graph homomorphism $\Phi : H \to K_{\mathbb{N}}$ which maps edges to red edges, and whose image contains $\mathcal{F}_i$ for infinitely many i. This is enough to prove Theorem 3.2, because for infinitely many large enough values of i we have

$$\begin{aligned}\frac{|\Phi(V(H)) \cap [n_i]|}{n_i} \geq & \frac{|V(\mathcal{F}_i) \cap [n_i]|}{n_i} \geq \frac{b}{a'}\frac{|V(W) \cap [n_i]|}{n_i} - \frac{(r+s)n_{i-1}}{n_i} \\ \geq & \frac{b}{a'}\bar{d}(W) - o(1).\end{aligned}$$

We will define Φ in steps. On every step, we will define the image of finitely many vertices of H. After every step, the following conditions must hold. Let u, v be two adjacent vertices in some component $\mathcal{C}$ of H, such that $\Phi(v)$ is defined and $\Phi(u)$ is not. Then:

- If $\kappa(\mathcal{C}) \neq a$, then $\Phi(v) \in R_{\kappa(\mathcal{C})}$.
- If $\kappa(\mathcal{C}) = a$ and $\Psi(v) = a$, then $\Phi(v) \in R_a$.
- If $\kappa(\mathcal{C}) = a$ and $\Psi(v) \neq a$, then $\Phi(v) \in B_{\Psi(v)}$ and $\Psi(u) < \Psi(v)$.

The algorithm will consist of two operations that alternate: defining the image of a vertex $v \in V(H)$ and adding some $\mathcal{F}_i$ to the image of Φ. If we identify $V(H)$ with $\mathbb{N}$, and always apply the first operation to the least vertex v with undefined $\Phi(v)$, at the end of the algorithm $\Phi(v)$ will be defined for every vertex in $V(H)$.

Define the image of a vertex $v \in V(H)$: Suppose first that $v \in \mathcal{C}$ and $\kappa(\mathcal{C}) = k \neq a$. Let $w_1, \dots, w_q$ be the neighbors of v which have $\Phi(w_i)$ defined. By our invariant, the vertices $\Phi(w_i)$ all have shade R_k, and therefore there are infinitely many vertices x in shade R_k which are connected to every $\Phi(w_i)$ through a red edge. Select one such x which is not yet in the image of Φ, and set $\Phi(v) = x$.

Now suppose that $\kappa(\mathcal{C}) = a$. Let $\Psi(v) = k$. For every edge uw of H, define an orientation $\overrightarrow{uw}$ such that $\Psi(u) < \Psi(w)$. Let T be the set of vertices that can be reached from v in this orientation. Because T is connected, does not contain a path of length greater than a, and the degree of every vertex is finite, by König's lemma T is finite. Also, T does not have an oriented cycle. Observe that, by our invariant, if $\overrightarrow{uw}$ is an edge and $\Phi(u)$ is defined, then $\Phi(w)$ is defined.

Now define $\Phi(w)$ for every $w \in T$ for which the image is still undefined, in decreasing order of $\Psi(w)$. If $\Psi(w) = a$, choose an arbitrary vertex $x \in R_a$ which is not yet the image of any vertex and set $\Phi(w) = x$. If $\Psi(w) = k < a$, then for every $w' \in N^+(w)$ the image $\Phi(w')$ is defined and in $B_{k+1} \cup \cdots \cup B_{a-1} \cup R_a$. By the properties of a-good colorings, there are infinitely many vertices[3] $x \in B_k$ which are connected to every $\Phi(w')$ through a red edge. Choose one which is not yet in the image of Φ, and set $\Phi(w) = x$.

Add some set $\mathcal{F}_i$ to the image: Select some $\mathcal{F}_i$ which is so far disjoint with the image of Φ. For each $K^R_{r,s}$ component $Z \subseteq \mathcal{F}_i \cap S_j$, choose a doubly independent set $I \subseteq V(H)$ in a component $\mathcal{C}$ with $\kappa(\mathcal{C}) = j$, such that no vertex from $I \cup N(I)$ has a defined image. If $V(Z) = X \cup Y$ with $|X| = r$ blue and $|Y| = s$ red, then set Φ to be bijective from I to X, and injective from $N(I)$ to Y. The vertices v of $\mathcal{F}_i \cap S_j$ that remain outside of the image at this point all have shade R_j. For each of them, choose a vertex w with $\Psi(w) = a$ in a component $\mathcal{C}$ with $\kappa(\mathcal{C}) = j$ (there are infinitely many of these vertices), whose image is not yet defined, and set $\Phi(w) = v$. After doing this for every vertex in $\mathcal{F}_i \cap S_j$ for every $j \in J'$, the set $\mathcal{F}_i$ is contained in the image.

After both steps are applied alternatingly infinitely many times, the image of Φ is a monochromatic red graph $H' \subseteq K_{\mathbb{N}}$ which contains infinitely many sets $\mathcal{F}_i$, and therefore $\bar{d}(H') \geq b/a'\bar{d}(W)$. □

To prove the upper bound of Theorem 1.7(ii)b, we just need the following variant of Lemma 3.6. The proof is then analogous to the proof of Theorem 3.2, except we bypass completely the use of Lemma 3.5, and we only need that in the a-good coloring we have $\max\{\bar{d}(R), \bar{d}(B)\} \geq 1/2$.

Lemma 3.12. *Let $\chi : E(K_{\mathbb{N}}) \to \{R, B\}$ be an edge-coloring, let $a \geq a' \geq b$ be positive integers. Let $\mathbb{N} \to \{R_1, \ldots, R_a, B_1, \ldots, B_a, X\}$ be an a-good coloring in which at most a' shades of each color are non-empty. Let $C \in \{R, B\}$. Let H be a graph with chromatic number a and at least b infinite components. Then there exists a monochromatic $H' \subseteq K_{\mathbb{N}}$ of color C, $H' \simeq H$, with $\bar{d}(H') \geq b/a'\bar{d}(C)$.*

Proof. Let $\Psi : V(H) \to [a]$ be a proper coloring, in which in every component of H the most common color is a. Without loss of generality suppose that C is red. As in the

[3] If $N^+(w)$ is empty, how do we know that B_k is infinite? Because $\kappa(\mathcal{C}) = a$, we know that R_a is not empty. By the properties of a-good colorings, every vertex $y \in R_a$ has infinitely many red neighbors in B_k, and in particular B_k is infinite.

proof of Lemma 3.6, there exists J' with $|J'| \leq b$, such that $\bar{d}(\cup_{j\in J'} R_j) \geq b/a'$, and all R_j with $J \in J'$ are infinite. Let $\mathcal{F} = \cup_{j\in J'} R_j$. Define a function κ from the components of H to J' for which the pre-image of every value contains infinitely many vertices. The algorithm now alternates between **define the image of a vertex** $v \in V(H)$, as above, and **add a vertex of** $\mathcal{F}$ **to the image**. At the end of the procedure, we obtain a red $H' \subseteq K_{\mathbb{N}}$ isomorphic to H which contains $\mathcal{F}$, and thus has density at least $\bar{d}(\mathcal{F}) \geq b/a'\bar{d}(R)$.

Add a vertex of $\mathcal{F}$ to the image: Let $v \in \mathcal{F}$ be a vertex in R_j. Choose a vertex w in a component $\mathcal{C}$ with $\kappa(\mathcal{C}) = j$, such that no vertex in $w \cup N(w)$ has a defined image and with $\Psi(w) = a$, and set $\Phi(w) = v$. □

3.3 Bounds for particular families of graphs

The goal of this section is to prove the remaining results from Section 1.2.1 and Section 1.2.2.

We will start with the proof of Theorem 1.10, which will later imply Theorem 1.11 and, in turn, this will imply Corollary 1.12, Corollary 1.13 and Corollary 1.14.

Proof of Theorem 1.10. The upper bound follows from Theorem 1.9. We will show that, for every $\epsilon > 0$, we have $\rho(H) \geq f(\lambda) - \epsilon$. Our goal is to show that H satisfies the condition of Theorem 3.2 for $a = 2$, $b = 1$, a certain coloring Ψ and some doubly independent sets I'_i. Let $\Psi : V(H) \to \{1, 2\}$ be a proper coloring. Choose $\lambda' > \lambda$ such that $f(\lambda') > f(\lambda) - \epsilon$ (it exists by continuity of f). There exist infinitely many pairwise disjoint independent sets I_i, all with the same size, such that $\frac{|N(I_i)|}{|I_i|} \leq \lambda'$ (by the condition from the statement). Partition each set I_i into non-empty sets $I_{i,1}, \dots, I_{i,k_i}$, where each vertex v is classified according to its color by Ψ and the component it belongs to. If two vertices v, w have a common neighbor, then they are in the same component and $\Psi(u) = \Psi(v)$. For this reason, $|N(I_i)| = \sum_{j=1}^{k_i} |N(I_{i,j})|$. There exists some τ_i such that

$$\frac{|N(I_{i,\tau_i})|}{|I_{i,\tau_i}|} \leq \frac{\sum_{j=1}^{k_i} |N(I_{i,j})|}{\sum_{j=1}^{k_i} |I_{i,j}|} = \frac{|N(I_i)|}{|I_i|} \leq \lambda'$$

Set $I'_i = I_{i,\tau_i}$. Set $r_i = |I'_i|$ and $s_i = |N(I'_i)|$. There is a pair (r, s) satisfying $(r, s) = (r_i, s_i)$ for infinitely many values of i. Considering only the values of i for which this equality holds, we have our set of independent sets. Note that, because H is bipartite, $N(I'_i)$ is monochromatic and thus independent, meaning that I'_i is doubly independent. If $\Psi(I'_i) = 2$ does not hold for infinitely many i, replace Ψ with $\bar{\Psi} = 3 - \Psi$ (the opposite two-coloring). We can now apply Theorem 3.2 to obtain $\rho(H) \geq f(s/r) \geq f(\lambda')$. □

To deduce Theorem 1.11 from Theorem 1.10, we need to show that, in both graphs with infinite orbits and forests, the condition in the statement of Theorem 1.10 holds.

Proof of Theorem 1.11. Let $\lambda = \liminf_{n\to\infty} \frac{\mu(H,n)}{n}$. Fix $\lambda' > \lambda$. We will show that, in both cases, there exist infinitely many pairwise disjoint independent sets $I_1, I_2, \ldots$, all with the same size, with $\frac{|N(I_i)|}{|I_i|} \leq \lambda'$.

For graphs with infinite orbits: Choose n such that $\frac{\mu(H,n)}{n} < \lambda'$. Let I be an independent set of size n with $|N(I)| = \mu(H,n)$. We will show that there exists an infinitely family of automorphisms $\sigma_i \in \mathrm{Aut}(H)$ such that the sets $\sigma_i(I)$ are pairwise disjoint. Then we can take $I_i = \sigma_i(I)$ to conlude the proof. We proceed by induction on n. For $n = 1$, if $I = \{v\}$, this is equivalent to the orbit of v being infinite.

Suppose that the result is true for $n-1$. Suppose that we have already found $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that the sets $\sigma_i(I)$ are pairwise disjoint. Let $X = \cup_{i=1}^{k} \sigma_i(I)$. We will construct $\sigma_{k+1} \in \mathrm{Aut}(H)$ such that $\sigma_{k+1}(I)$ is disjoint from X. Choose $v \in I$. By the induction hypothesis, there is an infinite family $\{\tau_i\}_{i=1}^{\infty} \subseteq \mathrm{Aut}(H)$ such that the sets $\tau_i(I - v)$ are pairwise disjoint We can suppose that the sets $\tau_i(I - v)$ are also disjoint with X. If $\tau_i(v) \notin X$ for some i, then we can take $\sigma_{k+1} = \tau_i$, and we are done. Therefore, assume that $\tau_i(v) \in X$ for every i. By pigeonhole principle, there exists w such that $\tau_i(v) = w$ for infinitely many i. Choose $\phi \in \mathrm{Aut(H)}$ such that $\phi(w) \notin X$ (it exists because the orbit of w is infinite). The set $\phi^{-1}(X)$ intersects finitely many sets $\tau_i(I - v)$, therefore there exists some i with $\tau_i(I - v)$ disjoint from $\phi^{-1}(X)$ and $\tau_i(v) = w$. Putting this together, $\phi(\tau_i(I))$ is disjoint from X, as we wanted.

For forests: The following lemma will be used to find independent sets of bounded size with bounded expansion within larger independent sets:

Lemma 3.13. *For every $\lambda' > \lambda$ there exists $M = M(\lambda, \lambda')$ such that, for every independent set I in a forest with $|N(I)| \leq \lambda |I|$, there exists $I' \subseteq I$ with $|N(I')| \leq \lambda' |I'|$ and $|I'| \leq M$.*

Knowing this lemma, for every $\lambda' > \lambda$ choose $\lambda'' < \lambda''' \in (\lambda, \lambda')$, and set $M = M(\lambda''', \lambda')$. Suppose that we have already constructed pairwise disjoint independent sets $I_1, I_2, \ldots, I_k$ with $|I_i| \leq M$ and $|N(I_i)| \leq \lambda' |I_i|$. We will find a new set I_{k+1}, disjoint from the others. Let $S = \cup_{i=1}^{k} I_i$. There exists n large enough such that $\frac{n}{n-|S|} \leq \frac{\lambda'''}{\lambda''}$. By definition of $\liminf$ and $\mu(H,n)$, there exists an independent set I with $|I| \geq n$ and $|N(I)| \leq \lambda'' |I|$. Then

$$|N(I \setminus S)| \leq |N(I)| \leq \lambda'' |I| \leq \lambda'' (|I \setminus S| + |S|) \leq \lambda''' |I \setminus S|$$

By our claim, there exists $I_{i+1} \subseteq I \setminus S$ such that $|I_{k+1}| \leq M$ and $|N(I_{k+1})| \leq \lambda'|I_{k+1}|$. Once we have constructed an infinite family of independent sets $I_1, I_2, \dots$, simply take a pair (r, s) which is equal to $(|I_i|, |N(I_i)|)$ for infinitely many i (which is possible because this pair can only take finitely many values), and we are done. $\square$

Proof of Lemma 3.13. Let $\delta = \delta(\lambda, \lambda') > 0$ be small enough, which we will fix later. Let F be the forest with vertex set $I \cup N(I)$ and containing only the edges between I and $N(I)$ in our original graph (in other words, do not include the edges between two vertices in $N(I)$). It is enough to prove our result in F. Denote $J = N(I)$. For every component of F take a vertex of I as the root.

There exists a set $S \subseteq V(F)$ with $|S| \leq \delta|V(F)|$, satisfying that every component of $F \setminus S$ has size at most δ^{-1}. Indeed, start with $S = \emptyset$ and consider the set U of vertices whose component in $F \setminus S$ contains at least δ^{-1} vertices (this set will be constantly updated). The rooted forest structure in F induces a rooted forest structure in $F \setminus S$. Let U' be the set of vertices in $V \setminus F$ which have at least $\delta^{-1} - 1$ descendants. If $U \neq \emptyset$ then $U' \neq \emptyset$, because the root of the largest component will be in U'. Select a minimal vertex v in U', and add it to S. This removes v and all its descendants from U, and thus reduces the size of U by at least δ^{-1}. After repeating this procedure at most $\delta|V(F)|$ times, U will be empty, so every component in $|F \setminus S|$ has size at most δ^{-1}.

Let X be the union of S and the parents of the vertices of $S \cap J$. This set has $|X| \leq 2|S| \leq 2\delta|V(F)|$, and every component of $F \setminus X$ is adjacent to at most one vertex in $X \cap J$, in which case it is the parent of the root. As a consequence, every component of $F \setminus (X \cap I)$ contains at most one vertex from $X \cap J$.

Let $\mathcal{C} = \{C_1, \dots, C_k\}$ be the components of $F \setminus (X \cap I)$. Then

$$\frac{\sum_{j=1}^k |C_i \cap J|}{\sum_{j=1}^k |C_i \cap I|} = \frac{|J|}{|I| - |X \cap I|} \leq \frac{|N(I)|}{|I| - 2\delta(|I| + |N(I)|)} \leq \frac{\lambda}{1 - 2\delta(1+\lambda)} =: \lambda''.$$

There exists some component C_i such that $|C_i \cap J| \leq \lambda''|C_i \cap I|$. If $C_i \cap I$ has size not greater than $M := 2\delta^{-1}$, then set $I' = C_i \cap I$ and we are done, because $N(I') \subseteq C_i \cap J$. Otherwise C_i has size greater than $2\delta^{-1}$, hence it must contain a (unique) vertex $v \in X \cap J$. Let $C'_1, C'_2, \dots, C'_r$ be the components obtained from C_i by removing v, labeled in decreasing order of $|C'_j \cap J|/|C'_j \cap I|$. Consider the minimum integer t such that $\sum_{j=1}^t |C'_j \cap I| \geq \delta^{-1}$. Because every component in $F \setminus X$ has size at most δ^{-1}, we have $\sum_{j=1}^t |C'_j \cap I| \leq \sum_{j=1}^{t-1} |C'_j \cap I| + \delta^{-1} \leq 2\delta^{-1} = M$. Set $I' = \cup_{j=1}^t (C'_i \cap I)$. Then

$$\frac{|N(I')|}{|I'|} = \frac{1 + \sum_{j=1}^t |C'_j \cap J|}{\sum_{j=1}^t |C'_j \cap I|} \leq \delta + \frac{\sum_{j=1}^r |C'_j \cap J|}{\sum_{j=1}^r |C'_j \cap I|} \leq \delta + \lambda''.$$

This proves Lemma 3.13, for $\delta > 0$ small enough such that $\delta + \lambda'' < \lambda'$. □

Next we will prove Corollaries 1.12 to 1.14 as direct applications of Theorem 1.11:

Proof of Corollary 1.12. We will show that $\mu(T_k, n) = kn$. For every independent set I of size n, the set of children of the vertices of I has size kn and is contained in $N(I)$, thus $|N(I)| \geq kn$. Equality can hold, for example for $I = \{v_1, \dots, v_n\}$ where v_1 is the root of T_k and v_{i+1} is a grandchild of v_i. We therefore have $\mu(T_k, n) = kn$. Since T_k is a forest, Theorem 1.11 applies and $\rho(T_k) = f(k)$. □

Proof of Corollary 1.13. Let I be an independent set. The set $I + (1, 0, 0, \dots)$ is contained in $N(I)$, so $|N(I)| \geq |I|$ and $\mu(\mathrm{Grid}_d, n) \geq n$ for all n. On the other hand, let I_k be the set of vertices in $[2k]^d$ with odd sum of coordinates. I_k is an independent set of size $(2k)^d/2$, and $I \cup N(I)$ is contained in $[2k+2]^d$. Since I and $N(I)$ are disjoint,

$$\frac{|N(I)|}{|I|} = \frac{|I \cup N(I)|}{|I|} - 1 \leq \frac{(2k+2)^d}{(2k)^d/2} - 1,$$

which tends to 1 as $k \to \infty$. We have $\liminf\limits_{n \to \infty} \frac{\mu(\mathrm{Grid_d}, n)}{n} = 1$. The graph Grid_d is vertex-transitive, so by Theorem 1.11 we have $\rho(\mathrm{Grid}_d) = f(1)$. □

Proof of Theorem 1.14. The graph $\omega \cdot F$ satisfies that every orbit of the automorphism group on $V(\omega \cdot F)$ is infinite (because it intersects every copy of F), so we are in the setting of Theorem 1.11. We need to show that $\liminf\limits_{n \to \infty} \frac{\mu(\omega \cdot F, n)}{n} = \min\limits_{I \subseteq V(F) \text{ indep.}} \frac{|N(I)|}{|I|}$.

Let I be an independent set in F that minimizes $\frac{|N(I)|}{|I|}$, and let $J \subseteq V(\omega \cdot F)$ be an independent sets of size n. Partition J into independent sets $J_1, J_2, \dots, J_m$, according to the component in which the vertices are contained. Then

$$\frac{|N(J)|}{|J|} = \frac{\sum_{i=1}^m |N(J_i)|}{\sum_{i=1}^m |J_i|} \geq \min \frac{|N(J_i)|}{|J_i|} \geq \frac{|N(I)|}{|I|}.$$

This implies that $\frac{\mu(\omega \cdot F, n)}{n} \geq \frac{|N(I)|}{|I|}$ for all n. Equality holds infinitely many times, since for all n divisible by $|I|$ we can take the union of the sets I in $\frac{n}{|I|}$ different copies of F. Therefore $\rho(\omega \cdot F) = \limsup\limits_{n \to \infty} \frac{\mu(\omega \cdot F, n)}{n} = \frac{|N(I)|}{|I|}$.

In an even cycle C_{2k}, each independent set I satisfies $|N(I)| \geq |I|$, because C_{2k} contains a perfect matching. Since each chromatic class in the bipartition satisfies $|N(I)| = |I|$, we have $\rho(\omega \cdot C_{2k}) = f(1)$. For $1 \leq a \leq b$, in $K_{a,b}$, every independent set I has size at most b, and its neighborhood has size at least a, and both inequalities are tight if I is the side of the bipartition with size b. Thus $\frac{|N(I)|}{|I|} \geq \frac{a}{b}$, and $\rho(\omega \cdot K_{a,b}) = f(a/b)$. □

Next we will deduce Theorem 1.15 from Theorem 3.2:

Proof of Theorem 1.15. Let $a = |V(F)|$, and let $b = a - 1$. Let $\Psi : V(F) \to [a]$ be a coloring that assigns the value a to every vertex in $N(I)$, and where the remaining vertices in F all get different values in $[a-1]$. Because I is doubly independent, this is a proper coloring. Ψ extends to a coloring of $\omega \cdot F$, by coloring all copies of F equally.

Let $I_1, I_2, \ldots$ be the sets I of all copies of F. Each I_i is contained in a component of F, $\Psi(N(I_i)) = a$ and the family of sets I_i is not concentrated in fewer than b components. Thus, by Theorem 3.2, setting $r = |I|$ and $s = |N(I)|$, we have $\rho(\omega \cdot F) \geq f\left(\frac{|N(I)|}{|I|}\right)$. ☐

Finally, we will prove Theorem 1.16 using a result of Burr, Erdős and Spencer [BES75] for the Ramsey number of $n \cdot F$:

Theorem 3.14 ([BES75]). *Let F_1, F_2 be two finite graphs without isolated vertices. The two-color Ramsey number $R(n \cdot F_1, n \cdot F_2)$ satisfies*

$$R(n \cdot F_1, n \cdot F_2) = (|V(F_1)| + |V(F_2)| - \min\{\alpha(F_1), \alpha(F_2)\})n + O(1),$$

where $\alpha(G)$ is the size of the largest independent set in G. In particular, $R(n \cdot F, n \cdot F) = (2|V(F)| - \alpha(F))n + O(1)$.

Proof of Theorem 1.16. Let $\chi : E(K_{\mathbb{N}}) \to \{R.B\}$ be a coloring. Let $n_1, n_2, \ldots$ be an increasing sequence of positive integers with $n_{i+1}/n_i \to \infty$. Let k_i be the maximum value such that $R(k_i \cdot F, k_i \cdot F) \leq n_{i+1} - n_i$. By Theorem 3.14, we have

$$k_i = \left(\frac{1}{2|V(F)| - \alpha(F)} + o(1)\right)(n_{i+1} - n_i) = \left(\frac{1}{2|V(F)| - \alpha(F)} + o(1)\right) n_{i+1}.$$

There exist a family $\mathcal{F}_i$ of k_i monochromatic disjoint copies of F with vertices in $[n_i + 1, n_{n+1}]$, all with the same color C_i. Choose a color C which is equal to C_i for infinitely many i. Then $H' = \bigcup_{C_i = C} \mathcal{F}_i$ is a copy of $\omega \cdot F$ with

$$\limsup_{n\to\infty} \frac{|V(H) \cap [n]|}{n} \geq \limsup_{i: C_i = C} \frac{k_i |V(F)|}{n_{i+1}} = \frac{|V(F)|}{2|V(F)| - \alpha(F)}.$$ ☐

Chapter 4

Ramsey density of infinite factors

In this chapter we will prove the results from Section 1.2.3. The main theorems here are Theorem 1.17 and Theorem 1.19, with Corollary 1.18 following from the former.

The results in this chapter represent joint work with József Balogh, and are adapted from [BL20].

4.1 Proof of Theorem 1.17

The proof of Lemma 1.17 uses the same overarching outline as Theorem 1.15: given an edge coloring of $K_{\mathbb{N}}$ we color the vertices of the graph while encoding some information, we find a smaller substructure in the resulting coloring and finally use the information in the vertices to extend this substructure into our desired graph.

The main difference comes from the choice of coloring. While the proof of Theorem 1.15 uses a coloring due to Elekes, Soukup, Soukup and Szentmiklóssy [ESSS17], here instead we will use a coloring that arises from a straightforward application of Ramsey's theorem. The reason for this change is that the coloring from [ESSS17] is good at finding "connections at infinity", allowing one to add infinitely many vertices to a component with little restriction as to how they are joined. Because the components of $\omega \cdot F$ all have bounded size, we do not need to worry about connecting more and more vertices into a component. Instead, we will be able to create each component of $\omega \cdot F$ in one or two steps.

The intermediate step of finding a smaller substructure is essentially the same. We will use the following lemma:

Lemma 4.1. *Let r and s be positive integers. For every $\epsilon > 0$ there exist $\tau > 0$, N with the following property: for every graph G on vertex set $[n]$, with $n > N$, such that $\delta(G) > (1-\tau)n$, in every coloring $\Psi : V(G) \cup E(G) \to \{R, B\}$ there exists a subgraph $F \subseteq G$, a color $C \in \{R, B\}$ and a value $k \in [\delta n, n]$ such that $|V(F) \cap [k]| \geq (f(s/r) - \epsilon)k$ and every component of F is of one of the following types:*

- *An isolated vertex of color C.*
- *A copy of $K_{r,s}$ in which the edges and the side of size s have color C, and the side of size r has the opposite color.*

This is a finitary version of Lemma 3.5. The same proof that was used there works here.

The next lemma will require the following definition, which will be the key information that will be encoded into our vertex coloring:

Definition 4.2. *Given a set X of vertices in an edge-colored graph, a real number $\epsilon > 0$, a natural number s and a color $C \in \{R, B\}$, we say that X is (C, ϵ, s)-adequate if every subset of X of size at least $\epsilon|X|$ contains a C-colored clique of size s.*

Observe that in particular, if a set X is (C, ϵ, s)-adequate, then it contains a disjoint family of C-colored copies of K_s covering at least $(1-\epsilon)|X|$ vertices. This is proved by considering a maximal family of such cliques.

Lemma 4.3. *For every $\epsilon > 0$ and $s \in \mathbb{N}$ there exist $T, N \in \mathbb{N}$ with the following property: for every $\Psi : E(K_n) \to \{R, B\}$ with $n > N$ there exists a partition $V(K_n) = V_1 \cup V_2 \cup \cdots \cup V_T$ into almost equal parts, and colors $C_1, \ldots, C_T \in \{R, B\}$ in which all but ϵT sets V_i are (C_i, ϵ, s)-adequate.*

Proof. Let $\alpha = \lceil 2\epsilon^{-1} \rceil s$ and $q = R(K_\alpha, K_\alpha)$. Take a maximal family $\mathcal{F}$ of disjoint monochromatic copies of K_α ($\mathcal{F}$ might contain cliques of different colors). The vertex set $V(K_n) \setminus V(\mathcal{F})$ does not contain any monochromatic K_α, so its size is at most q.

Let $T = \lceil 3\epsilon^{-1} \rceil$. Each of the sets $V_1, \ldots, V_T$ will have size either $\lfloor n/T \rfloor$ or $\lceil n/T \rceil$. Let $\beta = \lfloor \lfloor n/T \rfloor / \alpha \rfloor$, this is the number of α-cliques that fit into a set V_i (of the smaller type). Let $\mathcal{F}_R$ and $\mathcal{F}_B$ be the sets of red and blue cliques from $\mathcal{F}$, respectively.

Create $V_1, V_2, \ldots, V_T$, all initially empty. For $\lfloor |\mathcal{F}_R|/\beta \rfloor + \lfloor |\mathcal{F}_B|/\beta \rfloor$ of these sets, put β cliques from $\mathcal{F}$ of the same color into each. We call these sets pseudo-adequate, and we will later show that they are indeed adequate. Distribute the remaining vertices from K_n into the sets V_i so that the resulting sets are almost equal. The number of pseudo-adequate sets is

$$\left\lfloor \frac{|\mathcal{F}_R|}{\beta} \right\rfloor + \left\lfloor \frac{|\mathcal{F}_B|}{\beta} \right\rfloor \geq \left\lfloor \frac{|\mathcal{F}|}{\beta} \right\rfloor - 1 \geq \frac{\mathcal{F}}{\beta} - 2 \geq \frac{\frac{n-q}{\alpha}}{\beta} - 2 \geq T - \frac{q}{\alpha\beta} - 2 \geq T - 3$$

if $\beta \geq q$ (this will be the case if $n > q\alpha T$). That means that there are at most three sets which are not pseudo-adequate, which represents less than ϵT of the sets.

For each pseudo-adequate set V_i, we let C_i be the color of the cliques from $\mathcal{F}$ that were put into it. For the (up to three) remaining sets V_i, choose C_i arbitrarily. If V_i is pseudo-adequate, there are at most α vertices not in a copy of K_α from $\mathcal{F}$.

In a pseudo-adequate set V_i, take a subset $S \subseteq V_i$ of size at least $\epsilon|V_i|$. There are at least $\epsilon|V_i| - \alpha$ vertices in S which belong to a clique in $\mathcal{F}$, so by the pigeonhole principle S contains at least $\lceil \frac{\epsilon|V_i|-\alpha}{\beta} \rceil$ elements from some clique in $\mathcal{F}$. This is at least

$$\frac{\epsilon|V_i| - \alpha}{\beta} \geq \frac{\epsilon\lfloor \frac{n}{T} \rfloor - \alpha}{\left\lfloor \frac{\lfloor \frac{n}{T} \rfloor}{\alpha} \right\rfloor} \geq \frac{\frac{\epsilon}{2}\lfloor \frac{n}{T} \rfloor}{\frac{\lfloor \frac{n}{T} \rfloor}{\alpha}} = \frac{\alpha\epsilon}{2} \geq s$$

if $\alpha < \frac{\epsilon}{2}\left\lfloor \frac{n}{T} \right\rfloor$. This holds if $n > \lceil 2\epsilon^{-1}\alpha \rceil T$. We conclude that we can take $N = \max\{q\alpha T, \lceil 2\epsilon^{-1}\alpha \rceil T\}$. □

We will prove a finitary version of Theorem 1.17, from which the infinite version follows directly:

Lemma 4.4. *For every finite graph F and every $\epsilon > 0$ there exists N and $\tau > 0$ with the following property: for every $n > N$, for every coloring $\Psi : E(K_n) \to \{R, B\}$ there exists $t \in [\tau n, n]$ and a monochromatic family $\mathcal{F}$ of disjoint copies of F such that*

$$\frac{|V(\mathcal{F}) \cap [t]|}{t} \geq f\left(\frac{|V(F)|}{\alpha(F)} - 1\right) - \epsilon.$$

Proof of Theorem 1.17. Fix $\epsilon > 0$. Let $\Psi : E(K_\mathbb{N}) \to \{R, B\}$ be an edge-coloring. Consider a sequence $n_1 < n_2 < \dots$ of natural numbers with $\lim n_{i+1}/n_i \to \infty$. For each $i \geq 2$, consider the colored clique G_i on vertex set $(n_{i-1}, n_i]$, whose edge-coloring is Ψ_i, the one induced by Ψ. Now consider the graph G_i', obtained by substracting n_{i-1} from the label on each vertex.

By Lemma 4.4, there exists a fixed $\tau > 0$ such that, for every i large enough (enough to have $n_i - n_{i-1} > N$) there is a value $t_i \in [\tau(n_i - n_{i-1}), n_i - n_{i-1}]$, and a monochromatic family $\mathcal{F}_i'$ of disjoint copies of F in G_i' with color C_i, such that

$$\frac{|V(\mathcal{F}_i') \cap [t_i]|}{t_i} \geq f\left(\frac{|V(F)|}{\alpha(F)} - 1\right) - \epsilon.$$

For infinitely many values of i, the color C_i will be the same, which we denote C. Let $\mathcal{F}_i$ be family of copies of F in $K_{\mathbb{N}}$ obtained by adding n_{i-1} to every vertex in $\mathcal{F}_i'$. Then consider $\mathcal{F} = \bigcup_{C_i=C} \mathcal{F}_i$. Clearly $\mathcal{F}$ is a monochromatic copy of $\omega \cdot F$. We will show that it has upper density at least $f\left(|V(F)|/\alpha(F) - 1\right) - \epsilon$.

For every i with $C_i = C$, we have

$$\begin{aligned}\frac{|V(\mathcal{F}) \cap [t_i + n_{i-1}]|}{t_i + n_{i-1}} &\geq \frac{|V(\mathcal{F}_i') \cap [t_i]|}{t_i + n_{i-1}} \geq \left(f\left(\frac{|V(F)|}{\alpha(F)} - 1\right) - \epsilon\right)\frac{t_i}{t_i + n_{i-1}} \\ &\geq \left(f\left(\frac{|V(F)|}{\alpha(F)} - 1\right) - \epsilon\right)\frac{\tau n_i}{\tau n_i + n_{i-1}}.\end{aligned}$$

By taking the upper limit on the latter expression as $i \to \infty$, we conclude that $\bar{d}(\mathcal{F}) \geq f\left(|V(F)|/\alpha(F) - 1\right) - \epsilon$, and thus this value is a lower bound on $\rho(\omega \cdot F)$. Since this is valid for all $\epsilon > 0$, we conclude $\rho(\omega \cdot F) \geq f(|V(F)|/\alpha(F) - 1)$. □

In the proof of Lemma 4.4 we routinely omit rounding signs, except when they are part of a definition. All rounding errors are smaller than the effect that taking stronger constants.

Proof of Lemma 4.4. Let $\gamma = \gamma(\epsilon, F)$ be small enough (we will define how small later). Let $\ell = \lceil \gamma^{-1} \rceil$. Partition $[n]$ into ℓ almost equal intervals, labelled $I_1, I_2, \ldots, I_\ell$ from smallest to largest. Apply the colored variant of the Szemerédi regularity lemma [KS96] to this edge-colored graph K_n to find $M = M(\gamma)$ and a γ-regular partition $\{X_0, X_1, X_2, \ldots, X_m\}$, with $m \leq M$, that refines $\{I_1, \ldots, I_\ell\}$.

Now let $\kappa = \kappa(\epsilon, F)$ be small enough (κ will be chosen before γ, so we may suppose $\gamma \ll \kappa \ll \epsilon$). Lemma 4.3 gives $T, N'(\kappa, s')$, for $s' = |V(F)|$. If the size of all X_i with $i > 0$ is at least N' (which will happen if $n > (1-\gamma)^{-1} m N'$, and in particular is implied by $N > (1-\gamma)^{-1} M N'$) then by Lemma 4.3 we can subdivide each X_i into T parts $X_1^i, \ldots, X_T^i$, as in the statement of that lemma. Because the sizes of the X_i are all almost equal, the Tm sets X_j^i also have almost equal sizes. Color each set X_j^i with the color C_j^i that it receives from Lemma 4.3. Assuming that they are all non-empty (true if $N > (1-\gamma)^{-1} TM$) we can label them $Y_1, \ldots, Y_{Tm}$, with the property that $\min Y_1 < \min Y_2 < \ldots < \min Y_{Tm}$. We denote the color of Y_i by C_i.

If the pair (X_{i_1}, X_{i_2}) is γ-regular, then the pair $(X^{i_1}_{j_1}, X^{i_2}_{j_2})$ is $T\gamma$-regular. Construct an auxiliary colored graph J on $[Tm]$ as follows: the color of a vertex v is the color of Y_v. If the pair (Y_v, Y_w) is $T\gamma$-regular in the original graph, draw an edge vw in J, whose color is the densest color in the bipartite graph (Y_i, Y_j) (ties are broken arbitrarily). Because every X_i is γ-regular with at least $(1-\gamma)m$ other X_j, each Y_i is $T\gamma$-regular with at least $(1-\gamma)Tm$ other Y_j. The minimum degree of J is at least $(1-\gamma)Tm$.

Next we apply Lemma 4.1 to this totally colored graph. This produces a value $k \in [\delta Tm, Tm]$, a color C and a subgraph $J' \subseteq J$ in which every component is either an isolated vertex of color C or a $K_{r,s}$, for $r = \alpha(F)$ and $s = |V(F)| - \alpha(F)$, colored as in the statement of Lemma 4.1. These satisfy

$$\frac{|V(J') \cap [k]|}{k} \geq f\left(\frac{s}{r}\right) - \frac{\epsilon}{2}.$$

This assumes that the minimum degree condition is satisfied, which is true if $\gamma \ll \tau = \tau(\epsilon, F)$. We also have $\delta = \delta(\epsilon, F)$, with these two functions as in Lemma 4.1.

Now we return to our original graph on $[n]$ with a coloring given by Ψ. We will find in it a family $\mathcal{F}$ of vertex-disjoint copies of F, such that every copy is contained in parts Y_i corresponding to a component of J', according to the following restrictions:

- If the corresponding component of J' is an isolated vertex v, then there is no restriction: any copy of F in Y_v can be taken.
- If the corresponding component of J' is a $K_{r,s}$ on vertex classes $\{v_1, \dots, v_r\}$ and $\{v'_1, \dots, v'_s\}$, we only consider the copies of F which contain exactly r vertices in $Y_{v_1} \cup \cdots \cup Y_{v_r}$, and s vertices in $Y_{v'_1} \cup \cdots \cup Y_{v'_s}$.

Under these restrictions we consider an inclusion-maximal such family $\mathcal{F}$. Let $t = \min Y_k$. We will show that $\frac{|V(\mathcal{F}) \cap [t]|}{t}$ is large enough for our purposes.

When each set X_i is split into T sets $X_i^1, \dots, X_i^T$, there are at most κT of them which cannot be almost-partitioned into cliques $K_{s'}$ of the corresponding color. In these sets X_i^j, assuming they are isolated vertices in J, the family $\mathcal{F}$ will use at least $(1-\kappa)|X_i^j|$ vertices in each of them.

On the other hand, consider a component in J which is a copy of $K_{r,s}$ on vertex classes $\{v_1, \dots, v_r\}$ and $\{v'_1, \dots, v'_s\}$. There are at most κTm of these components which contain a set which is not adequate, in the sense of Lemma 4.3. We claim that, if all of the sets are adequate, then $\mathcal{F}$ contains at least a $1-\xi$ proportion of the vertices in $Y_{v_1} \cup \cdots \cup Y_{v_r} \cup Y_{v'_1} \cup \cdots \cup Y_{v'_s}$, for some ξ that will be defined later. Indeed, suppose that fewer than a

$1-\xi$ proportion of vertices is contained in $\mathcal{F}$. Then there are two sets, w.l.o.g. Y_{v_1} and $Y_{v_1'}$, such that $\frac{|Y_{v_1}\cap V(\mathcal{F})|}{|Y_{v_1}|}, \frac{|Y_{v_1'}\cap V(\mathcal{F})|}{|Y_{v_1'}|} < 1-\xi$. Let $W = Y_{v_1} \setminus V(\mathcal{F})$ and $W' = Y_{v_1'} \setminus V(\mathcal{F})$. Because the pair $(Y_{v_1}, Y_{v_1'})$ is $T\gamma$-regular, and has density at least $\frac{1}{2}$ in color C, the bipartite graph (W, W') has density at least $\frac{1}{2} - T\gamma > \frac{1}{4}$ in color C.

Next we will select $w_1, w_2, \dots, w_r$, which will be the vertices of our copy of F in Y_{v_1}. Select $w_1 \in W$ that maximizes the number of C-neighbors in W'. Then let $w_2 \in W\setminus\{w_1\}$ that maximizes the size of its C-neighborhood in the C-neighborhood of w_1 in W'. We proceed this way, each time picking the vertex that maximizes the common neighborhood with the previous choices. As long as $\xi 4^{-r} > T\gamma$ (which will be true because $\gamma \ll \xi, T^{-1}$), we can use regularity to ensure that the density of the C-colored edges between the common neighborhood and W is at least $\frac{1}{4}$.

Let Z be the common C-neighborhood of $w_1, w_2, \dots, w_r$ in W'. It has size at least $4^{-r}|W'| \geq \xi 4^{-r}|Y_{v_1'}|$. If $\xi 4^{-r} > \kappa$ (which again holds because $\kappa \ll \xi$), by the partition of Lemma 4.3 there is a clique of size $s' = r+s$ of color C in Z. We can take $w_1', w_2', \dots, w_s'$ in this clique as the remaining vertices of F. We have thus found a new copy of F, which contradicts the maximality of $\mathcal{F}$. This completes the proof that $\mathcal{F}$ contains at least a $1-\xi$ proportion of the vertices in $Y_{v_1} \cup \cdots \cup Y_{v_r} \cup Y_{v_1'} \cup \cdots \cup Y_{v_s'}$.

We are ready to estimate $\frac{|V(\mathcal{F})\cap[t]|}{t} = 1 - \frac{|[t]\setminus V(\mathcal{F})|}{t}$. We will do so by giving a lower bound on t, and an upper bound on the vertices of $[t]$ which do not belong to $\mathcal{F}$.

Since each set Y_i satisfies $\max Y_i - \min Y_i \leq \ell^{-1}n \leq \gamma n$ (because they are contained in some interval I_j), from $t = \min Y_k$ we have $Y_1 \cup \cdots \cup Y_k \subseteq [t+\gamma n]$. It is also true that $|Y_1 \cup \cdots \cup Y_{Tm}| = |[n] \setminus X_0| \geq (1-\gamma)n$. Since the $|Y_i|$ are almost equal, we have $|Y_i| \geq \frac{(1-\gamma)n}{Tm}$, and

$$t = |[t+\gamma n]| - \gamma n \geq k|Y_i| - \gamma n \geq \frac{(1-\gamma)n}{Tm}k - \gamma n.$$

Next we will estimate $|[t] \setminus V(\mathcal{F})|$. Every vertex z which belongs to $[t]$ but not to $V(\mathcal{F})$ is in at least one of the following classes:

- They are in X_0.
- They are in a class Y_v with $v \notin J'$.
- They are in a class Y_v, where v is an isolated vertex in J', but Y_v is not adequate as in Lemma 4.3.
- They are in a class Y_v, where v is an isolated vertex in J', and Y_v is adequate, but z is not in $\mathcal{F}$.

- They are in a class Y_v with v belonging to a $K_{r,s}$ in J', where some sets $Y_{v'}$ corresponding to a vertex of this component is not adequate.

- They are in a class Y_v with v belonging to a $K_{r,s}$ in J', and all sets $Y_{v'}$ corresponding to vertices of this component are adequate, but z is not in $\mathcal{F}$.

The sum of the number of vertices in each class can be upper-bounded by

$$\gamma n + \left(1 - f\left(\frac{s}{r}\right) + \frac{\epsilon}{2}\right) k \frac{n}{Tm} + \kappa n + \kappa n + (r+s)\kappa n + \xi n,$$

respectively. If ξ, κ and γ are chosen small enough after choosing ϵ, this can be upper-bounded by

$$|[t] \setminus V(\mathcal{F})| \leq \left(1 - f\left(\frac{s}{r}\right) + \frac{2\epsilon}{3}\right) k \frac{n}{Tm}.$$

Combining both bounds, we have

$$\begin{aligned}
\frac{|V(\mathcal{F}) \cap [t]|}{t} &= 1 - \frac{|[t] \setminus V(\mathcal{F})|}{t} \\
&\geq 1 - \frac{\left(1 - f\left(\frac{s}{r}\right) + \frac{2\epsilon}{3}\right) k \frac{n}{Tm}}{\frac{(1-\gamma)n}{Tm} k - \gamma n} \\
&= 1 - \frac{1 - f\left(\frac{s}{r}\right) + \frac{2\epsilon}{3}}{1 - \gamma - \gamma \frac{Tm}{k}} \\
&\geq 1 - \frac{1 - f\left(\frac{s}{r}\right) + \frac{2\epsilon}{3}}{1 - \gamma - \gamma \delta^{-1}} \\
&\geq f\left(\frac{s}{r}\right) - \epsilon
\end{aligned}$$

since $\gamma \ll \delta, \epsilon$. To conclude the proof of Lemma 4.4, we observe that there exists $\tau > 0$ such that $t \geq \tau n$, namely

$$t \geq \frac{(1-\gamma)n}{Tm} k - \gamma n \geq (\delta(1-\gamma) - \gamma)\, n. \qquad \square$$

Proof of Corollary 1.18. By the same argument as in Corollary 1.14, for every finite F we have that $\liminf_{n \to \infty} \frac{\mu(\omega \cdot F, n)}{n} = \min_{I \subseteq V(F) \text{ indep.}} \frac{|N(I)|}{|I|}$. Then, under the hypothesis of Corollary 1.18, the upper bound on $\rho(\omega \cdot F)$ from Theorem 1.9 is equal to $f\left(\frac{|N(I)|}{|I|}\right)$ for an independent set I with size $\alpha(F)$. Because I is a maximum independent set in F, we have $|N(I)| = |V(F)| - \alpha(F)$, and $f\left(\frac{|N(I)|}{|I|}\right) = f\left(\frac{|V(F)|}{\alpha(F)} - 1\right)$, which is the lower bound from Theorem 1.17.

In an odd cycle C_{2k+1}, every independent set I satisfies $|N(I)| \geq |I| + 1$, so $\frac{|N(I)|}{|I|} \geq 1 + \frac{1}{|I|} \geq 1 + \frac{1}{k} = \frac{k+1}{k} = \frac{|V(C_{2k+1})|}{\alpha(C_{2k+1})} - 1$. This means that we can apply the first part of Corollary 1.18 to odd cycles, to obtain $\rho(\omega \cdot C_{2k+1}) = f\left(\frac{k+1}{k}\right)$. For cliques K_k, the only non-empty independent sets are single vertices, so we have $\frac{|N(I)|}{|I|} = k - 1$. This means that $\rho(\omega \cdot K_k) = f(k-1)$. $\square$

4.2 Proof of Theorem 1.19

The proof of Theorem 1.19 borrows ideas from the proof by Burr, Erdős and Spencer [BES75] for the Ramsey number of n disjoint triangles, $R(n \cdot K_3, n \cdot K_3) = 5n$. The following configuration plays a crucial role in both results:

Definition 4.5. *A bowtie is an edge-colored graph on five vertices, formed by a red triangle and a blue triangle sharing a vertex.*

Lemma 4.6. *Let $\Psi : K_n \to \{R, B\}$, let F be the largest monochromatic family of disjoint triangles, and let F' be the largest family of disjoint bowties. Then $3|F| + 2|F'| \geq n - 5$.*

Proof. We start by noting that, if a two-edge-colored K_6 contains a vertex-disjoint red triangle and blue triangle, then it also contains a bowtie. This is because out of the nine edges between both triangles, there are at least five with the same color, w. l. o. g. red. By the pigeonhole principle, there is a vertex v in the blue triangle incident to at least two red edges. Then the original blue triangle plus the two red-neighbors of v form a bowtie.

Now consider the largest family F' of vertex-disjoint bowties. The set of vertices not in F' cannot contain triangles in both colors, by the argument above. Suppose w. l. o. g. that all remaining triangles are red. Let F'' be a maximal family of vertex-disjoint red triangles in $V(K_n) \setminus V(F')$. We know that $|V(F')| + |V(F'')| \geq n - 5$, because the remaining vertices do not contain a monochromatic triangle in either color. Now note that there is a family of disjoint red triangles in K_n of size $|F'| + |F''|$, obtained by taking the triangles in F'' and the red triangles in each bowtie of F'. Thus we get $|F| \geq |F'| + |F''|$, and

$$3|F| + 2|F'| \geq 3(|F'| + |F''|) + 2|F'| = 5|F'| + 3|F''| = |V(F')| + |V(F'')| \geq n - 5,$$

as we wanted to prove. $\square$

The proof of the following lemma is a simple (but slightly cumbersome) case analysis:

Lemma 4.7. *Let W_1, W_2 be two vertex-disjoint bowties in a red-blue edge-coloring of K_{10}. Then either there are two vertex-disjoint triangles of the same color using four vertices of W_1 or there are nine vertices containing two vertex-disjoint red triangles and two vertex-disjoint blue triangles.*

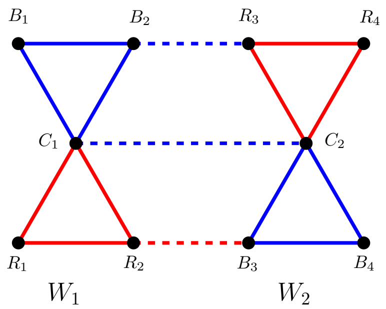

FIGURE 4.1: Labeling of the vertices of W_1 and W_2.

Proof. We label the vertices of W_1 and W_2 as in Figure 4.1. Whenever a target configuration from the statement appears, we will denote it by two or four triangles between square brackets. We suppose that the coloring is such that none of the configurations from the statement appears, and we will reach a contradiction.

Without loss of generality the edge C_1C_2 is blue. At least one of the edges B_2R_3 or B_2R_4 must be blue, because otherwise $[C_1R_1R_2, B_2R_3R_4]$. We will assume w. l. o. g. that B_2R_3 is blue. For the same reason, we assume that R_2B_3 is red. Then the color of some edges is forced:

- C_1R_3 is red, otherwise $[C_1R_1R_2, C_2R_3R_4, C_1B_2R_3, C_2B_3B_4]$.
- C_1B_3 is blue, otherwise $[C_1R_2B_3, C_2R_3R_4, C_1B_1B_2, C_2B_3B_4]$.
- B_1R_3 is red, otherwise $[C_1R_1R_2, C_2R_3R_4, B_1B_2R_3, C_1C_2B_3]$.
- B_1C_2 is blue, otherwise $[B_1C_2R_3, C_1R_1R_2]$.
- B_1R_4 is blue, otherwise $[B_1R_3R_4, C_1R_1R_2]$.
- B_2R_4 is red, otherwise $[C_1R_1R_2, C_2R_3R_4, B_1B_2R_4, C_1C_2B_3]$.
- B_2C_2 is blue, otherwise $[C_1R_1R_2, B_2C_2R_4]$.
- C_1R_4 is red, otherwise $[C_1R_1R_2, C_2R_3R_4, C_1B_1R_4, C_2B_3B_4]$.

At this point, we split our analysis into two cases, depending on the color of R_1B_1. If R_1B_1 is blue:

- R_1B_2 is red, otherwise $[B_1B_2R_1, C_1C_2B_3]$.
- R_1R_4 is red, otherwise $[B_1R_1R_4, C_1C_2B_2]$.
- R_2R_3 is blue, otherwise $[B_2R_1R_4, C_1R_2R_3]$.

But then we reach a contradiction with the color of R_2B_2. If it is red, then $[R_1R_2B_2, C_1R_3R_4]$. If it is blue, then $[R_2B_2R_3, C_1C_2B_1]$.

We suppose now that R_1B_1 is red. Then:

- B_1R_2 is blue, otherwise $[R_1R_2B_1, C_1R_3R_4]$.
- B_2R_2 is red, otherwise $[B_1B_2R_2, C_1C_2B_3]$.
- R_2R_4 is red, otherwise $[B_1R_2R_4, C_1C_2B_2]$.
- R_1R_3 is blue, otherwise $[B_1R_1R_3, C_1R_2R_4]$.

But then we reach a contradiction with the color of R_1B_2. If it is red, then $[R_1R_2B_2, C_1R_3R_4]$. If it is blue, then $[R_1R_3B_2, B_1C_1C_2]$. This concludes the case analysis. $\square$

We now have the necessary preparations for the proof of Theorem 1.19. Similarly to the proof of Theorem 1.17, we will here prove a finitary version. The reason that Theorem 1.19 follows from the lemma below is the same argument by which Theorem 1.17 follows from Lemma 4.4.

Lemma 4.8. *Let $\delta = \frac{4\sqrt{7}+2}{27} = 0.46603\ldots$ and $\gamma = 1 - \frac{1}{\sqrt{7}} = 0.62203\ldots$. For every $\Psi : E(K_n) \to \{R, B\}$ there exists $k \in \{\lfloor \delta n \rfloor, n\}$ and a monochromatic family of disjoint triangles F with $|V(F) \cap [k]| \geq \gamma k - 7$.*

Proof. Let F_1 be a maximum monochromatic family of disjoint triangles, and F_1' be a maximum family of disjoint bowties, both in $[\delta n]$. Let F_2, F_2' be the families defined similarly in $(\delta n, n]$. If $3|F_1| \geq \gamma\delta n - 7$, or $3|F_1| + 3|F_2'| \geq \gamma n - 7$, or $3|F_1'| + 3|F_2| \geq \gamma n - 7$, then we are done. Therefore we assume the opposite.

Suppose first that $|F_1'| \geq |F_2'|$. This leads to a contradiction, because, by Lemma 4.6, $0 < (3|F_1| + 2|F_1'| - \lfloor \delta n \rfloor + 5) + 2(3|F_2| + 2|F_2'| - (1 - \delta)n + 5) + (\gamma\delta n - 7 - 3|F_1|) + 2(\gamma n - 7 - 3|F_1'| - 3|F_2|) + 4(|F_1'| - |F_2'|) < (\gamma\delta + 2\gamma + \delta - 2)n - 5 < 0$. Therefore we will assume $|F_2'| \geq |F_1'|$.

Let $q = |F_1'|$. Let $(W_1^1, W_2^1), (W_1^2, W_2^2), \ldots, (W_1^q, W_2^q)$ be a family of pairs of bowties with $W_1^i \subseteq [\delta n]$ and $W_2^i \subseteq (\delta n, n]$. For every pair (W_1^i, W_2^i) there is either a set of two disjoint triangles of the same color which contain four vertices of W_1^i or nine vertices in $V(W_1^i \cup W_2^i)$ which contain two disjoint triangles of each color.

Let q_1 and q_2 be the number of pairs (W_1^i, W_2^i) for which we obtain the former or the latter, respectively. We have $q_1 + q_2 \geq q$. We can find a monochromatic family of triangles that uses $3(q - \frac{q_1}{2}) + 4\frac{q_1}{2} = 3q + \frac{q_1}{2}$ vertices from $[\delta n]$, and a monochromatic family of triangles that uses $6q_2 + \frac{3}{5}(n - 9q_2) = \frac{3}{5}(n + q_2)$ vertices from $[n]$. Therefore we must assume $3q + \frac{q_1}{2} < \gamma\delta n - 7$ and $\frac{3}{5}(n + q_2) < \gamma n - 7$. But then we have $0 < 7(3|F_1| + 2q - \lfloor \delta n \rfloor + 5) + 7(\gamma\delta n - 7 - 3|F_1|) + 4(\gamma\delta n - 7 - 3q - \frac{q_1}{2}) + \frac{10}{3}(\gamma n - 7 - \frac{3}{5}(n + q_2)) + 2(q_1 + q_2 - q) < (11\gamma\delta + \frac{10}{3}\gamma - 7\delta - 2)n - \frac{175}{3} < 0$, a contradiction. $\square$

Chapter 5

Ramsey density and list coloring

In this chapter we will prove the theorems from Section 1.2.4. The main results here are Theorem 1.21 and Theorem 1.22, with Theorem 1.20 following from the former.

5.1 Proof of Theorem 1.20 and Theorem 1.21

To prove Theorem 1.21, one needs to construct a list assignment L satisfying the property in the statement. In our construction, the assignment will be such that, for every pair of colors $\{C_1, C_2\}$, the graph formed by the edges with this list is the Rado graph, also known as the infinite random graph. This will be achieved by adapting a construction of Ackermann [Ack37].

One relevant part in this construction is that, for every vertex v and every list $\{C_1, C_2\}$, the set S of vertices that are connected to v through edges with list $\{C_1, C_2\}$ has positive density. This guarantees that, in every L-coloring, either $N_{C_1}(v)$ or $N_{C_2}(v)$ has positive density. As this holds for every pair of colors, it must be true that $N_C(v)$ has positive density for all but at most one color. We call this color, if it exists, the defective color of v. In our construction of $\mathcal{P}$, if a path misses v, then the path must be of its defective color.

Proof of Theorem 1.21. Given two positive integers i, j, we denote by $\mathrm{BIT}(i, j)$ the j-th bit in the binary expression of i, that is, the one corresponding to the value 2^{j-1}. We enumerate all the lists in $\binom{\mathcal{C}}{2}$ as $\ell_1, \ell_2, \ell_3 \ldots$ For any pair of positive integers $a < b$, the list on the edge ab is ℓ_i, where i is the minimum value that satisfies $\mathrm{BIT}(b, (2a-1)2^{i-1}) = 0$. Crucially, under this assignment, for any $a \neq a'$, the sets of binary digits of b that are checked to determine the lists assigned to ab and $a'b$ if $b > \max\{a, a'\}$ are disjoint.

We claim that this list assignment L satisfies the following property:

Proposition 5.1. *For any disjoint pair of sets $S_1, S_2 \subseteq V(K_{\mathbb{N}})$, both with positive density, the set of edges between S_1 and S_2 contains edges that get assigned all lists in $\binom{\mathcal{C}}{2}$ by L.*

Proof. Suppose that $\bar{d}(S_1), \bar{d}(S_2) \geq \epsilon > 0$. Consider a list ℓ_i. We will assume that there is no edge between S_1 and S_2 that takes the list ℓ_i, and derive a contradiction. Take $u_1, u_2, \ldots, u_r \in S_1$, where r will be defined later. Let S' be the set of vertices which are not connected to any of $u_1, u_2, \ldots, u_r$ through edges with list ℓ_i. Clearly $S_2 \subseteq S'$, so $\bar{d}(S') \geq \bar{d}(S_2) \geq \epsilon$.

If $v > \max\{u_1, u_2, \ldots, u_r\}$, whether $v \in S'$ or not depends exclusively on ir bits in the binary expression of v. Each vertex u_j acts on i bits, and the edge $u_j v$ takes the list ℓ_i if and only if those bits form the sequence $011\ldots1$. Thus we can deduce $\bar{d}(S') = \left(1 - 2^{-i}\right)^r$. But for r large enough, this density is less than ϵ, reaching the desired contradiction. $\square$

Next take an L-coloring Ψ. We will decide for which colors we will add a monochromatic path to $\mathcal{P}$. We say that a color C is broken if there exists a finite set S and a partition $V(K_{\mathbb{N}}) \setminus S = P_1 \cup P_2$ such that $\bar{d}(P_1), \bar{d}(P_2) > 0$ such that there is no edge with color C between P_1 and P_2.

We claim that there is at most one broken color. Indeed, suppose that C_1 and C_2 are broken, and let (P_1, P_2) and (P_1', P_2'), respectively, be the partitions of $V(K_{\mathbb{N}}) \setminus S$ that certify this fact (we can assume that the set S is the same in both partitions). For $a, b \in \{1, 2\}$, let $d_{a,b} = \bar{d}(P_a \cap P_b')$. As the sets P_1, P_2, P_1' and P_2' all have positive density, we have $d_{1,1} + d_{1,2}, d_{2,1} + d_{2,2}, d_{1,1} + d_{2,1}, d_{1,2} + d_{2,2} > 0$. W.l.o.g., we can suppose that both $d_{1,1}, d_{2,2} > 0$. But this implies that, by Proposition 5.1, there is an edge between $P_1 \cap P_1$ and $P_2 \cap P_2'$ with list $\{C_1, C_2\}$, which will goes across both partitions and receives a broken color. This is a contradiction, so there is at most one broken color.

We can now label all the non-broken colors as $C_1, C_2, C_3, \ldots$. For each i, we will create a monochromatic path Z_i with color C_i in which every vertex that is missed by Z_i has C_i as its defective color. Let $v_1 < v_2 < \ldots$ be the sequence of all vertices which do not have C_i as its defective color (this sequence may be finite or infinite). We take v_1 as the first vertex in Z_i. Suppose that we have constructed an initial segment of Z_i, ending at v_j. If Z_i already contains all vertices for which C_i is not broken, stop. Otherwise consider the minimum k such that v_k is not in Z_i. There must be a path from v_j to v_k in color C_i that avoids the already constructed Z_i, because otherwise there is a partition of

$V(K_{\mathbb{N}}) \setminus V(Z_i)$ that separates C_i-neighborhoods of v_j and v_k, contradicting Proposition 5.1. We can thus extend Z_i to contain v_k.

We proceed in this way to produce a finite or infinite path containing all v_j. Because each vertex has at most one defective color, each vertex is missed by at most one path, as we claimed. $\mathcal{P}$ cannot contain two or more finite paths, as there would be vertices missed by both of them. If $\mathcal{P}$ contains only one finite path, we remove it from $\mathcal{P}$. Thus we may suppose that all paths in $\mathcal{P}$ are infinite, completing the proof. $\square$

With a more careful construction, we can ensure that in fact every vertex is missed by exactly one path. The idea is as follows: the paths in colors C_i are constructed one by one. For each color C_i, we can predetermine a finite set X_i of vertices that are going to be avoided. Then Z_i will be missing all vertices in X_i, some vertices for which C_i is defective, and nothing else.

We start by adding the vertices v which do not have a defective color, or those whose defective color is broken, or those which share a defective color with all but finitely many other vertices, to distinct sets X_i. This represents all vertices which do not have a defective color for which an infinite path will be constructed. Then we construct the paths one by one, starting with Z_1. When we construct Z_i, we add the vertices $v \in Z_i$ for which C_i is a defective color to distinct sets X_j, with $j > i$. Following this procedure, we can check that every vertex is missed by exactly one path in $\mathcal{P}$.

Theorem 1.20 follows from Theorem 1.21 via a simple counting argument:

Proof of Theorem 1.20. We consider the same list assignment L as in Theorem 1.21. Let Ψ be an L-coloring, and $\epsilon > 0$. Consider the family $\mathcal{P}$ of paths from Theorem 1.21, and take k of them, for some $k > \epsilon^{-1}$. Denote these paths by $Z_1, Z_2, \ldots, Z_k$. We will show that one of these paths has density at least $1 - k^{-1} > 1 - \epsilon$.

Let n be a natural number. Each vertex in $[n]$ is missed by at most one path Z_j, so there is some i_n such that Z_{i_n} misses at most n/k vertices in $[n]$. This means that $|V(Z_{i_n}) \cap [n]| \geq (1 - 1/k)n$. There is some value of i such that $i = i_n$ for infinitely many n. Then we have

$$\bar{d}(Z_i) = \limsup_{n\to\infty} \frac{|V(Z_i) \cap [n]|}{n} \geq 1 - \frac{1}{k} > 1 - \epsilon. \qquad \square$$

5.2 Proof of Theorem 1.22

The idea for this proof is similar to the proof of the upper bound $\rho(P_\infty) \le 8/9$ given in [EG93] (which is also the basis for Theorem 1.9). However, we look at it from a different perspective this time.

The statement "there exists a two-coloring Ψ of $E(K_\mathbb{N})$ such that no monochromatic path has density higher than 8/9" is obviously equivalent to "there exists a two-coloring Ψ of $E(K_\mathbb{N})$ such that every path with density greater than 8/9 has edges in both colors". Indeed, the coloring Ψ from the second statement also satisfies the first statement, but so does its complement. This is the implication that we are interested in: given a list assignment, we will define which color *not* to choose, by using an auxilliary coloring (which does not depend on the lists) with the following property:

Lemma 5.2. *There exists a coloring $\Phi : E(K_\mathbb{N}) \to \mathcal{C}$ such that every infinite path P with density 1 contains edges in every color in $\mathcal{C}$.*

Proof. Let $C_1, C_2, C_3, \dots$ be the colors in $\mathcal{C}$. We will start by coloring the vertices of $K_\mathbb{N}$. We partition $\mathbb{N}$ into intervals $I_i = [3^{i-1}, 3^i)$. We color the vertices of I_i in color C_j, where j is the largest value such that 2^{j-1} divides i. For every edge uv with $u < v$, its color in Φ is the color of u.

Next consider a path P with density 1, and take a color C_j. We will show that P contains an edge in this color. Because $\bar{d}(P) = 1$, there exists some $t > 3^{2^j}$ such that $|V(P) \cap [t]| \ge \left(1 - 3^{-(2^j+1)}\right) t$. Let i be the largest value such that $i \equiv 2^{j-1} \pmod{2^j}$ and $3^i \le t$. This value satisfies $t < 3^{2^j} 3^i$, and so $|[t] \setminus V(P)| \le 3^{-(2^j+1)} t \le 3^{i-1}$. Since $|I_i| = 2 \cdot 3^{i-1}$ and $I_i \subseteq [t]$, this implies that $|V(P) \cap I_i| \ge 3^{i-1}$.

There must be a vertex $v \in V(P) \cap I_i$ whose succesor w in P is not in $[3^{i-1} - 1]$. We either have $w \in [3^i - 1]$ (in which case $w \in I_i$) or $w > v$. In the latter case the edge $vw \in P$ receives the color of v, which is C_j. In the former case vw receives the color of one of the endpoints, and both of them have color C_j. In any case, the edge $vw \in P$ receives the color C_j, completing our proof. $\square$

Now we can prove Theorem 1.22. Observe that Lemma 5.2 does not involve list coloring at all. Nevertheless, given a list assignment L, we will use the coloring Φ as an auxiliary tool to choose a color from each list.

Proof of Theorem 1.22. Let L be a list assignment, and let Φ be the coloring from Lemma 5.2. We will define an L-coloring Ψ as follows: for each edge uv, if $\Phi(uv) \in L(uv)$

we take $\Psi(uv)$ to be the color in $L(uv)$ other than $\Phi(uv)$. If $\Phi(uv) \notin L(uv)$, we choose $\Psi(uv)$ arbitrarily. With this we guarantee that $\Psi(uv) \neq \Phi(uv)$ for all $uv \in E(K_{\mathbb{N}})$.

Now for contradiction, assume that Ψ contains a monochromatic path P in color C_i. By Lemma 5.2, P contains an edge uv with $\Phi(uv) = C_i$. Thus we find $\Psi(uv) \neq C_i$, contradicting the fact that P is monochromatic in color C_i. $\square$

Chapter 6

Open problems

As a way to wrap up this thesis, in this chapter we will propose the conjectures and open problems related Ramsey density.

Let us start by addressing the most pressing question. Throughout this thesis we have mentioned a function $f(x)$ that connects the Ramsey density and the expansion of independent sets of a graph H (as in Theorems 1.9, 1.10, 1.11, 1.15, 1.17 and 3.2, and Corollaries 1.18, 1.12, 1.13 and 1.14), but we do not have a closed formula for this function, outside of the interval $[0, 1]$. We can only define $f(x)$ in terms of an optimization problem on Lipschitz functions.

We conjecture that the exact value of this function is the upper bound in (1.2):

Conjecture 6.1.

$$f(x) = \begin{cases} \frac{2x^2+3x+7+2\sqrt{x+1}}{4x^2+4x+9} & \text{for} \quad 0 \leq x < 3, \\ \\ \frac{x+1}{2x} & \text{for} \qquad x \geq 3. \end{cases}$$

When the contents of Chapter 3 appeared as an article [Lam20], one of the open questions that was asked was about the value of $\rho(\omega \cdot K_3)$, and to improve the bounds on (1.3). This question was the motivation for the work in Chapter 4, which answers it. Therefore we will ask about a graph which, arguably, is now the most interesting graph H for which $\rho(H)$ is still not known.

We already mentioned that Elekes et al. [ESSS17] proved that in every two-coloring of $E(K_{\mathbb{N}})$ the vertex set can be partitioned into at most four copies of P^2_∞ plus a finite set, and that DeBiasio and McKenney [DM19] observed that this implies $\rho(P^2_\infty) \geq 1/4$.

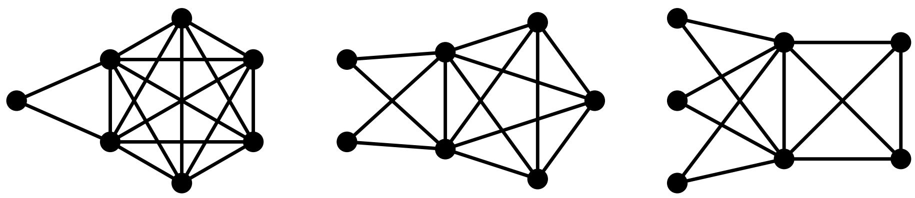

FIGURE 6.1: Three vertex-minimum graphs F for which $\rho(\omega \cdot F)$ is not known. If Conjecture 6.3 has an affirmative answer, their Ramsey densities are $f(2)$, $f(1)$ and $f(2/3)$, respectively. All other graphs on seven vertices with unknown $\rho(\omega \cdot F)$ are subgraphs of these three with the same minimum value of $|N(I)|/|I|$.

The best upper bound known is that given by Theorem 1.7, which is 1/2. It would be interesting to have more precision on the value.

Problem 6.2. *Improve either bound in* $1/4 \leq \rho(P_\infty^2) \leq 1/2$.

The square of the path poses an interesting challenge. Like in $\omega \cdot K_3$, this graph does not contain any non-empty doubly independent set, so the techniques from the proof of Theorem 3.2 are not necessarily of much help. On the other hand, unlike $\omega \cdot K_3$, the graph P_∞^2 is connected, which prevents us from using the same trick as in Theorem 1.17. If we restrict ourselves to looking for pieces of the square path in finite graphs, we still need to find a way of 'gluing' these pieces together into an infinite square path.

As we have seen, there are many different conditions that guarantee that the upper bound on $\rho(\omega \cdot F)$ from Theorem 1.9 is tight. This is the case if an independent set that minimizes $\frac{|N(I)|}{|I|}$ is maximal (in which case it matches Theorem 1.17) or if $N(I)$ is also independent (in which case it matches Theorem 1.15).

Remarkably, we do not know of any finite graph F for which Theorem 1.9 is not tight for $\omega \cdot F$. That leads to the following conjecture:

Conjecture 6.3. *For every finite graph F, we have*

$$\rho(\omega \cdot F) = f\left(\min_{\substack{I \text{ indep. in } F \\ I \neq \emptyset}} \frac{|N(I)|}{|I|} \right).$$

The smallest graphs F which are not covered by the previous cases have seven vertices, and are depicted in Figure 6.1.

In Theorem 1.19, we improve the lower bound on $f(2)$. There is no reason why the same argument cannot be used to improve the bound on $f(x)$ for all $x > 1$, but there is a catch: a straightforward generalization can only be applied for rational x, and even then the method appears to be scale-dependent. To clarify this, consider the graphs $F_1 = K_{a+b} \setminus K_b$ (the graph on $a + b$ vertices whose complement is an a-clique) and

$F_2 = K_{2(a+b)} \setminus K_{2b}$. They satisfy $\rho(\omega \cdot F_1) = f(a/b) = \rho(\omega \cdot F_2)$ by Corollary 1.18. However, it is not clear what the analogous for Lemma 4.7 should be in each case, and different versions might produce different bounds on $f(a/b)$, if applied to F_1 and F_2.

Problem 6.4. *What lower bounds on $f(x)$ can be obtained by adapting Lemma 4.7?*

Regarding the list coloring version of Ramsey density, so far we have only dealt with finding monochromatic infinite paths. It would be interesting to determine what happens when we look for other graphs H, like we did in Chapter 3 and Chapter 4. Let us consider locally finite graphs. As one can deduce from the proof of Theorem 1.11, there exists a locally finite tree T such that every two-coloring of $K_{\mathbb{N}}$ contains a monochromatic copy of T with density 1 (in fact, by Theorem 1.14 in [CDM20], there exists a copy of T that contains all but finitely many vertices of $K_{\mathbb{N}}$), so a generalization of Theorem 1.22, even for connected graphs, may require some extra conditions. However, there is no easy argument as for why Theorem 1.20 would fail for locally finite graphs in general. Thus we propose the following conjecture:

Conjecture 6.5. *There exists a list assignment L such that, for every locally finite graph H and every $\epsilon > 0$, every L-coloring contains a monochromatic copy of H with density at least $1 - \epsilon$.*

It is possible that the list assignment described in the proof of Theorem 1.21 is a valid choice of L for Conjecture 6.5.

Appendix A

Proof of Lemma 3.10

We will show that the parameters from Lemma 3.10 are essentially the same as Γ^+ and Γ^-, except the axes are rotated by 45 degrees. Thus we get that Lemma 3.10 is equivalent to the following lemma, which is just a compacity argument away from completing the proof:

Lemma A.1. *Let $\lambda \in (-1, 1)$ and $\epsilon > 0$. There exists $\gamma > 0$ with the following property. For every 1-Lipschitz function $g : [0, +\infty) \to \mathbb{R}$ with $g(0) = 0$, and every $m > 0$, there exists $t \in [\gamma m, m]$ such that*

$$\Gamma^+_\lambda(g, t) + \ell^-_\Gamma(g, t) \geq (h(\lambda) - \epsilon)t = \frac{2}{1-\lambda^2}\left(\frac{f\left(\frac{1+\lambda}{1-\lambda}\right)}{1 - f\left(\frac{1+\lambda}{1-\lambda}\right)} - \lambda\right) t - \epsilon t.$$

Proof of Lemma 3.10. Define the function $z : [0, +\infty) \to \mathbb{R}$ as follows: for every x, let $z(x) = g(y) - y$, where y is the unique value such that $x = g(y) + y$. This function is 1-Lipschitz: if x_1, x_2 are non-negative, and y_1, y_2 are the corresponding values of y, then

$$\begin{aligned} |z(x_1) - z(x_2)| &= |(g(y_1) - g(y_2)) - (y_1 - y_2)| \\ &\leq |(g(y_1) - g(y_2)) + (y_1 - y_2)| \\ &= |x_1 - x_2|, \end{aligned}$$

since $y_1 - y_2$ and $g(y_1) - g(y_2)$ do not have opposite signs.

For every $t > 0$, let $y_t = \ell^+_\lambda(g,t)$. By continuity of g, we have $g(\lambda y_t) - y_t = t$. Let $x_t = g(\lambda y_t) + \lambda y_t$. Then

$$
\begin{aligned}
z(x_t) =& g(\lambda y_t) - \lambda y_t \\
=& g(\lambda y_t) - \lambda y_t - \frac{2\lambda}{1+\lambda}\left(g(\lambda y_t) - y_t - t\right) \\
=& \frac{1-\lambda}{1+\lambda}\left(g(\lambda y_t) + \lambda y_t\right) + \frac{2\lambda}{1+\lambda}t \\
=& \frac{1-\lambda}{1+\lambda}x_t + \frac{2\lambda}{1+\lambda}t,
\end{aligned}
$$

which rearranges to $\frac{\lambda-1}{\lambda+1}x_t + z(x_t) = \frac{2\lambda}{1+\lambda}t$. Thus, $x_t \geq \Gamma^+_{\frac{\lambda-1}{\lambda+1}}(z, \frac{2\lambda}{1+\lambda}t)$. On the other hand, let $y'_t = \ell^-_\lambda(g,t)$. We have $y'_t - \frac{g(y'_t)}{\lambda} = t$. Let $x'_t = g(y'_t) + y'_t$. Then

$$
\begin{aligned}
z(x'_t) =& g(y'_t) - y'_t \\
=& g(y'_t) - y'_t + \frac{2\lambda}{1+\lambda}\left(y'_t - \frac{g(y'_t)}{\lambda} - t\right) \\
=& \frac{\lambda-1}{\lambda+1}\left(g(y'_t) + y'_t\right) - \frac{2\lambda}{1+\lambda}t \\
=& \frac{\lambda-1}{\lambda+1}x'_t - \frac{2\lambda}{1+\lambda}t,
\end{aligned}
$$

which rearranges to $\frac{\lambda-1}{\lambda+1}x'_t - z(x'_t) = \frac{2\lambda}{1+\lambda}t$. Thus, $x'_t \geq \Gamma^-_{\frac{\lambda-1}{\lambda+1}}\left(z, \frac{2\lambda}{1+\lambda}t\right)$. By Lemma A.1, for every m there exists a value $t \in [\gamma m, m]$, where γ depends only on λ and δ, such that

$$
\begin{aligned}
\left(h\left(\frac{\lambda-1}{\lambda+1}\right) - \delta\right)\frac{2\lambda}{1+\lambda}t \leq& x_t + x'_t \\
=& g(\lambda y_t) + \lambda y_t + g(y'_t) + y'_t \\
=& (y_t + t) + \lambda y_t + \lambda(y'_t - t) + y'_t \\
=& (\lambda+1)(y_t + y'_t) + (1-\lambda)t.
\end{aligned}
$$

Rearranging this inequality, substituting h and taking δ small enough we obtain $y_t + y'_t \geq (f(\lambda) - \epsilon)t$. $\square$

Finally, Lemma A.1 can easily be proved by showing that the restriction $t \in [\gamma m, m]$ is unnecessary:

Proof of Lemma A.1. Suppose that Lemma A.1 is false. For some $\lambda \in (-1,1)$ there are 1-Lipschitz functions $g_1, g_2, \ldots$ with $g_i(0) = 0$ such that $\frac{\Gamma^+_\lambda(g_i,t)+\Gamma^-(g_i,t)}{t} < h(\lambda) - \epsilon$ for every $t \in [\frac{m_i}{i}, m_i]$. By scaling each fraction as $\bar{g}_i(x) = \frac{\sqrt{i}}{m_i} g_i\left(\frac{m_i}{\sqrt{i}} x\right)$, we can assume that $m_i = \sqrt{i}$ for every $i \in \mathbb{N}$.

Because the sequence $g_i(x)$ is bounded for every x, there is a subsequence $g_{i_1}, g_{i_2}, \ldots$ which is uniformly convergent on every compact subset of $[0, +\infty)$, and has limit $g(x)$. This function is also 1-Lipschitz and has $g(0) = 0$, so by definition there exists $t > 1$ such that $\Gamma^+_\lambda(g, t) + \Gamma^-_\lambda(g, t) \geq (h(\lambda) - \epsilon/2)t$.

Let $q = \Gamma^+_\lambda(g, t)$. Since g satisfies that $\lambda x + g(x) < t$ for every $0 \leq x \leq q$, we also have $\lambda x + g_{i_j}(x) < (1 - \delta)t$ for all $x \in [0, q]$, all $\delta > 0$ and all $j > J(\delta)$ large enough, therefore $\Gamma^+_\lambda(g_{i_j}, (1 - \delta)t) > q$. For the same reason, $\Gamma^+_\lambda(g_{i_j}, (1 - \epsilon/4)t) > \Gamma^-_\lambda(g_{i_j}, (1 - \delta)t) > \Gamma^-_\lambda(g, t)$. We obtain the inequality

$$\frac{\Gamma^+_\lambda(g_{i_j}, (1-\delta)t) + \Gamma^-_\lambda(g_{i_j}, (1-\delta)t)}{(1-\delta)t} > \frac{\Gamma^+_\lambda(g, t) + \Gamma^-_\lambda(g, t)}{(1-\delta)t} \geq \frac{h(\lambda) - \epsilon/2}{1 - \delta} > h(\lambda) - \epsilon$$

for δ small enough. To reach a contradiction, simply observe that $(1-\delta)t \in \left[\frac{1}{\sqrt{i_j}}, \sqrt{i_j}\right] = \left[\frac{m_{i_j}}{i_j}, m_{i_j}\right]$ for all j large enough. $\square$

Appendix B

Properties of $f(\lambda)$

We will prove four propositions regarding $f(x)$. Propositions B.2 and B.5 together imply (1.2), while Proposition B.3 means that the upper bound is tight for $x \in [0, 1]$.

Proposition B.1. *f is non-increasing and continuous.*

Proof. Let $-1 < \gamma < \tau < 1$. Let $\epsilon > 0$. Choose g such that $\limsup \frac{\Gamma_\gamma^+(g,t)+\Gamma_\gamma^-(g,t)}{t} \leq h(\gamma) + \epsilon$. Let $z(t) = \Gamma_\gamma^+(g, t)$ and $\bar{z}(t) = \Gamma_\tau^+(g, t)$. We have $\gamma z(t) + g(z(t)) = t = \tau\bar{z}(t) + g(\bar{z}(t))$. Because $\tau z(t) + g(z(t)) > \gamma z(t) + g(z(t)) = t$, by definition of Γ_τ^+ we have $\bar{z}(t) \leq z(t)$. Thus

$$z(t) - \bar{z}(t) \geq g(z(t)) - g(\bar{z}(t)) = \tau\bar{z}(t) - \gamma z(t)$$

which rearranges to $(\tau + 1)\bar{z}(t) \leq (\gamma + 1)z(t)$. Similarly, if $z'(t) = \Gamma_\gamma^-(g, t)$ and $\bar{z}'(t) = \Gamma_\tau^-(g, t)$, then $(\tau + 1)z'(t) \leq (\gamma + 1)z'(t)$. We take upper limits.

$$\begin{aligned}
(\tau + 1)h(\tau) \leq& (\tau + 1) \limsup_{t\to\infty} \frac{\bar{z}(t) + \bar{z}'(t)}{t} \\
\leq& (\gamma + 1) \limsup_{t\to\infty} \frac{z(t) + z'(t)}{t} \\
\leq& (\gamma + 1)h(\gamma) + \epsilon
\end{aligned}$$

Since this inequality is valid for every $\epsilon > 0$, we find that the function $(\gamma + 1)h(\gamma)$ is non-increasing on γ. On the other hand, we have $t = \gamma z(t) + g(z(t)) \leq (\gamma + 1)z(t)$, or equivalently $z(t) \geq \frac{t}{\gamma+1}$. Similarly $z'(t) \geq \frac{t}{\gamma+1}$. This leads to $h(\gamma) \geq \frac{2}{\gamma+1}$. We conclude that the function $\frac{1-\gamma^2}{2}h(\gamma)+(\gamma+1)$ is non-increasing, because

$$\begin{aligned}\frac{1-\tau}{2}(\tau+1)h(\tau)+(\tau+1) \leq& \frac{1-\tau}{2}(\gamma+1)h(\gamma)+(\tau+1)\\ \leq& \frac{1-\tau}{2}(\gamma+1)h(\gamma)+(\tau+1)-\left(\frac{(\gamma+1)h(\gamma)}{2}-1\right)(\tau-\gamma)\\ =& \frac{1-\gamma}{2}(\gamma+1)h(\gamma)+(\gamma+1).\end{aligned}$$

After the change of variables, $\gamma=\frac{\lambda-1}{\lambda+1}$, this is equivalent to $f(\lambda)$ being non-decreasing.

To show that f is continuous, we will show that h is continuous. We will show that, for every $-1<\gamma<1$ and every $\epsilon>0$ there exists $\delta>0$ such that $h(\gamma-\delta)-h(\gamma)<\epsilon$. Because h is non-increasing, this implies continuity.

Choose $\xi>0$ very small, and choose g such that $\frac{\Gamma^+_\gamma(g,t)+\Gamma^-_\gamma(g,t)}{t}\leq h(\gamma)+\xi$ for t large enough. Let $z(t)=\Gamma^+_\gamma(g,t)$. We have $\gamma z(t)+g(z(t))=t$. If t is large enough, then $(\gamma-\delta)z(t)+g(z(t))\geq t-\delta z(t)\geq(1-h(\gamma)+\xi)t$. If $\bar{z}(t)=\Gamma^+_{\gamma-\delta}(g,t)$, then for t large enough we have $\bar{z}(t)\leq z\left(\frac{t}{1-(h(\gamma)+\xi)\delta}\right)$. As before, the same argument works for Γ^-, and produces (taking $\xi\to 0$)

$$h(\gamma-\delta)\leq\frac{h(\gamma)}{1-\delta h(\gamma)},$$

which is smaller than $h(\gamma)+\epsilon$ for δ small enough. □

Proposition B.2.

$$f(\lambda)\leq\begin{cases}\frac{2\lambda^2+3\lambda+7+2\sqrt{\lambda+1}}{4\lambda^2+4\lambda+9} & \textit{for}\quad 0\leq\lambda<3,\\ \\ \frac{\lambda+1}{2\lambda} & \textit{for}\quad x\geq 3.\end{cases}$$

Proof. Let $\gamma=\frac{\lambda-1}{\lambda+1}$. All we need to do is find a function $g(x)$ for which

$$\limsup_{t\to\infty}\frac{\Gamma^+_\gamma(g,t)+\Gamma^-_\gamma(g,t)}{t}\leq\begin{cases}\frac{2\gamma^2+2\gamma+8+\sqrt{32(1-\gamma)}}{(\gamma+1)^3} & \text{for}\quad\gamma\in\left(-1,\frac{1}{2}\right)\\ \\ \frac{2}{\gamma} & \text{for}\quad\gamma\in\left[\frac{1}{2},1\right),\end{cases}$$

to show an upper bound on $h(\gamma)$, and consequently on $f(\lambda)$.

For $\gamma\in[\frac{1}{2},1)$, take $g(x)=0$. Then $\Gamma^+_\gamma(g,t)=\Gamma^-_\gamma(g,t)=\frac{t}{\gamma}$.

For $\gamma\in(-1,\frac{1}{2})$, let $\sigma=\frac{1-\gamma+\sqrt{2(1-\gamma)}}{1+\gamma}$. We define g as follows: for every $x>0$, we have $|g(x)|=\min\{|x-\sigma^i|:i\in\mathbb{Z}\}$. $g(x)$ is non-negative if $x\in[\sigma^i,\sigma^{i+1})$ for some odd i

and non-positive otherwise. This creates a 1-Lipschitz function in which the derivative is always ± 1, and changes sign precisely at the values $x = \frac{\sigma^i + \sigma^{i+1}}{2}$.

This function g satisfies that $\Gamma^+_\gamma(g,t)$ and $\Gamma^-_\gamma(g,t)$ are piecewise linear, with sudden increases at the points at which the respective functions take value $\frac{\sigma^i+\sigma^{i+1}}{2}$. By symmetry, it is at these points that $\frac{\Gamma^+_\gamma + \Gamma^-_\gamma}{2}$ is maximized, and one can check that the value is

$$\lim_{t \to (t^*)^+} \frac{\Gamma^+_\gamma(g,t) + \Gamma^-_\gamma(g,t)}{t} = \frac{2(\gamma(\sigma+1) + \sigma^2 + 2\sigma - 1)}{(\gamma+1)((\gamma+1)\sigma + \gamma - 1)} = \frac{2\gamma^2 + 2\gamma + 8 + \sqrt{32(1-\gamma)}}{(\gamma+1)^3},$$

where $t^* = \gamma \frac{\sigma^i + \sigma^{i+1}}{2} + \frac{\sigma^{i+1} - \sigma^i}{2}$. □

The next proposition gives the exact value for $f(x)$ for $x \in [0,1]$. It can be seen as a generalization of Lemma 2.9. The problem of determining the value of $f(x)$ is similar, at least in appearence, to a certain generalization of the linear search problem, also known as the cow path problem. In its original form, this problem was solved by Beck and Newman [BN70]. There are many common ideas between their approach and our approach.

Proposition B.3.

$$f(\lambda) = \frac{2\lambda^2 + 3\lambda + 7 + 2\sqrt{\lambda+1}}{4\lambda^2 + 4\lambda + 9} \qquad \forall 0 \le \lambda \le 1$$

Proof. Let $\gamma = \frac{\lambda - 1}{\lambda + 1}$. If two 1-Lipschitz functions with $g_1(0) = g_2(0) = 0$ satisfy $|g_1(x) - g_2(x)| \le 1$ for every x, then $\Gamma^+_\gamma(g_1, t) = \min\{x : \gamma x + g_1(x) \ge t\} \ge \min\{x : \gamma x + g_2(x) \ge t - 1\} = \Gamma^+_\gamma(g_2, t-1)$. Similarly, $\Gamma^-_\gamma(g_1,t) \ge \Gamma^-_\gamma(g_2, t-1)$. This implies that

$$\limsup_{t\to\infty} \frac{\Gamma^+_\gamma(g_1,t) + \Gamma^-_\gamma(g_1,t)}{t} = \limsup_{t\to\infty} \frac{\Gamma^+_\gamma(g_2,t) + \Gamma^-_\gamma(g_2,t)}{t}$$

For this reason, we can focus our attention on continuous functions g which are piecewise linear, the slope of each piece is 1 or -1 and the length of each piece is at least $\frac{1}{2}$. Indeed, let g_1 be a 1-Lipschitz function with $g_1(0) = 0$. Define g_2 by starting at $g_2(0) = 0$. Take $g_2(x)$ in $[0, x_1]$, where x_1 is the minimum value for which $g_1(x_1) = x_1 - 1$. Then continue with slope -1 in the interval $[x_1, x_2]$ until the minimum value of x_2 for which we would have $g_2(x_2) = g_1(x_2) - 1$. Proceed alternating the sign of the slope every time g_2 reaches distance 1 from g_1. Clearly we always have $|g_1(x) - g_2(x)| \le 1$, and it is easy to check that each piece has length at least $\frac{1}{2}$ (it could be that some piece has infinite length).

We can further suppose that the first piece has slope 1. Let $\ell_1, \ell_2, \ldots$ be the length of the pieces, and x_i be the end of the i-th piece. The points x_i are local maxima if i is odd and minima if i is even. If for some odd i we have $\gamma x_i + g(x_i) \le \gamma x_{i-2} + g(x_{i-2})$

(which means that x_i does not equal $\Gamma^+_\gamma(g,t)$ for any t), we can "remove the peak" by extending the intervals $[x_{i-2}, x_{i-1}]$ and $[x_{i+1}, x_{i+2}]$ until they intersect. The new function $\bar{g}$ satisfies $\Gamma^+_\gamma(\bar{g},t) \le \Gamma^+_\gamma(g,t)$ and $\Gamma^-_\gamma(\bar{g},t) = \Gamma^-_\gamma(g,t)$, because $\Gamma^+_\gamma(g,t)$ will always lie on an increasing piece and $\Gamma^-_\gamma(g,t)$ lies on a decreasing piece.

We can similarly "remove a valley", if for some even i the value x_i does not equal $\Gamma^-_\gamma(g,t)$ for any t. Applying these two procedures repeatedly, always to the first peak or valley that can be removed, we produce a function in which every x_i is either $\Gamma^+_\gamma(g,t_i)$ for odd i or $\Gamma^-_\gamma(g,t_i)$ for even i. The function must still have infinitely many peaks and valleys, otherwise we have $\Gamma^+_\gamma(g,t) = \infty$ or $\Gamma^-_\gamma(g,t) = \infty$ for large enough t.

Let $\mu > \limsup_{t\to\infty} \frac{\Gamma^+_\gamma(g,t)+\Gamma^-_\gamma(g,t)}{t}$. After scaling, we can suppose that $\frac{\Gamma^+_\gamma(g,t)+\Gamma^-_\gamma(g,t)}{t} \le \mu$ for every $t \ge 1$ (because of the scaling, we might lose the property that every interval has length larger than $\frac{1}{2}$, but it will still be larger than some constant). Let us see the relation that the t_i must satisfy. We have

$$x_i = \sum_{j=1}^{i} \ell_i \qquad g(x_i) = \sum_{j=1}^{i} (-1)^{j+1} \ell_i \qquad t_i = \gamma x_i + (-1)^{i+1} g(x_i).$$

We can express x_i in terms of $t_1, t_2, \ldots, t_i$. Combining the identities above we have $t_i = \sum_{j=1}^{i} (\gamma + (-1)^{j-i})\ell_j$, and therefore $t_i + t_{i-1} = (\gamma+1)\ell_i + 2\gamma \sum_{j=1}^{i-1} \ell_j = (\gamma+1)x_i + (\gamma-1)x_{i-1}$, which produces the recursion $x_i = \frac{1-\gamma}{1+\gamma} x_{i-1} + \frac{1}{1+\gamma}(t_i + t_{i-1})$. Together with $x_1 = \frac{1}{1+\gamma} t_1$, the solution is

$$x_i = \frac{1}{1+\gamma} t_i + \sum_{j=1}^{i-1} \frac{2}{1-\gamma^2} \left(\frac{1-\gamma}{1+\gamma} \right)^{i-j} t_j.$$

Based on these values, we can compute $\Gamma^+_\gamma(g,t)$ and $\Gamma^-_\gamma(g,t)$. The former must lie in the interval $[x_{i-1}, x_i]$, where i is the smallest odd value for which $t_i \ge t$. Moreover, it is the unique value $x \in [x_{i-1}, x_i]$ for which $t = \gamma x + g(x) = \gamma x + (g(x_i) + x - x_i) = t_i - (\gamma+1)(x_i - x)$. If $m_o = m_o(\{t_i\}_{i=1}^{\infty}, t)$ is the smallest odd i such that $t_i \ge t$, then

$$\Gamma^+_\gamma(g,t) = t_{m_o} - (\gamma+1)(x_{m_0} - x) = \frac{1}{1+\gamma} t + \sum_{j=1}^{m_o - 1} \frac{2}{1-\gamma^2} \left(\frac{1-\gamma}{1+\gamma} \right)^{m_o - j} t_j.$$

Similarly, if $m_e = m_e(\{t_i\}_{i=1}^{\infty}, t)$ is the smallest even i such that $t_i \ge t$, then

$$\Gamma^-_\gamma(g,t) = \frac{1}{1+\gamma} t + \sum_{j=1}^{m_e - 1} \frac{2}{1-\gamma^2} \left(\frac{1-\gamma}{1+\gamma} \right)^{m_e - j} t_j.$$

Define

$$s(\{t_i\}_{i=1}^{\infty}, t) = \frac{2}{1+\gamma}t + \sum_{j=1}^{m_o-1} \frac{2}{1-\gamma^2}\left(\frac{1-\gamma}{1+\gamma}\right)^{m_o-j} t_j + \sum_{j=1}^{m_e-1} \frac{2}{1-\gamma^2}\left(\frac{1-\gamma}{1+\gamma}\right)^{m_e-j} t_j.$$

We are then interested in the lowest value that $\sup_{t\geq 1} \frac{s(\{t_i\}_{i=1}^{\infty},t)}{t}$ can take. Given a non-negative unbounded sequence $\{t_i\}_{i=1}^{\infty}$, let $i_1 < i_2 < \dots$ be the set of indices i for which $t_i \geq 1$ and $t_i > t_j$ for every $j < i$. One can check that $s(\{t_{i_j}\}_{j=1}^{\infty}, t) \leq s(\{t_i\}_{i=1}^{\infty}, t)$ for every $t \geq 1$ (this is because $\frac{1-\gamma}{1+\gamma} > 1$). Therefore, we can assume that t_i is increasing and $t_1 \geq 1$. In this case, and setting $t_0 = 0$,

$$\sup_{t\geq 1} \frac{s(\{t_i\}_{i=1}^{\infty}, t)}{t} \geq \sup_{i\geq 1} \frac{2}{1+\gamma} + \sum_{j=1}^{i+1} \frac{2}{1-\gamma^2}\left(\frac{1-\gamma}{1+\gamma}\right)^{i-j+2} \frac{t_j + t_{j-1}}{t_i},$$

since $\{m_o(\{t_i\}_{i=1}^{\infty}, t_i+\epsilon), m_e(\{t_i\}_{i=1}^{\infty}, t_i+\epsilon)\} = \{i+1, i+2\}$ for $\epsilon > 0$ small enough.

A sequence is called S-good if it satisfies

$$St_i \geq \frac{2}{1+\gamma}t_i + \sum_{j=1}^{i+1} \frac{2}{1-\gamma^2}\left(\frac{1-\gamma}{1+\gamma}\right)^{i-j+2} (t_j + t_{j-1}) \qquad \forall i \geq 1, \tag{B.1}$$

which is a necessary condition for $\sup_{t\geq 1} \frac{s(\{t_i\}_{i=1}^{\infty},t)}{t} \geq S$. Suppose that a sequence is S-good. Consider the recurring sequence where $T_1 = t_1$ and

$$ST_i = \frac{2}{1+\gamma}T_i + \sum_{j=1}^{i+1} \frac{2}{1-\gamma^2}\left(\frac{1-\gamma}{1+\gamma}\right)^{i-j+2} (T_j + T_{j-1}). \tag{B.2}$$

Observe that (B.2) is used to define T_{i+1} from the previous entries, rather than T_i.

Claim B.4. *If $\{t_i\}_{i=1}^{\infty}$ is S-good, we have $t_i \leq T_i$ for every $i \geq 1$.*

Proof of Claim. Suppose that there exists an index i for which $t_i > T_i$. Let u and v be the smallest indices such that $t_u \neq T_u$ and $t_v > T_v$, respectively. Clearly we have $u \geq 2$. We will prove our claim by induction on $v - u$. By definition of T_i we have $u, v > 1$. Combining (B.1) and (B.2) for $i = u - 1$ we see that we cannot have $v - u = 0$.

Let $\alpha = \frac{T_u - t_u}{t_1} > 0$. Consider the sequence $\{t_i'\}_{i=1}^{\infty}$, where $t_i' = t_i$ for $i < u$ and $t_i' = t_i + \alpha t_{i-u+1}$ for $i \geq u$. This sequence is still increasing. We have $t_1' = t_1$ and, by the choice of α we have $t_u = T_u$. Additionally, $t_v' \geq t_v > T_v$. This means that, if we define u' and v' analogously to u and v, then $u' > u$ and $v' \leq v$, which leads to $v' - u' < v - u$. Finally, observe that $\{t_i'\}$ is still good, since (B.1) is satisfied for $i < u$ (the equation becomes (B.2)) and for $i \geq u$ we have

$$\begin{aligned}
St'_i =& St_i + St_{i-u+1} \\
=& \frac{2}{1+\gamma}(t_i + t_{i-u+1}) + \sum_{j=1}^{i+1} \frac{2}{1-\gamma^2}\left(\frac{1-\gamma}{1+\gamma}\right)^{i-j+2} (t_j + t_{j-1} + t_{j-u+1} + t_{j-u}) \\
=& \frac{2}{1+\gamma} t'_i + \sum_{j=1}^{i+1} \frac{2}{1-\gamma^2}\left(\frac{1-\gamma}{1+\gamma}\right)^{i-j+2} (t'_j + t'_{j-1})
\end{aligned}$$

taking $t_i = 0$ for $i \le 0$. This completes the induction step. □

To complete the proof of Theorem B.3, we show that $T_i \le 0$ for some i if $S < \frac{2\gamma^2+2\gamma+8+\sqrt{32(1-\gamma)}}{(\gamma+1)^3}$. The reason is that the sequence T_i also satisfies the recursion

$$ST_i - \frac{1-\gamma}{1+\gamma} ST_{i-1} = \frac{2}{1+\gamma}\left(T_i - \frac{1-\gamma}{1+\gamma}T_{i-1}\right) + \frac{2}{1-\gamma^2}\frac{1-\gamma}{1+\gamma}(T_{i+1} - T_i).$$

which can be rewritten as $T_{i+1} + \alpha T_i + \beta T_{i-1} = 0$. If S is smaller than the claimed bound, then the roots of the polynomial $x^2 + \alpha x + \beta$ are not real. This implies that $T_i \le 0$ for some i (see for example [BW81]). This proves that $S \ge \frac{2\gamma^2+2\gamma+8+\sqrt{32(1-\gamma)}}{(\gamma+1)^3}$, which produces the desired value of $f(\lambda)$. □

For Proposition B.5, we note that the proofs of Theorem 1.9 and Theorem 1.16 do not use this proposition, so we avoid a circular argument.

Proposition B.5. *For $x > 1$ we have $f(x) \ge \frac{x+1}{2x+1}$.*

Proof. We will prove this statement for rational values of x, and it will follow for irrational values because f is continuous (Proposition B.1).

Let $x = s/r$. Let F be the graph on $r + s$ vertices whose complement is a clique on r vertices (hence $\alpha(F) = r$). We have $\mu(\omega \cdot F, n) = s\lceil n/r \rceil$, because an independent set I intersects at least $\lceil n/r \rceil$ components and has at least s neighbors in each. Combining Theorem 1.9 and Theorem 1.16 we find $f(s/r) \ge \rho(\omega \cdot F) \ge \frac{r+s}{r+2s}$, or $f(x) \ge \frac{x+1}{2x+1}$. □

Declaration of Authorship

I, Ander Lamaison Vidarte, declare that this thesis titled, "Upper density problems in infinite Ramsey theory" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ____

Date: ____

Bibliography

[Ack37] W. Ackermann, *Die widerspruchsfreiheit der allgemeinen mengenlehre*, Math. Ann. **114** (1937), 305–315.

[ABK+19] N. Alon, M. Bucić, T. Kalvari, E. Kuperwasser, and T. Szabó, *List ramsey numbers*, https://arxiv.org/abs/1902.07018 (2019).

[BL20] J. Balogh and A. Lamaison, *Ramsey upper density of infinite graph factors*, https://arxiv.org/abs/2010.13633 (2020).

[BN70] A. Beck and D. J. Newman, *Yet more on the linear search problem*, Israel J. Math. **8** (1970), 419–429.

[BW81] J. R. Burke and W. A. Webb, *Asymptotic behavior of linear recurrences*, Fibonacci Quart. **19** (1981), no. 4, 318–321.

[BES75] S. A. Burr, P. Erdős, and J. H. Spencer, *Ramsey theorems for multiple copies of graphs*, Trans. Amer. Math. Soc. **209** (1975).

[CDLL19] J. Corsten, L. DeBiasio, A. Lamaison, and R. Lang, *Upper density of monochromatic infinite paths*, Adv. Comb. (2019).

[CDM20] J. Corsten, L. DeBiasio, and P. McKenney, *Density of monochromatic infinite subgraphs ii*, https://arxiv.org/abs/2007.14277 (2020).

[DM19] L. DeBiasio and P. McKenney, *Density of monochromatic infinite subgraphs*, Combinatorica **39** (2019).

[ESSS17] M. Elekes, D. T. Soukup, L. Soukup, and Z. Szentmiklóssy, *Decompositions of edge-colored infinite complete graphs into monochromatic paths*, Discrete Math. **340** (2017).

[EG93] P. Erdős and F. Galvin, *Monochromatic infinite paths*, Discrete Math. **113** (1993).

[EH62] P. Erdős and A.ás Hajnal, *Some remarks concerning our paper "on the structure of set-mappings". non-existence of a two-valued σ-measure for the first uncountable inaccessible cardinal*, Acta Math. Hungar. **13** (1962), 223–226.

[ER50] P. Erdős and R. Rado, *A combinatorial theorem*, J. Lond. Math. Soc. **25** (1950), 249–255.

[ES35] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935).

[GG67] L. Gerencsér and A. Gyárfás, *On ramsey-type problems*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **10** (1967), 167–170.

[Gra07] R. Graham, *Some of my favorite problems in ramsey theory*, Integers **7** (2007), A15.

[Hax97] P. Haxell, *Partitioning complete bipartite graphs by monochromatic cycles*, Journal of Combinatorial Theory, Series B **69** (1997), no. 2, 210–218.

[KS96] J. Komlós and M. Simonovits, *Szemerédi's Regularity Lemma and its applications in graph theory*, Combinatorics, Paul Erdős is Eighty (D. Miklós, V. T. Sós, and T. Szőnyi, eds.), vol. 2, Bolyai Society Mathematical Studies, 1996, pp. 295–352.

[KO09] D. Kühn and D. Osthus, *Embedding large subgraphs into dense graphs*, Surveys in Combinatorics (2009), 137–168.

[Lam20] A. Lamaison, *Ramsey upper density of infinite graphs*, https://arxiv.org/abs/2003.06329 (2020).

[LSMW18] A. Lo, N. Sanhueza-Matamala, and G. Wang, *Density of monochromatic infinite paths*, Electron. J. Combin. **25** (2018).

[Rad78] R. Rado, *Monochromatic paths in graphs*, Ann. Discrete Math. **3** (1978), 191–194.

[Ram30] F. P. Ramsey, *On a problem of formal logic*, Proc. Lond. Math. Soc. **2** (1930).