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Partial stability analysis of stochastic differential equations with a general decay rate

Tomás Caraballo · **Faten Ezzine** ·
Mohamed Ali Hammami

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Abstract This paper is concerned with the almost sure partial practical stability of stochastic differential equations with general decay rate. We establish some sufficient conditions based upon the construction of appropriate Lyapunov functions. Finally, we provide a numerical example to demonstrate the efficiency of the obtained results.

Keywords Decay function · Itô formula · Lyapunov construction · Partial almost sure practical stability · Stochastic systems

Mathematics Subject Classification Primary 93E03 · Secondary 60H10

1 Introduction

Stochastic models are useful for modeling physical, biological, technical, as well as dynamical structures in engineering and mechanics in which notable uncertainty is present either by the intrinsic nature of the model or by the influence of external sources of noise.

The theory of stochastic differential equations in finite and infinite-dimensional spaces is already an established area of research. The corresponding study of the stability properties of solutions has received much attention during the last three decades. The reader is referred to [1,2], for more details.

In our present paper, we will analyze a quite general model of stochastic differential system which can cover a wide class of real models subjected to noise effects, and we emphasize that our results can be applied to many

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T. Caraballo (✉)
Depto. Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, c/ Tarfia s/n, 41012 Sevilla, Spain
e-mail: caraball@us.es

F. Ezzine · M. A. Hammami
Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Sfax, Tunisia
e-mail: MohamedAli.Hammami@fss.rnu.tn

interesting examples related to the analysis of stochastic stability in different kinds of structures (see, [3] for more details about the use of Lyapunov's technique to study this problem). In many situations, the stochastic model considered may not fulfill sufficient conditions ensuring the stability of solutions. However, it is still possible that the system is still stable with respect to a part of the unknowns. It is crucial to investigate whether it is still possible to ascertain some stability properties to some of the variables in the problem which is called "Partial Stability."

Partial stability has proved to have powerful applications in many branches of biotechnology, electro-magnetics, combustion systems, vibrations in rotating machinery, inertial navigation systems, spacecraft stabilization via gimballed gyroscopes, and/or flywheels. The reader is referred to [4–9], for more details.

The method of Lyapunov functions is one of the most powerful tools to investigate the stability of stochastic systems. Later on, by the development of the second method of Lyapunov, different authors have been working on partial stability with stochastic differential equations, see [6,8,10–15].

When the origin is not a trivial solution, we investigate the partial stability of the SDEs with respect to a small neighborhood of the origin. In this sense, we can analyze the ultimate boundedness of the solutions of the stochastic system or the possibility of proving the partial convergence of the solutions towards a small ball centered at the origin, this is the so-called "Partial Practical Stability." Several results on the stability of the nontrivial solution of stochastic systems are proposed in [16–18].

In the analysis of the asymptotic behavior of solutions to stochastic differential systems, one can find that a solution can be asymptotically stable but may not be exponential. Further, in the nonlinear and/or nonautonomous situations, it may happen that the stability cannot always be exponential but can even be sub or super-exponential, see [19]. For this reason, this paper deals with the analysis of almost sure partial practical stability of stochastic systems with general decay rate. Our main objective in this paper is to generalize the results in [18], which investigated the partial practical exponential stability of SDEs, to the partial practical stability with a general decay rate (including polynomial, logarithmic, sub-exponential and super-exponential, as well as exponential decay). We establish sufficient conditions ensuring the almost sure partial practical stability by using Lyapunov method. Further, one of our main purpose of this paper is to construct suitable Lyapunov functions to prove the partial practical stability with general decay rate. Moreover, we will illustrate the theory with an application example.

The arrangement of the paper is presented as follows. In Sect. 2, we evoke some essential preliminaries and results. In Sect. 3, some sufficient conditions ensuring almost sure partial practical stability with general decay rate are proven. In Sect. 4, we establish some sufficient conditions based on the construction of appropriate Lyapunov functions. In Sect. 5, we present an example to illustrate the theoretical findings. Finally, in Sect. 6, some conclusions and prospects are discussed.

2 Preliminaries

We consider the nonlinear stochastic differential equation in the following form:

$$\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + g(t, x(t))\mathrm{d}B_t, \quad t \geq 0, \tag{2.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $g : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times m}$, $x = (x_1, \dots, x_d)^{\mathrm{T}}$, and $B_t = (B_1(t), \dots, B_m(t))^{\mathrm{T}}$ is an m-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For the well-posedness of system (2.1), we impose that f and g satisfy the following assumptions:

$$||f(t, x)||^2 + ||g(t, x)||^2 \leq C_1(t)\big(1 + ||x||^2\big), \quad \forall t \geq 0, \quad x \in \mathbb{R}^d, \tag{2.2}$$

$$||f(t, x) - f(t, \widetilde{x})|| \vee ||g(t, x) - g(t, \widetilde{x})|| \leq C_2(t)||x - \widetilde{x}||, \quad \forall t \geq 0, \quad x, \widetilde{x} \in \mathbb{R}^d, \tag{2.3}$$

with $C_1(\cdot)$ and $C_2(\cdot)$ being nonnegative functions.

In the sequel, the same symbol is used to express the norm in $\mathbb{R}^d$ or in $\mathbb{R}^{d\times m}$.

Let $x = (x_1, x_2)^{\mathrm{T}}$, where $x_1 = (x_{1.1}, \dots, x_{1.d_1})^{\mathrm{T}} \in \mathbb{R}^{d_1}$, $x_2 = (x_{2.1}, \dots, x_{2.d_2})^{\mathrm{T}} \in \mathbb{R}^{d_2}$, $d_1 > 0$, $d_2 \geq 0$, $d_1 + d_2 = d$;

$||x_1|| = \sqrt{x_{1.1}{}^2 + \cdots + x_{1.d_1}{}^2}$, $\;||x_2|| = \sqrt{x_{2.1}{}^2 + \cdots + x_{2.d_2}{}^2}$, $\;||x|| = \sqrt{||x_1||^2 + ||x_2||^2}$.

Then, based on the previous conditions, there exists a unique global solution with initial value $x_0 \in \mathbb{R}^d$ defined in an interval $[t_0, T)$: $x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0))$ or simply $x(t) = (x_1(t), x_2(t))$ to express a solution to our system and, as we will be interested in analyzing the asymptotic behavior of solutions, we assume $T = +\infty$.

Definition 2.1 [20] Denote by $C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$ the family of all real-valued functions $\mathcal{V}(t, x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^d$ which are once continuously differentiable in t and twice in x.
Let $\mathcal{V} \in C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$, we set

$$\mathcal{V}_t(t, x) = \frac{\partial \mathcal{V}}{\partial t}(t, x)\,; \quad \mathcal{V}_x(t, x) = \Big(\frac{\partial \mathcal{V}}{\partial x_1}(t, x), \frac{\partial \mathcal{V}}{\partial x_2}(t, x)\Big); \quad \mathcal{V}_{xx}(t, x) = \Big(\frac{\partial^2 \mathcal{V}}{\partial x_i \partial x_j}(t, x)\Big)_{d \times d}.$$

We define operators $\mathcal{L}$ and $\mathcal{H}$ associated with Eq. (2.1) as follows: for $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$,

$$\mathcal{L}\mathcal{V}(t, x) = \mathcal{V}_t(t, x) + \mathcal{V}_x(t, x) f(t, x) + \frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}(t, x)\mathcal{V}_{xx}(t, x) g(t, x)],$$

and

$$\mathcal{H}\mathcal{V}(t, x) = ||\mathcal{V}_x(t, x) g(t, x)||^2.$$

Applying the well-known Itô formula [20], it follows for $x(\cdot)$, a solution to (2.1), that

$$\mathrm{d}\mathcal{V}(t, x(t)) = \mathcal{L}\mathcal{V}(t, x(t))\mathrm{d}t + \mathcal{V}_x(t, x(t)) g(t, x(t))\mathrm{d}B_t.$$

Now, we recall the following technical lemmas, which will be useful in our analysis.

Lemma 2.1 [20] *For all $x_0 \in \mathbb{R}^d$, such that $x_0 \neq 0$, we have*

$$\mathbb{P}\big(x(t; t_0, x_0) \neq 0, \ \ \forall t \geq t_0\big) = 1.$$

This means that almost all sample paths of any solution beginning from a nonzero state will never arrive at the origin.

The following lemma is known as the exponential martingale inequality and will play an important role in the analysis of some results in this paper.

Lemma 2.2 [20] *Let $g = (g_1, \dots, g_d) \in L^2(\mathbb{R}_+, \mathbb{R}^d)$, and τ, μ, η are positive constants. Then, for $t_0 \geq 0$,*

$$\mathbb{P}\Big(\sup_{t_0 \leq t \leq \tau}\Big[\int_{t_0}^{t} g(s)\mathrm{d}B_s - \frac{\mu}{2}\int_{t_0}^{t} ||g(s)||^2 \mathrm{d}s\Big] > \eta\Big) \leq \exp(-\mu\eta).$$

3 Partial practical stability of stochastic differential equations with general decay rate

We state the definition of partial convergence towards a small ball centered at the origin with a general decay function $\lambda(t)$. Mao [21] was the first who introduced the concept of stability with the polynomial decay rate. After that, this concept was generalized to the stability with general decay rate, see [17,19,22].

Definition 3.1 Let $\lambda(t)$ be a positive function defined for sufficiently large $t > 0$, such that $\lambda(t) \to +\infty$ as $t \to +\infty$. A solution $x(t) = (x_1(t), x_2(t))$ to system (2.1) is said to tend to the ball $\mathcal{B}_r := \{x \in \mathbb{R}^d : ||x|| \leq r\}$, $r > 0$ with respect to x_1 with decay function $\lambda(t)$ and order at least $\gamma > 0$, if its generalized Lyapunov exponent is less than or equal to $-\gamma$ with probability one, i.e.,

$$\lim_{t \to +\infty} \sup \frac{\ln(||x_1(t, t_0, x_0)|| - r)}{\ln \lambda(t)} \leq -\gamma, \ \text{a.s.} \tag{3.1}$$

If in addition, 0 is a solution to system (2.1), the zero solution is said to be almost surely practically asymptotically stable with respect to x_1 with decay function $\lambda(t)$ and order at least γ, if every solution to system (2.1) tends to the boundary of the ball $\mathcal{B}_r$ with respect to x_1 with decay function $\lambda(t)$ and order at least γ, for all $r > 0$ sufficiently small.

Remark 3.1 Clearly, replacing in the above definition the decay function $\lambda(t)$ by $O(\mathrm{e}^t)$ leads to the partial practical exponential stability, which is investigated in [18].

Remark 3.2 We should mention that the authors in [18] have established sufficient conditions for partial practical exponential stability while here we will prove the convergence to a small ball by using Lyapunov functions.

Definition 3.2 [18] The solutions of (2.1) are said to be globally uniformly bounded with probability one, if for each $\beta > 0$, there exists $c = c(\beta) > 0$ (independent of t_0), such that

$$\text{for every } t_0 \geq 0, \text{ and all } x_0 \in \mathbb{R}^d \text{ with } \quad ||x_0|| \leq \beta, \quad \sup_{t \geq t_0} ||x(t, t_0, x_0)|| \leq c(\beta), \quad \text{a.s.} \tag{3.2}$$

Now, we state and prove one of our main results in this paper.

Theorem 3.3 *Assume that there exist a function $\mathcal{V} \in C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)$, three continuous functions $\psi_1(t) \in \mathbb{R}$, $\psi_2(t) \geq 0$, $\rho(t) > 0$, some constants $q \in \mathbb{N}^*$, $m \geq 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \in \mathbb{R}$, and a small constant $\xi \geq 0$, such that for all $t \geq t_0$ and all $x = (x_1, x_2) \in \mathbb{R}^d$,*

($\mathcal{H}_1$) $\lambda(t)^m ||x_1||^q \leq \mathcal{V}(t, x)$,

($\mathcal{H}_2$) $\mathcal{L}\mathcal{V}(t, x) \leq \psi_1(t)\mathcal{V}(t, x) + \rho(t)$,

($\mathcal{H}_3$) $\mathcal{H}\mathcal{V}(t, x) \geq \psi_2(t)\mathcal{V}^2(t, x) + \xi$,

($\mathcal{H}_4$)
$$\lim_{t \to +\infty} \sup \frac{\int_T^t \psi_1(s)\mathrm{d}s}{\ln \lambda(t)} \leq \alpha_3, \quad \forall T \geq t_0,$$
$$\lim_{t \to +\infty} \inf \frac{\int_T^t \psi_2(s)\mathrm{d}s}{\ln \lambda(t)} \geq 2\alpha_1, \quad \forall T \geq t_0,$$
$$\lim_{t \to +\infty} \sup \frac{\ln(t)}{\ln \lambda(t)} \leq \frac{\alpha_2}{2},$$

($\mathcal{H}_5$)
$$\lim_{t \to +\infty} \sup \frac{t}{\ln \lambda(t)} = C \geq 0, \quad \lim_{t \to +\infty} \frac{\rho(t)}{\lambda(t)^m} = \zeta > 0.$$

Let $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, such that the corresponding solution $x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0))$ satisfies:

(i) $||x_1(t, t_0, x_0)|| > \left(\frac{\rho(t)}{\lambda(t)^m}\right)^{\frac{1}{q}}, \quad \forall t \geq t_0,$

(ii) *$x_2(t, t_0, x_0)$ is globally uniformly bounded with probability one.*

Then, for every $\beta \in (0, 1)$, it follows

$$\lim_{t \to +\infty} \sup \frac{\ln\left(||x_1(t, t_0, x_0)|| - \left(\frac{\rho(t)}{\lambda(t)^m}\right)^{\frac{1}{q}}\right)}{\ln \lambda(t)} \leq -\Big[m - \Big(\frac{1}{\beta}\alpha_2 + \alpha_3 - \alpha_1(1 - \beta) + C\Big)\Big], \ \textit{a.s.}$$

Proof For $x_0 \neq 0$, from Lemma 2.1 we deduce $x(t) = (x_1(t), x_2(t)) \neq 0$, $\forall t \geq t_0$ almost surely. Observe that

$$\lambda(t)^m ||x_1(t)||^q - \rho(t) = \lambda(t)^m \Big(||x_1(t)||^q - \frac{\rho(t)}{\lambda(t)^m}\Big) = \lambda(t)^m \Big(||x_1(t)||^q - \Big(\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}}\Big)^q\Big).$$

Based upon the following inequality:

$$a^q - b^q = (a - b)(a^{q-1} + a^{q-2}b + a^{q-3}b^2 + \cdots + a^0 b^{q-1}),$$

we deduce

$$\lambda(t)^m ||x_1(t)||^q - \rho(t) = \lambda(t)^m \Big(||x_1(t)|| - \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}} \Big) \sum_{k=1}^{q} ||x_1(t)||^{q-k} \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{k-1}{q}}. \tag{3.3}$$

Notice that, from condition $(\mathcal{H}_5)$, we have $\lim_{t\to+\infty} \rho(t)/\lambda(t)^m = \zeta > 0$. Thus, for $0 < \zeta_0 < \zeta$, there exists $T \geq t_0$ such that $\rho(t)/\lambda(t)^m \geq \zeta_0$, $\forall t \geq T$. Then, as we are also assuming that $||x_1(t)|| > (\rho(t)/\lambda(t)^m)^{\frac{1}{q}}$, $\forall t \geq 0$, it yields

$$\begin{aligned} \sum_{k=1}^{q} ||x_1(t)||^{q-k} \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{k-1}{q}} &= ||x_1(t)||^{q-1} + ||x_1(t)||^{q-2} \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}} + \cdots + \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{q-1}{q}} \\ &\geq \zeta' = q\zeta_0^{(q-1)/q}, \quad \forall t \geq T. \end{aligned}$$

It follows that that there exists t_0 such that

$$\lambda(t)^m ||x_1(t)||^q - \rho(t) \geq \lambda(t)^m \Big(||x_1(t)|| - \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}} \Big) \zeta', \quad \forall t \geq T.$$

Hence,

$$\mathcal{V}(t, x(t)) \geq \lambda(t)^m ||x_1(t)||^q \geq \lambda(t)^m ||x_1(t)||^q - \rho(t) \geq \lambda(t)^m \Big(||x_1(t)|| - \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}} \Big) \zeta'.$$

That is,

$$\zeta' \lambda(t)^m \Big(||x_1(t)|| - \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}} \Big) \leq \mathcal{V}(t, x(t)),$$

and

$$\ln(\zeta') + \ln\Big[\lambda(t)^m \Big(||x_1(t)|| - \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}} \Big) \Big] \leq \ln\Big[\mathcal{V}(t, x(t)) \Big].$$

Thus, we obtain

$$\ln(\zeta') + m \ln \lambda(t) + \ln\Big[||x_1(t)|| - \Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}} \Big] \leq \ln\Big[\mathcal{V}(t, x(t)) \Big], \quad \forall t \geq T.$$

Invoking Itô's formula for $\mathcal{V}(\cdot)$ along the trajectory $x(\cdot)$ of the stochastic system (2.1), we obtain $\forall t \geq T$,

$$\mathrm{d}\big(\ln(\mathcal{V}(t, x(t))) \big) = \frac{\mathcal{L}\mathcal{V}(t, x(t))}{\mathcal{V}(t, x(t))} \mathrm{d}t + \frac{\mathcal{V}_x(t, x(t)) g(t, x(t))}{\mathcal{V}(t, x(t))} \mathrm{d}B_t - \frac{1}{2} \frac{\mathcal{H}\mathcal{V}(t, x(t))}{\mathcal{V}^2(t, x(t))} \mathrm{d}t.$$

That is,

$$\int_T^t \mathrm{d}(\ln(\mathcal{V}(s, x(s))))\mathrm{d}s = \int_T^t \frac{\mathcal{L}\mathcal{V}(s, x(s))}{\mathcal{V}(s, x(s))} \mathrm{d}s + \int_T^t \frac{\mathcal{V}_x(s, x(s)) g(s, x(s))}{\mathcal{V}(s, x(s))} \mathrm{d}B_s - \frac{1}{2} \int_T^t \frac{\mathcal{H}\mathcal{V}(s, x(s))}{\mathcal{V}^2(s, x(s))} \mathrm{d}s.$$

Hence, we obtain

$$\ln(\mathcal{V}(t,x(t))) = \ln(\mathcal{V}(T,x(T))) + \int_T^t \frac{\mathcal{L}\mathcal{V}(s,x(s))}{\mathcal{V}(s,x(s))}\mathrm{d}s + \mathcal{M}(t) - \frac{1}{2}\int_T^t \frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s, \tag{3.4}$$

where

$$\mathcal{M}(t) = \int_T^t \frac{\mathcal{V}_x(s,x(s))g(s,x(s))}{\mathcal{V}(s,x(s))}\mathrm{d}B_s$$

is a continuous martingale with $\mathcal{M}(T) = 0$.
Using $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, it follows that, $\forall t \geq T$,

$$\ln(\mathcal{V}(t,x(t))) \leq \ln(\mathcal{V}(T,x(T))) + \int_T^t \psi_1(s)\mathrm{d}s + \int_T^t \frac{\rho(s)}{\lambda(s)^m ||x_1(s)||^q}\mathrm{d}s + \mathcal{M}(t) - \frac{1}{2}\int_T^t \frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s. \tag{3.5}$$

Consequently, we obtain

$$\ln(\mathcal{V}(t,x(t))) \leq \ln(\mathcal{V}(T,x(T))) + \int_T^t \psi_1(s)\mathrm{d}s + (t-T) + \mathcal{M}(t) - \frac{1}{2}\int_T^t \frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s. \tag{3.6}$$

Thanks to the exponential martingale inequality (2.2), we have

$$\mathbb{P}\left\{\omega : \sup_{T\leq t\leq \tau}\left[\mathcal{M}(t) - \frac{\mu}{2}\int_T^t \frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\right] > \eta\right\} \leq \mathrm{e}^{-\mu\eta},$$

for any positive constants μ, η, and τ. In particular, take $0 < \beta < 1$, and set

$$\mu = \beta, \quad \eta = \frac{2}{\beta}\ln(k-1), \quad \tau = k \in \mathbb{N}, \quad k > T+1.$$

We then apply the well-known Borel–Cantelli lemma to obtain that, for almost all $\omega \in \Omega$, $\exists\, k_0 = k(\varepsilon,\omega) > 0$, with

$$\mathcal{M}(t) \leq \frac{2}{\beta}\ln(k-1) + \frac{\beta}{2}\int_T^t \frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s, \quad \forall T \leq t \leq k,\ k \geq k_0(\varepsilon,\omega).$$

Substituting this into Eq. (3.6) and taking into account condition $(\mathcal{H}_3)$, we obtain

$$\ln(\mathcal{V}(t,x(t))) \leq \ln(\mathcal{V}(T,x(T))) + \frac{2}{\beta}\ln(k-1) + \int_T^t \psi_1(s)\mathrm{d}s + (t-T) - \frac{1-\beta}{2}\int_T^t \psi_2(s)\mathrm{d}s, \tag{3.7}$$

for $T \leq t \leq k,\ k \geq k_0(\varepsilon,\omega)$.
From condition $(\mathcal{H}_4)$, for $\varepsilon > 0$, $\exists \tilde{T} \geq T > T_0$, such that $\forall t > \tilde{T}$, we have

$$\int_T^t \psi_1(s)\mathrm{d}s \leq (\alpha_3+\varepsilon)\ln\lambda(t), \quad \int_T^t \psi_2(s)\mathrm{d}s \geq (2\alpha_1-\varepsilon)\ln\lambda(t), \quad \ln(t) \leq \frac{1}{2}(\alpha_2+\varepsilon)\ln\lambda(t).$$

Then, inequality (3.7) becomes

$$\ln(\mathcal{V}(t,x(t))) \leq \ln(\mathcal{V}(T,x(T))) + \frac{\alpha_2+\varepsilon}{\beta}\ln\lambda(t) + (\alpha_3+\varepsilon)\ln\lambda(t) - \frac{1-\beta}{2}(2\alpha_1-\varepsilon)\ln\lambda(t) + (t-T),$$

for $k-1 \leq t \leq k,\ k \geq k_0(\varepsilon,\omega)$, which yields, by using condition $(\mathcal{H}_5)$, that

$$\lim_{t\to+\infty}\sup\frac{\ln(\mathcal{V}(t,x(t)))}{\ln\lambda(t)} \leq \frac{\alpha_2+\varepsilon}{\beta} + (\alpha_3+\varepsilon) - \frac{1-\beta}{2}(2\alpha_1-\varepsilon) + C, \quad \text{a.s.}$$

Recall that, for $t \geq T$ and $q \in \mathbb{N}^\star$, we have

$$\ln\left(||x_1(t)|| - \left(\frac{\rho(t)}{\lambda(t)^m}\right)^{\frac{1}{q}}\right) \leq \ln(\mathcal{V}(t, x(t))) - m\ln\lambda(t) - \ln(\zeta').$$

Taking into account the fact that $\varepsilon > 0$ is arbitrary, we deduce that

$$\lim_{t\to+\infty}\sup\frac{\ln\left(||x_1(t)|| - \left(\frac{\rho(t)}{\lambda(t)^m}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\Big[m - \Big(\frac{1}{\beta}\alpha_2 + \alpha_3 - \alpha_1(1-\beta) + C\Big)\Big], \text{ a.s.,}$$

as required. □

Remark 3.4 Note that, in the above theorem, the decaying order of the solution depends on the parameter β. The crucial question imposed concerns the possibility of determining the largest value for γ_β. Indeed, in order to find the optimal valued $\gamma^\star = \sup_{0<\beta<1}\gamma_\beta$, we require to find out the minimum value $J^\star$ for the following function:

$$J(\beta) = \frac{1}{\beta}\alpha_2 + \alpha_3 - \alpha_1(1-\beta) + C$$

when the parameter $\beta \in (0, 1)$.

Hence, we deduce that the optimal value of $\gamma^\star$ will hold with $\gamma^\star = (m - J^\star)$. It is straightforward to check that

$$J^\star = \begin{cases} 2(\alpha_1\alpha_2)^{\frac{1}{2}} + \alpha_3 - \alpha_1 + C & \text{if } 0 \leq \alpha_2 < \alpha_1, \\ \alpha_2 + \alpha_3 + C & \text{if } \alpha_1 \leq \alpha_2. \end{cases}$$

Finally, we obtain

$$\gamma^\star = \begin{cases} m - \Big(2(\alpha_1\alpha_2)^{\frac{1}{2}} + \alpha_3 - \alpha_1 + C\Big) & \text{if } 0 \leq \alpha_2 < \alpha_1, \\ m - \Big(\alpha_2 + \alpha_3 + C\Big) & \text{if } \alpha_1 \leq \alpha_2. \end{cases}$$

In the next corollary, we will deduce the partial convergence to a ball with a general decay rate.

Corollary 3.5 *Assume that there exist a function $\mathcal{V} \in C^{1.2}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)$, three continuous functions $\psi_1(t) \in \mathbb{R}$, $\psi_2(t) \geq 0$, $\rho(t) > 0$, some constants $q \in \mathbb{N}^*$, $m \geq 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \in \mathbb{R}$, and a small constant $\xi \geq 0$, such that $\forall t \geq t_0$ and all $x = (x_1, x_2) \in \mathbb{R}^d$, assumptions $(\mathcal{H}_1) - (\mathcal{H}_5)$ are satisfied.*
Let $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, such that the corresponding solution $x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0))$ satisfies:

(i) $||x_1(t, t_0, x_0)|| > \Big(\rho(t)/\lambda(t)^m\Big)^{\frac{1}{q}}, \quad \forall t \geq t_0,$
(ii) $x_2(t, t_0, x_0)$ *is globally uniformly bounded with probability one.*

Then, if additionally there exists $\widetilde{\zeta} \geq \zeta > 0$, such that $||x_1(t, t_0, x_0)|| > \widetilde{\zeta}$, $\forall t \geq t_0$, it follows

$$\lim_{t\to+\infty}\sup\frac{\ln\left(||x_1(t, t_0, x_0)|| - \left(\widetilde{\zeta}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\gamma^\star, \quad a.s.,$$

where

$$\gamma^\star = \begin{cases} m - \Big(2(\alpha_1\alpha_2)^{\frac{1}{2}} + \alpha_3 - \alpha_1 + C\Big) & \text{if} \quad 0 \le \alpha_2 < \alpha_1, \\ m - \Big(\alpha_2 + \alpha_3 + C\Big) & \text{if} \quad \alpha_1 \le \alpha_2. \end{cases}$$

In particular, if $\gamma^\star > 0$, then the solution to system (2.1) tends to the ball $\mathcal{B}_r$, with $r = \big(\widetilde{\zeta}\big)^{\frac{1}{q}}$ almost surely with respect to x_1 with decay function $\lambda(t)$ and order at least $\gamma^\star$.

Proof Let $x_0 \neq 0$ in $\mathbb{R}^d$. It immediately follows from Lemma 2.1, $x(t) = (x_1(t), x_2(t)) \neq 0$, $\forall t \ge 0$ almost surely. Based upon Theorem 3.3, we have

$$\lim_{t\to+\infty} \sup \frac{\ln\left(||x_1(t)|| - \left(\frac{\rho(t)}{\lambda(t)^m}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \le -\gamma^\star, \quad \text{a.s.},$$

where

$$\gamma^\star = \begin{cases} m - \Big(2(\alpha_1\alpha_2)^{\frac{1}{2}} + \alpha_3 - \alpha_1 + C\Big), & \text{if} \quad 0 \le \alpha_2 < \alpha_1, \\ m - \Big(\alpha_2 + \alpha_3 + C\Big), & \text{if} \quad \alpha_1 \le \alpha_2. \end{cases}$$

Since, $\lim\limits_{t\to+\infty} \frac{\rho(t)}{\lambda(t)^m} = \zeta \le \widetilde{\zeta}$, then there exists $T \ge t_0$ such that $\frac{\rho(t)}{\lambda(t)^m} \le \widetilde{\zeta}$, $\forall t \ge T$. Consequently,

$$\lim_{t\to+\infty} \sup \frac{\ln\left(||x_1(t)|| - \big(\widetilde{\zeta}\big)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \le \lim_{t\to+\infty} \sup \frac{\ln\left(||x_1(t)|| - \left(\frac{\rho(t)}{\lambda(t)^m}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \le -\gamma^\star \quad \text{a.s.}$$

Hence, if $\gamma^\star > 0$, then the solution to system (2.1) tends to the ball $\mathcal{B}_r$, with $r = \Big(\widetilde{\zeta}\Big)^{\frac{1}{q}}$ with respect to x_1 with decay function $\lambda(t)$ and order at least $\gamma^\star$. □

Now, we will improve the statement of Theorem 3.3, when $\mathcal{HV}(t,x)$ is also bounded above.

Theorem 3.6 *Assume that there exist a function $\mathcal{V} \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+)$, four continuous functions $\psi_1(t) \in \mathbb{R}$, $\psi_2(t) \ge 0$, $\psi_3(t) \ge 0$, $\rho(t) > 0$ some constants $q \in \mathbb{N}^*$, $m \ge 0$, $\alpha_1 \ge 0$, $\alpha_2 \ge 0$, $\alpha_3 \in \mathbb{R}$, a small constant $\xi \ge 0$, such that for all $t \ge t_0$ and all $x = (x_1, x_2) \in \mathbb{R}^d$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_5)$ hold, and the following assumptions:*

$(\mathcal{H}_3')$ $\psi_2(t)\mathcal{V}^2(t,x) + \xi \le \mathcal{HV}(t,x) \le \psi_3(t)\mathcal{V}^2(t,x)$,

$(\mathcal{H}_4')$ $$\lim_{t\to+\infty}\sup \frac{\int_T^t \psi_1(s)\mathrm{d}s}{\ln\lambda(t)} \le \alpha_3, \quad \forall T \ge t_0,$$
$$\lim_{t\to+\infty}\inf \frac{\int_T^t \psi_2(s)\mathrm{d}s}{\ln\lambda(t)} \ge 2\alpha_1, \quad \forall T \ge t_0,$$
$$\lim_{t\to+\infty}\sup \frac{\int_T^t \psi_3(s)\mathrm{d}s}{\ln\lambda(t)} \le \alpha_2, \quad \forall T \ge t_0.$$

Let $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, such that the corresponding solution $x(t,t_0,x_0) = (x_1(t,t_0,x_0), x_2(t,t_0,x_0))$ satisfies:

(i) $||x_1(t,t_0,x_0)|| > \left(\frac{\rho(t)}{\lambda(t)^m}\right)^{\frac{1}{q}}$, $\quad \forall t \ge t_0$,

(ii) *$x_2(t,t_0,x_0)$ is globally uniformly bounded with probability one.*

Hence,

$$\lim_{t\to+\infty}\sup\frac{\ln\left(||x_1(t,t_0,x_0)||-\left(\frac{\rho(t)}{\lambda(t)^m}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)}\le-\big[m-(\alpha_3-\alpha_1+C)\big],\ a.s.$$

Proof Let $x_0\neq 0$ in $\mathbb{R}^d$. It follows from Lemma 2.1, $x(t)=(x_1(t),x_2(t))\neq 0$, $\forall t\geq 0$ almost surely. Applying the Itô formula once more implies again Eq. (3.4):

$$\ln(\mathcal{V}(t,x(t)))=\ln(\mathcal{V}(T,x(T)))+\int_T^t\frac{\mathcal{L}\mathcal{V}(s,x(s))}{\mathcal{V}(s,x(s))}\mathrm{d}s+\mathcal{M}(t)-\frac{1}{2}\int_T^t\frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s,$$

with

$$\mathcal{M}(t)=\int_T^t\frac{\mathcal{V}_x(s,x(s))g(s,x(s))}{\mathcal{V}(s,x(s))}\mathrm{d}B_s.$$

Taking into account assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_3')$, we obtain that, $\forall t\geq T$,

$$\ln(\mathcal{V}(t,x(t)))=\ln(\mathcal{V}(T,x(T)))+\mathcal{M}(t)+\int_T^t\psi_1(s)\mathrm{d}s+t-\frac{1}{2}\int_T^t\psi_2(s)\mathrm{d}s. \tag{3.8}$$

Based upon conditions $(\mathcal{H}_4')$ and $(\mathcal{H}_5)$, it follows that

$$\lim_{t\to+\infty}\sup\frac{\ln(\mathcal{V}(t,x(t)))}{\ln\lambda(t)}\le\lim_{t\to+\infty}\sup\frac{\mathcal{M}(t)}{\ln\lambda(t)}+(\alpha_3+\varepsilon)-\frac{1}{2}(2\alpha_1-\varepsilon)+C,\quad \text{a.s.}$$

Let us denote by $\langle\mathcal{M}(t)\rangle$ the quadratic variation process associated to $\mathcal{M}(t)$.
Based on our assumptions we deduce that $\mathcal{M}(t)$ is a local martingale vanishing at $t=T$.

Moreover, on account of condition $(\mathcal{H}_3')$, we obtain

$$\int_T^t\psi_2(s)\mathrm{d}s+\int_T^t\frac{\xi}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\le\langle\mathcal{M}(t)\rangle=\int_T^t\frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\le\int_T^t\psi_3(s)\mathrm{d}s.$$

Since $\alpha_1>0$, it follows that

$$\lim_{t\to+\infty}\langle\mathcal{M}(t)\rangle=+\infty.$$

Thanks to the strong law of large numbers (see, [20]), we obtain

$$\lim_{t\to+\infty}\frac{\mathcal{M}(t)}{\langle\mathcal{M}(t)\rangle}=0,\quad \text{a.s.}$$

Note that, for t large enough, we have

$$\frac{||\mathcal{M}(t)||}{\ln\lambda(t)}=\frac{||\mathcal{M}(t)||}{\langle\mathcal{M}(t)\rangle}\frac{\langle\mathcal{M}(t)\rangle}{\ln\lambda(t)}\le\frac{||\mathcal{M}(t)||}{\langle\mathcal{M}(t)\rangle}\frac{\int_0^t\psi_3(s)\mathrm{d}s}{\ln\lambda(t)}.$$

Assumption $(\mathcal{H}_4')$ allows us to conclude that

$$\lim_{t\to+\infty}\sup\frac{\mathcal{M}(t)}{\ln\lambda(t)}=0,\quad \text{a.s.}$$

Therefore,

$$\lim_{t\to+\infty}\sup\frac{\ln(\mathcal{V}(t,x(t)))}{\ln\lambda(t)}\le(\alpha_3+\varepsilon)-\frac{1}{2}(2\alpha_1-\varepsilon)+C,\quad \text{a.s.}$$

Since the constant $\varepsilon>0$, hence we infer that

$$\lim_{t\to+\infty}\sup\frac{\ln\Big(||x_1(t)||-\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}}\Big)}{\ln\lambda(t)}\le-\Big[m-(\alpha_3-\alpha_1+C)\Big],\quad \text{a.s.}$$

as required. □

Now, we aim at proving the partial convergence with a general decay rate.

Corollary 3.7 *Assume that there exist a function $\mathcal{V}\in C^{1.2}(\mathbb{R}_+\times\mathbb{R}^d,\mathbb{R}_+)$, four continuous functions $\psi_1(t)\in\mathbb{R}$, $\psi_2(t)\ge 0$, $\psi_3(t)\ge 0$, $\rho(t)>0$, some constants $q\in\mathbb{N}^*$, $m\ge 0$, $\alpha_1\ge 0$, $\alpha_2\ge 0$, $\alpha_3\in\mathbb{R}$, and a small constant $\xi\ge 0$, such that for all $t\ge t_0\ge 0$ and all $x=(x_1,x_2)\in\mathbb{R}^d$, assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3')$, $(\mathcal{H}_4')$, and $(\mathcal{H}_5)$ are satisfied.*
Let $x_0\in\mathbb{R}^d$, $x_0\neq 0$, such that the corresponding solution $x(t,t_0,x_0)=(x_1(t,t_0,x_0),x_2(t,t_0,x_0))$ satisfies:

(i) $||x_1(t,t_0,x_0)||>\big(\rho(t)/\lambda(t)^m\big)^{\frac{1}{q}},\quad \forall t\ge t_0,$
(ii) *$x_2(t,t_0,x_0)$ is globally uniformly bounded with probability one.*

Then, if further there exists $\widetilde{\zeta}\ge\zeta>0$ such that $||x_1(t,t_0,x_0)||>\widetilde{\zeta}$ $\forall t\ge t_0$, it follows

$$\lim_{t\to+\infty}\sup\frac{\ln\Big(||x_1(t,t_0,x_0)||-\big(\widetilde{\zeta}\big)^{\frac{1}{q}}\Big)}{\ln\lambda(t)}\le-\big[m-(\alpha_3-\alpha_1+C)\big],\ \textit{a.s.}$$

In particular, if $m>(\alpha_3-\alpha_1+C)$, then the solution to system (2.1) tends to the ball $\mathcal{B}_r$, with $r=\big(\widetilde{\zeta}\big)^{\frac{1}{q}}$ almost surely with respect to x_1 with decay function $\lambda(t)$ and order at least $\gamma_\beta=\big[m-(\alpha_3-\alpha_1+C)\big]$.

Proof Let $x_0\neq 0$ in $\mathbb{R}^d$, and use Lemma 2.1, we obtain $x(t)=(x_1(t),x_2(t))\neq 0$, $\forall t\ge 0$ almost surely. By using Theorem 3.6, it follows that

$$\lim_{t\to+\infty}\sup\frac{\ln\Big(||x_1(t)||-\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}}\Big)}{\ln\lambda(t)}\le-\Big[m-(\alpha_3-\alpha_1+C)\Big],\ \text{a.s.}$$

Since, we have $\lim_{t\to+\infty}\frac{\rho(t)}{\lambda(t)^m}=\zeta\le\widetilde{\zeta}$, then there exists $T\ge t_0$ such that $\frac{\rho(t)}{\lambda(t)^m}\le\widetilde{\zeta}$, $\forall t\ge T$. Consequently,

$$\lim_{t\to+\infty}\sup\frac{\ln\Big(||x_1(t)||-\big(\widetilde{\zeta}\big)^{\frac{1}{q}}\Big)}{\ln\lambda(t)}\le\lim_{t\to+\infty}\sup\frac{\ln\Big(||x_1(t)||-\big(\frac{\rho(t)}{\lambda(t)^m}\big)^{\frac{1}{q}}\Big)}{\ln\lambda(t)}$$

$$\leq -\left[m - (\alpha_3 - \alpha_1 + C)\right], \text{ a.s.}$$

Therefore, if $m > (\alpha_3 - \alpha_1 + C)$, then the solution to system (2.1) tends to the ball $\mathcal{B}_r$, with $r = \left(\widetilde{\zeta}\right)^{\frac{1}{q}}$ with respect to x_1 with decay function $\lambda(t)$ and order at least $\gamma = \left[m - (\alpha_3 - \alpha_1 + C)\right]$. □

4 Construction of suitable Lyapunov functions

The system (2.1) might be regarded as the following form:

$$\begin{cases} \mathrm{d}x_1(t) = f_1(t, x_1(t), x_2(t))\mathrm{d}t + g_1(t, x_1(t), x_2(t))\mathrm{d}B_t, \\ \mathrm{d}x_2(t) = f_2(t, x_1(t), x_2(t))\mathrm{d}t + g_2(t, x_1(t), x_2(t))\mathrm{d}B_t, \end{cases}$$

with the same initial condition $x(t_0) = x_0 = (x_{10}, x_{20})$, $f := (f_1, f_2)$, and $g := (g_1, g_2)$.

- $f_1 : \mathbb{R}_+ \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1}$, $\quad g_1 : \mathbb{R}_+ \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1 \times m}$.
- $f_2 : \mathbb{R}_+ \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$, $\quad g_2 : \mathbb{R}_+ \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2 \times m}$.

We assume that both conditions of existence and uniqueness of solutions (2.2) and (2.3) are satisfied.

Remark 4.1 In the previous part, we prove partial practical stability of stochastic systems with general decay. Now, our target is to construct an appropriate Lyapunov function, which satisfies all conditions of Corollary 3.5.

The construction of suitable Lyapunov function satisfying conditions of Corollary 3.5 is a strenuous task. Nevertheless, on some occasions it is not complicated to proceed with $\mathcal{V}(t, x) = \lambda(t)^m ||x_1||^2$, where $\lambda \in C^1(\mathbb{R}_+)$, $m \geq 0$.

Theorem 4.2 *Let $\phi_1(t) \in \mathbb{R}$, $\phi_2(t) \geq 0$, $\rho(t) > 0$ be three continuous functions. Assume that there exist constants $\widetilde{\phi}_1 \in \mathbb{R}$, $\widetilde{\phi}_2 \geq 0$, $\alpha_2 \geq 0$, such that $\forall t \geq t_0 \geq 0$, and all $x = (x_1, x_2) \in \mathbb{R}^d$,*

$(\mathcal{A}_1)$ $2\langle x_1, f_1(t, x)\rangle + \frac{1}{2}trace\Big(g_1^{\mathrm{T}}(t, x)g_1(t, x)\Big) \leq \phi_1(t)||x_1||^2 + \frac{\rho(t)}{\lambda(t)^m}$,

$(\mathcal{A}_2)$ $||g_1(t, x)x_1||^2 \geq \phi_2(t)||x_1||^4$,

$(\mathcal{A}_3)$ $\lim\limits_{t\to+\infty} \sup \frac{\int_T^t \phi_1(s)\mathrm{d}s}{\ln \lambda(t)} \leq \widetilde{\phi}_1, \quad \forall T \geq t_0,$

$\lim\limits_{t\to+\infty} \inf \frac{\int_T^t \phi_2(s)\mathrm{d}s}{\ln \lambda(t)} \geq \widetilde{\phi}_2, \quad \forall T \geq t_0,$

$\lim\limits_{t\to+\infty} \sup \frac{\ln(t)}{\ln \lambda(t)} \leq \frac{\alpha_2}{2},$

$(\mathcal{A}_4)$ $\lim\limits_{t\to+\infty} \sup \frac{t}{\ln \lambda(t)} = C \geq 0, \quad \lim\limits_{t\to+\infty} \frac{\rho(t)}{\lambda(t)^m} = \zeta > 0.$

Let $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, such that the corresponding solution $x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0))$ satisfies:

(i) $||x_1(t, t_0, x_0)|| > \left(\rho(t)/\lambda(t)^m\right)^{\frac{1}{q}}, \quad \forall t \geq t_0,$

(ii) $x_2(t, t_0, x_0)$ *is globally uniformly bounded with probability one.*

Then, if rather there exists $\widetilde{\zeta} \geq \zeta > 0$ such that $||x_1(t, t_0, x_0)|| > \widetilde{\zeta}$, $\forall t \geq t_0$, it follows

$$\lim_{t\to+\infty} \frac{\ln\left(||x_1(t, t_0, x_0)|| - \left(\widetilde{\zeta}\right)^{\frac{1}{2}}\right)}{\ln \lambda(t)} \leq -\upsilon, \quad a.s.,$$

with

$$\upsilon = \begin{cases} 2\widetilde{\phi}_2 - \Big(2(2\widetilde{\phi}_2\alpha_2)^{\frac{1}{2}} + \widetilde{\phi}_1 + C\Big) & \textit{if} \quad 0 \le \alpha_2 < 2\widetilde{\phi}_2, \\ -\Big(2\widetilde{\phi}_2 + \widetilde{\phi}_1 + C\Big) & \textit{if} \quad 2\widetilde{\phi}_2 \le \alpha_2. \end{cases}$$

In particular, if $\upsilon > 0$, then the solution to system (2.1) tends to the ball $\mathcal{B}_r$, with $r = \big(\widetilde{\zeta}\big)^{\frac{1}{2}}$ almost surely with respect to x_1 with decay function $\lambda(t)$ and order at least υ.

Proof Let $x_0 \neq 0$ in $\mathbb{R}^d$. From Lemma 2.1 it follows, $x(t) = (x_1(t), x_2(t)) \neq 0$, $\forall t \ge 0$ almost surely. Consider $\mathcal{V}(t, x(t)) = \lambda(t)^m ||x_1(t)||^2$, then we get

$$\begin{aligned} \mathcal{L}\mathcal{V}(t, x(t)) &= m\lambda'(t)\lambda(t)^{m-1}||x_1(t)||^2 + 2\lambda(t)^m \langle x_1(t), f_1(t, x(t))\rangle + \lambda(t)^m \mathrm{trace}\Big(g_1^{\mathrm{T}}(t, x(t))g_1(t, x(t))\Big) \\ &\le m\lambda'(t)\lambda(t)^{m-1}||x_1(t)||^2 + \lambda(t)^m\Big(\phi_1(t)||x_1(t)||^2 + \frac{\rho(t)}{\lambda(t)^m}\Big) \\ &= \Big(m\frac{\lambda'(t)}{\lambda(t)} + \phi_1(t)\Big)\mathcal{V}(t, x(t)) + \rho(t). \end{aligned}$$

As well as

$$\mathcal{H}\mathcal{V}(t, x(t)) = 4\lambda^{2m}(t)||g_1(t, x(t))\, x_1(t)||^2 \ge 4\phi_2(t)\mathcal{V}^2(t, x(t)).$$

Thus, setting

$$\psi_1(t) = \Big(m\frac{\lambda'(t)}{\lambda(t)} + \phi_1(t)\Big), \quad \psi_2(t) = 4\phi_2(t),$$

we deduce

$$\lim_{t\to+\infty} \sup \frac{\int_T^t \psi_1(s)\mathrm{d}s}{\ln\lambda(t)} \le m + \widetilde{\phi}_1, \quad \lim_{t\to+\infty} \inf \frac{\int_T^t \psi_2(s)\mathrm{d}s}{\ln\lambda(t)} \ge 4\widetilde{\phi}_2.$$

Finally, Corollary 3.5 allows us to conclude that

$$\lim_{t\to+\infty} \frac{\ln\Big(||x_1(t)|| - \big(\widetilde{\zeta}\big)^{\frac{1}{2}}\Big)}{\ln\lambda(t)} \le -\upsilon, \quad \text{a.s.,}$$

where

$$\upsilon = \begin{cases} 2\widetilde{\phi}_2 - \Big(2(2\widetilde{\phi}_2\alpha_2)^{\frac{1}{2}} + \widetilde{\phi}_1 + C\Big) & \text{if} \quad 0 \le \alpha_2 < 2\widetilde{\phi}_2, \\ -\Big(2\widetilde{\phi}_2 + \widetilde{\phi}_1 + C\Big) & \text{if} \quad 2\widetilde{\phi}_2 \le \alpha_2. \end{cases}$$

Thus, if $\upsilon > 0$, then the solution to system (2.1) tends to the ball $\mathcal{B}_r$, with $r = \big(\widetilde{\zeta}\big)^{\frac{1}{2}}$ almost surely with respect to x_1 with decay function $\lambda(t)$ and order at least υ. □

Now, our goal is to construct a suitable Lyapunov function when the function $\mathcal{H}\mathcal{V}(t, x)$ is bounded above through appropriate term. This means that our task is to find an adequate Lyapunov function which satisfies all conditions of

Corollary 3.7. We content ourselves with a particular case in which the Lyapunov function $\mathcal{V}(t,x)$ can be considered as $\lambda(t)^m||x_1||^2$ with $\lambda \in C^1(\mathbb{R}_+)$, and $m \geq 0$.

Theorem 4.3 *Let $\phi_1(t) \in \mathbb{R}$, $\phi_2(t) \geq 0$, $\phi_3(t) \geq 0$, $\rho(t) > 0$, be four continuous functions. Assume that there exist constants $\widetilde{\phi}_1 \in \mathbb{R}$, $\widetilde{\phi}_2 \geq 0$, $\widetilde{\phi}_3 \geq 0$, such that $\forall t \geq t_0$ and all $x = (x_1, x_2) \in \mathbb{R}^d$, $(\mathcal{A}_1)$, $(\mathcal{A}_4)$ hold, and the following assumptions:*

$(\mathcal{A}_2')$ $\phi_2(t)||x_1||^4 \leq ||g_1(t,x)x_1||^2 \leq \phi_3(t)||x_1||^4,$

$(\mathcal{A}_3')$
$$\lim_{t\to+\infty} \sup \frac{\int_T^t \phi_1(s)\mathrm{d}s}{\ln\lambda(t)} \leq \widetilde{\phi}_1, \quad \forall T \geq t_0,$$
$$\lim_{t\to+\infty} \inf \frac{\int_T^t \phi_2(s)\mathrm{d}s}{\ln\lambda(t)} \geq \widetilde{\phi}_2, \quad \forall T \geq t_0,$$
$$\lim_{t\to+\infty} \sup \frac{\int_T^t \phi_3(s)\mathrm{d}s}{\ln\lambda(t)} \leq \widetilde{\phi}_3, \quad \forall T \geq t_0.$$

Let $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, such that the corresponding solution $x(t,t_0,x_0) = (x_1(t,t_0,x_0), x_2(t,t_0,x_0))$ satisfies:

(i) $||x_1(t,t_0,x_0)|| > \left(\rho(t)/\lambda(t)^m\right)^{\frac{1}{2}}, \quad \forall t \geq t_0,$

(ii) *$x_2(t,t_0,x_0)$ is globally uniformly bounded with probability one.*

Then, if rather there exists $\widetilde{\zeta} \geq \zeta > 0$, such that $||x_1(t,t_0,x_0)|| > \widetilde{\zeta}$, $\forall t \geq t_0$, it follows

$$\lim_{t\to+\infty} \frac{\ln\left(||x_1(t,t_0,x_0)|| - \left(\widetilde{\zeta}\right)^{\frac{1}{2}}\right)}{\ln\lambda(t)} \leq -\left[2\widetilde{\phi}_2 - (\widetilde{\phi}_1 + C)\right], \ a.s.$$

Proof Let $x_0 \neq 0$ in $\mathbb{R}^d$. Using Lemma 2.1, $x(t) = (x_1(t), x_2(t)) \neq 0$, $\forall t \geq 0$ almost surely.
Define the Lyapunov function in the following form,

$$\mathcal{V}(t,x(t)) = \lambda(t)^m||x_1(t)||^2.$$

Applying the Itô formula, we obtain

$$\begin{aligned}\mathcal{L}\mathcal{V}(t,x(t)) &= m\lambda'(t)\lambda(t)^{m-1}||x_1(t)||^2 + 2\lambda(t)^m\langle x_1(t), f_1(t,x(t))\rangle + \lambda(t)^m \mathrm{trace}\Big(g_1^{\mathrm{T}}(t,x(t))g_1(t,x(t))\Big)\\ &\leq m\lambda'(t)\lambda(t)^{m-1}||x_1(t)||^2 + \lambda(t)^m\left(\phi_1(t)||x_1(t)||^2 + \frac{\rho(t)}{\lambda(t)^m}\right)\\ &= \left(m\frac{\lambda'(t)}{\lambda(t)} + \phi_1(t)\right)\mathcal{V}(t,x(t)) + \rho(t).\end{aligned}$$

As well as

$$4\phi_2(t)\mathcal{V}^2(t,x(t)) \leq \mathcal{H}\mathcal{V}(t,x(t)) = 4\lambda^{2m}(t)||g_1(t,x(t))\ x_1(t)||^2 \leq 4\phi_3(t)\mathcal{V}^2(t,x(t)).$$

Setting

$$\psi_1(t) = \left(m\frac{\lambda'(t)}{\lambda(t)} + \phi_1(t)\right), \quad \psi_2(t) = 4\phi_2(t), \quad \psi_3(t) = 4\phi_3(t).$$

Therefore, we obtain

$$\lim_{t\to+\infty} \sup \frac{\int_T^t \psi_1(s)\mathrm{d}s}{\ln\lambda(t)} \leq m + \widetilde{\phi}_1, \quad \lim_{t\to+\infty} \inf \frac{\int_T^t \psi_2(s)\mathrm{d}s}{\ln\lambda(t)} \geq 4\widetilde{\phi}_2, \quad \lim_{t\to+\infty} \sup \frac{\int_T^t \psi_3(s)\mathrm{d}s}{\ln\lambda(t)} \leq 4\widetilde{\phi}_3.$$

Hence, from Corollary 3.7 one can deduce that

$$\lim_{t\to+\infty} \frac{\ln\left(||x_1(t)|| - \left(\widetilde{\xi}\right)^{\frac{1}{2}}\right)}{\ln\lambda(t)} \leq -\left[2\widetilde{\phi}_2 - (\widetilde{\phi}_1 + C)\right], \quad \text{a.s.}$$

The theorem is proved. □

5 Example

To show the validity of our results, let us consider the following numerical example given by

$$\begin{cases} \mathrm{d}z_1(t) = (atz_1 + e^{-z_2})\mathrm{d}t + bt^{\frac{1}{2}} z_1 \mathrm{d}B_1(t) \\ \mathrm{d}z_2(t) = -\sin^2(z_1)z_2\mathrm{d}t + \sqrt{2}\sin(z_1)z_2\mathrm{d}B_2(t) \end{cases} \tag{5.1}$$

where $a, b \in \mathbb{R}$ and $z(t) = (z_1(t), z_2(t))^T \in \mathbb{R}^2$ with initial value $z_0 = (z_{1_0}, z_{2_0})$.

Let $\mathcal{V} := z_1^2$, we obtain

$$\begin{aligned}\mathcal{L}\mathcal{V}(t,z) &= 2z_1(atz_1 + \mathrm{e}^{-z_2}) + b^2tz_1^2 \leq (2a+b^2)tz_1^2 + 2z_1 \leq (2a+b^2)tz_1^2 + 2|z_1| \\ &\leq (2a+b^2)tz_1^2 + z_1^2 + 1 = [(2a+b^2)t+1]z_1^2 + 1.\end{aligned}$$

This means that we can set

$\psi_1(t) = [(2a+b^2)t+1], \quad \rho(t) = 1.$

On the other hand, we have

$$\mathcal{H}\mathcal{V}(t,z) = 4b^2t||z_1||^4.$$

Hence, we see that $\psi_2(t) = \psi_3(t) = 4b^2t$. Next, our target is to verify that the solution of the sub-system to the variable z_2 is globally uniformly bounded with probability one. So, let $\mathcal{V}(t, z_2) = z_2^2$. Then, we can easily check that

$$\mathcal{L}\mathcal{V}(t,z_2) = -2\sin^2(z_1)z_2^2 + 2\sin^2(z_1)z_2^2 = 0.$$

Theorem 2.2 in [20] provides the solution of the sub-system to the variable z_2 is globally uniformly stable in probability, which in turn implies $z_2(t)$ is globally uniformly bounded with probability one, as shown in Figure 2. Taking, $\lambda(t) = \exp(t^2),\ m = 0,\ q = 2$. We can easily check that assumptions in Corollary 3.7 hold with $\alpha_1 = b^2,\ \alpha_2 = 2b^2,\ \alpha_3 = a + b^2/2,\ \rho(t) = 1,\ \xi = C = 0$, and therefore $-[m - (\alpha_3 - \alpha_1)] = (a - b^2/2)$. Finally, we deduce that the solution to system (5.1) tends to the ball $\mathcal{B}_r$ with respect to x_1 with decay function $\lambda(t) = \exp(t^2),\ r = 1$ and order at least $b^2/2 - a$ provided b is large enough (in fact, whenever $b^2 > 2a$), as we can see in Fig. 1, for $a = 1/4$ and $b = 1$.

Remark 5.1 We remark that it is not possible to apply the results established in [16] to deduce the practical stability in all variables for system (5.1) since the state z_2 is globally uniformly bounded with probability one but not attractive.

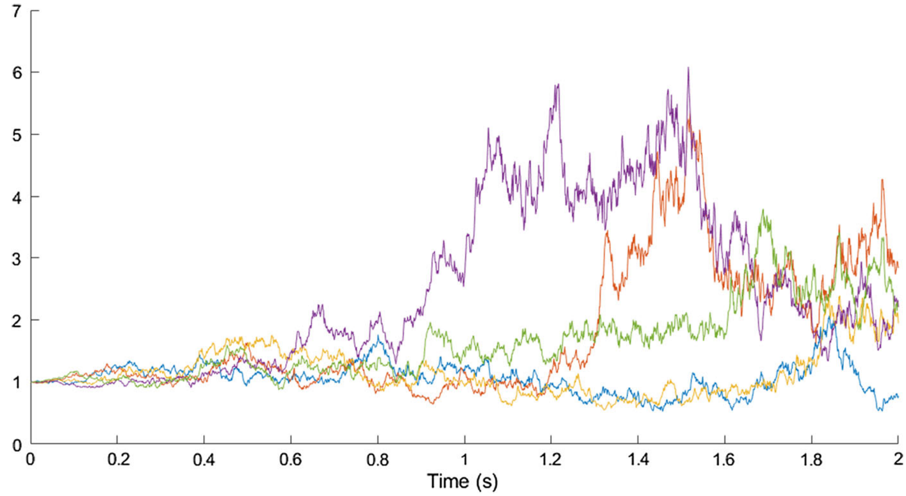

Fig. 1 Time evolution of the state $z_1(t)$, with five different Brownian motions

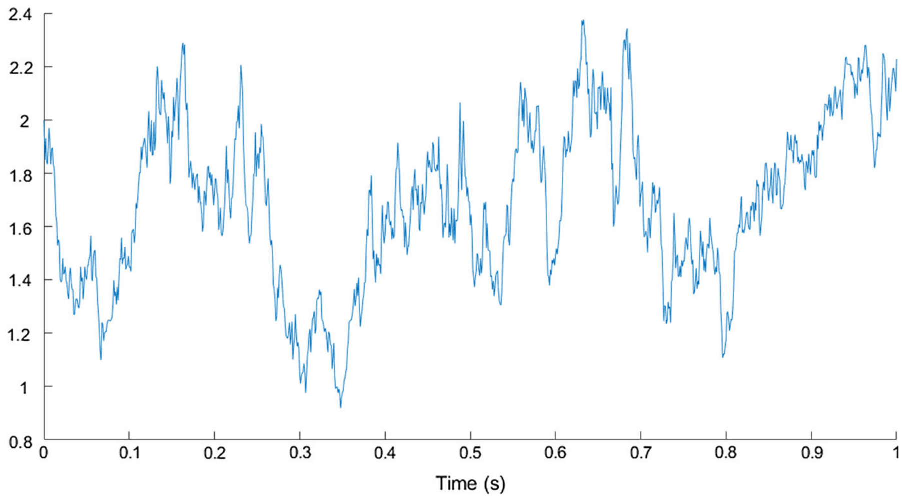

Fig. 2 Time evolution of the state $z_2(t)$

6 Conclusion

In our manuscript, we dealt with the partial practical stability analysis with general decay rate of stochastic systems. The main technical tool for deriving stability results is Lyapunov method. An illustrative example showed the effectiveness of the proposed approach. In [23], Caraballo et al. investigated the practical exponential stability of impulsive stochastic functional differential equations. For our next prospect research, we will try to extend our results to the case of impulsive equations.

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Appendix

Some details of calculus used in the proof of Theorem 3.3 are provided in what follows:

Appendix A: Details of calculus of Eq. (3.3)

$$\begin{aligned}\lambda(t)^m||x_1(t)||^q-\rho(t)&=\lambda(t)^m\Big(||x_1(t)||^q-\Big(\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}}\Big)^q\Big)\\&=\lambda(t)^m\Big(||x_1(t)||-\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}}\Big)\Big(||x_1(t)||^{q-1}+||x_1(t)||^{q-2}\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}}\\&\quad+\cdots+\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{q-1}{q}}\Big)\\&=\lambda(t)^m\Big(||x_1(t)||-\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{1}{q}}\Big)\sum_{k=1}^{q}||x_1(t)||^{q-k}\Big(\frac{\rho(t)}{\lambda(t)^m}\Big)^{\frac{k-1}{q}}.\end{aligned}$$

Appendix B: Details of calculus of Eq. (3.5)

$$\begin{aligned}\ln(\mathcal{V}(t,x(t)))&\le\ln(\mathcal{V}(T,x(T)))+\int_T^t\frac{\psi_1\mathcal{V}(s,x(s))+\rho(s)}{\mathcal{V}(s,x(s))}\mathrm{d}s+\mathcal{M}(t)-\frac{1}{2}\int_T^t\frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\\&\le\ln(\mathcal{V}(T,x(T)))+\int_T^t\psi_1(s)\mathrm{d}s+\int_T^t\frac{\rho(s)}{\mathcal{V}(s,x(s))}\mathrm{d}s+\mathcal{M}(t)-\frac{1}{2}\int_T^t\frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\\&\le\ln(\mathcal{V}(T,x(T)))+\int_T^t\psi_1(s)\mathrm{d}s+\int_T^t\frac{\rho(s)}{\lambda(s)^m||x_1(s)||^q}\mathrm{d}s+\mathcal{M}(t)-\frac{1}{2}\int_T^t\frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s.\end{aligned}$$

Appendix C: Details of calculus of Eq. (3.7)

$$\begin{aligned}\ln(\mathcal{V}(t,x(t)))&\le\ln(\mathcal{V}(T,x(T)))+\int_T^t\psi_1(s)\mathrm{d}s+(t-T)+\frac{2}{\beta}\ln(k-1)+\frac{\beta}{2}\int_T^t\frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\\&\quad-\frac{1}{2}\int_T^t\frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\\&\le\ln(\mathcal{V}(T,x(T)))+\int_T^t\psi_1(s)\mathrm{d}s+(t-T)+\frac{2}{\beta}\ln(k-1)-\frac{1-\beta}{2}\int_T^t\frac{\mathcal{H}\mathcal{V}(s,x(s))}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\\&\le\ln(\mathcal{V}(T,x(T)))+\int_T^t\psi_1(s)\mathrm{d}s+(t-T)+\frac{2}{\beta}\ln(k-1)-\frac{1-\beta}{2}\int_T^t\psi_2(s)\mathrm{d}s\\&\quad-\frac{1-\beta}{2}\int_T^t\frac{\xi}{\mathcal{V}^2(s,x(s))}\mathrm{d}s\\&\le\ln(\mathcal{V}(T,x(T)))+\frac{2}{\beta}\ln(k-1)+\int_T^t\psi_1(s)\mathrm{d}s+(t-T)-\frac{1-\beta}{2}\int_T^t\psi_2(s)\mathrm{d}s.\end{aligned}$$

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