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Stability of fractionally dissipative 2D quasi-geostrophic equation with infinite delay

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Abstract In this paper, fractionally dissipative 2D quasi-geostrophic equations with an external force containing infinite delay is considered in the space H^s with $s \geq 2 - 2\alpha$ and $\alpha \in (\frac{1}{2}, 1)$. First, we investigate the existence and regularity of solutions by Galerkin approximation and the energy method. The continuity of solutions with respect to initial data and the uniqueness of solutions are also established. Then we prove the existence and uniqueness of a stationary solution by the Lax-Milgram theorem and the Schauder fixed point theorem. Using the classical Lyapunov method, the construction method of Lyapunov functionals and the Razumikhin-Lyapunov technique, we analyze the local stability of stationary solutions. Finally, the polynomial stability of stationary solutions is verified in a particular case of unbounded variable delay.

Keywords Quasi-geostrophic equation · Infinite delay · Stationary solution · Stability · Asymptotic stability · Polynomial stability

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1 Introduction

We consider the following 2D quasi-geostrophic equation with infinite delay in the periodic domain $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\begin{cases} \theta_t + \kappa(-\Delta)^\alpha\theta + u \cdot \nabla\theta = f + g(t, \theta_t), \text{ in } (0,\infty) \times \mathbb{T}^2, \\ \theta(t,x) = \varphi(t,x), \ t \in (-\infty, 0], \ x \in \mathbb{T}^2, \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1)$, $\kappa > 0$, $\theta = \theta(t,x)$ represents the potential temperature, f is a nondelayed external force independent of time, g is the external force containing some hereditary characteristics, and $u = (u_1, u_2)$ is the velocity field determined by θ via the following relation:

$$u = (u_1, u_2) = (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}), \quad \text{and} \quad (-\Delta)^{\frac{1}{2}}\psi = -\theta. \tag{1.2}$$

The fractionally dissipative 2D quasi-geostrophic equation describes a kind of dynamics of large-scale phenomena in the atmosphere and ocean, see [31] for more details. Due to potential applications in meteorology, 2D quasi-geostrophic equations have been receiving much attention over the last decades, see, e.g. [13, 16–18, 30, 32, 34, 36] and the references therein. The existence, uniqueness and regularity of the quasi-geostrophic equation have been considered in [12, 14, 15, 19, 20]. The asymptotic analysis of the systems, as a key method to explore the evolution of the systems in the future, has also been investigated in [11, 34], where the large time behavior of solutions has been discussed by several decay estimates. For the existence and regularity of attractors of the quasi-geostrophic equation, we refer the reader to [10, 18, 21]. It is well known that stability theory is an important issue in the study of the asymptotic behaviour of the systems. The asymptotic stability for the weak solution of the quasi-geostrophic equation with respect to large perturbation has been investigated in [17, 32]. The existence and uniqueness of steady-state solutions to the quasi-geostrophic equation with finite energy (L^2 norm), and the nonlinear stability of such a solution have been proved in [13].

Here we are interested in a quasi-geostrophic model containing some hereditary features in the forcing term. These situations may appear, for instance, when the evolution of the systems depends not only on the present state of the systems, but also on the history of the solutions, see [28, 30, 36], for instance. There has, however, been little mention of the existence and stability of stationary solutions to the quasi-geostrophic equation with infinite delay in the space H^s even in the case of the non-delayed quasi-geostrophic equation.

The quasi-geostrophic equation exhibits strikingly similar features (singularities) as 3D Navier-Stokes equations. 2D Navier-Stokes models with hereditary characteristics were proposed in [4], and developed in [2, 3, 5, 24], where the existence, uniqueness and stability of stationary solutions have been established. For the 3D case, the stationary problem has been analyzed for a globally modified version of Navier-Stokes equations with delay terms within a locally Lipschitz operator in [26, 27].

In this paper, we aim to study the stability of the quasi-geostrophic equation with unbounded variable delay in the phase space $\mathscr{C}(H^s)$ with $s \geq 2-2\alpha$ and $\alpha \in (\frac{1}{2}, 1)$, see Section 2 for more details. The first purpose of this paper is to investigate the existence and uniqueness of the solution to Eq. (1.1), which are proved by the classic Galerkin approximation and the energy method combined with auxiliary L^p estimates, $p > \frac{2}{2\alpha-1}$. Compared to some recent works for the Navier-Stokes equation, the essential difficulty is the lack of cancellation property for the quadratic type of nonlinear term in H^s. On the other hand, the problems become harder thanks to the dissipative term $(-\Delta)^{\alpha}$, $\frac{1}{2} < \alpha < 1$, and the nonlinear term $u \cdot \nabla\theta$. In order to overcome these difficulties, we give some complicated estimates on solutions by using appropriate commutator estimates. Then we introduce some new suitable assumptions to guarantee the existence, uniqueness and regularity of stationary solutions to Eq. (1.1) by the Lax-Milgram theorem and the Schauder fixed point theorem.

Lyapunov's method is a fundamental tool to study the stability of systems. Along with the birth of functional differential equations, Razumikhin, and Kolmanovskii together with Shaikhet have proposed the method of Razumikhin type and the construction method of Lyapunov functionals, respectively, which have been successfully used to investigate the stability of delay evolution equations, for instance, [6, 22, 29]. Finally, we use different methods to deduce different sufficient conditions ensuring the stability of stationary solutions to Eq. (1.1), and discuss the polynomial stability of the quasi-geostrophic equation with proportional delay.

The article is organized as follows. In Section 2, we introduce some notations, and briefly recall some useful estimates and relevant mathematical preliminaries from functional analysis. The existence and uniqueness of solutions to Eq. (1.1) are considered in Section 3. In Section 4, the existence and uniqueness of stationary solutions to Eq. (1.1) are established, and several methods are used to show the stability of stationary solutions.

2 Preliminaries

Throughout this manuscript and notation, we consider mean-zero (zero average) solutions of Eq. (1.1), that is

$$\int_{\mathbb{T}^2} \theta(t,x)dx = 0, \text{ for any } t \in \mathbb{R}.$$

Denote $\Lambda \equiv (-\Delta)^{\frac{1}{2}}$. The fractional Laplacian Λ^s, with $s \in \mathbb{R}$ can be defined by

$$\widehat{\Lambda^s f}(k) = |k|^s \widehat{f}(k),$$

where $\widehat{f}$ denotes the Fourier transform of f, i.e.,

$$\widehat{f}(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x) e^{-ikx} dx.$$

For $1 \le p \le \infty$, L^p denotes the Banach space of pth-power integrable functions if $p < \infty$ and the essentially bounded functions when $p = \infty$. The following standard notations are used:

$$\|f\|_{L^p}^p = \int_{\mathbb{T}^2} |f|^p dx, \quad \|f\|_{L^\infty} = \text{ess} \sup_{x \in \mathbb{T}^2} |f(x)|.$$

For any tempered distribution f on $\mathbb{T}^2$ and $s \in \mathbb{R}$, we define

$$\|f\|_{H^s}^2 = \|\Lambda^s f\|_{L^2}^2 = \sum_{k \in \mathbb{Z}^2} |k|^{2s} |\widehat{f}(k)|^2, \tag{2.1}$$

and H^s denotes the Sobolev space of all f for which $\|f\|_{H^s}$ is finite. In particular, we denote by $H := H^0 = L^2$. For $1 \le p \le \infty$ and $s \in \mathbb{R}$, the space $H^{s,p}$ is a subspace of L^p, consisting of all f which can be written in the form $f = \Lambda^{-s} g$, $g \in L^p$, and the $H^{s,p}$ norm of f is defined by

$$\|f\|_{H^{s,p}} = \|\Lambda^s f\|_{L^p}.$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product of L^2. Given a Banach space X and its dual X', we also denote the dual pairing between X and X' by $\langle \cdot, \cdot \rangle$, unless noted otherwise.

The equality relating u to θ in (1.2) can be rewritten in terms of periodic Riesz transforms as:

$$u = (\partial_{x_2} \Lambda^{-1} \theta, -\partial_{x_1} \Lambda^{-1} \theta) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) \equiv \mathcal{R}^\perp \theta,$$

where $\mathcal{R}_j$, $j = 1, 2$ denote the Riesz transforms defined by

$$\widehat{\mathcal{R}_j f}(k) = -i \frac{k_j}{|k|} \widehat{f}(k), \quad k \in \mathbb{Z}^2 \setminus \{0\}.$$

The following result can be obtained by the fact that the Riesz transforms commute with $(-\Delta)^l$ and the boundedness of the Riesz transforms in L^p, see [35, Chapter III] for more details.

Lemma 1 *Let $1 < p < \infty$ and $l \ge 0$. Then there exists a constant $C(l,p)$ such that*

$$\|(-\Delta)^l u\|_{L^p} \le C(l,p) \|(-\Delta)^l \theta\|_{L^p}. \tag{2.2}$$

If $p = 2$, the inequality (2.2) *can be strengthened to*

$$\|(-\Delta)^l u\|_{L^2} = \|(-\Delta)^l \theta\|_{L^2}. \tag{2.3}$$

We recall some important estimates which will be used frequently in the section below.

Lemma 2 (Commutator Estimate) *Suppose that $s > 0$ and $p \in (1, \infty)$. If $f, g \in \mathcal{S}$, the Schwartz class, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C_1(\|\nabla f\|_{L^{p_1}}\|g\|_{H^{s-1,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{L^{p_4}}) \tag{2.4}$$

and

$$\|\Lambda^s(fg)\|_{L^p} \le C_2(\|f\|_{L^{p_1}}\|g\|_{H^{s,p_2}} + \|f\|_{H^{s,p_3}}\|g\|_{L^{p_4}}), \tag{2.5}$$

with $p_2, p_3 \in (1, \infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Lemma 3 (Fractional L^p Poincaré) *For $\alpha \in [0, 1]$, $p \ge 2$, $\phi \in H^{2\alpha}$ with $|\phi|^{p-1} \in H^\alpha$, the following estimate holds:*

$$\int_{\mathbb{T}^2} (-\Delta)^\alpha \phi \ \mathrm{sgn}\phi \ |\phi|^{p-1} dx \ge \frac{4(p-1)}{p^2} \int_{\mathbb{T}^2} \left[(-\Delta)^{\frac{\alpha}{2}}(|\phi|^{\frac{p}{2}})\right]^2 dx. \tag{2.6}$$

Lemma 3 is a version of the famous Kato-Beurling-Deny inequality, which has been proven in [8]. When p is even, we can also refer to [9,12] for more details.

Next, we recall some spaces which allow us to deal with infinite delay. Given real numbers $a < b$, we denote by $C([a, b]; X)$ the Banach space of all continuous functions from $[a, b]$ into X equipped with sup norm. For any given number $T > 0$, if a function $x \in C((-\infty, T]; X)$, for each $t \in [0, T]$, we denote by $x_t \in C((-\infty, 0]; X)$ the function defined by $x_t(r) = x(t + r)$, $\forall r \in (-\infty, 0]$. In this paper, we aim to establish well-posedness and stability results for 2D quasi-geostrophic equation with infinite delay in the phase space

$$\mathscr{C}(X) = \{\psi \in C((-\infty, 0]; X) : \lim_{t \to -\infty} \psi(t) \text{ exists in } X\},$$

which is endowed with the norm

$$\|\psi\|_{\mathscr{C}(X)} = \sup_{t \in (-\infty, 0]} \|\psi(t)\|_X.$$

To describe the conditions imposed on the delay term g, we introduce some notations. Let $\mathscr{X}$ and $\mathscr{Y}$ be two Banach spaces. We denote by $\mathrm{Lip}_0(\mathscr{X}, \mathscr{Y})$ the collection of all mappings $g : [0, T] \times \mathscr{X} \to \mathscr{Y}$ satisfying the following conditions:

- for any $\xi \in \mathscr{X}$, the mapping $t \in [0, T] \to g(t, \xi) \in \mathscr{Y}$ is measurable;
- for each $t \in [0, T]$, $g(t, 0) = 0$;
- there exists a constant $L_g > 0$ such that

$$\|g(t, \xi) - g(t, \eta)\|_{\mathscr{Y}} \le L_g \|\xi - \eta\|_{\mathscr{X}}, \quad \forall t \in [0, T] \text{ and } \forall x, y \in \mathscr{X}.$$

Remark 1 It is clear that if $g \in \mathrm{Lip}_0(\mathscr{X}, \mathscr{Y})$, then for any $\xi \in \mathscr{X}$,

$$\|g(t, \xi)\|_{\mathscr{Y}} = \|g(t, \xi) - g(t, 0)\|_{\mathscr{Y}} \le L_g \|\xi\|_{\mathscr{X}}, \ \forall t \in [0, T],$$

and therefore $|g(\cdot, \xi)| \in L^\infty(0, T)$.

Throughout the paper, we denote by C a real positive constant which can vary from a line to another and even in the same line. If the constant C depends on some variable x, we denote it by C_x.

3 Existence, regularity and uniqueness of solutions

In this section, we first study the existence and regularity of weak solutions to Eq. (1.1). Next the continuity of solutions with respect to initial datum is established and the uniqueness of solutions follows directly.

Definition 1 For any given number $T > 0$ and initial datum $\varphi \in \mathscr{C}(H^s)$, a weak solution θ to Eq. (1.1) in the interval $(-\infty, T)$ is a function $\theta \in C((-\infty, T]; H^s) \cap L^2(0, T; H^{s+\alpha})$ with $\theta_0 = \varphi$ such that, for all $v \in H^{s+\alpha}$, the equality

$$\frac{d}{dt}\langle \theta(t), v\rangle + \kappa\langle(-\Delta)^\alpha \theta(t), v\rangle + \langle u(t)\cdot\nabla\theta(t), v\rangle = \langle f, v\rangle + \langle g(t, \theta_t), v\rangle,$$

holds true in the sense of scalar distribution $\mathcal{D}'(0, T)$.

The existence and regularity of weak solutions to Eq. (1.1) can be proved by the classic Galerkin method.

Theorem 1 *Fix $\alpha \in (\frac{1}{2}, 1)$, $s \geq 2 - 2\alpha$ and $p > \frac{2}{2\alpha-1}$. Assume that $f \in H^{s-\alpha} \cap L^p$ and $g \in Lip_0(\mathscr{C}(H^s), H^s) \cap Lip_0(\mathscr{C}(L^p), L^p)$. Then for any given $T > 0$ and $\varphi \in \mathscr{C}(H^s) \cap \mathscr{C}(L^p)$, there exists a weak solution θ of Eq.* (1.1) *in the sense of Definition 1. Furthermore, if $f \in H^s \cap L^p$ and $\varphi \in \mathscr{C}(H^s) \cap \mathscr{C}(L^p)$ with $\varphi(0) \in H^{s+\alpha}$, then $\theta \in C([0, T]; H^{s+\alpha}) \cap L^2(0, T; H^{s+2\alpha})$.*

Proof We will split the proof into several steps.
Step 1. The Galerkin approximation. By the classical spectral theory of elliptic operators, it follows that $-\Delta$ possesses a sequence of eigenvalues $\{\lambda_i\}_{i=1}^\infty$ and a corresponding family of eigenfunctions $\{e_i\}_{i=1}^\infty$, which forms an orthonormal basis of H. Denote by P_n the projection from H onto the linear span P_nH of $\{e_1, \ldots, e_n\}$, i.e.,

$$P_n\theta = \sum_{i=1}^n \theta_i e_i \quad \text{for} \quad \theta = \sum_{i=1}^\infty \theta_i e_i.$$

We define the n-th Galerkin approximation of Eq. (1.1) as the following ODE system with infinite delay:

$$\begin{cases} \frac{d}{dt}\theta^n(t) + \kappa(-\Delta)^\alpha\theta^n(t) + P_n(u^n(t)\cdot\nabla\theta^n(t)) = P_nf + P_ng(t, \theta^n_t),\ t \geq 0, \\ \theta^n(t) = P_n\varphi(t),\ t \in (-\infty, 0], \end{cases} \tag{3.1}$$

with $u^n = \mathcal{R}^\perp\theta^n$ satisfying $\nabla\cdot u^n = 0$.
Step 2. A priori estimates. Multiplying Eq. (3.1) by $\mathrm{sgn}\theta^n(t)|\theta^n(t)|^{p-1}$, with $p > \frac{2}{2\alpha-1}$, and taking the inner product in L^2, we have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\mathbb{T}^2}|\theta^n(t)|^p dx + \kappa\int_{\mathbb{T}^2}(-\Delta)^\alpha\theta^n(t)\mathrm{sgn}\theta^n(t)|\theta^n(t)|^{p-1}dx \\ &\quad = \int_{\mathbb{T}^2} P_nf\mathrm{sgn}\theta^n(t)|\theta^n(t)|^{p-1}dx + \int_{\mathbb{T}^2} P_ng(t, \theta^n_t)\mathrm{sgn}\theta^n(t)|\theta^n(t)|^{p-1}dx, \end{aligned} \tag{3.2}$$

since the term $u^n \cdot \nabla\theta^n$ vanishes in the calculation due to the fact:

$$\int_{\mathbb{T}^2} \sum_{i=1}^{2} u_i^n(t) \frac{\partial \theta^n(t)}{\partial x_i} \mathrm{sgn}\theta^n(t)|\theta^n(t)|^{p-1} dx = \frac{1}{p} \int_{\mathbb{T}^2} \sum_{i=1}^{2} u_i^n(t) \frac{\partial |\theta^n(t)|^p}{\partial x_i} dx$$
$$= -\frac{1}{p} \int_{\mathbb{T}^2} |\theta^n(t)|^p \sum_{i=1}^{2} \frac{\partial u_i^n(t)}{\partial x_i} dx = 0. \tag{3.3}$$

It follows from (2.6) that

$$\kappa \int_{\mathbb{T}^2} (-\Delta)^\alpha \theta^n(t) \mathrm{sgn}\theta^n(t)|\theta^n(t)|^{p-1} dx \geq 0.$$

Then using the condition $g \in \mathrm{Lip}_0(\mathscr{C}(L^p), L^p)$ and the Young inequality, (3.2) can be transformed as follows:

$$\begin{aligned}
&\frac{d}{dt}\|\theta^n(t)\|_{L^p}^p \leq \|\theta^n(t)\|_{L^p}^p + C_p\|P_n f\|_{L^p}^p + C_p\|P_n g(t,\theta_t^n)\|_{L^p}^p \\
&\quad \leq \|\theta^n(t)\|_{L^p}^p + C_p\|f\|_{L^p}^p + C_p\|g(t,\theta_t^n)\|_{L^p}^p \\
&\quad \leq \|\theta^n(t)\|_{L^p}^p + C_p\|f\|_{L^p}^p + C_p L_g\|\theta_t^n\|_{\mathscr{C}(L^p)}^p \\
&\quad \leq \|\theta^n(t)\|_{L^p}^p + C_p\|f\|_{L^p}^p + C_p L_g \left(\|P_n\varphi\|_{\mathscr{C}(L^p)}^p + \sup_{0\leq\tau\leq t}\|\theta^n(\tau)\|_{L^p}^p \right).
\end{aligned}$$

Integrating the above inequality between 0 and $t \in [0,T]$, then replacing t by $\tau \in [0,t]$ and taking the supremum over $[0,t]$, we have

$$\begin{aligned}
\sup_{0\leq\tau\leq t}\|\theta^n(\tau)\|_{L^p}^p &\leq \|\theta^n(0)\|_{L^p}^p + \int_0^t \|\theta^n(\sigma)\|_{L^p}^p d\sigma + C_p T\|f\|_{L^p}^p \\
&\quad + C_p L_g \int_0^t \left(\|P_n\varphi\|_{\mathscr{C}(L^p)}^p + \sup_{0\leq\tau\leq \sigma}\|\theta^n(\tau)\|_{L^p}^p \right) d\sigma \\
&\leq (1 + C_p L_g T)\|\varphi\|_{\mathscr{C}(L^p)}^p + C_p T\|f\|_{L^p}^p + (1 + C_p L_g) \int_0^t \sup_{0\leq\tau\leq \sigma}\|\theta^n(\tau)\|_{L^p}^p d\sigma.
\end{aligned}$$

Applying the Gronwall lemma results in

$$\sup_{0\leq\tau\leq t}\|\theta^n(\tau)\|_{L^p}^p \leq \left((1 + C_p L_g T)\|\varphi\|_{\mathscr{C}(L^p)}^p + C_p T\|f\|_{L^p}^p \right) e^{(1+C_p L_g)t}.$$

Thus for any given $T > 0$, there exists a constant $C_{p,T}$ such that

$$\sup_{0\leq\tau\leq t}\|\theta^n(\tau)\|_{L^p} \leq C_{p,T}, \quad \forall t \in [0,T],\ \forall n \geq 1. \tag{3.4}$$

Applying Λ^s to Eq. (3.1) and taking the inner product in L^2 with $\Lambda^s\theta^n$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta^n(t)\|_{H^s}^2 + \kappa\|\theta^n(t)\|_{H^{s+\alpha}}^2 &= -\langle \Lambda^s P_n(u^n(t)\cdot\nabla\theta^n(t)), \Lambda^s\theta^n(t)\rangle \\
&\quad + \langle \Lambda^s P_n f, \Lambda^s\theta^n(t)\rangle + \langle \Lambda^s P_n g(t,\theta_t^n), \Lambda^s\theta^n(t)\rangle \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{3.5}$$

Note that P_n commutes with Λ^s on H^s, and P_n is self-adjoint in L^2, then

$$\begin{aligned}|I_1| &= |\langle P_n\Lambda^s(u^n(t)\cdot\nabla\theta^n(t)),\Lambda^s\theta^n(t)\rangle|\\ &= |\langle \Lambda^s(u^n(t)\cdot\nabla\theta^n(t)),P_n\Lambda^s\theta^n(t)\rangle|\\ &= |\langle \Lambda^s(u^n(t)\cdot\nabla\theta^n(t)),\Lambda^s\theta^n(t)\rangle|. \end{aligned}\tag{3.6}$$

Taking into account that $\nabla\cdot u^n=0$, by the Schwarz inequality and Lemmas 1 and 2, we obtain

$$\begin{aligned}|I_1| &\le \|\Lambda^{s+1-\alpha}(u^n(t)\theta^n(t))\|_{L^2}\|\Lambda^{s+\alpha}\theta^n(t)\|_{L^2}\\ &\le C\|u^n(t)\|_{L^p}\|\theta^n(t)\|_{H^{s+1-\alpha,q}}\|\theta^n(t)\|_{H^{s+\alpha}}\\ &\quad + C\|u^n(t)\|_{H^{s+1-\alpha,q}}\|\theta^n(t)\|_{L^p}\|\theta^n(t)\|_{H^{s+\alpha}}\\ &\le C\|\theta^n(t)\|_{L^p}\|\theta^n(t)\|_{H^{s+1-\alpha,q}}\|\theta^n(t)\|_{H^{s+\alpha}}\\ &\le C\|\theta^n(t)\|^{1+\eta}_{H^{s+\alpha}}\|\theta^n(t)\|^{1-\eta}_{L^2}\|\theta^n(t)\|_{L^p}, \end{aligned}\tag{3.7}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, and we have used the Nirenberg-Gagliardo inequality (cf. [7]):

$$\|\theta^n(t)\|_{H^{s+1-\alpha,q}}\le C\|\theta^n(t)\|^{\eta}_{H^{s+\alpha}}\|\theta^n(t)\|^{1-\eta}_{L^2},$$

where $\eta=\frac{s+2-\alpha-\frac{2}{q}}{s+\alpha}$. Since $1<1+\eta<2$ and $L^p\subset L^2$, in view of the Young inequality, (3.7) can be further estimated by

$$\begin{aligned}|I_1| &\le C\|\theta^n(t)\|^{1+\eta}_{H^{s+\alpha}}\|\theta^n(t)\|^{2-\eta}_{L^p}\\ &\le \frac{\kappa}{4}\|\theta^n(t)\|^2_{H^{s+\alpha}}+C\|\theta^n(t)\|^{\frac{4-2\eta}{1-\eta}}_{L^p}. \end{aligned}\tag{3.8}$$

For the term I_2, since P_n commutes with Λ^s on H^s and P_n is self-adjoint in L^2, by the Schwarz inequality and the Young inequality we have

$$\begin{aligned}|I_2| &= |\langle P_n\Lambda^s f,\Lambda^s\theta^n(t)\rangle| = |\langle \Lambda^s f,\Lambda^s\theta^n(t)\rangle|\\ &\le \|f\|_{H^{s-\alpha}}\|\theta^n(t)\|_{H^{s+\alpha}}\le \frac{\kappa}{4}\|\theta^n(t)\|^2_{H^{s+\alpha}}+C\|f\|^2_{H^{s-\alpha}}, \end{aligned}\tag{3.9}$$

where $H^{s-\alpha}$ denotes the dual space of $H^{\alpha-s}$ for the case $s-\alpha<0$. For the last term, arguing as in (3.9), in view of $g\in \mathrm{Lip}_0(\mathscr{C}(H^s),H^s)$, we find that

$$\begin{aligned}|I_3| &= |\langle P_n\Lambda^s g(t,\theta^n_t),\Lambda^s\theta^n(t)\rangle| = |\langle \Lambda^s g(t,\theta^n_t),\Lambda^s\theta^n(t)\rangle|\\ &\le \|g(t,\theta^n_t)\|_{H^s}\|\theta^n(t)\|_{H^s}\le L_g\|\theta^n_t\|^2_{\mathscr{C}(H^s)}. \end{aligned}\tag{3.10}$$

Inserting (3.8)-(3.10) into (3.5) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\theta^n(t)\|^2_{H^s}+\frac{\kappa}{2}\|\theta^n(t)\|^2_{H^{s+\alpha}}\\ &\quad\le C\|\theta^n(t)\|^{\frac{4-2\eta}{1-\eta}}_{L^p}+C\|f\|^2_{H^{s-\alpha}}+L_g\|\theta^n_t\|^2_{\mathscr{C}(H^s)}. \end{aligned}$$

Integrating the above inequality and using (3.4), we infer that

$$\|\theta^n(t)\|_{H^s}^2 + \kappa \int_0^t \|\theta^n(\sigma)\|_{H^{s+\alpha}}^2 d\sigma \le \|\theta^n(0)\|_{H^s}^2 + C_T(1 + \|f\|_{H^{s-\alpha}}^2) + 2L_g \int_0^t \|\theta_\sigma^n\|_{\mathscr{C}(H^s)}^2 d\sigma. \tag{3.11}$$

Hence,

$$\begin{aligned}\|\theta_t^n\|_{\mathscr{C}(H^s)}^2 &\le \max\{\sup_{-\infty<\tau\le 0} \|\theta^n(\tau)\|_{H^s}^2, \sup_{0\le\tau\le t} \|\theta^n(\tau)\|_{H^s}^2\} \\ &\le \|\varphi\|_{\mathscr{C}(H^s)}^2 + C_T(1 + \|f\|_{H^{s-\alpha}}^2) + 2L_g \int_0^t \|\theta_\sigma^n\|_{\mathscr{C}(H^s)}^2 d\sigma,\end{aligned}$$

and applying the Gronwall lemma,

$$\|\theta_t^n\|_{\mathscr{C}(H^s)}^2 \le \Big(\|\varphi\|_{\mathscr{C}(H^s)}^2 + C_T(1 + \|f\|_{H^{s-\alpha}}^2)\Big) e^{2L_g t}.$$

Then we obtain the following estimates. Firstly, there exists a constant C_T such that

$$\|\theta_t^n\|_{\mathscr{C}(H^s)}^2 \le C_T, \quad \forall t \in [0, T],\ \forall n \ge 1, \tag{3.12}$$

which implies that

$$\{\theta^n\} \text{ is bounded in } L^\infty(0, T; H^s). \tag{3.13}$$

Secondly, it follows from (3.11) and (3.12) that

$$\begin{aligned}\kappa \int_0^T \|\theta^n(\sigma)\|_{H^{s+\alpha}}^2 d\sigma &\le \|\theta^n(0)\|_{H^s}^2 + C_T(1 + \|f\|_{H^{s-\alpha}}^2) + 2L_g \int_0^T \|\theta_\sigma^n\|_{\mathscr{C}(H^s)}^2 d\sigma \\ &\le \|\theta^n(0)\|_{H^s}^2 + C_T(1 + \|f\|_{H^{s-\alpha}}^2) + 2L_g C_T,\end{aligned}$$

and consequently there exists another constant C_T such that

$$\|\theta^n\|_{L^2(0,T;H^{s+\alpha})}^2 \le C_T, \quad \forall t \in [0, T],\ \forall n \ge 1. \tag{3.14}$$

Finally, we prove the boundedness of $\{(\theta^n)'\}$. From Lemmas 1 and 2, we find that

$$\begin{aligned}\|\Lambda^{s-\alpha}(u^n \cdot \nabla\theta^n)\|_{L^2} &= \|\Lambda^{s-\alpha}\nabla(u^n\theta^n)\|_{L^2} \le \|\Lambda^{s+1-\alpha}(u^n\theta^n)\|_{L^2} \\ &\le C(\|u^n\|_{L^{p_1}}\|\theta^n\|_{H^{s+1-\alpha,p_2}} + \|u^n\|_{H^{s+1-\alpha,p_2}}\|\theta^n\|_{L^{p_1}}) \\ &\le C\|\theta^n\|_{H^s}\|\theta^n\|_{H^{s+\alpha}},\end{aligned}$$

where $p_1 = \frac{2}{2\alpha-1}$, $p_2 = \frac{1}{1-\alpha}$, and we have used the fact $\nabla \cdot u^n = 0$ and the Sobolev embeddings $H^s \subset L^{p_1}$, $H^{s+\alpha} \subset H^{s+1-\alpha,p_2}$. Then, by using the generalized Poincaré inequality:

$$\lambda_1^{\beta-\gamma} \int_{\mathbb{T}^2} [(-\Delta)^{\frac{\gamma}{2}}\phi]^2 dx \le \int_{\mathbb{T}^2} [(-\Delta)^{\frac{\beta}{2}}\phi]^2 dx, \quad \forall \beta > \gamma, \tag{3.15}$$

where λ_1 denotes the first eigenvalue of $-\Delta$ in $\mathbb{T}^2$, we deduce from (3.1) that

$$\|(\theta^n)'\|_{H^{s-\alpha}} \leq \kappa\|\theta^n\|_{H^{s+\alpha}} + C\|\theta^n\|_{H^s}\|\theta^n\|_{H^{s+\alpha}} + \|f\|_{H^{s-\alpha}} + \lambda_1^{-\frac{\alpha}{2}}\|g(t,\theta^n_t)\|_{H^s}.$$

This, together with the assumption on g and (3.12)-(3.14), implies that

$$\{(\theta^n)'\} \text{ is bounded in } L^2(0,T;H^{s-\alpha}). \tag{3.16}$$

Step 3. Approximation of initial datum in $\mathscr{C}(H^s)$. Let us prove that

$$P_n\varphi \to \varphi \quad \text{in } \mathscr{C}(H^s). \tag{3.17}$$

Assume on the contrary that this is not the case, then there exists $\varepsilon > 0$ and a subsequence $\{\tau_n\} \subset (-\infty, 0]$, such that

$$\|P_n\varphi(\tau_n) - \varphi(\tau_n)\|_{H^s} > \varepsilon, \quad \forall n \geq 1. \tag{3.18}$$

Case 1: $\tau_n \to \tau$ as $n \to \infty$. Thanks to the continuity of φ, we deduce

$$\begin{aligned}\|P_n\varphi(\tau_n) - \varphi(\tau_n)\|_{H^s} &\leq \|P_n\varphi(\tau_n) - P_n\varphi(\tau)\|_{H^s} + \|P_n\varphi(\tau) - \varphi(\tau)\|_{H^s}\\ &\quad + \|\varphi(\tau) - \varphi(\tau_n)\|_{H^s} \to 0 \text{ as } n \to \infty.\end{aligned}$$

Case 2: $\tau_n \to -\infty$ as $n \to \infty$. By the definition of $\mathscr{C}(H^s)$, the limit of $\varphi(\tau_n)$ exists in H^s. Let $x = \lim_{\tau\to-\infty}\varphi(\tau)$. Then we obtain

$$\begin{aligned}\|P_n\varphi(\tau_n) - \varphi(\tau_n)\|_{H^s} &\leq \|P_n\varphi(\tau_n) - P_nx\|_{H^s} + \|P_nx - x\|_{H^s}\\ &\quad + \|x - \varphi(\tau_n)\|_{H^s} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

We see that both cases contradict (3.18), and thus (3.17) holds true.

Step 4. Energy method and compactness result. From the assumption on the operator g, (3.13), (3.14) and (3.16), using the compactness theorem (see, e.g. [25]), we conclude that there exist a subsequence $\{\theta^n\}$ (after relabeling), an element $\theta \in L^\infty(0,T;H^s) \cap L^2(0,T;H^{s+\alpha})$ with $\theta' \in L^2(0,T;H^{s-\alpha})$ and $\xi \in L^2(0,T;H^s)$ such that

$$\begin{cases}\theta^n \rightharpoonup \theta & \text{weakly star in } L^\infty(0,T;H^s),\\ \theta^n \rightharpoonup \theta & \text{weakly in } L^2(0,T;H^{s+\alpha}),\\ \{\theta^n\}' \rightharpoonup \theta' & \text{weakly in } L^2(0,T;H^{s-\alpha}),\\ \theta^n \to \theta & \text{strongly in } L^2(0,T;H^s),\\ g(\cdot,\theta^n_\cdot) \rightharpoonup \xi & \text{weakly in } L^2(0,T;H^s),\end{cases} \tag{3.19}$$

and $\theta \in C([0,T];H^s)$, see also [33, Corollary 7.3] for more details.

Using (3.19), we deduce that

$$\theta^n(t) \to \theta(t) \quad \text{in } H^s \text{ a.e. } t \in [0,T], \tag{3.20}$$

which is not enough for our purpose. Indeed, our goal is to prove that

$$\theta^n \to \theta \quad \text{in } C([0,T];H^s). \tag{3.21}$$

If this were not true, then taking into account that $\theta \in C([0,T];H^s)$, there would exist $\varepsilon > 0$, a value $t_0 \in [0,T]$, a subsequence (relabelled the same) $\{\theta^n\}$ and $t_n \in [0,T]$ with $\lim_{n\to\infty} t_n = t_0$ such that

$$\|\theta^n(t_n) - \theta(t_0)\|_{H^s} \geq \varepsilon, \quad \forall n \geq 1.$$

In order to prove that this is absurd, we will use an energy method.

Observe that the following energy inequality holds for all θ^n and $\forall r,t \in [0,T]$:

$$\begin{aligned} &\frac{1}{2}\|\theta^n(t)\|_{H^s}^2 + \frac{\kappa}{2}\int_r^t \|\theta^n(\sigma)\|_{H^{s+\alpha}}^2 d\sigma \\ &\quad \leq \frac{1}{2}\|\theta^n(r)\|_{H^s}^2 + \int_r^t \langle \Lambda^s f, \Lambda^s \theta^n(\sigma)\rangle d\sigma + C_0(t-r), \end{aligned} \tag{3.22}$$

with $C_0 = CC_{p,T}^{\frac{4-2\eta}{1-\eta}} + \frac{L_g^2 C_T}{\lambda_1^\alpha \kappa}$, where C and η are the constants appearing in (3.8), $C_{p,T}$ is the constant appearing in (3.4), and $\frac{L_g^2 C_T}{\lambda_1^\alpha \kappa}$ corresponds to the following estimate:

$$\begin{aligned} \int_r^t \langle \Lambda^s g(\sigma,\theta^n_\sigma), \Lambda^s\theta^n(\sigma)\rangle d\sigma &\leq \frac{\lambda_1^\alpha \kappa}{4}\int_r^t \|\theta^n(\sigma)\|_{H^s}^2 d\sigma + \frac{1}{\lambda_1^\alpha\kappa}\int_r^t \|g(\sigma,\theta^n_\sigma)\|_{H^s}^2 d\sigma \\ &\leq \frac{\kappa}{4}\int_r^t \|\theta^n(\sigma)\|_{H^{s+\alpha}}^2 d\sigma + \frac{L_g^2 C_T}{\lambda_1^\alpha \kappa}(t-r), \end{aligned}$$

where C_T is the constant appearing in (3.12).

On the other hand, note that for any $v \in C^\infty([0,T];H^{s+\alpha})$,

$$\begin{aligned} &\int_0^T \langle u^n(t)\cdot\nabla\theta^n(t) - u(t)\cdot\nabla\theta(t), v(t)\rangle dt \\ &= \int_0^T \langle (u^n(t)-u(t))\cdot\nabla\theta^n(t), v(t)\rangle + \langle u(t)\cdot\nabla(\theta^n(t)-\theta(t)), v(t)\rangle dt \\ &= -\int_0^T \langle (u^n(t)-u(t))\theta^n(t), \nabla v(t)\rangle + \langle u(t)(\theta^n(t)-\theta(t)), \nabla v(t)\rangle dt \\ &\leq \int_0^T \|u^n(t)-u(t)\|_{L^2}\|\theta^n(t)\|_{L^{p_4}}\|\nabla v(t)\|_{L^{p_3}} dt \\ &\quad + \int_0^T \|u(t)\|_{L^{p_4}}\|\theta^n(t)-\theta(t)\|_{L^2}\|\nabla v(t)\|_{L^{p_3}} dt \\ &\leq C\|v\|_{L^\infty(0,T;H^{s+\alpha})}\|\theta^n\|_{L^2(0,T;H^{s+\alpha})}\|\theta^n-\theta\|_{L^2(0,T;H^s)} \\ &\quad + C\|v\|_{L^\infty(0,T;H^{s+\alpha})}\|\theta\|_{L^2(0,T;H^{s+\alpha})}\|\theta^n-\theta\|_{L^2(0,T;H^s)} \ \to 0, \end{aligned}$$

where $p_3 = \frac{2}{\alpha}$, $p_4 = \frac{2}{1-\alpha}$, and we have used $\nabla\cdot u = \nabla\cdot u^n = 0$, Lemma 1, (3.19) and the Sobolev embeddings $H^s \subset L^2$, $H^{s+\alpha} \subset H^{1,p_3}$, $H^{s+\alpha} \subset L^{p_4}$.

Then by (3.19), passing to the limit in (3.1), we obtain that $\theta \in C([0,T];H^s)$ is a solution of a similar problem to Eq. (1.1), namely, for $\forall v \in H^{s+\alpha}$,

$$\frac{d}{dt}\langle\theta(t),v\rangle + \kappa\langle(-\Delta)^\alpha\theta(t),v\rangle + \langle u(t)\cdot\nabla\theta(t),v\rangle = \langle f,v\rangle + \langle\xi(t),v\rangle,$$

with the initial datum $\theta(0)=\varphi(0)$, fulfilled in the sense of scalar distribution $\mathcal{D}'(0,T)$. Therefore, it satisfies the energy equality

$$\begin{aligned}
&\frac{1}{2}\|\theta(t)\|_{H^s}^2 + \kappa\int_r^t \|\theta(\sigma)\|_{H^{s+\alpha}}^2 d\sigma \\
&\quad = \frac{1}{2}\|\theta(r)\|_{H^s}^2 - \int_r^t \langle\Lambda^s(u(\sigma)\cdot\nabla\theta(\sigma)),\Lambda^s\theta(\sigma)\rangle d\sigma \\
&\qquad + \int_r^t \langle\Lambda^s f,\Lambda^s\theta(\sigma)\rangle d\sigma + \int_r^t \langle\Lambda^s\xi(\sigma),\Lambda^s\theta(\sigma)\rangle d\sigma.
\end{aligned}$$

Thanks to (3.4), we find that

$$\int_r^t \|\theta^n(\sigma)\|_{L^p}^\beta d\sigma < \infty,$$

where $\beta=\frac{4-2\eta}{1-\eta}$, which implies that $\{\theta^n\}$ converges weakly to θ in $L^\beta(r,t;L^p)$. Then, similar to arguments of (3.6)-(3.8), we have

$$\begin{aligned}
&\int_r^t \langle\Lambda^s(u(\sigma)\cdot\nabla\theta(\sigma)),\Lambda^s\theta(\sigma)\rangle d\sigma \le \frac{\kappa}{4}\int_r^t \|\theta(\sigma)\|_{H^{s+\alpha}}^2 d\sigma + C\int_r^t \|\theta(\sigma)\|_{L^p}^\beta d\sigma \\
&\quad \le \frac{\kappa}{4}\int_r^t \|\theta(\sigma)\|_{H^{s+\alpha}}^2 d\sigma + C\liminf_{n\to\infty}\int_r^t \|\theta^n(\sigma)\|_{L^p}^\beta d\sigma \\
&\quad \le \frac{\kappa}{4}\int_r^t \|\theta(\sigma)\|_{H^{s+\alpha}}^2 d\sigma + CC_{p,T}^\beta(t-r), \quad 0\le r\le t\le T,
\end{aligned}$$

where C is the constant appearing in (3.8) and $C_{p,T}$ is the constant appearing in (3.4). In addition, from the last convergence in (3.19) and the Young inequality, we obtain

$$\begin{aligned}
&\int_r^t \langle\Lambda^s\xi(\sigma),\Lambda^s\theta(\sigma)\rangle d\sigma \le \frac{\lambda_1^\alpha\kappa}{4}\int_r^t \|\theta(\sigma)\|_{H^s}^2 d\sigma + \frac{1}{\lambda_1^\alpha\kappa}\int_r^t \|\xi(\sigma)\|_{H^s}^2 d\sigma \\
&\quad \le \frac{\kappa}{4}\int_r^t \|\theta(\sigma)\|_{H^{s+\alpha}}^2 d\sigma + \frac{1}{\lambda_1^\alpha\kappa}\liminf_{n\to\infty}\int_r^t \|g(\sigma,\theta_\sigma^n)\|_{H^s}^2 d\sigma \\
&\quad \le \frac{\kappa}{4}\int_r^t \|\theta(\sigma)\|_{H^{s+\alpha}}^2 d\sigma + \frac{L_g^2 C_T}{\lambda_1^\alpha\kappa}(t-r), \quad 0\le r\le t\le T.
\end{aligned}$$

Hence, θ also satisfies inequality (3.22) with the same constant C_0.

We consider the functions $J_n, J : [0,T] \to \mathbb{R}$ defined by

$$J_n(t) = \frac{1}{2}\|\theta^n(t)\|_{H^s}^2 - \int_0^t \langle \Lambda^s f, \Lambda^s \theta^n(\sigma)\rangle d\sigma - C_0 t,$$
$$J(t) = \frac{1}{2}\|\theta(t)\|_{H^s}^2 - \int_0^t \langle \Lambda^s f, \Lambda^s \theta(\sigma)\rangle d\sigma - C_0 t,$$

where C_0 is the constant given in (3.22). From (3.22) and the analogous inequality for θ, it is clear that J_n and J are non-increasing and continuous functions. Moreover, by (3.19) and (3.20),

$$J_n(t) \to J(t), \quad \text{a.e. } t \in [0,T]. \tag{3.23}$$

Now, we are ready to prove that

$$\theta^n(t_n) \to \theta(t_0) \quad \text{in } H^s. \tag{3.24}$$

Observe that

$$\begin{aligned}\|\theta^n(t_2) - \theta^n(t_1)\|_{H^{s-\alpha}} &\le \int_{t_1}^{t_2} \|(\theta^n)'(t)\|_{H^{s-\alpha}} dt \\ &\le \|(\theta^n)'\|_{L^2(t_1,t_2;H^{s-\alpha})}(t_2 - t_1)^{1/2}, \quad \forall t_1, t_2 \in [0,T].\end{aligned}$$

By (3.16), we find that $\{\theta^n\}$ is equicontinuous on $[0,T]$ with values in $H^{s-\alpha}$. Since the injection of H^s into $H^{s-\alpha}$ is compact, using (3.12) and the equicontinuity in $H^{s-\alpha}$, the Arzelà-Ascoli theorem ensures that

$$\theta^n \to \theta \quad \text{in } C([0,T]; H^{s-\alpha}). \tag{3.25}$$

This, jointly with (3.12), implies that

$$\theta^n(t_n) \rightharpoonup \theta(t_0) \quad \text{weakly in } H^s, \tag{3.26}$$

where we have used (3.25) to identify which is the weak limit. Then we infer that

$$\|\theta(t_0)\|_{H^s} \le \liminf_{n\to\infty} \|\theta^n(t_n)\|_{H^s}.$$

Therefore, if we show that

$$\limsup_{n\to\infty} \|\theta^n(t_n)\|_{H^s} \le \|\theta(t_0)\|_{H^s}, \tag{3.27}$$

we obtain that $\lim_{n\to\infty} \|\theta^n(t_n)\|_{H^s} = \|\theta(t_0)\|_{H^s}$, which together with (3.26) implies (3.24).

For the case $t_0 = 0$, (3.27) follows directly from Step 3 and (3.22). So we may assume that $t_0 > 0$. This is important, since we will approach this value t_0 from the left by a sequence $\{\widetilde{t}_k\}$, i.e., $\lim_{k\to\infty} \widetilde{t}_k \nearrow t_0$, with $\{\widetilde{t}_k\}$ being values such that (3.23) holds. Since J is continuous at t_0, for any $\varepsilon > 0$, there is k_ε such that

$$|J(\widetilde{t}_k) - J(t_0)| < \frac{\varepsilon}{2}, \quad \forall k \geq k_\varepsilon.$$

Note that t_k satisfies (3.23), taking $n \geq n(k_\varepsilon)$, it is possible to obtain

$$|J_n(\widetilde{t}_{k_\varepsilon}) - J(\widetilde{t}_{k_\varepsilon})| < \frac{\varepsilon}{2}, \quad \forall n \geq n(k_\varepsilon).$$

Taking $n \geq n(k_\varepsilon)$ such that $t_n > \widetilde{t}_{k_\varepsilon}$, as J_n is non-increasing, we have

$$J_n(t_n) - J(t_0) \leq |J_n(\widetilde{t}_{k_\varepsilon}) - J(\widetilde{t}_{k_\varepsilon})| + |J(\widetilde{t}_{k_\varepsilon}) - J(t_0)| < \varepsilon.$$

It can also be deduced from (3.19) that

$$\int_0^{t_n} \langle \Lambda^s f, \Lambda^s \theta^n(\sigma)\rangle d\sigma \to \int_0^{t_0} \langle \Lambda^s f, \Lambda^s \theta(\sigma)\rangle d\sigma,$$

so we conclude that (3.27) holds. Thus, (3.24) and finally (3.21) are also true, as we wanted to check. Furthermore, thanks to Step 3, we deduce that

$$\begin{aligned}
&\sup_{-\infty<\tau\leq 0} \|\theta^n(t+\tau) - \theta(t+\tau)\|_{H^s} \\
&= \max\left\{ \sup_{-\infty<\tau\leq 0} \|P_n\varphi(\tau) - \varphi(\tau)\|_{H^s}, \sup_{0\leq\tau\leq t} \|\theta^n(\tau) - \theta(\tau)\|_{H^s} \right\} \\
&= \max\left\{ \|P_n\varphi - \varphi\|_{\mathscr{C}(H^s)}, \sup_{0\leq\tau\leq t} \|\theta^n(\tau) - \theta(\tau)\|_{H^s} \right\} \to 0,
\end{aligned}$$

which implies that

$$\theta_t^n \to \theta_t \quad \text{in } \mathscr{C}(H^s), \ \forall t \leq T.$$

By the assumption on g, we obtain

$$\int_0^T \|g(t,\theta_t^n) - g(t,\theta_t)\|_{H^s}^2 dt \leq L_g^2 \int_0^T \|\theta_t^n - \theta_t\|_{\mathscr{C}(H^s)}^2 dt \to 0,$$

thanks to the Lebesgue dominated convergence theorem. Thus, we can finally pass to the limit in (3.1) and conclude that θ solves Eq. (1.1).

Step 5. Regularity of solutions. Once that $f \in H^s \cap L^p$ and $\varphi \in \mathscr{C}(H^s) \cap \mathscr{C}(L^p)$ with $\varphi(0) \in H^{s+\alpha}$, it is immediate to gain the strong regularity for the solution θ to Eq. (1.1). Indeed, by replacing s with $s+\alpha$ in Step 2, we conclude that $\theta^n \in L^\infty(0,T;H^{s+\alpha}) \cap L^2(0,T;H^{s+2\alpha})$. Thus the regularity of solutions follows from the similar arguments above. □

The following lemma shows the continuity of solutions with respect to initial data.

Lemma 4 *Under the assumptions of Theorem 1, the solution to Eq.* (1.1) *is continuous with respect to the initial value.*

Namely, denoting $\theta^{(i)}$, for $i = 1, 2$, the corresponding solution to initial data $\varphi^{(i)} \in \mathscr{C}(H^s) \cap \mathscr{C}(L^p)$, the following estimate holds:

$$\begin{aligned}
&\|\theta_t^{(1)} - \theta_t^{(2)}\|^2_{\mathscr{C}(H^s)} \\
&\quad \leq \|\varphi^{(1)} - \varphi^{(2)}\|^2_{\mathscr{C}(H^s)} e^{\int_0^t (C\|\theta^{(1)}(\sigma)\|^2_{H^{s+\alpha}} + C\|\theta^{(2)}(\sigma)\|^2_{H^{s+\alpha}} + 2L_g) d\sigma}.
\end{aligned} \tag{3.28}$$

Proof Let $w(t) = \theta^{(1)}(t) - \theta^{(2)}(t)$. We can easily derive an evolution system for w:

$$\begin{aligned}
\frac{d}{dt} w(t) + \kappa(-\Delta)^\alpha w(t) &= -(u^{(1)}(t) \cdot \nabla\theta^{(1)}(t) - u^{(2)}(t) \cdot \nabla\theta^{(2)}(t)) \\
&\quad + g(t, \theta_t^{(1)}) - g(t, \theta_t^{(2)}),
\end{aligned}$$

where $u^{(1)} = \mathcal{R}^\perp \theta^{(1)}$ and $u^{(2)} = \mathcal{R}^\perp \theta^{(2)}$. Applying Λ^s to the above system and taking the inner product in L^2 with $\Lambda^s w$, in view of bilinearity of $u^{(1)} \cdot \nabla\theta^{(1)}$ and $u^{(2)} \cdot \nabla\theta^{(2)}$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{H^s} &+ \kappa\|w(t)\|^2_{H^{s+\alpha}} = -\langle \Lambda^s(u^{(1)}(t) \cdot \nabla w(t)), \Lambda^s w(t)\rangle \\
&- \langle \Lambda^s((u^{(1)}(t) - u^{(2)}(t)) \cdot \nabla\theta^{(2)}(t)), \Lambda^s w(t)\rangle \\
&+ \langle \Lambda^s(g(t, \theta_t^{(1)}) - g(t, \theta_t^{(2)})), \Lambda^s w(t)\rangle \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{3.29}$$

Note that $\langle u^{(1)}(t) \cdot \nabla(\Lambda^s w(t)), \Lambda^s w(t)\rangle = 0$, ∇ and Λ^s are commutable ([19] or [16, Remark 5.3]). Then we can make use of Lemmas 1-2 to obtain

$$\begin{aligned}
|J_1| &= |\langle \Lambda^s(u^{(1)}(t) \cdot \nabla w(t)) - u^{(1)}(t) \cdot \nabla(\Lambda^s w(t)), \Lambda^s w(t)\rangle| \\
&= |\langle \Lambda^s(u^{(1)}(t) \cdot \nabla w(t)) - u^{(1)}(t) \cdot \Lambda^s(\nabla w(t)), \Lambda^s w(t)\rangle| \\
&\leq \|\Lambda^s(u^{(1)}(t) \cdot \nabla w(t)) - u^{(1)}(t) \cdot \Lambda^s(\nabla w(t))\|_{L^2}\|\Lambda^s w(t)\|_{L^2} \\
&\leq C(\|u^{(1)}(t)\|_{H^{1,p_3}}\|w(t)\|_{H^{s,p_4}} + \|u^{(1)}(t)\|_{H^{s,p_4}}\|w(t)\|_{H^{1,p_3}})\|w(t)\|_{H^s} \\
&\leq C\|\theta^{(1)}(t)\|_{H^{s+\alpha}}\|w(t)\|_{H^{s+\alpha}}\|w(t)\|_{H^s},
\end{aligned} \tag{3.30}$$

where $p_3 = \frac{2}{\alpha}$, $p_4 = \frac{2}{1-\alpha}$ and we have used the Sobolev embeddings $H^{s+\alpha} \subset H^{1,p_3}$ and $H^{s+\alpha} \subset H^{s,p_4}$. For J_2, similar to the arguments of (3.30), we have

$$\begin{aligned}
|J_2| &\leq |\langle \Lambda^s((u^{(1)}(t) - u^{(2)}(t)) \cdot \nabla\theta^{(2)}(t)) \\
&\quad - (u^{(1)}(t) - u^{(2)}(t)) \cdot \nabla(\Lambda^s\theta^{(2)}(t)), \Lambda^s w(t)\rangle| \\
&\quad + |\langle (u^{(1)}(t) - u^{(2)}(t)) \cdot \nabla(\Lambda^s\theta^{(2)}(t)), \Lambda^s w(t)\rangle| \\
&\leq C\|w(t)\|_{H^{s+\alpha}}\|\theta^{(2)}(t)\|_{H^{s+\alpha}}\|w(t)\|_{H^s} \\
&\quad + |\langle (u^{(1)}(t) - u^{(2)}(t)) \cdot \nabla(\Lambda^s\theta^{(2)}(t)), \Lambda^s w(t)\rangle|.
\end{aligned} \tag{3.31}$$

By the Hölder inequality and Lemmas 1-2, in view of the fact that $\nabla \cdot u^{(1)} = \nabla \cdot u^{(2)} = 0$, we estimate the second term in the right-hand side of (3.31) as follows:

$$\begin{aligned}
&|\langle (u^{(1)}(t) - u^{(2)}(t)) \cdot \nabla(\Lambda^s \theta^{(2)}(t)), \Lambda^s w(t)\rangle| \\
&\quad = |\langle \nabla((u^{(1)}(t) - u^{(2)}(t))\Lambda^s \theta^{(2)}(t)), \Lambda^s w(t)\rangle| \\
&\quad \le \|\Lambda^{1-\alpha}((u^{(1)}(t) - u^{(2)}(t))\Lambda^s \theta^{(2)}(t))\|_{L^2} \|\Lambda^{s+\alpha} w(t)\|_{L^2} \\
&\quad \le C\|u^{(1)}(t) - u^{(2)}(t)\|_{L^{p_1}} \|\theta^{(2)}(t)\|_{H^{s+1-\alpha,p_2}} \|w(t)\|_{H^{s+\alpha}} \\
&\qquad + \|u^{(1)}(t) - u^{(2)}(t)\|_{H^{1-\alpha,p_3}} \|\theta^{(2)}(t)\|_{H^{s,p_4}} \|w(t)\|_{H^{s+\alpha}} \\
&\quad \le C\|w(t)\|_{H^s} \|\theta^{(2)}(t)\|_{H^{s+\alpha}} \|w(t)\|_{H^{s+\alpha}}, \qquad (3.32)
\end{aligned}$$

where $p_1 = \frac{2}{2\alpha-1}$, $p_2 = \frac{1}{1-\alpha}$, $p_3 = \frac{2}{\alpha}$, $p_4 = \frac{2}{1-\alpha}$, and we have used the Sobolev embeddings $H^s \subset L^{p_1}$, $H^{s+\alpha} \subset H^{s+1-\alpha,p_2}$, $H^s \subset H^{1-\alpha,p_3}$, $H^{s+\alpha} \subset H^{s,p_4}$. Substituting (3.32) into (3.31) gives

$$|J_2| \le C\|\theta^{(2)}(t)\|_{H^{s+\alpha}} \|w(t)\|_{H^{s+\alpha}} \|w(t)\|_{H^s}. \qquad (3.33)$$

For J_3, using the Hölder inequality and the assumption on g, we deduce that

$$\begin{aligned}
|J_3| &\le \|g(t, \theta_t^{(1)}) - g(t, \theta_t^{(2)})\|_{H^s} \|w(t)\|_{H^s} \\
&\le L_g \|w_t\|_{\mathscr{C}(H^s)} \|w(t)\|_{H^s} \le L_g \|w_t\|^2_{\mathscr{C}(H^s)}. \qquad (3.34)
\end{aligned}$$

Combining estimates (3.30), (3.33) and (3.34), by the Young inequality, it follows from (3.29) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{H^s} + \frac{1}{2}\kappa\|w(t)\|^2_{H^{s+\alpha}} \\
&\quad \le C(\|\theta^{(1)}(t)\|^2_{H^{s+\alpha}} + \|\theta^{(2)}(t)\|^2_{H^{s+\alpha}})\|w(t)\|^2_{H^s} + L_g\|w_t\|^2_{\mathscr{C}(H^s)}. \qquad (3.35)
\end{aligned}$$

After integrating (3.35) between 0 and t, if we substitute t by $\sigma \in [0, t]$ and take the supremum, we obtain

$$\begin{aligned}
\sup_{0\le\sigma\le t} \|w(\sigma)\|^2_{H^s} &\le \|w(0)\|^2_{H^s} \\
&\quad + C\int_0^t (\|\theta^{(1)}(\sigma)\|^2_{H^{s+\alpha}} + \|\theta^{(2)}(\sigma)\|^2_{H^{s+\alpha}})\|w(\sigma)\|^2_{H^s} d\sigma \\
&\quad + 2L_g \int_0^t \|w_\sigma\|^2_{\mathscr{C}(H^s)} d\sigma. \qquad (3.36)
\end{aligned}$$

Note that

$$\begin{aligned}
\|w_t\|_{\mathscr{C}(H^s)} &= \sup_{-\infty<\sigma\le 0} \|w(t+\sigma)\|_{H^s} \\
&= \max\left\{ \sup_{-\infty<\sigma\le 0} \|\varphi^{(1)}(\sigma) - \varphi^{(2)}(\sigma)\|_{H^s}, \sup_{0\le\sigma\le t} \|w(\sigma)\|_{H^s} \right\} \\
&= \max\left\{ \|\varphi^{(1)} - \varphi^{(2)}\|_{\mathscr{C}(H^s)}, \sup_{0\le\sigma\le t} \|w(\sigma)\|_{H^s} \right\}. \qquad (3.37)
\end{aligned}$$

Then we conclude from (3.36) that

$$\begin{aligned}\|w_t\|^2_{\mathscr{C}(H^s)} &\leq \|\varphi^{(1)} - \varphi^{(2)}\|^2_{\mathscr{C}(H^s)} \\ &+ \int_0^t (C\|\theta^{(1)}(\sigma)\|^2_{H^{s+\alpha}} + C\|\theta^{(2)}(\sigma)\|^2_{H^{s+\alpha}} + 2L_g)\|w_\sigma\|^2_{\mathscr{C}(H^s)} d\sigma.\end{aligned}$$

Thus the estimate (3.28) follows from the Gronwall lemma. □

The uniqueness of weak solutions follows immediately from Lemma 4.

Theorem 2 *Under the assumptions of Theorem 1, the weak solution obtained in Theorem 1 is unique.*

Proof Let $\theta^{(1)}$ and $\theta^{(2)}$ be two solutions of Eq. (1.1). Since they have the same initial datum φ, it follows from Lemma 4 that $\|\theta_t^{(1)} - \theta_t^{(2)}\|^2_{\mathscr{C}(H^s)} = 0$, and thus the proof is complete. □

4 Asymptotic behavior of solutions

In this section we consider the long time behavior of solutions in a neighborhood of a stationary solution to Eq. (1.1) when the delay term has a special form. First of all, we establish the existence, uniqueness and regularity of s-tationary solutions under some restrictions. Then we present various methods to obtain the stability properties: the Lyapunov function method, the construction method of appropriate Lyapunov functionals and the Razumikhin-Lyapunov technique. Finally, the polynomial stability of stationary solutions is addressed in the case of proportional variable delays.

4.1 Existence, uniqueness and regularity of stationary solutions

In order to investigate the properties of stationary solutions to Eq. (1.1), we assume that g fulfills

(H) Let $i : H^s \to \mathscr{C}(H^s)$ be the trivial immersion given by $i(\theta) = \widetilde{\theta}$ with $\widetilde{\theta}(t) = \theta$ for all $t \leq 0$. Then

$$g(t_1, \xi) = g(t_2, \xi), \quad \text{for any } t_1, t_2 \in \mathbb{R}^+ \text{ and all } \xi \in i(H^s).$$

For example, given measurable functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ and $G : H^s \to H^s$, we can define $g : \mathbb{R}^+ \times \mathscr{C}(H^s) \to H^s$ by $g(t, \xi) := G(\xi(-\rho(t)))$. It is clear that g satisfies the above assumption (H).

If (H) holds, for $g \in \mathrm{Lip}_0(\mathscr{C}(H^s), H^s)$, we define $\widetilde{g} : H^s \to H^s$ as $\widetilde{g}(\theta) = g(0, i(\theta))$. Then $\widetilde{g} = g|_{\mathbb{R}^+ \times i(H^s)}$ is of course autonomous, and it also belongs to $\mathrm{Lip}_0(\mathscr{C}(H^s), H^s)$ with the same Lipschitz constant as g.

Definition 2 A stationary solution to Eq. (1.1) is a function $\theta^* \in H^{s+\alpha}$ such that

$$\kappa(-\Delta)^\alpha \theta^* + \mathcal{R}^\perp \theta^* \cdot \nabla \theta^* = f + \widetilde{g}(\theta^*). \tag{4.1}$$

We now establish the existence, uniqueness and regularity of stationary solutions to Eq. (1.1) as stated in the next result.

Theorem 3 *Fix $\alpha \in (\frac{1}{2}, 1)$ and $s \geq 2 - 2\alpha$. Let $g \in Lip_0(\mathscr{C}(H^s), H^s)$ and the condition (H) be satisfied. Suppose that*

$$\kappa > \lambda_1^{-\alpha} L_g \quad \text{and} \quad \Delta := (\kappa - \lambda_1^{-\alpha} L_g)^2 - 4\Upsilon \|f\|_{H^{s-\alpha}} > 0, \tag{4.2}$$

where $\Upsilon = 4C_2 M_1 M_2 (C(0, p_5) + C(\frac{s+1-\alpha}{2}, p_6))$ and the related constants will be given later on. Then

(i) for all $f \in H^{s-\alpha}$, there exists at least one solution to Eq. (4.1);
(ii) for all $f \in H^{s-\alpha}$, the solution to Eq. (4.1) is unique;
(iii) if $f \in H^s$, the solution to Eq. (4.1) belongs to $H^{s+2\alpha}$.

Proof (i) Let a positive number R be fixed later on, and set

$$\mathscr{B} = \{z \in H^{s+\alpha} : \|z\|_{H^{s+\alpha}} \leq R\}.$$

It is clear that $\mathscr{B}$ is a closed and convex subset of $H^{s+\alpha}$. Then for each $z \in \mathscr{B}$, by the Lax-Milgram theorem there exists a unique $\theta \in H^{s+\alpha}$ such that

$$\kappa\langle \Lambda^{s+\alpha}\theta, \Lambda^{s+\alpha} v\rangle + \langle \Lambda^s(\mathcal{R}^\perp z \cdot \nabla\theta), \Lambda^s v\rangle = \langle \Lambda^s f, \Lambda^s v\rangle + \langle \Lambda^s \widetilde{g}(z), \Lambda^s v\rangle, \tag{4.3}$$

with $v \in H^{s+\alpha}$. Hence we define a mapping $\mathcal{L} : z \in \mathscr{B} \to \theta \in H^{s+\alpha}$. The rest of the proof can be divided into three steps.

Step 1. The mapping $\mathcal{L}$ maps $\mathscr{B}$ into itself.

Taking $v = \theta$ in (4.3), it follows that

$$\kappa\|\theta\|_{H^{s+\alpha}} \leq \|\Lambda^{s-\alpha}(\mathcal{R}^\perp z \cdot \nabla\theta)\|_{L^2} + \|f\|_{H^{s-\alpha}} + \|\widetilde{g}(z)\|_{H^{s-\alpha}}. \tag{4.4}$$

By Lemmas 1 and 2, in view of $\nabla \cdot (\mathcal{R}^\perp z) = 0$, we deduce that

$$\begin{aligned}
&\|\Lambda^{s-\alpha}(\mathcal{R}^\perp z \cdot \nabla\theta)\|_{L^2} \leq \|\Lambda^{s+1-\alpha}(\mathcal{R}^\perp z\theta)\|_{L^2} \\
&\quad \leq C_2(\|\mathcal{R}^\perp z\|_{L^{p_5}}\|\theta\|_{H^{s+1-\alpha,p_6}} + \|\mathcal{R}^\perp z\|_{H^{s+1-\alpha,p_6}}\|\theta\|_{L^{p_5}}) \\
&\quad \leq C_2(C(0,p_5)\|z\|_{L^{p_5}}\|\theta\|_{H^{s+1-\alpha,p_6}} + C(\frac{s+1-\alpha}{2}, p_6)\|z\|_{H^{s+1-\alpha,p_6}}\|\theta\|_{L^{p_5}}) \\
&\quad \leq C_2 M_1 M_2 \left(C(0,p_5) + C(\frac{s+1-\alpha}{2}, p_6)\right)\|z\|_{H^{s+\alpha}}\|\theta\|_{H^{s+\alpha}} \\
&\quad := \Upsilon\|z\|_{H^{s+\alpha}}\|\theta\|_{H^{s+\alpha}},
\end{aligned} \tag{4.5}$$

where $p_5 = \frac{2}{2\alpha^- - 1}$, $p_6 = \frac{1}{1-\alpha^-}$ with $\alpha^- \in (\frac{1}{2}, \alpha)$, $C(0, p_5)$ and $C(\frac{s+1-\alpha}{2}, p_6)$ are the constants appearing in Lemma 1, C_2 is the constant appearing in

Lemma 2, M_i, $i = 1, 2$, are the constants related with the following Sobolev embedding inequalities:

$$\|z\|_{L^{p_5}} \leq M_1 \|z\|_{H^{s+\alpha}}, \quad \|z\|_{H^{s+1-\alpha, p_6}} \leq M_2 \|z\|_{H^{s+\alpha}}.$$

Thanks to the assumption on g, we have

$$\|\widetilde{g}(z)\|_{H^{s-\alpha}} \leq \lambda_1^{-\frac{\alpha}{2}} \|\widetilde{g}(z)\|_{H^s} \leq \lambda_1^{-\frac{\alpha}{2}} L_g \|z\|_{H^s} \leq \lambda_1^{-\alpha} L_g \|z\|_{H^{s+\alpha}}. \tag{4.6}$$

Inserting (4.5)-(4.6) into (4.4) results in

$$\kappa \|\theta\|_{H^{s+\alpha}} \leq \Upsilon \|z\|_{H^{s+\alpha}} \|\theta\|_{H^{s+\alpha}} + \|f\|_{H^{s-\alpha}} + \lambda_1^{-\alpha} L_g \|z\|_{H^{s+\alpha}}. \tag{4.7}$$

Note that $\Delta = (\kappa - \lambda_1^{-\alpha} L_g)^2 - 4\Upsilon \|f\|_{H^{s-\alpha}} > 0$ and $\frac{(\kappa - \lambda_1^{-\alpha} L_g) - \sqrt{\Delta}}{2\Upsilon} < \frac{\kappa}{\Upsilon}$. We can pick $0 < R < \frac{\kappa}{\Upsilon}$ such that $(\kappa - \lambda_1^{-\alpha} L_g) R - \Upsilon R^2 \geq \|f\|_{H^{s-\alpha}}$. By (4.7), the mapping $\mathcal{L}$ defined by (4.3), maps $\mathscr{B}$ into $\mathscr{B}$.

Step 2. The mapping $\mathcal{L}$ is continuous in $\mathscr{B}$.

Let $z_i \in \mathscr{B}$ and $\theta_i \in \mathscr{B}$ be such that

$$\kappa \langle \Lambda^{s+\alpha} \theta_i, \Lambda^{s+\alpha} v \rangle + \langle \Lambda^s (\mathcal{R}^\perp z_i \cdot \nabla \theta_i), \Lambda^s v \rangle = \langle \Lambda^s f, \Lambda^s v \rangle + \langle \Lambda^s \widetilde{g}(z_i), \Lambda^s v \rangle,$$

with $v \in H^{s+\alpha}$, where $i = 1, 2$. Taking $v = \theta_1 - \theta_2$ and using the bilinearity of $\mathcal{R}^\perp z_i \cdot \nabla \theta_i$, we find that

$$\begin{aligned} \kappa \|\theta_1 - \theta_2\|_{H^{s+\alpha}}^2 &\leq |\langle \Lambda^s (\mathcal{R}^\perp z_1 \cdot \nabla (\theta_1 - \theta_2)), \Lambda^s (\theta_1 - \theta_2) \rangle| \\ &\quad + |\langle \Lambda^s ((\mathcal{R}^\perp z_1 - \mathcal{R}^\perp z_2) \cdot \nabla \theta_2), \Lambda^s (\theta_1 - \theta_2) \rangle| \\ &\quad + |\langle \Lambda^s (\widetilde{g}(z_1) - \widetilde{g}(z_2)), \Lambda^s (\theta_1 - \theta_2) \rangle| \\ &:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{4.8}$$

Modifying slightly the arguments in (4.5), in view of the Hölder inequality, we deduce that

$$\begin{aligned} \mathcal{I}_1 &\leq \|\Lambda^{s-\alpha} (\mathcal{R}^\perp z_1 \cdot \nabla (\theta_1 - \theta_2))\|_{L^2} \|\Lambda^{s+\alpha} (\theta_1 - \theta_2)\|_{L^2} \\ &\leq \Upsilon \|z_1\|_{H^{s+\alpha}} \|\theta_1 - \theta_2\|_{H^{s+\alpha}}^2, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \mathcal{I}_2 &\leq \|\Lambda^{s-\alpha} ((\mathcal{R}^\perp z_1 - \mathcal{R}^\perp z_2) \cdot \nabla \theta_2)\|_{L^2} \|\Lambda^{s+\alpha} (\theta_1 - \theta_2)\|_{L^2} \\ &\leq C_2 M_2 C(0, p_5) \|z_1 - z_2\|_{L^{p_5}} \|\theta_2\|_{H^{s+\alpha}} \|\theta_1 - \theta_2\|_{H^{s+\alpha}} \\ &\quad + C_2 M_1 C(\frac{s+1-\alpha}{2}, p_6) \|z_1 - z_2\|_{H^{s+1-\alpha, p_6}} \|\theta_2\|_{H^{s+\alpha}} \|\theta_1 - \theta_2\|_{H^{s+\alpha}}. \end{aligned} \tag{4.10}$$

By the Hölder inequality and the assumption on g, we have

$$\begin{aligned} \mathcal{I}_3 &\leq \|\widetilde{g}(z_1) - \widetilde{g}(z_2)\|_{H^{s-\alpha}} \|\theta_1 - \theta_2\|_{H^{s+\alpha}} \\ &\leq \lambda_1^{-\frac{\alpha}{2}} \|\widetilde{g}(z_1) - \widetilde{g}(z_2)\|_{H^s} \|\theta_1 - \theta_2\|_{H^{s+\alpha}} \\ &\leq \lambda_1^{-\frac{\alpha}{2}} L_g \|z_1 - z_2\|_{H^s} \|\theta_1 - \theta_2\|_{H^{s+\alpha}}. \end{aligned} \tag{4.11}$$

Substituting (4.9)-(4.11) into (4.8) gives

$$\begin{aligned}\kappa\|\theta_1-\theta_2\|^2_{H^{s+\alpha}} &\le \Upsilon\|z_1\|_{H^{s+\alpha}}\|\theta_1-\theta_2\|^2_{H^{s+\alpha}}\\ &+C_2M_2C(0,p_5)\|z_1-z_2\|_{L^{p_5}}\|\theta_2\|_{H^{s+\alpha}}\|\theta_1-\theta_2\|_{H^{s+\alpha}}\\ &+C_2M_1C(\frac{s+1-\alpha}{2},p_6)\|z_1-z_2\|_{H^{s+1-\alpha,p_6}}\|\theta_2\|_{H^{s+\alpha}}\|\theta_1-\theta_2\|_{H^{s+\alpha}}\\ &+\lambda_1^{-\frac{\alpha}{2}}L_g\|z_1-z_2\|_{H^s}\|\theta_1-\theta_2\|_{H^{s+\alpha}}. \end{aligned} \tag{4.12}$$

Note that $\|z_i\|_{H^{s+\alpha}} \le R < \frac{\kappa}{\Upsilon}$. Hence we can take $0<\varepsilon<\kappa-\Upsilon R$ such that

$$\begin{aligned}(\kappa-\Upsilon R-\varepsilon)\|\theta_1-\theta_2\|^2_{H^{s+\alpha}} &\le C_\varepsilon\|z_1-z_2\|^2_{L^{p_5}}\|\theta_2\|^2_{H^{s+\alpha}}\\ &+C_\varepsilon\|z_1-z_2\|^2_{H^{s+1-\alpha,p_6}}\|\theta_2\|^2_{H^{s+\alpha}}+C_\varepsilon\|z_1-z_2\|^2_{H^s}, \end{aligned} \tag{4.13}$$

thanks to the Young inequality. Since $H^{s+\alpha}\subset L^{p_5}$, $H^{s+\alpha}\subset H^{s+1-\alpha,p_6}$ and $H^{s+\alpha}\subset H^s$ are continuous embeddings, the continuity of the mapping $z\mapsto\theta$ in $\mathscr{B}$ follows from (4.13).

Step 3. $\mathcal{L}(\mathscr{B})$ is sequentially compact.

To ensure the sequential compactness of the set $\mathcal{L}(\mathscr{B})$, it suffices to prove that any sequence in $\mathcal{L}(\mathscr{B})$ has a convergent subsequence. Let $\{\theta^n\}\in\mathcal{L}(\mathscr{B})$ be given arbitrarily. Then there exists a sequence $\{z^n\}\in\mathscr{B}$ such that $\theta^n=\mathcal{L}(z^n)$. Since the embeddings $H^{s+\alpha}\subset L^{p_5}$, $H^{s+\alpha}\subset H^{s+1-\alpha,p_6}$ and $H^{s+\alpha}\subset H^s$ are compact (see, e.g. [7, Chapter 1]), we can find a subsequence z^{n_i} converging in L^{p_5}, $H^{s+1-\alpha,p_6}$ and H^s. By (4.13), we conclude that

$$\begin{aligned}(\kappa-\Upsilon R-\varepsilon)\|\theta^{n_i}-\theta^{n_j}\|^2_{H^{s+\alpha}} &\le C_\varepsilon\|z^{n_i}-z^{n_j}\|^2_{L^{p_5}}\|\theta^{n_j}\|^2_{H^{s+\alpha}}\\ &+C_\varepsilon\|z^{n_i}-z^{n_j}\|^2_{H^{s+1-\alpha,p_6}}\|\theta^{n_j}\|^2_{H^{s+\alpha}}+C_\varepsilon\|z^{n_i}-z^{n_j}\|^2_{H^s}\to 0 \end{aligned} \tag{4.14}$$

when i and j tend to infinity. It follows that the subsequence $\{\theta^{n_i}\}$ is a Cauchy sequence in $H^{s+\alpha}$, and therefore it is convergent as desired.

By the Schauder fixed point theorem, Step 1-Step 3 ensure the existence of a fixed point in $\mathscr{B}$, which clearly is a stationary solution to Eq. (4.1).

(*ii*) Arguing as in (4.4)-(4.7), the solution θ to Eq. (4.1) satisfies

$$\kappa\|\theta\|_{H^{s+\alpha}} \le \Upsilon\|\theta\|^2_{H^{s+\alpha}}+\|f\|_{H^{s-\alpha}}+\lambda_1^{-\alpha}L_g\|\theta\|_{H^{s+\alpha}}.$$

Note that $\Delta=(\kappa-\lambda_1^{-\alpha}L_g)^2-4\Upsilon\|f\|_{H^{s-\alpha}}>0$, then we deduce that $\|\theta\|_{H^{s+\alpha}}\le\beta_1$ or $\|\theta\|_{H^{s+\alpha}}\ge\beta_2$, where $\beta_1=\frac{(\kappa-\lambda_1^{-\alpha}L_g)-\sqrt{\Delta}}{2\Upsilon}$ and $\beta_2=\frac{(\kappa-\lambda_1^{-\alpha}L_g)+\sqrt{\Delta}}{2\Upsilon}$. According to the proof of (*i*), we know that the solution θ to Eq. (4.1) lies in $\mathscr{B}$ and consequently,

$$\|\theta\|_{H^{s+\alpha}}\le\beta_1. \tag{4.15}$$

Consider two solutions θ_1 and θ_2 to Eq. (4.1), similar to the arguments of (4.8), (4.9) and (4.11), we have

$$\begin{aligned}\kappa\|\theta_1-\theta_2\|^2_{H^{s+\alpha}} &\le |\langle \Lambda^s(\mathcal{R}^\perp\theta_1\cdot\nabla(\theta_1-\theta_2)),\Lambda^s(\theta_1-\theta_2)\rangle| \\ &\quad + |\langle \Lambda^s((\mathcal{R}^\perp\theta_1-\mathcal{R}^\perp\theta_2)\cdot\nabla\theta_2),\Lambda^s(\theta_1-\theta_2)\rangle| \\ &\quad + |\langle \Lambda^s(\widetilde{g}(\theta_1)-\widetilde{g}(\theta_2)),\Lambda^s(\theta_1-\theta_2)\rangle| \\ &\le \Upsilon(\|\theta_1\|_{H^{s+\alpha}}+\|\theta_2\|_{H^{s+\alpha}})\|\theta_1-\theta_2\|^2_{H^{s+\alpha}} \\ &\quad + \lambda_1^{-\frac{\alpha}{2}}L_g\|\theta_1-\theta_2\|_{H^s}\|\theta_1-\theta_2\|_{H^{s+\alpha}} \\ &\le \Upsilon(\|\theta_1\|_{H^{s+\alpha}}+\|\theta_2\|_{H^{s+\alpha}})\|\theta_1-\theta_2\|^2_{H^{s+\alpha}} + \lambda_1^{-\alpha}L_g\|\theta_1-\theta_2\|^2_{H^{s+\alpha}} \\ &\le 2\Upsilon\beta_1\|\theta_1-\theta_2\|^2_{H^{s+\alpha}} + \lambda_1^{-\alpha}L_g\|\theta_1-\theta_2\|^2_{H^{s+\alpha}}.\end{aligned}$$

Therefore,

$$\sqrt{\Delta}\|\theta_1-\theta_2\|^2_{H^{s+\alpha}} \le 0,$$

and the uniqueness of solutions to Eq. (4.1) follows.

(iii) If $f\in H^s\cap L^p$, then every solution θ^* to Eq. (4.1) is also a solution to Eq. (1.1), but with initial datum $\varphi(t)=\theta^*$ for $t\in(-\infty,0]$. Similar to Step 5 in Theorem 1, the strong regularity of θ^* follows immediately. □

As commented in the introduction of this section, our goal for the rest of our current work is to analyze the stability properties of stationary solutions to Eq. (1.1). For reader's convenience, we restate the following notions taken from [6].

Definition 3 *(i)* A stationary solution θ^* to Eq. (1.1) is *stable* if for any $\varepsilon>0$ there exists $\delta>0$ such that if $\varphi\in\mathscr{C}(H^s)$ satisfies $\|\varphi-i(\theta^*)\|_{\mathscr{C}(H^s)}<\delta$, then the solution $\theta(\cdot;\varphi)$ to Eq. (1.1) exists for all $t\ge0$ and satisfies $\|\theta(t;\varphi)-\theta^*\|_{H^s}<\varepsilon$ for any $t\ge0$.

(ii) A stationary solution θ^* to Eq. (1.1) is *attractive* if there exists $\widetilde{\delta}>0$ such that if $\varphi\in\mathscr{C}(H^s)$ satisfies $\|\varphi-i(\theta^*)\|_{\mathscr{C}(H^s)}<\widetilde{\delta}$, then the solution $\theta(\cdot;\varphi)$ to Eq. (1.1) exists for all $t\ge0$ and satisfies $\lim_{t\to\infty}\|\theta(t;\varphi)-\theta^*\|_{H^s}=0$.

(iii) A stationary solution θ^* to Eq. (1.1) is *asymptotically stable* if it is stable and attractive.

4.2 Local stability: Lyapunov function method

In this subsection we use a direct approach to prove that if the stationary solution to Eq. (1.1) exists, then it is stable in H^s with the topology induced by L^p, $p>\frac{2}{2\alpha-1}$, under some additional assumptions.

Theorem 4 *Fix $\alpha\in(\frac{1}{2},1)$, $s\ge2-2\alpha$ and $p>\frac{2}{2\alpha-1}$. Consider $f\in H^{s-\alpha}\cap L^p$ and $G\in Lip_0(H^s,H^s)\cap Lip_0(L^p,L^p)$. Let the delay forcing term g be given by*

$g(t,\theta_t) = G(\theta(t-\rho(t)))$, with $\rho \in C^1([0,\infty),\mathbb{R}^+)$ such that $\rho^* = \sup_{t\geq 0}\rho'(t) < 1$. Suppose (4.2) *and*

$$\frac{4\kappa(p-1)}{p} \geq pK\beta_1 + \frac{L_g(p-(p-1)\rho^*)}{\lambda_1^\alpha(1-\rho^*)} \tag{4.16}$$

hold true, where $\beta_1 = \frac{(\kappa-\lambda_1^{-\alpha}L_g)-\sqrt{\Delta}}{2\Upsilon}$ is as shown in Theorem 3 and K is given later on. Then for any solution $\theta^* \in H^{s+\alpha}$ to Eq. (4.1) *(whose existence is guaranteed by Theorem 3), and any $\varphi \in \mathscr{C}(H^s)\cap\mathscr{C}(L^p)$, the solution to Eq.* (1.1) *satisfies that for all $t \geq 0$,*

$$\|\theta(t)-\theta^*\|_{L^p}^p \leq \|\varphi(0)-\theta^*\|_{L^p}^p \quad \text{for } \rho(0)=0,$$

and

$$\|\theta(t)-\theta^*\|_{L^p}^p \leq \max\left\{1,\frac{L_g}{1-\rho^*}\right\}\left(\|\varphi(0)-\theta^*\|_{L^p}^p + \|\varphi - i(\theta^*)\|_{L^p(-\rho(0),0;L^p)}^p\right)$$

for $\rho(0) > 0$.

Proof Let θ be the solution to Eq. (1.1) and θ^* be the solution to Eq. (4.1). Set $w(t) = \theta(t)-\theta^*$, we observe that

$$\begin{aligned}&\frac{d}{dt}w(t) + \kappa(-\Delta)^\alpha w(t) + \mathcal{R}^\perp\theta(t)\cdot\nabla\theta(t) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^* \\ &\quad = G(\theta(t-\rho(t))) - G(\theta^*).\end{aligned}$$

Multiplying the above equation by $|w(t)|^{p-1}\mathrm{sgn}w(t)$ with $p > \frac{2}{2\alpha-1}$, and taking the inner product in L^2, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|w(t)\|_{L^p}^p &= -\kappa\int_{\mathbb{T}^2}(-\Delta)^\alpha w(t)|w(t)|^{p-1}\mathrm{sgn}w(t)dx \\ &\quad - \int_{\mathbb{T}^2}(\mathcal{R}^\perp\theta(t)\cdot\nabla\theta(t) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*)|w(t)|^{p-1}\mathrm{sgn}w(t)dx \\ &\quad + \int_{\mathbb{T}^2}(G(\theta(t-\rho(t))) - G(\theta^*))|w(t)|^{p-1}\mathrm{sgn}w(t)dx \\ &:= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.\end{aligned} \tag{4.17}$$

Using Lemma 3, we deduce that

$$\mathcal{J}_1 \leq -\frac{4\kappa(p-1)}{p^2}\int_{\mathbb{T}^2}\left[(-\Delta)^{\frac{\alpha}{2}}(|w(t)|^{\frac{p}{2}})\right]^2dx = -\frac{4\kappa(p-1)}{p^2}\||w(t)|^{\frac{p}{2}}\|_{H^\alpha}^2. \tag{4.18}$$

Since the quadratic nonlinear term is bilinear, by similar arguments to (3.3) we have

$$\begin{aligned}\mathcal{J}_2 &= -\int_{\mathbb{T}^2} \mathcal{R}^\perp\theta(t)\cdot\nabla w(t)|w(t)|^{p-1}\mathrm{sgn}w(t)dx\\ &\quad -\int_{\mathbb{T}^2} \mathcal{R}^\perp w(t)\cdot\nabla\theta^*|w(t)|^{p-1}\mathrm{sgn}w(t)dx\\ &= -\int_{\mathbb{T}^2} \mathcal{R}^\perp w(t)\cdot\nabla\theta^*|w(t)|^{p-1}\mathrm{sgn}w(t)dx.\end{aligned}$$

Thanks to the Hölder inequality and Lemma 1, the term $\mathcal{J}_2$ is bounded by

$$\begin{aligned}\mathcal{J}_2 &\le \|\mathcal{R}^\perp w(t)\|_{L^{2p}}\|\nabla\theta^*\|_{L^2}\||w(t)\|_{L^{2p}}^{p-1} \le C(0,2p)\|w(t)\|_{L^{2p}}^p\|\nabla\theta^*\|_{L^2}\\ &= C(0,2p)\||w(t)|^{\frac{p}{2}}\|_{L^4}^2\|\theta^*\|_{H^1} \le K\||w(t)|^{\frac{p}{2}}\|_{H^\alpha}^2\|\theta^*\|_{H^{s+\alpha}},\end{aligned}\tag{4.19}$$

with $K = C(0,2p)M_3M_4$, where M_3 and M_4 are the constants associated with the following Sobolev embedding inequalities:

$$\||w(t)|^{\frac{p}{2}}\|_{L^4}^2 \le M_3\||w(t)|^{\frac{p}{2}}\|_{H^\alpha}^2 \quad\text{and}\quad \|\theta^*\|_{H^1} \le M_4\|\theta^*\|_{H^{s+\alpha}}.$$

By the Hölder inequality, the Young inequality and the assumption on g, we have

$$\begin{aligned}\mathcal{J}_3 &\le \|G(\theta(t-\rho(t))) - G(\theta^*)\|_{L^p}\|w(t)\|_{L^p}^{p-1}\\ &\le L_g\|\theta(t-\rho(t)) - \theta^*\|_{L^p}\|w(t)\|_{L^p}^{p-1}\\ &\le \frac{L_g}{p}\|w(t-\rho(t))\|_{L^p}^p + \frac{\lambda_1^{-\alpha}L_g(p-1)}{p}\||w(t)|^{\frac{p}{2}}\|_{H^\alpha}^2,\end{aligned}\tag{4.20}$$

where we have used the following embedding inequality:

$$\|w(t)\|_{L^p}^p = \||w(t)|^{\frac{p}{2}}\|_{L^2}^2 \le \lambda_1^{-\alpha}\||w(t)|^{\frac{p}{2}}\|_{H^\alpha}^2.\tag{4.21}$$

Note that under the assumptions of Theorem 3, the solution to Eq. (4.1) satisfies $\|\theta^*\|_{H^{s+\alpha}} \le \beta_1$ (see (4.15)), then we conclude from (4.17)-(4.20) that

$$\begin{aligned}\frac{d}{dt}\|w(t)\|_{L^p}^p \le \left(-\frac{4\kappa(p-1)}{p} + pK\beta_1 + \lambda_1^{-\alpha}L_g(p-1)\right)\||w(t)|^{\frac{p}{2}}\|_{H^\alpha}^2\\ + L_g\|w(t-\rho(t))\|_{L^p}^p.\end{aligned}\tag{4.22}$$

Taking $\sigma = r-\rho(r)$, we infer that

$$\begin{aligned}L_g\int_0^t \|w(r-\rho(r))\|_{L^p}^p dr &= L_g\int_{-\rho(0)}^{t-\rho(t)} \|w(\sigma)\|_{L^p}^p\frac{1}{1-\rho'}d\sigma\\ &\le \frac{L_g}{1-\rho^*}\int_{-\rho(0)}^t \|w(\sigma)\|_{L^p}^p d\sigma.\end{aligned}$$

For the case $\rho(0) > 0$, using the embedding inequality (4.21) again and integrating (4.22) over $[0, t]$, we have

$$\begin{aligned}\|w(t)\|_{L^p}^p &\le \|w(0)\|_{L^p}^p \\ &+ \left(-\frac{4\kappa(p-1)}{p} + pK\beta_1 + \lambda_1^{-\alpha} L_g(p-1) + \frac{\lambda_1^{-\alpha} L_g}{1-\rho^*}\right) \int_0^t \||w(\sigma)|^{\frac{p}{2}}\|_{H^\alpha}^2 d\sigma \\ &+ \frac{L_g}{1-\rho^*} \int_{-\rho(0)}^0 \|w(\sigma)\|_{L^p}^p d\sigma, \end{aligned} \tag{4.23}$$

which, together with (4.16), implies that

$$\|w(t)\|_{L^p}^p \le \|w(0)\|_{L^p}^p + \frac{L_g}{1-\rho^*} \int_{-\rho(0)}^0 \|w(\sigma)\|_{L^p}^p d\sigma.$$

For another case $\rho(0) = 0$, the assertion of this theorem follows by neglecting the last term in (4.23), and thus the proof is complete. □

4.3 Local stability: Construction method of Lyapunov functionals

In the previous subsection we have showed the stability of stationary solutions to Eq. (1.1) by the Lyapunov function method. However, sometimes we can construct Lyapunov functionals rather than Lyapunov functions. In this paragraph we will analyze the stability of stationary solutions to Eq. (1.1) by constructing appropriate Lyapunov functionals.

Theorem 5 *Fix $\alpha \in (\frac{1}{2}, 1)$, $s \ge 2-2\alpha$ and $p > \frac{2}{2\alpha-1}$. Consider $f \in H^{s-\alpha} \cap L^p$ and $G \in Lip_0(H^s, H^s) \cap Lip_0(L^p, L^p)$. Let the delay forcing term g be given by $g(t, \theta_t) = G(\theta(t-\rho(t)))$, with $\rho \in C^1([0,\infty), \mathbb{R}^+)$ such that $\rho^* = \sup_{t\ge 0} \rho'(t) < 1$. If the conditions (4.2) and*

$$\frac{4\kappa(p-1)}{p} \ge pK\beta_1 + \frac{2L_g}{\lambda_1^\alpha} \frac{(p-1)^{\frac{p-1}{p}}}{(1-\rho^*)^{\frac{1}{p}}} \tag{4.24}$$

hold true, where β_1 and K are as shown in Theorem 4, then the stationary solution to Eq. (1.1) is stable, i.e., for any $\varphi \in \mathscr{C}(H^s) \cap \mathscr{C}(L^p)$, the solution $\theta(\cdot;\varphi)$ to Eq. (1.1) satisfies that for all $t \ge 0$,

$$\|\theta(t) - \theta^*\|_{L^p}^p \le \gamma \|\varphi - i(\theta^*)\|_{\mathscr{C}(L^p)}^p, \tag{4.25}$$

where $c = L_g\left((1-\rho^*)(p-1)\right)^{\frac{p-1}{p}}$ and $\gamma = 1 + \frac{c\rho(0)}{1-\rho^*}$.

Proof We construct $\Theta : [0,\infty) \times \mathscr{C}(L^p) \to \mathbb{R}^+$ in the form

$$\Theta(t, \varphi) = \|\varphi(0)\|_{L^p}^p + \frac{c}{1-\rho^*} \int_{-\rho(t)}^0 \|\varphi(\sigma)\|_{L^p}^p d\sigma,$$

where $c > 0$ is a constant to be determined later on, such that $\varTheta$ is a Lyapunov functional. Denoting by $\varTheta(t) = \varTheta(t, \theta_t(\cdot;\varphi) - \theta^*)$, where $\theta_t(\cdot;\varphi)$ is the solution to Eq. (1.1) with initial datum φ and θ^* is the stationary solution to Eq. (1.1), we have

$$\varTheta(t) = \|\theta(t) - \theta^*\|_{L^p}^p + \frac{c}{1-\rho^*}\int_{t-\rho(t)}^t \|\theta(\sigma) - \theta^*\|_{L^p}^p d\sigma. \tag{4.26}$$

By using (1.1) we deduce from (4.26) that

$$\begin{aligned}\frac{d}{dt}\varTheta(t) &= p\int_{\mathbb{T}^2}\frac{d}{dt}(\theta(t)-\theta^*)|\theta(t)-\theta^*|^{p-1}\mathrm{sgn}(\theta(t)-\theta^*)dx \\ &\quad + \frac{c}{1-\rho^*}\|\theta(t)-\theta^*\|_{L^p}^p - \frac{c(1-\rho'(t))}{1-\rho^*}\|\theta(t-\rho(t))-\theta^*\|_{L^p}^p \\ &= -p\kappa\int_{\mathbb{T}^2}(-\varDelta)^\alpha(\theta(t)-\theta^*)|\theta(t)-\theta^*|^{p-1}\mathrm{sgn}(\theta(t)-\theta^*)dx \\ &\quad - p\int_{\mathbb{T}^2}(\mathcal{R}^\perp\theta(t)\cdot\nabla\theta(t) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*)|\theta(t)-\theta^*|^{p-1}\mathrm{sgn}(\theta(t)-\theta^*)dx \\ &\quad + p\int_{\mathbb{T}^2}(G(\theta(t-\rho(t))) - G(\theta^*))|\theta(t)-\theta^*|^{p-1}\mathrm{sgn}(\theta(t)-\theta^*)dx \\ &\quad + \frac{c}{1-\rho^*}\|\theta(t)-\theta^*\|_{L^p}^p - \frac{c(1-\rho'(t))}{1-\rho^*}\|\theta(t-\rho(t))-\theta^*\|_{L^p}^p.\end{aligned}$$

Arguing as in (4.17)-(4.20), we conclude that

$$\begin{aligned}\frac{d}{dt}\varTheta(t) &\le \left(-\frac{4\kappa(p-1)}{p} + pK\beta_1\right)\||\theta(t)-\theta^*|^{\frac{p}{2}}\|_{H^\alpha}^2 \\ &\quad + pL_g\|\theta(t-\rho(t))-\theta^*\|_{L^p}\|\theta(t)-\theta^*\|_{L^p}^{p-1} \\ &\quad + \frac{c}{1-\rho^*}\|\theta(t)-\theta^*\|_{L^p}^p - c\|\theta(t-\rho(t))-\theta^*\|_{L^p}^p \\ &\le \left(-\frac{4\kappa(p-1)}{p} + pK\beta_1 + \lambda_1^{-\alpha}\left((p-1)\left(\frac{L_g}{c^{1/p}}\right)^{\frac{p}{p-1}} + \frac{c}{1-\rho^*}\right)\right) \\ &\quad \times \||\theta(t)-\theta^*|^{\frac{p}{2}}\|_{H^\alpha}^2,\end{aligned}$$

where we have used the Young inequality and the following embedding inequality:

$$\|\theta(t)-\theta^*\|_{L^p}^p = \||\theta(t)-\theta^*|^{\frac{p}{2}}\|_{L^2}^2 \le \lambda_1^{-\alpha}\||\theta(t)-\theta^*|^{\frac{p}{2}}\|_{H^\alpha}^2. \tag{4.27}$$

Minimizing the coefficient in brackets in the right-hand side, which is attained for $c = L_g\left((1-\rho^*)(p-1)\right)^{\frac{p-1}{p}}$, we deduce that

$$\frac{d}{dt}\varTheta(t) \le \left(-\frac{4\kappa(p-1)}{p} + pK\beta_1 + \frac{2L_g}{\lambda_1^\alpha}\frac{(p-1)^{\frac{p-1}{p}}}{(1-\rho^*)^{\frac{1}{p}}}\right)\||\theta(t)-\theta^*|^{\frac{p}{2}}\|_{H^\alpha}^2,$$

which, together with condition (4.24), implies that $\Theta(t)$ is non-increasing. On the other hand, it follows from (4.26) that

$$\Theta(t) \geq \|\theta(t)-\theta^*\|_{L^p}^p \quad \text{and} \quad \Theta(0) \leq \gamma\|\varphi - i(\theta^*)\|_{\mathscr{C}(L^p)}^p,$$

with $\gamma = 1 + \frac{c\rho(0)}{1-\rho^*}$. Then

$$\|\theta(t)-\theta^*\|_{L^p}^p \leq \Theta(t) \leq \Theta(0) \leq \gamma\|\varphi - i(\theta^*)\|_{\mathscr{C}(L^p)}^p.$$

Therefore, we obtain the desired stability directly, i.e.,

$$\|\theta(t)-\theta^*\|_{L^p}^p \leq \gamma\|\varphi - i(\theta^*)\|_{\mathscr{C}(L^p)}^p.$$

The proof is complete. □

Remark 2 The asymptotic stability result can also be extended to cover the case in which we do not need the conditions $s > 2-2\alpha$ and $\frac{2}{p} \geq 1-s$. In fact, if the conditions (4.2) and

$$\frac{4\kappa(p-1)}{p} > pK\beta_1 + \frac{2L_g}{\lambda_1^\alpha}\frac{(p-1)^{\frac{p-1}{p}}}{(1-\rho^*)^{\frac{1}{p}}}$$

hold true, then the stationary solution to Eq. (1.1) is asymptotically stable in the sense that for any $\varphi \in \mathscr{C}(H^s)\cap\mathscr{C}(L^p)$, the solution $\theta(\cdot;\varphi)$ to Eq. (1.1) satisfies (4.25) and

$$\lim_{n\to\infty}\|\theta(t+n)-\theta^*\|_{L^p}^p = 0 \quad \text{for a.e. } t \in \mathbb{R}^+. \tag{4.28}$$

Here we prove (4.28) only in the case $t \in [0,1]$, since the other case can be obtained similarly. By the embedding (4.27), there exists a constant $\widetilde{\gamma} > 0$ such that

$$\frac{d}{dt}\Theta(t) \leq -\widetilde{\gamma}\||\theta(t)-\theta^*|^{\frac{p}{2}}\|_{H^\alpha}^2 \leq -\lambda_1^\alpha\widetilde{\gamma}\|\theta(t)-\theta^*\|_{L^p}^p.$$

Integrating the above inequality from 0 to ∞, it follows from (4.26) that

$$\int_0^\infty \|\theta(t)-\theta^*\|_{L^p}^p dt \leq \frac{\gamma}{\lambda_1^\alpha\widetilde{\gamma}}\|\varphi - i(\theta^*)\|_{\mathscr{C}(L^p)}^p. \tag{4.29}$$

Observe that

$$\begin{aligned}\int_0^1 \sum_{n=1}^\infty \|\theta(t+n)-\theta^*\|_{L^p}^p dt &= \sum_{n=1}^\infty \int_0^1 \|\theta(t+n)-\theta^*\|_{L^p}^p dt \\ &= \sum_{n=1}^\infty \int_n^{n+1} \|\theta(t)-\theta^*\|_{L^p}^p dt = \int_1^\infty \|\theta(t)-\theta^*\|_{L^p}^p dt < \infty.\end{aligned}$$

This implies that for a.e. $t \in [0,1]$, $\sum_{n=1}^\infty \|\theta(t+n)-\theta^*\|_{L^p}^p$ is convergent, and thus (4.28) follows immediately.

4.4 Local stability: Razumikhin technique

We would like to mention that we need the differentiability of the variable delay in the previous two cases. However, it is possible to relax this restriction and prove a stability result for more general delay terms by using a different method, namely, the Razumikhin method, which is also widely used in dealing with the stability properties of functional differential equations. But this approach requires some kind of continuity concerning both the operators in the model and the solutions.

Theorem 6 *Fix* $\alpha \in (\frac{1}{2}, 1)$*,* $s \geq 2 - 2\alpha$ *and* $p > \frac{2}{2\alpha - 1}$*. Consider* $f \in H^s \cap L^p$ *and* $g \in Lip_0(\mathscr{C}(H^s), H^s) \cap Lip_0(\mathscr{C}(L^p), L^p)$ *satisfying the condition (H). Assume that the mapping* $t \in [0, \infty) \to g(t, \xi) \in H^s \cap L^p$ *is continuous for all* $\xi \in \mathscr{C}(H^s) \cap \mathscr{C}(L^p)$*. Moreover, suppose that there is a stationary solution* θ^* *to Eq.* (1.1) *such that*

$$\begin{aligned} &- \kappa \langle \Lambda^s (-\Delta)^\alpha (\psi(0) - \theta^*), \Lambda^s(\psi(0) - \theta^*) \rangle \\ &\quad - \langle \Lambda^s (\mathcal{R}^\perp \psi(0) \cdot \nabla \psi(0) - \mathcal{R}^\perp \theta^* \cdot \nabla \theta^*), \Lambda^s(\psi(0) - \theta^*) \rangle \\ &\quad + \langle \Lambda^s (g(t, \psi) - g(t, \theta^*)), \Lambda^s(\psi(0) - \theta^*) \rangle < 0, \quad t \geq 0, \end{aligned} \tag{4.30}$$

whenever $\psi \in \mathscr{C}(H^s)$*, with* $\psi(0) \in H^{s+\alpha}$*, satisfies*

$$\|\psi - i(\theta^*)\|_{\mathscr{C}(H^s)} = \|\psi(0) - \theta^*\|_{H^s}. \tag{4.31}$$

Then, for any $\varphi \in \mathscr{C}(H^s)$ *with* $\varphi(0) \in H^{s+\alpha}$*, the solution* $\theta(\cdot; \varphi)$ *to Eq.* (1.1) *satisfies*

$$\|\theta(t; \varphi) - \theta^*\|_{H^s} \leq \|\varphi - i(\theta^*)\|_{\mathscr{C}(H^s)}, \quad \forall t \geq 0. \tag{4.32}$$

Proof The case $\varphi = \theta^*$ is trivial. Thus we assume that $\varphi \neq \theta^*$. We argue by contradiction. Suppose that there exists an initial datum $\varphi \in \mathscr{C}(H^s)$ with $\varphi(0) \in H^{s+\alpha}$ and $\varphi \neq \theta^*$, such that (4.32) is not true. Then, there exists $t > 0$ such that $\|\theta(t; \varphi) - \theta^*\|_{H^s} > \|\varphi - i(\theta^*)\|_{\mathscr{C}(H^s)}$. Denoting

$$\sigma_0 = \inf\{t > 0 : \|\theta(t; \varphi) - \theta^*\|_{H^s} > \|\varphi - i(\theta^*)\|_{\mathscr{C}(H^s)}\},$$

we obtain for all $0 \leq t \leq \sigma_0$ that

$$\|\theta(t; \varphi) - \theta^*\|_{H^s} \leq \|\theta(\sigma_0; \varphi) - \theta^*\|_{H^s} = \|\varphi - i(\theta^*)\|_{\mathscr{C}(H^s)}, \tag{4.33}$$

and there is a sequence $\{t_k\}_{k \geq 1} \subset (\sigma_0, \infty)$ such that $t_k \downarrow \sigma_0$ as $k \to \infty$ and

$$\|\theta(t_k; \varphi) - \theta^*\|_{H^s} > \|\theta(\sigma_0; \varphi) - \theta^*\|_{H^s}. \tag{4.34}$$

On the other hand, by (4.33) we find that

$$\sup_{\tau \leq 0} \|\theta(\sigma_0 + \tau; \varphi) - \theta^*\|_{H^s} = \|\theta_{\sigma_0} - \theta^*\|_{\mathscr{C}(H^s)} = \|\theta(\sigma_0; \varphi) - \theta^*\|_{H^s},$$

which, in view of assumptions (4.30)-(4.31) with $\psi = \theta_{\sigma_0}$, immediately implies that

$$
\begin{aligned}
&-\kappa\langle \Lambda^s(-\Delta)^\alpha(\theta(\sigma_0;\varphi)-\theta^*), \Lambda^s(\theta(\sigma_0;\varphi)-\theta^*)\rangle \\
&\quad - \langle \Lambda^s(\mathcal{R}^\perp\theta(\sigma_0;\varphi)\cdot\nabla\theta(\sigma_0;\varphi) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*), \Lambda^s(\theta(\sigma_0;\varphi)-\theta^*)\rangle \\
&\quad + \langle \Lambda^s(g(t,\theta_{\sigma_0}(\cdot;\varphi)) - g(t,\theta^*)), \Lambda^s(\theta(\sigma_0;\varphi)-\theta^*)\rangle < 0.
\end{aligned}
$$

By the continuity of the operators, there exists $\varepsilon > 0$ such that for all $t \in [\sigma_0, \sigma_0+\varepsilon]$

$$
\begin{aligned}
&-\kappa\langle \Lambda^s(-\Delta)^\alpha(\theta(t;\varphi)-\theta^*), \Lambda^s(\theta(t;\varphi)-\theta^*)\rangle \\
&\quad - \langle \Lambda^s(\mathcal{R}^\perp\theta(t;\varphi)\cdot\nabla\theta(t;\varphi) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*), \Lambda^s(\theta(t;\varphi)-\theta^*)\rangle \\
&\quad + \langle \Lambda^s(g(t,\theta_t(\cdot;\varphi)) - g(t,\theta^*)), \Lambda^s(\theta(t;\varphi)-\theta^*)\rangle < 0.
\end{aligned}
$$

Setting $w(t) = \theta(t;\varphi) - \theta^*$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|_{H^s}^2 &= -\kappa\langle \Lambda^s(-\Delta)^\alpha w(t), \Lambda^s w(t)\rangle \\
&\quad - \langle \Lambda^s(\mathcal{R}^\perp\theta(t;\varphi)\cdot\nabla\theta(t;\varphi) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*), \Lambda^s w(t)\rangle \\
&\quad + \langle \Lambda^s(g(t,\theta_t) - g(t,\theta^*)), \Lambda^s w(t)\rangle < 0,
\end{aligned}
$$

for all $t \in [\sigma_0, \sigma_0+\varepsilon]$. Therefore, taking $t_{k_\varepsilon} \subset (\sigma_0, \sigma_0+\varepsilon]$ and integrating the above inequality from σ_0 to t_{k_ε}, we infer that $\|w(t_{k_\varepsilon};\varphi)\|_{H^s} < \|w(\sigma_0;\varphi)\|_{H^s}$, which contradicts (4.34). Thus (4.32) holds true. □

Remark 3 The condition (4.30) is satisfied when the diffusivity coefficient κ is sufficiently large. This condition can be read as: a strong dissipativity ensures the stability of the stationary solution to Eq. (1.1). Furthermore, a sufficient condition which implies (4.30) but easier to check in applications is given in the next result.

Corollary 1 *Fix $\alpha \in (\frac{1}{2}, 1)$, $s \geq 2-2\alpha$ and $p > \frac{2}{2\alpha-1}$. Consider $f \in H^s \cap L^p$ and $g \in Lip_0(\mathscr{C}(H^s), H^s) \cap Lip_0(\mathscr{C}(L^p), L^p)$ satisfying the condition (H). In addition to the condition (4.2), assume that*

$$
\kappa > \lambda_1^{-\alpha} L_g + 2\Upsilon\|\varphi(0)\|_{H^{s+\alpha}} - \sqrt{\Delta}, \tag{4.35}
$$

for some $\varphi \in \mathscr{C}(H^s)$ with $\varphi(0) \in H^{s+\alpha}$. Then the solution $\theta(t;\varphi)$ to Eq. (1.1) satisfies

$$
\|\theta(t;\varphi) - \theta^*\|_{H^s} \leq \|\varphi - i(\theta^*)\|_{\mathscr{C}(H^s)}, \quad \forall t \geq 0,
$$

where θ^* is the stationary solution to Eq. (1.1).

Proof The existence and uniqueness of the stationary solution to Eq. (1.1) are guaranteed by Theorem 3. In the following, we check that condition (4.35) implies (4.30)-(4.31) in Theorem 6. Suppose that $\varphi \in \mathscr{C}(H^s)$, with $\varphi(0) \in$

$H^{s+\alpha}$, is close to some stationary solution θ^* (but not equal, otherwise it is trivial) and satisfies

$$\|\varphi - i(\theta^*)\|^2_{\mathscr{C}(H^s)} = \|\varphi(0) - \theta^*\|^2_{H^s}.$$

Now we verify that (4.30) holds. Indeed,

$$\begin{aligned}
&-\kappa\langle \Lambda^s(-\Delta)^\alpha(\varphi(0)-\theta^*), \Lambda^s(\varphi(0)-\theta^*)\rangle \\
&\quad - \langle \Lambda^s(\mathcal{R}^\perp\varphi(0)\cdot\nabla\varphi(0) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*), \Lambda^s(\varphi(0)-\theta^*)\rangle \\
&\quad + \langle \Lambda^s(g(t,\varphi)-g(t,\theta^*)), \Lambda^s(\varphi(0)-\theta^*)\rangle \\
&\le -\kappa\|\varphi(0)-\theta^*\|^2_{H^{s+\alpha}} \\
&\quad - \langle \Lambda^s(\mathcal{R}^\perp\varphi(0)\cdot\nabla\varphi(0) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*), \Lambda^s(\varphi(0)-\theta^*)\rangle \\
&\quad + L_g\|\varphi-\theta^*\|_{\mathscr{C}(H^s)}\|\varphi(0)-\theta^*\|_{H^s} \\
&\le (-\kappa + \lambda_1^{-\alpha}L_g)\|\varphi(0)-\theta^*\|^2_{H^{s+\alpha}} \\
&\quad + |\langle \Lambda^s(\mathcal{R}^\perp\varphi(0)\cdot\nabla\varphi(0) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*), \Lambda^s(\varphi(0)-\theta^*)\rangle|.
\end{aligned} \tag{4.36}$$

Using the bilinearity of the nonlinear term and arguing as in (4.5), we have

$$\begin{aligned}
&|\langle \Lambda^s(\mathcal{R}^\perp\varphi(0)\cdot\nabla\varphi(0) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*), \Lambda^s(\varphi(0)-\theta^*)\rangle| \\
&\quad \le |\langle \Lambda^s(\mathcal{R}^\perp(\varphi(0)-\theta^*)\cdot\nabla\varphi(0), \Lambda^s(\varphi(0)-\theta^*)\rangle| \\
&\qquad + |\langle \Lambda^s(\mathcal{R}^\perp\theta^*\cdot\nabla(\varphi(0)-\theta^*), \Lambda^s(\varphi(0)-\theta^*)\rangle| \\
&\quad \le \|\Lambda^{s-\alpha}(\mathcal{R}^\perp(\varphi(0)-\theta^*)\cdot\nabla\varphi(0))\|_{L^2}\|\Lambda^{s+\alpha}(\varphi(0)-\theta^*)\|_{L^2} \\
&\qquad + \|\Lambda^{s-\alpha}(\mathcal{R}^\perp\theta^*\cdot\nabla(\varphi(0)-\theta^*))\|_{L^2}\|\Lambda^{s+\alpha}(\varphi(0)-\theta^*)\|_{L^2} \\
&\quad \le \Upsilon\|\varphi(0)\|_{H^{s+\alpha}}\|\varphi(0)-\theta^*\|^2_{H^{s+\alpha}} + \Upsilon\|\theta^*\|_{H^{s+\alpha}}\|\varphi(0)-\theta^*\|^2_{H^{s+\alpha}}.
\end{aligned} \tag{4.37}$$

Thanks to the boundedness of the stationary solution θ^* (see (4.15)), it follows from (4.36) and (4.37) that

$$\begin{aligned}
&-\kappa\langle \Lambda^s(-\Delta)^\alpha(\varphi(0)-\theta^*), \Lambda^s(\varphi(0)-\theta^*)\rangle \\
&\quad - \langle \Lambda^s(\mathcal{R}^\perp\varphi(0)\cdot\nabla\varphi(0) - \mathcal{R}^\perp\theta^*\cdot\nabla\theta^*), \Lambda^s(\varphi(0)-\theta^*)\rangle \\
&\quad + \langle \Lambda^s(g(t,\varphi)-g(t,\theta^*)), \Lambda^s(\varphi(0)-\theta^*)\rangle \\
&\le \left(-\kappa + \lambda_1^{-\alpha}L_g + \Upsilon\|\varphi(0)\|_{H^{s+\alpha}} + \frac{1}{2}(\kappa - \lambda_1^{-\alpha}L_g - \sqrt{\Delta})\right)\|\varphi(0)-\theta^*\|^2_{H^{s+\alpha}},
\end{aligned}$$

which is negative by (4.35). Thus, (4.31) holds and therefore (4.32) as well. □

4.5 Polynomial stability for a special case

As mentioned in [24], it may not be possible to prove exponential stability of stationary solutions for the evolution equations with unbounded variable delays. However, we can prove the polynomial stability of stationary solutions in a particular case of unbounded variable delay.

Theorem 7 *Fix* $p > \frac{2}{2\alpha-1}$. *Consider Eq.* (1.1) *with* $f = 0$ *and* $g(t, \theta_t) := L_g\theta(\lambda t)$, *where* $0 < \lambda < 1$, $L_g \in \mathbb{R}$ *and* $\kappa > \frac{p^2}{4(p-1)}\lambda_1^{-\alpha}|L_g|$. *Then the origin is the unique stationary solution and any solution* θ *converges to zero polynomially, namely, there exists a constant* $\mu < 0$ *such that*

$$\|\theta(t)\|_{L^p}^p < C\|\theta(0)\|_{L^p}^p(1+t)^\mu, \quad t \geq 0, \tag{4.38}$$

where μ *satisfies* $|L_g|(p-1) - \frac{4\lambda_1^\alpha\kappa(p-1)}{p} + |L_g|\lambda^\mu = 0$.

Proof The existence and uniqueness of origin as stationary solution are guaranteed by Theorem 3. Analogous to the arguments of (3.2), we derive

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\theta(t)\|_{L^p}^p + \kappa\int_{\mathbb{T}^2}(-\Delta)^\alpha\theta(t)|\theta(t)|^{p-1}\mathrm{sgn}\theta(t)dx\\ &\quad= L_g\int_{\mathbb{T}^2}\theta(\lambda t)|\theta(t)|^{p-1}\mathrm{sgn}\theta(t)dx.\end{aligned}$$

Using Lemma 3 and the Young inequality, we deduce that

$$\frac{d}{dt}\|\theta(t)\|_{L^p}^p + \frac{4\lambda_1^\alpha\kappa(p-1)}{p}\|\theta(t)\|_{L^p}^p \leq |L_g|(p-1)\|\theta(t)\|_{L^p}^p + |L_g|\|\theta(\lambda t)\|_{L^p}^p.$$

where we have used the following embedding inequality:

$$\frac{4\lambda_1^\alpha\kappa(p-1)}{p}\|\theta(t)\|_{L^p}^p = \frac{4\lambda_1^\alpha\kappa(p-1)}{p}\||\theta(t)|^{\frac{p}{2}}\|_{L^2}^2 \leq \frac{4\kappa(p-1)}{p}\||\theta(t)|^{\frac{p}{2}}\|_{H^\alpha}^2.$$

Setting $\omega(t) := \|\theta(t)\|_{L^p}^p$, we have

$$\omega'(t) \leq \left(-\frac{4\lambda_1^\alpha\kappa(p-1)}{p} + |L_g|(p-1)\right)\omega(t) + |L_g|\omega(\lambda t).$$

Note that the first term on the right-hand side of the above inequality is negative, then we deduce from Lemmas 3.5 and 3.6 in [1] that

$$\omega(t) \leq C\omega(0)(1+t)^\mu,$$

where $|L_g|(p-1) - \frac{4\lambda_1^\alpha\kappa(p-1)}{p} + |L_g|\lambda^\mu = 0$. Since $\lambda^\mu = \frac{4\lambda_1^\alpha\kappa(p-1)}{p|L_g|} - (p-1) > 1$, then $\mu < 0$, and consequently the estimate (4.38) holds true. □

Remark 4 It is worth mentioning that Theorem 7 improves all the stability results established previously for this special delay term, since any solution to Eq. (1.1) converges polynomially to zero as long as $\kappa > \frac{p^2}{4(p-1)}\lambda_1^{-\alpha}|L_g|$.

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