Properties of hadrons in the quark model

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Abstract

This work focuses on developing a model that predicts the existence of some of the baryons measured experimentally, as well as explaining properties such as spin, parity and mass. This is done in the context of the quark model in quantum mechanics. The mathematical background is provided by the theory of groups and representations, presented briefly in the first part of the project. After that, the states describing the different degrees of freedom for the baryons are obtained with their corresponding symmetry properties. The symmetry of the states is important in order to construct the complete wavefunction for the system since baryons are fermions, thus they follow the Pauli exclusion principle. Finally, baryons are identified following experimental data and the results derived throughout the work.

Resumen

Este trabajo se centra en desarrollar un modelo predictivo de la existencia de algunos bariones medidos experimentalmente, así como explicar propiedades de estos como el espín, la paridad y la masa. Esto se hace en el contexto del modelo de quarks en mecánica cuántica. La teoría de grupos y representaciones provee el marco matemático y se presenta brevemente en la primera parte del projecto. Después, los estados que describen los diferentes grados de libertad para bariones se obtienen con sus correspondientes propiedades de simetría. La simetría de los estados es importante a la hora de construir la función de onda completa debido a que los bariones son fermiones, por lo que están sujetos al principio de exclusión de Pauli. Finalmente, se identifican bariones siguiendo los datos experimentales y los resultados obtenidos a lo largo del trabajo.

Contents

Chapter 1

Introduction and methodology

Similarly to how Rutherford discovered that the atom is composed of smaller parts (an electron shell and the nucleus in the centre) with the gold foil experiment, physicists in the 1960s, utilizing the first major accelerators, discovered that electron beams were scattered when fired at protons, revealing that the proton had scattering centres within. This was not a surprise, since the dozens of new particles discovered at the time suggested that there existed more elementary particles that composed them. At first, these new elementary particles were given the name of “partons”. Later, with the success and the experimental verification of the quark model and quantum chromodynamics, partons were matched to quarks and gluons (the particle that carries the strong force).

In this context, it was found that many of these new discovered particles (mesons and baryons) could be arranged into geometrical patterns called multiplets. Particles in a specific multiplet had the same spin and parity, and approximately the same mass. This suggested that there was some degeneracy behind the fact that different particles described by different quantum states had the same properties. Degeneracy is something that generally arises whenever there is a certain symmetry of the system. For example, in the non-relativistic central potential of the hydrogen atom, energy levels for the electron do not depend on the orbital quantum number because the hamiltonian has spherical symmetry, meaning that the system is invariant under rotations of any angle and around any axis. Hence the importance of group theory (the branch of mathematics that studies symmetry) to understand the nooks of particle physics and the properties of the most fundamental constituents of nature.

This project focuses on the task of deducing how the mentioned patterns arise and what particles (specifically baryons) fit into them. In order to achieve this, we briefly introduce the basics of group theory and the properties of groups, and provide some references for anyone interested in a deeper understanding of the mathematics behind some of the results. This is done in chapter two. In the third chapter, we start by addressing the quark degrees of freedom inside the baryon and constructing the baryon wavefunction following the results derived in chapter two and the Pauli exclusion principle. Multiplets will naturally appear during this process, and their number and corresponding properties will be evaluated. Finally, we will identify the baryons that appear in these multiplets in order to ascertain that the obtained multiplets predict what baryons are found in nature along with some of their properties.

Chapter 2

General concepts of group theory

We will first give a general idea of what groups are as well as present some concepts and properties of groups that are going to be useful later on in our study about hadrons in the quark model. The focus will be on the permutation groups S_n. Then, we will introduce group representations and Clebsch-Gordan series which will be of use to evaluate the symmetry properties of the quantum states.

2.1 Definition of a group

Let us first define what groups are in terms of certain objects and a composition law, referred to as "multiplication",[1] whereby two members of a group combine to give a third. This abstract definition will become more apparent after discussing the permutation group.

A *group* G is a set of elements $\{g_1, g_2, ...\}$ with a law of composition (multiplication) which assigns to each ordered pair $g_1, g_2 \in G$ another element, written g_1g_2, of G [1]. The law of composition satisfies the following conditions:

- Closure.

  For all $g_1, g_2 \in G$, the product g_1g_2 is also a member of G.

- Associative law.

  For all $g_1, g_2, g_3 \in G$

$$g_1(g_2g_3) = (g_1g_2)g_3. \tag{2.1}$$

[1]Keep in mind that, in general, this has nothing to do with the conventional multiplication.

- Unit element.

  G contains an element called the identity element and denoted by e, such that for all $g \in G$

$$ge = eg = g. \tag{2.2}$$

- Existence of inverse.

  For all $g \in G$ there is an element denoted by g^{-1}, such that

$$gg^{-1} = g^{-1}g = e. \tag{2.3}$$

The order of a group is the number of elements in the set G. Note that the order of the elements of the pair g_1, g_2 is important because, in general, $g_1g_2 \neq g_2g_1$. If $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G$ the group is called *Abelian*.

2.2 The permutation group S_n

The permutation group is going to be our main tool in this work. It consists of the permutations of n objects, or the labels of those objects, and is of order $n!$. In this case, multiplication of two permutations is defined as successive application.

We will use the so-called *cyclic notation* [1] to write permutations. To understand this notation in practice, let us use the group S_2 as an example. This group contains two objects, so it is of order $2! = 2$. It consists of the identity permutation, written as ( ) or e and the *transposition* $1 \leftrightarrow 2$, or (1 2) in cycle notation. This is called a 2-cycle. The "1" represents a label designing the first object, and the "2" represents the second. These two numbers in parenthesis are a way of writing such transposition, or in other words, the action of permuting the first object with the second. In higher order groups, more complex cycles can be found. For example, in the group S_3 there are two 3-cycles consisting of permuting three objects. One of these 3-cycles is (1 2 3). The way of reading this is, from left to right, $1 \to 2$, $2 \to 3$ and $3 \to 1$. So the first object—or label—goes to the position of the second, the second to the third and the third to the first.

To multiply two permutations, we follow each label from right to left:

$$(1\,2)(1\,2\,3) = (2\,3). \tag{2.4}$$

In this example, $1 \to 2 \to 1$, $2 \to 3 \to 3$ and $3 \to 1 \to 2$. Multiplication of two permutations is not commutative. It is important to notice that 1-cycles are always omitted, so we do not write $(1)(2\,3)$.

Permutations can also be interpreted as operators. For example, the 2-cycle $(1\,2)$ can be expressed as an operator P_{12} that acts on a function depending on two generic variables, permuting them. We will write these operators as P_i where the sub-index i indicates the labels that are being permuted as in the cyclic notation.

One important concept is the multiplication table of a group. The multiplication table is a compact way of specifying a finite group by writing all of the possible products between two elements of the group. The multiplication table for S_2 is:

	e	$(1\,2)$
e	e	$(1\,2)$
$(1\,2)$	$(1\,2)$	e

Table 2.1: Multiplication table for S_2.

Here the convention is that the elements on the first column are on the left when doing the product, and elements on the first row are on the right. For S_3 the multiplication table is bigger since there are $3! = 6$ elements: $S_3 = \{e; (1\,2), (1\,3), (2\,3); (1\,2\,3), (1\,3\,2)\}$.

	e	$(1\,2)$	$(1\,3)$	$(2\,3)$	$(1\,2\,3)$	$(1\,3\,2)$
e	e	$(1\,2)$	$(1\,3)$	$(2\,3)$	$(1\,2\,3)$	$(1\,3\,2)$
$(1\,2)$	$(1\,2)$	e	$(1\,3\,2)$	$(1\,2\,3)$	$(2\,3)$	$(1\,3)$
$(1\,3)$	$(1\,3)$	$(1\,2\,3)$	e	$(1\,3\,2)$	$(1\,2)$	$(2\,3)$
$(2\,3)$	$(2\,3)$	$(1\,3\,2)$	$(1\,2\,3)$	e	$(1\,3)$	$(1\,2)$
$(1\,2\,3)$	$(1\,2\,3)$	$(1\,3)$	$(2\,3)$	$(1\,2)$	$(1\,3\,2)$	e
$(1\,3\,2)$	$(1\,3\,2)$	$(2\,3)$	$(1\,2)$	$(1\,3)$	e	$(1\,2\,3)$

Table 2.2: Multiplication table for S_3.

The multiplication table completely defines the group. Notice that each element appears once and only once in each column and each row.

2.3 Group representations

In this section we are going to define the concept of representations and give some results without proof that are going to be useful later on.

A *matrix representation* of dimension n of the abstract group G is defined as a set of square, non-singular matrices that satisfy the multiplication table of the group when they are multiplied by the conventional rules of matrix multiplication. In other words, it is a mapping under which each element of the group is associated with a matrix, $g \mapsto D(g)$, preserving the group structure:

$$D(g_1 g_2) = D(g_1)D(g_2). \tag{2.5}$$

The mapping is necessarily into the set of non-singular matrices, since each matrix must be invertible: $D(g^{-1}) = (D(g))^{-1}$.

A representation of dimension n is a representation whose matrices are $n \times n$. There are an infinite number of representations for a given group. A valid representation for the group S_2 is, for example:

$$D^{(S)}(e) = D^{(S)}(12) = 1 \tag{2.6}$$

which is a 1×1 representation called the trivial or *totally symmetric representation* (hence the superscript) and obviously satisfies the multiplication table.

Two $n \times n$ representations are equivalent if their matrices are related by the same similarity transformation M:

$$D^{(\mu)}(g) = M^{-1} D^{(\nu)}(g) M \tag{2.7}$$

$\forall g \in G$, M independent of g. Here the superscripts μ and ν indicate any two representations of the group G.

The idea of equivalent representations is related to the fact that the matrices of the two representations describe the same linear transformation in an n-dimensional vector space, but referred to different bases [1]. In linear algebra, $D^{(\nu)}$ would be a certain linear transformation in a vector space and M would be a change of basis matrix that transforms $D^{(\nu)}$ into $D^{(\mu)}$. Thus, from the perspective of representation theory, equivalent representations are regarded as being essentially the same.

A representation of dimension $n + m$ is *reducible* if $D(g)$ takes the form

$$D(g) = \begin{pmatrix} A(g) & C(g) \\ O & B(g) \end{pmatrix} \tag{2.8}$$

$\forall g \in G$, where A, C and B are submatrices of dimension $m \times m$, $m \times n$ and $n \times n$

respectively, and O denotes a null matrix of dimension $n \times m$.

For finite groups it can be shown [1] that under equivalence C can be taken to be a null matrix.[2] The representation is then said to be *completely reducible* or *decomposable* and the matrix $D(g)$ is said to be in block form since it consists of two smaller square matrices along the diagonal and the rest are zeros. We write then[3]

$$D(g) = A(g) \oplus B(g). \tag{2.9}$$

It may be that the representations A and B are themselves decomposable, in which case it is natural to continue the process which will terminate when we reach the level of irreducible representations, i.e. representations that cannot be further reduced. While there is no limit to the number and dimensions of reducible representations, it turns out that the irreducible representations (irreps from now on) can be classified and enumerated. Irreps are of central importance in the application of group theory to physical problems and are going to be the main tool in order to predict the existence of baryons (three-quark systems), since the states describing them are going to have symmetry properties related to the permutation group S_3.

2.4 Direct product of representations. Clebsch-Gordan series

Suppose we have two matrix representations for a certain group G, one of dimension n and the other of dimension m. The *direct product* of two matrices $D^{(\mu)}(g)$ and $D^{(\nu)}(g)$ is a matrix $D^{(\mu)}(g) \otimes D^{(\nu)}(g)$ whose elements are all the possible products of an element of $D^{(\mu)}(g)$ with an element of $D^{(\nu)}(g)$. Thus, the dimension of the new matrix is $n + m$.

As an example, consider two different representations of dimension $n = m = 2$. Let $D^{(\mu)}(g)$ and $D^{(\nu)}(g)$ be two matrices belonging to the two representations respectively and associated with the element g of the group:

$$D^{(\mu)}(g) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad D^{(\nu)}(g) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \tag{2.10}$$

[2]This is Maschke's theorem.

[3]The symbol + is used in some texts.

The direct product of these two matrices is

$$D^{(\mu)}(g) \otimes D^{(\nu)}(g) = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \tag{2.11}$$

which is a matrix of dimension $2 + 2 = 4$.

Calculating all the possible direct products of two matrices for each element of the group yields a third representation of the same group G. The new representation is reducible in general even if the two original representations were irreducible. The decomposition of the new representation into its irreducible components is called the *Clebsch-Gordan series*:

$$D^{(\mu)} \otimes D^{(\nu)} = \sum_{\oplus} a_\sigma D^{(\sigma)} \tag{2.12}$$

where the coefficients a_σ give the number of times that the irrep $D^{(\sigma)}$ is repeated in the decomposition.

Clebsch-Gordan series will be important in order to obtain the symmetry of the baryon wavefunctions. The states describing the three-quark system will be the product of states associated with different degrees of freedom, and each one of these states will have some symmetry under interchange of their labels related to some irrep. The product state will have a symmetry given by the symmetry properties of the direct product of representations.

2.5 Application to the group S_2. Young diagrams

Let us now deduce the irreps of S_2. To do this we are going to proceed by applying permutation operators to a set of states called *basis states* [2]. These states represent eigenstates for some unspecified quantum operator. The states can represent, for example, the spin of some system of particles. The "basis" term refers to the fact that the states in the set are orthonormal and can generate representations if we act with the operators of the group. Since S_2 is of order two, the states in this case can be written with two labels such as, say, a and b.

Let $\{|ab\rangle, |ba\rangle\}$ be a two-dimensional set of basis states. This set will generate a two-dimensional representation. We will denote as P_e and P_{12} the operators corresponding to the identity element and the transposition $1 \leftrightarrow 2$, respectively. It is clear that:

$$P_e |ab\rangle = |ab\rangle, P_e |ba\rangle = |ba\rangle. \tag{2.13}$$

Thus:

$$P_e(|ab\rangle, |ba\rangle) = (|ab\rangle, |ba\rangle) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.14}$$

Similarly, operating with P_{12} on the basis states produces:

$$P_{12} |ab\rangle = |ba\rangle, P_{12} |ba\rangle = |ab\rangle \tag{2.15}$$

so we can write:

$$P_{12}(|ab\rangle, |ba\rangle) = (|ab\rangle, |ba\rangle) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.16}$$

We have now a two-dimensional representation of the group S_2 that we will denote as $D^{(2)}$. In order for this to be a valid representation it must satisfy the multiplication table of the group, as one can easily check.

To obtain the irreps of S_2, we have to find a matrix M such that the similarity transformation $MD^{(2)}(g)M^{-1}$ results in a matrix consisting of blocks along the main diagonal for every representation matrix $D^{(2)}(g)$ belonging to $D^{(2)}$. Let

$$M = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.17}$$

Then:

$$M^{-1}D^{(2)}(12)M = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.18}$$

and trivially:

$$M^{-1}D^{(2)}(e)M = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.19}$$

The matrix M diagonalizes in block form the two representation matrices of $D^{(2)}$.

Now all we have left is to recognize the irreps of S_2. Looking at the entries of the two matrices $D^{(2)}(e)$ and $D^{(2)}(12)$, there are three blocks with a 1 and one block with a -1. The 1s in $D^{(2)}(e)$ form the completely symmetric irrep given in (2.6):

$$D^{(S)}(e) = D^{(S)}(12) = 1 \tag{2.20}$$

and the other 1 with the -1 in $D^{(2)}(12)$ form the *antisymmetric representation* $D^{(A)}$:

$$D^{(A)}(e) = 1, \;\; D^{(A)}(12) = -1. \tag{2.21}$$

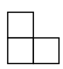
Hence the representation $D^{(2)}$ can be reduced into the two one-dimensional representations $D^{(S)}$ and $D^{(A)}$:

$$D^{(2)} = D^{(S)} \oplus D^{(A)}. \tag{2.22}$$

Note that there has not been any mention to the dimension and number of irreps; the procedure automatically gives this information. Since S_2 is of order two, we can only form two matrices for any representation of the group, and irreps are no exception. Generating a two-dimensional representation is particularly useful in this case since one can easily see by inspection that there are two one-dimensional irreps.

The same results could have been achieved utilizing the so-called *Young diagrams* [1, 2]. Young diagrams are a handy tool (especially when analyzing more complicated groups) to not only deduce the number and dimension of irreps for a given permutation group, but also to generate these irreps.

A Young diagram for a permutation group S_n consists of an arrangement of n cells in rows and columns following the rule that each row must contain no more boxes than the row above, so a figure such as



(2.23)

is not a legitimate Young diagram [1]. Each Young diagram represents an irrep of the group, so the number of Young diagrams that can be built equals the number of irreps. To obtain the dimensions of the irreps we build *Young tableaux*. For a specific Young diagram, a Young tableau can be obtained numbering each cell with an integer starting from 1 up to n. If the numbers increase from up to down and left to right, the tableau is called a *standard tableau*. More than one standard tableau can exist for a given Young diagram, and the number of standard tableaux gives the dimension of the irrep represented by that Young diagram.

To refer to a Young diagram we write $[\lambda_1\lambda_2...\lambda_h]$ where λ_i is the number of columns in the ith row and h is the total number of rows. For S_2 there are only two Young diagrams:

$$[2]\ \square\square \qquad [11] \equiv [1^2]\ \begin{array}{c}\square\\ \square\end{array} \tag{2.24}$$

and it is easy to see that there is only one Young tableau for each of these two Young diagrams. This already tells us that S_2 has two one-dimensional irreps. To generate the irreps, we would first need to define what Young operators are.

For any standard tableau let P be a permutation which interchanges numbers in one row. This is called a *horizontal permutation*. Similarly, a *vertical permutation* Q interchanges numbers in one column. We define then the *symmetrizing* and *antisymmetrizing* operators as

$$S = \sum_P P, \tag{2.25}$$

$$A = \sum_Q (-1)^q Q \tag{2.26}$$

where q is the parity of the permutation Q. The sum in (2.25) is over all horizontal permutations and the one at (2.26) is over all vertical permutations. The product

$$Y = AS \tag{2.27}$$

is known as a *Young operator* [2]. Each Young operator acting on a state projects a basis state for the irrep associated with that tableau. Once we have the set of basis states, acting with the permutation operators generates the corresponding representations. However, the basis states differ with respect to the ones used before and the process of obtaining the representations is also somewhat different. As a practical example, we are going to get the irreps of S_3 with this method.

2.6 Application to the group S_3

For S_3 there are three possible Young diagrams which means that there are three different irreps. These Young diagrams are:

$$[3]\ \ydiagram{3} \qquad [21]\ \ydiagram{2,1} \qquad [1^3]\ \ydiagram{1,1,1} \tag{2.28}$$

For the [3] and $[1^3]$ Young diagrams there is only one possible standard tableau, so these irreps are one-dimensional. For [21] there are two possible standard tableaux:

$$[21]\ \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \tag{2.29}$$

which means that the irrep associated with [21] is two-dimensional.

For $n = 3$, the most general function we can work with is $|abc\rangle$. This means that our set of states, applying every possible permutation, is $\{|abc\rangle, |acb\rangle, |bac\rangle, |bca\rangle, |cab\rangle, |cba\rangle\}$. Let P_e, P_{12}, P_{13}, P_{23}, P_{123} and P_{132} be the operators associated with each element of the group.

For the Young tableau [3], the associated Young operator is:

$$Y_{[3]} = S_{[3]} = \sum_P P \tag{2.30}$$

where the sum goes through all the possible horizontal permutations of the elements. Since the Young diagram only has one row, the antisymmetrizing operator does not contribute. We expect then to find a completely symmetric state. Acting with this operator on one of the states of the set results in:

$$\sum_P P\,|abc\rangle = |abc\rangle + |bac\rangle + |cba\rangle + |acb\rangle + |cab\rangle + |bca\rangle \equiv |S\rangle\,. \tag{2.31}$$

The basis function in (2.31) generates the irrep associated with the Young diagram [3].[4] It is easy to see that acting with any of the elements of S_3 on $|S\rangle$ permutes each ket so that

$$P_i\,|S\rangle = |S\rangle \tag{2.32}$$

[4]As stated before, basis states have to be normalized so we would have to multiply by $\frac{1}{\sqrt{6}}$, but this is sometimes not necessary.

with $i = e, 12, ..., 132$. This means that the irrep generated by the basis function $|S\rangle$ is the totally symmetric representation:

$$D^{(S)}(e) = D^{(S)}(12) = D^{(S)}(13) = D^{(S)}(23) = D^{(S)}(123) = D^{(S)}(132) = 1. \tag{2.33}$$

For $[1^3]$ the Young operator is:

$$Y_{[1^3]} = A_{[1^3]} = \sum_{Q} (-1)^q Q \tag{2.34}$$

where the sum again goes through all the possible vertical permutations. Since the Young diagram for $[1^3]$ only has one column, the symmetrizing operator does not contribute. In this case we expect to obtain an antisymmetric state:

$$\sum_{Q} (-1)^q Q \, |abc\rangle = |abc\rangle - |acb\rangle - |bac\rangle + |bca\rangle + |cab\rangle - |cba\rangle \equiv |A\rangle \, . \tag{2.35}$$

Now, any of the two 3-cycles acting on $|A\rangle$ leave it unchanged. For example:

$$P_{123} \, |A\rangle = |cab\rangle - |bac\rangle - |cba\rangle + |abc\rangle + |bca\rangle - |acb\rangle = |A\rangle \, . \tag{2.36}$$

However, acting with any of the three 2-cycles applies a minus sign:

$$P_{13} \, |A\rangle = |cba\rangle - |bca\rangle - |cab\rangle + |acb\rangle + |bac\rangle - |abc\rangle = - \, |A\rangle \tag{2.37}$$

so the resulting irrep is

$$D^{(A)}(e) = D^{(A)}(123) = D^{(A)}(132) = 1, \;\; D^{(A)}(12) = D^{(A)}(13) = D^{(A)}(23) = -1. \tag{2.38}$$

We have to proceed now to the two-dimensional irrep, but before that we are going to change the set of states. We could keep working with the same set as before and obtain the same results, but in the case of the two-dimensional irrep it can get complicated to calculate the coefficients for the matrix representation. It is easier instead if we set two of the elements equal to each other, e.g. $|aab\rangle$. Doing this reduces the dimension of the set to 3 and makes it easier to obtain the irrep.

Let us consider the tableau

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{2.39}$$

The Young operator in this case is:

$$Y_{[132]} = A_{12} S_{13} \tag{2.40}$$

where A_{12} is the antisymmetrizing operator with respect to indices 1 and 2 and S_{13} is the symmetrizing operator with respect to indices 1 and 3. Thus:

$$Y_{[132]} |aab\rangle = A_{12} S_{13} |aab\rangle = A_{12}(|aab\rangle + |baa\rangle) = |baa\rangle - |aba\rangle \,. \tag{2.41}$$

Normalizing:

$$|M_1\rangle = \frac{1}{\sqrt{2}}(|baa\rangle - |aba\rangle). \tag{2.42}$$

To obtain the other basis state let us first apply the operator P_{23}[5] upon $|M_1\rangle$ and force the result to be a linear combination of the two basis states where the coefficients are entries of the corresponding matrix:

$$P_{23} |M_1\rangle = \frac{1}{\sqrt{2}}(|baa\rangle - |aab\rangle) = C_1 |M_1\rangle + C_2 |M_2\rangle \,. \tag{2.43}$$

If we now consider that $|M_1\rangle$ and $|M_2\rangle$ have to be orthonormal, multiplying by $\langle M_1|$ to the left of (2.43) gives:

$$C_1 = \frac{1}{\sqrt{2}} \langle M_1| (|baa\rangle - |aab\rangle) \tag{2.44}$$

and considering that the states are orthonormal themselves we get:

$$C_1 = \frac{1}{2}. \tag{2.45}$$

We can do the same but multiplying this time by $\langle M_2|$ to get:

$$C_2 = \frac{1}{\sqrt{2}} \langle M_2| (|baa\rangle - |aab\rangle). \tag{2.46}$$

[5] We could apply another operator, for example, P_{12}, but this would not help us to obtain the other state.

Going back to equation (2.43) and plugging in (2.42) and (2.45):

$$\frac{1}{\sqrt{2}}(|baa\rangle - |aab\rangle) = \frac{1}{2\sqrt{2}}(|baa\rangle - |aba\rangle) + C_2 |M_2\rangle . \tag{2.47}$$

Thus:

$$\left.\begin{aligned} C_2 \langle aab|M_2\rangle &= -\frac{1}{\sqrt{2}} \\ C_2 \langle aba|M_2\rangle &= \frac{\sqrt{2}}{4} \\ C_2 \langle baa|M_2\rangle &= \frac{\sqrt{2}}{4} \end{aligned}\right\} \Longrightarrow \frac{\langle baa|M_2\rangle}{\langle aba|M_2\rangle} = 1, \quad \frac{\langle aab|M_2\rangle}{\langle baa|M_2\rangle} = -2. \tag{2.48}$$

Now, considering that:

$$|M_2\rangle = \langle aab|M_2\rangle |aab\rangle + \langle aba|M_2\rangle |aba\rangle + \langle baa|M_2\rangle |baa\rangle \tag{2.49}$$

and

$$\langle aab|M_2\rangle^2 + \langle aba|M_2\rangle^2 + \langle baa|M_2\rangle^2 = 1 \tag{2.50}$$

we get for the second basis state:

$$|M_2\rangle = \frac{1}{\sqrt{6}}(|baa\rangle + |aba\rangle - 2|aab\rangle). \tag{2.51}$$

So our set of basis states is $\left\{\frac{1}{\sqrt{2}}(|baa\rangle - |aba\rangle), \frac{1}{\sqrt{6}}(|aba\rangle + |baa\rangle - 2|aab\rangle)\right\}$. We can now calculate C_2 with any of the relations at (2.48):

$$C_2 = \frac{\sqrt{3}}{2}. \tag{2.52}$$

The other two coefficients are obtained applying the P_{23} operator on the state $|M_2\rangle$ with the advantage that we now know explicitly the two basis states. Therefore:

$$P_{23}|M_2\rangle = \frac{1}{\sqrt{6}}(|aab\rangle + |baa\rangle - 2|aba\rangle) = C_3|M_1\rangle + C_4|M_2\rangle \tag{2.53}$$

and following the same steps as before results in:

$$C_3 = \frac{\sqrt{3}}{2}, \quad C_4 = -\frac{1}{2} \tag{2.54}$$

so the representation matrix for the element (23) takes the form:

$$D^{(M)}(23) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{2.55}$$

Computing the rest of the representation matrices is fairly straightforward. One just has to apply the corresponding operators on states $|M_1\rangle$ and $|M_2\rangle$, write the result as a linear combination of these states and then find the coefficients taking advantage of the orthonormality of the states. The representation obtained this way is a two-dimensional irreducible representation corresponding to the Young diagram [21], as stated at the beginning of this section:

$$D^{(M)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^{(M)}(13) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad D^{(M)}(23) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \tag{2.56}$$

$$D^{(M)}(12) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^{(M)}(132) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad D^{(M)}(123) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

The notation used for the irreps should be more apparent now. The completely symmetric irrep $D^{(S)}$ for S_3 is generated by the state $|S\rangle$ given in (2.31),[6] which is a completely symmetric state, i.e. interchanging any two labels leaves the state unchanged. The antisymmetric irrep $D^{(A)}$ is generated by $|A\rangle$ given in (2.35) which is an antisymmetric state under interchange of any two labels as seen earlier (interchanging two labels corresponds to applying a 2-cycle permutation). Finally, the mixed symmetry irrep $D^{(M)}$ is generated by the basis $\{|M_1\rangle, |M_2\rangle\}$ which are states with mixed symmetry, meaning that they have defined symmetry under interchange of two specific labels—1 and 2 in this case—but no defined symmetry otherwise. From now on we will refer to completely symmetric states as just symmetric and antisymmetric under 2-cycle permutations as just antisymmetric.

Irrep	Young diagram	Basis	Symmetry
$D^{(S)}$	[3]	$\{\lvert S\rangle\}$	Symmetric
$D^{(A)}$	$[1^3]$	$\{\lvert A\rangle\}$	Antisymmetric
$D^{(M)}$	[21]	$\{\lvert M_1\rangle, \lvert M_2\rangle\}$	Mixed symmetry

Table 2.3: Irreps of S_3.

[6] Again, with the corresponding normalization.

2.6.1 Clebsch-Gordan series of S_3

It is important that we obtain the Clebsch-Gordan series of S_3 for our objective to construct the baryon wavefunctions as the product of wavefunctions of the different degrees of freedom. Each wavefunction associated with a certain degree of freedom is going to have some symmetry under interchange of their labels given by the symmetry properties of S_3 and the baryon wavefunction (as we will see later) has to be antisymmetric, respecting the Pauli principle. We have seen in section 2.6 that there are three possible symmetries for the basis states, one for each irrep: symmetric, associated with the Young diagram [3]; antisymmetric, related to $[1^3]$; and with mixed symmetry corresponding to the two-dimensional irrep [21]. In this section we will determine what symmetry arises when we do the direct product of representations.

The direct product of the symmetric irrep $D^{(S)}$ with itself is trivially the symmetric irrep, or in other words:

$$D^{(S)} \otimes D^{(S)} = D^{(S)}. \tag{2.57}$$

The direct products $D^{(S)} \otimes D^{(A)}$, $D^{(S)} \otimes D^{(M)}$ and $D^{(A)} \otimes D^{(A)}$ are very easy to calculate too, and they are equal to $D^{(A)}$, $D^{(M)}$ and $D^{(S)}$, respectively. The combinations $D^{(A)} \otimes D^{(M)}$ and $D^{(M)} \otimes D^{(M)}$ are harder to obtain, so we will centre our attention on them.

Equations (2.58) give the representation matrices for $D^{(A)} \otimes D^{(M)}$. The resulting representation is the same as $D^{(M)}$ but for a minus sign for every 2-cycle.

$$D^{(A)}(e) \otimes D^{(M)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^{(A)}(12) \otimes D^{(M)}(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.58}$$

$$D^{(A)}(13) \otimes D^{(M)}(13) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad D^{(A)}(23) \otimes D^{(M)}(23) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$D^{(A)}(123) \otimes D^{(M)}(123) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad D^{(A)}(132) \otimes D^{(M)}(132) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

The question now is whether this is the same representation as $D^{(M)}$ or not. For this we have to show that for a certain similarity transformation, the representation above becomes $D^{(M)}$, which would mean that they are equivalent. Such a transformation has to be such that $D^{(A)}(123) \otimes D^{(M)}(123)$ and $D^{(A)}(132) \otimes D^{(M)}(132)$ remain the same and the matrices corresponding to the 2-cycles are multiplied by a negative sign.

Let (2.59) be a similarity transformation. This transformation meets the required conditions.

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.59}$$

Some examples are:

$$M^{-1} D^{(A)}(12) \otimes D^{(M)}(12) M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = D^{(M)}(12), \tag{2.60}$$

$$M^{-1} D^{(A)}(123) \otimes D^{(M)}(123) M = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = D^{(M)}(123). \tag{2.61}$$

Hence:

$$D^{(A)} \otimes D^{(M)} = D^{(M)}. \tag{2.62}$$

Let us evaluate now the direct product $D^{(M)} \otimes D^{(M)}$. The direct product of two 2×2 matrices is a 4×4 matrix. For example, for the elements (1 2) and (2 3) we have:

$$D^{(M)}(12) \otimes D^{(M)}(12) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.63}$$

$$D^{(M)}(23) \otimes D^{(M)}(23) = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ \sqrt{3} & -1 & 3 & -\sqrt{3} \\ \sqrt{3} & 3 & -1 & -\sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix}. \tag{2.64}$$

The matrix at (2.63) is already in block form, but the one at (2.64) is not.[7] Again, we have to see if there exists a similarity transformation that reduces the four-dimensional

[7]We will not write the rest of the matrices for the sake of brevity, but one can easily see that, except for the matrix corresponding to the identity element and the one at (2.63), the matrices have a form similar to (2.64).

representation given by $D^{(M)} \otimes D^{(M)}$. In this case, the similarity transformation is:

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad N^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{2.65}$$

and applying it to the matrices above results in:

$$N^{-1} D^{(M)}(12) \otimes D^{(M)}(12) N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.66}$$

$$N^{-1} D^{(M)}(23) \otimes D^{(M)}(23) N = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.67}$$

We have to be careful now. It might seem at first glance that the matrix at (2.66) is made up of four different 1×1 blocks, but this is actually not the case. The 2×2 block on the upper-left of such matrix corresponds to $D^{(A)}(12) \otimes D^{(M)}(12)$. This can be easily understood if we look at (2.67), where the 2×2 block is clearly $D^{(A)}(23) \otimes D^{(M)}(23)$. This suggests that the four-dimensional representation reduces partially to $D^{(M)}$. The two 1×1 blocks on the lower-right of both matrices suggest that $D^{(S)}$ and $D^{(A)}$ are also at play here. It can be proved that there appear three -1s—corresponding to the 2-cycles of $D^{(A)}$—and the matrices corresponding to the 3-cycles contain 2×2 blocks associated with $D^{(M)}$ that are not multiplied by a negative sign. This means that this representation is reduced into $D^{(S)}$, $D^{(A)}$ and $D^{(A)} \otimes D^{(M)}$, but as we recently proved the latter is equivalent to $D^{(M)}$. Thus:

$$D^{(M)} \otimes D^{(M)} = D^{(S)} \oplus D^{(A)} \oplus D^{(M)}. \tag{2.68}$$

Table 2.4 shows the resulting symmetry when combining two representations of S_3 with either defined (symmetric or antisymmetric) or mixed symmetry.

	S ($D^{(S)}$)	A ($D^{(A)}$)	M ($D^{(M)}$)
S ($D^{(S)}$)	S	A	M
A ($D^{(A)}$)	A	S	M
M ($D^{(M)}$)	M	M	$S \oplus A \oplus M$

Table 2.4: Symmetry properties of the direct product of two S_3 representations.

Chapter 3

Structure of hadrons in the quark model

In the last century, with the development of quantum mechanics and the arrival of the first accelerators, the amount of discovered particles made physicists feel the need to classify them attending to certain criteria. In this context, hadrons were defined as particles that feel the strong interaction. Some examples are protons, neutrons, pions, etc.

Among this particles, one finds a large number of cases where a certain amount of them exist with the same spin, parity and similar masses, the only distinction being the different magnitudes of their electrical charge and strangeness. These families can be organized in multiplets following what is called the Eightfold Way [3]. The Eightfold Way was introduced by Murray Gell-Mann and Yuval Ne'eman in 1961 as a way to arrange mesons (two-quark systems) and baryons into geometrical patterns, according to their charge and strangeness. The eight lightest baryons fit into an hexagonal array, with two particles at the centre. This is called an octet. The baryon octet is formed by the nucleons (proton and neutron), three sigmas, two cascades and one lambda.

Apart from the octets, there are also triangular arrays called decuplets containing 10 particles, as the name indicates. One example is the baryon decuplet, with four deltas, three sigmas, two cascades and one omega.

One may wonder, why do hadrons form this geometric patterns? Gell-Mann and Zweig provided an explanation in 1964 when they independently proposed that hadrons are actually composed of more elementary parts called quarks. At first, quarks came in three types (or "flavours"): the u quark (for "up"), the d quark (for "down") and the s quark (for "strange")[1]. By combining these particles it is possible to obtain many properties of the hadrons and explain how the geometrical patterns arise.

Group theory allows us to predict not only these multiplets but the particles they contain as well with the help of the quark model. Our task in the following sections is to predict what baryons are contained in these multiplets attending mainly to the properties of the permutation group S_3 derived in section 2.6 and the Pauli principle.

3.1 Quark degrees of freedom in the hadron

To construct the wavefunction for the system of three quarks, we need to know what their degrees of freedom are and specify the wavefunction for each one with their respective symmetry. This section will briefly describe the quark degrees of freedom in order to provide some theoretical background for the next sections when we will explicitly construct the complete wavefunction of the system as the product of wavefunctions of the different variables.

Quark degrees of freedom: flavour

Baryons are composed by three quarks, so they can be described as qqq states, where each q refers to one of the quarks. As stated above, quarks come in three types given in Table 3.1 with their corresponding properties.

Quark	Charge	Strangeness
u	$\frac{2}{3}$	0
d	$-\frac{1}{3}$	0
s	$-\frac{1}{3}$	-1

Table 3.1: Properties of the u, d and s quarks.

The flavour wavefunction specifies the quark content of the hadron at hand. We will see later that it can be symmetric, antisymmetric or a mixture depending on the quark content. For example, the system uud can be either a proton or a Δ^+. The former is the mixed symmetry state flavour-wise, which means that the state has an structure similar to (2.42) or (2.51). As for the Δ^+, it corresponds to the symmetric state, meaning that the wavefunction is symmetric under interchange of any pair of quarks.

[1]We will refer to these as the u, d and s quarks, but keep in mind that the correct concept is that they are different states or flavours of the quark.

Quark degrees of freedom: spin

Quarks are fermions, so their intrinsic spin must be half-integer [4]. In particular, they all have spin $\frac{1}{2}$. This can also be seen if one considers, for example, the proton. In the last section we mentioned that the proton is a mixed flavour state formed by the quarks *uud*. We also know that it is a fermion with spin $\frac{1}{2}$. It is therefore impossible for the quarks to be bosons with integer spin since three of them could not produce a fermion. This is the reason why we need to take into account the Pauli exclusion principle in order to construct the baryon wavefunction.

The spin wavefunction specifies the intrinsic spin states of the quarks that compose the hadron. Since spin is an angular momentum, it follows the addition rules of angular momenta in quantum mechanics. In the baryon case, the three $\frac{1}{2}$ spin quarks can add to either $\frac{3}{2}$ or $\frac{1}{2}$. We will see that for $S = \frac{3}{2}$ the spin wavefunction is symmetric while for $S = \frac{1}{2}$ there are two pairs of mixed symmetry states.

Quark degrees of freedom: spatial

The spatial wavefunction is associated with the motion of the quarks within the hadron. It is determined by the orbital angular momentum of the system. A baryon is a three body system, which means that there are two relative angular momenta. On the ground state, the two angular momenta are 0 so the spatial wavefunction is symmetric. Thus, the total angular momentum of the baryon comes solely from the intrinsic spin contribution of the three quarks.

In later sections we will see what happens to the spatial wavefunction when we give enough energy to the system for it to promote to the first excited state.

Quark degrees of freedom: colour

It turns out that if we only account for the spin, flavour and spatial degrees of freedom, the resulting wavefunction of some particles is symmetric under interchange of any of the quarks' variables. This violates the Pauli exclusion principle since no two quarks can occupy the same state. In 1964, O. W. Greenberg suggested that there is one variable left that physicists were not taking into consideration: colour [3].

Colour is another property that quarks posses, like charge or strangeness. Greenberg proposed that quarks not only come in three flavours but each of these comes in three colours, usually referred to as *red, green* and *blue*. Adding this new degree of freedom vanishes the problem with the Pauli principle. How? One may wonder. This has to do with the terminology at use here. Of course, the term "colour" has nothing to do with its ordinary meaning, but it has stayed with us because it cleverly states one simple fact about the particles that are found in nature; they are all colourless. By "colourless" we mean that either the total amount of each colour is zero or that the three colours are present in equal amounts. The latter case mimics the fact that in optics, light rays of the three primary colours combine to make white.[2] Baryons exist in nature because they are three-quark systems, so each quark can have a different colour from the rest in order for the whole composite to be colourless. We will see later that this fact uniquely determines the colour wavefunction for baryons and all of this can be justified attending to properties of the strong force.

3.2 Baryons

Table 3.2 [5] lists the possible combinations of the quark *uds* that come up after three selections have been made (for now we will take no account of the order in which they were selected, hence *uud*, *udu* and *duu* are equivalent). The charge and strangeness of the resulting baryons are shown in columns three and four, respectively. In column five we give examples of baryons with these quantum numbers. If we have chosen *uuu* then

Quarks	Symmetry	Charge	Strangeness	Examples
uuu	S	2		Δ^{++}
uud	S M	1		$\Delta^{+} P$
udd	S M	0	0	$\Delta^{0} N$
ddd	S	-1		Δ^{-}
uus	S M	1		$\Sigma^{+*}\Sigma^{+}$
uds	S M M A	0	-1	$\Sigma^{0*}\Sigma^{0}\Lambda^{0}\Lambda(1405)$
dds	S M	-1		$\Sigma^{-*}\Sigma^{-}$
uss	S M	0		$\Xi^{0*}\Xi^{0}$
dss	S M	-1	-2	$\Xi^{-*}\Xi^{-}$
sss	S	-1	-3	Ω^{-}

Table 3.2: Systems of three quarks with three flavours.

we know that we have a Δ^{++}, but if we choose *uud* then how do we distinguish Δ^{+} from

[2] This also explains, for example, how no isolated quarks have been observed yet.

proton (other than by its spin)? For *uds* there are four possibilities.

The way we distinguish between them is related to the symmetry properties of the states in column two. Specifically—if we now worry about the order in which the quarks were selected—then, what happens if we change the quarks selected first and second around to second and first? Clearly if all were the same—*uuu* for instance—then you get the same state and this is called symmetric, labelled S. In fact one can always define an S combination whatever the three quarks content may be, hence 10 such states. This can simply be done by acting with the symmetrizing operator on the state. If at least one quark differs from the rest you can write a mixed symmetric state (M) and there are eight of these (*uds* comes in two ways since there are two choices for the "different quark"). Finally if all three are distinct one can form a single antisymmetric state (A) under interchange of any pair of quarks.

This brings us to the ideas of symmetry properties of states under interchange of their labels that was discussed in section 2.6. In order to get the complete wavefunction for the three-quark system we have to find the symmetric, antisymmetric and mixed symmetry states for each degree of freedom with the help of Young operators. These symmetries are associated with a certain irrep of S_3 that we will refer to by writing the corresponding Young diagram following the information in Table 2.3. The complete wavefunction is then the product of the states and its symmetry will be determined by the Clebsch-Gordan series of the direct product of two irreps of S_3.

3.2.1 S_3 symmetry of the spin wavefunction

It was already discussed that quarks have to be fermions with half-integer spin. More precisely, they carry spin $\frac{1}{2}$. This means that each quark can occupy two states, given by the two possible values of the spin projection $m_s = \frac{1}{2}$ or $m_s = -\frac{1}{2}$. These projections correspond to the "spin up" ($\uparrow$) or "spin down" ($\downarrow$) states. Thus, there are eight possible states for the three quark system: $(\uparrow\uparrow\uparrow), (\uparrow\uparrow\downarrow), (\uparrow\downarrow\uparrow), (\downarrow\uparrow\uparrow), (\uparrow\downarrow\downarrow), (\downarrow\uparrow\downarrow), (\downarrow\downarrow\uparrow)$ and $(\downarrow\downarrow\downarrow)$. We also saw that the three spins couple together to give either spin $\frac{3}{2}$ or $\frac{1}{2}$. In this section we will see the S_3 symmetry of the spin states constructing explicitly the symmetric and mixed symmetry states for each combination.

System of two particles with spin $\frac{1}{2}$

There are four possibilities, given in the first column of Table 3.3 [5]. Under interchange of the labels one and two, three states are symmetric and one is antisymmetric.

1st	2nd	$1 \leftrightarrow 2$ interchange	
$\uparrow$	$\uparrow$	$\uparrow\uparrow$	
$\uparrow$	$\downarrow$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$
$\downarrow$	$\uparrow$		
$\downarrow$	$\downarrow$	$\downarrow\downarrow$	
		symmetric	antisymmetric

Table 3.3: Symmetry states for spin (two particles).

In the group theory notation we write $(2\text{ states})_1 \times (2\text{ states})_2 = 2 \otimes 2 \to 3 \oplus 1$, showing that there are 3 symmetric and 1 antisymmetric combinations.

A familiar example of this is combining two states with spin $\frac{1}{2}$ to form a state of spin 1 or 0:

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0. \tag{3.1}$$

Rewriting this in terms of the $(2S+1)$ states one would have

$$2 \otimes 2 = 3 \oplus 1. \tag{3.2}$$

In fact, we can obtain the Table 3.3 by referring to the Clebsch-Gordan coefficients for combining the two spin $\frac{1}{2}$ states and this is identical to the separation made above into three states ($S = 1$) and one state ($S = 0$).

System of three particles with spin $\frac{1}{2}$

We saw at the beginning of this section that there are eight possible combinations. In order to obtain the spin states and relate them to the symmetry properties of S_3, let us first consider the simplest example: $(\uparrow\uparrow\uparrow)$. This is the case where the three spins are "up", so the z component for the total spin is $S_z = \frac{3}{2}$.

Usually, spin states are written with notation $|s_1 s_{1z}; s_2 s_{2z}; s_3 s_{3z}\rangle \to |S S_z\rangle$ so we could write this state as $(\uparrow\uparrow\uparrow) = \left|\frac{3}{2}\frac{3}{2}\right\rangle$. Instead, we will generally write states with the arrow notation we have been using since we are going to need to keep track of each monoparticular state. In this manner, we write $(\uparrow\uparrow\uparrow) = |\uparrow\uparrow\uparrow\rangle$.

It is clear that if we apply the antisymmetrizing operator upon $|\uparrow\uparrow\uparrow\rangle$ the result would be zero, since each term is the same as the rest so they would cancel out in pairs due to the $(-1)^q$ factor. In fact, this happens whatever the state is as long as it contains at least two equal labels. Since the Young operator corresponding to the [21] diagram contains an antisymmetrizing operator with respect to indices one and two, applying it to our state will return zero too. Hence, the only operator that would give something different from zero is the symmetrizing operator corresponding to the [3] diagram. Evidently, since the z component of the spin is $\frac{1}{2}$ for every particle (which means that we are constructing the wavefunction for $S_z = \frac{3}{2}$) the symmetrizing operator applied to $|\uparrow\uparrow\uparrow\rangle$ results in the same state, which is a symmetric one.[3] This is also what happens if we consider the $|\downarrow\downarrow\downarrow\rangle$ state, with $S_z = -\frac{3}{2}$.

What happens if one spin differs from the other two? For example, selecting two spins up and one down gives three possible combinations, namely $|\uparrow\downarrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$, all with $S_z = \frac{1}{2}$. This is exactly the same problem that we addressed when we obtained the irreps of S_3. Since Young operators do not care about how we name the labels, the spin states are going to be exactly the ones that we obtained as generators for the irreps of S_3 but changing the labels a to $\uparrow$ and $b = c$ to $\downarrow$. So, for the [3] Young diagram associated with the symmetrizing operator given in (2.30) we get one symmetric state:

$$S_{[3]} |\uparrow\uparrow\downarrow\rangle \rightarrow \left|\frac{3}{2}\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle). \tag{3.3}$$

Indeed, this is the state we get when making two labels at equation (2.31) equal to each other.

Consider now the [21] diagram associated with the Young operator given in (2.40). We expect to get two states with the same structure as the ones at (2.42) and (2.51):

$$Y_{[132]} |\uparrow\uparrow\downarrow\rangle \rightarrow \left|\frac{1}{2}\frac{1}{2}\right\rangle_{A_{12}} = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle), \tag{3.4}$$

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle_{S_{12}} = \frac{1}{\sqrt{6}}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2\,|\uparrow\uparrow\downarrow\rangle) \tag{3.5}$$

where the sub-indices A_{12} and S_{12} indicate that the state is antisymmetric under interchange of the labels one and two and symmetric under interchange of the same labels,

[3]There is a factor 6 that appears when applying the operator but it vanishes after normalizing.

respectively. These states have no simple symmetry under (1 3) or (2 3) interchange. In fact, interchanging labels (1 3) or (2 3) yields a state that is a linear combination of the already mentioned states, hence "mixed" symmetry.

The physical interpretation of the second state is clear once we consider that there are two ways of forming an $S = \frac{1}{2}$ state when we couple three particles with spin $\frac{1}{2}$. We saw in section 3.2.1 that coupling two particles with spin $\frac{1}{2}$ can yield two results, either 1 or 0. If they couple to 0, then adding another particle with the same spin can only yield spin $\frac{1}{2}$, but if they couple to 1 then there are again two possibilities: $\frac{1}{2}$ and $\frac{3}{2}$. So there are two different paths that give the same spin when coupling three particles, one with $S_{12} = 0$ (antisymmetric)—where the sub-index indicates that that is the spin of the first two particles without adding the third—and one with $S_{12} = 1$ (symmetric). Thus, there are two spin states with mixed symmetry associated with $S = \frac{1}{2}$.

One may wonder, what if we repeat this process but starting with another state? If we apply the Young operator on any of the other two states what we get is either the same state at (3.4) or one that differs in a negative sign, which as we know is irrelevant from the point of view of quantum mechanics. This also makes sense from the group theory perspective, since all we are doing here is repeat the process of obtaining the basis states that generate the irreps of S_3. Therefore, any other state obtained by, for example, making use of the other Young tableau or starting from another generic state different from $|aab\rangle$ has to be a linear combination of basis states.

All we have left now are the states that can be obtained by combining one spin "up" and two spins "down", namely $|{\uparrow\downarrow\downarrow}\rangle$, $|{\downarrow\uparrow\downarrow}\rangle$ and $|{\downarrow\downarrow\uparrow}\rangle$. These states are defined by a z component $S_z = -\frac{1}{2}$. Evidently, there is going to be another symmetric state similar to (3.3):

$$\left|\frac{3}{2} - \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|{\uparrow\downarrow\downarrow}\rangle + |{\downarrow\downarrow\uparrow}\rangle + |{\downarrow\uparrow\downarrow}\rangle). \tag{3.6}$$

Now, for the mixed symmetry states we could use Young operators as we have been doing up until now, but in this case we are going to proceed in a different fashion by utilizing ladder operators [3]. In quantum mechanics, ladder operators for the spin are defined as follows:

$$S_\pm |SS_z\rangle = \hbar\sqrt{S(S+1) - S_z(S_z \pm 1)}\, |S(S_z \pm 1)\rangle\,. \tag{3.7}$$

S_+ is called the rising operator because it increases S_z by $\hbar$ and S_- the lowering operator because it decreases S_z by $\hbar$.[4] In the case of one fermion with spin $s = \frac{1}{2}$, this means:

$$S_+ \left|\frac{1}{2} - \frac{1}{2}\right\rangle = \hbar \left|\frac{1}{2}\frac{1}{2}\right\rangle, \quad S_- \left|\frac{1}{2}\frac{1}{2}\right\rangle = \hbar \left|\frac{1}{2} - \frac{1}{2}\right\rangle \tag{3.8}$$

as one can easily check using (3.7). We can then use the lowering operator to obtain the $S_z = -\frac{1}{2}$ states starting from the $S_z = \frac{1}{2}$ ones. To do this, we have to keep in mind that, since the total spin of the system is the sum of the individual spins of each quark, the lowering operator S_- is the sum of the lowering operators for each quark independently. Acting with S_- upon (3.4) yields:

$$S_- \left|\frac{1}{2}\frac{1}{2}\right\rangle_{A_{12}} = \frac{1}{\sqrt{2}}(s_-^{(1)} + s_-^{(2)} + s_-^{(3)})(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle); \tag{3.9}$$

$$\left|\frac{1}{2} - \frac{1}{2}\right\rangle_{A_{12}} = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle). \tag{3.10}$$

This wavefunction is antisymmetric under interchange of the labels one and two, same as (3.4). Doing the same to (3.5) gives:

$$S_- \left|\frac{1}{2}\frac{1}{2}\right\rangle_{S_{12}} = \frac{1}{\sqrt{6}}(s_-^{(1)} + s_-^{(2)} + s_-^{(3)})(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2\,|\uparrow\uparrow\downarrow\rangle); \tag{3.11}$$

$$\left|\frac{1}{2} - \frac{1}{2}\right\rangle_{S_{12}} = -\frac{1}{\sqrt{6}}(|\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle - 2\,|\downarrow\downarrow\uparrow\rangle) \tag{3.12}$$

which is symmetric under interchange of the labels one and two. Notice that working with ladder operators gives the same result as Young operators would.

We have now constructed all the possible spin wavefunctions for the system of three quarks and studied their symmetry. We have obtained in total four symmetric states ($S = \frac{3}{2}$) and two pairs of mixed symmetry states ($S = \frac{1}{2}$).

3.2.2 S_3 symmetry of the flavour wavefunction

For the flavour wavefunction we consider the three possible states of the quark—u ("up"), d ("down") and s ("strange") as discussed previously in section 3.1. There are $3^3 = 27$ possibilities: *uuu, uud, udu, duu, udd, ... , sss* which we combine to give symmetric, antisymmetric and mixed symmetry combinations. We can write all of the possible com-

[4] We will not explicitly write $\hbar$ in our calculations. This can be justified by either defining dimensionless ladder operators or by working in natural units, where $\hbar = 1$.

binations of three quarks with the Dirac notation as we did with spin, signaling that they are the building blocks for the flavour wavefunctions of the baryons. As in section 3.2.1, we can apply Young operators to get the wavefunctions with defined symmetry under interchange of certain labels.

To get the symmetric states, we apply the symmetrizing operator on the 27 possible combinations. This yields a total of 10 different symmetric states, since acting with the symmetrizing operator on, say, $|dds\rangle$ and $|dsd\rangle$ results in the same state. As an example, let us consider the state $|uud\rangle$:

$$S_{[3]}\,|uud\rangle = 2(|uud\rangle + |udu\rangle + |duu\rangle) \xrightarrow{\text{Normalizing}} \frac{1}{\sqrt{3}}(|uud\rangle + |udu\rangle + |duu\rangle) \tag{3.13}$$

and this is the same that we would get if we apply the symmetrizing operator on $|udu\rangle$ or $|duu\rangle$.

Going back to the beginning of section 3, we talked about the Eightfold Way and the existence of multiplets. In particular, decuplets and octets, and we showed some examples. The particles that form these multiplets have the same spin, parity and approximately the same mass. What we just got from considering all the possible combinations of three quarks that can form a baryon and constructing the symmetric flavour states is the decuplet. Each of the 10 flavour states corresponds to a certain particle belonging to the decuplet.

Out of the 27 combinations, there are 17 states left that can be organized into two octets (16 states) plus one singlet. The existence of the singlet state makes perfect sense if one realises that of all the possible combinations, there is only one antisymmetric state that can be obtained when applying the antisymmetrizing operator. This means that the rest are either symmetric—the decuplet—or mixed symmetry states—the two octets. Obtaining the antisymmetric state is easy. We just have to apply the antisymmetrizing operator associated with the $[1^3]$ Young diagram upon $|uds\rangle$:

$$A_{[1^3]}\,|uds\rangle = |uds\rangle - |dus\rangle - |sdu\rangle - |usd\rangle + |sud\rangle + |dsu\rangle \tag{3.14}$$

and normalizing as always:

$$|A\rangle = \frac{1}{\sqrt{6}}(|uds\rangle - |dus\rangle - |sdu\rangle - |usd\rangle + |sud\rangle + |dsu\rangle) \tag{3.15}$$

which is the state that we called $|A\rangle$ for antisymmetric in equation (2.35).

Having calculated the decuplet and the singlet, we can now proceed to the octets. Naturally, we have to work now with the two-dimensional irrep of S_3, which is generated by the two basis states $|M_1\rangle$ and $|M_2\rangle$ in equations (2.42) and (2.51). The Young operator associated with the Young tableau at (2.39) is $A_{12}S_{13}$. Acting with this operator on any of the symmetric states, namely, $|uuu\rangle$, $|ddd\rangle$ and $|sss\rangle$ returns zero due to the antisymmetrizing operator with respect to indices one and two.

Twelve of the 16 mixed symmetry states are obtained by simply changing the labels a and b in $|M_1\rangle$ and $|M_2\rangle$ to u, d or s, depending on the combination of quarks the Young operator is applied to. As an example, let us consider the combination uud. The two states coming from this combination are obtained by changing a to u—since this is the label that appears twice—and b to d:

$$|P_1\rangle = \frac{1}{\sqrt{2}}(|duu\rangle - |udu\rangle), \ \ |N_1\rangle = \frac{1}{\sqrt{6}}(|udu\rangle + |duu\rangle - 2\,|uud\rangle). \tag{3.16}$$

For ssd we get:

$$|P_2\rangle = \frac{1}{\sqrt{2}}(|dss\rangle - |sds\rangle), \ \ |N_2\rangle = \frac{1}{\sqrt{6}}(|sds\rangle + |dss\rangle - 2\,|ssd\rangle) \tag{3.17}$$

and so on. From the 12 states obtained in this manner, six of them (the $|P_i\rangle$ states, $i = 1, ..., 6$) are antisymmetric with respect to interchange of the labels one and two and the other six symmetric with respect to interchange of the same labels (the $|N_i\rangle$ states).

Notice that each multiplet is also characterized by the symmetry of the states. The decuplet derived above contains symmetric flavour states. The singlet is an antisymmetric state flavour-wise. Meanwhile, the two octets are formed by mixed symmetry flavour states. This holds true precisely because we have derived these multiplets the same way as we calculated the basis states that generate the irreps of S_3 using Young tableaux and the elements of the group. Gell-Mann and Zweig provided a physical explanation on why mesons and baryons form these geometrical patterns when they proposed the existence of quarks, and group theory not only confers these patterns a more mathematical and subtle meaning but also allows us to understand the symmetry properties of the states involved. Understanding these properties is crucial from the perspective of physics in order to construct states that respect the Pauli exclusion principle.

We have $10 + 1 + 12 = 23$ states, so there are four left. Each octet is composed of six states that will sit in the vertices of the hexagon, and the remaining two are in the middle. There are two independent baryon octets, so the four states must be the ones that sit in the middle of the two octets. To get these states we have to start by applying the Young operator $A_{12}S_{13}$ upon $|uds\rangle$, the last combination that we have yet to use:

$$A_{12}S_{13}\,|uds\rangle = A_{12}(|uds\rangle + |sdu\rangle) = |uds\rangle - |dus\rangle + |sdu\rangle - |dsu\rangle$$

$$\implies |T_1\rangle = \frac{1}{2}(|uds\rangle - |dus\rangle + |sdu\rangle - |dsu\rangle). \tag{3.18}$$

This state, as one can easily check, is antisymmetric under interchange of the labels one and two, so it belongs to the same octet as the $|P_i\rangle$ states:

$$P_{12}\,|T_1\rangle = \frac{1}{2}(|dus\rangle - |uds\rangle\,|dsu\rangle - |sdu\rangle) = -\,|T_1\rangle\,. \tag{3.19}$$

The orthogonal wavefunction of $|T_1\rangle$ can be obtained acting with, for example, P_{13}:

$$P_{13}\,|T_1\rangle = \frac{1}{2}(|sdu\rangle - |sud\rangle + |uds\rangle - |usd\rangle) = C_1\,|T_1\rangle + C_2\,|Q_1\rangle\,. \tag{3.20}$$

Multiplying by $\langle T_1|$ to the left and distributing:

$$C_1 = \frac{1}{2}. \tag{3.21}$$

We can now isolate $C_2\,|Q_1\rangle$ by substracting $C_1\,|T_1\rangle$ and then normalize to get $|Q_1\rangle$:

$$\begin{aligned}(P_{13} - C_1)\,|T_1\rangle &= \frac{1}{2}(|sdu\rangle - |sud\rangle + |uds\rangle - |usd\rangle)\\ &- \frac{1}{4}(|uds\rangle - |dus\rangle + |sdu\rangle - |dsu\rangle)\\ &= \frac{1}{4}(|uds\rangle + |dus\rangle + |dsu\rangle + |sdu\rangle) - \frac{1}{2}(|usd\rangle + |sud\rangle)\end{aligned}$$

$$\implies |Q_1\rangle = \frac{1}{\sqrt{12}}[|uds\rangle + |dus\rangle + |dsu\rangle + |sdu\rangle - 2(|usd\rangle + |sud\rangle)]. \tag{3.22}$$

This state is symmetric under interchange of labels one and two:

$$P_{12}\,|Q_1\rangle = \frac{1}{\sqrt{12}}[|dus\rangle + |uds\rangle + |sdu\rangle + |dsu\rangle - 2(|sud\rangle + |usd\rangle)] = |Q_1\rangle \tag{3.23}$$

so it belongs to the octet with mixed (1 2) symmetry states. One can check that $|T_1\rangle$ and $|Q_1\rangle$ are indeed orthonormal.

To get the last two states we can repeat the process but starting with another combination, for example, $|usd\rangle$. Acting with the Young operator upon this state yields:

$$A_{12}S_{13}\,|usd\rangle = A_{12}(|usd\rangle + |dsu\rangle) = |usd\rangle - |sud\rangle + |dsu\rangle - |sdu\rangle \tag{3.24}$$

so our third state is:

$$|T_2\rangle = \frac{1}{2}(|usd\rangle - |sud\rangle + |dsu\rangle - |sdu\rangle). \tag{3.25}$$

Applying the P_{13} operator and solving as before results in:

$$|Q_2\rangle = \frac{1}{\sqrt{12}}[|usd\rangle + |dsu\rangle + |sud\rangle + |sdu\rangle - 2(|dus\rangle + |uds\rangle)]. \tag{3.26}$$

It is easy to prove that these four states are the only linearly independent states. Repeating the process again for any of the other four combinations that have not been used—$|dsu\rangle$, $|dus\rangle$, $|sdu\rangle$ and $|sud\rangle$—results in states that are linear combinations of the four states above. For example, if we now use $|dus\rangle$ as our starting point we get:

$$A_{12}S_{13}\,|dus\rangle = |dus\rangle - |uds\rangle + |sud\rangle - |usd\rangle \tag{3.27}$$

but this state is not independent. Adding $|T_1\rangle$ and $|T_2\rangle$ results in:

$$|T_1\rangle + |T_2\rangle \propto |uds\rangle - |dus\rangle + |usd\rangle - |sud\rangle \tag{3.28}$$

which is the “new” state but with a negative sign.

To summarize, we have got one decuplet with symmetric flavour states or [3] symmetry, one singlet with an antisymmetric flavour state or $[1^3]$ symmetry and two octets with mixed symmetry flavour states or [21] symmetry.

3.2.3 S_3 symmetry of the colour wavefunction

Having derived the flavour states, the colour case is straightforward. There are three possible colour states for each quark as stated in section 3.1. Some examples of colour states for baryons are $|rrb\rangle, |gbr\rangle, |bbb\rangle, |gbb\rangle$, etc. However, not all of these states are possible. The first, third and fourth states listed have two or three quarks with the same colour, but as we already discussed only colourless combinations are found in nature, so the only valid state is the second.

Quantum chromodynamics provides an explanation on why this is the case [3]. Studying the quark-quark interaction and assuming that they have different flavours, it is found that there exist a triplet—the antisymmetric combinations—and a sextet—the symmetric combinations. The force for the triplet combinations is attractive, while for the sextet it is repulsive. Of course, two-quark states are not found in nature (mesons are quark-antiquark pairs) but they tell us something about baryons. In the baryon case, the problem of finding the colour wavefunctions is identical to the flavour case but changing the labels *u, d* and *s* to *r, b* and *g*. We expect then to get the same structure when it comes to the symmetry of the states, so one decuplet, one singlet and two octets. The decuplet is symmetric, so every pair of quarks is in the (symmetric) sextet state which corresponds to a repulsive force. The singlet is antisymmetric, so every pair of quarks is in the (antisymmetric) triplet state—hence an attractive force. As for the octets (or mixed symmetry states) we expect to find some attraction and some repulsion. Only in the singlet configuration we find complete attraction between the three quarks. This explains why natural occurring particles are colourless—the strong potential favors these combinations over the decuplet and the octets.

Following the discussion above, we can see that although we have a colour decuplet, a singlet and two octets, nature chooses only the singlet state. This state can be obtained applying the antisymmetrizing operator:

$$|C\rangle = \frac{1}{\sqrt{6}}(|rgb\rangle - |rbg\rangle - |grb\rangle - |bgr\rangle + |gbr\rangle + |brg\rangle). \tag{3.29}$$

In this manner, since the colour state is the same for all baryons and is antisymmetric under any 2-cycle permutation, if we now restrict ourselves to symmetric combinations of the spin, spatial and flavour states we will obtain an antisymmetric state describing the whole system of three quarks, which not only respects the Pauli principle but also reproduces the wavefunctions of particles that are seen in nature.

3.2.4 S_3 symmetry of the spatial wavefunction

The last piece in our puzzle to construct the baryon wavefunctions is the spatial state. We saw in section 3.1 that this degree of freedom describes the motion of the quarks. We are going to consider a non-relativistic harmonic oscillator model [5], so quarks can be in the ground state, which is the lowest energy level, or in an excited state. If we consider that the quarks are in the ground state, then the orbital angular momentum of the system L is equal to 0 and the parity $(-1)^{l_1}(-1)^{l_2} = (-1)^{l_1+l_2} = (-1)^L = 1$ is positive, or in other words $L^\pi = 0^+$.[5] Therefore, the total angular momentum $\mathbf{J} = \mathbf{S} + \mathbf{L}$ is just the coupling of the intrinsic spins of each quark. Naturally, if the three quarks are all in the ground state, then their spatial wavefunction is completely symmetric. In the following section we will see what happens when we excite one quark to the first excited state.

First excited state

The harmonic oscillator states can be characterized with energy levels, starting from the ground state, or (1s) as in the shell model notation and going up to higher energy levels. The first excited state would be the $(1p_m)$ state since we are going from $L = 0$ to $L = 1$ and the m accounts for the three possible projections. Therefore, we have two quarks that remain in the ground state and the last quark is in the first excited state:

$$(1s)(1s)(1p_m) = (1s)^2(1p_m) \rightarrow L^\pi = 1^-. \tag{3.30}$$

Since one state now differs from the others, we expect to find symmetric and mixed symmetry states as in the spin case. Let us denote the ground state as $|(1s)^3\rangle \equiv |000\rangle$. This is the (symmetric) $(1s)^3$ state that we talked about earlier. We have to keep in mind that, since we are talking about orbital angular momentum, we have to choose a reference frame with some origin, e.g. the centre of mass.

The centre of mass reference frame is defined as the reference frame where

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 = 0 \tag{3.31}$$

where m_1, m_2 and m_3 are the masses of the quarks and $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are their position vectors. However, as a first approximation we can consider that the mass of the baryon

[5] L is obtained by coupling l_1 and l_2 which are two relative angular momenta in the three-quark system.

is just the combined masses of its quarks. This would include in the mass of the quarks the interactions between them as well as the kinetic energy that they have. In this approximation, the three quarks that compose the baryon are considered to contribute the same to the mass of the baryon, so $m_1 = m_2 = m_3$ and (3.31) reduces to [5]:

$$\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = 0. \tag{3.32}$$

Now, the state at (3.30) can be written as $|(1s)^2(1p_m)\rangle$ and following the notation above it can be either $|001\rangle$, $|010\rangle$ or $|100\rangle$, depending on which of the quarks is excited. From one of these combinations it is possible to obtain one symmetric state applying the symmetrizing operator:

$$|\psi_S\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \tag{3.33}$$

and two states with mixed symmetry:

$$|\psi_{A_{12}}\rangle = \frac{1}{\sqrt{2}}(|100\rangle - |010\rangle), \tag{3.34}$$

$$|\psi_{S_{12}}\rangle = \frac{1}{\sqrt{6}}(|010\rangle + |100\rangle - 2\,|001\rangle). \tag{3.35}$$

The wavefunctions for these states can be obtained projecting in position space by multiplying to the left by the bra $\langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|$. The symmetric combination is then:

$$\psi_S(L=1, m; \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{\sqrt{3}}(\langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|001\rangle + \langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|010\rangle + \langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|100\rangle) \tag{3.36}$$

and the mixed symmetry ones are:

$$\psi_{A_{12}}(L=1, m; \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{\sqrt{2}}(\langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|100\rangle - \langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|010\rangle), \tag{3.37}$$

$$\psi_{S_{12}}(L=1, m; \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{\sqrt{6}}(\langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|010\rangle + \langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|100\rangle - 2\,\langle \mathbf{r}_1\mathbf{r}_2\mathbf{r}_3|001\rangle). \tag{3.38}$$

It can be proven [3, 5] by expressing the symmetric state in Cartesian coordinates that it vanishes in the centre of mass reference frame.

However, one may wonder what happens if we choose another reference frame in which the symmetric state does not vanish. This state would exist, but it would not be a valid state describing an internal excitation of the system. These excitations have to be described by states that are invariant under spacial translations, i.e., the state can not depend on where the system is (hence internal). Thus, the symmetric state exists in general but is not a genuine internal excitation of the three-quark system, but rather a state describing the displacement of the system as a whole with their centre of mass rotating around the observer with one unit of angular momentum. This is known as an "spurious state".

We finally have all the pieces for constructing the baryon wavefunctions. The following sections will tackle the problem of combining the different states for each degree of freedom to obtain antisymmetric wavefunctions of the entire system and identifying these wavefunctions with the baryons found in nature.

3.3 Selection of S_3 representations that give rise to an antisymmetric wavefunction

The wavefunction for the three-quark system can be expressed as the product of the spin, flavour, colour and spatial wavefunctions:

$$\Psi = \psi_{spin}\psi_{flavour}\psi_{colour}\psi_{space}. \tag{3.39}$$

The colour wavefunction is always completely antisymmetric, as we saw in section 3.2.3. Thus, if we want the complete wavefunction to be antisymmetric in order to respect the Pauli principle, the product of the flavour, spin and spatial wavefunctions has to be completely symmetric, so that $(D_{spin} \otimes D_{flavour} \otimes D_{space}) \otimes D^{(A)}_{colour} = D^{(S)} \otimes D^{(A)}_{colour} = D^{(A)}$. We will start by studying the case where the three quarks are in the ground state, so the spatial wavefunction is completely symmetric, and then move onto the first excited state.

3.3.1 Antisymmetric wavefunctions in the ground state

Quarks in the ground state are all going to have $L^\pi = 0^+$. Since the spatial wavefunction is completely symmetric, what we are looking for here is that the product of the spin and flavour wavefunctions is also completely symmetric. Going back to sections 3.2.1 and 3.2.2 we see that for the flavour we can have the three types of symmetry while spin only has symmetric and mixed symmetry states. Working out each product, the only combinations that give rise to a symmetric flavour$\otimes$spin state (or $D^{(S)}$ representation) are $D^{(S)} \otimes D^{(S)}$ and $D^{(M)} \otimes D^{(M)}$. The former corresponds to the decuplet with $J = \frac{3}{2}$ and the latter to the octet with $J = \frac{1}{2}$, both with positive parity. Notice that there is no singlet compatible with the ground state, so every singlet state is going to be an excited system.

3.3.2 Antisymmetric wavefunctions in the first excited state

For the first excited state, the spatial wavefunctions are mixed symmetry states, so [21]. The angular momentum L is the coupling of l_1 and l_2, but in this case $l_1 = 0$, $l_2 = 1$ or the other way around. Therefore, $L = 1$ and the parity is negative, $(-1)^L = -1$. We will start by coupling the spin—which can be either [3] or [21]—with the orbital angular momentum, then add flavour and force the result to be symmetric so when we add colour the total wavefunction is antisymmetric.

Let us begin with the symmetric spin states. These states have spin $S = \frac{3}{2}$. Coupling spin with angular momentum results in states with defined values of S and L. However, there are multiple independent states with the same values of these quantum numbers which means that there is some degeneracy. For example, we can multiply $|\uparrow\uparrow\uparrow\rangle$ and $|\psi_{A_{12}}\rangle$ to get a state with $S = \frac{3}{2}$ and $L = 1$, but we could do the same with $\left|\frac{3}{2}\frac{1}{2}\right\rangle$ and $|\psi_{A_{12}}\rangle$ which would produce a product state with the same spin and angular momentum as the first one. What breaks this degeneracy is the spin-orbit interaction. This interaction couples the spin and orbital angular momentum to give a total angular momentum, $\mathbf{J} = \mathbf{S} + \mathbf{L}$, following the usual rules of addition of angular momenta. This breaks the degeneracy since now there are three possible values for J, $J = \frac{1}{2}$, $J = \frac{3}{2}$ and $J = \frac{5}{2}$.

Regarding the symmetry properties, we have the direct product between $[3]_{spin} \equiv [3]_s$ and $[21]_{orbital} \equiv [21]_o$, which as we already know is $[3]_s \otimes [21]_o = [21]_{os}$. This means that when adding flavour—which can have the three possible symmetries—the only way that we can get a symmetric representation is with another mixed symmetry representation. Thus, according to equation (2.68), $[21]_f \otimes [21]_{os} = [3] \oplus [1^3] \oplus [21]$, so the only valid combination is flavour octet with the total angular momenta written above.

If we now choose the mixed symmetry spin states, coupling with the spatial states yields $J = \frac{1}{2}, \frac{3}{2}$ and $[21]_s \otimes [21]_o = [3]_{os} \oplus [1^3]_{os} \oplus [21]_{os}$, so the range of possibilities increases with respect to the previous case:

- For $[3]_{os}$ the only possibility is $[3]_f$, so that $[3]_{os} \otimes [3]_f = [3]$. This is a decuplet.
- For $[1^3]_{os}$ we have $[1^3]_{os} \otimes [1^3]_f = [3]$, which is a singlet.
- For $[21]_{os}$ the only possibility is $[21]_{os} \otimes [21]_f = [3] \oplus [1^3] \oplus [21]$, which corresponds to an octet.

Table 3.4 summarizes the resulting multiplets for 1 quanta of excitation derived in this section, with their corresponding properties.

S_3 representation	Multiplet (symmetry)	L	S	J^π
$[21]_{os} \otimes [21]_f$	octet (M)	1	$\frac{3}{2}$	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
$[3]_{os} \otimes [3]_f$	decuplet (S)	1	$\frac{1}{2}$	$\frac{1}{2}^-, \frac{3}{2}^-$
$[1^3]_{os} \otimes [1^3]_f$	singlet (A)	1	$\frac{1}{2}$	$\frac{1}{2}^-, \frac{3}{2}^-$
$[21]_{os} \otimes [21]_f$	octet (M)	1	$\frac{1}{2}$	$\frac{1}{2}^-, \frac{3}{2}^-$

Table 3.4: Multiplets for 1 quanta.

3.4 Prediction of baryons for given flavour, J and parity for 0 and 1 quanta

Now that we have the multiplets for every value of the total angular momentum and parity with their corresponding symmetry for 0 and 1 quanta, we can identify the baryons that sit in these multiplets attending to certain assumptions and observations, following the list of baryons from the Particle Data Group [6] (with this model we will only predict baryons with up to $J = \frac{5}{2}$):

- We know how many baryons are in each multiplet and their quark content. As stated at the beginning of chapter 3, each decuplet contains four deltas, three sigmas, two cascades and one omega and each octet contains two nucleons, three sigmas, two cascades and one lambda. Singlets can only contain lambda particles.
- Since each multiplet is defined by its total angular momentum and parity, for a given multiplet we will only look for particles with the corresponding properties.

- Knowing the quark content of each baryon means that we can determine their strangeness and electric charge. This is important because these two properties change between particles of a given multiplet, so knowing them allows us to place particles in their corresponding position in a multiplet.

- Particles of like charge lie along the downward-sloping diagonal lines of the multiplets. Particles with the same strangeness lie in the horizontal lines [3].

- The strange quark adds about 150 MeV of mass to the three-quark system. This means that if we identify a specific baryon with a known amount of strange quarks (i.e. with a known strangess), baryons with one more strange quark are going to be approximately 150 MeV heavier.

- In general, baryons with a certain value of J^π corresponding to the $S = \frac{1}{2}$ multiplets are lighter than the baryons with the same value of J^π, but $S = \frac{3}{2}$.

Let us begin with the multiplets associated with 0 quanta. We have one decuplet with $J^\pi = \frac{3}{2}^+$ and one octet with $J^\pi = \frac{1}{2}^+$. A reasonable starting point would be to locate the proton and neutron. In the ground state, the nucleons have $J^\pi = \frac{1}{2}^+$. Nucleons are particles with 0 strangeness, so they have to sit on the top vertices of the octet. The neutron has no electric charge, while the proton has 1 unit of charge. Thus, the neutron is on the top-left corner and the proton on the top-right corner. The proton's mass is approximately $m_p = 938$ MeV (rounded to the unit) and the neutron is a bit heavier, with $m_n = 940$ MeV.

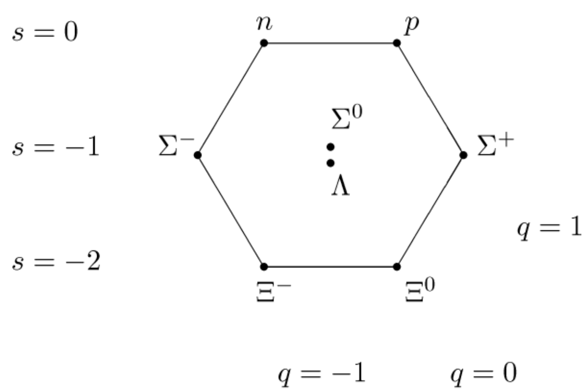


Fig. 3.1: Baryon octet for $J^\pi = \frac{1}{2}^+$.

With one strange quark (or -1 strangeness) we have the sigma and lambda baryons.

There are three sigmas: the Σ^+, which is a composite of *uus* quarks; the Σ^0, with *uds* combinations; and the Σ^-, with *dds*. These baryons along with the lambda with $J^\pi = \frac{1}{2}^+$ are in the middle of the octet. In terms of the masses, the Σ^- is a little heavier than the Σ^0 which is in turn a bit heavier than the Σ^+. This is because the d quark is heavier than the u quark. The Σ^+'s mass is around 1189 MeV, which is approximately 249 MeV more than the neutron.

Lastly, the cascades, with -2 strangeness, sit on the lower corners of the octet. There are two cascades (also known as the Xi baryons): the Ξ^0 which is a combination of *uss* quarks; and the Ξ^-, where the u quark of the Ξ^0 baryon is replaced by a d quark.[6] The Ξ^0 is the lighter of the two, with a mass equal to 1315 MeV, 126 MeV more than the Σ^+.

For the decuplet, we start by identifying the deltas since they are baryons with 0 strangeness. There are four delta particles that sit in the top side of the triangle: Δ^-, Δ^0, Δ^+ and Δ^{++}, and their quark content is *ddd*, *udd*, *uud* and *uuu* respectively. Their mass is roughly 1232 MeV.

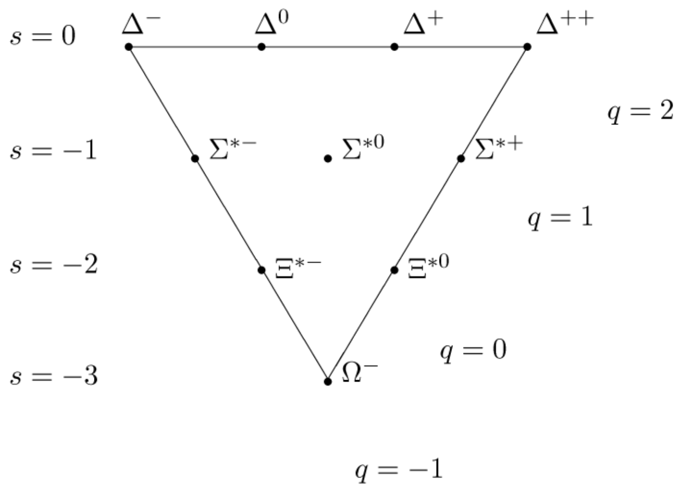


Fig. 3.2: Baryon decuplet for $J^\pi = \frac{3}{2}^+$.

After the delta baryons, there are three sigmas as in the case of the octet. The three sigmas compatible with $J^\pi = \frac{3}{2}^+$ are denoted as $\Sigma(1385)$, where the number in parenthesis indicates the approximate mass of the family[7] of baryons. This notation is used for baryons that decay with the strong interaction.

[6]Notice the nomenclature. We call "Xi" particles to baryons whose quark content is xss, where x is either an up or a down quark. This is the same for the other baryons.

[7]"Family" here refers to the three sigma baryons, Σ^-, Σ^0 and Σ^+. We refer to all the baryons of the same type just by writing the name—Σ in this case—without specifying the electric charge.

Next we have the two cascades and, with 3 strange quarks, the omega baryon. The Xi baryons compatible with the decuplet with $J^\pi = \frac{3}{2}^+$ are $\Xi(1530)$, while the omega baryon is just Ω^-.

The procedure to identify the baryons for the multiplets with 1 quanta given in Table 3.4 is the same. We start by identifying the lightest baryons—the delta baryons in the case of the decuplets and the nucleons in the case of the octets—and then look for baryons with decreasing strangeness (or increasing mass) compatible with the total angular momentum and parity of the multiplet. In Table 3.5 a tentative assignment of the observed baryons to the multiplets is done.

Multiplet	S	J^π	Identified baryons
decuplet	$\frac{3}{2}$	$\frac{3}{2}^+$	Δ^-, Δ^0, Δ^+, Δ^{++}, Σ^{*-}, Σ^{*0}, Σ^{*+}, Ξ^{*-}, Ξ^{*0}, Ω^-
octet	$\frac{1}{2}$	$\frac{1}{2}^+$	n, p, Σ^-, Σ^+, Σ^0, Λ, Ξ^-, Ξ^0
decuplet	$\frac{1}{2}$	$\frac{1}{2}^-$	$\Delta(1620)$, $\Sigma(1750)$
decuplet	$\frac{1}{2}$	$\frac{3}{2}^-$	$\Delta(1700)$, $\Sigma(1910)$
singlet	$\frac{1}{2}$	$\frac{1}{2}^-$	$\Lambda(1405)$
singlet	$\frac{1}{2}$	$\frac{3}{2}^-$	$\Lambda(1520)$
octet	$\frac{1}{2}$	$\frac{1}{2}^-$	$N(1535)$, $\Sigma(1620)$, $\Lambda(1670)$
octet	$\frac{1}{2}$	$\frac{3}{2}^-$	$N(1520)$, $\Sigma(1670)$, $\Lambda(1690)$, $\Xi(1820)$
octet	$\frac{3}{2}$	$\frac{1}{2}^-$	$N(1650)$, $\Sigma(1900)$, $\Lambda(1800)$
octet	$\frac{3}{2}$	$\frac{3}{2}^-$	$N(1700)$, $\Sigma(1910)$, $\Lambda(2050)$
octet	$\frac{3}{2}$	$\frac{5}{2}^-$	$N(1675)$, $\Sigma(1775)$, $\Lambda(1830)$

Table 3.5: Identified baryons for 0 (first two rows) and 1 quanta.

Notice that some multiplets are incomplete. For example, the cascades and omega particles for the decuplets with negative parity are missing. This is because baryons with higher masses are more difficult to obtain experimentally. Nonetheless, if our model is correct then this would be a sign that there are baryons that fit in these multiplets and have not been discovered yet.

In Table 3.6 baryons found experimentally and composed of quarks u, d and s are listed. Baryons identified with the model are coloured in blue.

p	$\frac{1}{2}^+$	$\Delta(1232)$	$\frac{3}{2}^+$	Λ	$\frac{1}{2}^+$	Σ^+	$\frac{1}{2}^+$	Ξ^0	$\frac{1}{2}^+$
n	$\frac{1}{2}^+$	$\Delta(1600)$	$\frac{3}{2}^+$	Λ	$\frac{1}{2}^-$	Σ^0	$\frac{1}{2}^+$	Ξ^-	$\frac{1}{2}^+$
$N(1440)$	$\frac{1}{2}^+$	$\Delta(1620)$	$\frac{1}{2}^-$	$\Lambda(1405)$	$\frac{1}{2}^-$	Σ^-	$\frac{1}{2}^+$	$\Xi(1530)$	$\frac{3}{2}^+$
$N(1520)$	$\frac{3}{2}^-$	$\Delta(1700)$	$\frac{3}{2}^-$	$\Lambda(1520)$	$\frac{3}{2}^-$	$\Sigma(1385)$	$\frac{3}{2}^+$	$\Xi(1620)$	
$N(1535)$	$\frac{1}{2}^-$	$\Delta(1750)$	$\frac{1}{2}^+$	$\Lambda(1600)$	$\frac{1}{2}^+$	$\Sigma(1580)$	$\frac{3}{2}^-$	$\Xi(1690)$	
$N(1650)$	$\frac{1}{2}^-$	$\Delta(1900)$	$\frac{1}{2}^-$	$\Lambda(1670)$	$\frac{1}{2}^-$	$\Sigma(1620)$	$\frac{1}{2}^-$	$\Xi(1820)$	$\frac{3}{2}^-$
$N(1675)$	$\frac{5}{2}^-$	$\Delta(1905)$	$\frac{5}{2}^+$	$\Lambda(1690)$	$\frac{3}{2}^-$	$\Sigma(1660)$	$\frac{1}{2}^+$	$\Xi(1950)$	
$N(1680)$	$\frac{5}{2}^+$	$\Delta(1910)$	$\frac{1}{2}^+$	$\Lambda(1710)$	$\frac{1}{2}^+$	$\Sigma(1670)$	$\frac{3}{2}^-$	$\Xi(2030)$	$\geqslant \frac{5}{2}?$
$N(1700)$	$\frac{3}{2}^-$	$\Delta(1920)$	$\frac{3}{2}^+$	$\Lambda(1800)$	$\frac{1}{2}^-$	$\Sigma(1750)$	$\frac{1}{2}^-$	$\Xi(2120)$	
$N(1710)$	$\frac{1}{2}^+$	$\Delta(1930)$	$\frac{5}{2}^-$	$\Lambda(1810)$	$\frac{1}{2}^+$	$\Sigma(1775)$	$\frac{5}{2}^-$	$\Xi(2250)$	
$N(1720)$	$\frac{3}{2}^+$	$\Delta(1940)$	$\frac{3}{2}^-$	$\Lambda(1820)$	$\frac{5}{2}^+$	$\Sigma(1780)$	$\frac{3}{2}^+$	$\Xi(2370)$	
$N(1860)$	$\frac{5}{2}^+$	$\Delta(1950)$	$\frac{7}{2}^+$	$\Lambda(1830)$	$\frac{5}{2}^-$	$\Sigma(1880)$	$\frac{1}{2}^+$	$\Xi(2500)$	
$N(1875)$	$\frac{3}{2}^-$	$\Delta(2000)$	$\frac{5}{2}^+$	$\Lambda(1890)$	$\frac{3}{2}^+$	$\Sigma(1900)$	$\frac{1}{2}^-$		
$N(1880)$	$\frac{1}{2}^+$	$\Delta(2150)$	$\frac{1}{2}^-$	$\Lambda(2000)$	$\frac{1}{2}^-$	$\Sigma(1910)$	$\frac{3}{2}^-$	Ω^-	$\frac{3}{2}^+$
$N(1895)$	$\frac{1}{2}^-$	$\Delta(2200)$	$\frac{7}{2}^-$	$\Lambda(2050)$	$\frac{3}{2}^-$	$\Sigma(1915)$	$\frac{5}{2}^+$	$\Omega(2012)^-$	$?^-$
$N(1900)$	$\frac{3}{2}^+$	$\Delta(2300)$	$\frac{9}{2}^+$	$\Lambda(2070)$	$\frac{3}{2}^+$	$\Sigma(1940)$	$\frac{3}{2}^+$	$\Omega(2250)^-$	
$N(1990)$	$\frac{7}{2}^+$	$\Delta(2350)$	$\frac{5}{2}^-$	$\Lambda(2080)$	$\frac{5}{2}^-$	$\Sigma(2010)$	$\frac{3}{2}^-$	$\Omega(2380)^-$	
$N(2000)$	$\frac{5}{2}^+$	$\Delta(2390)$	$\frac{7}{2}^+$	$\Lambda(2085)$	$\frac{7}{2}^+$	$\Sigma(2030)$	$\frac{7}{2}^+$	$\Omega(2470)^-$	
$N(2040)$	$\frac{3}{2}^+$	$\Delta(2400)$	$\frac{9}{2}^-$	$\Lambda(2100)$	$\frac{7}{2}^-$	$\Sigma(2070)$	$\frac{5}{2}^+$		
$N(2060)$	$\frac{5}{2}^-$	$\Delta(2420)$	$\frac{11}{2}^+$	$\Lambda(2110)$	$\frac{5}{2}^+$	$\Sigma(2080)$	$\frac{3}{2}^+$		
$N(2100)$	$\frac{1}{2}^+$	$\Delta(2750)$	$\frac{13}{2}^-$	$\Lambda(2325)$	$\frac{3}{2}^-$	$\Sigma(2100)$	$\frac{7}{2}^-$		
$N(2120)$	$\frac{3}{2}^-$	$\Delta(2950)$	$\frac{15}{2}^+$	$\Lambda(2350)$	$\frac{9}{2}^+$	$\Sigma(2160)$	$\frac{1}{2}^-$		
$N(2190)$	$\frac{7}{2}^-$			$\Lambda(2585)$		$\Sigma(2230)$	$\frac{3}{2}^+$		
$N(2220)$	$\frac{9}{2}^+$					$\Sigma(2250)$			
$N(2250)$	$\frac{9}{2}^-$					$\Sigma(2455)$			
$N(2300)$	$\frac{1}{2}^+$					$\Sigma(2620)$			
$N(2570)$	$\frac{5}{2}^-$					$\Sigma(3000)$			
$N(2600)$	$\frac{11}{2}^-$					$\Sigma(3170)$			
$N(2700)$	$\frac{13}{2}^+$								

Table 3.6: List of experimentally measured baryons with their corresponding J and parity. From left to right: nucleons, deltas, lambdas, sigmas, cascades and omegas.

Chapter 4

Conclusions and further work

In chapter 3 the states for the system of three quarks were obtained attending to the different variables: spin, flavour, colour and spatial. Each degree of freedom gave some information about the three-quark system within the baryons and the symmetry properties of the states, which was crucial in the path of obtaining the multiplets associated with 0 and 1 quanta and identifying baryons compatible with these multiplets. The baryons with positive parity ($L = 0$) were explicitly shown in their corresponding multiplets, while the remaining baryons with negative parity were presented in Table 3.5.

It is safe to say that the multiplets derived with the help of group theory and the permutation group S_3 correctly predict the existence of baryons compatible with the given values of spin and parity. There are, however, baryons that are not currently present in some multiplets because of the lack of experimental evidence. Nonetheless, we can make an educated guess following the reasoning of the mass of the strange quark and predict the existence of the missing baryons. In this manner, one can suggest the existence of, say, a strong decaying Xi baryon with $J^\pi = \frac{1}{2}^-$, a mass of approximately 2000 MeV and a completely symmetric flavour wavefunction.

There are also many more things that can be done following the results in this project. For example, we can evaluate the constituent mass of the u, d and s quarks for each multiplet, or the spin-orbit interaction between quarks comparing states with $J = \frac{1}{2}$ and $J = \frac{3}{2}$, all of this with very good results compared with the experimental values.

However, any model in physics has limitations and this is no exception. The model described in this project assumes that the $SU(3)_F$ symmetry is valid. $SU(3)$ is the special unitary group, a group consisting of unitary matrices with determinant 1 [5].

The subscript F stands for flavour. The $SU(3)_F$ group describes the symmetry related to the mass of the different flavours of the quark stating that the u, d and s quarks have approximately the same mass. If that is the case, then baryons would appear in decuplets, octets and singlets as we have seen. But this is not the case since we know that the strange quark is considerably heavier than quarks u and d, hence breaking the $SU(3)_F$ symmetry. This means that baryons found in nature could be, in general, a mixture of states belonging to the decuplet, octet and singlet. This mixture would not occur for nucleons since they only appear in octets as well as delta and omega particles that only appear in decuplets, but it could happen in the case of lambdas, sigmas and the cascades. This mixture of multiplets already occurs for mesons such as the η and η', which are mixtures of octet and singlet states.

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