Transportation inequalities for coupled systems of stochastic delay evolution equations with a fractional Brownian motion

Abstract

We prove an existence and uniqueness result of mild solution for a system of stochastic semilinear differential equations with fractional Brownian motions and Hurst parameter $H < 1/2$. Our approach is based on Perov's fixed point theorem, and we establish the transportation inequalities, with respect to the uniform distance, for the law of the mild solution.

Keywords and phrases: Transportation inequality, Girsanov transformation, generalized Banach space, fixed point, fractional Brownian motion.
AMS (MOS) Subject Classifications: 60H15, 60G22.

1 Introduction

The existence and uniqueness of solutions of stochastic differential equations have been significantly studied by many researchers (for instance, see [1–6] just to mention a few). Stochastic differential equations are used as models in many different applications from the real world. This is due to a combination of uncertainties, complexities, and ignorance on our part which inevitably cloud our mathematical modeling process [7,8]. This interest is due to the fact that there are many applications of this theory to various applied fields such as problems arising in mechanics, medicine and biology, economics, electronics and telecommunication etc. For a discussion of such applications, one may refer to [2,9].

During the past few decades, the research of coupled systems has received considerable interest, since they have come to play an important role in mechanics, electrical engineering, and biological systems (see [10–12] and references therein).

Some phenomena can be better described by coupled systems. For example, in epidemiology, the migration of migratory birds from all over the world may bring some infectious diseases, then the transmission rate of infectious diseases will increase with a sea of migratory birds migrating. Furthermore, considering the existence of random disturbance and time delays, the investigation of stochastic systems of delay evolution equations with a fractional Brownian motion is of great significance and worthy to study further.

It is known that many different arguments have been developed to establish the transportation inequalities. Among others, the Girsanov transformation argument introduced in [13] has been efficiently applied, see, e.g., [14] for infinite-dimensional dynamical systems, [15] for time-inhomogeneous diffusions, [16] for multi-valued SDEs and singular SDEs, [17] for neutral functional SDEs.

Recently, Saussereau [18] established Talagrand's $T_1(C)$ and $T_2(C)$ inequalities for the law of the solution of a stochastic differential equation driven by a fractional Brownian motion, Li and Luo [19] proved the quadratic transportation inequalities for the law of the mild solution of stochastic functional partial differential equations and neutral partial differential equations of retarded type driven by fractional Brownian motion with Hurst parameter $H > 1/2$, while to the best of our knowledge, there is no paper dealing with the existence of solution and the property $T_2(C)$ for coupled systems of stochastic evolution equations driven by a fractional Brownian motion with $H < 1/2$. The existence and the transportation inequalities for the law of the mild solution of neutral stochastic differential equation with bounded variable delay driven by a fractional Brownian motion with Hurst parameter $H < 1/2$ has been examined by Boufoussi and Hajji [20]. Following this line, in this paper we study the existence and uniqueness of solutions, and we investigate the property $T_2(C)$ for the law of the mild solution of the following coupled stochastic functional equations with finite delay driven by a fractional Brownian motion with $H < 1/2$:

$$\begin{cases} dx(t) &= (A_1x(t) + f_1(t, x_t, y_t))dt + \sigma_1(t)dB^H(t), \ t \in J = [0, T], \\ dy(t) &= (A_2y(t) + f_2(t, x_t, y_t))dt + \sigma_2(t)dB^H(t), \ t \in J, \\ x(t) &= \phi_1(t), \quad t \in J_0 = [-r, 0], \\ y(t) &= \phi_2(t), \quad t \in J_0. \end{cases} \tag{1.1}$$

The states $x(\cdot), y(\cdot)$ take values in a real separable Hilbert space $\mathcal{U}$ with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, where $\{A_i, \ i = 1, 2\}$ are the infinitesimal generators of analytic semigroups of bounded linear operators $\{S_i(t), t \geq 0\}$, B^H is a fractional Brownian motion on a real and separable Hilbert space $\mathcal{K}$, with Hurst parameter $H < 1/2$, and with respect to a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ furnished with a family of continuous and increasing $\sigma-$algebras $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Fix $T > 0$ and let $\mathbb{Q}$ be another probability measure on $\mathcal{F}_T$. We say that $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}|_{\mathcal{F}_T}$ (the restriction of $\mathbb{P}$ to $\mathcal{F}_T$) and write $\mathbb{Q} \ll \mathbb{P}$ if

$$\mathbb{P}(A) = 0 \Longrightarrow \mathbb{Q}(A) = 0 \text{ for all } A \in \mathcal{F}_T.$$

Also $r > 0$ is the maximum delay. As for x_t, y_t we mean the segment solution which is defined in the usual way, that is, if $x(\cdot, \cdot) : [-r, T] \times \Omega \to \mathcal{U}$ and $y(\cdot, \cdot) : [-r, T] \times \Omega \to \mathcal{U}$, then for any $t \geq 0$, $x_t(\cdot, \cdot) : [-r, 0] \times \Omega \to \mathcal{U}$ is given by

$$x_t(\theta, \omega) = x(t + \theta, \omega), \ \text{ for } \theta \in [-r, 0], \ \omega \in \Omega.$$

Before describing the properties fulfilled by operators f_i, σ_i , we need to introduce some notation and describe some spaces.

We define $\mathcal{D}_0$ as the space of all continuous processes $\varphi : [-r, 0] \times \Omega \to \mathcal{U}$ such that $\varphi(\theta, \cdot)$ is $\mathcal{F}_0$-measurable for each $\theta \in [-r, 0]$ and $\sup\limits_{\theta \in [-r,0]} \mathbb{E}|\varphi(\theta)|^2 < \infty$. In the space $\mathcal{D}_0$, we consider the norm:

$$||\varphi||^2_{\mathcal{D}_0} = \sup_{\theta \in [-r,0]} \mathbb{E}|\varphi(\theta)|^2.$$

Next, we denote by $C(a,b;L^2(\Omega;\mathcal{U})) = C(a,b;L^2(\Omega,\mathcal{F},\mathbb{P};\mathcal{U}))$ the Banach space of all continuous functions from $[a,b]$ into $L^2(\Omega;\mathcal{U})$. Now, for a given $T>0$, and for given initial data $(\phi_1,\phi_2)\in\mathcal{D}_0\times\mathcal{D}_0$ for our problem, we define, for $i=1,2$,

$$\mathcal{D}_T^i = \{z\in C(-r,T;L^2(\Omega;\mathcal{U})) \text{ with } z(t)=\phi_i(t), t\in[-r,0] \text{ and } \sup_{[0,T]}\mathbb{E}(|z(t)|^2)<\infty\},$$

with the metric induced by the norm

$$\|z\|_{\mathcal{D}_T^i} = \sup_{t\in[-r,T]}\sqrt{(\mathbb{E}(|z(t)|^2)} \le \sup_{t\in[0,T]}\sqrt{(\mathbb{E}(|z(t)|^2)} + ||\phi_i(\cdot)||_{\mathcal{D}_0},$$

which ensures that $\mathcal{D}_T^i$ is a complete metric space.

Together with our initial data $(\phi_1,\phi_2)\in\mathcal{D}_0\times\mathcal{D}_0$, we will consider another real separable Hilbert space $\mathcal{K}$ and suppose that $B_Q^H = B^H$ is a $\mathcal{K}$-valued fractional Brownian motion with increment covariance given by a nonnegative trace class operator Q (see next section for more details), and let us denote by $L(\mathcal{K},\mathcal{U})$ the space of all bounded, continuous and linear operators from $\mathcal{K}$ into $\mathcal{U}$.
Assume $f_i : J\times\mathcal{D}_0\times\mathcal{D}_0 \longrightarrow \mathcal{U}$ and $\sigma_i : J\to L_Q^0(\mathcal{K},\mathcal{U})$. Here, $L_Q^0(\mathcal{K},\mathcal{U})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{U}$, which will be also defined in the next section.

Let us now consider the kinds of inequalities we will deal with. To measure distances between probability measures, we use the transportation distance, also called Wasserstein distance. Let (E,d) be a metric space equipped with the σ-field $\mathcal{B}$, such that $d(\cdot,\cdot)$ is $\mathcal{B}\otimes\mathcal{B}$-measurable. Given $p\ge 1$ and two probability measures μ and ν on E, we define the Wasserstein distance of order p between μ and ν by

$$W_p^d(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)}\left(\int_{E\times E} d(x,y)^p d\pi(x,y)\right)^{\frac{p}{2}},$$

where $\Pi(\mu,\nu)$ denotes the totality of probability measures on $E\times E$ with the marginal μ and ν. The relative entropy of ν with respect to μ is defined as

$$\mathrm{H}(\nu|\mu) = \begin{cases} \displaystyle\int \log\frac{d\nu}{d\mu}d\nu, & \nu\ll\mu \\ +\infty & \text{otherwise.}\end{cases}$$

The probability measure μ satisfies the L^p-transportation inequality on (E,d) if there exists a constant $C\ge 0$ such that for any probability measure ν,

$$W_p^d(\mu,\nu) \le \sqrt{2C\mathrm{H}(\nu|\mu)}.$$

As usual, we write $\mu\in T_p(C)$ for this relation. The properties $T_2(C)$ are of particular interest. We will investigate the properties $T_2(C)$ for the law of mild solutions to stochastic delay evolution equations driven by fractional Brownian motion with Hurst parameter $H < 1/2$ under the L^2 metric and the uniform one as well.

The aim of this paper is to study the existence and the properties $T_2(C)$ of mild solutions of semilinear systems of stochastic differential equations with fractional Brownian motion. The content is organized as follows. In Section 2, we introduce all the background material used in this paper such as stochastic calculus and some properties of generalized Banach spaces. In Section 3, we state and prove our main results by using Perov's fixed point type theorem in generalized Banach spaces. In Section 4, we investigate the properties $T_2(C)$ for law of the solution of stochastic delay evolution equations driven by fractional Brownian motion with Hurst parameter $H < 1/2$ under the L^2 metric and the uniform metric. Finally, we present an example to illustrate the efficiency of the obtained result in Section 5.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which will be used throughout this paper. In particular, we consider fractional Brownian motion as well as the Wiener integral with respect to it. We also establish some important results which will be needed throughout the paper.

Definition 2.1. *Given $H \in (0,1)$, a continuous centered Gaussian process $\beta^H = \{\beta^H(t), t \in \mathbb{R}\}$, with the covariance function*

$$R_H(t,s) = E[\beta^H(t)\beta^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), t, s \in \mathbb{R}$$

is called a two−sided one−dimensional fractional Brownian motion, and H is the Hurst parameter.

Moreover β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s)d\beta(s), \tag{2.1}$$

where $\beta = \{\beta(t) : t \in [0,T]\}$ is a Wiener process, and $K_H(t,s)$ is a square integrable kernel given by (see [21])

$$\begin{aligned} K_H(t,s) &= c_H\Bigg[\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} \\ &\quad -(H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\Bigg], \end{aligned} \tag{2.2}$$

for $H < \frac{1}{2}$ and $t > s$, where $c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H,H+\frac{1}{2})}}$ and $\beta(\cdot,\cdot)$ is the Beta function (we will use this notation for the beta function since no confusion is possible with that of Brownian motion).
We set $K_H(t,s) = 0$ if $t \leq s$. And from (2.2), it follows that:

$$|K_H(t,s)| \leq 2c_H\left((t-s)^{H-1/2} + s^{H-1/2}\right). \tag{2.3}$$

In the sequel, we will use the following inequality:

$$\left|\frac{\partial K_H}{\partial t}(t,s)\right| \leq c_H\left(\frac{1}{2} - H\right)(t-s)^{H-3/2}. \tag{2.4}$$

Let us consider the operator $K_{H,T}^*$ from $\mathcal{U}$ to $L^2([0,T])$ defined by

$$(K_H^*\varphi)(s) = K_H(T,s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s))\frac{\partial K_H}{\partial r}(r,s)dr. \tag{2.5}$$

We refer to [21] for the proof of the fact that $K_{H,T}^*$ is an isometry between $\mathcal{U}$ and $L^2([0,T])$. Moreover for any $\varphi \in \mathcal{U}$, we have

$$\int_0^T \varphi(s)d\beta^H(s) := \beta^H(\varphi) = \int_0^T (K_{H,T}^*\varphi)(t)d\beta(t).$$

We also have for $0 \leq t \leq T$

$$\int_0^t \varphi(s)d\beta^H(s) := \int_0^T (K_{H,T}^*\varphi\chi_{[0,t]})(s)d\beta(s) = \int_0^t (K_{H,t}^*\varphi)(s)d\beta(s),$$

where $K_{H,t}^*$ is defined in the same way as in (2.5) with t instead of T. In the next, we will use the notation K_H^* without specifying the parameter $t \in [0,T]$.

Let $Q \in L(\mathcal{K},\mathcal{U})$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty}\lambda_n < \infty$, where $\lambda_n \geq 0$ $(n = 1,2,\cdots)$ are non-negative real numbers and $\{e_n\}$ $(n = 1,2,\cdots)$ is a complete orthonormal basis in $\mathcal{K}$. We define the infinite-dimensional fBm on $\mathcal{K}$ with covariance Q as follows:

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n^H(t), \quad t \geq 0,$$

where β_n^H are real, independent fBms.

To define Wiener integrals with respect to the Q-fBm, we introduce the space $L_Q^0 := L_Q^0(\mathcal{K},\mathcal{U})$ of all Q-Hilbert–Schmidt operators $\varphi : \mathcal{K} \longrightarrow \mathcal{U}$. We recall that $\varphi \in L(\mathcal{K},\mathcal{U})$ is called a Q-Hilbert–Schmidt operator, if

$$\|\varphi\|_{L_Q^0}^2 := \sum_{n=1}^{\infty}\|\sqrt{\lambda_n}\varphi e_n\|^2 < \infty.$$

Now, let $\sigma_i(\cdot)$, $s \in [0,T]$, be a function with values in L_Q^0. The Wiener integral of σ_i with respect to B^H is defined by the following:

$$\begin{aligned}
\int_0^t \sigma_i(s) dB^H(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \sigma_i(s) e_n d\beta_n^H(s) \\
&= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} (K_H^*(\sigma_i e_n))(s) d\beta_n(s), \qquad (2.6)
\end{aligned}$$

where β_n is the standard Brownian motion used to represent β_n^H as in (2.1), and the above series is well defined when $\sum_{n=1}^{\infty} \lambda_n \|K_H^*(\sigma_i e_n)\|^2 < \infty$.

Definition 2.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable in $\mathbb{R}^2$ is a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathbb{R} \times \mathbb{R}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}))$. Notice that this mapping possesses the form of a pair of two real random variables X and Y,*

$$\begin{aligned}
(X,Y) \;:\; & \Omega \to \mathbb{R}^2 \\
& \omega \longmapsto (X(\omega), Y(\omega))
\end{aligned}$$

The joint law of X and Y is the measure of $\mathbb{P}$ by (X,Y), in other words, the measure $\mathbb{P}_{(X,Y)}$ on $\mathbb{R}^2$ defined by

$$\begin{aligned}
\mathbb{P}_{(X,Y)}(B_1 \times B_2) \quad &= \mathbb{P}((X,Y)^{-1}(B_1 \times B_2)) \\
&= \mathbb{P}((X)^{-1}(B_1) \cap (Y)^{-1}(B_2)) \\
&= \mathbb{P}(\{\omega \in \Omega | X(\omega) \in B_1 \text{ and } Y(\omega) \in B_2\})
\end{aligned}$$

holds for all B_1, B_2 in $\mathbb{R}$. We notice

$$\mathbb{P}(X \in B_1 \, and \, Y \in B_2) = \mathbb{P}_{(X,Y)}(B_1 \times B_1)$$

We call marginal laws of (X,Y) the laws of X and Y which are the measures images $\mathbb{P}_X$ and $\mathbb{P}_Y$ of $\mathbb{P}_{(X,Y)}$ by canonical projections.

Definition 2.3. *The real-valued random variables X and Y are said to be independent if for all borelians B_1 and B_2 of $\mathbb{P}$, we have*

$$\mathbb{P}(X \in B_1 \, and \, Y \in B_2) = \mathbb{P}(X \in B_1)\mathbb{P}(Y \in B_2)$$

This is equivalent to say that the joint law $\mathbb{P}_{(X,Y)}$ is the measure produced by the marginal laws

$$\mathbb{P}_{(X,Y)} = \mathbb{P}_X \otimes \mathbb{P}_Y.$$

We see that stochastic independence can be reinterpreted as a rule to compute the joint distribution of two random variables from their marginal distribution. More precisely, their joint distribution can be computed as a product of their marginal distributions. This product is associative and can also be iterated to compute the joint distribution of more than two independent random variables.

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metric space by Perov [22] in 1964, Perov and Kibenko [23] and Precup [24]. Let us recall now some useful definitions and results.

Definition 2.4. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call the pair (X, d) a generalized metric space. For $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, we denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces. If $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$.

Definition 2.5. *A generalized metric space (X, d), where $d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_n(x, y) \end{pmatrix}$. is complete if for every $i = 1, \ldots, n$, (X, d_i) is a complete metric space.*

Definition 2.6. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc. (i.e.$|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$).*

Definition 2.7. *We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the absolute value property if*

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Some examples of matrices convergent to zero

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$;

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max(a, b) < 1$;

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1$.

We can recall now a fixed point theorem in a complete generalized metric space.

Theorem 2.1. *[22]Let (X, d) be a complete generalized metric space with $d : X \times X \longrightarrow \mathbb{R}^n$ and let $N : X \longrightarrow X$ be such that*

$$d(N(x), N(y)) \leq Md(x, y)$$

for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \longrightarrow 0$ as $k \longrightarrow \infty$, then N has a unique fixed point $x_* \in X$

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(N(x_0), x_0)$$

for every $x_0 \in X$ and $k \geq 1$.

3 Existence and Uniqueness of a Solution

In this section, we study the existence and uniqueness of a mild solution for (1.1). First, we will list the following hypotheses which will be imposed in our main theorem. For this equation, we assume that the following conditions hold.

(H_1) A_i is the infinitesimal generator of an analytic semigroup of bounded linear operators $S_i(t)$, $t \geq 0$ and there exists a constant M such that $\{\|S_i(t)\|^2 \leq M\}$ for all $t \geq 0$.

(H_2) There exist constants a_{f_i}, $b_{f_i} \in \mathbb{R}^+$ for each $i = 1, 2$ such that

$$\int_0^t \|f^i(s, x_s, y_s) - f^i(s, \overline{x}_s, \overline{y}_s)\|^2_{\mathcal{U}} ds \leq a_{f_i} \int_{-r}^t \|x(s) - \overline{x}(s)\|^2_{\mathcal{U}} ds + b_{f_i} \int_{-r}^t \|y(s) - \overline{y}(s)\|^2_{\mathcal{U}} ds,$$

for all x, y, $\overline{x}$, $\overline{y} \in C(-r, T; \mathcal{U})$.

(H_3) The function $\sigma_i : J \to L^0_Q(\mathcal{K}, \mathcal{U})$ satisfies

$$\int_0^T \|\sigma_i(s)\|^2_{L^0_Q} ds < \infty, \quad i = 1, 2. \tag{3.1}$$

Now, we state the following definition of mild solution to our problem.

Definition 3.1. *A $\mathcal{U}$–valued process $u(t) = (x(t), y(t))$ is called a mild solution of (1.1) with respect to the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, if $(x, y) \in C(-r, T; L^2(\Omega_1; \mathcal{U})) \times C(-r, T; L^2(\Omega_2; \mathcal{U}))$, $(x(t), y(t)) = (\phi_1(t), \phi_2(t))$ for $t \in [-r, 0]$, and, for each $t \in [0, T]$, $u(t)$ satisfies the following integral equation:*

$$\begin{cases} x(t) = S_1(t)\phi_1(0) + \int_0^t S_1(t-s) f_1(s, x_s, y_s) ds \\ \qquad + \int_0^t S_1(t-s)\sigma_1(s) dB^H(s), \quad \mathbb{P}-a.s, \quad t \in J, \\ \\ y(t) = S_2(t)\phi_2(0) + \int_0^t S_2(t-s) f_2(s, x_s, y_s) ds \\ \qquad + \int_0^t S_2(t-s)\sigma_2(s) dB^H(s), \quad \mathbb{P}-a.s, \quad t \in J. \end{cases} \tag{3.2}$$

The following lemma proves that the stochastic integrals in (3.2) are well defined.

Lemma 3.1. *Under assumptions (H_1), (H_3) on A_i and σ_i, the stochastic integrals in (3.2) are well defined and satisfy:*

$$\mathbb{E}\left\| \int_0^t S_i(t-s)\sigma_i(s) dB^H(s) \right\|^2 \leq \widetilde{C} t^{2H}, \quad t > 0$$

where $\widetilde{C}$ is a positive constant depending on H, σ_i, and M.

Proof. By (2.5) and (2.6), we have

$$\begin{aligned} \mathbb{E}\left\| \int_0^t S_i(t-s)\sigma_i(s) dB^H(s) \right\|^2 &= \sum_{n=1}^{\infty} \mathbb{E}\left\| \int_0^t \sqrt{\lambda_n}(K_H^*(S_i(t-s)\sigma_i(s)e_n)\beta_n(s) \right\|^2 ds \\ &= \sum_{n=1}^{\infty} \lambda_n \int_0^t \|(K_H^*(S_i(t-s)\sigma_i(s)e_n\|^2 ds \\ &\leq 2\sum_{n=1}^{\infty} \lambda_n \int_0^t \|K_H(t,s) S_i(t-s)\sigma_i(s)e_n\|^2 ds \\ &\quad +4\sum_{n=1}^{\infty} \lambda_n \int_0^t \left\| \int_s^t S(t-r)\sigma_i(r)e_n \frac{\partial K_H}{\partial r}(r,s) dr \right\|^2 ds \\ &\quad +4\sum_{n=1}^{\infty} \lambda_n \int_0^t \left\| \int_s^t S_i(t-s)\sigma_i(s)e_n \frac{\partial K_H}{\partial r}(r,s) dr \right\|^2 ds \\ &\leq I_1 + I_2 + I_3. \end{aligned} \tag{3.3}$$

We estimate the various terms of the right-hand side of (3.3) separately. For the first term, we have by applying inequality (2.3):

$$
\begin{aligned}
I_1 &= 2\sum_{n=1}^{\infty}\lambda_n\int_0^t \|K(t,s)S_i(t-s)\sigma_i(s)e_n\|^2 \\
&\leq 16Mc_H^2\sum_{n=1}^{\infty}\lambda_n\int_0^t((t-s)^{2H-1}+s^{2H-1})\|\sigma_i(s)e_n\|^2 ds. \\
&\leq 16Mc_H^2\int_0^t((t-s)^{2H-1}+s^{2H-1})\|\sigma_i(s)\|^2_{L_Q^0} ds. \\
&\leq 16Mc_H^2\widetilde{\sigma}_i\frac{t^{2H}}{H}. \qquad (3.4)
\end{aligned}
$$

where $\widetilde{\sigma}_i := \sup_{t\in[0,T]}\|\sigma_i(t)\|^2_{L^0_{Q_i}}$.

For the second term, we obtain by inequality (2.4):

$$
\begin{aligned}
I_2 &= 4\sum_{n=1}^{\infty}\lambda_n\int_0^t\left\|\int_s^t S(t-r)\sigma_i(r)e_n\frac{\partial K}{\partial r}(r,s)dr\right\|^2 ds \\
&\leq 4Mc_H^2\sum_{n=1}^{\infty}\lambda_n\int_0^t\left(\int_s^t\left(\left[\frac{1}{2}-H\right](r-s)^{H-\frac{3}{2}}\right)\|\sigma_i(r)e_n\|dr\right)^2 ds. \\
&\leq 4Mc_H^2\widetilde{\sigma}_i\int_0^t\left(\int_s^t(r-s)^{H-\frac{3}{2}}dr\right)^2 ds. \\
&\leq \frac{4Mc_H^2\widetilde{\sigma}_i}{\left(H-\frac{1}{2}\right)^2}\int_0^t(t-s)^{2H-1}ds. \\
&\leq \frac{2Mc_H^2\widetilde{\sigma}_i}{\left(H-\frac{1}{2}\right)^2}\frac{t^{2H}}{H}. \qquad (3.5)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_3 &= 4\sum_{n=1}^{\infty}\lambda_n\int_0^t\left\|\int_s^t S_i(t-s)\sigma_i(s)e_n\frac{\partial K}{\partial r}(r,s)dr\right\|^2 ds \\
&\leq \frac{2Mc_H^2\widetilde{\sigma}_i}{\left(H-\frac{1}{2}\right)^2}\frac{t^{2H}}{H}. \qquad (3.6)
\end{aligned}
$$

Inequalities (3.4), (3.5), (3.6) together imply the desired estimate. □

For our main consideration of problem (1.1), a Perov fixed point theorem is used to investigate the existence and uniqueness of mild solution for our system of stochastic differential equations.

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *are satisfied. If the matrix*

$$M_{trice} \ = \begin{pmatrix} \sqrt{Ma_{f_1}T^2} & \sqrt{Mb_{f_1}T^2} \\ \sqrt{Ma_{f_2}T^2} & \sqrt{Mb_{f_2}T^2} \end{pmatrix}$$

converges to zero, then problem (1.1) *has a unique solution.*

Proof. Let us consider operator $N : \mathcal{D}_T^1 \times \mathcal{D}_T^2 \to \mathcal{D}_T^1 \times \mathcal{D}_T^2$ defined by

$$N(x, y) = (N_1(x, y), N_2(x, y)), \ (x, y) \in \mathcal{D}_T^1 \times \mathcal{D}_T^2,$$

where

$$N_1(x, y) = \begin{cases} \phi_1(t), & t \in [-r, 0] \\ S_1(t)\phi_1(0) + \displaystyle\int_0^t S_1(t-s) f_1(s, x_s, y_s) ds \\ + \displaystyle\int_0^t S_1(t-s)\sigma_1(s) dB^H(s) \ , \mathbb{P} - a.s, \quad t \in J \end{cases}$$

and

$$N_2(x, y) = \begin{cases} \phi_2(t), & t \in [-r, 0] \\ S_2(t)\phi_2(0) + \displaystyle\int_0^t S_2(t-s) f_2(s, x_s, y_s) ds \\ + \displaystyle\int_0^t S_2(t-s)\sigma_2(s) dB^H(s) \ , \mathbb{P} - a.s, \quad t \in J \end{cases}$$

We shall use Theorem 2.1 to prove that N has a fixed point. Indeed, let $(x, y), (\overline{x}, \overline{y}) \in \mathcal{D}_T^1 \times \mathcal{D}_T^2$. Then we have for each $t \in [0, T]$

$$\begin{aligned} &\mathbb{E}|N_1(x(t), y(t)) - N_1(\overline{x}(t), \overline{y}(t))|^2 \\ &\quad \le \int_0^t \|S_1(t-s)\|^2 ds \mathbb{E} \int_0^t [f_1(s, x_s, y_s) - f_1(s, \overline{x}_s, \overline{y}_s)]^2 ds \\ &\quad \le tMa_{f_1} \int_0^t \mathbb{E}|x(s) - \overline{x}(s)|^2 ds + tMb_{f_1} \int_0^t \mathbb{E}|y(s) - \overline{y}(s)|^2 ds \\ &\quad \le tMa_{f_1} \int_0^t \sup_{\tau\in[0,s]} \mathbb{E}|x(\tau) - \overline{x}(\tau)|^2 ds + tMb_{f_1} \int_0^t \sup_{\tau\in[0,s]} \mathbb{E}|y(\tau) - \overline{y}(\tau)|^2 ds \end{aligned}$$

and therefore, since $(x, y) = (\overline{x}, \overline{y})$ over the interval $[-r, 0]$, by taking supremum in the above inequality,

$$\|N_1(x, y) - N_1(\overline{x}, \overline{y})\|^2_{\mathcal{D}_T^1} \le Ma_{f_1}T^2\|x - \overline{x}\|^2_{\mathcal{D}_T^1} + Mb_{f_1}T^2\|y - \overline{y}\|^2_{\mathcal{D}_T^2}.$$

Similarly we have

$$\begin{aligned} &\mathbb{E}|N_2(x(t), y(t)) - N_2(\overline{x}(t), \overline{y}(t))|^2 \\ &\le tMa_{f_2} \int_0^t \sup_{\tau\in[0,s]} \mathbb{E}|x(\tau) - \overline{x}(\tau)|^2 ds + tMb_{f_2} \int_0^t \sup_{\tau\in[0,s]} \mathbb{E}|y(\tau) - \overline{y}(\tau)|^2 ds. \end{aligned}$$

Therefore,

$$\|N_2(x,y)-N_2(\overline{x},\overline{y})\|^2_{\mathcal{D}^2_T} \le Ma_{f_2}T^2\|x-\overline{x}\|^2_{\mathcal{D}^1_T} + Mb_{f_2}T^2\|y-\overline{y}\|^2_{\mathcal{D}^2_T}.$$

Hence

$$\begin{aligned}
\|N(x,y)-N(\overline{x},\overline{y})\|_{\mathcal{D}_T} &= \begin{pmatrix} \|N_1((x,y)-N_1(\overline{x},\overline{y})\|_{\mathcal{D}^1_T} \\ \|N_2(x,y)-N_2(\overline{x},\overline{y})\|_{\mathcal{D}^2_T} \end{pmatrix} \\
&\le \begin{pmatrix} \sqrt{Ma_{f_1}T^2} & \sqrt{Mb_{f_1}T^2} \\ \sqrt{Ma_{f_2}T^2} & \sqrt{Mb_{f_2}T^2} \end{pmatrix} \begin{pmatrix} \|x-\overline{x}\|_{\mathcal{D}^1_T} \\ \|y-\overline{y}\|_{\mathcal{D}^2_T} \end{pmatrix}.
\end{aligned}$$

Therefore

$$\|N(x,y)-N(\overline{x},\overline{y})\|_{\mathcal{D}_T} \le M_{trice} \begin{pmatrix} \|x-\overline{x}\|_{\mathcal{D}^1_T} \\ \|y-\overline{y}\|_{\mathcal{D}^2_T} \end{pmatrix}, \text{ for all}, (x,y),(\overline{x},\overline{y}) \in \mathcal{D}^1_T \times \mathcal{D}^2_T.$$

From Perov's fixed point theorem, the mapping N has a unique fixed point $(x,y) \in \mathcal{D}^1_T \times \mathcal{D}^2_T$ which is the unique solution of problem (1.1). □

4 Transportation Inequalities

In this section, we study the property $T_2(C)$ for the law of the mild solution of problem (1.1) on the space $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2 = C([0,T],\mathcal{U}) \times C([0,T],\mathcal{U})$, endowed with the uniform metric d_∞. Precisely, we have the following theorem:

Theorem 4.1. *Assume that* (H_1) *and* (H_2) *hold, and let* $\mathbb{P}_{\phi_1} \otimes \mathbb{P}_{\phi_2}$ *be the law of* $(x(\phi_1,.), y(\phi_2,.))$ *which is the solution process of problem* (1.1). *Then the probability measure* $\mathbb{P}_{\phi_i}$ *satisfies* $T_2(C)$ *on the metric space* $C([0,T],\mathcal{U})$ *with the metric* d_∞ *given by*

$$d_\infty(\eta_1,\eta_2) = \sup_{t\in[0,T]} \|\eta_1(t)-\eta_2(t)\| \quad \eta_1,\eta_2 \in C([0,T],\mathcal{U})$$

Proof. Let $\mathbb{P}_\phi := \mathbb{P}_{\phi_1} \otimes \mathbb{P}_{\phi_2}$ be the law of $(x(t,\phi_1), y(t,\phi_2))$ on $\mathcal{E} := C([0,T],\mathcal{U}) \times C([0,T],\mathcal{U})$ and let $\mathbb{Q} := \mathbb{Q}_1 \otimes \mathbb{Q}_2$ be any probability measure on $\mathcal{E}$ such that $\mathbb{Q}_i \ll P_{\phi_i}$. Define $\widetilde{\mathbb{Q}}_1 := \frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}(x(.,\phi_1))\mathbb{P}$. Let us first remark that $\frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}(x(.,\phi_1))$ is an $\mathcal{F}^{H_1}_T$ -measurable random variable. Since $\mathbb{Q}_1$ is a probability measure on $\mathcal{E}_1$ and the law of x under $\mathbb{P}$ is $\mathbb{P}_{\phi_1}$, then

$$\int_{\Omega_1} \frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}(x)d\mathbb{P} = \int_{\mathcal{E}_1} \frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}(w)d\mathbb{P}_{\phi_1}(w) = \mathbb{Q}_1(\mathcal{E}_1) = 1.$$

Then $\frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}(x(.,\phi_1))$ is integrable and the process $M_t = \mathbb{E}(\frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}(x)|\mathcal{F}^{H_1}_t)$, $0 \le t \le T$ is an $\mathcal{F}^{H_1}_t$ -martingale that we can and will choose to be continuous.
The first part of the proof follows the arguments of [25]. The idea is to express the

finiteness of the entropy by means of the energy of the drift arising from the Girsanov transform of a well chosen probability measure. Recalling the definition of entropy and adopting a measure-transformation argument

$$H(\widetilde{\mathbb{Q}}|\mathbb{P}) = \begin{pmatrix} H_1(\widetilde{\mathbb{Q}}_1|\mathbb{P}) \\ H_2(\widetilde{\mathbb{Q}}_2|\mathbb{P}) \end{pmatrix}$$

and

$$H(\mathbb{Q}|\mathbb{P}_\phi) = \begin{pmatrix} H_1(\mathbb{Q}_1|\mathbb{P}_{\phi_1}) \\ H_2(\mathbb{Q}_2|\mathbb{P}_{\phi_2}) \end{pmatrix}$$

$$\begin{aligned} H_1(\widetilde{\mathbb{Q}}_1|\mathbb{P}) &= \int_{\Omega_1} \log\left(\frac{d\widetilde{\mathbb{Q}}_1}{d\mathbb{P}_1}\right) d\widetilde{\mathbb{Q}_1} \\ &= \int_{\Omega_1} \log\left(\frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}(x(.,\phi_1)\right)\frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}(x(.,\phi_1))d\mathbb{P} \\ &= \int_{\mathcal{E}_1} \log\left(\frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}}\right)\frac{d\mathbb{Q}_1}{d\mathbb{P}_{\phi_1}} d\mathbb{P}_{\phi_1} \\ &= H_1(\mathbb{Q}_1|\mathbb{P}_{\phi_1}). \end{aligned}$$

and

$$\begin{aligned} H_2(\widetilde{\mathbb{Q}}_2|\mathbb{P}) &= \int_{\Omega_2} \log\left(\frac{d\widetilde{\mathbb{Q}}_2}{d\mathbb{P}_2}\right) d\widetilde{\mathbb{Q}_2} \\ &= \int_{\Omega_2} \log\left(\frac{d\mathbb{Q}_2}{d\mathbb{P}_{\phi_2}}(y(.,\phi_2)\right)\frac{d\mathbb{Q}_2}{d\mathbb{P}_{\phi_2}}(y(.,\phi_2))d\mathbb{P} \\ &= \int_{\mathcal{E}_2} \log\left(\frac{d\mathbb{Q}_2}{d\mathbb{P}_{\phi_2}}\right)\frac{d\mathbb{Q}_2}{d\mathbb{P}_{\phi_2}} d\mathbb{P}_{\phi_2} \\ &= H_2(\mathbb{Q}_2|\mathbb{P}_{\phi_2}). \end{aligned}$$

Following [25], there exists a predictable process $h(t)_{0\le t\le T} \in \mathcal{U}$ with

$$\int_0^T \|h(s)\|^2_{\mathcal{U}} ds < +\infty, \qquad \mathbb{P} - a.s.,$$

such that:

$$H_1(\mathbb{Q}_1|\mathbb{P}_{\phi_1}) = \frac{1}{2}\mathbb{E}_{\widetilde{\mathbb{Q}}_1} \int_0^T \|h(s)\|ds \ , \quad H_2(\mathbb{Q}_2|\mathbb{P}_{\phi_2})) = \frac{1}{2}\mathbb{E}_{\widetilde{\mathbb{Q}}_2} \int_0^T \|h(s)\|ds.$$

By the Girsanov theorem [26], the process $(\widetilde{B}(t))_{t\in[0,T]}$ defined by

$$\widetilde{B}(t) = B(t) - \int_0^t h(s)ds$$

is a Brownian motion under $\widetilde{\mathbb{Q}}_i$ and is associated (thanks to the transfer principle) with the $\widetilde{\mathbb{Q}}_i$ fractional Brownian motion $(\widetilde{B}_i^H(t))_{t\in[0,T]}$ defined by

$$\begin{aligned}\widetilde{B}^H(t) &= \int_{[0,t]} K_H(t,s)d\widetilde{B}(s) \\ &= \int_{[0,t]} K_H(t,s)dB(s) - \int_{[0,t]} K_H(t,s)h(s)ds \\ &= \int_{[0,t]} K_H(t,s)dB(s) - (K_Hh)(t),\end{aligned}$$

where the operator K_H is defined by

$$(K_Hh)(t) := \int_{[0,t]} K_H(t,s)h(s)ds.$$

Consequently, under the measure $\widetilde{\mathbb{Q}}_1 \otimes \widetilde{\mathbb{Q}}_2$, the process $\{u(t,\phi) = (x(t,\phi_1), y(t,\phi_2)\}_{t\in[0,T]}$ satisfies

$$\begin{cases} d(x(t)) &= (A_1x(t) + f_1(t,x_t,y_t))dt + \sigma_1(s)d(K_Hh)(t) + \sigma_1(t)dB^H(t),\ t\in J=[0,T], \\ d(y(t)) &= (A_2y(t) + f_2(t,x_t,y_t))dt + \sigma_2(s)d(K_Hh)(t) + \sigma_2(t)dB^H(t),\ t\in J, \\ x(t) &= \phi_1(t), \quad t\in J_0=[-r,0], \\ y(t) &= \phi_2(t), \quad t\in J_0, \end{cases} \tag{4.1}$$

We now consider the solution $(\bar{x},\bar{y})$ (under $(\widetilde{\mathbb{Q}}_1 \otimes \widetilde{\mathbb{Q}}_2)$ of the following equation

$$\begin{cases} d(\bar{x}(t)) &= (A_1\bar{x}(t) + f_1(t,\bar{x}_t,\bar{y}_t))dt + \sigma_1(t)d\widetilde{B}^H(t),\ t\in[0,T] \\ d(\bar{y}(t)) &= (A_2\bar{y}(t) + f_2(t,\bar{x}_t,\bar{y}_t))dt + \sigma_2(t)d\widetilde{B}^H(t),\ t\in[0,T] \\ x(t) &= \phi_1(t), \quad t\in[-r,0] \\ y(t) &= \phi_2(t), \quad t\in[-r,0] \end{cases}$$

By Theorem 3.1, under $\widetilde{\mathbb{Q}}$, the law of the process $\{\bar{u}(t,\phi) = (\bar{x}(t,\phi_1), \bar{y}(t,\phi_1)),\ t\in[0,T]\}$ is $\mathbb{P}_\phi = \mathbb{P}_{\phi_1}\otimes\mathbb{P}_{\phi_2}$. Thus, $\{(x(t),\bar{x}(t)),(y(t),\bar{y}(t)),\ t\in[0,t_1]\}$, under $\widetilde{\mathbb{Q}}$ is a coupling of $(\mathbb{Q},\mathbb{P}_\phi)$ and it follows that:

$$[W_2^{d_\infty}(\mathbb{Q},\mathbb{P}_\phi)]^2 \le \mathbb{E}_{\widetilde{\mathbb{Q}}}(d_\infty(u,\bar{u}))^2 = \mathbb{E}_{\widetilde{\mathbb{Q}}}(\sup_{t\in[0,T]}\|u(t)-\bar{u}(t)\|^2),$$

where

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}(d_\infty(u,\bar{u}))^2 = \begin{pmatrix} \mathbb{E}_{\widetilde{\mathbb{Q}}_1}(d_\infty(x,\bar{x}))^2 \\ \mathbb{E}_{\widetilde{\mathbb{Q}}_2}(d_\infty(y,\bar{y}))^2 \end{pmatrix}.$$

and

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}(\sup_{t\in[0,T]}\|u(t)-\bar{u}(t)\|^2) = \begin{pmatrix} \mathbb{E}_{\widetilde{\mathbb{Q}}_1}(\sup_{t\in[0,T]}\|x(t)-\bar{x}(t)\|^2) \\ \mathbb{E}_{\widetilde{\mathbb{Q}}_2}(\sup_{t\in[0,T]}\|y(t)-\bar{y}(t)\|^2) \end{pmatrix},$$

where we also use the basic inequality

$$(a+b)^2 \leq 2a^2 + 2b^2.$$

We now estimate the distance between u and $\bar{u}$ with respect to d_∞.

$$\begin{aligned}
\|x(t)-\bar{x}(t)\|^2 &= \Big\| \int_0^t S(t-s)\Big(f_1(s,x_s,y_s) - f_1(s,\bar{x}_s,\bar{y}_s)\Big)ds \\
&+ \int_0^t S(t-s)\sigma_1(s)d(K_H h_1)(s)\Big\|^2 \\
&\leq 2\Big\| \int_0^t S(t-s)\Big(f_1(s,x_s,y_s - f_1(s,\bar{x}_s,\bar{y}_s)\Big)ds\Big\|^2 \\
&+ 2\Big\| \int_0^t S(t-s)\sigma_1(s)d(K_H h_1)(s)\Big\|^2 \\
&:= 2(J_1 + J_2).
\end{aligned}$$

By condition (H_1),

$$\begin{aligned}
J_1 =& \Big\| \int_0^t S(t-s)\Big(f_1(s,x_s,y_s) - f_1(t,\bar{x}_s,\bar{y}_s\Big)ds\Big\|^2 \\
\leq& T\int_0^t \Big\|S(t-s)\Big(f_1(s,x_s,y_s) - f_1(t,\bar{x}_s,\bar{y}_s)\Big)\Big\|^2 ds \\
\leq& TM\int_{-r}^t \Big\|f_1(s,x(s),y(s) - f_1(s,\bar{x}(s),\bar{y}(s))\Big\|^2 ds.
\end{aligned}$$

Hence

$$J_1 \leq TMa_{f_1}\int_0^t \|x(s)-\overline{x}(s)\|^2 ds + TMb_{f_1}\int_0^t \|y(s)-\overline{y}(s)\|^2 ds. \tag{4.2}$$

Here, we used $x = y$ over the interval $[-r, 0]$.
On the other hand, since $h_1 \in L^2(0,T;\mathcal{U})$, by inequality (2.3) and Hölder's inequality, we can obtain

$$\begin{aligned}
J_2 \leq& M\widetilde{\sigma}_1 \Big\|\int_0^t K_H(t,s)h(s)ds\Big\|^2 \\
\leq& M\widetilde{\sigma}_1 \int_0^t K_H^2(t,s)ds\int_0^t \|h(s)\|^2 ds \\
\leq& 8M\widetilde{\sigma}_1 c_H^2 \frac{t^{2H}}{H}\int_0^t \|h(s)\|^2 ds
\end{aligned} \tag{4.3}$$

Substituting (4.2) and (4.3), we have

$$\begin{aligned}
\sup_{s\in[0,t]} \|x(s)-\bar{x}(s)\|^2 \leq\;& 2TMa_{f_1}\int_0^t \sup_{\theta\in[0,s]}\|x(\theta)-\overline{x}(\theta)\|^2 ds \\
&+2TMb_{f_1}\int_0^t \sup_{\theta\in[0,s]}\|y(\theta)-\overline{y}(\theta)\|^2 ds \\
&+16M\widetilde{\sigma}_1 c_H^2 \tfrac{t^{2H}}{H}\int_0^t \|h(s)\|^2 ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}\sup_{s\in[0,t]}\|y(s)-\bar{y}(s)\|^2 &\leq 2TMa_{f_2}\int_0^t\sup_{\theta\in[0,s]}\|x(\theta)-\overline{x}(\theta)\|^2ds\\ &+2TMb_{f_2}\int_0^t\sup_{\theta\in[0,s]}\|y(\theta)-\overline{y}(\theta)\|^2ds\\ &+16M\widetilde{\sigma}_2c_H^2\tfrac{t^{2H}}{H}\int_0^t\|h(s)\|^2ds,\end{aligned}$$

where

$$C_f=\max\{2TM(a_{f_1}+a_{f_2}),2TM(b_{f_1}+b_{f_1})\},\quad\text{and}\quad C_\sigma=16Mc_H^2\frac{t^{2H}}{H}\max\{\widetilde{\sigma_1},\widetilde{\sigma_2}\}.$$

Therefore,

$$\begin{aligned}\sup_{s\in[0,t]}\Big(\|x(s)-\bar{x}(s)\|^2+\|y(s)-\bar{y}(s)\|^2\Big) &\leq C_f\int_0^t\sup_{\theta\in[0,s]}\Big(\|x(\theta)-\overline{x}(\theta)\|^2+\|y(\theta)-\overline{y}(\theta)\|^2\Big)ds\\ &+\ 2C_\sigma\int_0^t\|h(s)\|^2ds.\end{aligned}$$

Then, Gronwall's lemma implies that

$$\sup_{s\in[0,t]}\Big(\|x(s)-\bar{x}(s)\|^2+\|y(s)-\bar{y}(s)\|^2\Big)\leq 2C_\sigma\exp(C_fT)\int_0^T\|h(s)\|^2ds.$$

Consequently,

$$\sup_{s\in[0,t]}\|x(s)-\bar{x}(s)\|^2\leq 2C_\sigma\exp(C_fT)\int_0^T\|h(s)\|^2ds,$$

and

$$\sup_{s\in[0,t]}\|y(s)-\bar{y}(s)\|^2\leq 2C_\sigma\exp(C_fT)\int_0^T\|h(s)\|^2ds,$$

which implies that

$$[W_2^{d_\infty}(\mathbb{Q}_1,\mathbb{P}_{\phi_1})]^2\leq C_\sigma\exp(C_fT)\mathbb{E}_{\widetilde{\mathbb{Q}}_1}\left(\int_0^T\|h(s)\|^2ds\right)\leq 2CH_1(\mathbb{Q}_1|\mathbb{P}_{\phi_1}).$$

Similarly,

$$[W_2^{d_\infty}(\mathbb{Q}_2,\mathbb{P}_{\phi_2})]^2\leq 2C_\sigma\exp(C_fT)\mathbb{E}_{\widetilde{\mathbb{Q}}_2}\left(\int_0^T\|h(s)\|^2ds\right)\leq 2CH_2(\mathbb{Q}_2|\mathbb{P}_{\phi_2}).$$

where

$$C=2C_\sigma\exp(C_fT).$$

The desired inequality and the proof are complete. □

5 An example

In this section we present an example to illustrate the usefulness and applicability of our results. We consider a case with finite fractional Brownian motion.

Example 5.1. *Consider the following stochastic partial differential equation with delay effects*

$$\begin{cases} d(u(t,x)) = \frac{\partial^2}{\partial x^2}u(t,x) + (1-\alpha_1 u(t-\tau,x)(\sin t + \sin(\sqrt{2}t))) \\ \qquad -\beta_1 v(t-\tau,x)(\cos t + \cos(\sqrt{2}t))) + \sigma(t)\frac{dB^H}{dt}, \quad t\in[0,T], \quad 0\le x\le\pi, \\ d(v(t,x)) = \frac{\partial^2}{\partial x^2}v(t,x) + (-\alpha_2 u(t-\tau,x)(\cos t + \cos(\sqrt{2}t))) \\ \qquad -\beta_2 v(t-\tau,x)(\sin t + \sin(\sqrt{2}t))) + \sigma(t)\frac{dB^H}{dt}, \quad t\in[0,T], \quad 0\le x\le\pi, \\ u(t,0) = u(t,\pi) = 0, \quad t\in[0,T], \\ v(t,0) = v(t,\pi) = 0, \quad t\in[0,T], \\ u(t,x) = \phi_1(t,x), \quad t\in[-\tau,0], \quad 0\le x\le\pi, \\ v(t,x) = \phi_2(t,x), \quad t\in[-\tau,0], \quad 0\le x\le\pi, \end{cases} \tag{5.1}$$

where $\alpha_i, \beta_i > 0$ and $\tau > 0$, B^H denotes a fractional Brownian motion. Now, we rewrite this system into the abstract form (1.1)*.*

Take first

$$f(t,\phi_{1,t},\phi_{2,t}) = 1-\alpha_1(\phi_{1,t}(-\tau)(\sin t+\sin(\sqrt{2}t))) - \beta_1(\phi_{2,t}(-\tau)(\cos t+\cos(\sqrt{2}t))),$$

$$g(t,\phi_{1,t},\phi_{2,t}) = -\alpha_1(\phi_{2,t}(-\tau)(\cos t+\cos(\sqrt{2}t))) - \beta_2(\phi_{2,t}(-\tau)(\sin t+\sin(\sqrt{2}t))).$$

Now $\mathcal{K} = \mathcal{U} = L^2([0,\pi])$, and define the operator $A_1 = A_2 = A$ by $Au = u''$, with domain $D(A) = \{u \in \mathcal{U}, u'' \in \mathcal{U} \quad and \quad u(0) = u(\pi) = 0\}$.

Then, it is well known that

$$Az = -\sum_{n=1}^{\infty} n^2 \langle z, e_n\rangle e_n, \quad z\in\mathcal{U},$$

and A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\ge0}$ on $\mathcal{U}$, which is given by

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n, \quad u\in\mathcal{U},$$

and $e_n(u) = (2/\pi)^{1/2}\sin(nu), n = 1,2,\cdots$, is the orthogonal set of eigenvectors of A. Since the analytic semigroup $\{S(t)\}$, $t \in J$, is compact, and there exists a constant $M \ge 1$ such that $\|S(t)\|^2 \le M$.

In order to define the operator $Q : \mathcal{K} \longrightarrow \mathcal{K}$, we choose a sequence $\{\sigma_n\}_{n\ge1} \subset \mathbb{R}^+$, set $Qe_n = \sigma_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B^H(s)$ by

$$B^H = \sum_{n=1}^{\infty} \sqrt{\sigma_n} \gamma_n^H(t) e_n,$$

where $H \in (0, 1/2)$, and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent. Thus, one has

$$\|f(t,\phi,\psi) - f(t,\bar{\phi},\bar{\psi})\|^2 \leq 8\alpha_1\|\phi - \bar{\phi}\|_{\mathcal{D}_0} + 8\beta_1\|\psi - \bar{\psi}\|_{\mathcal{D}_0},$$

and

$$\|g(t,\phi,\psi) - g(t,\bar{\phi},\bar{\psi})\|^2 \leq 8\alpha_2\|\phi - \bar{\phi}\|_{\mathcal{D}_0} + 8\beta_2\|\psi - \bar{\psi}\|_{\mathcal{D}_0}.$$

Thanks to these assumptions, it is straightforward to check that (H_1) and (H_2) hold. Let

$$M_{trice} \quad = 2\sqrt{2} \begin{pmatrix} \sqrt{\alpha_1 M T^2} & \sqrt{\beta_1 M T^2} \\ & \\ \sqrt{\alpha_2 M T^2} & \sqrt{\beta_2 M T^2} \end{pmatrix}$$

where M is defined in above. If α_i, β_i are such that the matrix M_{trice} converges to zero, then assumptions in Theorem 3.1 are fulfilled, and we can conclude that system (5.1) has a unique solution.

Acknowledgements. The research of T.C. has been partially supported by FEDER and the Spanish Ministerio de Ciencia, Innovación y Universidades project PGC2018-096540-B-I00, and Junta de Andalucía (Spain) under projects US-1254251 and P18-FR-4509.

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