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On terminal value problems for bi-parabolic equations driven by Wiener process and fractional Brownian motions

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Abstract. In this paper, we study two terminal value problems (TVPs) for stochastic bi-parabolic equations perturbed by standard Brownian motion and fractional Brownian motion with Hurst parameter $h \in (\frac{1}{2}, 1)$ separately. For each problem, we provide a representation for the mild solution and find the space where the existence of the solution is guaranteed. Additionally, we show clearly that the solution of each problem is not stable, which leads to the ill-posedness of each problem. Finally, we propose two regularization results for both considered problems by using the filter regularization method.

Keywords: Bi-parabolic equation, standard Brownian motion, fractional Brownian motion, terminal value problem, ill-posedness

1. Introduction

Let $\mathcal{J} := (0, T)$, where $T > 0$ is the final time of observation. Let $\mathcal{O} \subset \mathbb{R}^n$, $n \geqslant 1$, be a bounded domain with a smooth boundary in the case $n > 1$, and let $H := L^2(\mathcal{O})$. In this paper, we study the following terminal value problems (TVPs) for two stochastic bi-parabolic equations driven by standard and fractional Brownian motion (fBm) respectively.

- **TVP for a bi-parabolic equation perturbed by standard Brownian motion.** Our first problem is aimed to determine $u(t) = u(t, \cdot)$, $t \in \overline{\mathcal{J}}_* := \overline{\mathcal{J}} \setminus \{0\}$, satisfying

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{A})^2 u(t) = F(t) + G(t)\dot{W}(t), & t \in \mathcal{J}, \\ u(t)|_{\partial\mathcal{O}} = 0, & t \in \mathcal{J}, \\ \frac{\partial}{\partial t} u(t)|_{t=T} = 0, \qquad u(T) = u_f, \end{cases} \tag{1}$$

whereupon $\{W(t)\}_{t\in\overline{\mathcal{J}}}$ is an H-valued Q-Wiener process defined on a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\overline{\mathcal{J}}})$. The term $\dot{W}(t) = \frac{\partial W(t)}{\partial t}$ is used to describe a white noise.

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- **TVP for a bi-parabolic equation driven by fractional Brownian motion.** Our purpose in the second problem is to determine $u(t) = u(t, \cdot)$, $t \in \overline{\mathcal{J}}_*$, satisfying

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{A})^2 u(t) = F(t) + G(t)\dot{W}^h(t), & t \in \mathcal{J}, h \in (\frac{1}{2}, 1), \\ u(t)|_{\partial\mathcal{O}} = 0, & t \in \mathcal{J}, \\ \frac{\partial}{\partial t} u(t)|_{t=T} = 0, \qquad u(T) = u_f, \end{cases} \tag{2}$$

whereupon $\{W^h(t)\}_{t\in\overline{\mathcal{J}}}$ is an H-valued Q-fractional Brownian motion, with Hurst parameter $h \in (\frac{1}{2}, 1)$, defined on the filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\overline{\mathcal{J}}})$. Here, the term $\dot{W}^h(t) = \frac{\partial W^h(t)}{\partial t}$ stands for a fractional noise.

In the previous two problems, the operator $\mathcal{A} = -\Delta$ is the negative Laplacian defined on $H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ and noting that $(\frac{\partial}{\partial t} + \mathcal{A})^2 u(t) = \frac{\partial^2}{\partial t^2} u(t) + 2\mathcal{A}\frac{\partial}{\partial t} u(t) + \mathcal{A}^2 u(t)$, $u_f : \Omega \to H$ is called the terminal value, $F : \overline{\mathcal{J}} \times \Omega \to H$ is a linear source function and G is a mapping coming from $\overline{\mathcal{J}} \times \Omega$ to the space $L_0^2(H)$ defined latter.

Let us now introduce the connection between the bi-parabolic equation and the classical parabolic equation and the importance of TVP (or called backward) models. For the classical parabolic equation, the literature is traditional and pretty huge due to its theoretical interest. We can list here some works concerned with terminal value problems for classical parabolic equations [12,24,32]. It is the fact that those classical equations cannot describe accurately the procedure of heat conduction [15,20]. Therefore, some more flexible models including bi-parabolic equations have appeared to describe this phenomenon better [1,2,11,30,33]. For more details about successful applications of bi-parabolic equations, the readers can refer to [16]. As regards TVP's perspective, this model plays a significant role in some practical areas, where we only have the final status $u(T)$ instead of the initial one $u(0)$ and need to recover the previous distribution $u(t)$, for $t \in [0, T)$. For example, in several engineering problems, it is undeniable that determining the previous data of the physical field from its present state is an essential problem.

Recently, there are many works concerned with terminal value problems for bi-parabolic equations in the deterministic case, where u_f, F are deterministic functions and there is no appearance of the stochastic term $G(t)\dot{W}(t)$ (res. $G(t)\dot{W}^h(t)$) as in the first equations of (1), (2). For the homogeneous case (when $F = 0$), Lakhdari [22] showed that the such problem is ill-posed and proposed a regularizing strategy based on the Kozlov-Mazýa iteration method to approximate the solution. Zhang, in [39], established a conditional stability of Holder type and used a modified regularization method to overcome the ill-posedness in this case. For the non-homogeneous problem, the very last paper [27] investigated the deterministic TVP with two cases of source function including linear and nonlinear sources.

Although there have been many studies on TVPs for bi-parabolic equations in the deterministic case, to the best of our knowledge, TVPs for stochastic bi-parabolic equations driven by Wiener process and fractional Brownian motion have not been investigated in the literature, which are contained in the topic of inverse problems for stochastic partial differential equations (SPDEs). This is the motivation leading to our study here. In what follows, we will list some works on inverse problems for SPDEs in recent years. Ibragimov, in [18], considered the problem of estimating coefficients for SPDEs driven by Wiener process. Q. Lü, in [25], studied two different inverse problems for stochastic parabolic equations driven by standard Brownian motion by establishing a global Carleman estimate. In [26], Q. Lü continued to consider the well-posedness of some linear and semilinear TVP for SPDEs with general filtration, without using the Martingale Representation Theorem. In 2017, Yuan and co-authors [37,38] solved some inverse source problems and TVPs for stochastic wave and parabolic equations. More recently, Xiaoli Feng [14] and Pingping Niu [28] investigated inverse problems for two stochastic fractional diffusion equations driven by standard and fractional Brownian motion separately.

The main contributions and difficulties of this paper are as follows. Due to the appearances of the stochastic integrals in the representations of the solutions to (1) and (2), the considered stochastic problems become more difficult than the deterministic cases and it is required to use stochastic analysis techniques to deal with. After stating the existence of the solution in $C^{\nu}(\overline{\mathcal{J}}, L^2(\Omega, \dot{H}^{\sigma}))$ for each of two problems, we show clearly the instability of the solution by the strategy as follows. We provide concrete spaces (34)–(36) and prove that, if the approximation for the exact data of each problem belongs to those spaces, then it can lead to a large error in the solution. This is one of our new results in the present paper. Furthermore, since the considered stochastic problems are not well-posed, we apply the filter regularization method to construct regularized approximate solutions. Under strong conditions for the solutions of two problems (see Theorem 4.1 and Theorem 4.2), which require a quick decay in the Fourier coefficients of the final datum u_f, the regularized solutions we constructed are convergent to the sought solutions. We also show some concrete examples to illustrate our regularization results.

The rest of the present paper is organized as follows. We prepare some notations and preliminaries in Section 2. In Section 3, we state the existence of the solution of each TVP and then prove that it is instable, which is the reason making the ill-posedness. The regularized solutions for both considered problems are proposed in Section 4 by using the filter method. Furthermore, convergence rates of those approximate solutions are proved. Finally, some materials including the definitions of fBm, Wiener integral with respect to fBm, properties of the solution operators, etc., are recalled in Appendix A and Appendix B.

2. Preliminaries

Let σ be a non-negative number. By $\dot{H}^{\sigma}$ we denote the space of H-valued function θ such that $\|\theta\|^2_{\dot{H}^{\sigma}} := \sum_{k\in\mathbb{Z}^*} \lambda_k^{2\sigma}(\theta, e_k)^2 < \infty$, where $(\cdot,\cdot)$ is the usual inner product in H, e_k and λ_k are taken from the orthonormal basis $\{e_k\}_{k\in\mathbb{Z}^+}$ of H and the Dirichlet eigenvalues $\{\lambda_k\}_{k\in\mathbb{Z}^+}$ which form an infinite sequence tending to infinity

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Let us denote by $\dot{H}^{-\sigma}$ the dual space of $\dot{H}^{\sigma}$. Then, the fractional operator $\mathcal{A}^{\sigma} = (-\Delta)^{\sigma} : \dot{H}^{\frac{\sigma}{2}} \to \dot{H}^{-\frac{\sigma}{2}}$ can be defined as (see [13,21])

$$A^{\sigma}\theta := \sum_{k\in\mathbb{Z}^*} \lambda_k^{\sigma}(\theta, e_k)e_k, \quad \theta \in \dot{H}^{\sigma/2}.$$

It is observed that $\dot{H}^0 \equiv H$ and $\|\theta\|_{\dot{H}^{\sigma}} = \|\mathcal{A}^{\sigma}\theta\|_H$.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\overline{\mathcal{J}}})$ be a filter probability space with a normal filtration $\{\mathcal{F}_t\}_{t\in\overline{\mathcal{J}}}$ satisfying standard assumptions (the filtration is right continuous, all $\mathbb{P}$-null sets belong to $\mathcal{F}_0$). For $p \geqslant 1$, we introduce $L^p(\Omega, \dot{H}^{\sigma})$ the space of all $\dot{H}^{\sigma}$-valued random variables $\mathcal{X}$ such that

$$\|\mathcal{X}\|_{L^p(\Omega,\dot{H}^{\sigma})} := \left(\mathbb{E}\|\mathcal{X}\|^p_{\dot{H}^{\sigma}}\right)^{\frac{1}{p}} < \infty.$$

By $\mathcal{L}(H)$ we denote the space of all bounded linear operators from H to H. Let $\mathcal{Q} \in \mathcal{L}(H)$ be a non-negative self-adjoint operator defined by $\mathcal{Q}e_k = \Lambda_k e_k$ with finite trace $\mathrm{tr}(\mathcal{Q}) = \sum_{k\in\mathbb{Z}^*} \Lambda_k < \infty$, where $\Lambda_k \geqslant 0$, $n \in \mathbb{Z}^*$. The standard Brownian motion $\{W(t)\}_{t\in\overline{\mathcal{J}}}$ (res. fractional Brownian motion $\{W^h(t)\}_{t\in\overline{\mathcal{J}}}$) with covariance $\mathcal{Q}$ (see [10,19,40]) can be defined as

$$W(t) = W_{\mathcal{Q}}(t) = \sum_{k\in\mathbb{Z}^*} \mathcal{Q}^{1/2} e_k \xi_k(t) = \sum_{k\in\mathbb{Z}^*} \Lambda_k^{1/2} e_k \xi_k(t), \tag{3}$$

$$W^h(t) = W^h_{\mathcal{Q}}(t) = \sum_{k\in\mathbb{Z}^*} \mathcal{Q}^{1/2} e_k \xi^h_k(t) = \sum_{k\in\mathbb{Z}^*} \Lambda_k^{1/2} e_k \xi^h_k(t), \tag{4}$$

whereupon $\xi_k(t)$ (res. $\xi^h_k(t)$) are independent one-dimensional Brownian motions (res. independent one-dimensional fractional Brownian motions). Let $L^2_0(H, \dot{H}^\sigma)$ be the space of all linear bounded operators $R : H \to \dot{H}^\sigma$ such that $R\mathcal{Q}^{1/2}$ is a Hillbert-Schmidt operator from H to $\dot{H}^\sigma$ with the norm

$$\|R\|_{L^2_0(H,\dot{H}^\sigma)} := \left(\sum_{k\in\mathbb{Z}^*} \big\| \mathcal{Q}^{1/2} R e_k \big\|^2_{\dot{H}^\sigma} \right)^{1/2} < \infty.$$

For short, we denote $L^2_0(H) := L^2_0(H, H) = L^2_0(H, \dot{H}^0)$. For more details about the fractional Brownian motions and the Wiener integral with respect to the fractional Brownian motions, one can see Appendix A.

Next, we present the definitions of mild solutions to TVP (1), TVP (2). The readers can find in Appendix B the way we construct them. Furthermore, the definitions of well-posed and ill-posed problems are proposed.

Definition 2.1 (Mild solution of TVP (1))**.** An H-valued process $\{u(t)\}_{t\in\overline{\mathcal{J}}}$ satisfying the following equation almost surely

$$u(t) = \mathcal{S}_1(t, T)u_f + \int_t^T \mathcal{S}_2(t, s)F(s)\,ds + \int_t^T \mathcal{S}_2(t, s)G(s)\,dW(s), \tag{5}$$

is called a mild solution of TVP (1), where (see Appendix B)

$$\mathcal{S}_1(t, T) := \sum_{k\in\mathbb{Z}^*} \big(1 - (T - t)\lambda_k\big) e^{(T-t)\lambda_k} (\cdot, e_k)e_k, \qquad \mathcal{S}_2(t, s) := \sum_{k\in\mathbb{Z}^*} (s - t)e^{(s-t)\lambda_k} (\cdot, e_k)e_k.$$

Definition 2.2 (Mild solution of TVP (2))**.** An H-valued process $\{\overline{u}(t)\}_{t\in\overline{\mathcal{J}}}$ satisfying the following equation almost surely

$$\overline{u}(t) = \mathcal{S}_1(t, T)u_f + \int_t^T \mathcal{S}_2(t, s)F(s)\,ds + \int_0^T \mathcal{S}_2(t, s)G(s)\,dW^h(s) - \int_0^t \mathcal{S}_2(t, s)G(s)\,dW^h(s), \tag{6}$$

is called a mild solution of TVP (2), where $\mathcal{S}_1(t, T)$, $\mathcal{S}_2(t, s)$ are defined in Definition 2.1.

Definition 2.3 (Well-posed and ill-posed problems)**.** According to Jacques Hadamard [17], a problem is said to be well-posed if it satisfies the following conditions

i) it has a solution,
ii) the solution is unique,
iii) the solution is stable, i.e. it depends continuously on data.

Problems that are not well-posed in the sense of Hadamard are termed ill-posed.

In the next section, we will show that the existence of the solutions of problems (1), (2) is guaranteed on $C(\overline{\mathcal{J}}; L^2(\Omega, \dot{H}^\sigma))$, with $\sigma \geqslant 0$, under strict assumptions for the data (u_f, F, G). However, it is unfortunate that those solutions are not stable on $C(\overline{\mathcal{J}}; L^2(\Omega, \dot{H}^\sigma))$, which makes the considered problems be ill-posed. For clarity, let us explain the instability of the solution of Problem (1). For Problem (2), this property can be understood similarly. Assume that (u_f, F, G) is noised by observed data $(\widetilde{u}^\varepsilon_f, \widetilde{F}^\varepsilon, \widetilde{G}^\varepsilon)$ which contains small errors as

$$\|\widetilde{u}^\varepsilon_f - u_f\|_{L^2(\Omega,H)} \leqslant \varepsilon, \qquad \|\widetilde{F}^\varepsilon - F\|_{L^p(\mathcal{J};L^2(\Omega,H))} \leqslant \varepsilon, \qquad \|\widetilde{G}^\varepsilon - G\|_{L^q(\mathcal{J};L^2(\Omega,L^2_0(H)))} \leqslant \varepsilon,$$

where ε, p, q are some positive constants such that $\varepsilon \geqslant 0$, $p > 1$, $q > 2$. Let $\widetilde{u}^\varepsilon$ be the solution of (1) with respect to $(\widetilde{u}^\varepsilon_f, \widetilde{F}^\varepsilon, \widetilde{G}^\varepsilon)$. Then, the error between $\widetilde{u}^\varepsilon$ and u, namely $\|\widetilde{u}^\varepsilon - u\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))}$, would not tend to zero as $\varepsilon \to 0^+$. For short, it can be said that the solution u does not depend continuously on the data (u_f, F, G). One can see clearly this property of the solution in Theorem 3.3 in Section 3.2.

3. The ill-posedness of two problems on $C(\overline{\mathcal{J}}; L^2(\Omega, \dot{H}^\sigma))$

3.1. The existence of the solution of each TVP

In this subsection, we attempt to find the spaces where we obtain the existence of solutions to both TVPs. For two non-negative numbers a, b, let us introduce a pair of spaces

$$\mathbb{V}^b_a := \left\{\theta \in H \text{ s.t. } \|\theta\|^2_{\mathbb{V}^b_a} := \sum_{k\in\mathbb{Z}^*} \lambda_k^{2a} e^{2b\lambda_k}(\theta, e_k)^2 < \infty\right\},$$

which is known as the Gevrey-type space [7], and

$$\mathbb{W}^b_a := \left\{R \in L^2_0(H) \text{ s.t. } \|R\|^2_{\mathbb{W}^b_a} := \sum_{k\in\mathbb{Z}^*} \left\|\mathcal{Q}^{1/2} Re_k\right\|^2_{\mathbb{V}^b_a} < \infty\right\}.$$

If $a = b = 0$, then $\mathbb{V}_{a,b}$ and $\mathbb{W}_{a,b}$ turn to be H and $L^2_0(H)$ respectively. Let σ be a non-negative number. To obtain the existence of the solution to each of two TVPs (1) and (2), we need the following strong assumptions for the data (u_f, F, G).

(H1) $u_f \in L^2(\Omega, \mathbb{V}^T_{\sigma+1})$,
(H2) $F \in L^p(\mathcal{J}; L^2(\Omega, \mathbb{V}^T_{\sigma+1}))$, for some $p > 1$,
(H3) $G \in L^q(\mathcal{J}; L^2(\Omega, \mathbb{W}^T_{\sigma+1}))$, for some $q > 2$,
(H4) $G \in L^r(\mathcal{J}; L^2(\Omega, \mathbb{W}^T_{\sigma+1}))$, for some $r > \frac{2}{h-1/2}$.

Before establishing the existence of solution to both TVPs, we prepare some necessary lemmas. The following ones will provide some needed properties for all terms in the right-hand sides of equations (5), (6).

Lemma 3.1. *Let us consider $\sigma \geqslant 0$ and $t, s \in \overline{\mathcal{J}}$. Assume that u_f, F satisfy Assumptions (H1), (H2). Then, there hold*

$$\left\| \mathcal{S}_1(t,T)u_f \right\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant C_1 \|u_f\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}, \tag{7}$$

$$\left\| \int_t^T \mathcal{S}_2(t,s)F(s)\,ds \right\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant \lambda_1^{-1} T^{(2p-1)/p} \|F\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))}, \tag{8}$$

where C_1 is defined in Lemma B.1. Furthermore, if $\delta > 0$ is small enough, then

$$\left\| \big(\mathcal{S}_1(t+\delta,T) - \mathcal{S}_1(t,T)\big)u_f \right\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant T\delta \|u_f\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}, \tag{9}$$

$$\left\| \int_{t+\delta}^T \mathcal{S}_2(t+\delta,s)F(s)\,ds - \int_t^T \mathcal{S}_2(t,s)F(s)\,ds \right\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant C_2 \delta^{(p-1)/p} \|F\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))}, \tag{10}$$

where $C_2 := C_2(\mathcal{A},T) = T^{\frac{p-1}{p}}(2\lambda_1^{-1} + T)$.

Lemma 3.2. *Let $\sigma \geqslant 0$ and $t, s \in \overline{\mathcal{J}}$. Assume that G satisfies Assumption (H3). Then, there holds*

$$\left\| \int_t^T \mathcal{S}_2(t,s)G(s)\,dW(s) \right\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant C_3 \|G\|_{L^q(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}. \tag{11}$$

Furthermore, if $\delta > 0$ is small enough, then

$$\begin{aligned} &\left\| \int_{t+\delta}^T \mathcal{S}_2(t+\delta,s)G(s)\,dW(s) - \int_t^T \mathcal{S}_2(t,s)G(s)\,dW(s) \right\|_{L^2(\Omega,\dot{H}^\sigma)} \\ &\quad \leqslant C_4 \delta^{\frac{q-2}{2q}} \|G\|_{L^q(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}, \end{aligned} \tag{12}$$

where $C_3 := C_3(\mathcal{A},T,q) = \lambda_1^{-1} T^{(q-2)/(2q)}$, $C_4 := C_4(\mathcal{A},T,q) = \lambda_1^{-1} T + (\lambda_1^{-1} + T) T^{(q-2)/(2q)}$.

Lemma 3.3. *Given $\sigma \geqslant 0$ and $t, s \in \overline{\mathcal{J}}$. Assume that G satisfies Assumption (H4). Then, there exists a positive constant $C_5 = C_5(\mathcal{A},T,h,r)$ such that*

$$\left\| \int_0^T \mathcal{S}_2(t,s)G(s)\,dW^h(s) \right\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant C_5 \|G\|_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}. \tag{13}$$

Furthermore, if $\delta > 0$ is small enough, then there exists a positive constant $C_6 = C_6(\mathcal{A},T,h,r)$ such that

$$\left\| \int_0^T \big(\mathcal{S}_2(t+\delta,s) - \mathcal{S}_2(t,s)\big)G(s)\,dW^h(s) \right\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant C_6 \delta \|G\|_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}. \tag{14}$$

Lemma 3.4. *Let $\sigma \geqslant 0$ and $t, s \in \overline{\mathcal{J}}$. Assume that G satisfies Assumption (H4). If $\delta > 0$ is small enough, then there exists a positive constant $C_7 = C_7(\mathcal{A}, T, h, r)$ such that*

$$\begin{aligned} &\left\| \int_0^{t+\delta} \mathcal{S}_2(t+\delta, s) G(s)\, dW^h(s) - \int_0^t \mathcal{S}_2(t, s) G(s)\, dW^h(s) \right\|_{L^2(\Omega, \dot{H}^\sigma)} \\ &\quad \leqslant C_7 \delta^{\min\{1-h, \frac{1}{2}(h - \frac{3}{2} + \frac{r-2}{r})\}} \|G\|_{L^r(\mathcal{J}; L^2(\Omega, \mathbb{W}^T_{\sigma+1}))}. \end{aligned} \tag{15}$$

Proof of Lemma 3.1. We begin with the proof for (7) and (8). Since u_f satisfies Assumption (H1), Lemma B.1 yields $\|\mathcal{S}_1(t, T) u_f\|_{L^2(\Omega, \dot{H}^\sigma)} \leqslant C_1 \|u_f\|_{L^2(\Omega, \mathbb{V}^T_{\sigma+1})}$. The estimate (9) can be prove easily by applying the property (78) of Lemma B.1. For the second term, by Assumption (H2) and Lemma B.1, one has

$$\begin{aligned} \left\| \int_t^T \mathcal{S}_2(t, s) F(s)\, ds \right\|_{L^2(\Omega, \dot{H}^\sigma)} &\leqslant \int_t^T \big\| \mathcal{S}_2(t, s) F(s) \big\|_{L^2(\Omega, \dot{H}^\sigma)}\, ds \\ &\leqslant \lambda_1^{-1} \int_t^T |s - t| \big\| F(s) \big\|_{L^2(\Omega, \mathbb{V}^T_{\sigma+1})}\, ds. \end{aligned}$$

By applying the Hölder inequality, one arrives at

$$\begin{aligned} \left\| \int_t^T \mathcal{S}_2(t, s) F(s)\, ds \right\|_{L^2(\Omega, \dot{H}^\sigma)} &\leqslant \lambda_1^{-1} \left(\int_t^T |s - t|^{\frac{p}{p-1}}\, ds \right)^{\frac{p-1}{p}} \left(\int_t^T \big\| F(s) \big\|^p_{L^2(\Omega, \mathbb{V}^T_{\sigma+1})}\, ds \right)^{\frac{1}{p}} \\ &\leqslant \lambda_1^{-1} T^{(2p-1)/p} \|F\|_{L^p(\mathcal{J}; L^2(\Omega, \mathbb{V}^T_{\sigma+1}))}. \end{aligned}$$

We next prove the two latter estimates (9) and (10). Firstly, it is obvious that

$$\begin{aligned} &\left\| \int_{t+\delta}^T \mathcal{S}_2(t+\delta, s) F(s)\, ds - \int_t^T \mathcal{S}_2(t, s) F(s)\, ds \right\|_{L^2(\Omega, \dot{H}^\sigma)} \\ &\quad \leqslant \left\| \int_{t+\delta}^T \big(\mathcal{S}_2(t+\delta, s) - \mathcal{S}_2(t, s) \big) F(s)\, ds \right\|_{L^2(\Omega, \dot{H}^\sigma)} + \left\| \int_t^{t+\delta} \mathcal{S}_2(t, s) F(s)\, ds \right\|_{L^2(\Omega, \dot{H}^\sigma)} \\ &\quad \leqslant \int_{t+\delta}^T \big\| \big(\mathcal{S}_2(t+\delta, s) - \mathcal{S}_2(t, s) \big) F(s) \big\|_{L^2(\Omega, \dot{H}^\sigma)}\, ds + \int_t^{t+\delta} \big\| \mathcal{S}_2(t, s) F(s) \big\|_{L^2(\Omega, \dot{H}^\sigma)}\, ds \\ &\quad =: (I) + (II). \end{aligned}$$

By applying property (10) of Lemma B.1 and the Hölder inequality, we deduce that

$$\begin{aligned} (I) &\leqslant \delta \int_{t+\delta}^T \big(\lambda_1^{-1} + s \big) \big\| F(s) \big\|_{L^2(\Omega, \mathbb{V}^T_{\sigma+1})}\, ds \\ &\leqslant \delta \left(\int_t^T \big(\lambda_1^{-1} + T \big)^{\frac{p}{p-1}}\, ds \right)^{\frac{p-1}{p}} \left(\int_t^T \big\| F(s) \big\|^p_{L^2(\Omega, \mathbb{V}^T_{\sigma+1})}\, ds \right)^{\frac{1}{p}} \\ &\leqslant \delta T^{\frac{p-1}{p}} \big(\lambda_1^{-1} + T \big) \|F\|_{L^p(\mathcal{J}; L^2(\Omega, \mathbb{V}^T_{\sigma+1}))}. \end{aligned}$$

Similarly, for the second term,

$$(II) \leqslant \lambda_1^{-1}\left(\int_t^{t+\delta} |s-t|^{\frac{p}{p-1}}\,ds\right)^{\frac{p-1}{p}}\left(\int_t^{t+\delta} \|F(s)\|^p_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}\,ds\right)^{\frac{1}{p}}$$
$$\leqslant \delta^{(p-1)/p}\lambda_1^{-1}T\|F\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))},$$

Hence, for $\delta > 0$ small enough, we conclude that

$$\left\|\int_{t+\delta}^T \mathcal{S}_2(t+\delta,s)F(s)\,ds - \int_t^T \mathcal{S}_2(t,s)F(s)\,ds\right\|_{L^2(\Omega,\dot{H}^\sigma)}$$
$$\leqslant \delta^{(p-1)/p}T\big(2\lambda_1^{-1}+T\big)\|F\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))}.$$

This completes the proof. □

Proof of Lemma 3.2. We begin with the proof of (11). By using the Itô isometry,

$$\left\|\int_t^T \mathcal{S}_2(t,s)G(s)\,dW(s)\right\|_{L^2(\Omega,\dot{H}^\sigma)} = \left(\int_t^T \mathbb{E}\big\|\mathcal{A}^\sigma\mathcal{S}_2(t,s)G(s)\big\|^2_{L_0^2(H)}\,ds\right)^{1/2}$$
$$= \left(\int_t^T \sum_{k\in\mathbb{Z}^*}\mathbb{E}\big\|\mathcal{Q}^{1/2}\mathcal{S}_2(t,s)G(s)e_k\big\|^2_{\dot{H}^\sigma}\,ds\right)^{1/2}.$$

One can estimate the above term by using Assumption (H2), Lemma B.1 and the Hölder inequality as follows

$$\begin{aligned}\left\|\int_t^T \mathcal{S}_2(t,s)G(s)\,dW(s)\right\|_{L^2(\Omega,\dot{H}^\sigma)} &\leqslant \lambda_1^{-1}\left(\int_t^T \sum_{k\in\mathbb{Z}^*}\mathbb{E}\big\|\mathcal{Q}^{1/2}G(s)e_k\big\|^2_{\mathbb{V}^T_{\sigma+1}}\,ds\right)^{1/2}\\ &= \lambda_1^{-1}\left(\int_t^T \|G(s)\|^2_{L^2(\Omega,\mathbb{W}^T_{\sigma+1})}\,ds\right)^{1/2}\\ &\leqslant \lambda_1^{-1}\left(\int_t^T ds\right)^{(q-2)/(2q)}\left(\int_t^T \|G(s)\|^q_{L^2(\Omega,\mathbb{W}^T_{\sigma+1})}\,ds\right)^{1/q}\\ &\leqslant \lambda_1^{-1}T^{(q-2)/(2q)}\|G\|_{L^q(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}. \end{aligned} \tag{16}$$

Next, we will show that (12) holds. By the triangle inequality and the Itô isometry, one can see

$$\left\|\int_{t+\delta}^T \mathcal{S}_2(t+\delta,s)G(s)\,dW(s) - \int_t^T \mathcal{S}_2(t,s)G(s)\,dW(s)\right\|_{L^2(\Omega,\dot{H}^\sigma)}$$
$$\leqslant \left\|\int_{t+\delta}^T \big(\mathcal{S}_2(t+\delta,s)-\mathcal{S}_2(t,s)\big)G(s)\,dW(s)\right\|_{L^2(\Omega,\dot{H}^\sigma)} + \left\|\int_t^{t+\delta} \mathcal{S}_2(t,s)G(s)\,dW(s)\right\|_{L^2(\Omega,\dot{H}^\sigma)}$$

$$
\begin{aligned}
&= \left(\int_{t+\delta}^{T} \left\| \mathcal{A}^{\sigma} \big(\mathcal{S}_2(t+\delta, s) - \mathcal{S}_2(t,s) \big) G(s) \right\|_{L_0^2(H)}^2 ds \right)^{1/2} \\
&\quad + \left(\int_{t}^{t+\delta} \left\| \mathcal{A}^{\sigma} \mathcal{S}_2(t,s) G(s) \right\|_{L_0^2(H)}^2 ds \right)^{1/2} \\
&=: (III) + (IV).
\end{aligned}
$$

The previous two terms can be estimated thanks to Lemma B.1 and a similar technique to the one for (16)

$$
\begin{aligned}
(III) &= \left(\int_{t}^{T} \sum_{k \in \mathbb{Z}^*} \mathbb{E} \left\| \mathcal{Q}^{1/2} \big(\mathcal{S}_2(t+\delta, s) - \mathcal{S}_2(t,s) \big) G(s) e_k \right\|_{\dot{H}^{\sigma}}^2 ds \right)^{1/2} \\
&\leqslant \delta \big(\lambda_1^{-1} + T \big) \left(\int_{t}^{T} \sum_{k \in \mathbb{Z}^*} \mathbb{E} \left\| \mathcal{Q}^{1/2} G(s) e_k \right\|_{\mathbb{V}_{\sigma+1}^T}^2 ds \right)^{1/2} \\
&\leqslant \delta \big(\lambda_1^{-1} + T \big) T^{(q-2)/(2q)} \| G \|_{L^q(\mathcal{J}; L^2(\Omega, \mathbb{W}_{\sigma+1}^T))},
\end{aligned}
$$

and

$$
\begin{aligned}
(IV) &\leqslant \lambda_1^{-1} T \left(\int_{t}^{t+\delta} \sum_{k \in \mathbb{Z}^*} \mathbb{E} \left\| \mathcal{Q}^{1/2} G(s) e_k \right\|_{\mathbb{V}_{\sigma+1}^T}^2 ds \right)^{1/2} \\
&\leqslant \lambda_1^{-1} T \left(\int_{t}^{t+\delta} ds \right)^{(q-2)/(2q)} \left(\int_{t}^{\delta} \| G(s) \|_{L^2(\Omega, \mathbb{W}_{\sigma+1}^T)}^q ds \right)^{1/q} \\
&\leqslant \lambda_1^{-1} T \delta^{(q-2)/(2q)} \| G \|_{L^q(\mathcal{J}; L^2(\Omega, \mathbb{W}_{\sigma+1}^T))}.
\end{aligned}
$$

From all above arguments, we conclude that

$$
\begin{aligned}
&\left\| \int_{t+\delta}^{T} \mathcal{S}_2(t+\delta, s) G(s)\, dW(s) - \int_{t}^{T} \mathcal{S}_2(t,s) G(s)\, dW(s) \right\|_{L^2(\Omega, \dot{H}^{\sigma})} \\
&\quad \leqslant \delta^{(q-2)/(2q)} \big(\lambda_1^{-1} T + \big(\lambda_1^{-1} + T \big) T^{(q-2)/(2q)} \big) \| G \|_{L^q(\mathcal{J}; L^2(\Omega, \mathbb{W}_{\sigma+1}^T))}.
\end{aligned}
$$

This completes the proof. □

Proof of Lemma 3.3. • *Step 1*. This step is aimed to prove estimate (13). By the representation (75), we have

$$
\begin{aligned}
\left\| \int_{0}^{T} \mathcal{S}_2(t,s) G(s)\, dW^h(s) \right\|_{L^2(\Omega, \dot{H}^{\sigma})}^2 &= \mathbb{E} \left\| \sum_{k \in \mathbb{Z}^*} \int_{0}^{T} K_{h,T}^* \big(\mathcal{S}_2(t, \cdot) G(\cdot) \mathcal{Q}^{1/2} e_k \big)(s)\, d\xi_k(s) \right\|_{\dot{H}^{\sigma}}^2 \\
&= \sum_{k \in \mathbb{Z}^*} \int_{0}^{T} \mathbb{E} \left\| K_{h,T}^* \big(\mathcal{S}_2(t, \cdot) G(\cdot) \mathcal{Q}^{1/2} e_k \big)(s) \right\|_{\dot{H}^{\sigma}}^2 ds, \qquad (17)
\end{aligned}
$$

where we have used the Itô isometry. We now find an upper bound for $\mathbb{E}\|\mathcal{Q}^{1/2} \times K_{h,T}^*(\mathcal{S}_2(t,\cdot)G(\cdot)e_k)(s)\|_{\dot{H}^\sigma}^2$. From the observations (72), (73), we can see

$$\begin{aligned}
&\mathbb{E}\big\|K_{h,T}^*\big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|_{\dot{H}^\sigma}^2 \\
&\quad = \mathbb{E}\left\|\int_s^T \big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(\mu)\frac{\partial K_h}{\partial \mu}(\mu,s)\,d\mu\right\|_{\dot{H}^\sigma}^2 \\
&\quad = c_h^2\left(h-\frac{1}{2}\right)^2\mathbb{E}\left\|\int_s^T \mathcal{S}_2(t,\mu)G(\mu)\mathcal{Q}^{1/2}e_k\left(\frac{\mu}{s}\right)^{h-\frac{1}{2}}(\mu-s)^{h-\frac{3}{2}}\,d\mu\right\|_{\dot{H}^\sigma}^2 \\
&\quad \leqslant c_h^2\left(h-\frac{1}{2}\right)^2 s^{1-2h}\mathbb{E}\left[\int_s^T \mu^{h-\frac{1}{2}}(\mu-s)^{h-\frac{3}{2}}\big\|\mathcal{S}_2(t,\mu)G(\mu)\mathcal{Q}^{1/2}e_k\big\|_{\dot{H}^\sigma}\,d\mu\right]^2.
\end{aligned} \tag{18}$$

Applying the Hölder inequality, we arrive at

$$\begin{aligned}
\mathbb{E}\big\|K_{h,T}^*\big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|_{\dot{H}^\sigma}^2 &\leqslant c_h^2\left(h-\frac{1}{2}\right)^2 s^{1-2h}\left(\int_s^T \mu^{2h-1}(\mu-s)^{h-\frac{3}{2}}\,d\mu\right) \\
&\quad \times\left(\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|\mathcal{S}_2(t,\mu)G(\mu)\mathcal{Q}^{1/2}e_k\big\|_{\dot{H}^\sigma}^2\,d\mu\right).
\end{aligned} \tag{19}$$

On the other hand,

$$\int_s^T \mu^{2h-1}(\mu-s)^{h-\frac{3}{2}}\,d\mu \leqslant \int_0^T \mu^{2h-1}(\mu-s)^{h-\frac{3}{2}}\,d\mu = T^{3(h-\frac{1}{2})}\frac{\Gamma(2h)\Gamma(h-1/2)}{\Gamma(3h-1/2)}. \tag{20}$$

Setting $M_1 := M_1(h,T) = c_h(h-\frac{1}{2})T^{\frac{3}{2}(h-\frac{1}{2})}(\frac{\Gamma(2h)\Gamma(h-1/2)}{\Gamma(3h-1/2)})^{1/2}$, we obtain

$$\begin{aligned}
&\mathbb{E}\big\|K_{h,T}^*\big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|_{\dot{H}^\sigma}^2 \\
&\quad \leqslant M_1^2 s^{1-2h}\left(\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|\mathcal{S}_2(t,\mu)G(\mu)\mathcal{Q}^{1/2}e_k\big\|_{\dot{H}^\sigma}^2\,d\mu\right) \\
&\quad \leqslant \lambda_1^{-2}T^2M_1^2 s^{1-2h}\left(\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\mathcal{Q}^{1/2}e_k\big\|_{\mathbb{V}_{\sigma+1}^T}^2\,d\mu\right),
\end{aligned} \tag{21}$$

where we have used the property (77). From (17), (21), we deduce that

$$\begin{aligned}
&\left\|\int_0^T \mathcal{S}_2(t,s)G(s)\,dW^h(s)\right\|_{L^2(\Omega,\dot{H}^\sigma)}^2 \\
&\quad \leqslant \lambda_1^{-2}T^2M_1^2\int_0^T s^{1-2h}\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\big\|_{\mathbb{W}_{\sigma+1}^T}^2\,d\mu\,ds.
\end{aligned} \tag{22}$$

To bound the right-hand side of the above inequality, we will use the following estimate obtained by applying the Hölder inequality

$$\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\big\|^2_{\mathbb{W}^T_{\sigma+1}}\,d\mu \leqslant \left(\int_s^T (\mu-s)^{\frac{r}{r-2}(h-\frac{3}{2})}\,d\mu\right)^{\frac{r-2}{r}}\left(\int_s^T \big\|G(\mu)\big\|^r_{L^2(\Omega,\mathbb{W}^T_{\sigma+1})}\,d\mu\right)^{\frac{2}{r}}$$
$$\leqslant \left(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{r}}\|G\|^2_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}. \tag{23}$$

Now, combining (22), (23) and noting that $\int_0^T s^{1-2h}\,ds = \frac{T^{2-2h}}{2-2h}$, we conclude that

$$\left\|\int_0^T \mathcal{S}_2(t,s)G(s)\,dW^h(s)\right\|^2_{L^2(\Omega,\dot{H}^\sigma)}$$
$$\leqslant \lambda_1^{-2}\frac{T^{4-2h}}{2-2h}M_1^2\left(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{r}}\|G\|^2_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}. \tag{24}$$

• *Step 2*. Now we prove (14). Here, we note that property (79) is used instead of (77). Firstly, we have a similar formula as in (17)

$$\left\|\int_0^T \big(\mathcal{S}_2(t+\delta,s)-\mathcal{S}_2(t,s)\big)G(s)\,dW^h(s)\right\|^2_{L^2(\Omega,\dot{H}^\sigma)}$$
$$=\sum_{k\in\mathbb{Z}^*}\int_0^T \mathbb{E}\big\|K^*_{h,T}\big(\big(\mathcal{S}_2(t+\delta,\cdot)-\mathcal{S}_2(t,\cdot)\big)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\,ds.$$

By using a similar way as in (18)–(21), one can easily check that

$$\mathbb{E}\big\|K^*_{h,T}\big(\big(\mathcal{S}_2(t+\delta,s)-\mathcal{S}_2(t,s)\big)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}$$
$$\leqslant M_1^2 s^{1-2h}\left(\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|\big(\mathcal{S}_2(t+\delta,s)-\mathcal{S}_2(t,s)\big)G(\mu)\mathcal{Q}^{1/2}e_k\big\|^2_{\dot{H}^\sigma}\,d\mu\right)$$
$$\leqslant \big(\lambda_1^{-1}+T\big)^2\delta^2M_1^2s^{1-2h}\left(\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\mathcal{Q}^{1/2}e_k\big\|^2_{\mathbb{V}^T_{\sigma+1}}\,d\mu\right). \tag{25}$$

By the observations (22), (24), we deduce that

$$\left\|\int_0^T \big(\mathcal{S}_2(t+\delta,s)-\mathcal{S}_2(t,s)\big)G(s)\,dW^h(s)\right\|^2_{L^2(\Omega,\dot{H}^\sigma)}$$
$$\leqslant \big(\lambda_1^{-1}+T\big)^2\delta^2\frac{T^{2-2h}}{2-2h}M_1^2\left(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{r}}\|G\|^2_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}. \tag{26}$$

We now complete the proof. □

Proof of Lemma 3.4. In what follows we prove (15). From the representation (76), we can see

$$\begin{aligned}
&\int_0^{t+\delta} \mathcal{S}_2(t+\delta,s)G(s)\,dW^h(s) - \int_0^t \mathcal{S}_2(t,s)G(s)\,dW^h(s) \\
&\quad= \sum_{k\in\mathbb{Z}^*}\int_t^{t+\delta} K^*_{h,t+\delta}\big(\mathcal{S}_2(t+\delta,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\,d\xi_k(s) \\
&\qquad- \sum_{k\in\mathbb{Z}^*}\int_0^t \big(K^*_{h,t+\delta}\big(\mathcal{S}_2(t+\delta,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s) - K^*_{h,t}\big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big)\,d\xi_k(s) \\
&\quad=: (V) + (VI).
\end{aligned} \tag{27}$$

To show (15) holds, we now turn our attention to estimate (V) and (VI). It is obvious that

$$\big\|(V)\big\|^2_{L^2(\Omega,\dot{H}^\sigma)} = \sum_{k\in\mathbb{Z}^*}\int_t^{t+\delta} \mathbb{E}\big\|K^*_{h,t+\delta}\big(\mathcal{S}_2(t+\delta,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\,ds.$$

By using a similar argument as in (18)–(21), one arrives at

$$\begin{aligned}
&\mathbb{E}\big\|K^*_{h,t+\delta}\big(\mathcal{S}_2(t+\delta,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma} \\
&\quad\leqslant c_h^2\Big(h-\frac{1}{2}\Big)^2 s^{1-2h}\bigg(\int_s^{t+\delta}\mu^{2h-1}(\mu-s)^{h-\frac{3}{2}}\,d\mu\bigg)\times \\
&\qquad\times\bigg(\int_s^{t+\delta}(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|\mathcal{S}_2(t+\delta,\mu)G(\mu)\mathcal{Q}^{1/2}e_k\big\|^2_{\dot{H}^\sigma}\,d\mu\bigg) \\
&\quad\leqslant \lambda_1^{-2}T^2M_1^2 s^{1-2h}\bigg(\int_s^{t+\delta}(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\mathcal{Q}^{1/2}e_k\big\|^2_{\mathbb{V}^T_{\sigma+1}}\,d\mu\bigg).
\end{aligned}$$

By a similar way as in (22)–(24) and noting that $\int_t^{t+\delta} s^{1-2h}\,ds \leqslant \frac{(t+\delta)^{2(1-h)}-t^{2(1-h)}}{2(1-h)} \leqslant \frac{\delta^{2(1-h)}}{2(1-h)}$ since $h\in(\frac{1}{2},1)$, one obtains

$$\begin{aligned}
\big\|(V)\big\|^2_{L^2(\Omega,\dot{H}^\sigma)} &\leqslant \lambda_1^{-2}T^2M_1^2\int_t^{t+\delta} s^{1-2h}\int_s^{t+\delta}(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\big\|^2_{\mathbb{W}^T_{\sigma+1}}\,d\mu\,ds \\
&\leqslant \lambda_1^{-2}T^2M_1^2\frac{\delta^{2(1-h)}}{2(1-h)}\bigg(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\bigg)^{\frac{r-2}{r}}\|G\|^2_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}.
\end{aligned} \tag{28}$$

We now continue with an estimate for the second term (VI). By the observation

$$\begin{aligned}
\big\|(VI)\big\|^2_{L^2(\Omega,\dot{H}^\sigma)} = \sum_{k\in\mathbb{Z}^*}\int_0^t \mathbb{E}\big\|&K^*_{h,t+\delta}\big(\mathcal{S}_2(t+\delta,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s) \\
&- K^*_{h,t}\big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\,ds,
\end{aligned} \tag{29}$$

it is necessary to estimate the expectation under the integral sign. From the observations (73), (72), we have

$$\begin{aligned}
&\mathbb{E}\big\| K^*_{h,t+\delta}\big(\mathcal{S}_2(t+\delta,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s) - K^*_{h,t}\big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\\
&\quad = \mathbb{E}\bigg\| \int_s^{t+\delta} \big(\mathcal{S}_2(t+\delta,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(\mu)\frac{\partial K_h}{\partial \mu}(\mu,s)\,d\mu\\
&\qquad - \int_s^t \big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(\mu)\frac{\partial K_h}{\partial \mu}(\mu,s)\,d\mu\bigg\|^2_{\dot{H}^\sigma}\\
&\quad \leqslant 2\mathbb{E}\bigg\| \int_t^{t+\delta} \big(\mathcal{S}_2(t+\delta,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(\mu)\frac{\partial K_h}{\partial \mu}(\mu,s)\,d\mu\bigg\|^2_{\dot{H}^\sigma}\\
&\qquad + 2\mathbb{E}\bigg\| \int_s^t \big(\big(\mathcal{S}_2(t+\delta,\cdot)-\mathcal{S}_2(t,\cdot)\big)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(\mu)\frac{\partial K_h}{\partial \mu}(\mu,s)\,d\mu\bigg\|^2_{\dot{H}^\sigma}\\
&\quad =: 2\mathcal{E}_1 + 2\mathcal{E}_2.
\end{aligned} \tag{30}$$

One can estimate the term $\mathcal{E}_1$ by a similar argument as in (18)–(21). In this way, one arrives at

$$\begin{aligned}
\mathcal{E}_1 &\leqslant M_1^2 s^{1-2h}\bigg(\int_t^{t+\delta}(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|\mathcal{S}_2(t+\delta,\mu)G(\mu)\mathcal{Q}^{1/2}e_k\big\|^2_{\dot{H}^\sigma}\,d\mu\bigg)\\
&\leqslant \lambda_1^{-2}T^2M_1^2 s^{1-2h}\bigg(\int_t^{t+\delta}(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\mathcal{Q}^{1/2}e_k\big\|^2_{\mathbb{V}^T_{\sigma+1}}\,d\mu\bigg).
\end{aligned} \tag{31}$$

The last term $\mathcal{E}_2$ can be estimated as in (25)

$$\mathcal{E}_2 \leqslant \big(\lambda_1^{-1}+T\big)^2\delta^2M_1^2 s^{1-2h}\bigg(\int_s^t(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\mathcal{Q}^{1/2}e_k\big\|^2_{\mathbb{V}^T_{\sigma+1}}\,d\mu\bigg). \tag{32}$$

From (29)–(32), we deduce that

$$\begin{aligned}
\big\|(VI)\big\|^2_{L^2(\Omega,\dot{H}^\sigma)} &\leqslant 2\lambda_1^{-2}T^2M_1^2\int_0^t s^{1-2h}\int_t^{t+\delta}(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\big\|^2_{\mathbb{W}^T_{\sigma+1}}\,d\mu\,ds\\
&\quad + 2\big(\lambda_1^{-1}+T\big)^2\delta^2M_1^2\int_0^t s^{1-2h}\int_s^t(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\big\|^2_{\mathbb{W}^T_{\sigma+1}}\,d\mu\,ds.
\end{aligned}$$

On the other hand, a similar argument as (23) yields

$$\int_t^{t+\delta}(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\big\|^2_{\mathbb{W}^T_{\sigma+1}}\,d\mu \leqslant \left(\frac{\delta^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{r}}\|G\|^2_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))},$$

where we have used the fact that $(t+\delta-s)^{\frac{r}{r-2}(h-\frac{3}{2})+1}-(t-s)^{\frac{r}{r-2}(h-\frac{3}{2})+1}\leqslant\delta^{\frac{r}{r-2}(h-\frac{3}{2})+1}$, since $\frac{r}{r-2}(h-\frac{3}{2})+1\in(0,1)$. Similarly, one can check that

$$\int_s^t(\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|G(\mu)\big\|^2_{\mathbb{W}^T_{\sigma+1}}\,d\mu\leqslant\left(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{r}}\|G\|^2_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}.$$

By the above arguments and recalling that $\int_0^t s^{1-2h}\,ds\leqslant\frac{T^{2-2h}}{2-2h}$, we deduce that there exists a positive constant $M_2=M_2(\mathcal{A},h,T,r)$

$$\big\|(VI)\big\|^2_{L^2(\Omega,\dot{H}^\sigma)}\leqslant M_2^2\big(\delta^2+\delta^{h-\frac{3}{2}+\frac{r-2}{r}}\big)\|G\|^2_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}. \tag{33}$$

Now, combining (27), (28), (33), we conclude that there exists a positive constant $C_7=C_7(\mathcal{A},h,T,r)$ such that (15) holds. □

From the four above lemmas, we can state the existences of the solutions to TVPs (1) and (2) in the following theorems.

Theorem 3.1. *Let $\sigma\geqslant 0$. Assume that u_f, F, G satisfy Assumptions (H1), (H2), (H3). Then, TVP (1) has a solution $u\in C^{\nu_1}(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))$ satisfying*

$$\big\|u(t+\delta)-u(t)\big\|_{L^2(\Omega,\dot{H}^\sigma)}\leqslant C_9\delta^{\nu_1}\big(\|u_f\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}+\|F\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))}+\|G\|_{L^q(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}\big),$$

where $\nu_1=\min\{\frac{p-1}{p},\frac{q-2}{2q}\}$. Furthermore, the following regularity property holds

$$\sup_{t\in\overline{\mathcal{J}}}\big\|u(t)\big\|_{L^2(\Omega,\dot{H}^\sigma)}\leqslant C_{10}\big(\|u_f\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}+\|F\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))}+\|G\|_{L^q(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}\big).$$

Here, $C_9=C_9(\mathcal{A},T,q)$, $C_{10}=C_{10}(\mathcal{A},T,q)$ are two positive constants.

Theorem 3.2. *Let $\sigma\geqslant 0$. Assume that u_f, F, G satisfy Assumptions (H1), (H2), (H4). Then, TVP (2) has a solution $\overline{u}\in C^{\nu_2}(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))$ satisfying*

$$\begin{aligned}&\big\|\overline{u}(t+\delta)-\overline{u}(t)\big\|_{L^2(\Omega,\dot{H}^\sigma)}\\&\quad\leqslant C_{11}\delta^{\nu_2}\big(\|u_f\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}+\|F\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))}+\|G\|_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}\big),\end{aligned}$$

where $\nu_2=\min\{\frac{p-1}{p},1-h,\frac{1}{2}(h-\frac{3}{2}+\frac{r-2}{r})\}$. Furthermore, the following regularity property holds

$$\sup_{t\in\overline{\mathcal{J}}}\big\|\overline{u}(t)\big\|_{L^2(\Omega,\dot{H}^\sigma)}\leqslant C_{12}\big(\|u_f\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}+\|F\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))}+\|G\|_{L^r(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))}\big).$$

Here, $C_{11}=C_{11}(\mathcal{A},T,h,r)$, $C_{12}=C_{12}(\mathcal{A},T,h,r)$ are two positive constants.

Proof of Theorem 3.1. For $\delta > 0$ small enough, we have from (5) that

$$\begin{aligned}
&\left\|u(t+\delta)-u(t)\right\|_{L^2(\Omega,\dot H^\sigma)}\\
&\quad\leqslant \left\|\left(\mathcal{S}_1(t+\delta,T)-\mathcal{S}_1(t,T)\right)u_f\right\|_{L^2(\Omega,\dot H^\sigma)}\\
&\qquad+\left\|\int_{t+\delta}^T \mathcal{S}_2(t+\delta,s)F(s)\,ds-\int_t^T \mathcal{S}_2(t,s)F(s)\,ds\right\|_{L^2(\Omega,\dot H^\sigma)}\\
&\qquad+\left\|\int_{t+\delta}^T \mathcal{S}_2(t+\delta,s)G(s)\,dW(s)-\int_t^T \mathcal{S}_2(t,s)G(s)\,dW(s)\right\|_{L^2(\Omega,\dot H^\sigma)},
\end{aligned}$$

and that

$$\begin{aligned}
\left\|u(t)\right\|_{L^2(\Omega,\dot H^\sigma)} &\leqslant \left\|\mathcal{S}_1(t,T)u_f\right\|_{L^2(\Omega,\dot H^\sigma)}+\left\|\int_t^T \mathcal{S}_2(t,s)F(s)\,ds\right\|_{L^2(\Omega,\dot H^\sigma)}\\
&\quad+\left\|\int_t^T \mathcal{S}_2(t,s)G(s)\,dW(s)\right\|_{L^2(\Omega,\dot H^\sigma)}.
\end{aligned}$$

Applying Lemma 3.1 and Lemma 3.2, one can easily prove the two results of Theorem 3.1. □

Proof of Theorem 3.2. Using a similar way as in the proof of Theorem 3.1 and applying Lemma 3.1, Lemma 3.3, Lemma 3.4, one can easily obtain the two results of Theorem 3.2. □

3.2. The instability of the solution of each TVP

The following pair of theorems will show that both TVPs we are studying are ill-posed on $C(\overline{\mathcal{J}}; L^2(\Omega,\dot H^\sigma))$, $\sigma \geqslant 0$, in the sense of Hadamard. For $k \in \mathbb{Z}^*$, $p \geqslant 2$, we define some necessary spaces as follows

$$\mathbb{U}_k := \left\{u_f^{(k)} = \alpha_k e_k \rho_k\right\}, \tag{34}$$

$$\mathbb{F}_k := \left\{F^{(k)} : [0,T]\times\Omega \to H,\ F^{(k)}(t) = \beta_k e_k \xi_k(t)\right\}, \tag{35}$$

$$\mathbb{G}_k := \left\{G^{(k)} : [0,T]\times\Omega \to L_0^2(H),\ G^{(k)}(t)\theta = \gamma_k(t) e_k(\theta,e_k)\varrho_k,\ \theta\in H\right\} \tag{36}$$

where $\alpha_k, \beta_k, \gamma_k$ are taken from sequences $\{\alpha_k\}_{k\in\mathbb{Z}^*}$, $\{\beta_k\}_{k\in\mathbb{Z}^*}$, $\{\gamma_k\}_{k\in\mathbb{Z}^*}$ satisfying

i) $\lim_{k\to\infty}\alpha_k = 0$, $\lim_{k\to\infty}\alpha_k\lambda_k^\sigma|1-T\lambda_k|e^{T\lambda_k} = \infty$,
ii) $\lim_{k\to\infty}\beta_k\lambda_k^{2\sigma+1}e^{T\lambda_k} = 0$,
iii) $\lim_{k\to\infty}\lambda_k^{2\sigma+1}e^{T\lambda_k}(\int_0^T \gamma_k^q(s)\,ds)^{\frac{1}{q}} = 0$

Here, ρ_k, ϱ_k are random variables with standard normal distribution and $\xi_k(t)$ are Wiener noises.

Theorem 3.3. *Let $\sigma \geqslant 0$, $q > 2$, $k \in \mathbb{Z}^+$. Assume that $(u_f^{(k)}, F^{(k)}, G^{(k)}) \in \mathbb{U}_k\times\mathbb{F}_k\times\mathbb{G}_k$. Then, there holds*

$$\lim_{k\to\infty}\left\|u_f^{(k)}\right\|_{L^2(\Omega,H)} = \lim_{k\to\infty}\left\|F^{(k)}\right\|_{L^p(\mathcal{J};L^2(\Omega,H))} = \lim_{k\to\infty}\left\|G^{(k)}\right\|_{L^q(\mathcal{J};L^2(\Omega,L_0^2(H)))} = 0. \tag{37}$$

However, the solution $u^{(k)}$ of TVP (1) with respect to the data $(u_f^{(k)}, F^{(k)}, G^{(k)})$ satisfies

$$\lim_{k\to\infty} \| u^{(k)} \|_{C(\overline{\mathcal{J}}, L^2(\Omega, \dot{H}^\sigma))} = \infty. \tag{38}$$

Theorem 3.4. *Let $\sigma \geqslant 0$, $r > \frac{2}{h-1/2}$, $k \in \mathbb{Z}^+$. Assume that $(u_f^{(k)}, F^{(k)}, G^{(k)}) \in \mathbb{U}_k \times \mathbb{F}_k \times \mathbb{G}_k$. Then, there holds*

$$\lim_{k\to\infty} \| u_f^{(k)} \|_{L^2(\Omega,H)} = \lim_{k\to\infty} \| F^{(k)} \|_{L^p(\mathcal{J};L^2(\Omega,H))} = \lim_{k\to\infty} \| G^{(k)} \|_{L^r(\mathcal{J};L^2(\Omega,L_0^2(H)))} = 0. \tag{39}$$

However, the solution $\overline{u}^{(k)}$ of TVP (2) with respect to the data $(u_f^{(k)}, F^{(k)}, G^{(k)})$ satisfies

$$\lim_{k\to\infty} \| \overline{u}^{(k)} \|_{C(\overline{\mathcal{J}}, L^2(\Omega, \dot{H}^\sigma))} = \infty. \tag{40}$$

Remark 3.1. We can give here an example for $(u_f^{(k)}, F^{(k)}, G^{(k)}) \in \mathbb{U}_k \times \mathbb{F}_k \times \mathbb{G}_k$ as follows

$$\begin{aligned}
&u_f^{(k)} = \alpha_k e_k \rho_k, \quad \text{with } \alpha_k = \lambda_k^{-\tau_1},\\
&F^{(k)}(t) = \beta_k e_k \xi_k(t), \quad \text{with } \beta_k = \lambda_k^{-(2\sigma+1+\tau_2)} e^{-T\lambda_k},\\
&G^{(k)}(t)\theta = \gamma_k(t) e_k (\theta, e_k) \varrho_k, \quad \theta \in H, \text{with } \gamma_k(t) = \lambda_k^{-(2\sigma+1+\tau_3)} e^{-T\lambda_k} \varpi(t),
\end{aligned}$$

where $\tau_1, \tau_2, \tau_3 > 0$, and $\varpi : \overline{\mathcal{J}} \to \mathbb{R}^+$ is a continuous function.

Remark 3.2. From Theorem 3.3 (res. Theorem 3.4), it is clear that the solution of TVP (1) (res. TVP (2)) does not depend continuously on the data. In other words, the solutions of two problems are unstable, which leads to their ill-posedness.

Proof of Theorem 3.3. We begin with the proof of (37). Since $(u_f^{(k)}, F^{(k)}, G^{(k)}) \in \mathbb{U}_k \times \mathbb{F}_k \times \mathbb{G}_k$, it is obvious that

$$\| u_f^{(k)} \|_{L^2(\Omega,H)} = \alpha_k \| e_k \|_H \left(\mathbb{E}\left[\rho_k^2\right]\right)^{1/2} = \alpha_k \longrightarrow 0, \quad \text{as } k \to \infty,$$

and

$$\begin{aligned}
\lim_{k\to\infty} \| F^{(k)} \|_{L^p(\mathcal{J};L^2(\Omega,H))} &= \lim_{k\to\infty} \left(\int_0^T \left(\mathbb{E} \| F^{(k)}(s) \|_H^2\right)^{q/2} ds \right)^{\frac{1}{q}} = \lim_{k\to\infty} \left(\int_0^T \left(\mathbb{E}\left[\xi_k^2(s)\right]\right)^{q/2} ds \right)^{\frac{1}{q}} \beta_k \\
&= \lim_{k\to\infty} \left(\frac{T^{q/2+1}}{q/2+1} \right)^{\frac{1}{q}} \beta_k = 0,
\end{aligned} \tag{41}$$

where we have used the properties that $\|e_k\|_H = 1$, $\mathbb{E}[\rho_k^2] = 1$, and $\mathbb{E}[\xi_k(t)\xi_l(t)] = 0$, for $k \neq l$. We also have

$$\lim_{k\to\infty}\|G^{(k)}\|_{L^q(\mathcal{J};L^2(\Omega,L_0^2(H)))} = \lim_{k\to\infty}\left(\int_0^T \left(\mathbb{E}\|G^{(k)}(s)\|^2_{L_0^2(H)}\right)^{\frac{q}{2}} ds\right)^{\frac{1}{q}}$$
$$= \lim_{k\to\infty}\left(\int_0^T \left(\sum_{l\in\mathbb{Z}^*}\mathbb{E}\|\mathcal{Q}^{\frac{1}{2}}G^{(k)}(s)e_l\|^2_H\right)^{\frac{q}{2}} ds\right)^{\frac{1}{q}}. \tag{42}$$

Since $\sum_{l\in\mathbb{Z}^*}\mathbb{E}\|\mathcal{Q}^{\frac{1}{2}}G^{(k)}(s)e_l\|^2_H = \gamma_k^2(s)\mathbb{E}\varrho_k^2 = \gamma_k^2(s)$, we deduce that

$$\lim_{k\to\infty}\|G^{(k)}\|_{L^q(\mathcal{J};L^2(\Omega,L_0^2(H)))} = \lim_{k\to\infty}\left(\int_0^T \gamma_k^q(s)\,ds\right)^{\frac{1}{q}} = 0. \tag{43}$$

Next, we aim to prove (38). From the equation (5), we have the following representation for $u^{(k)}$

$$u^{(k)}(t) = \mathcal{S}_1(t,T)u_f^{(k)} + \int_t^T \mathcal{S}_2(t,s)F^{(k)}(s)\,ds + \int_t^T \mathcal{S}_2(t,s)G^{(k)}(s)\,dW(s).$$

For the first term on the right-hand side,

$$\|\mathcal{S}_1(\cdot,T)u_f^{(k)}\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} = \sup_{t\in\overline{\mathcal{J}}}\left(\sum_{l\in\mathbb{Z}^*}\mathbb{E}\left(u_f^{(k)},e_l\right)^2\lambda_l^{2\sigma}\left(1-(T-t)\lambda_l\right)^2 e^{2(T-t)\lambda_l}\right)^{\frac{1}{2}}$$
$$= \alpha_k\lambda_k^\sigma \sup_{t\in\overline{\mathcal{J}}}\left|1-(T-t)\lambda_k\right|e^{(T-t)\lambda_k}$$
$$\geqslant \alpha_k\lambda_k^\sigma|1-T\lambda_k|e^{T\lambda_k},$$

which implies that $\lim_{k\to\infty}\|\mathcal{S}_1(\cdot,T)u_f^{(k)}\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} = \infty$. By a similar way as in (41), one can check that

$$\lim_{k\to\infty}\|F^{(k)}\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))} = \lim_{k\to\infty}\left(\int_0^T \left(\mathbb{E}\|F^{(k)}(s)\|^2_{\mathbb{V}^T_{\sigma+1}}\right)^{\frac{q}{2}} ds\right)^{\frac{1}{q}}$$
$$= \lim_{k\to\infty}\left(\frac{T^{q/2+1}}{q/2+1}\right)^{\frac{1}{q}}\beta_k\lambda_k^{2\sigma+1}e^{T\lambda_k} = 0.$$

This associated with estimate (8) allow that

$$\left\|\int_t^T \mathcal{S}_2(t,s)F^{(k)}(s)\,ds\right\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant \lambda_1^{-1}T^{(2p-1)/p}\|F^{(k)}\|_{L^p(\mathcal{J};L^2(\Omega,\mathbb{V}^T_{\sigma+1}))} \to 0, \quad \text{as } k\to\infty.$$

By a similar way as in (42)–(43),

$$\lim_{k\to\infty}\|G^{(k)}\|_{L^q(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))} = \lim_{k\to\infty}\lambda_k^{2\sigma+1}e^{T\lambda_k}\left(\int_0^T \gamma_k^q(s)\,ds\right)^{\frac{1}{q}} = 0.$$

This associated with the estimate (11) imply

$$\left\| \int_t^T \mathcal{S}_2(t,s) G^{(k)}(s)\, dW(s) \right\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant C_3 \| G^{(k)} \|_{L^q(\mathcal{J};L^2(\Omega,\mathbb{W}^T_{\sigma+1}))} \to 0, \quad \text{as } k \to \infty.$$

By all above arguments and noting that

$$\begin{aligned} \| u^{(k)} \|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \geqslant{} & \| \mathcal{S}_1(\cdot, T) u_f^{(k)} \|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} - \left\| \int_t^T \mathcal{S}_2(t,s) F^{(k)}(s)\, ds \right\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \\ & - \left\| \int_t^T \mathcal{S}_2(t,s) G^{(k)}(s)\, dW(s) \right\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))}, \end{aligned}$$

we conclude that $\lim_{k\to\infty} \|u^{(k)}\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} = \infty$. □

Proof of Theorem 3.4. The estimate (40) can be proved easily by using a similar way as in the proof of (38). We note that the strategy here is to use (13) instead of (11). □

4. Regularization for each TVP

Physically, in most situations of reality, we cannot obtain exactly the data (u_f, F, G). From the observations, we can only have the contaminated data (denote by $(\widetilde{u}^\varepsilon_f, \widetilde{F}^\varepsilon, \widetilde{G}^\varepsilon)$) which contain small errors. Assume that $(\widetilde{u}^\varepsilon_f, \widetilde{F}^\varepsilon, \widetilde{G}^\varepsilon)$ satisfies the model

$$\| \widetilde{u}^\varepsilon_f - u_f \|_{L^2(\Omega,H)} \leqslant \varepsilon, \qquad \| \widetilde{F}^\varepsilon - F \|_{L^p(\mathcal{J};L^2(\Omega,H)} \leqslant \varepsilon, \qquad \| \widetilde{G}^\varepsilon - G \|_{L^q(\mathcal{J};L^2(\Omega,L^2_0(H)))} \leqslant \varepsilon, \quad (44)$$

where $\varepsilon > 0$ is the noisy level and p, q will be specified latter.

Since TVP (1) is ill-posedness, it is required to establish an approximate solution (called regularized solution), denoted by $\mathcal{U}^{\alpha,\epsilon}$, such that $\|\mathcal{U}^{\alpha,\epsilon} - u\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))}$ tends to zero as $\varepsilon \to 0^+$. Similarly, we also construct a regularized solution for TVP (2), denoted by $\overline{\mathcal{U}}^{\alpha,\epsilon}$, such that $\|\overline{\mathcal{U}}^{\alpha,\epsilon} - \overline{u}\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))}$ tends to zero as $\varepsilon \to 0^+$.

4.1. Regularization for TVP (1)

We now use a regularization method, named filter method (see [36]), to establish a regularized solution for TVP (1). From Section 3, we remark that the reason makes our problem be ill-posed is that both operators $\mathcal{S}_1(t,T)$, $\mathcal{S}_2(t,s)$ are not bounded in $\mathcal{L}(L^2(\Omega,\dot{H}^\sigma), L^2(\Omega,\dot{H}^\sigma))$. Hence, our strategy here is to replace those operators by new approximate ones, which are bounded in $\mathcal{L}(L^2(\Omega,\dot{H}^\sigma), L^2(\Omega,\dot{H}^\sigma))$.

Let $\sigma \geqslant 0$ and $\alpha = \alpha(\varepsilon)$ be a positive number. Precisely, in this subsection, we construct a regularized solution as follows

$$\mathcal{U}^{\alpha,\varepsilon}(t) = \widetilde{\mathcal{S}}_{\alpha,1}(t,T)\widetilde{u}^\varepsilon_f + \int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\widetilde{F}^\varepsilon(s)\, ds + \int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\widetilde{G}^\varepsilon(s)\, dW(s).$$

Here, operators $\widetilde{\mathcal{S}}_{\alpha,1}(t,T)$, $\widetilde{\mathcal{S}}_{\alpha,2}(t,s)$ are defined by

$$\widetilde{\mathcal{S}}_{\alpha,1}(t,T)\theta := \sum_{k\in\mathbb{Z}^*}(\theta,e_k)\zeta_{\alpha,k}\big(1-(T-t)\lambda_k\big)e^{(T-t)\lambda_k}e_k, \tag{45}$$

$$\widetilde{\mathcal{S}}_{\alpha,2}(t,s)\theta := \sum_{k\in\mathbb{Z}^*}(\theta,e_k)\zeta_{\alpha,k}(s-t)e^{(s-t)\lambda_k}e_k, \quad \theta\in H, t,s\in\overline{\mathcal{J}}, \tag{46}$$

where the kernels $\zeta_{\alpha,k}$ (called filter kernels), with $k\in\mathbb{Z}^+$, will be specified later. The following lemma states the convergence rate of the regularized solution.

Theorem 4.1. *Let $\sigma\geqslant 0$, $\alpha=\alpha(\epsilon)>0$. Suppose that (44) holds for some $p>1$, $q\geqslant 2$, and there exists $E>0$ such that $\sup_{t\in\overline{\mathcal{J}}}\|u(t)\|_{L^2(\Omega,\dot{H}^{\sigma+\eta})}\leqslant E$, for some $\eta>0$. Assume that the filter kernels $\zeta_{\alpha,k}$ satisfy*

(S1) *both $|\zeta_{\alpha,k}|\lambda_k^{\sigma+1}e^{T\lambda_k}$ and $|\zeta_{\alpha,k}|\lambda_k^{\sigma}e^{T\lambda_k}$ are bounded by some $\Pi_{\alpha,1}>0$ independent of k,*
(S2) *$|\zeta_{\alpha,k}-1|\leqslant\Pi_{\alpha,2}\lambda_k^{\eta}$, where $\Pi_{\alpha,2}$ is some positive constant independent of k.*

Assume further that $\Pi_{\alpha,1}$, $\Pi_{\alpha,2}$ satisfy $\lim_{\varepsilon\to0^+}(\Pi_{\alpha,1}\varepsilon)=\lim_{\varepsilon\to0^+}\Pi_{\alpha,2}=0$. Then, the following error estimate holds

$$\big\|\mathcal{U}^{\alpha,\epsilon}-u\big\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^{\sigma}))}\leqslant\big(C_1+T^{\frac{2p-1}{p}}+T^{\frac{3q-2}{2q}}\big)\Pi_{\alpha,1}\varepsilon+\Pi_{\alpha,2}E. \tag{47}$$

Consequently, $\lim_{\varepsilon\to0^+}\|\mathcal{U}^{\alpha,\varepsilon}-u\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^{\sigma}))}=0$.

Remark 4.1 (Example 1 for the filter kernels $\zeta_{\alpha,k}$)**.** Let $\sigma\geqslant0$ and $\alpha(\varepsilon)=T\log^{-1}(\varepsilon^{-\vartheta})$, with some $\vartheta<\frac{1}{\sigma+2}$. Suppose that (44) holds for some $p>1$, $q>2$, and there exists $E>0$ such that $\sup_{t\in\overline{\mathcal{J}}}\|u(t)\|_{L^2(\Omega,\dot{H}^{\sigma+\eta})}\leqslant E$, for some $\eta>0$. If we choose the filter kernels $\zeta_{\alpha,k}$ as follows

$$\zeta_{\alpha,k}=1,\quad \text{if } \lambda_k\leqslant\alpha^{-1},\qquad \zeta_{\alpha,k}=0,\quad \text{if } \lambda_k>\alpha^{-1},$$

then $\zeta_{\alpha,k}$ satisfies conditions (S1), (S2) with $\Pi_{\alpha,1}=\max\{1,\lambda_1^{-1}\}\alpha^{-(\sigma+1)}e^{T\alpha^{-1}}$ and $\Pi_{\alpha,2}=\alpha^{\eta}$. Furthermore, it is clear that $\lim_{\varepsilon\to0^+}(\Pi_{\alpha,1}\varepsilon)=\lim_{\varepsilon\to0^+}\Pi_{\alpha,2}=0$. As a consequence of Theorem 4.1, we obtain $\lim_{\varepsilon\to0^+}\|\mathcal{U}^{\alpha,\varepsilon}-u\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^{\sigma}))}=0$. Here the condition $\vartheta<\frac{1}{\sigma+2}$ is proposed to guarantee $\lim_{\varepsilon\to0^+}(\Pi_{\alpha,1}\varepsilon)=\lim_{\varepsilon\to0^+}\Pi_{\alpha,2}=0$.

Remark 4.2 (Example 2 for the filter kernels $\zeta_{\alpha,k}$)**.** Let $\sigma\geqslant0$ and $\alpha=\alpha(\varepsilon)=\varepsilon^{\vartheta}$, for some $0<\vartheta<1$. Suppose that (44) holds for some $p>1$, $q\geqslant2$, and there exists $E>0$ such that $\sup_{t\in\overline{\mathcal{J}}}\|u(t)\|_{L^2(\Omega,\dot{H}^{\eta})}\leqslant E$, for $\eta=\sigma+1$. If we choose the filter kernels $\zeta_{\alpha,k}$ as follows

$$\zeta_{\alpha,k}=\big(1+\alpha\lambda_k^{\sigma+1}e^{\lambda_k^{\sigma+1}[1+T\lambda_1^{-\sigma}]}\big)^{-1},$$

then $\zeta_{\alpha,k}$ satisfies conditions (S1), (S2) with $\Pi_{\alpha,1}=\frac{[1+T\lambda_1^{-\sigma}]/\alpha}{\log([1+T\lambda_1^{-\sigma}]/\alpha)}$ and $\Pi_{\alpha,2}=\frac{1+T\lambda_1^{-\sigma}}{\log([1+T\lambda_1^{-\sigma}]/\alpha)}$. Indeed, by using the inequality $e^{-c_1\lambda}+c_2\lambda\geqslant\frac{\log(c_1/c_2)}{c_1/c_2}$, for $0<c_2<ec_1$, $\lambda>0$, we have

$$|\zeta_{\alpha,k}|\lambda_k^{\sigma+1}e^{T\lambda_k}\leqslant|\zeta_{\alpha,k}|e^{\lambda_k^{\sigma+1}[1+T\lambda_1^{-\sigma}]}=\big(e^{-\lambda_k^{\sigma+1}[1+T\lambda_1^{-\sigma}]}+\alpha\lambda_k^{\sigma+1}\big)^{-1}\leqslant\frac{[1+T\lambda_1^{-\sigma}]/\alpha}{\log([1+T\lambda_1^{-\sigma}]/\alpha)},$$

and

$$|\zeta_{\alpha,k} - 1| = \alpha\lambda_k^{\sigma+1}\big(e^{-\lambda_k^{\sigma+1}[1+T\lambda_1^{-\sigma}]} + \alpha\lambda_k^{\sigma+1}\big)^{-1} \leqslant \lambda_k^{\sigma+1}\frac{1+T\lambda_1^{-\sigma}}{\log([1+T\lambda_1^{-\sigma}]/\alpha)}.$$

Furthermore, it is clear that $\lim_{\varepsilon\to 0^+}(\Pi_{\alpha,1}\varepsilon) = \lim_{\varepsilon\to 0^+}\Pi_{\alpha,2} = 0$. As a consequence of Theorem 4.1, we obtain $\lim_{\varepsilon\to 0^+}\|\mathcal{U}^{\alpha,\varepsilon} - u\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} = 0$.

Remark 4.3. The strong assumption $\sup_{t\in\overline{\mathcal{J}}}\|u(t)\|_{L^2(\Omega,\dot{H}^{\sigma+\eta})} \leqslant E$ implies a very quick decay of the Fourier coefficients of the final datum u_f (one can see clearly from the equation (74)). Now, let us turn our attention to $u_f^{(k)}$ constructed in (34). It can be seen that there is a reverse trend here, where we do not observe a significant decrease in the corresponding Fourier coefficients, which makes the instability of the solution.

Proof. Step 1. In this step, our goal is to prove that if conditions (S1), (S2) hold, then we have

$$\big\|\widetilde{\mathcal{S}}_{\alpha,1}(t,T)\big\|_{L^2(\Omega,H)\to L^2(\Omega,\dot{H}^\sigma)} \leqslant C_1\Pi_{\alpha,1}, \qquad \big\|\widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big\|_{L^2(\Omega,H)\to L^2(\Omega,\dot{H}^\sigma)} \leqslant T\Pi_{\alpha,1}, \tag{48}$$

for every $t, s \in \overline{\mathcal{J}}$. For $\theta \in L^2(\Omega, H)$, $t \in \overline{\mathcal{J}}$, it is easy to see that

$$\begin{aligned}
\mathbb{E}\big\|\widetilde{\mathcal{S}}_{\alpha,1}(t,T)\theta\big\|_{\dot{H}^\sigma}^2 &= \sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2\sigma}\zeta_{\alpha,k}^2\big(1-(T-t)\lambda_k\big)^2 e^{2(T-t)\lambda_k} \\
&\leqslant \sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2\sigma}\zeta_{\alpha,k}^2\big(1+(T-t)^2\lambda_k^2\big)e^{2(T-t)\lambda_k} \\
&\leqslant \big(\lambda_1^{-2}+(T-t)^2\big)\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\zeta_{\alpha,k}^2\lambda_k^{2(\sigma+1)}e^{2T\lambda_k} \\
&\leqslant \big(\lambda_1^{-2}+T^2\big)\Pi_{\alpha,1}^2\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2,
\end{aligned}$$

which implies that $\|\widetilde{\mathcal{S}}_{\alpha,1}(t,T)\theta\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant C_1\Pi_{\alpha,1}\|\theta\|_{L^2(\Omega,H)}$. Similarly, for $t, s \in \overline{\mathcal{J}}$,

$$\begin{aligned}
\big\|\widetilde{\mathcal{S}}_{\alpha,2}(t,s)\theta\big\|_{L^2(\Omega,\dot{H}^\sigma)}^2 = \mathbb{E}\big\|\widetilde{\mathcal{S}}_{\alpha,2}(t,s)\theta\big\|_{\dot{H}^\sigma}^2 &= \sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2\sigma}\zeta_{\alpha,k}^2(s-t)^2e^{2(s-t)\lambda_k} \\
&\leqslant T^2\Pi_{\alpha,1}^2\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2,
\end{aligned}$$

which shows that $\|\widetilde{\mathcal{S}}_{\alpha,2}(t,s)\theta\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant T\Pi_{\alpha,1}\|\theta\|_{L^2(\Omega,H)}$. We finish Step 1 here.

Step 2. In this step we prove the error estimate (47). Firstly, it is clear that

$$\mathcal{U}^{\alpha,\varepsilon}(t) - u(t) = \mathcal{V}_1^{\alpha,\varepsilon}(t) + \mathcal{V}_2^{\alpha,\varepsilon}(t), \tag{49}$$

where we set $\mathcal{V}_1^{\alpha,\varepsilon}(t)$ and $\mathcal{V}_2^{\alpha,\varepsilon}(t)$ as follows

$$\begin{aligned}\mathcal{V}_1^{\alpha,\varepsilon}(t) &= \widetilde{\mathcal{S}}_{\alpha,1}(t,T)\big(\widetilde{u}_f^{\varepsilon} - u_f\big) + \int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{F}^{\varepsilon}(s) - F(s)\big)\,ds \\ &\quad + \int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{G}^{\varepsilon}(s) - G(s)\big)\,dW(s), \end{aligned}\tag{50}$$

$$\begin{aligned}\mathcal{V}_2^{\alpha,\varepsilon}(t) &= \big(\widetilde{\mathcal{S}}_{\alpha,1}(t,T) - \mathcal{S}_1(t,T)\big)u_f + \int_t^T \big(\widetilde{\mathcal{S}}_{\alpha,2}(t,s) - \mathcal{S}_2(t,s)\big)F(s)\,ds \\ &\quad + \int_t^T \big(\widetilde{\mathcal{S}}_{\alpha,2}(t,s) - \mathcal{S}_2(t,s)\big)G(s)\,dW(s). \end{aligned}\tag{51}$$

Now, we use the properties in (48) to estimate the first term $\mathcal{V}_1^{\alpha,\varepsilon}(t)$. It follows from (50) that

$$\begin{aligned}\big\|\mathcal{V}_1^{\alpha,\varepsilon}(t)\big\|_{L^2(\Omega,\dot{H}^{\sigma})} &\leqslant \big\|\widetilde{\mathcal{S}}_{\alpha,1}(t,T)\big(\widetilde{u}_f^{\varepsilon} - u_f\big)\big\|_{L^2(\Omega,\dot{H}^{\sigma})} + \left\|\int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{F}^{\varepsilon}(s) - F(s)\big)\,ds\right\|_{L^2(\Omega,\dot{H}^{\sigma})} \\ &\quad + \left\|\int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{G}^{\varepsilon}(s) - G(s)\big)\,dW(s)\right\|_{L^2(\Omega,\dot{H}^{\sigma})} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

By using the first property in (48), we obtain

$$I_1 \leqslant \big\|\widetilde{\mathcal{S}}_{\alpha,1}(t,T)\big\|_{L^2(\Omega,H)\to L^2(\Omega,\dot{H}^{\sigma})}\big\|\widetilde{u}_f^{\varepsilon} - u_f\big\|_{L^2(\Omega,H)} \leqslant C_1\Pi_{\alpha,1}\varepsilon. \tag{52}$$

The second property in (48) allows that

$$\begin{aligned} I_2 &\leqslant \int_t^T \big\|\widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{F}^{\varepsilon}(s) - F(s)\big)\big\|_{L^2(\Omega,\dot{H}^{\sigma})}\,ds \\ &\leqslant \int_t^T \big\|\widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big\|_{L^2(\Omega,H)\to L^2(\Omega,\dot{H}^{\sigma})}\big\|\widetilde{F}^{\varepsilon}(s) - F(s)\big\|_{L^2(\Omega,H)}\,ds \\ &\leqslant T\Pi_{\alpha,1}\int_t^T \big\|\widetilde{F}^{\varepsilon}(s) - F(s)\big\|_{L^2(\Omega,H)}\,ds \end{aligned}$$

Applying the Hölder inequality,

$$\begin{aligned} I_2 &\leqslant T\Pi_{\alpha,1}\left(\int_t^T ds\right)^{\frac{p-1}{p}}\left(\int_t^T \big\|\widetilde{F}^{\varepsilon}(s) - F(s)\big\|_{L^2(\Omega,H)}^p\,ds\right)^{\frac{1}{p}} \\ &\leqslant T^{\frac{2p-1}{p}}\Pi_{\alpha,1}\big\|\widetilde{F}^{\varepsilon} - F\big\|_{L^p(\mathcal{J};L^2(\Omega,H))} \leqslant T^{\frac{2p-1}{p}}\Pi_{\alpha,1}\varepsilon. \end{aligned}\tag{53}$$

For the last term, we can estimate by using the Itô isometry and the second property in (48)

$$
\begin{aligned}
I_3^2 &= \int_t^T \mathbb{E}\big\| \mathcal{A}^\sigma \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{G}^\varepsilon(s) - G(s)\big)\big\|^2_{L_0^2(H)}\,ds \\
&= \int_t^T \sum_{k\in\mathbb{Z}^*} \mathbb{E}\big\| \mathcal{A}^\sigma \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{G}^\varepsilon(s) - G(s)\big)\mathcal{Q}^{\frac{1}{2}} e_k\big\|^2_H\,ds \\
&\leqslant T^2\Pi_{\alpha,1}^2 \int_t^T \sum_{k\in\mathbb{Z}^*} \mathbb{E}\big\| \big(\widetilde{G}^\varepsilon(s) - G(s)\big)\mathcal{Q}^{\frac{1}{2}} e_k\big\|^2_H\,ds \\
&\leqslant T^2\Pi_{\alpha,1}^2 \int_t^T \mathbb{E}\big\|\widetilde{G}^\varepsilon(s) - G(s)\big\|^2_{L_0^2(H)}\,ds. \qquad (54)
\end{aligned}
$$

The Hölder inequality once more implies

$$
\begin{aligned}
I_3 &\leqslant T\Pi_{\alpha,1}\bigg(\int_t^T ds\bigg)^{\frac{q-2}{2q}} \bigg(\int_t^T \big(\mathbb{E}\big\|\widetilde{G}^\varepsilon(s) - G(s)\big\|^2_{L_0^2(H)}\big)^{\frac{q}{2}}\,ds\bigg)^{\frac{1}{q}} \\
&\leqslant T^{\frac{3q-2}{2q}}\Pi_{\alpha,1}\big\|\widetilde{G}^\varepsilon - G\big\|_{L^q(\mathcal{J};L^2(\Omega,L_0^2(H)))} \leqslant T^{\frac{3q-2}{2q}}\Pi_{\alpha,1}\varepsilon, \quad \text{for } q>2. \qquad (55)
\end{aligned}
$$

In the case of $q=2$, we have from (54) that

$$
I_3 \leqslant T\Pi_{\alpha,1}\big\|\widetilde{G}^\varepsilon - G\big\|_{L^q(\mathcal{J};L^2(\Omega,L_0^2(H)))} \leqslant T^{\frac{3q-2}{2q}}\Pi_{\alpha,1}\varepsilon.
$$

From (52), (53), (55), we deduce that

$$
\big\|\mathcal{V}_1^{\alpha,\varepsilon}(t)\big\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant \big(C_1 + T^{\frac{2p-1}{p}} + T^{\frac{3q-2}{2q}}\big)\Pi_{\alpha,1}\varepsilon.
$$

Since the right-hand side of the above inequality does not depend on the variable t, we see that

$$
\big\|\mathcal{V}_1^{\alpha,\varepsilon}\big\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant \big(C_1 + T^{\frac{2p-1}{p}} + T^{\frac{3q-2}{2q}}\big)\Pi_{\alpha,1}\varepsilon. \qquad (56)
$$

Next, we estimate the last term $\mathcal{V}_2^{\alpha,\varepsilon}(t)$. From (51) and the formulas of the operators $\widetilde{\mathcal{S}}_{\alpha,1}(t,T)$, $\widetilde{\mathcal{S}}_{\alpha,2}(t,s)$, $\mathcal{S}_1(t,T)$, $\mathcal{S}_2(t,s)$, we observe that

$$
\begin{aligned}
\mathcal{V}_2^{\alpha,\varepsilon}(t) &= \sum_{k\in\mathbb{Z}^*}\big(1-(T-t)\lambda_k\big)e^{(T-t)\lambda_k}(\zeta_{\alpha,k}-1)(u_f,e_k)e_k \\
&\quad + \sum_{k\in\mathbb{Z}^*}\bigg(\int_t^T (s-t)e^{(s-t)\lambda_k}\big(F(s),e_k\big)\,ds\bigg)(\zeta_{\alpha,k}-1)e_k \\
&\quad + \sum_{k\in\mathbb{Z}^*}\bigg(\Lambda_k^{1/2}\int_0^T (s-t)e^{(s-t)\lambda_k}G(s)\,d\xi_k(s) - \Lambda_k^{1/2}\int_0^t (s-t)e^{(s-t)\lambda_k}G(s)\,d\xi_k(s)\bigg) \\
&\quad \times (\zeta_{\alpha,k}-1)e_k. \qquad (57)
\end{aligned}
$$

This together with (74) gives $\mathcal{V}_2^{\alpha,\varepsilon}(t) = \sum_{k\in\mathbb{Z}^*}(u(t), e_k)(\zeta_{\alpha,k} - 1)e_k$, which follows that

$$\begin{aligned}\left\|\mathcal{V}_2^{\alpha,\varepsilon}(t)\right\|^2_{L^2(\Omega,\dot{H}^\sigma)} &= \sum_{k\in\mathbb{Z}^*} \lambda_k^{2\sigma}\mathbb{E}\big(u(t), e_k\big)^2|\zeta_{\alpha,k} - 1|^2 \leqslant \Pi^2_{\alpha,2}\sum_{k\in\mathbb{Z}^*}\mathbb{E}\big(u(t), e_k\big)^2\lambda_k^{2(\sigma+\eta)} \\ &= \Pi^2_{\alpha,2}\left\|u(t)\right\|^2_{L^2(\Omega,\dot{H}^{\sigma+\eta})}.\end{aligned}$$

Since $\|u(t)\|_{L^2(\Omega,\dot{H}^{\sigma+\eta})} \leqslant E$, the last inequality implies

$$\left\|\mathcal{V}_2^{\alpha,\varepsilon}\right\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant \Pi_{\alpha,2}E. \tag{58}$$

Combining (49), (56), (58) and using the triangle inequality, we deduce that

$$\left\|\mathcal{U}^{\alpha,\varepsilon} - u\right\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant \big(C_1 + T^{\frac{2p-1}{p}} + T^{\frac{3q-2}{2q}}\big)\Pi_{\alpha,1}\varepsilon + \Pi_{\alpha,2}E.$$

By the conditions $\lim_{\varepsilon\to 0^+}(\Pi_{\alpha,1}\varepsilon) = \lim_{\varepsilon\to 0^+}\Pi_{\alpha,2} = 0$, we conclude that $\|\mathcal{U}^{\alpha,\varepsilon} - u\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))}$ tends to zero as $\varepsilon \to 0^+$. □

4.2. Regularization for TVP (2)

Let us construct a regularized solution for TVP (2) as follows

$$\begin{aligned}\overline{\mathcal{U}}^{\alpha,\varepsilon}(t) &= \widetilde{\mathcal{S}}_{\alpha,1}(t,T)\widetilde{u}_f^\varepsilon + \int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\widetilde{F}^\varepsilon(s)\,ds \\ &\quad + \int_0^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\widetilde{G}^\varepsilon(s)\,dW^h(s) - \int_0^t \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\widetilde{G}^\varepsilon(s)\,dW^h(s),\end{aligned}$$

where the operators $\widetilde{\mathcal{S}}_{\alpha,1}(t,T)$, $\widetilde{\mathcal{S}}_{\alpha,2}(t,s)$ are of the forms (45), (46) respectively. The following lemma investigates the convergence rate of this regularized solution.

Theorem 4.2. *Let $\sigma \geqslant 0$, $\alpha = \alpha(\epsilon) > 0$. Suppose that (44) holds for some $p > 1$, $q = r > \frac{2}{h-1/2}$, and there exists $\overline{E} > 0$ such that $\sup_{t\in\overline{\mathcal{J}}}\|\overline{u}(t)\|_{L^2(\Omega,\dot{H}^{\sigma+\overline{\eta}})} \leqslant \overline{E}$, for some $\overline{\eta} > 0$. Assume that the filter kernels $\zeta_{\alpha,k}$ satisfy*

(S1) *both $|\zeta_{\alpha,k}|\lambda_k^{\sigma+1}e^{T\lambda_k}$ and $|\zeta_{\alpha,k}|\lambda_k^{\sigma}e^{T\lambda_k}$ are bounded by some $\overline{\Pi}_{\alpha,1} > 0$ independent of k,*
(S2) *$|\zeta_{\alpha,k} - 1| \leqslant \overline{\Pi}_{\alpha,2}\lambda_k^{\overline{\eta}}$, where $\overline{\Pi}_{\alpha,2}$ is some positive constant independent of k.*

Assume further that $\overline{\Pi}_{\alpha,1}$, $\overline{\Pi}_{\alpha,2}$ satisfy $\lim_{\varepsilon\to 0^+}(\overline{\Pi}_{\alpha,1}\varepsilon) = \lim_{\varepsilon\to 0^+}\overline{\Pi}_{\alpha,2} = 0$. Then, there exists a positive constant M_0 independent of α, ε such that the following error estimate holds

$$\|\overline{\mathcal{U}}^{\alpha,\epsilon} - \overline{u}\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant M_0\overline{\Pi}_{\alpha,1}\varepsilon + \overline{\Pi}_{\alpha,2}\overline{E}. \tag{59}$$

Consequently, $\lim_{\varepsilon\to 0^+}\|\overline{\mathcal{U}}^{\alpha,\varepsilon} - \overline{u}\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} = 0.$

Remark 4.4.

i) Notice that the result in Theorem 4.2 needs a more strict assumption for the data than the result in Theorem 4.1. The first result only needs (44) holds for $q > 2$ while the second result needs it holds for $q = r > \frac{2}{h-1/2}$, which is greater than 2.

ii) One can easily give some similar examples for the filter kernels $\zeta_{\alpha,k}$, $k \in \mathbb{Z}^+$, as in Remark 4.1 and Remark 4.2.

Remark 4.5. The strong assumption $\sup_{t\in\overline{\mathcal{J}}} \|\overline{u}(t)\|_{L^2(\Omega,\dot{H}^{\sigma+\overline{\eta}})} \leqslant \overline{E}$ implies a very quick decay of the Fourier coefficients of the final datum u_f. By contract, in the case $u_f^{(k)}$ is constructed in (34), we do not see a significant decrease in the corresponding Fourier coefficients, which makes the instability of the solution.

Proof. Let us split $\overline{\mathcal{U}}^{\alpha,\varepsilon}(t) - \overline{u}(t)$ into two terms as follows

$$\overline{\mathcal{U}}^{\alpha,\varepsilon}(t) - \overline{u}(t) = \overline{\mathcal{V}}_1^{\alpha,\varepsilon}(t) + \overline{\mathcal{V}}_2^{\alpha,\varepsilon}(t), \tag{60}$$

where $\overline{\mathcal{V}}_1^{\alpha,\varepsilon}(t)$ and $\overline{\mathcal{V}}_2^{\alpha,\varepsilon}(t)$ are defined by

$$\begin{aligned}\overline{\mathcal{V}}_1^{\alpha,\varepsilon}(t) &= \widetilde{\mathcal{S}}_{\alpha,1}(t,T)\big(\widetilde{u}_f^{\varepsilon} - u_f\big) + \int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{F}^{\varepsilon}(s) - F(s)\big)\,ds \\ &\quad + \underbrace{\int_0^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{G}^{\varepsilon}(s) - G(s)\big)\,dW^h(s)}_{=:J_1(t)} \\ &\quad - \underbrace{\int_0^t \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{G}^{\varepsilon}(s) - G(s)\big)\,dW^h(s)}_{=:J_2(t)},\end{aligned} \tag{61}$$

$$\begin{aligned}\overline{\mathcal{V}}_2^{\alpha,\varepsilon}(t) &= \big(\widetilde{\mathcal{S}}_{\alpha,1}(t,T) - \mathcal{S}_1(t,T)\big)u_f + \int_t^T \big(\widetilde{\mathcal{S}}_{\alpha,2}(t,s) - \mathcal{S}_2(t,s)\big)F(s)\,ds \\ &\quad + \int_0^T \big(\widetilde{\mathcal{S}}_{\alpha,2}(t,s) - \mathcal{S}_2(t,s)\big)G(s)\,dW^h(s) \\ &\quad - \int_0^t \big(\widetilde{\mathcal{S}}_{\alpha,2}(t,s) - \mathcal{S}_2(t,s)\big)G(s)\,dW^h(s).\end{aligned} \tag{62}$$

By similar estimates to those in (52) and (53), we have the following ones for two first terms in the right-hand side of (61)

$$\big\|\widetilde{\mathcal{S}}_{\alpha,1}(t,T)\big(\widetilde{u}_f^{\varepsilon} - u_f\big)\big\|_{L^2(\Omega,\dot{H}^{\sigma})} \leqslant C_1\overline{\Pi}_{\alpha,1}\varepsilon, \tag{63}$$

$$\left\|\int_t^T \widetilde{\mathcal{S}}_{\alpha,2}(t,s)\big(\widetilde{F}^{\varepsilon}(s) - F(s)\big)\,ds\right\|_{L^2(\Omega,\dot{H}^{\sigma})} \leqslant T^{\frac{2p-1}{p}}\overline{\Pi}_{\alpha,1}\varepsilon. \tag{64}$$

Now, we estimate the third term $J_1(t)$. It can be seen that

$$\begin{aligned}\|J_1(t)\|^2_{L^2(\Omega,\dot{H}^\sigma)} &= \mathbb{E}\bigg\|\sum_{k\in\mathbb{Z}^*}\int_0^T K^*_{h,T}\big(\widetilde{\mathcal{S}}_{\alpha,2}(t,\cdot)\big(\widetilde{G}^\varepsilon(\cdot)-G(\cdot)\big)\mathcal{Q}^{1/2}e_k\big)(s)\,d\xi_k(s)\bigg\|^2_{\dot{H}^\sigma}\\ &= \sum_{k\in\mathbb{Z}^*}\int_0^T \mathbb{E}\big\|K^*_{h,T}\big(\widetilde{\mathcal{S}}_{\alpha,2}(t,\cdot)\big(\widetilde{G}^\varepsilon(\cdot)-G(\cdot)\big)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\,ds. \end{aligned}\tag{65}$$

The expectation under the integral sign can be estimated as follows

$$\begin{aligned}&\mathbb{E}\big\|K^*_{h,T}\big(\widetilde{\mathcal{S}}_{\alpha,2}(t,\cdot)\big(\widetilde{G}^\varepsilon(\cdot)-G(\cdot)\big)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\\ &\quad= \mathbb{E}\bigg\|\int_s^T (\widetilde{\mathcal{S}}_{\alpha,2}(t,\cdot)\big(\widetilde{G}^\varepsilon(\cdot)-G(\cdot)\mathcal{Q}^{1/2}e_k\big)(\mu)\frac{\partial K_h}{\partial\mu}(\mu,s)\,d\mu\bigg\|^2_{\dot{H}^\sigma}\\ &\quad= c_h^2\Big(h-\frac{1}{2}\Big)^2\mathbb{E}\bigg\|\int_s^T \widetilde{\mathcal{S}}_{\alpha,2}(t,\mu)\big(\widetilde{G}^\varepsilon(\mu)-G(\mu)\big)\mathcal{Q}^{1/2}e_k\Big(\frac{\mu}{s}\Big)^{h-\frac{1}{2}}(\mu-s)^{h-\frac{3}{2}}\,d\mu\bigg\|^2_{\dot{H}^\sigma}\\ &\quad\leqslant c_h^2\Big(h-\frac{1}{2}\Big)^2 s^{1-2h}\mathbb{E}\bigg[\int_s^T \mu^{h-\frac{1}{2}}(\mu-s)^{h-\frac{3}{2}}\big\|\widetilde{\mathcal{S}}_{\alpha,2}(t,\mu)\big(\widetilde{G}^\varepsilon(\mu)-G(\mu)\big)\mathcal{Q}^{1/2}e_k\big\|_{\dot{H}^\sigma}\,d\mu\bigg]^2.\end{aligned}$$

Applying the Hölder inequality and using properties (20), (48), we obtain

$$\begin{aligned}&\mathbb{E}\big\|K^*_{h,T}\big(\widetilde{\mathcal{S}}_{\alpha,2}(t,\cdot)\big(\widetilde{G}^\varepsilon(\cdot)-G(\cdot)\big)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\\ &\quad\leqslant c_h^2\Big(h-\frac{1}{2}\Big)^2 s^{1-2h}\bigg(\int_s^T \mu^{2h-1}(\mu-s)^{h-\frac{3}{2}}\,d\mu\bigg)\\ &\qquad\times\bigg(\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|\widetilde{\mathcal{S}}_{\alpha,2}(t,\mu)\big(\widetilde{G}^\varepsilon(\mu)-G(\mu)\big)\mathcal{Q}^{1/2}e_k\big\|^2_{\dot{H}^\sigma}\,d\mu\bigg)\\ &\quad\leqslant c_h^2\Big(h-\frac{1}{2}\Big)^2 s^{1-2h}T^{3(h-\frac{1}{2})}\frac{\Gamma(2h)\Gamma(h-1/2)}{\Gamma(3h-1/2)}\times\\ &\qquad\times T^2\overline{\Pi}^2_{\alpha,1}\bigg(\int_s^T (\mu-2)^{h-\frac{3}{2}}\mathbb{E}\big\|\big(\widetilde{G}^\varepsilon(\mu)-G(\mu)\big)\mathcal{Q}^{1/2}e_k\big\|^2_H\,d\mu\bigg).\end{aligned}$$

Setting $M_3 := c_h(h-\frac{1}{2})T^{\frac{3}{2}h+\frac{1}{4}}(\frac{\Gamma(2h)\Gamma(h-1/2)}{\Gamma(3h-1/2)})^{1/2}$, we see that

$$\begin{aligned}&\mathbb{E}\big\|K^*_{h,T}\big(\widetilde{\mathcal{S}}_{\alpha,2}(t,\cdot)\big(\widetilde{G}^\varepsilon(\cdot)-G(\cdot)\big)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\\ &\quad\leqslant M_3^2\overline{\Pi}^2_{\alpha,1}s^{1-2h}\bigg(\int_s^T (\mu-2)^{h-\frac{3}{2}}\mathbb{E}\big\|\big(\widetilde{G}^\varepsilon(\mu)-G(\mu)\big)\mathcal{Q}^{1/2}e_k\big\|^2_H\,d\mu\bigg).\end{aligned}\tag{66}$$

From (65) and (66),

$$\|J_1(t)\|^2_{L^2(\Omega,\dot{H}^\sigma)} \leqslant M_3^2\overline{\Pi}^2_{\alpha,1}\int_0^T s^{1-2h}\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|\widetilde{G}^\varepsilon(\mu)-G(\mu)\big\|^2_{L^2_0(H)}\,d\mu\,ds.$$

On the other hand, by a similar technique as in (23),

$$\int_s^T (\mu-s)^{h-\frac{3}{2}}\mathbb{E}\big\|\widetilde{G}^\varepsilon(\mu)-G(\mu)\big\|^2_{L_0^2(H)}\,d\mu \leqslant \left(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{r}}\big\|\widetilde{G}^\varepsilon-G\big\|^2_{L^r(\mathcal{J};L^2(\Omega,L_0^2(H)))}$$
$$\leqslant \left(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{r}}\varepsilon^2.$$

From two last inequalities, we obtain

$$\big\|J_1(t)\big\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant M_3\overline{\Pi}_{\alpha,1}\frac{T^{1-h}}{\sqrt{2-2h}}\left(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{2r}}\varepsilon. \tag{67}$$

Next, we estimate the last term in the right-hand side of (61). It is obvious that

$$\big\|J_2(t)\big\|^2_{L^2(\Omega,\dot{H}^\sigma)} = \mathbb{E}\left\|\sum_{k\in\mathbb{Z}^*}\int_0^t K^*_{h,t}\big(\widetilde{\mathcal{S}}_{\alpha,2}(t,\cdot)\big(\widetilde{G}^\varepsilon(\cdot)-G(\cdot)\big)\mathcal{Q}^{1/2}e_k\big)(s)\,d\xi_k(s)\right\|^2_{\dot{H}^\sigma}$$
$$= \sum_{k\in\mathbb{Z}^*}\int_0^t \mathbb{E}\big\|K^*_{h,t}\big(\widetilde{\mathcal{S}}_{\alpha,2}(t,\cdot)\big(\widetilde{G}^\varepsilon(\cdot)-G(\cdot)\big)\mathcal{Q}^{1/2}e_k\big)(s)\big\|^2_{\dot{H}^\sigma}\,ds. \tag{68}$$

Proceeding as for $J_1(t)$, we can bound the term $J_2(t)$ as

$$\big\|J_2(t)\big\|_{L^2(\Omega,\dot{H}^\sigma)} \leqslant M_3\overline{\Pi}_{\alpha,1}\frac{t^{1-h}}{\sqrt{2-2h}}\left(\frac{t^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{2r}}\varepsilon$$
$$\leqslant M_3\overline{\Pi}_{\alpha,1}\frac{T^{1-h}}{\sqrt{2-2h}}\left(\frac{T^{\frac{r}{r-2}(h-\frac{3}{2})+1}}{\frac{r}{r-2}(h-\frac{3}{2})+1}\right)^{\frac{r-2}{2r}}\varepsilon. \tag{69}$$

Combining (61), (63), (64), (67), (69), we deduce that there exists a positive constant M_0 independent of α, ε such that

$$\big\|\overline{\mathcal{V}}_1^{\alpha,\varepsilon}\big\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant M_0\overline{\Pi}_{\alpha,1}\varepsilon. \tag{70}$$

For the last term $\overline{\mathcal{V}}_2^{\alpha,\varepsilon}(t)$, from (62) and the formulas of the operators $\widetilde{\mathcal{S}}_{\alpha,1}(t,T)$, $\widetilde{\mathcal{S}}_{\alpha,2}(t,s)$, $\mathcal{S}_1(t,T)$, $\mathcal{S}_2(t,s)$ and using a similar argument as in (57), we deduce

$$\overline{\mathcal{V}}_2^{\alpha,\varepsilon}(t) = \sum_{k\in\mathbb{Z}^*}\big(\overline{u}(t),e_k\big)(\zeta_{\alpha,k}-1)e_k,$$

which follows that

$$\big\|\overline{\mathcal{V}}_2^{\alpha,\varepsilon}(t)\big\|^2_{L^2(\Omega,\dot{H}^\sigma)} = \sum_{k\in\mathbb{Z}^*}\lambda_k^{2\sigma}\mathbb{E}\big(\overline{u}(t),e_k\big)^2|\zeta_{\alpha,k}-1|^2 \leqslant \overline{\Pi}^2_{\alpha,2}\sum_{k\in\mathbb{Z}^*}\mathbb{E}\big(\overline{u}(t),e_k\big)^2\lambda_k^{2(\sigma+\overline{\eta})},$$

which leads to

$$\left\|\overline{\mathcal{V}}_2^{\alpha,\varepsilon}\right\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant \overline{\Pi}_{\alpha,2} \sup_{t\in\overline{T}}\left\|\overline{u}(t)\right\|_{L^2(\Omega,\dot{H}^{\sigma+\overline{\eta}})} \leqslant \overline{\Pi}_{\alpha,2}\overline{E}. \tag{71}$$

Combining (60), (70), (71) and using the triangle inequality, we deduce that

$$\left\|\overline{\mathcal{U}}^{\alpha,\varepsilon} - \overline{u}\right\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))} \leqslant M_0\overline{\Pi}_{\alpha,1}\varepsilon + \overline{\Pi}_{\alpha,2}\overline{E}.$$

By the conditions $\lim_{\varepsilon\to 0^+}(\overline{\Pi}_{\alpha,1}\varepsilon) = \lim_{\varepsilon\to 0^+}\overline{\Pi}_{\alpha,2} = 0$, we conclude that $\|\overline{\mathcal{U}}^{\alpha,\varepsilon} - \overline{u}\|_{C(\overline{\mathcal{J}};L^2(\Omega,\dot{H}^\sigma))}$ tends to zero as $\varepsilon \to 0^+$. The proof is completed. □

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Appendix A

In this appendix, we recall the definition of a one-dimensional fractional Brownian motion and then introduce the Wiener integral with respect to an fBm.

A one-dimensional fBm $\{\xi^h(t)\}_{t\geqslant 0}$, with $h \in (0,1)$, is a continuous and centered Gaussian process with covariance function $R_h(t_1,t_2) = \frac{1}{2}(t_1^{2h}+t_2^{2h}-|t_1-t_2|^{2h})$. One can see that, if $h = \frac{1}{2}$, then $\{\xi^h(t)\}_{t\geqslant 0}$ becomes the standard Brownian motion.

By $\mathcal{E}$ and $\overline{\mathcal{E}}$, we define the space of step functions ψ on $\overline{\mathcal{J}}$ and the closure of $\mathcal{E}$ endowed with the product $(\chi_{[0,t_1]},\chi_{[0,t_2]})_{\overline{\mathcal{E}}} = R_h(t_1,t_2)$ respectively. Let $\beta(\cdot,\cdot)$ be the Beta function [31]. For $\frac{1}{2} < h < 1$, we introduce the following kernel, which will be used to present the relation between the fBm and the standard Brownian motion latter.

$$K_h(t_1,t_2) = c_h\int_{t_2}^{t_1}(\mu-t_2)^{h-\frac{3}{2}}\left(\frac{\mu}{t_2}\right)^{h-\frac{1}{2}}d\mu, \quad c_h = \sqrt{\frac{h(2h-1)}{\beta(2-2h,h-1/2)}}.$$

We refer to [4] the following derivative of $K_h(t_1,t_2)$ with respect to the first variable t_1

$$\frac{\partial K_h}{\partial t_1}(t_1,t_2) = c_h\left(h-\frac{1}{2}\right)\left(\frac{t_1}{t_2}\right)^{h-\frac{1}{2}}(t_1-t_2)^{h-\frac{3}{2}}. \tag{72}$$

It is known from [4,10,29] that the fBm $\xi^h(t)$ has the following relation with the standard Brownian motion $\xi(t)$

$$\xi^h(t) = \int_0^t K_h(t,s)\,d\xi(s).$$

Constructing the operator $K_{h,T}^{*} : \overline{\mathcal{E}} \to L^2(\overline{\mathcal{J}})$, with $\frac{1}{2} < h < 1$, as follows

$$\big(K_{h,T}^{*}\psi\big)(t) = \int_t^T \psi(\mu)\frac{\partial K_h}{\partial \mu}(\mu, t)\,d\mu, \tag{73}$$

then $K_{h,T}^{*}$ is an isometry from $\overline{\mathcal{E}}$ to $L^2(\overline{\mathcal{J}})$ [4,10,29]. From [4,10,29], it is known that

$$\int_0^T \psi(s)\,d\xi^h(s) = \int_0^T K_{h,T}^{*}\psi(s)\,d\xi(s).$$

Consider the fBm $\{W^h(t)\}_{t\in\mathcal{J}}$ defined as in (4). We can define the Wiener integral of $\phi : \overline{\mathcal{J}} \to L_0^2(H)$ with respect to $W^h(t)$ [4,10,29] as

$$\int_0^T \phi(s)\,dW^h(s) = \sum_{k\in\mathbb{Z}^*} \phi(s)\mathcal{Q}^{1/2}e_k\,d\xi_k^h(s) = \sum_{k\in\mathbb{Z}^*} K_{h,T}^{*}\big(\phi\mathcal{Q}^{1/2}e_k\big)(s)\,d\xi_k(s).$$

For some other works concerned with the stochastic above integral, the readers can refer to [5,6,8,9,23,34,35].

Appendix B

In this appendix we propose representations for the solutions to TVP (1), TVP (2), and some useful estimates for the solution operators.

We now find a representation for the solution of TVP (1) in the form $u(t) = \sum_{k\in\mathbb{Z}^*}(u(t), e_k)e_k$. From the first equation of (1) and $\mathcal{A}e_k = \lambda_k e_k$, we have

$$\frac{\partial^2}{\partial t^2}\big(u(t), e_k\big) + 2\lambda_k\frac{\partial}{\partial t}\big(u(t), e_k\big) + \lambda_k^2\big(u(t), e_k\big) = \big(F(t), e_k\big) + G(t)\Lambda_k^{1/2}\xi_k(t).$$

Setting $u_k(t) := (u(t), e_k)$, it can be seen that the above equation is a second order differential equation in the form

$$u_k''(t) + 2\lambda_k u_k'(t) + \lambda_k^2 u_k(t) = \big(F(t), e_k\big) + G(t)\Lambda_k^{1/2}\xi_k(t).$$

By using the method of variation of constants and noting that $u_k'(T) = 0$, we arrive at

$$\begin{aligned} u_k(t) = {} & \big(1 - (T-t)\lambda_k\big)e^{(T-t)\lambda_k}u_k(T) + \int_t^T (s-t)e^{(s-t)\lambda_k}\big(F(s), e_k\big)\,ds \\ & + \Lambda_k^{1/2}\int_0^T (s-t)e^{(s-t)\lambda_k}G(s)\,d\xi_k(s) - \Lambda_k^{1/2}\int_0^t (s-t)e^{(s-t)\lambda_k}G(s)\,d\xi_k(s). \end{aligned}$$

Now, noting that $u_k(T) = (u(T), e_k) = (u_f, e_k)$ since $u(T) = u_f$, we obtain

$$u_k(t) = \big(1 - (T-t)\lambda_k\big)e^{(T-t)\lambda_k}(u_f, e_k) + \int_t^T (s-t)e^{(s-t)\lambda_k}\big(F(s), e_k\big)\, ds + \Lambda_k^{1/2}\int_0^T (s-t)e^{(s-t)\lambda_k}G(s)\, d\xi_k(s) - \Lambda_k^{1/2}\int_0^t (s-t)e^{(s-t)\lambda_k}G(s)\, d\xi_k(s). \tag{74}$$

Notice that the two last terms in the right-hand side of (74) can be combined as

$$\Lambda_k^{1/2}\int_0^T (s-t)e^{(s-t)\lambda_k}G(s)\, d\xi_k(s) - \Lambda_k^{1/2}\int_0^t (s-t)e^{(s-t)\lambda_k}G(s)\, d\xi_k(s) = \Lambda_k^{1/2}\int_t^T (s-t)e^{(s-t)\lambda_k}G(s)\, d\xi_k(s).$$

For $t, s \in \overline{\mathcal{J}}$, we define the following operators

$$\mathcal{S}_1(t,T) := \sum_{k\in\mathbb{Z}^*}\big(1-(T-t)\lambda_k\big)e^{(T-t)\lambda_k}(\cdot, e_k)e_k, \qquad \mathcal{S}_2(t,s) := \sum_{k\in\mathbb{Z}^*}(s-t)e^{(s-t)\lambda_k}(\cdot, e_k)e_k.$$

Then, a representation for the solution of TVP (1) is obtained as

$$u(t) = \mathcal{S}_1(t,T)u_f + \int_t^T \mathcal{S}_2(t,s)F(s)\, ds + \int_t^T \mathcal{S}_2(t,s)G(s)\, dW(s).$$

By a similar way as above, a representation for the solution of TVP (2) can be found as

$$\overline{u}(t) = \mathcal{S}_1(t,T)u_f + \int_t^T \mathcal{S}_2(t,s)F(s)\, ds + \int_0^T \mathcal{S}_2(t,s)G(s)\, dW^h(s) - \int_0^t \mathcal{S}_2(t,s)G(s)\, dW^h(s),$$

where we note that the two last terms have two different explicit representations as

$$\int_0^T \mathcal{S}_2(t,s)G(s)\, dW^h(s) = \sum_{k\in\mathbb{Z}^*}\int_0^T \mathcal{S}_2(t,s)G(s)\mathcal{Q}^{1/2}e_k\, d\xi_k^h(s) = \sum_{k\in\mathbb{Z}^*}\int_0^T K_{h,T}^*\big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\, d\xi_k(s), \tag{75}$$

$$\int_0^t \mathcal{S}_2(t,s)G(s)\, dW^h(s) = \sum_{k\in\mathbb{Z}^*}\int_0^t \mathcal{S}_2(t,s)G(s)\mathcal{Q}^{1/2}e_k\, d\xi_k^h(s) = \sum_{k\in\mathbb{Z}^*}\int_0^t K_{h,t}^*\big(\mathcal{S}_2(t,\cdot)G(\cdot)\mathcal{Q}^{1/2}e_k\big)(s)\, d\xi_k(s). \tag{76}$$

Next, the following lemma presents upper bounds for the solution operators $\mathcal{S}_1(t,T)$ and $\mathcal{S}_2(t,s)$, for $t,s\in\overline{\mathcal{J}}$.

Lemma B.1. *Given $\sigma\geqslant 0$. Then, for $t,s\in\overline{\mathcal{J}}$, there holds*

$$\big\|\mathcal{S}_1(t,T)\big\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})\to L^2(\Omega,\dot H^\sigma)}\leqslant C_1,\qquad \big\|\mathcal{S}_2(t,s)\big\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})\to L^2(\Omega,\dot H^\sigma)}\leqslant \lambda_1^{-1}|s-t|,\tag{77}$$

where we set $C_1:=C_1(\mathcal{A},T)=\sqrt{\lambda_1^{-2}+T^2}$. Furthermore, if $\delta>0$ is small enough, then

$$\big\|\mathcal{S}_1(t+\delta,T)-\mathcal{S}_1(t,T)\big\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})\to L^2(\Omega,\dot H^\sigma)}\leqslant T\delta,\tag{78}$$

$$\big\|\mathcal{S}_2(t+\delta,s)-\mathcal{S}_2(t,s)\big\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})\to L^2(\Omega,\dot H^\sigma)}\leqslant \big(\lambda_1^{-1}+T\big)\delta.\tag{79}$$

Proof. We begin with the first result (77). For $\theta\in L^2(\Omega,\dot H^\sigma)$, $t\in\overline{\mathcal{J}}$, it is easy to see that

$$\begin{aligned}\mathbb{E}\big\|\mathcal{S}_1(t,T)\theta\big\|^2_{\dot H^\sigma}&=\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2\sigma}\big(1-(T-t)\lambda_k\big)^2e^{2(T-t)\lambda_k}\\&\leqslant\big(\lambda_1^{-2}+(T-t)^2\big)\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2(\sigma+1)}e^{2T\lambda_k}\\&\leqslant\big(\lambda_1^{-2}+T^2\big)\mathbb{E}\|\theta\|^2_{\mathbb{V}^T_{\sigma+1}},\end{aligned}$$

which implies that $\|\mathcal{S}_1(t,T)\theta\|_{L^2(\Omega,\dot H^\sigma)}\leqslant C_1\|\theta\|_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}$. Similarly, for $t,s\in\overline{\mathcal{J}}$, one has

$$\begin{aligned}\big\|\mathcal{S}_2(t,s)\theta\big\|^2_{L^2(\Omega,\dot H^\sigma)}=\mathbb{E}\big\|\mathcal{S}_2(t,s)\theta\big\|^2_{\dot H^\sigma}&=\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2\sigma}(s-t)^2e^{2(s-t)\lambda_k}\\&\leqslant\lambda_1^{-2}(s-t)^2\|\theta\|^2_{L^2(\Omega,\mathbb{V}^T_{\sigma+1})}.\end{aligned}$$

We next prove the second result (78). For $t\in\overline{\mathcal{J}}$ and δ small enough, one can see

$$\begin{aligned}&\mathbb{E}\big\|\big(\mathcal{S}_1(t+\delta,T)-\mathcal{S}_1(t,T)\big)\theta\big\|^2_{\dot H^\sigma}\\&\quad=\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2\sigma}\big|\big(1-\big(T-(t+\delta)\big)\lambda_k\big)e^{(T-(t+\delta))\lambda_k}-\big(1-(T-t)\lambda_k\big)e^{(T-t)\lambda_k}\big|^2\\&\quad=\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2\sigma}\left|\int_t^{t+\delta}\frac{d}{dz}\big(\big(1-(T-z)\lambda_k\big)e^{(T-z)\lambda_k}\big)\,dz\right|^2\\&\quad\leqslant\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2(\sigma+1)}\left|\int_t^{t+\delta}e^{(T-z)\lambda_k}(T-z)\,dz\right|^2.\end{aligned}$$

Since $e^{(T-z)\lambda_k}(T-z)\leqslant Te^{T\lambda_k}$, it is clear that

$$\mathbb{E}\big\|\big(\mathcal{S}_1(t+\delta,T)-\mathcal{S}_1(t,T)\big)\theta\big\|^2_{\dot H^\sigma}\leqslant T^2\delta^2\sum_{k\in\mathbb{Z}^*}\mathbb{E}(\theta,e_k)^2\lambda_k^{2(\sigma+1)}e^{2T\lambda_k}=T^2\delta^2\mathbb{E}\|\theta\|^2_{\mathbb{V}^T_{\sigma+1}}.$$

Then (78) holds. Similarly, for $t, s \in \overline{\mathcal{J}}$ and δ small enough,

$$\begin{aligned}
&\mathbb{E}\big\|\big(\mathcal{S}_2(t+\delta, s) - \mathcal{S}_2(t, s)\big)\theta\big\|^2_{\dot{H}^\sigma} \\
&\quad = \sum_{k\in\mathbb{Z}^*} \mathbb{E}(\theta, e_k)^2 \lambda_k^{2\sigma} \big|\big(s-(t+\delta)\big)e^{(s-(t+\delta))\lambda_k} - (s-t)e^{(s-t)\lambda_k}\big|^2 \\
&\quad = \sum_{k\in\mathbb{Z}^*} \mathbb{E}(\theta, e_k)^2 \lambda_k^{2\sigma} \left|\int_t^{t+\delta} \frac{d}{dz}\big((s-z)e^{(s-z)\lambda_k}\big)\, dz\right|^2 \\
&\quad \leqslant \sum_{k\in\mathbb{Z}^*} \mathbb{E}(\theta, e_k)^2 \lambda_k^{2\sigma} \left|\int_t^{t+\delta} \big(1+\lambda_k(s-z)\big)e^{(s-t)\lambda_k}\, dz\right|^2.
\end{aligned}$$

Since $(1+\lambda_k(s-z))e^{(s-t)\lambda_k} \leqslant (\lambda_1^{-1}+T)\lambda_k e^{T\lambda_k}$, one deduces that

$$\mathbb{E}\big\|\big(\mathcal{S}_2(t+\delta, s) - \mathcal{S}_2(t, s)\big)\theta\big\|^2_{\dot{H}^\sigma} \leqslant \big(\lambda_1^{-1}+T\big)^2\delta^2\mathbb{E}\|\theta\|^2_{\mathbb{V}^T_{\sigma+1}}.$$

This leads to (79). We now complete the proof. □

References

[1] K.A. Ames and B. Straughan, *Non-standard and Improperly Posed Problems*, Academic Press, New York (NY), 1997.

[2] J.-L. Auriault, The paradox of Fourier heat equation: A theoretical refutation, *Internat. J. Engrg. Sci.* **118** (2017), 82–88. doi:10.1016/j.ijengsci.2017.06.006.

[3] G. Bao, S.-N. Chow, P. Li and H. Zhou, Numerical solution of an inverse medium scattering problem with a stochastic source, *Inverse Problems* **26** (2010), 074014, 23 pp. doi:10.1088/0266-5611/26/7/074014.

[4] F. Biagini, Y. Hu, B. Øksendal and T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Probability and Its Applications (New York), Springer-Verlag London, Ltd., London, 2008, xii+329 pp. ISBN: 978-1-85233-996-8.

[5] A. Boudaoui, T. Caraballo and A. Ouahab, Existence of mild solutions to stochastic delay evolution equations with a fractional Brownian motion and impulses, *Stoch. Anal. Appl.* **33**(2) (2015), 244–258. doi:10.1080/07362994.2014.981641.

[6] B. Boufoussi and S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space, *Statist. Probab. Lett.* **82**(8) (2012), 1549–1558. doi:10.1016/j.spl.2012.04.013.

[7] C. Cao, M.A. Rammaha and E.S. Titi, The Navier Stokes equations on the rotating 2-D sphere: Gevrey regularity and asymptotic degrees of freedom, *Z. Angew. Math. Phys.* **50**(3) (1999), 341–360. doi:10.1007/PL00001493.

[8] T. Caraballo, A.M.M. Duran and F. Rivero, Asymptotic behaviour of a non-classical and non-autonomous diffusion equation containing some hereditary characteristic, *Discrete Contin. Dyn. Syst. Ser. B* **22**(5) (2017), 1817–1833.

[9] T. Caraballo, M.J. Garrido-Atienza and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Anal.* **74**(11) (2011), 3671–3684. doi:10.1016/j.na.2011.02.047.

[10] T. Caraballo and A.M. Márquez-Durán, Existence, uniqueness and asymptotic behavior of solutions for a nonclassical diffusion equation with delay, *Dyn. Partial Differ. Equ.* **10**(3) (2013), 267–281. doi:10.4310/DPDE.2013.v10.n3.a3.

[11] A.S. Carasso, Bochner subordination, logarithmic diffusion equations, and blind deconvolution of hubble space telescope imagery and other scientifc data, *SIAM J. Imaging Sci.* **3** (2010), 954–980. doi:10.1137/090780225.

[12] J. Cheng and J.J. Liu, A quasi Tikhonov regularization for a two-dimensional backward heat problem by a fundamental solution, *Inverse Probl.* **24**(6) (2008), 065012, 18 pp. doi:10.1088/0266-5611/24/6/065012.

[13] L. Debbi, Well-posedness of the multidimensional fractional stochastic Navier–Stokes equations on the torus and on bounded domains, *J. Math. Fluid Mech.* **18**(1) (2016), 25–69. doi:10.1007/s00021-015-0234-5.

[14] X. Feng, P. Li and X. Wang, An inverse random source problem for the time fractional diffusion equation driven by a fractional Brownian motion, *Inverse Problems* (2019).

[15] G. Fichera, Is the Fourier theory of heat propagation paradoxical?, *Rend. Circ. Mat. Palermo (2)* **41**(1) (1992), 5–28. doi:10.1007/BF02844459.

[16] V.I. Fushchich, A.S. Galitsyn and A.S. Polubinskii, A new mathematical model of heat conduction processes, *Ukrainian Math. J.* **42**(2) (1990), 210–216 (Russian); translated from *Ukrain. Mat. Zh.* **42**(2) (1990), 237–245. doi:10.1007/BF01071016.

[17] J. Hadamard, *Lectures on Cauchy's Problems in Linear Partial Differential Equations*, Dover Publications, New York, 1953.

[18] I.A. Ibragimov and R.Z. Khas'minskii, Estimation problems for coefficients of stochastic partial differential equations, Part I, *Theory Probab. Appl.* **43** (1999), 370–387. doi:10.1137/S0040585X97976982.

[19] B. Jin, Y. Yan and Z. Zhou, Numerical approximation of stochastic time-fractional diffusion, *ESAIM Math. Model. Numer. Anal.* **53**(4) (2019), 1245–1268. doi:10.1051/m2an/2019025.

[20] L. Joseph and D.D. Preziosi, Heat waves, *Rev. Mod. Phys.* **61** (1989), 41. doi:10.1103/RevModPhys.61.41.

[21] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.

[22] A. Lakhdari and N. Boussetila, An iterative regularization method for an abstract ill-posed biparabolic problem, *Bound. Value Probl.* **2015** (2015), 55, 17 pp.

[23] K. Li, Stochastic delay fractional evolution equations driven by fractional Brownian motion, *Math. Methods Appl. Sci.* **38**(8) (2015), 1582–1591. doi:10.1002/mma.3169.

[24] J.J. Liu, Numerical solution of forward and backward problem for 2-D heat conduction equation, *J. Comput. Appl. Math.* **145**(2) (2002), 459–482. doi:10.1016/S0377-0427(01)00595-7.

[25] Q. Lü, Carleman estimate for stochastic parabolic equations and inverse stochastic parabolic problems, *Inverse Problems* **28**(4) (2012), 045008, 18 pp.

[26] Q. Lü and X. Zhang, Well-posedness of backward stochastic differential equations with general filtration, *J. Differential Equations* **254**(8) (2013), 3200–3227. doi:10.1016/j.jde.2013.01.010.

[27] H.T. Nguyen, M. Kirane, N.D.H. Quoc and V.A. Vo, Approximation of an inverse initial problem for a biparabolic equation, *Mediterr. J. Math.* **15**(1) (2018), Paper No. 18, 18 pp. doi:10.1007/s00009-017-1053-0.

[28] P. Niu, T. Helin and Z. Zhang, An inverse random source problem in a stochastic fractional diffusion equation, *Inverse Problems* (2019).

[29] D. Nualart, *The Malliavin Calculus and Related Topics*, 2nd edn, Springer-Verlag, Berlin, 2006.

[30] L.E. Payne, On a proposed model for heat conduction, *IMA J. Appl. Math.* **71**(4) (2006), 590–599. doi:10.1093/imamat/hxh112.

[31] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[32] Z. Qian, C.L. Fu and R. Shi, A modified method for a backward heat conduction problem, *Appl. Math. Comput.* **185**(1) (2007), 564–573.

[33] P. Radu, G. Todorova and B. Yordanov, Diffusion phenomenon in Hilbert spaces and applications, *J. Differential Equations* **250**(11) (2011), 4200–4218. doi:10.1016/j.jde.2011.01.024.

[34] D. Ruan and J. Luo, The existence, uniqueness, and controllability of neutral stochastic delay partial differential equations driven by standard Brownian motion and fractional Brownian motion, *Discrete Dyn. Nat. Soc.* **2018** (2018), Article ID 7502514.

[35] S. Tindel, C.A. Tudor and F. Viens, Stochastic evolution equations with fractional Brownian motion, *Probab. Theory Relat. Fields* **127** (2003), 186–204. doi:10.1007/s00440-003-0282-2.

[36] N.H. Tuan, M. Kirane, B. Bin-Mohsin and P.T.M. Tam, Filter regularization for final value fractional diffusion problem with deterministic and random noise, *Comput. Math. Appl.* **74**(6) (2017), 1340–1361. doi:10.1016/j.camwa.2017.06.014.

[37] G. Yuan, Determination of two kinds of sources simultaneously for a stochastic wave equation, *Inverse Problems* **31**(8) (2015), 085003, 13 pp.

[38] G. Yuan, Conditional stability in determination of initial data for stochastic parabolic equations, *Inverse Problems* **33**(3) (2017), 035014, 26 pp.

[39] H.W. Zhang and X.J. Zhang, Stability and regularization method for inverse initial value problem of biparabolic equation, *Open Acc. Library J.* **2** (2015), e1542.

[40] G. Zou and B. Wang, Stochastic Burgers' equation with fractional derivative driven by multiplicative noise, *Comput. Math. Appl.* **74**(12) (2017), 3195–3208. doi:10.1016/j.camwa.2017.08.023.