Synchronization on a limit cycle of multi-agent systems governed by discrete-time switched affine dynamics

M. Serieye * **C. Albea** ** **A. Seuret** * **M. Jungers** ***

* *LAAS-CNRS, Univ. de Toulouse, UPS, LAAS, 7 avenue du Colonel Roche, F-31400 Toulouse, France.*
(e-mail : mserieye,aseuret@laas.fr)
** *Univ. de Sevilla, Camino de los Descubrimientos, s/n 41092 Sevilla, Spain.*
(e-mail : albea@us.es)
*** *Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France*
e-mail : marc.jungers@univ-lorraine.fr.

Abstract: This paper addresses the problem of synchronizing a group of interacting discrete-time switched affine systems with centralized control laws. A first time-dependent control law is obtained directly, and then, two other state-dependent control laws are proposed to improve performance. The different methods are based on recent literature on switched affine systems and are evaluated on an academical example with a multi-agent system.

Keywords: Control of switched systems ; Lyapunov methods ; Stabilization of nonlinear systems ; Multi-agent systems ; Complete graph ; Consensus ; State synchronization

1. INTRODUCTION

Over the last few decades, problems of consensus and synchronization in multi-agent systems have aroused a great deal of interest in the systems and control community, mainly motivated by a wide range of applications in physics, biology and engineering. The nature of these problematics is to reach an agreement collectively about some quantity of interest. The authors in (Scardovi and Sepulchre, 2008) characterize the terms *consensus* and *synchronization* frequently used in the context of multi-agent system. In (Sinafar et al., 2020a), the authors resumed this as a categorization of multi-agent problems: the complexity in network topology and the complexity in agents dynamics. In the case of consensus control problems, the focus is on the communication constraints rather than the individual system dynamics. Hence, it is common to find in the literature works where the dynamics of agents are described by integrators (Shi et al., 2013); by double-integrators (Olfati-Saber et al., 2007; Ren and Beard, 2008); or by linear systems (Fax and Murray, 2004; Wieland et al., 2008). The challenges are then to deal with the communication graph features where the connection can be time-dependent (Moreau, 2005) ; the agents can suffer from communication delay in the network (Olfati-Saber and Murray, 2004) or switching topology (Cervantes-Herrera et al., 2012; Wang and Han, 2011). In contrast to consensus, the emphasis of the research on synchronization is on the individual dynamics rather than the communication limitations. It is possible to find in the literature works on nonlinear multi-agent systems (Zhang et al., 2018; Liu and Jiang, 2013) or studies on robust synchronization guarantees (Dal Col et al., 2018). However, most of the existing works have set aside nonlinear systems with switching topology, yet relevant in the case of microgrid where agents can represent the electronic power converters (Albea, 2021; De Persis et al., 2018; Meng et al., 2015).

Switched systems, as a subclass of hybrid systems (Goebel et al., 2012), consist in the association of a finite set of dynamics with a switching rule that assignes at each time instant which mode is active (Liberzon, 2003). This kind of systems have generated a rich study concerning their stability to the origin (Daafouz et al., 2002; Sun and Ge, 2011), stabilization (Geromel and Colaneri, 2006) or stabilizability (Fiacchini et al., 2016). Investigations on limit cycles for switched affine systems were mainly done in the continuous-time domain (Johansson et al., 1997; Rubensson et al., 1998) and recently, results have been developed in the discrete-time domain (Egidio et al., 2020; Serieye et al., 2020).

Since modal control is more challenging, especially for nonlinear systems, only few contributions can be found in the literature. Recently, works on multi-agent systems where agents are switched affine systems have aroused. In (Sinafar et al., 2020b,a), the authors have proposed switching control laws based on averaged models or based on linear combination of state matrices. These assumptions are not necessary when considering the stabilization of switched affine systems as it has been proposed in (Serieye et al., 2020). Hence, through this paper, we are aiming at extending the results proposed in (Serieye et al., 2020) in the case of synchronization of a group of switched affine systems interacting thanks to an all-to-all communication configuration. This work has for ambition to be a first step to distributed control of switched affine multi-agent systems.

The paper is organized as follows: in Section 2 the problem is formulated by presenting the switched affine systems and the

⋆ This work has been partially funded under grant projet ANR-18-CE40-0022-01 (Projet HISPALIS), projet PDI2019-105890RJ-100 (Project HYCAM) and projet ANR 18-CE40-0010-02 (Project HANDY).

multi-agent system. In Section 3, one can find some preliminaries on limit cycle useful afterwards. Then, centralized switching control laws are proposed in Section 4 where proofs are given in addition. Finally, before concluding, an academical example offers illustrations on the effectiveness of our method.

Notations: Throughout the paper, $\mathbb{N}$ denotes the natural numbers; $\mathbb{R}$, the real numbers; $\mathbb{R}^n$, the n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$, the set of all real $n \times m$ matrices and $\mathbb{S}^n$ the set of symmetric matrices in $\mathbb{R}^{n\times n}$. For any n and m in $\mathbb{N}$, matrices I_n, $\mathbb{1}_n$ and $\mathbf{0}_{n,m}$ denote the identity matrix of $\mathbb{R}^{n\times n}$, the vector in $\mathbb{R}^n$, whose components are all equal to 1 and the null matrix of $\mathbb{R}^{n\times m}$, respectively. When no confusion is possible, the subscripts of these matrices that precise the dimension, will be omitted. For any matrix M of $\mathbb{R}^{n\times n}$, the notation $M \succ 0$, $(M \prec 0)$ means that M is symmetric positive (negative) definite and $\det(M)$ represents its determinant. The symbol $\otimes$ denotes the Kronecker product, for further details on its properties, please refer to (Horn and Johnson, 1994). $\|\cdot\|$ denotes the Euclidean norm. For a symmetric positive definite matrix P and a vector x, we denote $\|x\|_P = \sqrt{x^\top P x}$, the weighted norm.
Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a undirected graph. $\mathcal{V}$ and $\mathcal{E}$ denote respectively the set of nodes and the set of edges. The elements $\{e_{i,j}\}$ of $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ specify the incidence relation between distinct pairs of nodes (v_i, v_j). We denote with $\mathcal{N}$ the index set of $\mathcal{V}$ and with $|\mathcal{N}|$ its cardinality.

2. PROBLEM FORMULATION

2.1 Switched affine agents

Consider a set of the discrete-time switched affine systems or agents governed by

$$\begin{aligned} x_{\ell,k+1} &= A_{\sigma_{\ell,k}} x_{\ell,k} + B_{\sigma_{\ell,k}}, \quad \forall \ell \in \mathcal{N}, \\ \sigma_{\ell,k} &= u_k \in \mathbb{K} \end{aligned} \tag{1}$$

where, for each $\ell \in \mathcal{N}$, $x_{\ell,k} \in \mathbb{R}^n$ is the state vector considered at time k in $\mathbb{N}$. Variable $\sigma_{\ell,k} \in \mathbb{K}$ characterizes the active mode among the K subsystems and matrices $A_{\sigma_{\ell,k}}$ and $B_{\sigma_{\ell,k}}$ present appropriate dimension. One feature of this class of systems relies on the fact that the selection of the active mode $\sigma_{\ell,k}$ is the only possible control action available at each agent, achieved through the selection of the input u_k.

2.2 Multi-agent representation

Consider a group of $|\mathcal{N}|$ homogeneous switched systems where each agent $i \in \mathcal{N}$ follows the dynamic given by (1) and has, *a priori*, its own switching rule $\sigma_{i,k}$. A compact representation of such multi-agent systems can be defined with an extended state vector $\mathbf{x}_k = \begin{bmatrix} x_{1,k} & \dots & x_{|\mathcal{N}|,k} \end{bmatrix}^\top \in \mathbb{R}^{n|\mathcal{N}|}$.

Then, it is possible to express the collective dynamic as follows

$$\mathbf{x}_{k+1} = \mathbf{A}(\boldsymbol{\sigma}_k)\mathbf{x}_k + \mathbf{B}(\boldsymbol{\sigma}_k). \tag{2}$$

In this representation, the active modes are gathered in a single vector $\boldsymbol{\sigma}_k = [\sigma_{1,k}, \dots, \sigma_{|\mathcal{N}|,k}]^\top \in \mathbb{K}^{|\mathcal{N}|}$ and the matrices $\mathbf{A}(\boldsymbol{\sigma}_k)$ and $\mathbf{B}(\boldsymbol{\sigma}_k)$ of system (2) are given by

$$\begin{aligned} \mathbf{A}(\boldsymbol{\sigma}_k) &= \mathrm{diag}(A_{\sigma_{1,k}}, \dots, A_{\sigma_{|\mathcal{N}|,k}}), \\ \mathbf{B}(\boldsymbol{\sigma}_k) &= \begin{bmatrix} B_{\sigma_{1,k}}^\top, \dots, B_{\sigma_{|\mathcal{N}|,k}}^\top \end{bmatrix}^\top. \end{aligned} \tag{3}$$

The stabilizability of multi-agent systems can depend on the interaction between agents. We aim at designing a centralized switching control law that synchronizes some homogeneous systems. In this view, we consider the following assumption

Assumption 1. The graph $\mathcal{G}$ is complete, which roughly speaking means that all agents are connected with all agents.

From Assumption 1, one can simplify the control law as a global switching control law, *i.e.* every agent is controlled with the same switching law $u(\mathbf{x}_k) \in \mathbb{K}$ and $\boldsymbol{\sigma}_k = \mathbb{1}_{|\mathcal{N}|} \otimes u(\mathbf{x}_k)$.

Therefore, the dynamics of the overall system are reduced to

$$\mathbf{A}(\boldsymbol{\sigma}_k) = I_{|\mathcal{N}|} \otimes A_{\sigma_k} \quad \text{and} \quad \mathbf{B}(\boldsymbol{\sigma}_k) = \mathbb{1}_{|\mathcal{N}|} \otimes B_{\sigma_k}. \tag{4}$$

The objective throughout this paper is to design a suitable shared control law that ensures the global asymptotic state synchronization of system (2), *i.e.* that all agents converge to the same trajectory as k tends to infinity. More precisely, this means that

$$\lim_{k\to+\infty} \|x_{\ell,k} - x_{\ell',k}\| = 0, \quad \forall (\ell, \ell') \in \mathcal{N}^2. \tag{5}$$

More importantly, the objective is to design a state-dependent control law that is based on the model of the agent (1), not on the collective dynamics (2). Indeed there exist many contributions dealing with the stabilization of such a class of systems such as in (Deaecto and Geromel, 2016; Hetel and Fridman, 2013). Most of them relies on the resolution of LMI conditions. Therefore, if the number of agents is too large, solving such a problem can be computationally difficult.

As a first step in this direction, Assumption 1 is made so that some first contributions are provided in the context of the synchronization of a set of switched affine systems. The relaxation of this assumption is kept to future work.

3. PRELIMINARIES ON LIMIT CYCLES OF SWITCHED AFFINE SYSTEMS

3.1 Limit cycles

This subsection clarifies the notion of limit cycle considered in this paper. Indeed, it is known that hybrid systems can exhibit a periodic behaviour. Likewise, we are interested in the time-varying steady states of switched affine systems which seem to have such behaviour when the control action is constrained by, for example, periodic updates (Serieye et al., 2020). A limit cycle generally refers to an isolated closed trajectory (Sun, 2008; Strogatz, 1994) or to a limit set which is a closed orbit (Rubensson et al., 1998). To fit with the hybrid nature of system (1), we first need to give few definitions.

Definition 1. A cycle, ν, of a switched affine system refers to a periodic switching function from $\mathbb{N}$ to $\mathbb{K}$. We define

$$\begin{aligned} &\mathrm{N}_\nu = \min N \in \mathbb{N}^* \text{ s.t. } \nu(i+N) = \nu(i), \quad \forall i \in \mathbb{N} \\ &\mathbb{D}_\nu = \{1, \dots, \mathrm{N}_\nu\} \end{aligned}$$

where N_ν stands for the minimum period of the sequence ν and $\mathbb{D}_\nu$ is its minimum domain.

Definition 2. Denote the set of cycles from $\mathbb{N}$ to $\mathbb{K}$ by

$$C := \{\nu : \mathbb{N} \to \mathbb{K}, \text{ s.t. } \exists N \in \mathbb{N}^*, \ \forall i \in \mathbb{N}, \ \nu(i+N) = \nu(i)\}.$$

We introduce the following modulo notation to ease the readability :

$$\lfloor i \rfloor_\nu = ((i-1) \bmod \mathrm{N}_\nu) + 1,$$

for any $i \in \mathbb{N}$, $i \geq 1$. That is, in particular, $\lfloor i \rfloor_\nu = i$, for any $i \in \mathbb{D}_\nu$ and $\lfloor \mathrm{N}_\nu + 1 \rfloor_\nu = 1$.

Definition 3. For a given cycle $\nu \in C$, if there exists an isolated closed orbit $\mathbb{N} \to \mathbb{K} \times \mathbb{R}^n$, $k \mapsto (\sigma_k, x_k)$ which is a solution of (1) such that $\sigma_k = \nu(k + \delta)$ (where δ is a possible shift), then

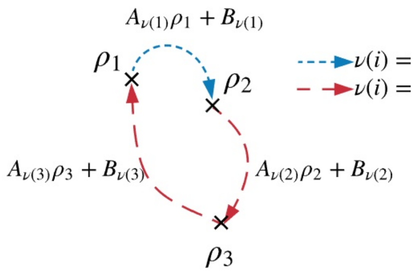

Fig. 1. Illustrative example of Definition 3 .

this hybrid trajectory is a limit cycle where its components are given by

$$\rho_{\lfloor i+1 \rfloor_\nu} = A_{\nu(i)}\rho_i + B_{\nu(i)}, \quad \forall i \in \mathbb{D}_\nu. \tag{6}$$

Figure 1 illustrates the previous definition, where one can see that the application of the cycle ν. Equations (6) can be written in a more compact formulation as follows

$$(I_{n\mathrm{N}_\nu} - \mathbb{A}_\nu)\rho := \mathbb{B}_\nu, \tag{7}$$

where $\rho = \left[\rho_1^\top, \rho_2^\top, \ldots, \rho_{\mathrm{N}_\nu}^\top\right]^\top$ and where matrices $\mathbb{A}_\nu$ and $\mathbb{B}_\nu$ are given by

$$\mathbb{A}_\nu = \begin{bmatrix} 0 & \ldots & 0 & A_{\nu(N_\nu)} \\ A_{\nu(1)} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & A_{\nu(N_\nu-1)} & 0 \end{bmatrix}, \quad \mathbb{B}_\nu = \begin{bmatrix} B_{\nu(N_\nu)} \\ B_{\nu(1)} \\ \vdots \\ B_{\nu(N_\nu-1)} \end{bmatrix}.$$

In the sequel, we will first present several results related to the asymptotic stability of the limit cycle defined by a given cycle ν selected a priori.

3.2 Stabilization to a limit cycle

In this part, the problem of stabilizing a single agent governed by (1) to limit cycle is addressed. Therefore in this section, for the sake of simplicity, the subscript related to the agent will be omitted (i.e. $x_{\ell,k} = x_k$). There are few recent contributions on the subject (Egidio et al., 2020; Serieye et al., 2020) where the authors share similar methods and come up with different switching control laws. We invoke here (Egidio et al., 2020, Theorem 2) to adduce some features about the results the author have proposed.

Theorem 1. For a given cycle ν in C, assume there exist matrices P_i in $\mathbb{S}^n$, $i \in \mathbb{D}_\nu$ solution to the following inequalities

$$P_i \succ 0, \quad A_{\nu(i)}^\top P_{\lfloor i+1 \rfloor_\nu} A_{\nu(i)} - P_i \prec 0, \quad \forall i \in \mathbb{D}_\nu. \tag{8}$$

Then, the following statements hold:

(i) Eq. (7) admits a unique solution ρ, defining the limit cycle.
(ii) Attractor $\mathcal{A}_\nu = \bigcup_{i\in\mathbb{D}_\nu}\{\rho_i\}$, is globally exponentially stable for system (1) with the periodic, *time-dependent*, switching control law

$$u(k) = \nu(k+\delta), \quad \delta \in \mathbb{D}_\nu, \tag{9}$$

(iii) Moreover,

$$\lim_{k\to+\infty} \|x_k - \rho_{\nu(k+\delta)}\| = 0.$$

This theorem refers to the stability analysis of periodic systems referring to the works of (Bolzern and Colaneri, 1988). In the present paper, the last item (iii) has been added with respect to (Egidio et al., 2020, Theorem 2).

Proof. We will do the proof of the theorem item by item.
Proof of (i): Let us consider $\bar{P} = \mathrm{diag}\,(P_i)_{i=1,\ldots,\mathrm{N}_\nu}$ solution to (8). Therefore, we can see that

$$\mathbb{A}_\nu^\top \bar{P} \mathbb{A}_\nu - \bar{P} = \mathrm{diag}\left(A_{\nu(i)}^\top P_{\lfloor i+1 \rfloor_\nu} A_{\nu(i)} - P_i\right)_{i=1,\ldots,\mathrm{N}_\nu} \prec 0.$$

Hence, matrix $\mathbb{A}_\nu$ is Schur stable. This implies, in particular, that 1 is not an eigenvalue of $\mathbb{A}_\nu$. Consequently, matrix $I_{nN} - \mathbb{A}_\nu$ is nonsingular and equation (7) admits a unique solution.

Proof of (ii): Let us consider, without loss of generality, that $\delta = 0$ and the following *time-dependent* Lyapunov function candidate, V_{td} given by

$$V_{td}(x_k, k) := (x_k - \rho_{\nu(k)})^\top P_{\nu(k)} (x_k - \rho_{\nu(k)}), \quad \forall x_k \in \mathbb{R}^n, \tag{10}$$

and its forward increment given by $\Delta V_{td}(x_k, k) = V_{td}(x_{k+1}, k+1) - V_{td}(x_k, k)$. Note that the expression of $V_{td}(x_{k+1}, k+1)$ exposes the term x_{k+1} which can be manipulated as follows by invoking (6)

$$x_{k+1} = A_{\nu(k)}x_k + B_{\nu(k)} = A_{\nu(k)}(x_k - \rho_{\nu(k)}) + \underbrace{A_{\nu(k)}\rho_{\nu(k)} + B_{\nu(k)}}_{=\rho_{\lfloor k+1 \rfloor_\nu}}. \tag{11}$$

Hence, we can obtain the following expression of $\Delta V_{td}(x_k, k)$

$$\Delta V_{td}(x_k, k) = (x_k - \rho_{\nu(k)})^\top \left(A_{\nu(k)}^\top P_{\lfloor k+1 \rfloor_\nu} A_{\nu(k)} - P_{\nu(k)}\right)(x_k - \rho_{\nu(k)}).$$

The latter expression together with condition (8) respected allow us to conclude that the Lyapunov function is decreasing for all $x_k \in \mathbb{R}^n \backslash \mathcal{A}_\nu$ which proves that $\mathcal{A}_\nu$ is globally exponentially stable for system (1).

Proof of (iii): This item simply results from the fact that V tends to zero as k tends to infinity. □

Remark 1. The convergence rate of the solution to system (1) to the limit cycle is characterized by the eigenvalues of the monodromy matrix defined by $\prod_{i\in\mathbb{D}_\nu} A_{\nu(i)}$.

The following theorem presents an alternative control law for the same system. The difference is that the control law does not depend on the time but is only computed based on the value of the state. This theorem is taken from (Serieye et al., 2020).

Theorem 2. For a given cycle ν, assume there exist matrices $\{P_i\}_{i\in\mathbb{D}_\nu}$ in $\mathbb{S}^n$, solution to (8). Then, the following statements hold:

(i) Eq. (7) admits a unique solution ρ, defining the limit cycle.
(ii) Attractor $\mathcal{A}_\nu = \bigcup_{i\in\mathbb{D}_\nu}\{\rho_i\}$ is globally exponentially stable for system (1) where $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ are given by the solution to (7) with the *state-feedback* control law

$$u(x) \in \left\{\nu(\theta),\ \theta \in \underset{i\in\mathbb{D}_\nu}{\mathrm{argmin}}\,(x-\rho_i)^\top P_i (x-\rho_i)\right\} \subset \mathbb{K}, \tag{12}$$

(iii) Moreover, if the $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ of a limit cycle associated to $\nu \in C$ are two by two different, then there exist $k_0 \in \mathbb{N}$ and an integer $\delta \in \mathbb{D}_\nu$ such that

$$u(x_k) = \nu(k+\delta), \quad \forall k \geq k_0. \tag{13}$$

and consequently

$$\lim_{k\to+\infty} \|x_k - \rho_{\nu(k+\delta)}\| = 0.$$

Remark 2. It is worth noting that the previous theorem relies on the same LMI condition (8) but the contribution and interpretation are different.

Proof. The proof of each item is performed below except the one of (i), which is the same as in Theorem 1.

Proof of (ii): Consider the *state-dependent* Lyapunov function, V_{sd}, given by

$$V_{sd}(x_k) = \min_{i \in \mathbb{D}_\nu} (x_k - \rho_i)^\top P_i (x_k - \rho_i), \forall x_k \in \mathbb{R}^n. \quad (14)$$

In order to prove (ii), let us note from the control law (12) that, at any time k, we have $V_{sd}(x_k) = (x_k - \rho_\theta)^\top P_\theta (x_k - \rho_\theta)$. Likewise, $V_{sd}(x_{k+1})$ is

$$\begin{aligned} V_{sd}(x_{k+1}) &= \min_{j \in \mathbb{D}_\nu} \left(x_{k+1} - \rho_j\right)^\top P_j \left(x_{k+1} - \rho_j\right) \\ &\leq (x_{k+1} - \rho_{\lfloor \theta+1 \rfloor_\nu})^\top P_{\lfloor \theta+1 \rfloor_\nu} (x_{k+1} - \rho_{\lfloor \theta+1 \rfloor_\nu}), \end{aligned}$$

which is the main difference with respect to the proof of Theorem 1. Then, the manipulations, as in (11), yield

$$\Delta V_{sd}(x_k) \leq (x_k - \rho_\theta)^\top \left(A_{\nu(\theta)}^\top P_{\lfloor \theta+1 \rfloor_\nu} A_{\nu(\theta)} - P_\theta\right) (x_k - \rho_\theta),$$

where matrices P_i's are solutions to (8). Therefore, the global exponential stabilization of the closed-loop system (1),(12) to attractor $\mathcal{A}_\nu$ is proven.

Proof of (iii): The proof of this item is taken from (Serieye et al., 2020, Theorem 1) but is recalled here for the sake of consistency. The idea is to first prove that the state-dependent control law (12) becomes ultimately periodic, i.e. that (13) holds. The proof is obtained by showing that there exists a sufficiently small scalar $\epsilon > 0$ (to be determined in this proof), such that we have the implication: $x \in \mathcal{S}_\epsilon = \{x \in \mathbb{R}^n,\ V(x) \leq \epsilon^2\}$ and $\theta \in \arg\min_{i \in \mathbb{D}_\nu} (x - \rho_i)^\top P_i (x - \rho_i)$ implies

$$(x - \rho_{\lfloor \theta+1 \rfloor_\nu})^\top P_{\lfloor \theta+1 \rfloor_\nu} (x - \rho_{\lfloor \theta+1 \rfloor_\nu}) < (x - \rho_j)^\top P_j (x - \rho_j), \quad (15)$$

for all $j \in \mathbb{D}_\nu$, with $j \neq \lfloor \theta+1 \rfloor_\nu$, that is the solution of the next minimization problem (12) is $\lfloor \theta+1 \rfloor_\nu$. To sum up, k_0 is related to the time to reach the level set $\mathcal{S}_\epsilon$, which is always possible to reach thanks to the convergence of the Lyapunov function to zero. The shift δ is determined thanks to the solution θ of the minimization law at time k_0, that is on the initial condition x_0 and the selection of the previous switchings. First, notice that thanks to the equivalence of weighted norms, there exist constants $c_{i,j} > 0$, $\forall (i, j) \in \mathbb{D}_\nu$, such that

$$\|x\|_{P_i} \leq c_{i,j} \|x\|_{P_j}, \quad (16)$$

(for instance, select $c_{i,j} \geq \sqrt{\lambda_M(P_i)/\lambda_m(P_j)}$). Thanks to LMIs (8) and $x \in \mathcal{S}_\epsilon$, we have

$$\|x_{k+1} - \rho_{\lfloor \theta+1 \rfloor_\nu}\|_{P_{\lfloor \theta+1 \rfloor_\nu}} \leq \|x_k - \rho_\theta\|_{P_\theta} \leq \epsilon, \quad (17)$$

and from (6)

$$\begin{aligned} \|x_{k+1} - \rho_{\lfloor \theta+1 \rfloor_\nu}\|_{P_\theta} &= \|A_{\nu(\theta)}(x_k - \rho_\theta)\|_{P_\theta} \leq \|A_{\nu(\theta)}\|_{P_\theta} \|x_k - \rho_\theta\|_{P_\theta}, \\ &\leq \|A_{\nu(\theta)}\|_{P_\theta} \epsilon \end{aligned} \quad (18)$$

hold, where $\|A_{\nu(\theta)}\|_{P_\theta}$ denotes the matrix norm induced by the weighted norm $\| \cdot \|_{P_\theta}$. That yields, due to the triangular inequality and relations (16),

$$\begin{aligned} \|\rho_{\lfloor \theta+1 \rfloor_\nu} - \rho_j\|_{P_\theta} - \|A_{\nu(\theta)}\|_{P_\theta}\epsilon &\leq \|\rho_{\lfloor \theta+1 \rfloor_\nu} - \rho_j\|_{P_\theta} - \|A_{\nu(\theta)}(x_k - \rho_\theta)\|_{P_\theta}, \\ &\leq \|\rho_{\lfloor \theta+1 \rfloor_\nu} - \rho_j + A_{\nu(\theta)}(x_k - \rho_\theta)\|_{P_\theta}, \\ &\leq \|x_{k+1} - \rho_j\|_{P_\theta}, \\ &\leq c_{\theta,j} \|x_{k+1} - \rho_j\|_{P_j}, \quad \forall j \in \mathbb{D}_\nu. \end{aligned} \quad (19)$$

Since the $\{\rho_i\}_{i \in \mathbb{D}_\nu}$ of a limit cycle associated to $\nu \in C$ are two by two different, it is always possible to find a positive scalar ϵ such that the strict inequalities $0 < c_{\theta,j}\epsilon < \|\rho_{\lfloor \theta+1 \rfloor_\nu} - \rho_j\|_{P_\theta} - \|A_{\nu(\theta)}\|_{P_\theta}\epsilon$ hold for any $j \in \mathbb{D}_\nu$, $j \neq \lfloor \theta+1 \rfloor_\nu$. Combining the two latter inequalities leads to $\epsilon < \|x_{k+1} - \rho_j\|_{P_j}$, for all $j \in \mathbb{D}_\nu \backslash \{\lfloor \theta+1 \rfloor_\nu\}$. Merging these inequalities yields, for all $j \in \mathbb{D}_\nu \backslash \{\lfloor \theta+1 \rfloor_\nu\}$

$$\|x_{k+1} - \rho_{\lfloor \theta+1 \rfloor_\nu}\|_{P_{\lfloor \theta+1 \rfloor_\nu}} \leq \epsilon < \|x_{k+1} - \rho_j\|_{P_j}. \quad (20)$$

The proof ends by applying item (iii) of Theorem 1, since the control law becomes ultimately periodic. □

4. SYNCHRONIZATION OF SWITCHED AFFINE SYSTEMS

The previous theorems allow stabilizing the set of agents to the same limit cycle (ν, ρ). This means, that each agent with one of both control laws, can individually converge to the limit cycle (ν, ρ). However, these theorems will not ensure the synchronization of the agent along the limit cycle. To do so, one has to include to the control law some level of cooperation. As mentioned in the introduction, in this section, we will propose three first solutions to solve this problem, consisting of a centralized control solution. The first time-dependent one can be seen as a natural extension of Theorem 1. Two additional solutions, which rely on the state-dependent control law are also presented.

4.1 Time-dependent open-loop control using a centralized control law

The main idea of the following theorem is to impose the same periodic (open-loop) switching control law to every agent. This first control comes direct from Theorem 1.

Theorem 3. For a given cycle $\nu \in C$, assume there exist $\{P_i\}_{i \in \mathbb{D}_\nu}$ in $\mathbb{S}^n$ for $i \in \mathbb{D}_\nu$ solution to (8). Then, the following statements hold:

(i) Eq. (7) admits a unique solution ρ, defining the limit cycle.
(ii) The periodic, *time-dependent*, switching control law (9) exponentially stabilizes the multi agent system (2) to $\mathcal{A}_\nu := \bigcup_{i \in \mathbb{D}_\nu} \{\mathbb{1}_{|\mathcal{N}|} \otimes \rho_i\}$, and, consequently, the agents are synchronized, i.e.

$$\lim_{k \to +\infty} \|x_{\ell,k} - x_{\ell',k}\| = 0, \quad \forall (\ell, \ell') \in \mathcal{N}.$$

Proof. This theorem is a direct application of Theorem 1 to the case of multi-agent systems. The synchronization is ensured since all the agents converge exponentially to $\rho_{\nu(k+\delta)}$, with the same δ for all agents. □

The centralized time-dependent control law presented in the previous theorem is finally quite simple. It is indeed a centralized solution since all the agents share the same time shift δ. However, as it will be presented in the example section, this centralized solution leads to poor performances, because it is limited by the fact that the control law is finally open loop. In the sequel, two additional solutions will be presented that are state-dependent, leading potentially to better performances.

4.2 First state-dependent control law

The first centralized solution consists in building a min-switching control law that gathers and unifies the min-switching law of all the agents. This is presented in the next theorem.

Theorem 4. For a given cycle $\nu \in C$, assume there exist P_i in $\mathbb{S}^n$ for $i \in \mathbb{D}_\nu$ solution to (8). Then, the following statements hold

(i) Eq. (7) admits a unique solution ρ, defining the limit cycle.
(ii) The centralized, *state-dependent*, switching control law

$$u(\mathbf{x}_k) \in \left\{ \nu(\theta),\ \theta \in \underset{i \in \mathbb{D}_\nu}{\operatorname{argmin}} (\mathbf{x}_k - \boldsymbol{\rho}_i)^\top \mathbf{P}_i (\mathbf{x}_k - \boldsymbol{\rho}_i) \right\} \subset \mathbb{K}, \quad (21)$$

where $\boldsymbol{\rho}_i = \mathbb{1}_{|\mathcal{N}|} \otimes \rho_i$ and $\mathbf{P}_i = I_{|\mathcal{N}|} \otimes P_i$ for all $i \in \mathcal{N}$, ensures that attractor $\boldsymbol{\mathcal{A}}_\nu := \bigcup_{i\in\mathbb{D}_\nu}\{\mathbb{1}_{|\mathcal{N}|} \otimes \rho_i\}$ is globally exponentially stable for system (2), and, consequently, the agents are synchronized, i.e.

$$\lim_{k\to+\infty} \|x_{\ell,k} - x_{\ell',k}\| = 0, \quad \forall(\ell,\ell') \in \mathcal{N}.$$

Proof. The proof of item (i) is as in Theorem 1.
Proof of (ii): Consider the *state-dependent* Lyapunov function, V_{sd}, given by

$$V_{sd}(\mathbf{x}_k) := \min_{i\in\mathbb{D}_\nu} (\mathbf{x}_k - \boldsymbol{\rho}_i)^\top \mathbf{P}_i (\mathbf{x}_k - \boldsymbol{\rho}_i), \quad \forall \mathbf{x}_k \in \mathbb{R}^{n|\mathcal{N}|}. \tag{22}$$

This Lyapunov function is the same as in Theorem 2, but applied to the multi-agent system defined in (2), instead of the dynamics of one single agent (1) and with $\boldsymbol{\rho}_i = \mathbb{1}_{|\mathcal{N}|} \otimes \rho_i$ and $\mathbf{P}_i = I_{|\mathcal{N}|} \otimes P_i$ instead of ρ_i and P_i. From control law (23), we have $V(\mathbf{x}_k) = (\mathbf{x}_k - \boldsymbol{\rho}_\theta)^\top \mathbf{P}_\theta (\mathbf{x} - \boldsymbol{\rho}_\theta)$, and the min-switching control law ensures that

$$V(\mathbf{x}_{k+1}) \le \left(\mathbf{x}_{k+1} - \boldsymbol{\rho}_{\lfloor\theta+1\rfloor_\nu}\right)^\top \mathbf{P}_{\lfloor\theta+1\rfloor_\nu} \left(\mathbf{x}_{k+1} - \boldsymbol{\rho}_{\lfloor\theta+1\rfloor_\nu}\right).$$

Despite the presence of the Kronecker product, it is still possible to do similar manipulations as those performed in (11), achieving

$$\mathbf{x}_{k+1} = \mathbf{A}_\theta \mathbf{x}_k + \mathbf{B}_\theta = \mathbf{A}_\theta (\mathbf{x}_k - \boldsymbol{\rho}_\theta) + \mathbf{A}_\theta\boldsymbol{\rho}_\theta + \mathbf{B}_\theta.$$

The properties of the Kronecker product yield

$$\begin{aligned} \mathbf{A}_\theta\boldsymbol{\rho}_\theta + \mathbf{B}_\theta &= (I_{|\mathcal{N}|} \otimes A_\theta)(\mathbb{1}_{|\mathcal{N}|} \otimes \rho_\theta) + (\mathbb{1}_{|\mathcal{N}|} \otimes B_\theta) \\ &= (I_{|\mathcal{N}|}\mathbb{1}_{|\mathcal{N}|}) \otimes (A_\theta\rho_\theta) + (\mathbb{1}_{|\mathcal{N}|} \otimes B_\theta) \\ &= \mathbb{1}_{|\mathcal{N}|} \otimes (A_\theta\rho_\theta + B_\theta) = \mathbb{1}_{|\mathcal{N}|} \otimes \rho_{\lfloor\theta+1\rfloor_\nu} := \boldsymbol{\rho}_{\lfloor\theta+1\rfloor_\nu}, \end{aligned}$$

so that we get

$$\begin{aligned} \Delta V(\mathbf{x}_k) &\le (\mathbf{x}_k - \boldsymbol{\rho}_\theta)^\top \left(\mathbf{A}_\theta^\top \mathbf{P}_{\lfloor\theta+1\rfloor_\nu} \mathbf{A}_\theta - \mathbf{P}_\theta\right)(\mathbf{x}_k - \boldsymbol{\rho}_\theta), \\ &= (\mathbf{x}_k - \boldsymbol{\rho}_\theta)^\top I_{|\mathcal{N}|} \otimes \left(A_\theta^\top P_{\lfloor\theta+1\rfloor_\nu} A_\theta - P_\theta\right)(\mathbf{x}_k - \boldsymbol{\rho}_\theta), \end{aligned}$$

which is ensured to be negative definite from (8). This means that $\mathbf{x}_k$ converges to $\mathbb{1}_{|\mathcal{N}|} \otimes \rho_{u_k}$, or equivalently that, for all $(\ell,\ell') \in \mathcal{N}$, the triangular inequality ensures

$$\lim_{k\to+\infty} \|x_{\ell,k} - x_{\ell',k}\| \le \lim_{k\to+\infty} (\|x_{\ell,k} - \rho_{u_k}\| + \|x_{\ell',k} - \rho_{u_k}\|) = 0,$$

which concludes the proof. □

Remark 3. From item (iii) in Theorem 2, if the components of the limit cycles are two-by-two different, then there exists δ in $\mathbb{D}_\nu$ such that

$$\lim_{k\to+\infty} \|x_{\ell,k} - \rho_{\nu(k+\delta)}\| = 0, \quad \forall \ell \in \mathcal{N}.$$

4.3 Second state-dependent control law

An alternative centralized solution to the synchronization problem is presented in this second theorem

Theorem 5. For a given cycle $\nu \in C$, assume there exist P_i in $\mathbb{S}^n$ for $i \in \mathbb{D}_\nu$ solution to (8). Then, the following statements hold

(i) Eq. (7) admits a unique solution ρ, defining the limit cycle.
(ii) If the $\{\rho_i\}_{i\in\mathbb{D}_\nu}$ of a limit cycle associated to $\nu \in C$ are two by two different, then the centralized, *state-dependent*, switching control law

$$u(\mathbf{x}_k) \in \left\{\nu(\theta),\ \theta \in \operatorname*{argmin}_{i\in\mathbb{D}_\nu} (\bar{\mathbf{x}}_k - \rho_i)^\top P_i (\bar{\mathbf{x}}_k - \rho_i)\right\} \subset \mathbb{K}, \tag{23}$$

where $\bar{\mathbf{x}}_k = \frac{1}{|\mathcal{N}|}(\mathbb{1}_{|\mathcal{N}|} \otimes I_n)^\top \mathbf{x}_k$ denotes the mean of $\mathbf{x}_k$, ensures that attractor $\boldsymbol{\mathcal{A}}_\nu := \bigcup_{i\in\mathbb{D}_\nu}\{\mathbb{1}_{|\mathcal{N}|} \otimes \rho_i\}$ is globally exponentially stable for system (2) and, consequently, the agents are synchronized, i.e.

$$\lim_{k\to+\infty} \|x_{\ell,k} - x_{\ell',k}\| = 0, \quad \forall(\ell,\ell') \in \mathcal{N}.$$

Proof. Again, the proof of item (i) is as in Theorem 1.
Proof of (ii): Consider the change of variable

$$\bar{\mathbf{x}}_k = \frac{1}{|\mathcal{N}|}(\mathbb{1}_{|\mathcal{N}|} \otimes I_n)^\top \mathbf{x}_k.$$

The dynamics of the mean variable verifies

$$\begin{aligned} \bar{\mathbf{x}}_{k+1} &= \frac{1}{|\mathcal{N}|}(\mathbb{1}_{|\mathcal{N}|} \otimes I_n)^\top \mathbf{x}_{k+1} = \frac{1}{|\mathcal{N}|}(\mathbb{1}_{|\mathcal{N}|} \otimes I_n)^\top (\mathbf{A}_\theta\mathbf{x}_k + \mathbf{B}_\theta) \\ &= A_\theta \frac{1}{|\mathcal{N}|}(\mathbb{1}_{|\mathcal{N}|} \otimes I_n)^\top \mathbf{x}_k + B_\theta := A_\theta\bar{\mathbf{x}} + B_\theta. \end{aligned}$$

Then, following item (ii) of Theorem 2, one can conclude that the mean of the agent states converges exponentially to $\mathcal{A}_\nu := \bigcup_{i\in\mathbb{D}_\nu}\{\rho_i\}$. However, at this point, the stabilization of any agent to the attractor $\boldsymbol{\mathcal{A}}_\nu$ is not proven. Now, from item (ii) of Theorem 2, we know that the switching control law $u(\mathbf{x}_k)$ converges ultimately to ν or a possibly shifted version of it, i.e., that after a finite-time, a periodic switching control law is applied to each agent. Finally, from item (i) of Theorem 1, one gets that all agent states reach synchronization since the control law is synchronized. □

5. EXAMPLES

This section is devoted to compare the different results proposed in this paper. To do so, we select an example adapted from (Fiacchini et al., 2016) and (Serieye et al., 2020). In the latter, we proposed a solution to the stabilization of switched affine systems to limit cycle defined *a priori*, which has been resumed here to extend it to the synchronization of an homogeneous multi-agent system. Let us consider system (2), composed of three homogeneous switched affine systems described by the following matrices A_i

$$A_1 = \begin{bmatrix}1 & 0\\ 0 & 0.5\end{bmatrix},\ A_2 = \begin{bmatrix}4 & -4\\ 4 & 4\end{bmatrix},\ B_1 = \begin{bmatrix}0.01\\ 0\end{bmatrix},\ B_2 = \begin{bmatrix}0\\ 0\end{bmatrix}. \tag{24}$$

Finally, the all-to-all communication, in which each pair of agents is connected by a unique edge, is illustrated by the simple graph on Figure 2.

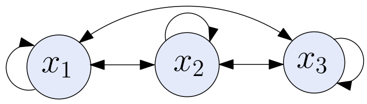

Fig. 2. All-to-all communication graph between 3 agents

A common assumption to ensure the asymptotic stabilization of switched affine systems is to have a linear combination of matrices A_i being Schur stable. Although such condition cannot be verified for the example considered, there exists a unique limit cycle (ν,ρ) where $\nu = \{1^{11}, 2^2\}$ as defined in (Fiacchini et al., 2016; Serieye et al., 2020) and as can be proven from Theorem 1. Then, the asymptotic stabilization to a limit cycle can be guaranteed.

On Figures 3, 4 the evolutions of the system variables obtained in simulation are illustrated, being the switching control laws those proposed in Section 4. From the left to the right, the simulations are performed with the centralized control laws given in Theorems 3, 4 and 5. Figure 3 shows the time-evolutions of the states and of the control inputs and, Figure 4

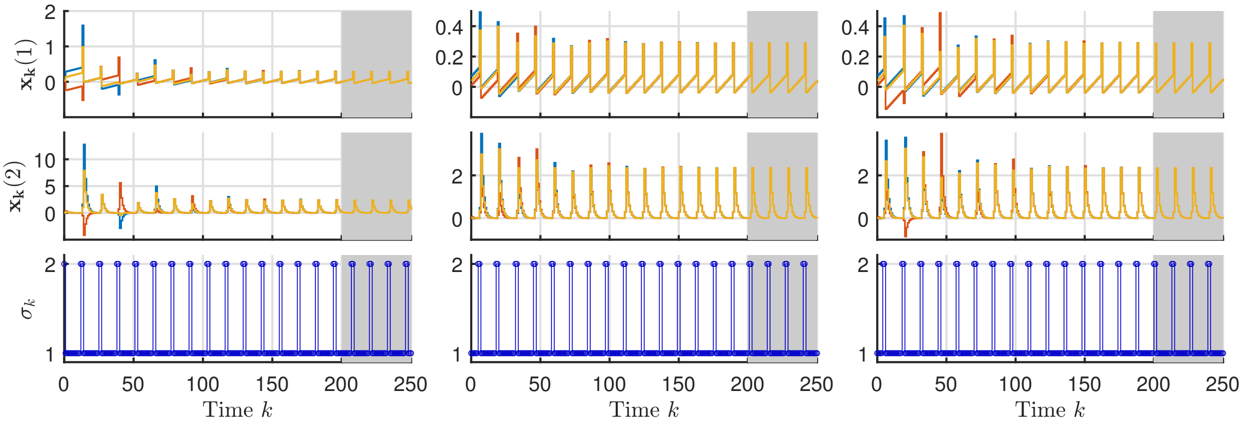

Fig. 3. From left to right, the different columns correspond to simulations results where is applied in first, the periodic control law (9); in second, the *all state*-dependent control law (21) and finally, the *mean state*-dependent control law (23). From top to bottom, the different rows correspond respectively to the evolution of the first state variable of each agent, the evolution of the second state variable of each agent and the switching signal σ_k.

depicts the state evolutions in the state plane. Note as the system converges to $\mathcal{A}_\nu$ with all proposed control laws, as it is highlighted in the second row subfigures, which correspond to the shadow area plotted in Figure 3.

The initial condition was selected such that the agents start from different positions in the state periodic sequence ρ of the limit cycle (ν,ρ), *i.e.* $\mathbf{x}_0 := \left[x_{1,0}^\top \; x_{2,0}^\top \; x_{3,0}^\top\right]^\top = \left[\rho_1^\top \; \rho_8^\top \; \rho_{11}^\top\right]^\top$. It is worth noting that even if each agent i starts from different positions in the limit cycle, the initial condition is not in $\mathcal{A}_\nu$.

Even though each control law ensures the synchronization of the multi-agent system (2), control law (9) provides a slower transient time than the ones generated from the others control laws proposed in Section 4.

6. CONCLUSION

In this paper, some centralized switching control laws have been given to synchronize a set of discrete-time switched affine systems in a given limit cycle. The controllers are obtained from the framework given in (Serieye et al., 2020) and (Egidio et al., 2020). On one hand, we propose a time-dependent control law, that is related to periodic systems. On the other hand, two state-dependent control laws that can improve performance with respect to the first one. An academical example shows the paper contribution. This work is a first step to progress on switched affine multi-agent system stabilization and suggests further study on distributed control for synchronization.

REFERENCES

Albea, C., 2021. Hybrid dynamical control based on consensus algorithms for current sharing in dc-bus microgrids. Nonlinear Analysis: Hybrid Systems 39, 100972.

Bolzern, P., Colaneri, P., 1988. The periodic Lyapunov equation. SIAM Journal on Matrix Analysis and Applications 9 (4), 499–512.

Cervantes-Herrera, A., Ruiz-Leon, J., Lopez-Limon, C., Ramirez-Trevino, A., 2012. A distributed control design for the output regulation and output consensus of a class of switched linear multi-agent systems. IEEE International Conference on Emerging Technologies and Factory Automation, ETFA.

Daafouz, J., Riedinger, P., Iung, C., 2002. Stability analysis and control synthesis for switched systems : A switched Lyapunov function approach. IEEE Transactions on Automatic Control 47, 1883–1887.

Dal Col, L., Tarbouriech, S., Zaccarian, L., 2018. $\mathcal{H}_\infty$ control design for synchronisation of identical linear multi-agent systems. International Journal of Control 91 (10), 2214–2229.

De Persis, C., Weitenberg, E. R., Dörfler, F., 2018. A power consensus algorithm for DC microgrids. Automatica 89, 364–375.

Deaecto, G., Geromel, J., 2016. Stability analysis and control design of discrete-time switched affine systems. IEEE Trans.on Automatic Control 62 (8), 4058–4065.

Egidio, L. N., Daiha, H. R., Deaecto, G. S., 2020. Global asymptotic stability of limit cycle and $\mathcal{H}_2/\mathcal{H}_\infty$ performance of discrete-time switched affine systems. Automatica 116, 108927.

Fax, J. A., Murray, R. M., 2004. Information flow and cooperative control of vehicle formations. IEEE Transactions on Automatic Control 49 (9), 1465–1476.

Fiacchini, M., Girard, A., Jungers, M., 2016. On the stabilizability of discrete-time switched linear systems: novel conditions and comparisons. IEEE Transactions on Automatic Control 61 (5), 1181–1193.

Geromel, J. C., Colaneri, P., July 2006. Stability and stabilization of discrete-time switched systems. International Journal of Control 79 (7), 719–728.

Goebel, R., Sanfelice, R. G., Teel, A. R., 2012. Hybrid Dynamical Systems: Modeling, Stability, and Robustness. Princeton University Press.

Hetel, L., Fridman, E., 2013. Robust sampled – data control of switched ffine systems. IEEE Transactions on Automatic Control 58 (11), 2922–2928.

Horn, R. A., Johnson, C. R., 1994. Topics in matrix analysis. Cambridge university press.

Johansson, K. H., Baranov, A., Åström, K. J., 1997. Limit cycles with chattering in relay feedback systems. In: Proceedings of the 36th IEEE Conference on Decision and Control. Vol. 4. pp. 3220–3225.

Liberzon, D., 2003. Switching in systems and control. Springer Science & Business Media.

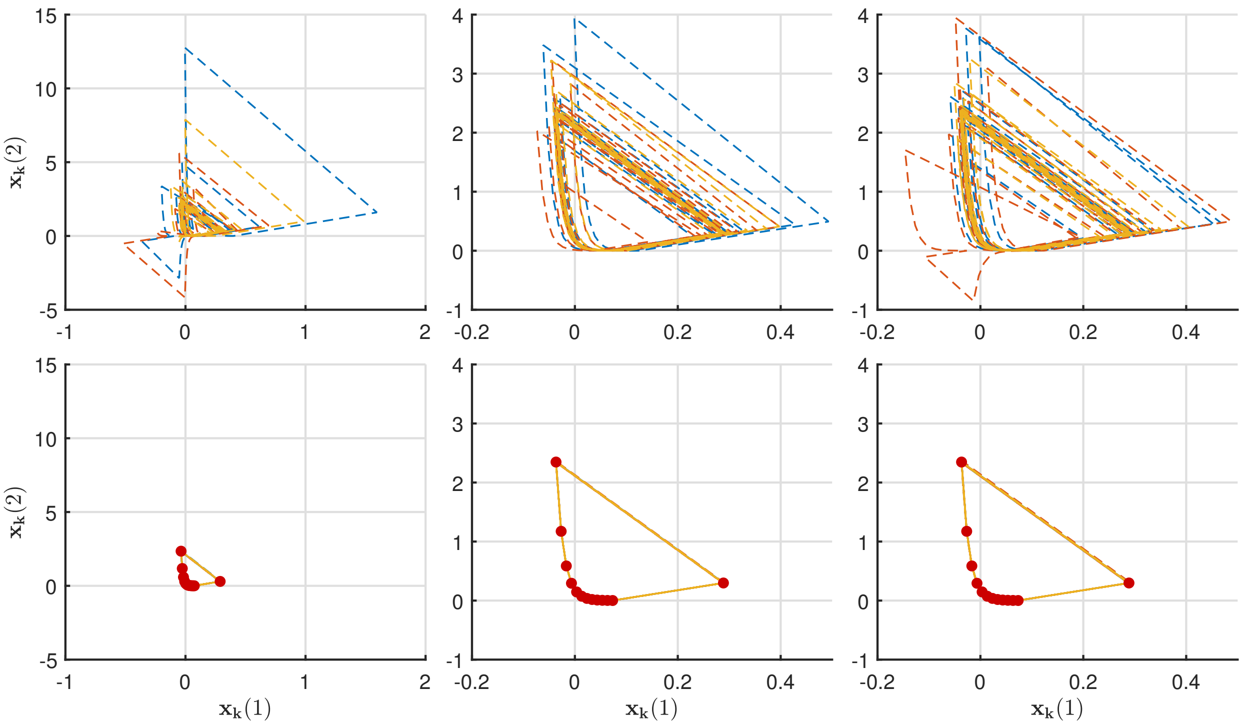

Fig. 4. From left to right, the different columns correspond to the trajectories in the phase plane from simulations results where is applied in first, the periodic control law (9); in second, the *all agent state*-dependent control law (21) and finally, the *agent mean state*-dependent control law (23). From top to bottom, the different rows correspond respectively to the state evolution for the complete simulation time and the state evolution corresponding to the shadow area represented in Fig. 3. The state periodic sequence of the limit cycle is represented on the second row by the red points.

Liu, T., Jiang, Z.-P., 2013. Distributed output-feedback control of nonlinear multi-agent systems. IEEE Transactions on Automatic Control 58 (11), 2912–2917.

Meng, L., Dragicevic, T., Roldán-Pérez, J., Vasquez, J. C., Guerrero, J. M., 2015. Modeling and sensitivity study of consensus algorithm-based distributed hierarchical control for dc microgrids. IEEE Transactions on Smart Grid 7 (3), 1504–1515.

Moreau, L., 2005. Stability of multiagent systems with time-dependent communication links. IEEE Transactions on Automatic Control 50 (2), 169–182.

Olfati-Saber, R., Fax, J. A., Murray, R. M., 2007. Consensus and cooperation in networked multi-agent systems. Proceedings of the IEEE 95 (1), 215–233.

Olfati-Saber, R., Murray, R. M., 2004. Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on Automatic Control 49 (9), 1520–1533.

Ren, W., Beard, R. W., 2008. Consensus algorithms for double-integrator dynamics. Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications, 77–104.

Rubensson, M., Lennartson, B., Pettersson, S., 1998. Convergence to limit cycles in hybrid systems – an example. IFAC Proceedings Volumes 31 (20), 683–688.

Scardovi, L., Sepulchre, R., 2008. Synchronization in networks of identical linear systems. In: 2008 47th IEEE Conference on Decision and Control. IEEE, pp. 546–551.

Serieye, M., Albea, C., Seuret, A., Jungers, M., 2020. Robust stabilization to limit cycles of switching discrete-time affine systems using control Lyapunov functions. Rapport LAAS-CNRS.

Serieye, M., Albea, C., Seuret, A., Jungers, M., 2020. Stabilization of switched affine systems via multiple shifted Lyapunov functions. In: IFAC World Congress.

Shi, G., Johansson, K. H., Hong, Y., 2013. Reaching an optimal consensus: Dynamical systems that compute intersections of convex sets. IEEE Transactions on Automatic Control 58 (3), 610–622.

Sinafar, B., Badamchizadeh, M. A., Kharrati, H., Baradarannia, M., 2020a. Distributed min-projection control: A switching consensus protocol for switched affine multi-agent systems. Journal of Vibration and Control.

Sinafar, B., Kharrati, H., Badamchizadeh, M. A., Baradarannia, M., 2020b. Distributed adaptive switching control of uncertain switched affine multi-agent systems. Nonlinear Analysis: Hybrid Systems 38, 100941.

Strogatz, S. H., 1994. Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering. Perseus Books.

Sun, Y.-J., 2008. Existence and uniqueness of limit cycle for a class of nonlinear discrete-time systems. Chaos, Solitons & Fractals 38 (1), 89–96.

Sun, Z., Ge, S. S., 2011. Stability Theory of Switched Dynamical Systems. Springer.

Wang, X., Han, F., 2011. Robust coordination control of switching multi-agent systems via output regulation approach. Kybernetika 47 (5).

Wieland, P., Kim, J.-S., Scheu, H., Allgöwer, F., 2008. On consensus in multi-agent systems with linear high-order agents. IFAC Proceedings Volumes 41 (2), 1541–1546.

Zhang, J., Lyu, M., Shen, T., Liu, L., Bo, Y., 2018. Sliding mode control for a class of nonlinear multi-agent system with time delay and uncertainties. IEEE Transactions on Industrial Electronics 65 (1), 865–875.