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Invariant sample measures and random Liouville type theorem for the two-dimensional stochastic Navier-Stokes equations ☆

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Abstract

In this article, we first prove some sufficient conditions guaranteeing the existence of invariant sample measures for random dynamical systems via the approach of global random attractors. Then we consider the two-dimensional incompressible Navier-Stokes equations with additive white noise as an example to show how to check the sufficient conditions for concrete stochastic partial differential equations. Our results generalize the Liouville type theorem to the random case and reveal that the invariance of the sample measures is a particular situation of the random Liouville type theorem.

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Keywords: Invariant sample measures; Random Liouville type theorem; Random dynamical system; Global random attractor; Stochastic Navier-Stokes equations

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1. Introduction

Invariant measures are one of the fundamental objective in the theory of turbulence. This is due to the fact that the measurements of some aspects, say the velocity, kinetic energy and turbulent boundary layer of the turbulent flows are indeed measurements of time-averaged quantities (see e.g. [13,20]). The invariant measures for deterministic evolution equations have been extensively studied, one can refer to [4,10,13,14,16,18,23,26] for well-posed systems and to [3,15,21,24,25] for ill-posed ones. Especially, Łukaszewicz, Real and Robinson [17] used the notion of Generalized Banach limit to construct the invariant measures for general continuous dynamical system on metric spaces. Later, Chekroun and Glatt-Holtz [10] improved the results of [17] to construct invariant measures for dissipative autonomous dynamical systems, and Łukaszewicz and Robinson [18] extended the result of [10] to construct invariant measures for dissipative non-autonomous dynamical systems. Recently, Zhao, Li and Caraballo [22] established some sufficient conditions ensuring the existence of trajectory statistical solutions for general evolution equations, including those systems which possess global weak solutions but without a known result of global uniqueness, say, the three-dimensional (3D) incompressible Navier-Stokes equations.

The original motivation of the current article is to investigate the invariant sample measures for dissipative random dynamical systems. Here we will adopt the definition and theory of random dynamical system from [11,27]. In this article, we let (X, d) be a separable and complete metric space and use $\mathcal{B}(\bullet)$ to denote the Borel σ-algebra over the space $\bullet$. Also let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\theta_t : \Omega \longmapsto \Omega, t \in \mathbb{R}\}$ be a family of measure preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$. If the mapping $(t, \omega) \mapsto \theta_t\omega$ is $\mathcal{B}(\mathbb{R} \times \mathcal{F}, \mathcal{F})$ measurable and $\{\theta_t\}_{t\in\mathbb{R}}$ satisfies the group property, then we call $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ a measurable dynamical system and $\{\theta_t\}_{t\in\mathbb{R}}$ the metric dynamical system over the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.1. ([11,27]) A family of mappings $\psi(t, \tau; \omega) : X \longmapsto X$, $-\infty < \tau < t < +\infty$, parameterized by $\omega \in \Omega$, is called a random dynamical system over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ with state space X, if it satisfies for $\mathbb{P}$ almost surely ($\mathbb{P}$ a.s. for short) $\omega \in \Omega$,

(a) $\psi(t, \tau; \omega)\psi(\tau, s; \omega)u = \psi(t, s; \omega)u$ for all $s \leqslant \tau \leqslant t$ and $u \in X$;
(b) $\psi(t, \tau; \omega)\cdot$ is continuous on X for all $\tau \leqslant t$;
(c) for all $t \in \mathbb{R}$, $u \in X$ the mapping $(s, \omega) \mapsto \psi(t, s; \omega)u$ is measurable from $((-\infty, t] \times \Omega, \mathcal{B}((-\infty, t] \times \mathcal{F})$ to $(X, \mathcal{B}(X))$;
(d) for all $s < t$ and $u \in X$, the mapping $\omega \mapsto \psi(t, s; \omega)u$ is measurable from $(\Omega, \mathcal{F})$ to $(X, \mathcal{B}(X))$.

For a given random dynamical system $\{\psi(t, \tau; \omega)\}_{t\geqslant\tau,\omega\in\Omega}$ over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ with state space X, we set

$$\phi(t - \tau, \theta_\tau\omega) = \psi(t, \tau; \omega) \text{ and } \Phi(t) : (\omega, u) \longmapsto (\theta_t\omega, \phi(t, \omega)u).$$

If $\phi(t, \omega)$ satisfies the so-called cocycle property (see e.g. [1]), then $\{\Phi(t)\}_{t\in\mathbb{R}}$ meets the semigroup property $\Phi(t + \tau) = \Phi(t)\Phi(\tau)$ (see [9]). $\{\Phi(t)\}_{t\in\mathbb{R}}$ is called the skew product on the extended phase space $\Omega \times X$. The random invariant measures, which plays the essential role in the theory of random dynamical systems, are intimately related to random attractors. If $\{\Phi(t)\}_{t\in\mathbb{R}}$ possesses a global random attractor $\mathbb{A} \subset \Omega \times X$, then $\mathbb{A}$ supports all the invariant measures μ of

$\{\Phi(t)\}_{t\in\mathbb{R}}$ on the product space $\Omega\times X$, and μ is invariant under the action of the skew product $\{\Phi(t)\}_{t\in\mathbb{R}}$, that is, $\Phi(t)\mu=\mu$ for all $t\in\mathbb{R}$. This result is indeed the same as the deterministic situation. At a glimpse, it seems that one recovers the approach (see e.g. [10,18,22]) of constructing the invariant measure for the deterministic dynamical system. In fact, it is not the case. There produces an additional difficulty, because in the space $\Omega\times X$ one can only utilize measurability on Ω without any topological tools available.

Notice that the invariant measure μ lift the probability measure $\mathbb{P}$, which is defined on the sample space Ω, into the extended phase space $\Omega\times X$, and the projection of μ onto Ω equals $\mathbb{P}$. It may be more convenient to work on the phase space X, rather than on the extended phase space $\Omega\times X$. The invariant property of μ on $\Omega\times X$ corresponds to the use of random measures $\omega\longmapsto\mu_\omega$ on X called **sample measures** (cf. [9]). In fact, we can establish that there exists a one-to-one correspondence between any μ_ω on X and any μ on $\Omega\times X$. Particularly, $\mu(\mathbb{A})=1$ if and only if $\mu_\omega(\mathbb{A}(\omega))=1$, in other words, each sample $\mathbb{A}(\omega)$ of $\mathbb{A}$ supports the sample measure μ_ω.

Definition 1.2. Let $\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$ be a random dynamical system over the measurable dynamical system $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ with state space X. A family of Borel probability measures $\{\mu_{\theta_t\omega}\}_{t\in\mathbb{R}}$ on X is called the invariant sample measures for $\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$, if for $\mathbb{P}$ a.s. $\omega\in\Omega$ and for all $E\in\mathcal{B}(X)$,

$$\mu_{\theta_t\omega}(E)=\mu_{\theta_\tau\omega}(\psi^{-1}(t,\tau;\omega)E),\quad t,\tau\in\mathbb{R},\quad t\geqslant\tau.$$

The main results of the current article are to present a general approach to construct the invariant sample measures for random dynamical systems, with application to stochastic partial differential equations (PDEs for short). Firstly, we establish some sufficient conditions guaranteeing the existence of invariant sample measures for random dynamical systems via global random attractors. Then we investigate the two-dimensional incompressible stochastic Naiver-Stokes equations, showing how to check the sufficient conditions for concrete stochastic PDEs. We prove the existence of invariant sample measures for the two-dimensional incompressible Navier-Stokes equations with additive white noise. Our results generalize the Liouville type theorem to the random case and reveal that the invariance of the sample measures is a particular situation of the random Liouville type theorem.

We want to point out that there exists an essential difference between the invariant sample measures and the invariant measures for stochastic PDEs. The invariant measures for stochastic PDEs have been extensively studied, see e.g. [6,19] and the references therein. To investigate the invariant measures for stochastic PDEs on its phase space $\mathcal{X}$, loosely speaking, one generally considers the associated Markov transition semigroup $\{P(t)\}_{t\geqslant 0}$ defined on the set $\mathcal{B}_b(\mathcal{X})$ of bounded Borel functions. Then the invariant measures for this stochastic PDEs refer to a probability measure ρ on $\mathcal{X}$ such that $P_t^*\rho=\rho$, $t\geqslant 0$, where $\{P_t^*\}_{t\geqslant 0}$ is the dual semigroup of $\{P_t\}_{t\geqslant 0}$. Here, our investigations rely heavily on the theory of infinite dimensional systems and functional analysis. We construct the invariant sample measures via the global random attractor of the random dynamical systems generated by the stochastic PDEs. Our proofs rely on the novel use of a general but elementary functional analysis, valid in any metric space, which concerns the growth of continuous functions in the neighborhood of random compact sets. We want to remark that our idea is inspired by that of [10,18,22], and our abstract result can also be applied to other dissipative stochastic PDEs including those on unbounded domains (see e.g. [2]).

The rest of the article is organized as follows. Section 2 is devoted to the proof of the sufficient conditions guaranteeing the existence of invariant sample measures for general random dynamical systems via the approach of global random attractors. In Section 3, we first recall some known results concerning the 2D stochastic Navier-Stokes equations, including the well-posedness and the existence of the global random attractor. Then we establish that the generated random dynamical system is continuous with respect to the initial time. Finally, we apply the abstract result obtained in Section 2 to the 2D stochastic Navier-Stokes equations. We prove the existence of the invariant sample measures and establish that the 2D stochastic Navier-Stokes equations satisfy the random Liouville type theorem. Moreover, we reveal that the invariance of the sample measures is exactly a particular situation of the random Liouville type theorem.

2. Sufficient condition guaranteeing the existence of invariant sample measures

In this section, we first recall some definitions relative to the random dynamical system. Then we prove the sufficient condition guaranteeing the existence of invariant sample measures for random dynamical system via the approach of random attractor.

We have introduced the separable and complete metric space (X, d) and its Borel σ-algebra $\mathcal{B}(X)$ over X, the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ with the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the metric dynamical system $\{\theta_t\}_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Besides these, we denote by $dist_X(\mathscr{A}, \mathscr{B}) = \sup\limits_{a\in\mathscr{A}} \inf\limits_{b\in\mathscr{B}} d(a, b)$ the Hausdorff semidistance between $\mathscr{A} \subset X$ and $\mathscr{B} \subset X$. Particularly, $dist_X(a, \mathscr{B}) = \inf\limits_{b\in\mathscr{B}} d(a, b)$ and $dist_X(a, b) = d(a, b)$. Also, we will use some other definitions relative to the random dynamical system. A random set can be regarded as a family of sets parameterized by the random parameter ω and satisfies some measurability property. Precisely, a random set B can be identified by the family of its ω-fibers $B(\omega)$, defined by

$$B(\omega) = \{u \in X : (x, \omega) \in B\}, \quad \omega \in \Omega.$$

As a random set $B \subset X \times \Omega$ possesses closed fibers, it is said to be a closed random set if and only if for every $u \in X$ the mapping $\omega \in \Omega \longmapsto dist_X(u, B(\omega))$ is measurable (cf. [7,8]). When the fibers of B are compact, B is called to be a random compact set.

Definition 2.1. Let $\{\psi(t, \tau; \omega)\}_{t\geqslant\tau,\omega\in\Omega}$ be a random dynamical system over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ with the state space (X, d). A random subset $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a global random attractor for $\{\psi(t, \tau; \omega)\}_{t\geqslant\tau,\omega\in\Omega}$ on (X, d), if the following conditions hold

(1) (Random compactness) For $\mathbb{P}$ a.s. $\omega \in \Omega$, $\mathcal{A}(\omega)$ is compact in X;
(2) (Invariance) For $\mathbb{P}$ a.s. $\omega \in \Omega$, $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant in the sense that

$$\psi(t, \tau; \omega)\mathcal{A}(\theta_\tau\omega) = \mathcal{A}(\theta_t\omega), \ \forall\tau \leqslant t;$$

(3) (Attracting property) For $\mathbb{P}$ a.s. $\omega \in \Omega$, for every $t \in \mathbb{R}$ and $B \subset X$ bounded, there holds

$$\lim_{\tau\to-\infty} dist_X(\psi(t, \tau; \omega)B, \mathcal{A}(\theta_t\omega)) = 0.$$

We now begin to prove two auxiliary lemmas which will play the key role when we construct the invariant sample measures. In the sequel, we use $\mathcal{C}(\bullet)$ to denote the set of continuous functions defined on the space $\bullet$.

Lemma 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{K(\omega)\}_{\omega\in\Omega}$ be a random compact subset of the separable and complete metric space (X, d). Then for every $g \in \mathcal{C}(X)$, for $\mathbb{P}$ a.s. $\omega \in \Omega$ there corresponds an $\epsilon_\omega > 0$ such that*

$$\sup_{v\in\mathcal{O}(K(\omega);\epsilon_\omega)} |g(v)| < +\infty,$$

where $\mathcal{O}(K(\omega); \epsilon_\omega) = \{v \in X : dist_X(v, K(\omega)) < \epsilon_\omega\}$.

Proof. Let $\{K(\omega)\}_{\omega\in\Omega}$ be a random compact subset of the separable and complete metric space (X, d). Without loss of generality, we consider a fixed $g \in \mathcal{C}(X)$ and a fixed $\omega \in \Omega$. Then for every $u \in K(\omega)$ one can pick $\delta(u, \omega)$ such that for every

$$v \in \mathcal{O}(u; \delta(u, \omega)) = \{v \in X : d(v, u) < \delta(u, \omega)\}$$

there holds $|g(u) - g(v)| < 1$. Choosing numbers $\delta(u, \omega)$ in this way, we obtain an open covering

$$\Xi_\omega = \big\{\mathcal{O}(u; \delta(u, \omega)/3) : u \in K(\omega)\big\}$$

for $K(\omega)$. Note that $K(\omega)$ is compact in X. We can extract from the open covering Ξ_ω a finite one

$$\Xi_\omega^{(m)} = \big\{\mathcal{O}(u_1; \delta(u_1, \omega)/3), \mathcal{O}(u_2; \delta(u_2, \omega)/3), \cdots, \mathcal{O}(u_m; \delta(u_m, \omega)/3)\big\}.$$

Choose

$$\epsilon_\omega = \min\big\{\delta(u_1, \omega)/3, \delta(u_2, \omega)/3, \cdots, \delta(u_m, \omega)/3\big\}, \quad c = 1 + \max_{1\leqslant j\leqslant m} |g(u_j)|.$$

Now for any given $v \in \mathcal{O}(K(\omega); \epsilon_\omega)$, we can pick $u \in K(\omega)$ so that $d(v, u) < 2\epsilon_\omega$. Also we can choose u_j meeting $d(u, u_j) < \delta(u_j, \omega)/3$ because $\Xi_\omega^{(m)}$ covers $K(\omega)$. Therefore, we have

$$d(v, u_j) \leqslant d(v, u) + d(u, u_j) < 2\epsilon_\omega + \delta(u_j, \omega)/3 \leqslant \delta(u_j, \omega)$$

and

$$|g(v)| \leqslant 1 + |g(u_j)| \leqslant 1 + \max_{1\leqslant j\leqslant m} |g(u_j)| = c.$$

This ends the proof. □

Lemma 2.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{K(\omega)\}_{\omega\in\Omega}$ a random compact subset of the separable and complete metric space (X, d), and let $g, h \in \mathcal{C}(X)$ satisfying for $\mathbb{P}$ a.s. $\omega \in \Omega$ $g(\xi) = h(\xi)$ for every $\xi \in \mathcal{K}(\omega)$. Then for every $\epsilon > 0$ there corresponds for $\mathbb{P}$ a.s. $\omega \in \Omega$ a $\gamma(\epsilon, \omega) > 0$ such that*

$$\sup_{v\in\mathcal{O}(K(\omega);\gamma(\epsilon,\omega))} |g(v) - h(v)| < \epsilon.$$

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{K(\omega)\}_{\omega\in\Omega}$ a random compact subset of the separable and complete metric space (X, d). Consider given $g, h \in \mathcal{C}(X)$. Fix $\epsilon > 0$ and $\omega \in \Omega$. For every $\xi \in K(\omega)$ we can pick $\gamma(\xi, \epsilon, \omega)$ yielding

$$|g(\xi) - g(v)| + |h(\xi) - h(v)| < \epsilon \quad \text{whenever} \quad v \in \mathcal{O}(\xi; \gamma(\xi, \epsilon, \omega)).$$

Obviously, $\{\mathcal{O}(\xi; \gamma(\xi, \epsilon, \omega)) : \xi \in K(\omega)\}$ is an open covering of $K(\omega)$. Due to the compactness of $K(\omega)$ in X, one can cover $K(\omega)$ with finite collection

$$\big\{\mathcal{O}(\xi_1; \gamma(\xi_1, \epsilon, \omega)/3), \mathcal{O}(\xi_2; \gamma(\xi_2, \epsilon, \omega)/3), \cdots, \mathcal{O}(\xi_k; \gamma(\xi_k, \epsilon, \omega)/3)\big\}.$$

Put $\gamma(\epsilon, \omega) = \min\limits_{1\leqslant j\leqslant k} \frac{\gamma(\xi_j,\epsilon,\omega)}{3}$ and we have

$$\mathcal{O}(K(\omega); \gamma(\epsilon, \omega)) \subset \mathcal{O}\Big(\big(\bigcup_{j=1}^{k} \mathcal{O}(\xi_j; \frac{\gamma(\xi_j, \epsilon, \omega)}{3}\big); \gamma(\epsilon, \omega)\Big) \subset \bigcup_{j=1}^{k} \mathcal{O}(\xi_j; \gamma(\xi_j, \epsilon, \omega)).$$

Now for any $v \in \mathcal{O}(K(\omega); \gamma(\epsilon, \omega))$, we may pick j such that $v \in \mathcal{O}(\xi_j; \gamma(\xi_j, \epsilon, \omega))$. Note that $g(\xi_j) = h(\xi_j)$. Hence

$$|g(v) - h(v)| \leqslant |g(v) - g(\xi_j)| + |h(\xi_j) - h(v)| < \epsilon.$$

The proof is complete. □

To state and prove the main result of this section, we need to recall the definition of generalized Banach limit.

Definition 2.2. ([13,18]) A generalized Banach limit is any linear functional, denoted by $\mathrm{LIM}_{t\to+\infty}$, defined on the space of all bounded real-valued functions on $[0, +\infty)$ and satisfying

(1) $\mathrm{LIM}_{t\to+\infty}\zeta(t) \geqslant 0$ for nonnegative functions $\zeta(\cdot)$ on $[0, +\infty)$;
(2) $\mathrm{LIM}_{t\to+\infty}\zeta(t) = \lim\limits_{t\to+\infty} \zeta(t)$ if the usual limit $\lim\limits_{t\to+\infty} \zeta(t)$ exists.

Let B_+ be the collection of all bounded real-valued functions on $[0, +\infty)$. For any generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$, the following useful property

$$|\mathrm{LIM}_{t\to+\infty}\zeta(t)| \leqslant \limsup_{t\to+\infty} |\zeta(t)|, \quad \forall \zeta(\cdot) \in B_+, \tag{2.1}$$

is presented in [13, (1.38)] and in [10, (2.3)].

Notice that we will consider the asymptotic behavior $\tau \to -\infty$ of $\psi(t,\tau;\omega)\bullet$. Therefore, we require generalized limits as $\tau \to -\infty$. For a given real-valued function ζ defined on $(-\infty, 0]$ and a given Banach limit $\mathrm{LIM}_{T\to+\infty}$, we define

$$\mathrm{LIM}_{t\to-\infty}\zeta(t) = \mathrm{LIM}_{t\to+\infty}\zeta(-t).$$

In the sequel, for a given Borel probability measure μ on X and a function $\varrho \in \mathcal{C}(X)$, we use $\int_X \varrho(u)\mathrm{d}\mu(u)$ to denote the Bochner integral.

The main result of this section reads as follows.

Theorem 2.1. *Let (X,d) be a complete metric space and $\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$ be a random dynamical system over the measurable dynamical system $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ with state space (X,d). Suppose that*

(i) *$\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$ possesses a global random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ on X;*
(ii) *for each given $t \in \mathbb{R}$, $u \in X$ and for $\mathbb{P}$ a.s. $\omega \in \Omega$, the X-valued mapping $\tau \longmapsto \psi(t,\tau;\omega)u$ is continuous and bounded on $(-\infty, t]$.*

Then for a given continuous mapping $v(\cdot) : \mathbb{R} \mapsto X$ and a generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$, there exists for $\mathbb{P}$ a.s. $\omega \in \Omega$ a family of Borel probability measures $\{\mu_{\theta_t\omega}\}_{t\in\mathbb{R}}$ on X such that the support of $\mu_{\theta_t\omega}$ is contained in $\mathcal{A}(\theta_t\omega)$ and

$$\begin{aligned}\int_X \varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u) &= \int_{\mathcal{A}(\theta_t\omega)} \varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u)\\ &= \mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \varrho(\psi(t,s;\omega)v(s))\mathrm{d}s \qquad (2.2)\\ &= \mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t\int_X \varrho(\psi(t,s;\omega)u)\mathrm{d}\mu_{\theta_s\omega}(u)\mathrm{d}s \qquad (2.3)\end{aligned}$$

for any nonnegative, real-valued continuous functional ϱ on X. Moreover, for $\mathbb{P}$ a.s. $\omega \in \Omega$, $\mu_{\theta_t\omega}$ is invariant under the action of the random dynamical system $\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$ in the sense that

$$\int_{\mathcal{A}(\theta_t\omega)} \varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u) = \int_{\mathcal{A}(\theta_\tau\omega)} \varrho(\psi(t,\tau;\omega)u)\mathrm{d}\mu_{\theta_\tau\omega}(u), \quad \forall t \geqslant \tau. \qquad (2.4)$$

Proof. Let $\mathrm{LIM}_{t\to+\infty}$ be a given generalized Banach limit, $v(\cdot) : \mathbb{R} \mapsto X$ a continuous map and $\varrho(\cdot)$ a nonnegative, real-valued continuous functional ϱ on X.

We first prove that for each given $t \in \mathbb{R}$ and for $\mathbb{P}$ a.s. $\omega \in \Omega$, the function $s \longmapsto \varrho(\psi(t,s;\omega)v(s))$ is bounded on $(-\infty, t]$. Indeed, we claim that there exists some negative t_0

sufficiently large such that, for $\mathbb{P}$ a.s. $\omega \in \Omega$, the function $s \longmapsto \varrho(\psi(t,s;\omega)v(s))$ is bounded on $(-\infty, t_0]$. Assume that this is not the case. Then, there is a sequence $\{s_n\}_{n=1}^{\infty}$ with $s_n \to -\infty$ as $n \to \infty$ such that

$$|\varrho(\psi(t,s_n;\omega)v(s_n))| \to +\infty, \quad n \to \infty. \tag{2.5}$$

Now by condition (i), the random dynamical system $\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$ possesses a global random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ on X. From the attracting property of the global random attractor, we see that, for $\mathbb{P}$ a.s. $\omega \in \Omega$ and every $\epsilon > 0$, there exists a time $s(\epsilon,\omega,t)$ such that

$$\psi(t,\tau;\omega)v(\tau) \in \mathcal{O}(\mathcal{A}(\theta_t\omega);\epsilon), \quad \forall \tau \leqslant s(\epsilon,\omega,t). \tag{2.6}$$

By Lemma 2.1, we can choose $\epsilon_\omega > 0$ such that

$$C_\omega = \sup_{\mathcal{O}(\mathcal{A}(\theta_t\omega);\epsilon_\omega)} |\varrho(u)| < +\infty. \tag{2.7}$$

Then (2.6) and (2.7) contradict with (2.5). At the same time, from condition (ii) we see that, for $\mathbb{P}$ a.s. $\omega \in \Omega$, the X-valued mapping $s \longmapsto \varrho(\psi(t,s;\omega)v(s))$ is continuous on $(-\infty, t]$. Thus it is bounded on each compact interval $[t_0, t]$.

Secondly, for each given $t \in \mathbb{R}$ and for $\mathbb{P}$ a.s. $\omega \in \Omega$, we define

$$\mathcal{L}_{\omega,v}(\varrho) = \mathrm{LIM}_{s\to-\infty} \frac{1}{t-s}\int_s^t \varrho(\psi(t,\eta;\omega)v(\eta))\mathrm{d}\eta \tag{2.8}$$

for nonnegative function $\varrho \in \mathcal{C}(X)$. Then, by the above analysis and the property of the generalized Banach limit, we see that the function

$$s \longmapsto \frac{1}{t-s}\int_s^t \varrho(\psi(t,\eta;\omega)v(\eta))\mathrm{d}\eta$$

is bounded on $(-\infty, t]$ for $\mathbb{P}$ a.s. $\omega \in \Omega$. Hence $\mathcal{L}_{\omega,v}(\varrho)$ defined by (2.8) is well defined as a positive linear functional on $\mathcal{C}(X)$ for $\mathbb{P}$ a.s. $\omega \in \Omega$.

Thirdly, we prove that, for $\mathbb{P}$ a.s. $\omega \in \Omega$, $\mathcal{L}_{\omega,v}(\varrho)$ depends only on the values of ϱ on $\mathcal{A}(\theta_t\omega)$. Factually, take nonnegative ϱ_1 and ϱ_2 in $\mathcal{C}(X)$ with $\varrho_1(\cdot) = \varrho_2(\cdot)$ on $\mathcal{A}(\theta_t\omega)$ for $\mathbb{P}$ a.s. $\omega \in \Omega$. Then for any $\epsilon > 0$, by Lemma 2.2 we can find for $\mathbb{P}$ a.s. $\omega \in \Omega$ a $\gamma(\epsilon,\omega) > 0$ such that

$$|\varrho_1(u) - \varrho_2(u)| < \epsilon/2, \quad \text{whenever } u \in \mathcal{O}(\mathcal{A}(\theta_t\omega);\gamma(\epsilon,\omega)). \tag{2.9}$$

By the attracting property of the global random attractor, we can pick s_0 such that

$$dist_X(\psi(t,s;\omega)v(s), \mathcal{A}(\theta_t\omega)) \leqslant \gamma(\epsilon,\omega) \text{ for all } s \leqslant s_0.$$

Then, for every $s \leqslant s_0$ there corresponds a $v_s \in \mathcal{A}(\theta_t\omega)$ such that

$$d(\psi(t,s;\omega)v(s), v_s) \leqslant \gamma(\epsilon,\omega).$$

Thus, by Lemma 2.2 we have for $s \leqslant s_0$ that

$$\begin{aligned}&|\varrho_1(\psi(t,s;\omega)v(s)) - \varrho_2(\psi(t,s;\omega)v(s))|\\ \leqslant&|\varrho_1(\psi(t,s;\omega)v(s)) - \varrho_1(v_s)| + |\varrho_1(v_s) - \varrho_2(v_s)| + |\varrho_2(\psi(t,s;\omega)v(s)) - \varrho_2(v_s)| < \epsilon.\end{aligned}$$

By condition (ii) and noticing that $v(\cdot) : \mathbb{R} \mapsto X$ is a continuous map, we find that

$$\sup_{\eta\in[s_0,t]} \{|\varrho_1(\psi(t,\eta;\omega)v(\eta))| + |\varrho_2(\psi(t,\eta;\omega)v(\eta))|\}$$

is bounded by a constant independent of s. Therefore, by the property of the generalized Banach limit, we have

$$\begin{aligned}|\mathcal{L}_{\omega,v}(\varrho_1-\varrho_2)| =&\Big|\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^t \big(\varrho_1(\psi(t,\eta;\omega)v(\eta)) - \varrho_2(\psi(t,\eta;\omega)v(\eta))\big)\mathrm{d}\eta\Big|\\ =&\Big|\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_s^{s_0} \big(\varrho_1(\psi(t,\eta;\omega)v(\eta)) - \varrho_2(\psi(t,\eta;\omega)v(\eta))\big)\mathrm{d}\eta\Big|\\ &+\Big|\mathrm{LIM}_{s\to-\infty}\frac{1}{t-s}\int_{s_0}^{t} \big(\varrho_1(\psi(t,\eta;\omega)v(\eta)) - \varrho_2(\psi(t,\eta;\omega)v(\eta))\big)\mathrm{d}\eta\Big|\\ \leqslant&\limsup_{s\to-\infty}\frac{s_0-s}{t-s}\epsilon\\ &+\limsup_{s\to-\infty}\frac{(t-s_0)}{t-s}\sup_{\eta\in[s_0,t]}\{|\varrho_1(\psi(t,\eta;\omega)v(\eta))| + |\varrho_2(\psi(t,\eta;\omega)v(\eta))|\}\\ =&\epsilon.\end{aligned}$$

By the arbitrariness of ϵ, we obtain $\mathcal{L}_{\omega,v}(\varrho_1-\varrho_2) = 0$.

Fourthly, for $\mathbb{P}$ a.s. $\omega \in \Omega$, we define $G_{\omega,v}(\varrho) = \mathcal{L}_{\omega,v}(\ell(\varrho)$ for $\varrho \in \mathcal{C}(\mathcal{A}(\theta_t\omega))$, where $\ell(\varrho)$ is the extension of ϱ from $\mathcal{C}(\mathcal{A}(\theta_t\omega))$ to $\mathcal{C}(X)$ given by the Tietze theorem (see [13, Theorem A.7]). Then $G_{\omega,v}(\cdot)$ is a positive linear functional on $\mathcal{C}(\mathcal{A}(\theta_t\omega))$. Notice that $\mathcal{A}(\theta_t\omega)$ is compact in X. $\mathcal{A}(\theta_t\omega)$ is obviously a locally compact topological space. By the Kakutani-Riesz Representation Theorem (see [13, Theorem A.1]), we obtain that there exists a unique positive, finite, Borel measure $\mu_{\theta_t\omega}$ on $\mathcal{A}(\theta_t\omega)$ such that

$$G_{\omega,v}(\varrho) = \int_{\mathcal{A}(\theta_\omega t)} \varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u). \tag{2.10}$$

We extend $\mu_{\theta_t\omega}$ by zero to Borel measure on X, which is still denoted by $\mu_{\theta_t\omega}$:

$$\mu_{\theta_t\omega}(E) = \mu_{\theta_t\omega}(E \cap \mathcal{A}(\theta_t\omega)), \quad E \in \mathcal{B}(X).$$

Therefore $\int\limits_X \varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u) = \int\limits_{\mathcal{A}(\theta_t\omega)} \varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u)$ and (2.2) is proved.

Finally, we prove (2.3) and (2.4). Fix t and τ with $t \geqslant \tau$. Consider some nonnegative $\varrho \in \mathcal{C}(X)$. By (2.2) we have for $\mathbb{P}$ a.s. $\omega \in \Omega$,

$$
\begin{aligned}
&\int\limits_{\mathcal{A}(\theta_t\omega)} \varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u) \\
=&\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{t-\gamma}\int\limits_{\gamma}^{t} \varrho(\psi(t,\eta;\omega)v(\eta))\mathrm{d}\eta \\
=&\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{t-\gamma}\int\limits_{\gamma}^{\tau} \varrho(\psi(t,\eta;\omega)v(\eta))\mathrm{d}\eta + \mathrm{LIM}_{\gamma\to-\infty}\frac{1}{t-\gamma}\int\limits_{\tau}^{t} \varrho(\psi(t,\eta;\omega)v(\eta))\mathrm{d}\eta.
\end{aligned}
$$

Since $[\tau, t]$ is compact in $\mathbb{R}$ and by condition (ii) the mapping $\eta \longmapsto |\varrho(\psi(t,\eta;\omega)v(\eta))|$ is continuous, we obtain by using the property of the generalized Banach limit that

$$
\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{t-\gamma}\int\limits_{\tau}^{t} \varrho(\psi(t,\eta;\omega)v(\eta))\mathrm{d}\eta = 0.
$$

Note that $\psi(t,\eta;\omega)\cdot$ is continuous on X and $\varrho \in \mathcal{C}(X)$. Hence $\varrho \circ (\psi(t,\eta;\omega)\cdot \in \mathcal{C}(X)$. By (2.2) and direct calculations, we arrive at

$$
\begin{aligned}
&\int\limits_{\mathcal{A}(\theta_t\omega)} \varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u) \\
=&\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{t-\gamma}\int\limits_{\gamma}^{\tau} \varrho(\psi(t,\eta;\omega)v(\eta))\mathrm{d}\eta \\
=&\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{\tau-\gamma}\int\limits_{\gamma}^{\tau} \varrho(\psi(t,\eta;\omega)v(\eta))\mathrm{d}\eta \\
=&\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{\tau-\gamma}\int\limits_{\gamma}^{\tau} \varrho(\psi(t,\tau;\omega)\psi(\tau,\eta;\omega)v(\eta))\mathrm{d}\eta \\
=&\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{\tau-\gamma}\int\limits_{\gamma}^{\tau} [\varrho\circ(\psi(t,\tau;\omega)]\big(\psi(\tau,\eta;\omega)v(\eta)\big)\mathrm{d}\eta \\
=&\int\limits_X [\varrho\circ(\psi(t,\tau;\omega)]u\mathrm{d}\mu_{\theta_\tau\omega}(u) = \int\limits_X \varrho(\psi(t,\tau;\omega)u)\mathrm{d}\mu_{\theta_\tau\omega}(u),
\end{aligned}
$$

and (2.4) is proved. Now by (2.2) and (2.4), we have

$$
\begin{aligned}
&\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_{\tau}^{t}\int_{X}\varrho(\psi(t,s;\omega)u)\mathrm{d}\mu_{\theta_s\omega}(u)\mathrm{d}s\\
=&\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_{\tau}^{t}\int_{\mathcal{A}(\theta_t\omega)}\varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u)\mathrm{d}s\\
=&\int_{\mathcal{A}(\theta_t\omega)}\varrho(u)\mathrm{d}\mu_{\theta_t\omega}(u).
\end{aligned}
$$

(2.3) is proved and this completes the proof of Theorem 2.1. □

3. Invariant sample measures and random Liouville type theorem for the two-dimensional Navier-Stokes equations with additive white noise

In this section, we will apply the abstract theory obtained in Section 2 to the two-dimensional Navier-Stokes equations with additive white noise. We will show that the random dynamical system $\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$ generated by the two-dimensional Navier-Stokes equations with additive white noise possesses a global random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ on the phase space H. Then we prove that for given t and $u_0\in H$, for $\mathbb{P}$ a.a. $\omega\in\Omega$ the H-valued mapping $\tau\mapsto\psi(t,\tau;\omega)u_0$ is continuous and bounded on $(-\infty,t]$, and thus we obtain the existence of the invariant sample measures $\{\mu_{\theta_t\omega}\}_{t\in\mathbb{R}}$ for $\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$ on H. Further, we establish that the invariant sample measures $\{\mu_{\theta_t\omega}\}_{t\in\mathbb{R}}$ satisfies the so-called random Liouville type theorem, and reveal that the invariance of the sample measures is a particular situation of the random Liouville type theorem.

We consider the following 2D Navier-Stokes equations with additive noise [11]

$$
\begin{cases}
\mathrm{d}u+(-\nu\Delta u+(u\cdot\nabla)u+\nabla p)\mathrm{d}t=f\mathrm{d}t+\sum\limits_{j=1}^{m}\varphi_j\mathrm{d}w_j, & x\in\mathbb{D},\\
\nabla\cdot u=0, \quad x\in\mathbb{D},\\
u|_{\partial\mathbb{D}}=0,\\
u|_{t=s}=u_s,
\end{cases}
\tag{3.1}
$$

where the unknowns are the velocity $u=(u_1,u_2)$ and the pressure p, u_s is the initial value, f is the time-independent external force, $\nu>0$ is the viscosity, $\mathbb{D}\subset\mathbb{R}^2$ is a bounded domain with suitable smooth boundary $\partial\mathbb{D}$, the functions φ_j, $j=1,2,\cdots,m$, for some positive integer m, are time independent which will be specified below. The random functions w_j, $j=1,2,\cdots,m$, are independent two-sided real valued Wiener processes on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, here $\Omega=\{\omega\in\mathcal{C}(\mathbb{R},\mathbb{R}^m):\omega(0)=0\}$, with $\mathbb{P}$ being a product measure of two Wiener measures on the negative and the positive time part of Ω. Then we have

$$
(w_1(t,\omega),w_2(t,\omega),\cdots,w_m(t,\omega))=\omega(t),\quad t\in\mathbb{R}.
$$

The metric dynamical system $\{\theta_t\}_{t\in\mathbb{R}}$ on $(\Omega,\mathcal{F},\mathbb{P})$ is defined as

$$\theta_t\omega(s) = \omega(s+t) - \omega(t), \quad t, s \in \mathbb{R}.$$

The stochastic Navier-Stokes equations (3.1) have been extensively studied, one can refer to [5,6,11] and the references therein. Here we just take the stochastic problem (3.1) as an example to show how to employ our abstract to the concrete stochastic PDEs.

Let us introduce the mathematical setting of our problem. In the following, we will use the following notations.
$L^q(\mathbb{D})$-the usual 2D Lebesgue space with norm $\|\cdot\|_{L^q(\mathbb{D})}$; particularly, $\|\cdot\|_{L^2(\mathbb{D})} = \|\cdot\|$;
$W^{m,q}(\mathbb{D})$ and $W_0^{m,q}(\mathbb{D})$-the usual 2D Sobolev space with norm $\|\cdot\|_{m,q}$;
$\mathcal{V} = \{g = (g_1, g_2) \in (\mathcal{C}_0^\infty(\mathbb{D}))^2 | \nabla \cdot g = 0\}$;
$H =$ the closure of $\mathcal{V}$ in L^2 with inner product $(\cdot,\cdot)$ and norm as in L^2, $\|\cdot\|_H = \|\cdot\|$;
$V =$ the closure of $\mathcal{V}$ in $W^{1,2}$ with norm $\|\cdot\|_V = \|\cdot\|_{1,2}$ and dual space V^*;
$\langle\cdot,\cdot\rangle$-the dual pairing between V^* and V.

We consider the operator $A : V \mapsto V'$ which is defined as

$$\langle Au, v\rangle = (\nabla u, \nabla v), \quad u, v \in V.$$

Denoting the domain $D(A) = W^{2,2}(\mathbb{D}) \cap V$, then $Au = -\mathcal{P}\Delta u$, $\forall u \in D(A)$, is the Stokes operator, where $\mathcal{P}$ is the Leray-Helmholtz projection from $L^2(\mathbb{D})$ onto H. At the same time, we define a trilinear form

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x, \quad u, v, w \in W_0^1(\mathbb{D}).$$

Since $V \subseteq W_0^1(\mathbb{D})$ is a closed subspace, $b(u, v, w)$ is continuous on $V \times V \times V$, and it is classical that

$$b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0, \quad \forall u, v, w \in V.$$

For any $u, v \in V$, $\langle B(u, v), w\rangle = b(u, v, w)$, $\forall\, w \in V$, defines a continuous function $B(u, v)$ on $V \times V$.

We assume that $\varphi_j \in W^{2,2}(\mathbb{D}) \cap W_0^{1,2}(\mathbb{D})$, and set $\Phi_j = \mathcal{P}\varphi_j$, $j = 1, 2, \cdots, m$. Then problem (3.1) can be written as the stochastic differential equations in H:

$$\mathrm{d}u + (\nu Au + B(u))\mathrm{d}t = f\,\mathrm{d}t + \sum_{j=1}^{m} \Phi_j \mathrm{d}w_j, \tag{3.2}$$

$$u|_{t=s} = u_s. \tag{3.3}$$

It is usual to translate the unknown in the investigation of problem (3.2)-(3.3). Here we recall the procedures in [11]. Let

$$z = \sum_{j=1}^{m} \Phi_j z_j$$

be the Ornstein-Uhlenbeck process with

$$z_j = \int_{-\infty}^{t} e^{-\alpha(t-s)} \mathrm{d}w_j(s),$$

where $\alpha > 0$ is a parameter which will be chosen sufficiently large in the proof of the existence of the global random attractor (see [11]). We all know that for $j = 1, 2, \cdots, m$, z_j is the stationary solution of the one-dimensional Itô equation

$$\mathrm{d}z_j = -\alpha z_j \mathrm{d}t + \mathrm{d}w_j, \tag{3.4}$$

and that z is a stationary process and its trajectories are continuous for $\mathbb{P}$ a.s. $\omega \in \Omega$. Set

$$v = u - z,$$

then problem (3.2)-(3.3) can be written as the form of the following problem with random parameter z

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \nu A v + B(v + z) = f + \alpha z - \nu A z, \tag{3.5}$$

$$v(t, \omega)|_{t=s} = v_s(\omega). \tag{3.6}$$

For the definition of solutions to problem (3.2)-(3.3) and problem (3.5)-(3.6), we can refer to [11,12]. To obtain the existence and uniqueness of solutions to problem (3.2)-(3.3), we assume (H) *For $j = 1, 2, \cdots, m$, $\Phi_j \in D(A)$ and that there exists a positive constant c such that*

$$|b(u, \Phi_j, u)| \leqslant c\|u\|^2, \quad \forall u \in H. \tag{3.7}$$

The following results have been proved in [11].

Proposition 3.1. *Let assumption* (H) *be satisfied and $f \in H$ be given.*

(1) *For each $\omega \in \Omega$, $s \in \mathbb{R}$ and $v_s(\omega) \in H$, problem* (3.5)-(3.6) *possesses a unique corresponding solution $v(\cdot, s; \omega, v_s)$ defined on $[s, +\infty)$.*
(2) *Let $v(t, s; \omega, v_s)$ be the solution of problem* (3.5)-(3.6) *with $v_s = u_s - z(s, \omega)$ at the initial time s. Then $u(t, s; \omega, u_s) = v(t, s; \omega, v_s) + z(t, \omega)$ is the solution of problem* (3.2)-(3.3) *corresponding to the initial value u_s. Define the family of operators $\{\psi(t, s; \omega)\}_{t \geqslant s, \omega \in \Omega}$ from H to H via*

$$\psi(t, s; \omega)u_s = u(t, s; \omega, u_s) = v(t, s; \omega, v_s) + z(t, \omega).$$

Then $\{\psi(t, s; \omega)\}_{t \geqslant s, \omega \in \Omega}$ generates a random dynamical system over the measurable dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ with the state space H, satisfying Definition 1.1 and for $\mathbb{P}$ a.s. $\omega \in \Omega$,

$$\psi(t, s; \omega)u = \psi(t - s, 0; \theta_s \omega)u, \quad \text{for all } s < t \text{ and } u \in H.$$

(3) *The random dynamical system* $\{\psi(t,s;\omega)\}_{t\geqslant s,\omega\in\Omega}$ *possesses a global random attractor* $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *in* H *satisfying Definition 2.1.*

We next investigate the continuity of the random dynamical system $\{\psi(t,s;\omega)\}_{t\geqslant s,\omega\in\Omega}$ with respect to the initial time s. For simplicity, we will employ the notation $a\lesssim b$ to mean that $a\leqslant cb$ for a universal constant $c>0$ that only depends on the parameters coming from the problem.

Lemma 3.1. *For given* $u_*\in H$*,* $t\in\mathbb{R}$*, and* $\mathbb{P}$ *a.s.* $\omega\in\Omega$*, the* H*-valued mapping* $s\mapsto\psi(t,s;\omega)u_*$ *is right continuous on* $(-\infty,t]$.

Proof. Notice that $\psi(t,s;\omega)u_*=v(t,s;\omega,u_*-z(s,\omega))+z(t,\omega)$, and for $\mathbb{P}$ a.s. $\omega\in\Omega$, $\|z(t,\omega)\|$ is continuous with respect to $t\in\mathbb{R}$. Hence we just need prove that

$$v(t,\cdot;\omega,v_*)=v(t,\cdot;\omega,u_*-z(\cdot,\omega))$$

is right continuous on $(-\infty,t]$. Without loss of generality, we fix $v_*\in H$, $\omega\in\Omega$ and $s_*\in\mathbb{R}$. Then we just need to prove that for any $\epsilon>0$ there exists a positive constant $\delta=\delta(\epsilon,s_*,\omega,v_*)$ such that for $\mathbb{P}$ a.a. $\omega\in\Omega$,

$$\|v(\tau,s;\omega,v_*)-v_*\|<\epsilon,\quad \text{whenever } s\in(s_*,s_*+\delta),\quad \tau\in(s,s_*+\delta), \tag{3.8}$$

where $v(\tau,s;\omega,v_*)$ is the solution of problem (3.5)-(3.6) with the initial value v_* at initial time s.

In fact, we observe that

$$\begin{aligned}\|v(\tau,s;\omega,v_*)-v_*\|^2=&\|v(\tau,s;\omega,v_*)\|^2-\|v_*\|^2-2(v(\tau,s;\omega,v_*)-v_*,v_*)\\=&\int_s^\tau\frac{\mathrm{d}}{\mathrm{d}\eta}\|v(\eta,s;\omega,v_*)\|^2\mathrm{d}\eta-2(v(\tau,s;\omega,v_*)-v_*,v_*).\end{aligned}\tag{3.9}$$

By the definition of z_j and z, we know (see [1, Proposition 4.3.3]) that the random variable $\|z_j(t,\omega)\|$ is tempered and satisfies

$$\sum_{j=1}^m\|z_j(t,\omega)\|^2\leqslant e^{\frac{\alpha|t|}{2}}r(\omega),\tag{3.10}$$

where $r(\cdot):\Omega\mapsto\mathbb{R}_+$ is a tempered function. From (3.7) we see that there exists a constant M_Φ such that

$$\max_{1\leqslant j\leqslant m}\|\Phi_j\|^2_{W^{2,2}(\mathbb{D})}\leqslant M_\Phi.$$

Thus, by the Gagliardo-Nirenberg inequality we have

$$\|z(t,\omega)\|_V^2\lesssim\sum_{j=1}^m\|z_j(t,\omega)\|^2\|\Phi_j\|^2_{W^{2,2}(\mathbb{D})}\leqslant M_\Phi e^{\frac{\alpha|t|}{2}}r(\omega).\tag{3.11}$$

Note that [11, (3.9)] has proved that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t,s;\omega,v_*)\|^2 \lesssim g(t,\omega) - \Big(1-\sum_{j=1}^{m}\|z_j(t,\omega)\|\Big)\|v(t,s;\omega,v_*)\|^2 \tag{3.12}$$

and

$$\begin{aligned}\|v(\eta,s;\omega,v_*)\|^2 \lesssim &\|v_*\|^2\exp(-\int_s^\eta(1-\sum_{j=1}^{m}\|z_j(\theta,\omega)\|)\mathrm{d}\theta)\\ &+\int_s^\eta g(\theta,\omega)\exp(-\int_\theta^\eta(1-\sum_{j=1}^{m}\|z_j(\xi,\omega)\|)\mathrm{d}\xi)\mathrm{d}\theta,\end{aligned} \tag{3.13}$$

where

$$g(t,\omega)=(\sum_{j=1}^{m}\|z_j(t,\omega)\|)\|z(t,\omega)\|^2+\|z(t,\omega)\|^2+\|z(t,\omega)\|_V^2+\|f\|^2.$$

Hence,

$$\int_s^\tau \frac{\mathrm{d}}{\mathrm{d}\eta}\|v(\eta,s;\omega,v_*)\|^2\mathrm{d}\eta \lesssim \int_s^\tau g(\eta,\omega)\mathrm{d}\eta - \int_s^\tau \Big(1-\sum_{j=1}^{m}\|z_j(\eta,\omega)\|\Big)\|v(\eta,s;\omega,v_*)\|^2\mathrm{d}\eta. \tag{3.14}$$

Now the trajectories of $\|z(\eta,\omega)\|^2$ and $\|z(\eta,\omega)\|_V^2$ are continuous for $\mathbb{P}$ a.s. $\omega\in\Omega$, $f\in H$ and

$$v(\cdot,s;\omega,v_*)\in\mathcal{C}([s,+\infty);H)\cap L^2_{\mathrm{loc}}([s,+\infty;V). \tag{3.15}$$

Also, (3.13) implies that

$$\begin{aligned}\sup_{s\in[s_*-1,s_*+1]}\|v(\eta,s;\omega,v_*)\|^2 \lesssim \sup_{s\in[s_*-1,s_*+1]}\Big\{&\|v_*\|^2\exp(-\int_s^\eta(1-\sum_{j=1}^{m}\|z_j(\theta,\omega)\|)\mathrm{d}\theta)\\ &+\int_s^\eta g(\theta,\omega)\exp(-\int_\theta^\eta(1-\sum_{j=1}^{m}\|z_j(\xi,\omega)\|)\mathrm{d}\xi)\mathrm{d}\theta\Big\},\end{aligned} \tag{3.16}$$

the right-hand side of which is independent of s. From these facts, we see from (3.14) that for any $\epsilon>0$ there exists a positive constant $\delta_1=\delta_1(\epsilon,s_*,v_*,\omega)$ such that

$$\int_s^\tau \frac{\mathrm{d}}{\mathrm{d}\eta}\|v(\eta,s;\omega,v_*)\|^2\mathrm{d}\eta \leqslant \epsilon^2/2, \quad \text{whenever } s\in(s_*,s_*+\delta_1),\quad \tau\in(s,s_*+\delta_1). \tag{3.17}$$

We next estimate the term $(v(\tau, s; \omega, v_*) - v_*, v_*)$ in (3.9). From (3.15) and (3.16), we see that there exists a positive $M(\omega, s_*, v_*)$ independent of s such that

$$\max_{s_*-1\leqslant\eta\leqslant s_*+1} \|v(\eta, s; \omega, v_*)\| \leqslant M(\omega, s_*, v_*), \quad \forall s \in [s_* - 1, \eta].$$

Then by the density of V in H, we find that for above ϵ there exists an element $\tilde{v} \in V$ such that $\|\tilde{v} - v_*\| \leqslant \frac{\epsilon^2}{8(M(\omega,s_*,v_*)+\|v_*\|)}$. Thus we have for $s \in (s_*, s_* + \delta_1)$ and $\tau \in (s, s_* + \delta_1)$ that

$$\begin{aligned} |(v(\tau, s; \omega, v_*) - v_*, v_*)| \leqslant& |(v(\tau, s; \omega, v_*) - v_*, \tilde{v})| + |(v(\tau, s; \omega, v_*) - v_*, \tilde{v} - v_*)| \\ \leqslant& |\langle v(\tau, s; \omega, v_*) - v_*, \tilde{v}\rangle| + \frac{\epsilon^2}{8}. \end{aligned} \tag{3.18}$$

We shall estimate the term $|\langle v(\tau, s; \omega, v_*) - v_*, \tilde{v}\rangle|$ in (3.18). Observe that

$$\begin{aligned} |\langle v(\tau, s; \omega, v_*) - v_*, v_*\rangle| =& \left|\langle \int_s^\tau \frac{\mathrm{d}}{\mathrm{d}\eta} v(\eta, s; \omega, v_*)\mathrm{d}\eta, \tilde{v}\rangle\right| \\ \lesssim& \|\tilde{v}\|_V \left(\int_s^\tau \|\frac{\mathrm{d}}{\mathrm{d}\eta} v(\eta, s; \omega, v_*)\|_{V^*}^2 \mathrm{d}\eta\right)^{1/2} (\tau - s)^{1/2}. \end{aligned} \tag{3.19}$$

By (3.5) and the embedding $V \hookrightarrow V^*$, we have

$$\begin{aligned} &\|\frac{\mathrm{d}v(\eta, s; \omega, v_*)}{\mathrm{d}\eta}\|_{V^*}^2 \\ \lesssim& \|Av(\eta, s; \omega, v_*)\|_{V^*}^2 + \|B(v(\eta, s; \omega, v_*) + z(\eta, \omega))\|_{V^*}^2 + \|f\|^2 + \|z(\eta, \omega)\|^2 + \|Az(\eta, \omega)\|_{V^*}^2. \end{aligned} \tag{3.20}$$

Now using the property of the operators A, B and the embedding $V \hookrightarrow H \hookrightarrow V^*$, we obtain

$$\begin{cases} \|Av(\eta, s; \omega, v_*)\|_{V^*}^2 = \|v(\eta, s; \omega, v_*)\|_V^2, \quad \|z(\eta, \omega)\|_{V^*}^2 \lesssim e^{\frac{\alpha}{2}|\eta|} r(\omega), \\ \|Az(\eta, s; \omega, v_*)\|_{V^*}^2 = \|z(\eta, \omega)\|^2 \lesssim e^{\frac{\alpha}{2}|\eta|} r(\omega), \\ \|B(v(\eta, s; \omega, v_*) + z(\eta, \omega))\|_{V^*}^2 \lesssim \|v(\eta, s; \omega, v_*)\|_V^2 + e^{\frac{\alpha}{2}|\eta|} r(\omega). \end{cases} \tag{3.21}$$

Inserting (3.20) and (3.21) into (3.19) gives

$$\begin{aligned} &|\langle v(\tau, s; \omega, v_*) - v_*, v_*\rangle| \\ \lesssim& \|\tilde{v}\|_V \left(\int_s^\tau (\|v(\eta, s, \omega, v_*)\|_V^2 + e^{\frac{\alpha}{2}|\eta|} r(\omega) + \|f\|^2)\mathrm{d}\eta\right)^{1/2} (\tau - s)^{1/2}. \end{aligned} \tag{3.22}$$

It then follows form (3.15), (3.16) and (3.22) that for above $\epsilon > 0$ there exists a positive constant $\delta_2 = \delta_2(\epsilon, s_*, v_*, \omega)$ such that for $\mathbb{P}$ a.s. $\omega \in \Omega$,

$$|\langle v(\tau, s; \omega, v_*) - v_*, \tilde{v}\rangle| \leqslant \frac{\epsilon^2}{8}, \quad \text{whenever } s \in (s_*, s_* + \delta_2), \quad \tau \in (s, s_* + \delta_2). \tag{3.23}$$

Picking $\delta = \min\{\delta_1, \delta_2\}$, we obtain (3.8) from (3.9), (3.17), (3.18) and (3.23). This ends the proof of Lemma 3.1. □

Similarly to Lemma 3.1, we can also prove that for given $u_* \in H$, $t \in \mathbb{R}$, and for $\mathbb{P}$ a.s. $\omega \in \Omega$, the H-valued mapping $s \mapsto \psi(t, s; \omega)u_*$ is left continuous on $(-\infty, t]$. Therefore, we conclude that for given $u_* \in H$, $t \in \mathbb{R}$, and for $\mathbb{P}$ a.s. $\omega \in \Omega$, the H-valued mapping $s \mapsto \psi(t, s; \omega)u_*$ is continuous on $(-\infty, t]$. From this continuity and the attracting property of the global random attractor, we see that H-valued mapping $s \mapsto \psi(t, s; \omega)u_*$ is bounded on $(-\infty, t]$. Now, thanks to the abstract theory of Theorem 2.1, we can claim that the 2D incompressible Navier-Stokes equations with additive white noise possesses a family of invariant sample measures on the phase space H. This result reads as follows.

Theorem 3.1. *Suppose that condition* (H) *is satisfied and $f \in H$. Let $\{\psi(t, \tau; \omega)\}_{t \geqslant \tau, \omega \in \Omega}$ be the random dynamical system generated by problem* (3.1)-(3.3) *over the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ with the state space H. Let $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ be the global random attractor guaranteed by Proposition 3.1(3). Then for a given generalized Banach limit $\mathrm{LIM}_{t \to +\infty}$ and given continuous function $v(\cdot) : \mathbb{R} \mapsto H$, there exists for $\mathbb{P}$ a.s. $\omega \in \Omega$ a family of Borel probability measures $\{\mu_{\theta_t \omega}\}_{t \in \mathbb{R}}$ on H such that the support of $\mu_{\theta_t \omega}$ is contained in $\mathcal{A}(\theta_t \omega)$ and*

$$\begin{aligned} \int_H \varrho(u) \mathrm{d}\mu_{\theta_t \omega}(u) &= \int_{\mathcal{A}(\theta_t \omega)} \varrho(u) \mathrm{d}\mu_{\theta_t \omega}(u) \\ &= \mathrm{LIM}_{\tau \to -\infty} \frac{1}{t - \tau} \int_\tau^t \varrho(\psi(t, s; \omega) v(s)) \mathrm{d}s \\ &= \mathrm{LIM}_{\tau \to -\infty} \frac{1}{t - \tau} \int_\tau^t \int_H \varrho(\psi(t, s; \omega) u) \mathrm{d}\mu_{\theta_s \omega}(u) \mathrm{d}s \end{aligned} \tag{3.24}$$

for any nonnegative, real-valued continuous functional ϱ on H. Moreover, for $\mathbb{P}$ a.s. $\omega \in \Omega$, $\mu_{\theta_t \omega}$ is invariant under the action of the random dynamical system $\{\psi(t, \tau; \omega)\}_{t \geqslant \tau, \omega \in \Omega}$ in the sense that

$$\int_{\mathcal{A}(\theta_t \omega)} \varrho(u) \mathrm{d}\mu_{\theta_t \omega}(u) = \int_{\mathcal{A}(\theta_\tau \omega)} \varrho(\psi(t, \tau; \omega) u) \mathrm{d}\mu_{\theta_\tau \omega}(u), \quad \forall t \geqslant \tau. \tag{3.25}$$

We next investigate the random Liouville type theorem for the 2D incompressible Navier-Stokes equations with additive white noise. To this end, we need the definition of the class of test functions. Write equation (3.2) as

$$\mathrm{d}u = F(u, t, \omega) = -(\nu A u + B(u)) \mathrm{d}t + f \mathrm{d}t + \sum_{j=1}^m \Phi_j \mathrm{d}w_j. \tag{3.26}$$

Definition 3.1. We define the class $\mathcal{T}$ of test functions as the set of nonnegative, real-valued continuous functionals $\Upsilon = \Upsilon(w)$ on H that are bounded on bounded subset of H and satisfy

(1) for any $w \in V$, the Fréchet derivative $\Upsilon'(w)$ exists: for each $w \in V$ there exists an element $\Upsilon'(w)$ such that

$$\frac{|\Upsilon(w+v) - \Upsilon(w) - (\Upsilon'(w), v)|}{\|v\|_V} \longrightarrow 0 \ \ as \ \ \|v\|_V \to 0, \quad v \in V;$$

(2) $\Upsilon'(w) \in V$ for all $w \in V$, and the mapping $w \longmapsto \Upsilon'(w)$ is continuous and bounded as a functional from V to V;
(3) for every global solution $u(t,\omega)$ of equation (3.2), there holds for $\mathbb{P}$ a.a. $\omega \in \Omega$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(u(t,\omega)) = \langle F(u,t,\omega), \Upsilon'(u)\rangle. \tag{3.27}$$

For the existence of functions satisfying Definition 3.1, one can refer to [22, Definition 2.5]. Here we omit the details.

Theorem 3.2. *Let the conditions of Theorem 3.1 hold. Then, for $\mathbb{P}$ a.s. $\omega \in \Omega$, the following random Liouville type equation*

$$\begin{aligned}&\int\limits_{\mathcal{A}(\theta_t\omega)} \Upsilon(u)\mathrm{d}\mu_{\theta_t\omega}(u) - \int\limits_{\mathcal{A}(\theta_\tau\omega)} \Upsilon(u)\mathrm{d}\mu_{\theta_\tau\omega}(u)\\ &= \int\limits_\tau^t \int\limits_H \langle F(u,\eta,\omega), \Upsilon'(u)\rangle \mathrm{d}\mu_{\theta_\eta\omega}(u)\mathrm{d}\eta, \quad \forall t \geqslant \tau,\end{aligned} \tag{3.28}$$

holds for all test functions $\Upsilon \in \mathcal{T}$.

Proof. Let $\Upsilon \in \mathcal{T}$ be given. By (3.27), we have for $\mathbb{P}$ a.s. $\omega \in \Omega$ that

$$\Upsilon(\psi(t,s;\omega)u) - \Upsilon(\psi(\tau,s;\omega)u) = \int\limits_\tau^t \langle F(u,\eta,\omega), \Upsilon'(u)\rangle \mathrm{d}\eta, \quad \forall t \geqslant \tau. \tag{3.29}$$

Now for any $s < \tau$, let $u_* \in H$ and $u(\eta,\omega) = \psi(\eta,s;\omega)u_*$ for $\eta \geqslant s$. By (3.29),

$$\Upsilon(\psi(t,s;\omega)u_*) - \Upsilon(\psi(\tau,s;\omega)u_*) = \int\limits_\tau^t \langle F(\psi(\eta,s;\omega)u_*,\eta,\omega), \Upsilon'(\psi(\eta,s;\omega)u_*)\rangle \mathrm{d}\eta. \tag{3.30}$$

Using (3.24), (3.30) and Fubini's theorem, we arrive at

$$
\begin{aligned}
&\int_H \Upsilon(u)\mathrm{d}\mu_{\theta_t\omega}(u) - \int_H \Upsilon(u)\mathrm{d}\mu_{\theta_\tau\omega}(u)\\
&= \int_{\mathcal{A}(\theta_t\omega)} \Upsilon(u)\mathrm{d}\mu_{\theta_t\omega}(u) - \int_{\mathcal{A}(\theta_\tau\omega)} \Upsilon(u)\mathrm{d}\mu_{\theta_\tau\omega}(u)\\
&=\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{\tau-\gamma}\int_\gamma^\tau\int_H (\Upsilon(\psi(t,s;\omega)u_*)) - \Upsilon(\psi(\tau,s;\omega)u_*))\mathrm{d}\mu_{\theta_s\omega}(u_*)\mathrm{d}s\\
&=\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{\tau-\gamma}\int_\gamma^\tau\int_H\int_\tau^t \langle F(\psi(\eta,s;\omega)u_*,\eta,\omega), \Upsilon'(\psi(\eta,s;\omega)u_*)\rangle \mathrm{d}\eta\mathrm{d}\mu_{\theta_s\omega}(u_*)\mathrm{d}s\\
&=\mathrm{LIM}_{\gamma\to-\infty}\frac{1}{\tau-\gamma}\int_\gamma^\tau\int_\tau^t\int_H \langle F(\psi(\eta,s;\omega)u_*,\eta,\omega), \Upsilon'(\psi(\eta,s;\omega)u_*)\rangle \mathrm{d}\mu_{\theta_s\omega}(u_*)\mathrm{d}\eta\mathrm{d}s.
\end{aligned}
\tag{3.31}
$$

Now using the invariance of the random dynamical system $\psi(\eta,s;\omega)=\psi(\eta,\tau;\omega)\psi(\tau,s;\omega)$ and (3.25), we obtain

$$
\begin{aligned}
&\int_H \langle F(\psi(\eta,s;\omega)u_*,\eta,\omega), \Upsilon'(\psi(\eta,s;\omega)u_*)\rangle \mathrm{d}\mu_{\theta_s\omega}(u_*)\\
&=\int_H \langle F(\psi(\eta,\tau;\omega)\psi(\tau,s;\omega)u_*,\eta,\omega), \Upsilon'(\psi(\eta,\tau;\omega)\psi(\tau,s;\omega)u_*)\rangle \mathrm{d}\mu_{\theta_s\omega}(u_*)\\
&=\int_H \langle F(\psi(\eta,\tau;\omega)u_*,\eta,\omega), \Upsilon'(\psi(\eta,\tau;\omega)u_*)\rangle \mathrm{d}\mu_{\theta_\tau\omega}(u_*),
\end{aligned}
$$

which is independent of s. It then follows from (3.31) that

$$
\begin{aligned}
&\int_{\mathcal{A}(\theta_t\omega)} \Upsilon(u)\mathrm{d}\mu_{\theta_t\omega}(u) - \int_{\mathcal{A}(\theta_\tau\omega)} \Upsilon(u)\mathrm{d}\mu_{\theta_\tau\omega}(u)\\
&=\int_\tau^t\int_H \langle F(\psi(\eta,\tau;\omega)u_*,\eta,\omega), \Upsilon'(\psi(\eta,\tau;\omega)u_*)\rangle \mathrm{d}\mu_{\theta_\tau\omega}(u_*)\mathrm{d}\eta\\
&=\int_\tau^t\int_H \langle F(u,\eta,\omega), \Upsilon'(u)\rangle \mathrm{d}\mu_{\theta_\eta\omega}(u)\mathrm{d}\eta.
\end{aligned}
$$

The proof is complete. □

The result of Theorem 3.2 can be regarded as the random Liouville type theorem. If the random statistical equilibrium has been reached by the addressed stochastic Navier-Stokes system, then the statistical informations do not change with time, that is $\Phi'(u(\cdot,\omega))=0$. In this situation, we follow from (3.25) and (3.28) that for $\mathbb{P}$ a.s. $\omega\in\Omega$

$$\begin{aligned}&\int\limits_{\mathcal{A}(\theta_t\omega)}\Upsilon(u)\mathrm{d}\mu_{\theta_t\omega}(u)\\&=\int\limits_{\mathcal{A}(\theta_\tau\omega)}\Upsilon(\psi(t,\tau;\omega)u)\mathrm{d}\mu_{\theta_\tau\omega}(u)=\int\limits_{\mathcal{A}(\theta_\tau\omega)}\Upsilon(u)\mathrm{d}\mu_{\theta_\tau\omega}(u),\quad \tau\in\mathbb{R}.\end{aligned}\tag{3.32}$$

(3.32) describes exactly the invariant property of the sample measures $\{\mu_{\theta_t\omega}\}_{t\in\mathbb{R}}$ under the action of the random dynamical system $\{\psi(t,\tau;\omega)\}_{t\geqslant\tau,\omega\in\Omega}$. It reveals that for $\mathbb{P}$ a.s. $\omega\in\Omega$, the shape of the global random attractor $\mathcal{A}(\theta_t\omega)$ could change randomly with the evolution of time from τ to t, along with the sample point $\omega\in\Omega$, but the measures of $\mathcal{A}(\theta_\tau\omega)$ and $\mathcal{A}(\theta_t\omega)$ coincide with each other. This is the random version of the Liouville Theorem in Statistical Mechanics. Thus we say that the invariant sample measures $\{\mu_{\theta_t\omega}\}_{t\in\mathbb{R}}$ of the stochastic Navier-Stokes equations satisfies a random Liouville type theorem.

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