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Mild Solutions to Time Fractional Stochastic 2D-Stokes Equations with Bounded and Unbounded Delay

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To Professor Russell A. Johnson, "In Memoriam"

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Abstract
In this paper, the well-posedness of stochastic time fractional 2D-Stokes equations of order $\alpha \in (0, 1)$ containig finite or infinite delay with multiplicative noise is established, respectively, in the spaces $C([-h, 0]; L^2(\Omega; L^2_\sigma))$ and $C((-\infty, 0]; L^2(\Omega; L^2_\sigma))$. The existence and uniqueness of mild solution to such kind of equations are proved by using a fixed-point argument. Also the continuity with respect to initial data is shown. Finally, we conclude with several comments on future research concerning the challenging model: time fractional stochastic delay 2D-Navier–Stokes equations with multiplicative noise. Hence, this paper can be regarded as a first step to study this challenging topic.

Keywords Well-posedness · Stochastic time fractional 2D-Stokes equations · Mild solution · Finite delay · Infinite delay · Multiplicative noise

Mathematics Subject Classification 35R11 · 35Q30 · 65F08 · 60H15 · 65F10

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1 Introduction

The well-posedness of flow problems in a viscous fluid is crucial for many areas of science and engineering, for example, the automotive and aerospace industries, as well as nanotechnology. In the latter case of microfluidic structures, we often encounter flow problems at moderate viscosities which arise, in the study of the modeling of various devices for the separation and manipulation of particles in microfluids systems [18], in the study of tumor tissue as a porous medium described by Darcy's law [11–13], etc. In applications such as these, the Stokes equations provide a first approximation of the more general Navier–Stokes equations in situations where the flow is nearly steady and slow, and has small velocity gradients, so the inertial effects can be ignored.

The classical integer order ($\alpha = 1$) primitive partial differential equations, which describe heat/wave propagation in a homogeneous medium, have been previously studied in [3,14–16, 22,24,25,27], whereas, it is mostly observed the heat/wave propagation exhibits subdiffusive behavior due to the complex inhomogeneity of the medium. In fact, there is a diversity of real-world systems which exhibit this type of phenomenon, from the mathematical point of view, the fractional derivative plays the role of characterizing the power-law behavior [19,20,28]. Hence, in recent decades, scientists have developed many new models that naturally involve fractional differential equations, which demonstrate the anomalous diffusion phenomenon appearing in the real-world successfully, see e.g., [1,6,8,9,17,19,20,23] and references therein.

Moreover, the problem we consider here uses the Caputo time fractional derivative, whose advantage is that the Caputo derivative of constant functions is zero. Thus, time-independent solutions are also solutions of the time-dependent problem [2]. Also, compared with Riemann–Liouville derivative [23], Caputo derivatives remove singularities at the origin and share many similarities with the classical derivatives so that they are suitable for initial value problems. P.M. Carvalho-Neto and G. Planas analyzed in [7] the following Navier–Stokes model with Caputo fractional derivative,

$$\begin{cases} \partial_t^\alpha u - \kappa \Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \mathbb{R}^N,\ t \geq 0, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^N,\ t \geq 0, \\ u(0,x) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where ∂_t^α is the Caputo fractional derivative of order $\alpha \in (0, 1)$ with respect to t (see Definition 1), u is the velocity field of the fluid, $\kappa > 0$ is the kinematic viscosity, p is the associated pressure, f is the external force and u_0 is an appropriate initial value, and $N \geq 2$. The authors in [7] analyzed the well-posedness of the problem, the existence and eventual uniqueness of mild solutions as well as their regularity in time.

However, in order to have a much better description of our model, it is sensible to consider some other features in the formulation of the equations. On the one hand, it is well known and accepted nowadays that, in physical systems of the real world, the different stochastic perturbations that originate from many natural sources are ubiquitous, most often, cannot be ignored. This leads us to consider some randomness in the model which can be described by some kind of white or colored noise or some other type of stochastic terms. On the other hand, it is also obvious that the future evolution of a system does not only depend on its current state, its past history does determine its future behavior too. Also, on those problems in which we intend to apply some control, it is very convenient to consider some delays or memory terms in the formulation [4,5].

Motivated by the previous considerations, our model can be more realistic if we introduce both features in the formulation. Needless to say, there are many choices in the type of noise, such as, Brownian motion/Wiener process, fractional Brownian motion, Lévy or Poisson ones, etc. In order to perform our analysis clearly and show how it works, we prefer to consider the classical and standard Brownian motion, because the problem can be easily handled mathematically, and serves as guide for more complicated expressions. Based on these advantages, there has been a growing interest in stochastic time fractional partial differential equations with delays. For instance, in more recent decades, the stochastic classical/time fractional partial differential equations have been extensively studied theoretically [10,19,20,26,30,31]. However, there appear to be fewer studies in the literatures related to the theoretical analysis of stochastic Stokes/Navier–Stokes equations driven by multiplicative noise with time fractional derivative, and as far as we know, no one dealt with delays. This is why we are strongly interested in the following problem

$$\begin{cases} \partial_t^\alpha u - \kappa \Delta u + u \cdot \nabla u + \nabla p = f(t, u_t) + g(t, u_t)\frac{dW(t)}{dt} & \text{in } \mathbb{R}^N,\ t \geq 0, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^N,\ t \geq 0, \\ u(t, x) = \varphi(t, x) & \text{in } \mathbb{R}^N,\ t \in [-h, 0], \end{cases} \tag{2}$$

where now f and g are external forcing terms containing some hereditary or delay characteristics, and φ is the initial datum in the interval of time $t \in [-h, 0]$, where h is a fixed positive number, and $W(t)$ is a standard scalar Brownian motion/ Wiener process on an underlying complete filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$.

Although our final and challenging goal is to analyze the well-posedness of mild solutions and asymptotic behavior of time fractional stochastic Navier–Stokes model with delay (2), there are some difficulties/troubles which suggest us to start by analyzing first a linearized version before we can tackle the complete problem. It is well known that when we deal with the integer time stochastic Navier–Stokes equations in the phase space $L^2(\Omega; C([0, T]; X))$, with the help of Itô's isometry and Burkholder–Davis–Gundy's inequality, a priori estimates can be handled smoothly. However, for time fractional stochastic Navier–Stokes equations, if the same phase space were adopted, we would face essential troubles: (a) Itô's isometry only holds true for the integer time derivative rather than time fractional derivative; (b) Burkholder–Davis–Gundy's inequality cannot be used since the integral is not a martingale (the main reason is the singular kernel appearing in the stochastic integral).

For this reason, in this first approach we will analyze the following time fractional stochastic delay incompressible flow problem, i.e., the non-stationary 2D-Stokes equations,

$$\begin{cases} \partial_t^\alpha u - \kappa \Delta u + \nabla p = f(t, u_t) + g(t, u_t)\frac{dW(t)}{dt} & \text{in } \mathbb{R}^2,\ t \geq 0, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^2,\ t \geq 0, \\ u(t, x) = \varphi(t, x) & \text{in } \mathbb{R}^2,\ t \in [-h, 0]. \end{cases} \tag{3}$$

In the deterministic case, the concept of weak (or variational) solution to the Navier–Stokes problem without delay was also analyzed [33]. However, the proof of this deterministic problem relies on direct estimates involving the time fractional derivative as well as the Fourier transform, while the stochastic case cannot be analyzed by similar techniques since the term containing noise only makes sense in integral form. For this reason, we carry out a program based on a fixed-point theorem which is different also from the one used in the papers [7,32]. We highlight that it might be possible to perform the technique in [33] to handle those cases containing a much simpler noise term in which the stochastic integral does not appear, for instance, when the noise has a special additive form. It is our objective to analyze these problems in future works.

We have structured our paper as follows. In the next section we briefly recall some relevant preliminaries. Section 3 is firstly devoted to the existence of local mild solutions of (3) in the case of bounded/finite delay. Next the continuous dependence on the initial data as well as the uniqueness of local mild solution is also proved. Futhermore, the existence and uniqueness of global mild solution are obtained in the Banach space $\mathcal{X}_2$. In Sect. 4, we investigate problem (3) with unbounded delay. Although the technique to prove the well-posedness of mild solution to such model is similar to the one used in Sect. 3, there are substantial differences in this unbounded delay case which justify a detailed study. Eventually, some conclusions and comments are included in Sect. 5.

2 Preliminaries

In this section we present basic notations related to stochastic theory, collect useful facts on Mittag-Leffler functions and establish the definition of the mild solution to problem (3). For more details, we refer to [6,7,21,23] and references therein.

2.1 Stochastic Theory and Notations

To begin we fix a stochastic basis, that is,

$$\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W),$$

where $\mathbb{P}$ is a probability measure on Ω and $\mathcal{F}$ is a σ-algebra. In order to avoid unnecessary complications below, we may assume that $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible subsets and $W(t) = W(\omega, t)$, $\omega \in \Omega$ is a standard 1-D Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$.

To set our problem (3) in the abstract framework, we consider the standard notation L^2_σ to describe the subspace of the divergence-free vector fields in L^2:

$$L^2_\sigma = \{u \in L^2 : \nabla \cdot u = 0 \ \text{ in } \ \mathbb{R}^2\}$$

with norm $\|\cdot\|_{L^2}$, where L^2 denotes the vector-valued Lebesgue space and for $u \in L^2$,

$$\|u\|^2_{L^2} = \sum_{j=1}^{2} \int_{\mathbb{R}^2} |u_j(x)|^2 dx.$$

Besides, let $S \subset \mathbb{R}$ and let X be a Banach space. We denote the space of continuous functions from S to X by $C(S; X)$ (equipped with its usual norm). $L^2(S; X)$ denotes the Banach space of L^2 integrable functions $u : S \to X$. $H^1(S; X) = W^{1,2}(S; X)$ is the subspace of $L^2(S; X)$ consisting of functions such that the weak derivative $\frac{\partial u}{\partial t}$ belongs to $L^2(S; X)$. Both spaces $L^2(S; X)$ and $W^{1,2}(S; X)$ are endowed with their standard norms. Moreover, we denote $a \wedge b = \min\{a, b\}$.

Consider a fixed $T > 0$, given $u : [-h, T] \to L^2_\sigma$, for each $t \in [0, T]$, we denote by u_t the function on $[-h, 0]$ via the relation

$$u_t(s) = u(t + s), \qquad s \in [-h, 0],$$

where $h > 0$ denotes the delay, when $h = \infty$, it denotes infinite or unbounded delay. Furthermore, let $L^2(\Omega; X)$ be the Hilbert space of X-valued random variable with norm $\|u(\cdot)\|^2_{L^2} = \mathbb{E}\|u(\cdot)\|^2$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}u = \int_\Omega u(\cdot) d\mathbb{P}$.

2.2 Fractional Setting and Mittag-Leffler Operators

We now recall some facts about the fractional calculus.

For $\alpha > 0$, define the function $g_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

where $\Gamma(\alpha)$ is Euler's Gamma function. Assume that $T > 0$, for a function $u \in L^1([0,T]; X)$, the Riemann–Liouville fractional integral of order α of u is given by

$$J_t^\alpha u(t) := g_\alpha * u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \qquad t \in [0,T].$$

Thus, based on the definition of Riemann–Liouville fractional integral operator, we present the Caputo fractional differential operator. For more details, we refer to the references [7,17, 23].

Definition 1 ([7, Definition 1]) Let $\alpha \in (0,1)$ and $T > 0$. Consider $u \in C([0,T]; X)$ such that the convolution $g_{1-\alpha} * u \in W^{1,1}([0,T]; X)$. The expression

$$D_t^\alpha u(t) := \frac{d}{dt}\left\{ J_t^{1-\alpha}[u(t) - u(0)] \right\} = \frac{d}{dt}\left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} [u(s) - u(0)] ds \right\}$$

is called the Caputo fractional derivative of order α of the function u.

In what follows, we present some properties of Mainardi function [6], denoted by M_α. This function is a particular case of the Wright type function introduced by Mainardi in [21]. More precisely, for $\alpha \in (0,1)$, the entire function $M_\alpha : \mathbb{C} \to \mathbb{C}$ is given by

$$M_\alpha(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n! \Gamma(1 - \alpha(1+n))}.$$

Some basic properties of the Mainardi function will be used further in this paper to obtain most of the estimates.

Proposition 1 ([7, Proposition 2]) *For $\alpha \in (0,1)$ and $-1 < r < \infty$, when we restrict M_α to the positive real line, it holds that*

$$M_\alpha(t) \ge 0 \ \textit{for all} \ t \ge 0, \ \ \textit{and} \ \ \int_0^\infty t^r M_\alpha(t) dt = \frac{\Gamma(r+1)}{\Gamma(\alpha r + 1)}.$$

The next results are classical computations done in the literatures related to the Mittag-Leffler operators, for instance [7]. To do this, let X be a Banach space and $-\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be the infinitesimal generator of an analytic semigroup $\{T(t) : t \ge 0\}$. Then, for each $\alpha \in (0,1)$, we define the Mittag-Leffler families $\{\mathbf{E}_\alpha(-t^\alpha \mathcal{A}) : t \ge 0\}$ and $\{\mathbf{E}_{\alpha,\alpha}(-t^\alpha \mathcal{A}) : t \ge 0\}$ by

$$\mathbf{E}_\alpha(-t^\alpha \mathcal{A}) = \int_0^\infty M_\alpha(s) T(st^\alpha) ds,$$

and

$$\mathbf{E}_{\alpha,\alpha}(-t^\alpha \mathcal{A}) = \int_0^\infty \alpha s M_\alpha(s) T(st^\alpha) ds.$$

It is interesting to notice that the Mainardi functions act as a bridge between the fractional and the classical abstract theories. This relation is based on the inversion of certain Laplace transform in order to obtain the fundamental solutions of the fractional diffusion-wave equation. Let us mention e.g., [6,7,19] and references therein.

The following lemma compiles the main assertions of the abstract theory of fractional calculus.

Lemma 1 ([7, Theorem 3]) *The operators* $\mathbf{E}_\alpha(-t^\alpha \mathcal{A})$ *and* $\mathbf{E}_{\alpha,\alpha}(-t^\alpha \mathcal{A})$ *are well defined from* X *to* X. *Moreover, for* $x \in X$ *it holds,*

(i) $\mathbf{E}_\alpha(-t^\alpha \mathcal{A})x|_{t=0} = x$;
(ii) *the vectorial functions* $t \to \mathbf{E}_\alpha(-t^\alpha \mathcal{A})x$ *and* $t \to \mathbf{E}_{\alpha,\alpha}(-t^\alpha \mathcal{A})x$ *are analytic from* $[0, \infty)$ *to* X.

Let us rewrite the time fractional stochastic 2D-Stokes delay differential equations (3) in an abstract form

$$\begin{cases} D_t^\alpha u = -Au + F(t, u_t) + G(t, u_t)\frac{dW(t)}{dt}, & t > 0, \\ u(t) = \varphi(t), & t \in [-h, 0], \end{cases} \tag{4}$$

where $A = -P\Delta = -\Delta P$, $F(t, u_t) = Pf(t, u_t)$ and $G(t, u_t) = Pg(t, u_t)$. Here, $P : L^2 \to L^2_\sigma$ is the Helmholtz–Leray projector and $A : D(A) \subset L^2_\sigma \to L^2_\sigma$ is the Stokes operator.

We end this section by recapitulating the properties of both families of Mittag-Leffler operators, which furnish the essential tools used throughout the whole article, see [7,15] for more details. Notice that the following lemmas hold true when the dimension is $N \geq 2$.

Lemma 2 ([7, Lemma 6]) *Consider* $\alpha \in (0, 1)$, *and* r_1, r_2 *real numbers satisfying*

$$1 < r_1 \leq r_2 < \infty \quad \textit{and} \quad r_2 N/(2r_2 + N) < r_1.$$

Then, for any $v \in L^{r_1}_\sigma$, *there exists a constant* $C = C(r_1, r_2, N, \alpha) > 0$ *such that*

$$\text{(i)} \qquad \|\mathbf{E}_\alpha(-t^\alpha A_{r_1})v\|_{L^{r_2}} \leq Ct^{-\alpha(N/r_1 - N/r_2)/2}\|v\|_{L^{r_1}}, \quad t > 0$$

and

$$\text{(ii)} \qquad \|\mathbf{E}_{\alpha,\alpha}(-t^\alpha A_{r_1})v\|_{L^{r_2}} \leq Ct^{-\alpha(N/r_1 - N/r_2)/2}\|v\|_{L^{r_1}}, \quad t > 0.$$

Remark 1 For simplicity we will consider the case $N = 2$ in our analysis, but the results hold true for $N \geq 2$ (see Remark 6 at the end of Sect. 4).

2.3 Definition of Mild Solution

Inspired by the arguments in [28] and references therein, we now make precise the notation of mild solution to problem (4), which is given by a fractional variation of constants formula involving the Mittag-Leffler families.

Definition 2 (*Mild solution*) Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{t\geq 0})$ be a fixed stochastic basis generated by a standard Brownian motion W, and $T > 0$. Consider $\alpha \in (0, 1)$ and an initial function φ, such that $\varphi(t, \cdot)$ is a $\mathcal{F}_0$-measurable random variable for all $t \leq 0$ (relative to $\mathcal{S}$). A mild solution to problem (4) on $[-h, T]$ is a stochastic process u such that $u(t) = \varphi(t)$, for $t \in [-h, 0]$, fulfilling

$$u(t) = \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) + \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds$$
$$+ \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s),\ \mathbb{P}\ a.s.,\ \text{for every } t\in[0,T]. \tag{5}$$

Remark 2 Notice that, the Stokes operator $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$. Hence, both Mittag-Leffler families $\mathbf{E}_\alpha(-t^\alpha A)$ and $\mathbf{E}_{\alpha,\alpha}(-t^\alpha A)$ are well defined.

Remark 3 It is worth mentioning that the analysis in this paper can be easily extended to the case in which system (3) is driven by Hilbert valued Brownian motions/Wiener processes in infinite dimensions, however we prefer to consider this simpler formulation for the sake of clarity to the reader.

3 Well-Posedness of Mild Solution to Problem (4) with Bounded Delay

In this section, the crucial well-posedness of fractional stochastic 2D-Stokes equation with bounded delay

$$\begin{cases} D_t^\alpha u = -Au + F(t,u_t) + G(t,u_t)\frac{dW(t)}{dt}, & t>0,\\ u(t) = \varphi(t), & t\in[-h,0], \end{cases} \tag{6}$$

will be justified, where h is a positive fixed constant (finite delay).

In order to apply the previous lemmas successfully, it is necessary to introduce suitable Banach spaces, which aim to capture the essence of the problem.

For any $\alpha \in (0,1)$ and fixed $T > 0$, consider the Banach space $\mathcal{X}_2$ which is the set of continuous function $u : [-h,T]\times\Omega \to L^2(\Omega; L^2_\sigma)$ equipped with its natural norm

$$\|u\|_{\mathcal{X}_2} = \left(\sup_{t\in[-h,T]} \mathbb{E}\|u(t)\|^2_{L^2}\right)^{\frac{1}{2}},$$

here we omit T in $\mathcal{X}_2$ but no confusion is possible.

Let us now state the hypotheses imposed on the external forcing terms in our problem.

(H_1) There exists a constant $L_f > 0$ such that the function $F : [0,\infty)\times C([-h,0]; L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$ satisfies

$$\int_0^t \mathbb{E}\|F(s,u_s)-F(s,v_s)\|^2_{L^2}ds \leq L_f \int_{-h}^t \mathbb{E}\|u(s)-v(s)\|^2_{L^2}ds,$$

for all $u, v \in C([-h,T]; L^2(\Omega; L^2_\sigma))$.

(H_2) There exists a constant $L_g > 0$ such that the function $G : [0,\infty)\times C([-h,0]; L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$ satisfies

$$\int_0^t \mathbb{E}\|G(s,u_s)-G(s,v_s)\|^2_{L^2}ds \leq L_g \int_{-h}^t \mathbb{E}\|u(s)-v(s)\|^2_{L^2}ds,$$

for all $u, v \in C([-h,T]; L^2(\Omega; L^2_\sigma))$.

Initially, we establish the local existence and uniqueness of mild solution to problem (6) by a fixed-point argument.

Theorem 1 *Let $\alpha \in (0,1)$. Assume that conditions (H_1)–(H_2) hold, and initial function $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$, such that $\varphi(t,\cdot)$ is a $\mathcal{F}_0$-measurable random variable for all $-h \le t \le 0$. Then there exists $T > 0$ (small enough) such that problem (6) admits a unique mild solution u in the sense of Definition 2 on $[-h,T]$.*

Proof To start off, we pick up an initial function $\varphi(t) \in C([-h,0]; L^2(\Omega; L^2_\sigma))$ such that $\|\varphi\|_{C([-h,0];L^2(\Omega;L^2_\sigma))}$ is small enough compared with R, precisely, we choose R such that

$$(3(C+1) + 2ChL_g)\|\varphi\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))} \le \frac{R^2}{2}.$$

Define the following space $\mathcal{B}^\varphi_R$ with $\alpha \in (0,1)$ and $R > 0$, for every $t \in [0,T]$:

$$\mathcal{B}^\varphi_R = \left\{ u \in C([-h,T]; L^2(\Omega; L^2_\sigma)) : u(t) = \varphi(t) \;\; \forall t \in [-h,0], \quad \|u\|_{\mathcal{X}_2} \le R \right\}.$$

As a preparation for our main result, with the choice of an initial value $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$, let us define the operator $\mathcal{L}$ on $\mathcal{B}^\varphi_R$ as follows,

$$(\mathcal{L}u)(t) = \begin{cases} \varphi(t), \qquad t \in [-h,0], \\ \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) + \displaystyle\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \\ \quad + \displaystyle\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s), \;\; t \in [0,T], \;\; \mathbb{P}\text{-}a.s. \end{cases} \tag{7}$$

Assertion 1: $\mathcal{L}u \in C([-h,T]; L^2(\Omega; L^2_\sigma))$, for each $u \in C([-h,0]; L^2(\Omega; L^2_\sigma))$.

Notice that $(\mathcal{L}u)(t) = \varphi(t)$ if $t \in [-h,0]$, and $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$. Therefore, we only need to check the continuity of $\mathcal{L}u$ on $[0,T]$. For any $t_1, t_2 \in [0,T]$, $\delta > 0$ small enough with $0 < |t_2 - t_1| < \delta$, by slightly modifying the proof of [7, Lemma 11], with the help of the analytical property of the Mittag-Leffler operators in time (see Lemma 1(ii)), the result holds true immediately.

Assertion 2: $\|\mathcal{L}u\|_{\mathcal{X}_2} \le R$, for sufficiently small T.

To this end, we have to prove that, for any $u \in \mathcal{B}^\varphi_R$,

$$\|\mathcal{L}u\|_{\mathcal{X}_2} = \left(\sup_{t\in[-h,T]} \mathbb{E}\|(\mathcal{L}u)(t)\|^2_{L^2} \right)^{\frac{1}{2}} \le R. \tag{8}$$

For $t \in [-h,0]$, we have

$$\mathbb{E}\|(\mathcal{L}u)(t)\|^2_{L^2} = \mathbb{E}\|\varphi(t)\|^2_{L^2} \le \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2_{L^2}. \tag{9}$$

If $t \in (0,T]$, it follows

$$\begin{aligned} \mathbb{E}\|(\mathcal{L}u)(t)\|^2_{L^2} &\le 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2_{L^2} \\ &\quad + 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \right\|^2_{L^2} \\ &\quad + 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s) \right\|^2_{L^2} \\ &:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{10}$$

We now estimate each term on the right hand side of (10).

For $\mathcal{I}_1$, by Lemma 2(i), it is obvious that

$$\mathcal{I}_1 = 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|_{L^2}^2 \le 3C\mathbb{E}\|\varphi(0)\|_{L^2}^2 \le 3C \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2. \tag{11}$$

For $\mathcal{I}_2$, by Lemma 2(i), (H_1), the Cauchy–Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}
\mathcal{I}_2 &= 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \right\|_{L^2}^2 \\
&\le 3C\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)\|_{L^2}ds\right)^2 \\
&\le 6Ct\left(\int_0^t \mathbb{E}\|F(s,u_s)-F(s,0)\|_{L^2}^2 ds + \int_0^t \mathbb{E}\|F(s,0)\|_{L^2}^2 ds\right) \\
&\le 6CL_f t\int_{-h}^t \mathbb{E}\|u(s)\|_{L^2}^2 ds + 6Ct\int_0^t \mathbb{E}\|F(s,0)\|_{L^2}^2 ds \\
&\le 6ChL_f t \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 + 6CL_f t\int_0^t \mathbb{E}\|u(s)\|_{L^2}^2 ds \\
&\quad + 6Ct^2 \sup_{s\in[0,t]} \mathbb{E}\|F(s,0)\|_{L^2}^2 \\
&\le 6ChL_f t \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 \\
&\quad + 6Ct^2\left(L_f R^2 + \sup_{s\in[0,t]} \mathbb{E}\|F(s,0)\|_{L^2}^2\right).
\end{aligned} \tag{12}$$

For $\mathcal{I}_3$, by Lemma 2(i), Itô's isometry and (H_2),

$$\begin{aligned}
\mathcal{I}_3 &= 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s) \right\|_{L^2}^2 \\
&\le 3C\int_0^t \mathbb{E}\|G(s,u_s)\|_{L^2}^2 ds \\
&\le 6C\int_0^t \mathbb{E}\|G(s,u_s)-G(s,0)\|_{L^2}^2 ds + 6C\int_0^t \mathbb{E}\|G(s,0)\|_{L^2}^2 ds \\
&\le 6CL_g\left(\int_{-h}^0 \mathbb{E}\|\varphi(s)\|_{L^2}^2 ds + \int_0^t \mathbb{E}\|u(s)\|_{L^2}^2 ds\right) \\
&\quad + 6C\int_0^t \mathbb{E}\|G(s,0)\|_{L^2}^2 ds \\
&\le 6ChL_g \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 + 6Ct\left(L_g R^2 + \sup_{s\in[0,t]} \mathbb{E}\|G(s,0)\|_{L^2}^2\right).
\end{aligned} \tag{13}$$

Substituting (11)–(13) into (10), combining with (9), it is obvious that

$$\begin{aligned}
\mathbb{E}\|(\mathcal{L}u)(t)\|_{L^2}^2 \le 3\left((C+1) + 2ChL_f t + 2ChL_g\right) \sup_{s\in[-h,0]} \mathbb{E}\|\varphi(s)\|_{L^2}^2 \\
+ 6Ct^2\left(L_f R^2 + \sup_{s\in[0,t]} \mathbb{E}\|F(s,0)\|_{L^2}^2\right)
\end{aligned}$$

$$+\,6Ct\left(L_gR^2+\sup_{s\in[0,t]}\mathbb{E}\|G(s,0)\|_{L^2}^2\right).$$

Consequently, thanks to the choice of R, we can choose T small enough such that

$$\begin{aligned}\|\mathcal{L}u\|_{\mathcal{X}_2}&=\left(\sup_{t\in[-h,T]}\mathbb{E}\|(\mathcal{L}u)(t)\|_{L^2}^2\right)^{\frac{1}{2}}\\&\le\Bigg(3((C+1)+2ChL_fT+2ChL_g)\sup_{t\in[-h,0]}\mathbb{E}\|\varphi(t)\|_{L^2}^2\\&\quad+6CT^2\left(L_fR^2+\sup_{t\in[0,T]}\mathbb{E}\|F(t,0)\|_{L^2}^2\right)\\&\quad+6CT\left(L_gR^2+\sup_{t\in[0,T]}\mathbb{E}\|G(t,0)\|_{L^2}^2\right)\Bigg)^{\frac{1}{2}}\le R.\end{aligned}\tag{14}$$

Assertion 3: Operator $\mathcal{L}:\mathcal{B}_R^{\varphi}\to\mathcal{B}_R^{\varphi}$ is a contraction.
To this end, for any $u,v\in\mathcal{B}_R^{\varphi}$, it follows that

$$\|\mathcal{L}u-\mathcal{L}v\|_{\mathcal{X}_2}:=\left(\sup_{t\in[-h,T]}\mathbb{E}\|(\mathcal{L}u)(t)-(\mathcal{L}v)(t)\|_{L^2}^2\right)^{\frac{1}{2}}.\tag{15}$$

For $t\in[-h,0]$, one has $(\mathcal{L}u)(t)=(\mathcal{L}v)(t)=\varphi(t)$. Thus, it is sufficient to consider the case $t\in[0,T]$. Observe that

$$\begin{aligned}&\mathbb{E}\,\|(\mathcal{L}u)(t)-(\mathcal{L}v)(t)\|_{L^2}^2\\&\le2\mathbb{E}\left\|\int_0^t\mathbf{E}_{\alpha,\alpha}(-(t-s)^{\alpha}A)(F(s,u_s)-F(s,v_s))ds\right\|_{L^2}^2\\&\quad+2\mathbb{E}\left\|\int_0^t\mathbf{E}_{\alpha,\alpha}(-(t-s)^{\alpha}A)(G(s,u_s)-G(s,v_s))dW(s)\right\|_{L^2}^2\\&:=\mathcal{J}_1+\mathcal{J}_2.\end{aligned}\tag{16}$$

For $\mathcal{J}_1$, by Lemma 2(i), (H_2), the Cauchy–Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{J}_1&=2\mathbb{E}\left\|\int_0^t\mathbf{E}_{\alpha,\alpha}(-(t-s)^{\alpha}A)(F(s,u_s)-F(s,v_s))ds\right\|_{L^2}^2\\&\le2C\mathbb{E}\left(\int_0^t\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^{\alpha}A)(F(s,u_s)-F(s,v_s))\|_{L^2}ds\right)^2\\&\le2CL_ft\int_{-h}^t\mathbb{E}\|u(s)-v(s)\|_{L^2}^2ds\\&=2CL_ft\int_0^t\mathbb{E}\|u(s)-v(s)\|_{L^2}^2ds\\&\le2CL_ft^2\sup_{s\in[0,t]}\mathbb{E}\|u(s)-v(s)\|_{L^2}^2.\end{aligned}\tag{17}$$

For $\mathcal{J}_2$, by Lemma 2(i), (H_2) and Itô's isometry, one has

$$\begin{aligned}\mathcal{J}_2 &= 2\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\right\|_{L^2}^2 \\ &\le 2CL_g \int_{-h}^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2 ds \\ &= 2CL_g \int_0^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2 ds \\ &\le 2CL_g t \sup_{s\in[0,t]} \mathbb{E}\|u(s)-v(s)\|_{L^2}^2.\end{aligned} \tag{18}$$

Hence, substituting (17)–(18) into (16), it follows that

$$\begin{aligned}\|\mathcal{L}u-\mathcal{L}v\|_{\mathcal{X}_2} &\le \left(2CT(L_fT+L_g)\sup_{t\in[0,T]}\mathbb{E}\|u(t)-v(t)\|_{L^2}^2\right)^{\frac{1}{2}} \\ &:= \mathcal{M}\|u(t)-v(t)\|_{\mathcal{X}_2},\end{aligned} \tag{19}$$

where

$$\mathcal{M}^2 = 2CT(L_fT+L_g).$$

Therefore, we can choose T small enough such that $0<\mathcal{M}<1$, in other words, we can choose T small enough such that operator $\mathcal{L}$ maps $\mathcal{B}_R^\varphi$ into itself, and it is a contraction as well. The Banach fixed-point theory yields that operator $\mathcal{L}$ possesses a fixed-point in $\mathcal{B}_R^\varphi$. Namely, problem (6) has a unique local mild solution on $[-h,T]$, the proof of this theorem is completed. □

Proposition 2 *Under the assumptions of Theorem 1, the mild solution to (6) is continuous with respect to the initial data $\varphi\in C([-h,0];L^2(\Omega;L_\sigma^2))$. In particular, if $u(t)$, $w(t)$ are the corresponding mild solutions on the interval $[-h,T]$, to the initial data ϕ and ψ, then the following estimate holds*

$$\|u-w\|_{\mathcal{X}_2} \le 3\|\phi-\psi\|_{C([-h,0];L^2(\Omega;L_\sigma^2))}\exp(3C(L_ft+L_g)t), \qquad \forall t\in[0,T].$$

Proof The result of this proposition is proved by similar arguments to those concerning the uniqueness of next theorem, so we omit the details here. □

In the following lines, a theorem will be considered to prove the global existence and uniqueness of mild solution to problem (6).

Theorem 2 *Assume the hypotheses of Theorem 1 hold. Then for every initial value $\varphi\in C([-h,0];L^2(\Omega;L_\sigma^2))$, the initial value problem (6) has a unique mild solution defined globally in the sense of Definition 2.*

Proof Initially, we assume that there exist two solutions, u and v on $[0,T_1]$ and $[0,T_2]$, respectively to problem (6). Next let us prove that $u=v$ on $[-h,T_1\wedge T_2]$. It is clear that $u(t)=v(t)=\varphi(t)$ on $[-h,0]$, so we only need to prove that $u(t)=v(t)$ for any $t\in[0,T_1\wedge T_2]$. Notice that

$$\|u-v\|_{\mathcal{X}_2}^2 := \sup_{t\in[-h,T_1\wedge T_2]}\mathbb{E}\|u(t)-v(t)\|_{L^2}^2. \tag{20}$$

Then one has

$$
\begin{aligned}
&\mathbb{E}\|u(t)-v(t)\|_{L^2}^2\\
&\le 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\right\|_{L^2}^2\\
&\qquad+2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\right\|_{L^2}^2\\
&:=I_1+I_2.
\end{aligned}
\tag{21}
$$

For I_1, by Lemma 2(i), (H_1) and the Cauchy–Schwarz inequality, it follows that

$$
\begin{aligned}
I_1&\le 2C\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|_{L^2}ds\right)^2\\
&\le 2C\mathbb{E}\left(\int_0^t \|F(s,u_s)-F(s,v_s)\|_{L^2}ds\right)^2\\
&\le 2CL_f t\int_0^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2ds\\
&\le 2CL_f t\int_0^t \sup_{\sigma\in[0,s]}\mathbb{E}\|u(\sigma)-v(\sigma)\|_{L^2}^2ds.
\end{aligned}
\tag{22}
$$

For I_2, by Lemma 2(i), (H_2) and Itô's isometry, we derive

$$
\begin{aligned}
I_2&\le 2\int_0^t \mathbb{E}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))\|_{L^2}^2ds\\
&\le 2CL_g\int_0^t \mathbb{E}\|u(s)-v(s)\|_{L^2}^2ds\\
&\le 2CL_g\int_0^t \sup_{\sigma\in[0,s]}\mathbb{E}\|u(\sigma)-v(\sigma)\|_{L^2}^2ds.
\end{aligned}
\tag{23}
$$

Substituting (22)–(23) into (21),

$$
\mathbb{E}\|u(t)-v(t)\|_{L^2}^2\le 2C\left(L_f t+L_g\right)\int_0^t \sup_{\sigma\in[0,s]}\mathbb{E}\|u(\sigma)-v(\sigma)\|_{L^2}^2ds.
$$

Denote by $\mathcal{M}_1=2C(L_f(T_1\wedge T_2)+L_g)$, then

$$
\sup_{t\in[-h,T_1\wedge T_2]}\mathbb{E}\|u(t)-v(t)\|_{L^2}^2\le \mathcal{M}_1\int_0^{T_1\wedge T_2}\left(\sup_{\sigma\in[-h,t]}\|u(\sigma)-v(\sigma)\|_{L^2}^2\right)dt,
$$

whence the Gronwall Lemma implies that

$$
\|u-v\|_{\mathcal{X}_2}=0.
$$

Therefore, $u=v$ on $[-h,T_1\wedge T_2]$ for every initial function $\varphi(t)$.

Now we prove that for each given $T>0$, the mild solution u to problem (6) is bounded with $\mathcal{X}_2$ norm. Taking into account Lemma 2(i), (H_1)–(H_2), Itô's isometry, the Cauchy–Schwarz inequality and Fubini's theorem, we have

$$
\begin{aligned}
\mathbb{E}\|u(t)\|_{L^2}^2 \leq\ & 3\mathbb{E}\left\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\right\|_{L^2}^2 \\
&+ 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\right\|_{L^2}^2 \\
&+ 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\right\|_{L^2}^2 \\
\leq\ & 3C \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 \\
&+ 6Ct\mathbb{E}\left(L_f \int_{-h}^t \|u(s)\|_{L^2}^2 ds + \int_0^t \|F(s,0)\|_{L^2}^2 ds\right) \\
&+ 6C\mathbb{E}\left(L_g \int_{-h}^t \|u(s)\|_{L^2}^2 ds + \int_0^t \|G(s,0)\|_{L^2}^2 ds\right) \\
\leq\ & 6C(1+L_f th + L_g h) \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 \\
&+ 6Ct^2 \sup_{s\in[0,t]} \mathbb{E}\|F(s,0)\|_{L^2}^2 + 6Ct \sup_{s\in[0,t]} \mathbb{E}\|G(s,0)\|_{L^2}^2 \\
&+ 6C(L_f t + L_g)\int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)\|_{L^2}^2 ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sup_{t\in[-h,T]} \mathbb{E}\|u(t)\|_{L^2}^2 \leq\ & 6((C+1) + CL_f Th + CL_g h) \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 \\
&+ 6CT^2 \sup_{t\in[0,T]} \mathbb{E}\|F(t,0)\|_{L^2}^2 \\
&+ 6CT \sup_{t\in[0,T]} \mathbb{E}\|G(t,0)\|_{L^2}^2 \\
&+ 6C(L_f T + L_g)\int_0^T \sup_{\sigma\in[0,t]} \mathbb{E}\|u(\sigma)\|_{L^2}^2 dt \\
:=\ & A(\varphi, T, F, G) + \mathcal{M}_2 \int_0^T \sup_{\sigma\in[-h,t]} \mathbb{E}\|u(\sigma)\|_{L^2}^2 dt,
\end{aligned}
$$

where we have used the notation

$$
\begin{aligned}
A(\varphi, T, F, G) :=\ & 6((C+1) + CL_f Th + CL_g h) \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2 \\
&+ 6CT^2 \sup_{t\in[0,T]} \mathbb{E}\|F(t,0)\|_{L^2}^2 + 6CT \sup_{t\in[0,T]} \mathbb{E}\|G(t,0)\|_{L^2}^2,
\end{aligned}
$$

and

$$
\mathcal{M}_2 := 6C(L_f T + L_g).
$$

Applying the Gronwall lemma, for any fixed $T > 0$ and all $t \in [0, T]$, we obtain

$$
\|u\|_{\mathcal{X}_2}^2 \leq A(\varphi, T, F, G)\exp(\mathcal{M}_2 T).
$$

Because of the arbitrariness of T, together with the conclusion of uniqueness of u on $[-h, T]$, it is straightforward that the mild solution u to problem (6) is defined globally. The proof of this theorem is complete. □

4 Well-Posedness of Mild Solution to Problem (4) with Unbounded Delay

In this section, let us consider the well-posedness of mild solution to the following stochastic time fractional 2D-Stokes equation with unbounded delay:

$$\begin{cases} D_t^\alpha u = -Au + F(t, u_t) + G(t, u_t)\frac{dW(t)}{dt}, & t > 0, \\ u(t) = \varphi(t), & t \in (-\infty, 0]. \end{cases} \tag{24}$$

Before going a step further to prove the main results, we first introduce a suitable space motivated by our unbounded delay. Let $\mathbb{H}$ be a separable Hilbert space, then the space $\mathcal{C}_X$ on $\mathbb{H}$ is defined as

$$\mathcal{C}_X(\mathbb{H}) = \{\varphi \in C((-\infty, 0]; \mathbb{H}) : \lim_{\theta \to -\infty} \varphi(\theta) \text{ exists in } \mathbb{H}\},$$

which is a Banach space equipped with the norm

$$\|\varphi\|_{\mathcal{C}_X} = \sup_{\theta \in (-\infty, 0]} \|\varphi(\theta)\|_{\mathbb{H}}.$$

Let us denote $\mathbb{R}_+ = [0, \infty)$ and enumerate now the assumptions on the delay terms F and G. Assume that $F, G : [0, \infty) \times \mathcal{C}_X(L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$, then

- (H_3) For any $\xi \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$, the mappings $[0, \infty) \ni t \to F(t, \xi) \in L^2(\Omega; L^2_\sigma)$ and $[0, \infty) \ni t \to G(t, \xi) \in L^2(\Omega; L^2_\sigma)$ are measurable.
- (H_4) $F(\cdot, 0) = 0,\ G(\cdot, 0) = 0$ (for simplicity).
- (H_5) There exist two constants L'_f and L'_g, such that for all $t \in [0, \infty)$, and for all ξ, $\eta \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$,

$$\|F(t, \xi) - F(t, \eta)\|_{L^2(\Omega; L^2_\sigma)} \le L'_f \|\xi - \eta\|_{\mathcal{C}_X(L^2(\Omega; L^2_\sigma))},$$
$$\|G(t, \xi) - G(t, \eta)\|_{L^2(\Omega; L^2_\sigma)} \le L'_g \|\xi - \eta\|_{\mathcal{C}_X(L^2(\Omega; L^2_\sigma))}.$$

At this point, some remarks are in order.

Remark 4 (i) Notice that in this unbounded delay case, assumptions (H_4) and (H_5) imposed on the delay terms are simply Lipschitz continuity while in the bounded delay case we need to impose (H_1) and (H_2) which are some kind of integral Lipschitz condition. The main reason is that in the current situation, we can use the estimate $\sup_{\theta \le 0} \|u_t(\theta)\|_{\mathbb{H}} \le \sup_{\theta \le 0} \|u_s(\theta)\|_{\mathbb{H}}$, if $s > t$, while in the bounded delay case this is not true. This will make our computations different in both cases. Also, this is why we will include the complete details in this section.

(ii) It is quite usual when dealing with unbounded delay differential equations, to adopt a different space for the initial data [29], namely,

$$C^\gamma(\mathbb{H}) = \{\varphi \in C((-\infty, 0]; \mathbb{H}) : \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \|\varphi(\theta)\|_{\mathbb{H}} < +\infty\}.$$

However, if we consider this space, then hypotheses (H_4) and (H_5) are not fulfilled when the delay in F or G is a variable delay one. For instance, $F(t, u_t) = F_0(u(t - \rho(t)))$, where ρ is a measurable function taking nonnegative values and $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is a Lipschitz function. Therefore, this new space, although it is a bit more restrictive than the usual one, allows us to consider more general delay terms in the functional formulation.

We can now prove our main results on well-posedness of mild solution to problem (24).

Theorem 3 *Let $\alpha \in (0,1)$, F and G satisfy assumptions (H_3)–(H_5). Then, for each initial function $\varphi \in C((-\infty,0];(L^2(\Omega;L^2_\sigma)))$, such that $\varphi(t,\cdot)$ is a $\mathcal{F}_0$-measurable random variable for all $t \le 0$, problem (24) admits a unique mild solution u in the sense of Definition 2 on $(-\infty,T]$, for $T>0$ small enough.*

Proof To start off, let us pick an initial function $\varphi(t) \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$ such that $\|\varphi\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}$ is small enough compared with R, namely, we choose R such that

$$3(C+1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \le \frac{R^2}{3}.$$

Define the following space $\mathcal{V}^\varphi_R$ with $\alpha \in (0,1)$, $R>0$

$$\mathcal{V}^\varphi_R = \Big\{u \in C((-\infty,T];L^2(\Omega;L^2_\sigma)):\ u(t)=\varphi(t) \text{ for } t\in(-\infty,0], \\ \text{and } u_t \in \mathcal{C}_X(L^2(\Omega;L^2_\sigma)) \text{ for } t\ge 0,\ \text{satisfying } \|u_t\|_{\mathcal{C}_X} \le R.\Big\}$$

As a preparation for handling the main result, with the choice of an initial value $\varphi(t) \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$, let us define the operator $\mathcal{K}$ on $\mathcal{V}^\varphi_R$ as follows,

$$(\mathcal{K}u)(t) = \begin{cases} \varphi(t), & t\in(-\infty,0], \\ \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) + \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds & \\ \quad + \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s), & t\in[0,T],\ \mathbb{P}\text{-}a.s. \end{cases} \tag{25}$$

Assertion 1: $\mathcal{K}u \in C((-\infty,T];L^2(\Omega;L^2_\sigma))$, for all $u \in C((-\infty,T];L^2(\Omega;L^2_\sigma))$.

Observe that, if $t\in(-\infty,0]$, then $(\mathcal{K}u)(t)=\varphi(t)$. Therefore, we only need to check the continuity of $\mathcal{K}u$ on $[0,T]$. For any $t_1,t_2\in[0,T]$, $\delta>0$ small enough with $0<|t_2-t_1|<\delta$, by slightly modifying the proof in [7, Lemma 11], with the help of the analyticity of Mittag-Leffler operators in time [see Lemma 1 (ii)], the result holds true immediately.

Assertion 2: $\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \le R$, for all $t\in[0,T]$ with sufficiently small T.

For every $u\in\mathcal{V}^\varphi_R$, we have to show that

$$\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} := \left(\sup_{\theta\in(-\infty,0]} \mathbb{E}\|(\mathcal{K}u)(t+\theta)\|^2_{L^2}\right)^{\frac12} \le R.$$

For $t\in(-\infty,0]$, we have

$$\mathbb{E}\|(\mathcal{K}u)(t)\|^2_{L^2} = \mathbb{E}\|\varphi(t)\|^2_{L^2} \le \sup_{t\in(-\infty,0]} \mathbb{E}\|\varphi(t)\|^2_{L^2}. \tag{26}$$

If $t+\theta\in(0,T]$, then it follows that

$$\begin{aligned} \mathbb{E}\|(\mathcal{K}u)(t)\|^2_{L^2} &\le 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2_{L^2} \\ &\quad + 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\right\|^2_{L^2} \\ &\quad + 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\right\|^2_{L^2} \\ &:= \mathcal{I}^1+\mathcal{I}^2+\mathcal{I}^3. \end{aligned} \tag{27}$$

We now estimate $\mathcal{I}^i (i = 1, 2, 3)$. For $\mathcal{I}^1$, by Lemma 2(i), it is obvious that

$$\mathcal{I}^1 = 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|_{L^2}^2 \leq 3C\mathbb{E}\|\varphi(0)\|_{L^2}^2 \leq 3C \sup_{t\in(-\infty,0]} \mathbb{E}\|\varphi(t)\|_{L^2}^2. \tag{28}$$

For $\mathcal{I}^2$, by Lemma 2(i), (H_1), the Cauchy–Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{I}^2 &= 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \right\|_{L^2}^2 \\ &\leq 3\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)\|_{L^2}ds\right)^2 \\ &\leq 3Ct\int_0^t \mathbb{E}\|F(s,u_s)-F(s,0)\|_{L^2}^2 ds \\ &\leq 3CL_f't\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega,L^2_\sigma))}ds \\ &\leq 3CL_f't^2R^2.\end{aligned} \tag{29}$$

For $\mathcal{I}^3$, by Lemma 2(i), Itô's isometry and (H_2),

$$\begin{aligned}\mathcal{I}^3 &= 3\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s) \right\|_{L^2}^2 \\ &\leq 3C\int_0^t \mathbb{E}\|G(s,u_s)-G(s,0)\|_{L^2}^2 ds \\ &\leq 3CL_g'\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\ &\leq 3CL_g'tR^2.\end{aligned} \tag{30}$$

Replacing (28)–(30) into (27), combining with (26), it is obvious that

$$\mathbb{E}\|(\mathcal{K}u)_t\|_{L^2}^2 \leq 3\left((C+1)\sup_{t\in(-\infty,0]}\mathbb{E}\|\varphi(t)\|_{L^2}^2 + Ct^2L_f'R^2 + CtL_g'R^2\right).$$

Consequently, due to the choice of R, we can choose T small enough such that

$$\begin{aligned}\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} &= \left(\sup_{\theta\in(-\infty,0]}\mathbb{E}\|(\mathcal{K}u)(t+\theta)\|_{L^2}^2\right)^{\frac{1}{2}} \\ &\leq 3\left((C+1)\|\varphi\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + CT^2L_f'R^2 + CTL_g'R^2\right)^{\frac{1}{2}} \leq R.\end{aligned}$$

Assertion 3: Operator $\mathcal{K} : \mathcal{V}_R^\varphi \to \mathcal{V}_R^\varphi$ is a contraction.

To this end, for any $u, v \in \mathcal{V}_R^\varphi$, it follows that

$$\|(\mathcal{K}u)_t - (\mathcal{K}v)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} := \left(\sup_{\theta\in(-\infty,0]}\mathbb{E}\|(\mathcal{K}u)(t+\theta)-(\mathcal{K}v)(t+\theta)\|_{L^2}^2\right)^{\frac{1}{2}}. \tag{31}$$

For $t \in (-\infty, 0]$, one has $(\mathcal{K}u)(t) = (\mathcal{K}v)(t) = \varphi(t)$. Thus, it is only needed to consider the case $t \in [0, T]$. Observe that

$$\mathbb{E}\,\|(\mathcal{K}u)(t) - (\mathcal{K}v)(t)\|_{L^2}^2$$

$$\leq 2\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds \right\|_{L^2}^2 \\ +2\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s) \right\|_{L^2}^2 \\ := \mathcal{J}^1 + \mathcal{J}^2. \tag{32}$$

For $\mathcal{J}^1$, by Lemma 2(i), (H_2), the Cauchy–Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned} \mathcal{J}^1 &= 2\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds \right\|_{L^2}^2 \\ &\leq 2\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|_{L^2} ds \right)^2 \\ &\leq 2CL'_f t \int_0^t \|u_s - v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} ds \\ &\leq 2CL'_f t^2 \|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}. \end{aligned} \tag{33}$$

For $\mathcal{J}^2$, by Lemma 2(i), (H_2) and Itô's isometry, one has

$$\begin{aligned} \mathcal{J}^2 &= 2\mathbb{E}\left\| \int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s) \right\|_{L^2}^2 \\ &\leq 2CL'_g \int_0^t \|u_s - v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} ds \\ &\leq 2CL'_g t \|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}. \end{aligned} \tag{34}$$

Hence, substituting (32)–(34) into (31), it follows that

$$\begin{aligned} \|(\mathcal{K}u)_t - (\mathcal{K}v)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} &\leq \left(2C(L'_f T^2 + L'_g T)\|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \right)^{\frac{1}{2}} \\ &:= \mathcal{W}\|u(t) - v(t)\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}, \end{aligned}$$

where

$$\mathcal{W}^2 = 2C(L'_f T^2 + L'_g T).$$

Therefore, we can choose T small enough such that $0 < \mathcal{W} < 1$, which means that, the operator $\mathcal{K}$ maps $\mathcal{V}_R^\varphi$ into itself, also it is a contraction. The Banach fixed-point theorem yields that the operator $\mathcal{K}$ has a fixed-point in $\mathcal{V}_R^\varphi$. Namely, the problem (24) has a unique local mild solution on $(-\infty, T]$. The proof of this theorem is completed. □

Proposition 3 *Under the assumptions of Theorem 3, the mild solution to (24) is continuous with respect to the initial data $\varphi \in C((-\infty, 0]; L^2(\Omega; L^2_\sigma))$. In particular, if $u(t)$, $w(t)$ are the corresponding mild solutions, on the interval $(-\infty, T]$, to the initial data ϕ and ψ, then the following estimate holds*

$$\|u_t - w_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq 3C\|\phi - \psi\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \exp(3C(L'_f t + L'_g)t), \ \forall t \in [0, T].$$

Proof This theorem can be proved by similar arguments to those concerning the uniqueness of next theorem, so we omit the details. □

The following result is concerned with the existence and uniqueness of the global mild solution to problem (24).

Theorem 4 *Assume the hypotheses of Theorem 3 hold. Then for every initial value* $\varphi \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$*, the initial value problem (24) has a unique mild solution defined globally in the sense of Definition 2.*

Proof Although the proof of this theorem follows the same lines as the case of bounded delay in Sect. 3, but with differences in the estimates, we prefer to include it here because the proof of Proposition 3 is similar to the uniqueness below.

Assume that there exist two solutions, u and v on $[0,T_1]$ and $[0,T_2]$, respectively to problem (24). Next let us prove that $u=v$ on $(-\infty,T_1\wedge T_2]$. It is notable that $u(t)=v(t)=\varphi(t)$ on $(-\infty,0]$, so we only need to prove that $u(t)=v(t)$ for any $t\in[0,T_1\wedge T_2]$. Observe that

$$\|u-v\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} := \sup_{t\in(-\infty,T_1\wedge T_2]} \mathbb{E}\|u(t)-v(t)\|^2_{L^2}. \tag{35}$$

On the one hand, it holds

$$\begin{aligned}
&\mathbb{E}\|u(t)-v(t)\|^2_{L^2}\\
&\le 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\right\|^2_{L^2}\\
&\quad + 2\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\right\|^2_{L^2}\\
&:= I^1+I^2.
\end{aligned} \tag{36}$$

For I^1, by Lemma 2(i), (H_1) and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
I^1 &\le 2\mathbb{E}\left(\int_0^t \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|_{L^2}ds\right)^2\\
&\le 2C\mathbb{E}\left(\int_0^t \|F(s,u_s)-F(s,v_s)\|_{L^2}ds\right)^2\\
&\le 2CL'_f t\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds.
\end{aligned} \tag{37}$$

For I^2, by Lemma 2(i), (H_2) and Itô's isometry, we derive

$$\begin{aligned}
I^2 &\le 2\int_0^t \mathbb{E}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))\|^2_{L^2}ds\\
&\le 2CL'_g\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds.
\end{aligned} \tag{38}$$

Substituting (37)–(38) into (36), it yields

$$\mathbb{E}\|u(t)-v(t)\|^2_{L^2} \le 2C\left(L'_f t+L'_g\right)\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds.$$

Denote by $\mathcal{W}_1 = 2C(L'_f(T_1\wedge T_2)+L'_g)$, we have

$$\|u-v\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \le \mathcal{W}_1\int_0^{T_1\wedge T_2} \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}dt.$$

The Gronwall Lemma implies that

$$\|u - v\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} = 0.$$

Therefore, $u = v$ on $(-\infty, T_1 \wedge T_2]$ for every initial function $\varphi(t)$.

Now we prove that for each given $T > 0$, the mild solution u to problem (24) is bounded with $\mathcal{C}_X(L^2(\Omega; L^2_\sigma))$ norm. Taking into account Lemma 2(i), (H_1)–(H_2), Itô's isometry, the Cauchy–Schwarz inequality and Fubini's theorem, we have

$$\begin{aligned}
\mathbb{E}\|u(t)\|^2_{L^2} &\leq 3\mathbb{E}\left\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\right\|^2_{L^2} \\
&\quad + 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\right\|^2_{L^2} \\
&\quad + 3\mathbb{E}\left\|\int_0^t \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\right\|^2_{L^2} \\
&\leq 3C\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + 3CL'_f t\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\
&\quad + 3CL'_g\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\
&\leq 3C\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + 3C(L'_f t + L'_g)\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds.
\end{aligned}$$

Applying the Gronwall lemma, for any fixed $T > 0$ and all $t \in [0, T]$,

$$\|u_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq (3C+1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\exp(3C(L'_f T + L'_g)T).$$

Because of the arbitrariness of T, together with the conclusion of uniqueness of u on $(-\infty, T]$, it is straightforward that the mild solution u to problem (24) is defined globally. This finishes the proof. □

Remark 5 The well-posedness results to problems (6) and (24) can be modified to the case that the driven process is an additive fractional Brownian motion, which is L^2_σ-value. Of course we need to redefine $G(t,\cdot)$ and impose certain assumptions similar to (H_2)–(H_5), see [29], for example.

Remark 6 Although we have performed our analysis for the stochastic time fractional 2D-Stokes delay differential equations, the results of sections 3 and 4 still hold true when the phase spaces are extended to $C([-h,0]; L^2(\Omega, L^N_\sigma))$ and $C((-\infty,0]; L^2(\Omega, L^N_\sigma))$ respectively, where $N > 2$ [7].

5 Conclusions and Final Remarks

In this paper we have considered a quite general time fractional stochastic Stokes model with finite and infinite delay and multiplicative Brownian motion. As we said, this is only a first approach to our goal concerning the case of stochastic time fractional delay Navier–Stokes with multiplicative noise. But, to that end, a new technique has to be designed because the fixed-point theorem used in our proofs is not appropriate to handle the nonlinear term: the appearance of expectation in the norm does not allow us to bound that term in an appropriate way as it is done in the deterministic case, specially for the contraction property. Therefore, this is a challenging problem to be analyzed shortly. However, it is not surprising that the problem cannot be analyzed with this technique since, to the best of our knowledge, even the integer time derivative system has not been solved for the multiplicative noise case, but only for the additive one. We plan to work on this first case and combine the ideas of both techniques to achieve our goal for the time fractional stochastic Navier–Stokes with delays.

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References

1. Alikhanov, A.A.: A priori estimates for solutions of boundary value problems for equations of fractional order. Differ. Equ. **46**, 660–666 (2010)
2. Allen, M., Caffarelli, L., Vasseur, A.: A parabolic problem with a fractional time derivative. Arch. Ration. Mech. Anal. **221**, 603–630 (2016)
3. Bonotto, E.M., Mesquita, J.G., Silva, R.P.: Global mild solutions for a nonautonomous 2D Navier–Stokes equations with impulses at variable times. J. Math. Fluid Mech. **20**, 801–818 (2018)
4. Caraballo, T., Real, J.: Navier–Stokes equations with delays. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457**, 2441–2453 (2014)
5. Caraballo, T., Han, X.Y.: A survey on Navier–Stokes models with delays: existence, uniqueness and asymptotic behavior of solutions. Discret. Contin. Dyn. Syst. Ser. S **8**, 1079–1101 (2015)
6. Carvalho-Neto, P. M.: Fractional differential equations: a novel study of local and global solutions in Banach spaces. PhD thesis, Universidade de São Paulo, São Carlos (2013)
7. Carvalho-Neto, P.M., Planas, G.: Mild solutions to the time fractional Navier–Stokes equations in $\mathbb{R}^N$. J. Differ. Equ. **259**, 2948–2980 (2015)
8. Chen, Z.Q., Kim, K.H., Kim, P.: Fractional time stochastic partial differnetial equations. Stoch. Process. Appl. **125**, 1470–1499 (2015)
9. Curtain, R.F., Falb, P.L.: Stochastic differential equations in Hilbert space. J. Differ. Equ. **10**, 412–430 (1971)
10. Debbi, L.: Well-posedness of the multidimensional fractional stochastic Navier–Stokes equations on the torus and on bounded domains. J. Math. Fluid Mech. **18**, 25–69 (2016)
11. Friedman, A., Hu, B.: Bifurcation from stability to instability for a free boundary problem modeling tumor growth by Stokes equation. J. Math. Anal. Appl. **327**, 643–664 (2007)
12. Friedman, A., Hu, B.: Bifurcation for a free boundary problem modeling tumor growth by Stokes equation. SIAM J. Math. Anal. **39**, 174–194 (2007)
13. Hao, W.R., Hauenstein, J.D., Hu, B., McCoy, T., Sommese, A.J.: Computing steady-state solutions for a free boundary problem modeling tumor growth by Stokes equation. J. Comput. Appl. Math. **237**, 326–334 (2013)
14. Haak, B.H., Kunstmann, P.C.: On Kato's method for Navier–Stokes equations. J. Math. Fluid Mech. **11**, 492–535 (2009)
15. Kato, T.: Strong L^p-solutions of the Navier–Stokes equation in R^m, with applications to weak solution. Math. Z. **187**, 471–480 (1984)
16. Kloeden, P.E., Valero, J.: The Kneser property of the weak solutions of the three dimensional Navier–Stokes equations. Discret. Contin. Dyn. Syst. **28**, 161–179 (2010)
17. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
18. Kirby, B.: Micro- and Nanoscale Fluid Mechanics: Transport in Microfluidic Devices. Cambridge University Press, Cambridge (2010)
19. Li, Y.J., Wang, Y.J., Deng, W.H.: Galerkin finite element approximations for stochastic space-time fractional wave equations. SIAM J. Numer. Anal. **55**, 3173–3202 (2017)
20. Liu, W., Röckner, M., da Silva, J.L.: Quasi-linear (stochastic) partial differential equations with time-fractional derivatives. SIAM J. Math. Anal. **50**, 2588–2607 (2018)
21. Mainardi, F.: On the initial value problem for the fractional diffusion-wave equation. Ser. Adv. Math. Appl. Sci. **23**, 246–251 (1994)

22. Marín-Rubio, P., Real, J., Valero, J.: Pullback attractors for a two-dimensional Navier–Stokes model in an infinite delay case. Nonlinear Anal. **74**, 2012–2030 (2011)
23. Podlubny, I.: Fractional Differential Equations. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)
24. Robinson, J.C.: Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors. Cambridge University Press, Cambridge (2001)
25. Schwab, C., Stevenson, R.: Fractional space-time variational formulations of (Navier-) Stokes equations. SIAM J. Math. Anal. **49**, 2442–2467 (2017)
26. Tang, H.: On the pathwise solutions to the Camassa–Holm equation with multiplicative noise. SIAM J. Math. Anal. **50**, 1322–1366 (2018)
27. Taniguchi, T.: The existence and asymptotic behaviour of energy solutions to stochastic 2D functional Navier–Stokes equations driven by Lévy processes. J. Math. Anal. Appl. **385**, 634–654 (2012)
28. Wang, R.N., Chen, D.H., Xiao, T.J.: Abstract fractional Cauchy problems with almost sectorial operators. J. Differ. Equ. **252**, 202–235 (2012)
29. Xu, J.H., Caraballo, T.: Long time behavior of fractional impulsive stochastic differential equations with infinite delay. Discrete Contin. Dyn. Syst. Ser. B **24**, 2719–2743 (2019)
30. Xu, P.F., Zeng, C.B., Huang, J.H.: Well-posedness of the time-space fractional stochastic Navier–Stokes equations driven by fractional Brownian motion. Math. Model. Nat. Phenom. **13**, 11 (2018)
31. Zou, G.A., Lv, G.Y., Wu, J.L.: Stochastic Navier–Stokes equations with Caputo derivative driven by fractional noises. J. Math. Anal. Appl. **461**, 595–609 (2018)
32. Zhou, Y., Peng, L.: On the time-fractional Navier–Stokes equations. Comput. Math. Appl. **73**, 874–891 (2017)
33. Zhou, Y., Peng, L.: Weak solutions of the time-fractional Navier–Stokes equations and optimal control. Comput. Math. Appl. **73**, 1016–1027 (2017)