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Research paper

On a terminal value problem for parabolic reaction–diffusion systems with nonlocal coupled diffusivity terms

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ABSTRACT

In this article, we are interested in investigating the nonlocal nonlinear reaction–diffusion system with final conditions. This problem is called backward in time problem, or terminal value problem which is understood as redefining the previous distributions when the distribution data at the terminal observation are known. There are three main goals presented in this paper. First, we prove that the problem is ill-posed (often called as unstable property) in the sense of Hadamard. Our next propose is to provide a modified quasi-reversibility model to stabilize the ill-posed problem. Using some techniques and tools of Faedo–Galerkin method, we prove the existence of the unique weak solution of the regularized problem. Further, we investigate error estimates between the sought solution and the regularized solution in $L^2(\Omega)-$ and $H^1(\Omega)-$ norms. The final aim of this paper is to give some numerical results to demonstrate that our method is useful and effective.

1. Introduction

Let T be a positive number and $\Omega \subset \mathbb{R}^n, n \geq 1$, be an open bounded domain with a smooth boundary Γ. Denote $Q_T = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$; Σ is called the lateral boundary of the cylinder Q_T. In this work, we consider the following nonlocal nonlinear parabolic coupled system of reaction–diffusion equations

$$\begin{cases} u_t = \mathcal{D}_1 \left(\mathbf{a}(u)(t), \mathbf{b}(v)(t)\right) \Delta u + F(x, t, u, v), & \text{in } Q_T, \\ v_t = \mathcal{D}_2 \left(\mathbf{c}(u)(t), \mathbf{d}(v)(t)\right) \Delta v + G(x, t, u, v), & \text{in } Q_T, \end{cases} \tag{1}$$

To complete the terminal-boundary value problem, we consider the terminal conditions

$$u(x, T) = \Psi(x), \quad v(x, T) = \Phi(x), \quad \text{in } \Omega, \tag{2}$$

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and Neumann boundary conditions

$$\frac{\partial u(x,t)}{\partial \sigma} = \frac{\partial v(x,t)}{\partial \sigma} = 0, \quad \text{on } \Sigma, \tag{3}$$

where σ is the outward unit normal to the boundary Γ. u and v are the population densities of two observed species. F and G are the reaction terms and $\mathcal{D}_1, \mathcal{D}_2$ are the diffusion coefficients, working on the functionals $\mathbf{a}(u)(t)$, $\mathbf{b}(v)(t)$, $\mathbf{c}(u)(t)$ and $\mathbf{d}(v)(t)$. The problem is nonlocal in the sense that the diffusion terms $\mathcal{D}_1$ and $\mathcal{D}_2$ depend on the entire populations, which mean the global quantity, rather than the local density. In this model, the interaction of two species is represented not only in the reaction terms, but also in the diffusion terms. The homogeneous Neumann condition in biological models mean that the boundary of the specimen domain observed is insulated. We can also consider the homogeneous Dirichlet boundary condition instead, and the method can be straightforwardly applied. In this paper, our main goal is to seek the initial values $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$ when we only know the measurements of the terminal data Ψ and Φ. We can give one practical application of the model (1) for locating the source of brain tumours. Let u and v be the normal (healthy) and abnormally growing normal tissue cells in the brain due to genetic and epigenetic events. In the perspective of (1), we assume that the movements of each kind of cells are dominantly influenced by the whole population of the corresponding type. The nonlinear source terms F and G can be considered as reactions, mortality rates and proliferation rates. For the last several decades, various types of equations have been employed as mathematical models describing physical, chemical, biological and ecological systems. Among them, one of the most successful systems is the reaction–diffusion system

$$\begin{cases} u_t = \mathcal{D}_1 \Delta u + F(x,t,u,v), \\ v_t = \mathcal{D}_2 \Delta v + G(x,t,u,v). \end{cases} \tag{4}$$

For instance,

- Activator–inhibitor FitzHugh - Nagumo model for propagation of electrical signals in neurons [1,2]

  $$F = au - bv + \alpha u^2 - \beta u^3, \quad G = cu - dv + m,$$

  where the constants a, b, c, d, β are positive and $m, \alpha \in \mathbb{R}$, u is the activator, v is the inhibitor. This model describes the control of the electrical potential across a cell membrane by the change of flow of the ionic channels;
- Fisher - Kolmogorov model for delay effects in the response of low-grade gliomas (LGG) to radiotherapy [3]

  $$F = \rho(1-u-v)u, \quad G = -\frac{\rho}{k}(1-u-v)v,$$

  where the constants ρ, k are positive, u is the tumour cells density, v is the density of cells irreversibly damaged by radiation;
- Diffusive Lotka–Volterra system [4] describes the relation between population densities u, v of interacting species

  $$F = au + buv, \quad G = cv + duv,$$

  where a, b, c, d are the constants;
- Turing model for biological pattern formation [5,6]. u, v are concentrations of activator, and substrate, respectively. The Turing model can display a variety of intricate spatial patterns that result from an interplay between local aggregation of u through autocatalysis, and rapid diffusion of v away from u-rich regions: $\mathcal{D}_v > \mathcal{D}_u$. Examples:

  - The Schnakenberg system

    $$F = k_1 - k_2 u + k_3 u^2 v, \quad G = k_4 - k_3 u^2 v,$$

    where $k_i,\ i = \overline{1,4}$ are positive constants;
  - The Gierer–Meinhardt system

    $$F = k_1 - k_2 u + k_3 \frac{u^2}{v}, \quad G = k_4 u^2 - k_5 v,$$

    where the constants $k_i > 0, i = \overline{1,5}$;
  - The Thomas system

    $$F = k_1 - k_2 u - \frac{k_5 uv}{k_6 + k_7 u + k_8 u^2}, \quad G = k_3 - k_4 v - \frac{k_5 uv}{k_6 + k_7 u + k_8 u^2},$$

    where $k_i, i = \overline{1,8}$ are positive constants;

- Model for exothermic chemical reaction occurring in a solid [7]

  $$F = Au^m \exp\left(\frac{-E}{Rv}\right), \quad G = Bu^m \exp\left(\frac{-E}{Rv}\right),$$

  where the constants $A, B, E, m, R > 0$. The rate of the reaction is determined by temperature v, through an Arrhenius law, and concentration of one key reagent u.

Some other models for reaction cross-diffusion systems have also been studied by J. A. Carrillo et al. [8–12]. Recently, nonlocal problems have attracted attention of many researchers because nonlocal terms allow to give more accurate results, (the measurement represents the average in a neighbourhood of a point). For instance, these problems arise in physics [13], engineering [14], and population dynamics [15]. When we consider the problem , we are not considering what the behaviour of the population is, if an area is overcrowded or isolated. Therefore, the nonlocal diffusion is imposed so that these effects are taken into account. In 1997, M. Chipot and B. Lovat [16] studied the nonlocal problem

$$\begin{cases} u_t - \mathcal{D}(\ell(u)(t))\Delta u = f, & \text{in } Q_T, \\ u = 0, & \text{on } \Sigma, \\ u = u_0, & \text{in } \Omega \times \{0\}, \end{cases}$$

where u is the density of population located at x at the time t, f is the external source, $\mathcal{D}$ is the diffusion rate. In the case of a migration of population, for instance of bacteria in a container [17], it is obvious that the environment is of prime importance and one will easily imagine that

$$\mathcal{D}(\ell(u)(t)) = \mathcal{D}\left(\int_{\Omega'} u\mathrm{d}x\right)(t),$$

i.e. the velocity of the migration depends on the total population in a subdomain Ω'. If one wants to model species having the tendency to leave crowded zones, a natural assumption would be to assume that a is an increasing function. On the other hand, if we are dealing with species attracted by the growing population in Ω', one will suppose a to decrease. Another justification of such a model lies also in the fact that in reality, for instance in the case of u being temperature of a conductor, the measurements are not made pointwise, but through some local average. Some similar models for nonlocal parabolic has been developed by M. Chipot et al. [18,19], T. Caraballo et al. [9,10,20], M. Burger [21–23].

Some authors [24–27] have studied the properties of the solution of a generalized model. In [28], Ferreira et al. considered a model with nonlocal coupled diffusivity terms

$$\begin{cases} u_t - \mathcal{D}_1(p(u)(t), q(v)(t))\Delta u = f_1(u, v), & \text{in } Q_T, \\ v_t - \mathcal{D}_2(r(u)(t), s(v)(t))\Delta v = f_2(u, v), & \text{in } Q_T, \\ u = u_0, \quad v = v_0, & \text{in } \Omega \times \{0\}. \end{cases}$$

Although the initial problems have been investigated by many authors, there are not many results for the inverse problems. Let us emphasize that the property of solution for the terminal value problem is very different to the initial value problem. Due to the smoothing effects of the parabolic operator, in fact, it is not possible, in general, to guarantee the existence of the solution for initial data which are not suitably regular. In addition, even when the solution possibly exists, the uniqueness cannot be ensured without additional assumptions on the operator. In his celebrated paper [29], John introduced the notion of well-behaved problem, which is now typical in the context of ill-posed problems. According to John, a problem is well-behaved if "only a fixed percentage of the significant digits need be lost in determining the solution from the data" [29]. More precisely we may say that a problem is well-behaved if its solutions in a space H depend Hölder continuously on the data belonging to a space K, provided they satisfy a prescribed bound.

To the best of our knowledge, there have not been any works related to the system (1)–(3). In this paper, we provide a modified quasi-reversibility (QR) method which was applied to construct the regularized problem. The QR approach was first introduced by Lattès and Lions [30]. The main idea of the method is stabilizing the ill-posed problem by using a small regularization parameter. Recently, a modified QR method was applied quite successfully to the following problem in [31]

$$\begin{cases} u_t(x,t) - \mathcal{D}(\ell(u)(t))\Delta u = F(u, x, t), & \text{in } Q_T, \\ \frac{\partial u}{\partial \sigma} = 0, & \text{on } \Sigma, \\ u = g, & \text{in } \Omega \times \{T\}. \end{cases} \tag{5}$$

The authors [31] considered the backward in time nonlocal nonlinear parabolic equation for the population density u.

In the same spirit, we use this method for the system (1)–(3). However, there are different views in our analysis compared to the paper [31]. In [31], the authors used the Banach fixed point theorem for local self-mappings to show the existence of local regularized solution over the layers $[t_{j+1}, t_j]$. These techniques for are interesting but complicated. However, for our models (1)–(3) this technique can hardly be applied. More details, in our case, we consider a coupled system, and this is more challenging since the length of the layers is somewhat difficult to compute. Therefore, we need a different and new way of thinking when considering the solution for the regularized problem. We thus use Faedo–Galerkin method and Aubin–Lions lemma as the main tools instead, and some techniques are required to modify the QR solution so that its existence condition holds. In addition, the error estimates are given not only in L^2-, but also in H^1-norm.

Our work is organized as follows. Section 2 contains some notations and assumptions used throughout the paper. In Section 3, an example is considered to show the instability of the inverse problem. The main results are in the next two sections. Section 4 focuses on the construction of approximate problem by a modified QR approach. First, we provide an approximation of locally Lipschitz reaction. Then, we prove the existence of the unique regularized solution using Faedo–Galerkin method and the Aubin–Lions lemma. Section 5 is devoted to the error estimates in L^2- and H^1- norms. In Section 6, we illustrate the theoretical results by numerically solving the regularized systems of two biological models. Finally, the conclusion is presented in Section 7.

2. Preliminaries

Let us denote by $\mathbb{U}$ the space of all functions in $H^1(\Omega)$, satisfying the Neumann boundary condition, with the $H^1(\Omega)$ -norm

$$\mathbb{U} = \left\{ u \in H^1(\Omega) : \frac{\partial u}{\partial \sigma} = 0, \quad \forall x \in \Gamma \right\}.$$

Throughout this paper, we denote the inner product in $L^2(\Omega)$ by $\langle \cdot, \cdot \rangle$. $H^{-1}(\Omega)$ denotes the dual space of $H^1(\Omega)$.

For a Banach space X, we denote by $L^p(0, T; X)$, $C([0, T]; X)$, $C^1(0, T; X)$ the Banach spaces

$$\begin{aligned}
\|u\|_{L^p(0,T;X)} &= \left(\int_0^T \|u(\cdot, t)\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,\\
\|u\|_{L^\infty(0,T;X)} &= \operatorname*{ess\,sup}_{t \in (0,T)} \|u(\cdot, t)\|_X < \infty,\\
\|u\|_{C([0,T];X)} &= \sup_{t \in [0,T]} \|u(\cdot, t)\|_X < \infty,\\
\|u\|_{C^1(0,T;X)} &= \|u\|_{C([0,T];X)} + \|u_t\|_{C([0,T];X)} < \infty.
\end{aligned}$$

$\{\lambda_p\}_{p=0}^\infty$ are eigenvalues of the Laplacian operator $-\Delta$ on the bounded domain Ω with Neumann boundary condition, and satisfy

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_p \le \cdots,$$

with $\lambda_p \to \infty$ when $p \to \infty$. $\{\mu_p\}_{p=0}^\infty \subset \mathbb{U}$ are eigenfunctions respectively, forming an orthonormal basis of $L^2(\Omega)$.

Let us introduce a space of Gevrey type $G_\Upsilon(\Omega)$ of index $\Upsilon > 0$, see [31] *e.g.*, defined by

$$G_\Upsilon(\Omega) = \left\{ u \in L^2(\Omega) : \sum_{p=0}^\infty e^{\Upsilon \lambda_p} u_p^2 < +\infty \right\},$$

with norm defined by

$$\|u\|_{G_\Upsilon(\Omega)} = \left(\sum_{p=0}^\infty e^{\Upsilon \lambda_p} u_p^2 \right)^{\frac{1}{2}}, \quad \text{where} \quad u_p = \langle u, \mu_p \rangle.$$

A couple (u, v) of functions $u(x, t)$ and $v(x, t) : \overline{Q}_T \to \mathbb{R}$, $(\overline{Q}_T = \overline{\Omega} \times [0, T])$ is called a function of two variables x, t

$$\begin{aligned}
&(u, v) : \overline{Q}_T \to \mathbb{R}^2\\
&(u, v)(x, t) = (u(x, t), v(x, t)).
\end{aligned}$$

Here, the norm of $(u, v) \in \mathbb{X}^2$ (for any space $\mathbb{X}$) is defined as

$$\|(u, v)\|_{\mathbb{X}^2} = \|u\|_{\mathbb{X}} + \|v\|_{\mathbb{X}}.$$

We state now some assumptions

(A_1) The reaction functions $F(x, t, u, v)$ and $G(x, t, u, v)$ are continuous with respect to t;

(A_2) There exist positive constants m and M such that

$$m \le \mathcal{D}_i(\vartheta, \nu) \le M, \quad \forall\, (\vartheta, \nu) \in \mathbb{R}^2, i = 1, 2;$$

(A_3) There exist positive constants $L_{\mathcal{D}_i}$ such that $\forall\ \vartheta_1, \vartheta_2, \nu_1, \nu_2 \in \mathbb{R}, i = 1, 2$

$$|\mathcal{D}_i(\vartheta_1, \nu_1) - \mathcal{D}_i(\vartheta_2, \nu_2)| \le L_{\mathcal{D}_i}(|\vartheta_1 - \vartheta_2| + |\nu_1 - \nu_2|);$$

(A_4) There exist $L_{\mathbf{a}}, L_{\mathbf{b}}, L_{\mathbf{c}}, L_{\mathbf{d}} > 0$, such that, for all $t \in [0, T]$, $(u_1, v_1), (u_2, v_2) \in [L^2(\Omega)]^2$

$$\begin{aligned}
|(\mathbf{a}(u_1) - \mathbf{a}(u_2))(\cdot, t)| &\le L_{\mathbf{a}} \|(u_1 - u_2)(\cdot, t)\|_{L^2(\Omega)},\\
|(\mathbf{b}(v_1) - \mathbf{b}(v_2))(\cdot, t)| &\le L_{\mathbf{b}} \|(v_1 - v_2)(\cdot, t)\|_{L^2(\Omega)},\\
|(\mathbf{c}(u_1) - \mathbf{c}(u_2))(\cdot, t)| &\le L_{\mathbf{c}} \|(u_1 - u_2)(\cdot, t)\|_{L^2(\Omega)},\\
|(\mathbf{d}(v_1) - \mathbf{d}(v_2))(\cdot, t)| &\le L_{\mathbf{d}} \|(v_1 - v_2)(\cdot, t)\|_{L^2(\Omega)}.
\end{aligned}$$

Denote $\mathcal{L}_{\max} = \max\{L_{\mathcal{D}_i}.L_{\mathbf{p}}\}$, $i = 1, 2$, $\mathbf{p} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$. Notice that if (A$_3$) and (A$_4$) hold, we have

$$|\mathcal{D}_1(\mathbf{a}(u_1), \mathbf{b}(v_1))(t) - \mathcal{D}_1(\mathbf{a}(u_2), \mathbf{b}(v_2))(t)| \leq \mathcal{L}_{\max}\| ((u_1, v_1) - (u_2, v_2))(\cdot, t)\|_{[L^2(\Omega)]^2},$$
$$|\mathcal{D}_2(\mathbf{c}(u_1), \mathbf{d}(v_1))(t) - \mathcal{D}_2(\mathbf{c}(u_2), \mathbf{d}(v_2))(t)| \leq \mathcal{L}_{\max}\| ((u_1, v_1) - (u_2, v_2))(\cdot, t)\|_{[L^2(\Omega)]^2}.$$

For abbreviation, from now on, denote

$$\mathcal{D}_1(\mathbf{a}(u)(t), \mathbf{b}(v)(t)) = \mathcal{D}_1(u, v), \quad \mathcal{D}_2(\mathbf{c}(u)(t), \mathbf{d}(v)(t)) = \mathcal{D}_2(u, v).$$

(A$_5$) Ψ and $\Phi \in L^2(\Omega)$ represent the exact data, whilst Ψ^ε and $\Phi^\varepsilon \in L^2(\Omega)$ represent the measured data with noise level $\varepsilon > 0$, such that

$$\|\Psi^\varepsilon - \Psi\|_{L^2(\Omega)} + \|\Phi^\varepsilon - \Phi\|_{L^2(\Omega)} \leq \varepsilon.$$

3. Instability of the backward problem

In order to ascertain the need for a regularization method, we take an example to illustrate the instability of the backward problem. Let the exact and perturbed final data

$$\Psi(x) = \Phi(x) = 0, \quad \Psi^{p_*}(x) = \Phi^{p_*}(x) = \frac{1}{\sqrt{\lambda_{p_*}}}\mu_{p_*}(x), \quad \text{for } p_* \in \mathbb{N}, \tag{6}$$

from which we observe that when $p_* \to \infty$, the noise terms disappear $\frac{1}{\sqrt{\lambda_{p_*}}}\mu_{p_*}(x) \to 0$. Due to the fact that proving the existence of solution to the inverse system is still an open problem, in this example, we take the reaction functions of the following special form

$$F = G = \sum_{p=1}^{\infty} e^{-\lambda_p TM}\lambda_p^{-1}\left(e^{-|\langle u(\cdot,t),\mu_p\rangle|} + e^{-|\langle v(\cdot,t),\mu_p\rangle|}\right)\mu_p(x).$$

The concept of weak solution is given by

Definition 3.1. A couple (u, v) is called a *weak solution* to the problem (1), (3), if for all $\varphi, \psi \in \mathbb{U}$, yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(\cdot, t), \varphi\rangle + \mathcal{D}_1(u, v)(t)\langle \nabla u(\cdot, t), \nabla\varphi\rangle = \langle F(\cdot, t, u, v), \varphi\rangle, \tag{7}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(\cdot, t), \psi\rangle + \mathcal{D}_2(u, v)(t)\langle \nabla v(\cdot, t), \nabla\psi\rangle = \langle G(\cdot, t, u, v), \psi\rangle. \tag{8}$$

Then we have the following lemma, which shows that even if the noise level goes to 0, the instability always happens backwards in time.

Lemma 3.2. *Suppose that* (A$_1$)–(A$_4$) *hold, then there exist weak solutions to the backward problem with the exact data* $(u_{\text{ex}}, v_{\text{ex}})$*, and with perturbed final data* $(\widetilde{u^{p_*}}, \widetilde{v^{p_*}})$ *for* $p_* \in \mathbb{N}$*. Moreover*

$$\|(\widetilde{u^{p_*}}, \widetilde{v^{p_*}}) - (u_{\text{ex}}, v_{\text{ex}})\|_{[C([0,T];L^2(\Omega))]^2} \geq \frac{2e^{m\lambda_{p_*}T}}{\lambda_{p_*}\left(1 + 4T^2\mathcal{L}_{\max} + 2\sqrt{2}T\lambda_1^{-1}\right)} \to \infty, \text{ if } p_* \to \infty.$$

Proof. In (7), let $\varphi = \mu_p$, we multiply both sides with $e^{-\lambda_p\int_t^T \mathcal{D}_1(u_{\text{ex}},v_{\text{ex}})(\eta)\mathrm{d}\eta}$, and then integrate from t to T, with the final condition $u_{\text{ex}}(x, T) = 0$, it yields

$$\langle u_{\text{ex}}(\cdot, t), \mu_p\rangle = -\int_t^T e^{\lambda_p\int_t^\eta \mathcal{D}_1(u_{\text{ex}},v_{\text{ex}})(\xi)\mathrm{d}\xi}e^{-\lambda_p TM}\lambda_p^{-1}\left(e^{-|u_{\text{ex},p}(\eta)|} + e^{-|v_{\text{ex},p}(\eta)|}\right)\mathrm{d}\eta,$$

where $u_{\text{ex},p}(t) = \langle u_{\text{ex}}(\cdot, t), \mu_p\rangle$, $v_{\text{ex},p}(t) = \langle v_{\text{ex}}(\cdot, t), \mu_p\rangle$.

Thus, by similar calculations, we represent the exact solution by the Fourier series

$$u_{\text{ex}}(x, t) = -\sum_{p=1}^{\infty}\left[\int_t^T e^{\lambda_p\left(\int_t^\eta \mathcal{D}_1(u_{\text{ex}},v_{\text{ex}})(\xi)\mathrm{d}\xi - TM\right)}\frac{e^{-|u_{\text{ex},p}(\eta)|} + e^{-|v_{\text{ex},p}(\eta)|}}{\lambda_p}\mathrm{d}\eta\right]\mu_p(x),$$
$$v_{\text{ex}}(x, t) = -\sum_{p=1}^{\infty}\left[\int_t^T e^{\lambda_p\left(\int_t^\eta \mathcal{D}_2(u_{\text{ex}},v_{\text{ex}})(\xi)\mathrm{d}\xi - TM\right)}\frac{e^{-|u_{\text{ex},p}(\eta)|} + e^{-|v_{\text{ex},p}(\eta)|}}{\lambda_p}\mathrm{d}\eta\right]\mu_p(x).$$

For the system with noisy data $\widetilde{u^{p_*}}(x,T) = \widetilde{v^{p_*}}(x,T) = \dfrac{1}{\lambda_{p_*}}\mu_{p_*}(x)$, for any $p_* \in \mathbb{N}$, we have

$$\begin{aligned}\widetilde{u^{p_*}}(x,t) = \mathcal{I}_1(\widetilde{u^{p_*}},\widetilde{v^{p_*}})(x,t) &= \frac{e^{\lambda_{p_*}\int_t^T \mathcal{D}_1(\widetilde{u^{p_*}},\widetilde{v^{p_*}})(\eta)\mathrm{d}\eta}}{\lambda_{p_*}}\mu_{p_*}(x)\\ &- \sum_{p=1}^{\infty}\left[\int_t^T e^{\lambda_p\left(\int_t^\eta \mathcal{D}_1(\widetilde{u^{p_*}},\widetilde{v^{p_*}})(\xi)\mathrm{d}\xi - TM\right)}\frac{e^{-|\widetilde{u_p^{p_*}}(\eta)|}+e^{-|\widetilde{v_p^{p_*}}(\eta)|}}{\lambda_p}\mathrm{d}\eta\right]\mu_p(x),\end{aligned}$$

$$\begin{aligned}\widetilde{v^{p_*}}(x,t) = \mathcal{I}_2(\widetilde{u^{p_*}},\widetilde{v^{p_*}})(x,t) &= \frac{e^{\lambda_{p_*}\int_t^T \mathcal{D}_2(\widetilde{u^{p_*}},\widetilde{v^{p_*}})(\eta)\mathrm{d}\eta}}{\lambda_{p_*}}\mu_{p_*}(x)\\ &- \sum_{p=1}^{\infty}\left[\int_t^T e^{\lambda_p\left(\int_t^\eta \mathcal{D}_2(\widetilde{u^{p_*}},\widetilde{v^{p_*}})(\xi)\mathrm{d}\xi - TM\right)}\frac{e^{-|\widetilde{u_p^{p_*}}(\eta)|}+e^{-|\widetilde{v_p^{p_*}}(\eta)|}}{\lambda_p}\mathrm{d}\eta\right]\mu_p(x),\end{aligned}$$

where we denote $\widetilde{w_p^{p_*}} = \langle \widetilde{w^{p_*}}, \mu_p\rangle$, with $\widetilde{w^{p_*}} \in L^2(\Omega)$.

We now prove the existence of the noisy solution by using the Banach fixed point theorem.

Let us define the operator $\mathbf{I}(\widetilde{u^{p_*}},\widetilde{v^{p_*}}) = \left(\mathcal{I}_1(\widetilde{u^{p_*}},\widetilde{v^{p_*}}), \mathcal{I}_2(\widetilde{u^{p_*}},\widetilde{v^{p_*}})\right)$

$$\mathbf{I}(\widetilde{u^{p_*}},\widetilde{u^{p_*}}) : \left[C([0,T];L^2(\Omega))\right]^2 \to \left[C([0,T];L^2(\Omega))\right]^2.$$

First we show that for any couples $(u_1,v_1),(u_2,v_2)$, $k\in\mathbb{N}$, it holds

$$\left\|\left(\mathbf{I}^k(u_1,v_1)-\mathbf{I}^k(u_2,v_2)\right)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \le \frac{\mathcal{C}_0^k(T-t)^k}{k!}\|(u_1,v_1)-(u_2,v_2)\|^2_{[C([0,T];L^2(\Omega))]^2}, \tag{9}$$

where

$$\mathcal{C}_0 = 12T\left(e^{2\lambda_{p_*}TM}\mathcal{L}^2_{\max} + 4\lambda_1^{-2}T^2\mathcal{L}^2_{\max} + 2\lambda_1^{-2}\right).$$

Indeed, we have

$$\mathcal{I}_1(u_1,v_1) - \mathcal{I}_1(u_2,v_2) = A_1 - A_2 - A_3, \text{ with} \tag{10}$$

$$\begin{aligned}A_1 &= \left(e^{\lambda_{p_*}\int_t^T \mathcal{D}_1(u_1,v_1)(\eta)\mathrm{d}\eta} - e^{\lambda_{p_*}\int_t^T \mathcal{D}_1(u_2,v_2)(\eta)\mathrm{d}\eta}\right)\frac{\mu_{p_*}(x)}{\lambda_{p_*}},\\ A_2 &= \sum_{p=1}^{\infty}\int_t^T\left(e^{\lambda_p\left(\int_t^\eta \mathcal{D}_1(u_1,v_1)\mathrm{d}\xi\right)} - e^{\lambda_p\left(\int_t^\eta \mathcal{D}_1(u_2,v_2)\mathrm{d}\xi\right)}\right)\frac{e^{-TM\lambda_p}\left(e^{-|u_{1p}(\eta)|}+e^{-|v_{1p}(\eta)|}\right)}{\lambda_p}\mathrm{d}\eta\mu_p(x),\\ A_3 &= \sum_{p=1}^{\infty}\int_t^T e^{\lambda_p\left(\int_t^\eta \mathcal{D}_1(u_2,v_2)(\xi)\mathrm{d}\xi - TM\right)}\frac{e^{-|u_{1p}(\eta)|}-e^{-|u_{2p}(\eta)|}+e^{-|v_{1p}(\eta)|}-e^{-|v_{2p}(\eta)|}}{\lambda_p}\mathrm{d}\eta\mu_p(x).\end{aligned}$$

First, A_1 is estimated as follows

$$\begin{aligned}\|A_1\|^2_{L^2(\Omega)} &= \frac{1}{\lambda^2_{p_*}}\left|\exp\left(\lambda_{p_*}\int_t^T \mathcal{D}_1(u_1,v_1)(\eta)\mathrm{d}\eta\right) - \exp\left(\lambda_{p_*}\int_t^T \mathcal{D}_1(u_2,v_2)(\eta)\mathrm{d}\eta\right)\right|^2\\ &\le \frac{1}{\lambda^2_{p_*}}e^{2\lambda_{p_*}TM}T\lambda^2_{p_*}\int_t^T\left|\mathcal{D}_1(u_1,v_1)(\eta)-\mathcal{D}_1(u_2,v_2)(\eta)\right|^2\mathrm{d}\eta\\ &\le e^{2\lambda_{p_*}TM}T\mathcal{L}^2_{\max}\int_t^T\left\|\left((u_1,v_1)-(u_2,v_2)\right)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta,\end{aligned} \tag{11}$$

where we have used hypotheses $(A_3),(A_4)$, Parseval's relation, Hölder's inequality, and the inequality $|e^y - e^z| \le \max\{e^y,e^z\}|y-z|$. In a similar way, we obtain estimation for A_2

$$\begin{aligned}\|A_2\|^2_{L^2(\Omega)} &\le \frac{4T^2}{\lambda_1^2}\sum_{p=1}^{\infty}\int_t^T\int_t^\eta\Big(\mathcal{D}_1(u_1,v_1)-\mathcal{D}_1(u_2,v_2)\Big)^2\mathrm{d}\xi\mathrm{d}\eta\\ &\le \frac{4T^3\mathcal{L}^2_{\max}}{\lambda_1^2}\int_t^T\left\|\left((u_1,v_1)-(u_2,v_2)\right)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta.\end{aligned} \tag{12}$$

For A_3, applying Parseval's relation and Hölder's inequality we obtain

$$\begin{aligned}\|A_3\|^2_{L^2(\Omega)} &\le \frac{T}{\lambda_1^2}\sum_{p=1}^{\infty}\int_t^T \left(e^{-|u_{1p}(\eta)|}-e^{-|u_{2p}(\eta)|}+e^{-|v_{1p}(\eta)|}-e^{-|v_{2p}(\eta)|}\right)^2 \mathrm{d}\eta \\ &\le \frac{2T}{\lambda_1^2}\sum_{p=1}^{\infty}\int_t^T \left(e^{-|u_{1p}(\eta)|}-e^{-|u_{2p}(\eta)|}\right)^2+\left(e^{-|v_{1p}(\eta)|}-e^{-|v_{2p}(\eta)|}\right)^2 \mathrm{d}\eta \\ &\le \frac{2T}{\lambda_1^2}\sum_{p=1}^{\infty}\int_t^T \left(|u_{1p}(\eta)-u_{2p}(\eta)|^2+|v_{1p}(\eta)-v_{2p}(\eta)|^2\right)\mathrm{d}\eta \\ &\le \frac{2T}{\lambda_1^2}\int_t^T \left\|\left((u_1,v_1)-(u_2,v_2)\right)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta.\end{aligned} \tag{13}$$

Combining (10)–(13), we have

$$\begin{aligned}\left\|\left(\mathcal{I}_1(u_1,v_1)-\mathcal{I}_1(u_2,v_2)\right)(\cdot,t)\right\|^2_{L^2(\Omega)} &\le 3\|A_1\|^2_{L^2(\Omega)}+3\|A_2\|^2_{L^2(\Omega)}+3\|A_3\|^2_{L^2(\Omega)} \\ &\le \frac{\mathcal{C}_0}{4}\int_t^T \left\|\left((u_1,v_1)-(u_2,v_2)\right)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta.\end{aligned} \tag{14}$$

With a similar proof for $\|\left(\mathcal{I}_2(u_1,v_1)-\mathcal{I}_2(u_2,v_2)\right)(\cdot,t)\|^2_{L^2(\Omega)}$, we can deduce that

$$\begin{aligned}&\left\|\left(\mathbf{I}(u_1,v_1)-\mathbf{I}(u_2,v_2)\right)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \\ &\le 2\left\|\left(\mathcal{I}_1(u_1,v_1)-\mathcal{I}_1(u_2,v_2)\right)(\cdot,t)\right\|^2_{L^2(\Omega)}+2\left\|\left(\mathcal{I}_2(u_1,v_1)-\mathcal{I}_2(u_2,v_2)\right)(\cdot,t)\right\|^2_{L^2(\Omega)} \\ &\le \mathcal{C}_0(T-t)\left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{[C[0,T];L^2(\Omega)]^2},\end{aligned}$$

which means that (9) holds for $k=1$.

Suppose that (9) holds for $k=N$. From (14), one has

$$\begin{aligned}&\left\|\left(\mathcal{I}_1^{N+1}(u_1,v_1)-\mathcal{I}_1^{N+1}(u_2,v_2)\right)(\cdot,t)\right\|^2_{L^2(\Omega)} \\ &\le \frac{\mathcal{C}_0}{4}\int_t^T \left\|\left(\mathbf{I}^N(u_1,v_1)-\mathbf{I}^N(u_2,v_2)\right)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta \\ &\le \frac{\mathcal{C}_0^{N+1}}{4}\int_t^T \frac{(T-\eta)^N}{N!}\mathrm{d}\eta\left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{[C[0,T];L^2(\Omega)]^2} \\ &\le \frac{\mathcal{C}_0^{N+1}}{4}\frac{(T-t)^{N+1}}{(N+1)!}\left\|(u_1,v_1)-(u_2,v_2)\right\|^2_{[C[0,T];L^2(\Omega)]^2}.\end{aligned} \tag{15}$$

By a similar argument, we obtain the estimation for $\mathcal{I}_2^{N+1}$, combining with induction principle, we deduce that (9) holds for all $k\in\mathbb{N}$. Notice that $C_0^k\frac{(T-t)^k}{k!}$ tends to 0 when $k\to\infty$, we can choose k, such that $\mathbf{I}^k$ is a contraction mapping. By Banach fixed point theorem, we conclude that $\exists!(u^*,v^*):\mathbf{I}^k(u^*,v^*)=(u^*,v^*)$. Taking $\mathbf{I}$ operator both sides, it gives

$$\mathbf{I}^{k+1}(u^*,v^*)=\mathbf{I}(u^*,v^*)=\mathbf{I}^k(\mathbf{I}(u^*,v^*)).$$

Due to the uniqueness of the fixed point of the operator $\mathbf{I}^k$, we have $\mathbf{I}(u^*,v^*)=(u^*,v^*)$. Consequently, the problem (1) with the noisy data Ψ^{p_*}, Φ^{p_*} has a unique weak solution $(\widetilde{u^{p_*}},\widetilde{v^{p_*}})$. It can be straightforwardly proved that the system with the exact data also possesses a unique weak solution $(u_{\text{ex}},v_{\text{ex}})$.

Next, we consider the difference between the exact solution and the perturbed one

$$\begin{aligned}&\widetilde{u^{p_*}}-u_{\text{ex}}=B_1-B_2-B_3,\quad \text{with} \\ &B_1=\frac{e^{\lambda_{p_*}\int_t^T\mathcal{D}_1(\widetilde{u^{p_*}},\widetilde{v^{p_*}})\mathrm{d}\eta}}{\lambda_{p_*}}\mu_{p_*} \\ &B_2=\sum_{p=1}^{\infty}\int_t^T\left(e^{\lambda_p\int_t^\eta\mathcal{D}_1(\widetilde{u^{p_*}},\widetilde{v^{p_*}})\mathrm{d}\xi}-e^{\lambda_p\int_t^\eta\mathcal{D}_1(u_{\text{ex}},v_{\text{ex}})\mathrm{d}\xi}\right)\frac{e^{-\lambda_pTM}\left(e^{-|\widetilde{u_p^{p_*}}(\eta)|}+e^{-|\widetilde{v_p^{p_*}}(\eta)|}\right)}{\lambda_p}\mathrm{d}\eta\mu_p(x) \\ &B_3=\sum_{p=1}^{\infty}\int_t^T e^{\lambda_p\left(\int_t^\eta\mathcal{D}_1(u_{\text{ex}},v_{\text{ex}})\mathrm{d}\xi-TM\right)}\frac{e^{-|\widetilde{u_p^{p_*}}(\eta)|}-e^{-|u_{\text{ex},p}(\eta)|}+e^{-|\widetilde{v_p^{p_*}}(\eta)|}-e^{-|v_{\text{ex},p}(\eta)|}}{\lambda_p}\mathrm{d}\eta\mu_p(x)\end{aligned}$$

Using hypothesis (A_2), we find that

$$\|B_1\|_{L^2(\Omega)} \geq \frac{e^{m\lambda_{p_*}(T-t)}}{\lambda_{p_*}}. \tag{16}$$

Applying Parseval's relation, Hölder's inequality, and $|e^y - e^z| \leq \max\{e^y, e^z\}|y-z|$, we get

$$\begin{aligned}\|B_2\|^2_{L^2(\Omega)} &\leq 4T^2 \sum_{p=1}^{\infty} \int_t^T \int_t^\eta \left(\mathcal{D}_1(\widetilde{u^{p_*}}, \widetilde{v^{p_*}})(\xi) - \mathcal{D}_1(u_{\rm ex}, v_{\rm ex})(\xi) \right)^2 d\xi\, d\eta \\ &\leq 4T^4 \mathcal{L}^2_{\max} \left\| (\widetilde{u^{p_*}}, \widetilde{v^{p_*}}) - (u_{\rm ex}, v_{\rm ex}) \right\|^2_{[C([0,T];L^2(\Omega))]^2}.\end{aligned} \tag{17}$$

Next, we estimate B_3 as follows

$$\begin{aligned}\|B_3\|^2_{L^2(\Omega)} &\leq \frac{T}{\lambda_1^2} \sum_{p=1}^{\infty} \int_t^T \left(e^{-|\widetilde{u_p^{p_*}}(\eta)|} - e^{-|u_{{\rm ex},p}(\eta)|} + e^{-|\widetilde{v_p^{p_*}}(\eta)|} - e^{-|v_{{\rm ex},p}(\eta)|} \right)^2 d\eta \\ &\leq \frac{2T}{\lambda_1^2} \sum_{p=1}^{\infty} \int_t^T \left(|\widetilde{u_p^{p_*}}(\eta) - u_{{\rm ex},p}(\eta)|^2 + |\widetilde{v_p^{p_*}}(\eta) - v_{{\rm ex},p}(\eta)|^2 \right) d\eta \\ &\leq \frac{2T^2}{\lambda_1^2} \left\| (\widetilde{u^{p_*}}, \widetilde{v^{p_*}}) - (u_{\rm ex}, v_{\rm ex}) \right\|^2_{[C([0,T];L^2(\Omega))]^2}.\end{aligned} \tag{18}$$

Thus, by combining (16)–(18), we obtain

$$\begin{aligned}&\| \left(\widetilde{u^{p_*}} - u_{\rm ex}\right)(\cdot, t)\|_{L^2(\Omega)} \\ &\geq \|B_1\|_{L^2(\Omega)} - \|B_2\|_{L^2(\Omega)} - \|B_3\|_{L^2(\Omega)} \\ &\geq \frac{e^{m\lambda_{p_*}(T-t)}}{\lambda_{p_*}} - \left(2T^2 \mathcal{L}_{\max} + \frac{\sqrt{2}T}{\lambda_1} \right) \|(\widetilde{u^{p_*}}, \widetilde{v^{p_*}}) - (u_{\rm ex}, v_{\rm ex})\|_{[C([0,T];L^2(\Omega))]^2}.\end{aligned} \tag{19}$$

By the same argument for $\|(\widetilde{v^{p_*}} - v_{\rm ex})(\cdot, t)\|_{L^2(\Omega)}$, and combination with (19), we deduce that

$$\begin{aligned}&\|(\widetilde{u^{p_*}}, \widetilde{v^{p_*}}) - (u_{\rm ex}, v_{\rm ex})\|_{[C([0,T];L^2(\Omega))]^2} \\ &\geq \frac{2e^{m\lambda_{p_*}T}}{\lambda_{p_*}} - \left(4T^2\mathcal{L}_{\max} + 2\sqrt{2}T\lambda_1^{-1}\right) \left\| (\widetilde{u^{p_*}}, \widetilde{v^{p_*}}) - (u_{\rm ex}, v_{\rm ex}) \right\|_{[C([0,T];L^2(\Omega))]^2}.\end{aligned}$$

This implies that

$$\|(\widetilde{u^{p_*}}, \widetilde{v^{p_*}}) - (u_{\rm ex}, v_{\rm ex})\|_{[C([0,T];L^2(\Omega))]^2} \geq \frac{2e^{m\lambda_{p_*}T}}{\lambda_{p_*}\left(1 + 4T^2\mathcal{L}_{\max} + 2\sqrt{2}T\lambda_1^{-1}\right)} \to \infty,$$

when $p_* \to \infty$. The proof is completed. □

4. Quasi-reversibility (QR) regularization

Since the instability of the system (1) has been shown, it is now worth constructing a stable regularized solution. In this work, we consider the problem with the assumption (A_6) that F and G are locally Lipschitz functions with the coefficients K_F^R and K_G^R, respectively, i.e.,

$$|F(x,t,u_1,v_1) - F(x,t,u_2,v_2)| \leq K_F^R(|u_1 - u_2| + |v_1 - v_2|), \tag{20}$$

$$|G(x,t,u_1,v_1) - F(x,t,u_2,v_2)| \leq K_G^R(|u_1 - u_2| + |v_1 - v_2|), \tag{21}$$

for all $(x,t) \in Q_T$, $u_i, v_i \in \mathbb{R}$, $i = 1, 2 : |u_i| + |v_i| \leq R$. Here, K_F^R and K_G^R only depend on R.

Notice that these Lipschitz coefficients tend to ∞, when $R \to \infty$, we cannot give the error estimate for the solutions with noisy data, and standard regularization techniques are thus not applicable. To overcome this issue, instead of the original functions F and G, we deal with the approximate sources F_{R^ε}, G_{R^ε}, which are given in the next section.

4.1. Approximation of the locally Lipschitz reaction terms

We employ two sequences of globally Lipschitz functions F_{R^ε} and G_{R^ε} to approximate the locally Lipschitz functions F and G, as follows

$$F_{R^\varepsilon}(x,t,u,v) = \begin{cases} F(x,t,u,v), & \text{if } |u| + |v| \leq R^\varepsilon, \\ F\left(x, t, \dfrac{R^\varepsilon u}{|u|+|v|}, \dfrac{R^\varepsilon v}{|u|+|v|}\right), & \text{if } |u| + |v| > R^\varepsilon, \end{cases} \tag{22}$$

$$G_{R^\varepsilon}(x,t,u,v)=\begin{cases} G(x,t,u,v), & \text{if } |u|+|v|\le R^\varepsilon,\\ G\left(x,t,\dfrac{R^\varepsilon u}{|u|+|v|},\dfrac{R^\varepsilon v}{|u|+|v|}\right), & \text{if } |u|+|v|>R^\varepsilon.\end{cases} \tag{23}$$

Here, $R^\varepsilon(\varepsilon)\to\infty$ when $\varepsilon\to 0$, and will be chosen later in Section 5 to obtain the convergence of the regularized solution. The next lemma shows the globally Lipschitz property of F_{R^ε} and G_{R^ε}.

Lemma 4.1. *Let F_{R^ε} and G_{R^ε} be given in (22), (23). Then, they are globally Lipschitz functions with respect to u and v, i.e., for all $(x,t)\in\overline{Q}_T$, $u_i, v_i\in\mathbb{R}$, $i=1,2$, we have*

$$|F_{R^\varepsilon}(x,t,u_1,v_1)-F_{R^\varepsilon}(x,t,u_2,v_2)|\le 2K_F^{R^\varepsilon}(|u_1-u_2|+|v_1-v_2|),$$
$$|G_{R^\varepsilon}(x,t,u_1,v_1)-G_{R^\varepsilon}(x,t,u_2,v_2)|\le 2K_G^{R^\varepsilon}(|u_1-u_2|+|v_1-v_2|).$$

Proof of Lemma 4.1. Since the similarity between F_{R^ε} and G_{R^ε}, we just consider F_{R^ε}.

Case 1. $|u_1|+|v_1|\le R^\varepsilon$, $|u_2|+|v_2|\le R^\varepsilon$. Thanks to (20), we have

$$\begin{aligned}|F_{R^\varepsilon}(x,t,u_1,v_1)-F_{R^\varepsilon}(x,t,u_2,v_2)|&=|F(x,t,u_1,v_1)-F(x,t,u_2,v_2)|\\ &\le K_F^{R^\varepsilon}(|u_1-u_2|+|v_1-v_2|).\end{aligned}$$

Case 2. $|u_1|+|v_1|\le R^\varepsilon$, $|u_2|+|v_2|>R^\varepsilon$, (the same proof is used for the case $|u_1|+|v_1|>R^\varepsilon$, $|u_2|+|v_2|\le R^\varepsilon$).

$$\begin{aligned}|F_{R^\varepsilon}(x,t,u_1,v_1)-F_{R^\varepsilon}(\cdot,t,u_2,v_2)|&=\left|F(x,t,u_1,v_1)-F\left(x,t,\frac{R^\varepsilon u_2}{|u_2|+|v_2|},\frac{R^\varepsilon v_2}{|u_2|+|v_2|}\right)\right|\\ &\le K_F^{R^\varepsilon}\left(\left|u_1-\frac{R^\varepsilon u_2}{|u_2|+|v_2|}\right|+\left|v_1-\frac{R^\varepsilon v_2}{|u_2|+|v_2|}\right|\right)\\ &\le K_F^{R^\varepsilon}\left[|u_1-u_2|+|v_1-v_2|+(|u_2|+|v_2|)\left(1-\frac{R^\varepsilon}{|u_2|+|v_2|}\right)\right]\\ &\le K_F^{R^\varepsilon}\,(|u_1-u_2|+|v_1-v_2|+|u_2|+|v_2|-|u_1|-|v_1|)\\ &\le 2K_F^{R^\varepsilon}(|u_1-u_2|+|v_1-v_2|).\end{aligned}$$

Case 3. $|u_1|+|v_1|>R^\varepsilon$, $|u_2|+|v_2|>R^\varepsilon$

$$\begin{aligned}&|F_{R^\varepsilon}(x,t,u_1,v_1)-F_{R^\varepsilon}(\cdot,t,u_2,v_2)|\\ &=\left|F\left(x,t,\frac{R^\varepsilon u_1}{|u_1|+|v_1|},\frac{R^\varepsilon v_1}{|u_1|+|v_1|}\right)-F\left(x,t,\frac{R^\varepsilon u_2}{|u_2|+|v_2|},\frac{R^\varepsilon v_2}{|u_2|+|v_2|}\right)\right|\\ &\le K_F^{R^\varepsilon}\left(\left|\frac{R^\varepsilon u_1}{|u_1|+|v_1|}-\frac{R^\varepsilon u_2}{|u_2|+|v_2|}\right|+\left|\frac{R^\varepsilon v_1}{|u_1|+|v_1|}-\frac{R^\varepsilon v_2}{|u_2|+|v_2|}\right|\right)\\ &\le K_F^{R^\varepsilon}\left[\frac{R^\varepsilon(|u_1-u_2|+|v_1-v_2|)}{|u_1|+|v_1|}+\frac{R^\varepsilon(|u_2|+|v_2|)}{|u_1|+|v_1|}-R^\varepsilon\right]\\ &\le 2K_F^{R^\varepsilon}(|u_1-u_2|+|v_1-v_2|).\end{aligned}$$

The proof of the lemma is completed.

It is easy to see that F_{R^ε}, G_{R^ε} satisfy

$$\|F_{R^\varepsilon}(\cdot,t,u_1,v_1)-F_{R^\varepsilon}(\cdot,t,u_2,v_2)\|_{L^2(\Omega)}\le\sqrt{8}K_F^{R^\varepsilon}\,\|((u_1,v_1)-(u_2,v_2))(\cdot,t)\|_{[L^2(\Omega)]^2},$$
$$\|G_{R^\varepsilon}(\cdot,t,u_1,v_1)-G_{R^\varepsilon}(\cdot,t,u_2,v_2)\|_{L^2(\Omega)}\le\sqrt{8}K_G^{R^\varepsilon}\,\|((u_1,v_1)-(u_2,v_2))(\cdot,t)\|_{[L^2(\Omega)]^2}.$$

Throughout this paper, denote

$$\max\left\{\sqrt{8}K_F^{R^\varepsilon},\sqrt{8}K_G^{R^\varepsilon}\right\}=K_{R^\varepsilon}.$$

Next, we establish a well-posed approximate system by using a modified QR method.

4.2. The existence and uniqueness of QR regularized solution

Consider the following system

$$\partial_t U^\varepsilon_{\text{reg}}=\mathcal{D}_1(U^\varepsilon_{\text{reg}},V^\varepsilon_{\text{reg}})(t)\Delta U^\varepsilon_{\text{reg}}+\widetilde{\mathbf{Q}}^\alpha_\varepsilon(U^\varepsilon_{\text{reg}})+F_{R^\varepsilon}(x,t,U^\varepsilon_{\text{reg}},V^\varepsilon_{\text{reg}}), \tag{24}$$
$$\partial_t V^\varepsilon_{\text{reg}}=\mathcal{D}_2(U^\varepsilon_{\text{reg}},V^\varepsilon_{\text{reg}})(t)\Delta V^\varepsilon_{\text{reg}}+\widetilde{\mathbf{Q}}^\alpha_\varepsilon(V^\varepsilon_{\text{reg}})+G_{R^\varepsilon}(x,t,U^\varepsilon_{\text{reg}},V^\varepsilon_{\text{reg}}), \tag{25}$$

accompanied with the final conditions

$$U^{\varepsilon}_{\text{reg}}(x,T) = \Psi^{\varepsilon}(x), \quad V^{\varepsilon}_{\text{reg}}(x,T) = \Phi^{\varepsilon}(x), \quad x \in \Omega, \tag{26}$$

and the Neumann boundary condition (3).

Here, $\alpha := \alpha(\varepsilon) > 0$ is the regularization parameter, satisfying $\alpha(\varepsilon) \to 0$ when $\varepsilon \to 0$, and will be chosen later. $\widetilde{\mathbf{Q}}^{\alpha}_{\varepsilon}$ is an operator given by

$$\widetilde{\mathbf{Q}}^{\alpha}_{\varepsilon}(w) = \sum_{p=0}^{\infty} \frac{\log\left(1 + \alpha e^{TM_0\lambda_p}\right)}{T} \langle w, \mu_p\rangle \mu_p(x), \tag{27}$$

where M_0 is a positive constant, satisfying $M_0 > M$, with M given in (A_2). This operator is also introduced in [31]. The idea of this method is that by adding an appropriate small term to the system, we transform our unbounded Laplacian operator into a bounded operator, which guarantees the stability of regularized solution. This operator depends on a positive regularization parameter $\alpha = \alpha(\varepsilon) \to 0$, and will be chosen later to obtain the convergence of the solution. In the case that the source terms F and G depend only on x and t, it is quite simple to get a clue, by using the classical QR method. However, when the problem has nonlinear reactions, calculating the eigenvalues are much more complex, as we can see in the form of $\widetilde{\mathbf{Q}}^{\alpha}_{\varepsilon}$. We now establish the existence of the weak solution to the problem (24)–(26) in the following theorem. As mentioned before, the proof follows the well-known Faedo–Galerkin method and Aubin–Lions lemma. At this stage, we may see that the globally Lipschitz property of $F_{R^{\varepsilon}}, G_{R^{\varepsilon}}$ is very handy.

Theorem 4.2. Suppose that $(\mathrm{A}_1) - (\mathrm{A}_4)$, (A_6) hold. Then, the problem (24)–(26) has a solution

$$(U^{\varepsilon}_{\text{reg}}, U^{\varepsilon}_{\text{reg}}) \in \left[C([0,T]; L^2(\Omega)) \cap L^2(0,T;\mathbb{U})\right]^2,$$

in the weak sense, i.e., for all $\varphi, \psi \in \mathbb{U}$, yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle U^{\varepsilon}_{\text{reg}}(\cdot,t), \varphi\rangle + \mathcal{D}_1(U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}})(t)\langle \nabla U^{\varepsilon}_{\text{reg}}(\cdot,t), \nabla\varphi\rangle \\
&\qquad = \left\langle \widetilde{\mathbf{Q}}^{\alpha}_{\varepsilon}(U^{\varepsilon}_{\text{reg}})(\cdot,t), \varphi\right\rangle + \left\langle F_{R^{\varepsilon}}(\cdot,t,U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}), \varphi\right\rangle, \qquad (28)\\
&\frac{\mathrm{d}}{\mathrm{d}t}\langle V^{\varepsilon}_{\text{reg}}(\cdot,t), \psi\rangle + \mathcal{D}_2\left(U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}\right)(t)\left\langle \nabla V^{\varepsilon}_{\text{reg}}(\cdot,t), \nabla\psi\right\rangle \\
&\qquad = \left\langle \widetilde{\mathbf{Q}}^{\alpha}_{\varepsilon}(V^{\varepsilon}_{\text{reg}})(\cdot,t), \psi\right\rangle + \left\langle G_{R^{\varepsilon}}(\cdot,t,U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}), \psi\right\rangle. \qquad (29)
\end{aligned}$$

Proof. Let us define the following operators

$$\begin{aligned}
&\widetilde{\mathbf{P}}^{\alpha}_{\varepsilon} w = \widetilde{\mathbf{Q}}^{\alpha}_{\varepsilon} w + M_0 \Delta w,\\
&\mathcal{B}_1(u,v)(t) = M_0 - \mathcal{D}_1(u,v)(t),\\
&\mathcal{B}_2(u,v)(t) = M_0 - \mathcal{D}_2(u,v)(t).
\end{aligned}$$

Notice that from $(\mathrm{A}_2) - (\mathrm{A}_4)$, for all $t \in [0,T]$, $(u,v), (u_1,v_1), (u_2,v_2) \in [L^2(\Omega)]^2$, we have

$$\begin{aligned}
0 < M_0 - M \le \mathcal{B}_1(u,v)(t) &\le M_0 - m,\\
0 < M_0 - M \le \mathcal{B}_2(u,v)(t) &\le M_0 - m,\\
\left|\mathcal{B}_i(u_1,v_1)(t) - \mathcal{B}_i(u_2,v_2)(t)\right| &= \left|\mathcal{D}_i(u_1,v_1) - \mathcal{D}_i(u_2,v_2)\right|\\
&\le \mathcal{L}_{\max}\left\|((u_1,v_1)-(u_2,v_2))(\cdot,t)\right\|_{[L^2(\Omega)]^2}.
\end{aligned}$$

The system (28)–(29) can be rewritten as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle U^{\varepsilon}_{\text{reg}}(\cdot,t), \varphi\rangle - \mathcal{B}_1(U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}})(t)\langle \nabla U^{\varepsilon}_{\text{reg}}(\cdot,t), \nabla\varphi\rangle \qquad (30)\\
&\qquad = \left\langle \widetilde{\mathbf{P}}^{\alpha}_{\varepsilon}(U^{\varepsilon}_{\text{reg}})(\cdot,t), \varphi\right\rangle + \langle F_{R^{\varepsilon}}(\cdot,t,U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}), \varphi\rangle,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\langle V^{\varepsilon}_{\text{reg}}(\cdot,t), \psi\rangle - \mathcal{B}_2(U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}})(t)\langle \nabla V^{\varepsilon}_{\text{reg}}(\cdot,t), \nabla\psi\rangle \qquad (31)\\
&\qquad = \left\langle \widetilde{\mathbf{P}}^{\alpha}_{\varepsilon}(V^{\varepsilon}_{\text{reg}})(\cdot,t), \psi\right\rangle + \left\langle G_{R^{\varepsilon}}(\cdot,t,U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}), \psi\right\rangle.
\end{aligned}$$

To obtain the boundedness of the regularized operator, the following technical lemma plays the key role.

Lemma 4.3.

1. *For any $w \in G_{\Upsilon}(\Omega)$, $\Upsilon \ge 2M_0T$, it yields*

$$\left\|\widetilde{\mathbf{Q}}^{\alpha}_{\varepsilon}(w)\right\|_{L^2(\Omega)} \le \frac{\alpha}{T}\|w\|_{G_{\Upsilon}(\Omega)}.$$

2. *For any $w \in L^2(\Omega)$, it yields*

$$\left\|\widetilde{\mathbf{P}}_\varepsilon^\alpha(w)\right\|_{L^2(\Omega)} \leq \frac{1}{T}\log\left(\frac{1}{\alpha}\right)\|w\|_{L^2(\Omega)}.$$

Proof of Lemma 4.3. Using Parseval's equality and the inequality $\log(1+a) \leq a, \forall a > 0$, we have

$$\begin{aligned}\left\|\widetilde{\mathbf{Q}}_\varepsilon^\alpha(w)\right\|_{L^2(\Omega)}^2 &= \frac{1}{T^2}\sum_{p=0}^{\infty}\log^2\left(1+\alpha e^{TM_0\lambda_p}\right)\langle w, \mu_p\rangle^2 \\ &\leq \frac{\alpha^2}{T^2}\sum_{p=0}^{\infty} e^{2TM_0\lambda_p}\langle w, \mu_p\rangle^2 \leq \frac{\alpha^2}{T^2}\|w\|_{G_\Upsilon(\Omega)}^2.\end{aligned}$$

For the second statement, using Parseval's equality, we have

$$\begin{aligned}\left\|\widetilde{\mathbf{P}}_\varepsilon^\alpha(w)\right\|_{L^2(\Omega)}^2 &= \frac{1}{T^2}\sum_{p=0}^{\infty}\left[\log\left(1+\alpha e^{TM_0\lambda_p}\right) - \log\left(e^{TM_0\lambda_p}\right)\right]^2\langle w, \mu_p\rangle^2 \\ &= \frac{1}{T^2}\sum_{p=0}^{\infty}\left[\log\left(\alpha + e^{-TM_0\lambda_p}\right)\right]^2\langle w, \mu_p\rangle^2 \leq \frac{1}{T^2}\log^2\left(\frac{1}{\alpha}\right)\|w\|_{L^2(\Omega)}^2.\end{aligned}$$

This ends the proof of Lemma 4.3.

Let us now provide the proof of Theorem 4.2. We divide the proof of the theorem into three steps.

Step 1. *Existence of the Galerkin approximate solution*

We begin with the construction of a sequence of weak approximate solutions by using Galerkin method: solving the projected problem in a finite dimensional subspace of $\mathbb{U}$. Consider the correspondingly $(n+1)-$ dimensional space $\mathbb{U}_{n+1} = \text{span}\,\langle\mu_0, \mu_1, \dots, \mu_n\rangle$. For each n, we search for an approximate solution (U^n, V^n) in the following form

$$U^n(x,t) = \sum_{p=0}^{n} U_{np}(t)\mu_p(x), \quad V^n(x,t) = \sum_{p=0}^{n} V_{np}(t)\mu_p(x),$$

where for all $\varphi, \psi \in \mathbb{U}_{n+1}$, the solution (U^n, V^n) satisfies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\langle U^n(\cdot,t), \varphi\rangle + \mathcal{B}_1\left(U^n, V^n\right)(t)\langle \Delta U^n(\cdot,t), \varphi\rangle \\ &\qquad = \left\langle \widetilde{\mathbf{P}}_\varepsilon^\alpha(U^n)(\cdot,t), \varphi\right\rangle + \langle F_{R^\varepsilon}(\cdot,t,U^n,V^n), \varphi\rangle,\end{aligned} \tag{32}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\langle V^n(\cdot,t), \psi\rangle + \mathcal{B}_2\left(U^n, V^n\right)(t)\langle \Delta V^n(\cdot,t), \psi\rangle \\ &\qquad = \left\langle \widetilde{\mathbf{P}}_\varepsilon^\alpha(V^n)(\cdot,t), \psi\right\rangle + \langle G_{R^\varepsilon}(\cdot,t,U^n,V^n), \psi\rangle,\end{aligned} \tag{33}$$

and the final conditions

$$U^n(x,T) = \sum_{p=0}^{n}\langle \Psi^\varepsilon, \mu_p\rangle\mu_p(x) = \Psi^n(x), \quad V^n(x,T) = \sum_{p=0}^{n}\langle \Phi^\varepsilon, \mu_p\rangle\mu_p(x) = \Phi^n(x). \tag{34}$$

Here, $\Psi^n \to \Psi^\varepsilon, \quad \Phi^n \to \Phi^\varepsilon$ strongly in $L^2(\Omega)$-norm. In another way, $U_{np}(t)$ and $V_{np}(t)$ are the solutions of the $2(n+1)$ non-linear ordinary differential equations

$$\begin{aligned}&\frac{\mathrm{d}U_{np}(t)}{\mathrm{d}t} - \lambda_p\mathcal{B}_1\left(U^n, V^n\right)U_{np}(t) = \left\langle \widetilde{\mathbf{P}}_\varepsilon^\alpha(U^n)(\cdot,t), \mu_p\right\rangle + \left\langle F_{R^\varepsilon}(\cdot,t,U^n,V^n), \mu_p\right\rangle, \\ &\frac{\mathrm{d}V_{np}(t)}{\mathrm{d}t} - \lambda_p\mathcal{B}_2\left(U^n, V^n\right)V_{np}(t) = \left\langle \widetilde{\mathbf{P}}_\varepsilon^\alpha(V^n)(\cdot,t), \mu_p\right\rangle + \left\langle G_{R^\varepsilon}(\cdot,t,U^n,V^n), \mu_p\right\rangle,\end{aligned}$$

for $p = \overline{0, n}$. Attending to the continuity of $\mathcal{D}_1, \mathcal{D}_2$ and of F, G (assumptions (A_1), (A_3), (A_4)), we can use Peano's theorem to have that the system (32)–(34) has a local solution (U^n, V^n) in some interval $[T_m, T]$ for $0 \leq T_m < T$. We now give a priori estimate for (U^n, V^n), to extend $[T_m, T]$ to the whole interval $[0, T]$.

In (32), taking $\varphi = U^n$, and then integrating from t to T, we have

$$\begin{aligned}&\|\Psi^n\|_{L^2(\Omega)}^2 - \|U^n(\cdot,t)\|_{L^2(\Omega)}^2 - 2\int_t^T \mathcal{B}_1\left(U^n, V^n\right)(\eta)\|\nabla U^n(\cdot,\eta)\|_{L^2(\Omega)}^2\mathrm{d}\eta \\ &= 2\underbrace{\int_t^T\left\langle \widetilde{\mathbf{P}}_\varepsilon^\alpha(U^n)(\cdot,\eta), U^n(\cdot,\eta)\right\rangle\mathrm{d}\eta}_{J_1} + 2\underbrace{\int_t^T\left\langle F_{R^\varepsilon}(\cdot,\eta,U^n,V^n), U^n(\cdot,\eta)\right\rangle\mathrm{d}\eta}_{J_2}.\end{aligned} \tag{35}$$

J_1 is estimated by applying Hölder's inequality and Lemma 4.3

$$J_1 \leq \int_t^T \left\|\widetilde{\mathbf{P}}_\varepsilon^\alpha(U^n)(\cdot,\eta)\right\|_{L^2(\Omega)}\left\|U^n(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \leq \frac{\log\left(\frac{1}{\alpha}\right)}{T}\int_t^T \left\|U^n(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta. \tag{36}$$

For J_2, using Hölder's inequality and the globally Lipschitz property of F_{R^ε}, it yields

$$\begin{aligned} J_2 &\leq \int_t^T \left\|F_{R^\varepsilon}(\cdot,\eta,U^n,V^n)\right\|_{L^2(\Omega)}\left\|U^n(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\leq \int_t^T \left(\left\|F_{R^\varepsilon}(\cdot,\eta,0,0)\right\|_{L^2(\Omega)} + K_{R^\varepsilon}\left\|(U^n,V^n)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\right)\left\|U^n(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\leq \frac{1}{2}\int_t^T \left\|F_{R^\varepsilon}(\cdot,\eta,0,0)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\int_t^T \left\|U^n(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta \\ &+ K_{R^\varepsilon}\int_t^T \left\|(U^n,V^n)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\left\|U^n(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta. \end{aligned} \tag{37}$$

Combining (35)–(37), gives

$$\begin{aligned} &\left\|U^n(\cdot,t)\right\|^2_{L^2(\Omega)} + 2(M_0-M)\int_t^T \left\|\nabla U^n(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta \\ &\leq C_1^* + C_2^*\int_t^T \left\|(U^n,V^n)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta. \end{aligned} \tag{38}$$

By a similar argument with V^n, adding the resulting inequality to (38), we have

$$\begin{aligned} &\left\|(U^n,V^n)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} + 2(M_0-M)\int_t^T \left\|(\nabla U^n,\nabla V^n)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta \\ &\leq C_1 + C_2\int_t^T \left\|(U^n,V^n)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta. \end{aligned} \tag{39}$$

Therefore, we obtain

$$\left\|(U^n,V^n)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \leq C_1 + C_2\int_t^T \left\|(U^n,V^n)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta.$$

Applying Gronwall's inequality, we arrive at

$$\left\|(U^n,V^n)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \leq C_1 e^{(T-t)C_2} \leq C_1 e^{TC_2}. \tag{40}$$

On the other hand, from (39) we have

$$\int_t^T \left\|(\nabla U^n,\nabla V^n)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta \leq \frac{\left(C_1 + C_2\int_t^T \left\|(U^n,V^n)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta\right)}{2(M_0-M)} \leq C_3. \tag{41}$$

By (40), (41), we deduce that

$$(U^n,V^n) \text{ are bounded in } \left[L^\infty(0,T;L^2(\Omega))\right]^2, \tag{42}$$

$$(U^n,V^n) \text{ are bounded in } \left[L^2(0,T;\mathbb{U})\right]^2. \tag{43}$$

Thus, from the Theory of the ODEs, we can extend the local solution to the interval $[0,T]$.

Remark 4.1. It is worth noting that if we use the operator $\widetilde{\mathbf{Q}}_\varepsilon^\alpha$ as in [31], from (35) we are not able to give the estimate of U^n, as well as V^n, in H^1 -norm, and hence, cannot apply the Aubin–Lions lemma. Therefore, we employ $\widetilde{\mathbf{P}}_\varepsilon^\alpha$, and some techniques have been used to modify the operator.

Step 2. *Convergence of the Galerkin approximate solutions to the QR solution*

The next step is to show that the sequence of Galerkin solutions converges to a function, which is the solution of our constructed QR problem, as $n\to\infty$. From (32) we have that

$$U_t^n = -\mathcal{B}_1\left(U^n,V^n\right)(t)\Delta U^n(x,t) + \widetilde{\mathbf{P}}_\varepsilon^\alpha(U^n)(x,t) + F_{R^\varepsilon}(x,t,U^n,V^n) \in H^{-1}.$$

Note that $-\mathcal{B}_1\left(U^n,V^n\right)(t)\Delta U^n(x,t)$ defines an element of H^{-1}, given by

$$\langle -\mathcal{B}_1\left(U^n,V^n\right)(t)\Delta U^n(\cdot,t),\varphi\rangle = \mathcal{B}_1\left(U^n,V^n\right)(t)\langle\nabla U^n(\cdot,t),\nabla\varphi\rangle,$$

for all $\varphi \in \mathbb{U}$. Thanks to (42), (43), Lemma 4.3, the globally Lipschitz property of $F_{R^\varepsilon}, G_{R^\varepsilon}, \mathcal{B}_i, (i=1,2)$, and the similarity between u, v, we obtain that

$$(U_t^n, V_t^n) \text{ are bounded in } \left[L^2(0,T;\mathbb{U}')\right]^2. \tag{44}$$

From (42)–(44), by the Banach–Alouglu theorem, we can extract subsequences $U_k^n = U^n, V_k^n = V^n$ (which are denoted by the same symbols) such that

$$U^n \rightharpoonup U_{\text{reg}}^\varepsilon, \quad V^n \rightharpoonup V_{\text{reg}}^\varepsilon \quad {}^*\text{-weakly in} \quad L^\infty(0,T;L^2(\Omega)), \tag{45}$$

$$U^n \rightharpoonup U_{\text{reg}}^\varepsilon, \quad V^n \rightharpoonup V_{\text{reg}}^\varepsilon \quad \text{weakly in} \quad L^2(0,T;\mathbb{U}), \tag{46}$$

$$U_t^n \rightharpoonup \partial_t U_{\text{reg}}^\varepsilon, \quad V_t^n \rightharpoonup \partial_t U_{\text{reg}}^\varepsilon \quad \text{weakly in } L^2(0,T;\mathbb{U}'). \tag{47}$$

On the other hand, $\mathbb{U} \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow \mathbb{U}'$. From (46), (47), using the Aubin–Lions compactness lemma, we have

$$U^n \to U_{\text{reg}}^\varepsilon, \quad V^n \to V_{\text{reg}}^\varepsilon \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{48}$$

Hence, by Riesz–Fischer's theorem, we can extract subsequences $U_k^n = U^n, V_k^n = V^n$ (which we denote by the same symbols), such that

$$U^n \to U_{\text{reg}}^\varepsilon, \quad V^n \to V_{\text{reg}}^\varepsilon \quad \text{a.e in } \overline{Q}_T. \tag{49}$$

Due to the continuity of $\mathcal{B}_i, i=1,2$, we have

$$\mathcal{B}_1(U^n,V^n) \to \mathcal{B}_1(U_{\text{reg}}^\varepsilon, V_{\text{reg}}^\varepsilon) \text{ strongly in } L^2(0,T),$$
$$\mathcal{B}_2(U^n,V^n) \to \mathcal{B}_2(U_{\text{reg}}^\varepsilon, V_{\text{reg}}^\varepsilon) \text{ strongly in } L^2(0,T).$$

Using the Riesz–Fischer theorem, we have up to some subsequences,

$$\mathcal{B}_1(U^n,V^n) \to \mathcal{B}_1(U_{\text{reg}}^\varepsilon, V_{\text{reg}}^\varepsilon) \text{ a.e in } \overline{Q}_T, \tag{50}$$

$$\mathcal{B}_2(U^n,V^n) \to \mathcal{B}_2(U_{\text{reg}}^\varepsilon, V_{\text{reg}}^\varepsilon) \text{ a.e in } \overline{Q}_T. \tag{51}$$

Due to the linearity and boundedness of $\widetilde{\mathbf{P}}_\varepsilon^\alpha$, we have

$$\widetilde{\mathbf{P}}_\varepsilon^\alpha(U^n) \to \widetilde{\mathbf{P}}_\varepsilon^\alpha(U_{\text{reg}}^\varepsilon) \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{52}$$

$$\widetilde{\mathbf{P}}_\varepsilon^\alpha(V^n) \to \widetilde{\mathbf{P}}_\varepsilon^\alpha(V_{\text{reg}}^\varepsilon) \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{53}$$

From the globally Lipschitz property of $F_{R^\varepsilon}, G_{R^\varepsilon}$, it follows

$$F_{R^\varepsilon}(x,t,U^n,V^n) \to F_{R^\varepsilon}(x,t,U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon) \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{54}$$

$$G_{R^\varepsilon}(x,t,U^n,V^n) \to G_{R^\varepsilon}(x,t,U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon) \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{55}$$

Combining (43), (44), (50)–(55), we can pass (32), (33) to the limit $n \to \infty$ to prove that (30), (31) hold in $D'(0,T)$ for all $\varphi, \psi \in \mathbb{U}$. By (46), we have that $U_{\text{reg}}^\varepsilon(t), V_{\text{reg}}^\varepsilon(t) \in \mathbb{U}$ for a.e. $t \in [0,T]$. Taking $\varphi = U_{\text{reg}}^\varepsilon$ in (30), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|U_{\text{reg}}^\varepsilon(\cdot,t)\right\|_{L^2(\Omega)}^2 - 2\mathcal{B}_1\left(U_{\text{reg}}^\varepsilon, V_{\text{reg}}^\varepsilon\right)(t)\left\|\nabla U_{\text{reg}}^\varepsilon(\cdot,t)\right\|_{L^2(\Omega)}^2\\ &= 2\left\langle \widetilde{\mathbf{P}}_\varepsilon^\alpha(U_{\text{reg}}^\varepsilon)(\cdot,t), U_{\text{reg}}^\varepsilon(\cdot,t)\right\rangle + \left\langle F_{R^\varepsilon}(\cdot,t,U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon), U_{\text{reg}}^\varepsilon(\cdot,t)\right\rangle,\end{aligned}$$

in $D'(0,T)$. Then by analogous arguments as for (U^n,V^n), but taking the supremum, we arrive at

$$U_{\text{reg}}^\varepsilon \text{ and } V_{\text{reg}}^\varepsilon \text{ are bounded in } C(0,T;L^2(\Omega)) \cap L^2(0,T;\mathbb{U}). \tag{56}$$

Therefore,

$$(U_{\text{reg}}^\varepsilon, V_{\text{reg}}^\varepsilon) \in \left[C(0,T;L^2(\Omega)) \cap L^2(0,T;\mathbb{U})\right]^2.$$

On the other hand, for a.e $t \in [0,T]$, we have

$$\left\langle \Psi^n, \varphi\right\rangle - \left\langle U^n(\cdot,t), \varphi\right\rangle = \int_t^T \left\langle \partial_\eta U^n(\cdot,\eta), \varphi\right\rangle \mathrm{d}\eta. \tag{57}$$

From (47), (48) and the fact that $\Psi^n \to \Psi^\varepsilon$ strongly in $L^2(\Omega)$, we can pass (57) to the limit $n \to \infty$ to obtain

$$\left\langle \Psi^\varepsilon, \varphi\right\rangle - \left\langle U_{\text{reg}}^\varepsilon(\cdot,t), \varphi\right\rangle = \int_t^T \left\langle \partial_\eta U_{\text{reg}}^\varepsilon(\cdot,\eta), \varphi\right\rangle \mathrm{d}\eta = \left\langle U_{\text{reg}}^\varepsilon(\cdot,T), \varphi\right\rangle - \left\langle U_{\text{reg}}^\varepsilon(\cdot,t), \varphi\right\rangle,$$

for a.e. $t \in [0,T]$. Thus, $U_{\text{reg}}^\varepsilon(x,T) = \Psi^\varepsilon(x)$. In a similar way, we have that $V_{\text{reg}}^\varepsilon(x,T) = \Phi^\varepsilon(x)$. This completes the proof of Step 2. It only remains to show that this QR solution is unique.

Step 3. *Uniqueness of the QR solution.*
Suppose that

$$(u_i, v_i) \in \Big[C(0, T; L^2(\Omega)) \cap L^2(0, T; \mathbb{U})\Big]^2,$$

$i = \overline{1, 2}$, are two weak solutions of the system (30), (31).
Define

$$\mathcal{U}(x, t) = (u_1 - u_2)(x, t), \quad \mathcal{V}(x, t) = (v_1 - v_2)(x, t).$$

It yields

$$\mathcal{U}(x, T) = \mathcal{V}(x, T) = 0.$$

From (30), we have

$$\begin{aligned}&\Big\langle \mathcal{U}_t(\cdot, t), \varphi \Big\rangle - \mathcal{B}_1(u_1, v_1) \Big\langle \nabla u_1(\cdot, t), \nabla\varphi \Big\rangle + \mathcal{B}_1(u_2, v_2) \Big\langle \nabla u_2(\cdot, t), \nabla\varphi \Big\rangle \\ &\quad = \Big\langle \widetilde{\mathbf{P}}^{\alpha}_{\varepsilon}(u_1)(\cdot, t) - \widetilde{\mathbf{P}}^{\alpha}_{\varepsilon}(u_2)(\cdot, t), \varphi \Big\rangle + \Big\langle F_{R^\varepsilon}(\cdot, t, u_1, v_1) - F_{R^\varepsilon}(\cdot, t, u_2, v_2), \varphi \Big\rangle.\end{aligned}$$

Consequently,

$$\begin{aligned}\Big\langle \mathcal{U}_t(\cdot, t), \varphi \Big\rangle - \mathcal{B}_1(u_1, v_1) \Big\langle \nabla\mathcal{U}(\cdot, t), \nabla\varphi \Big\rangle &= \Big(\mathcal{B}_1(u_1, v_1) - \mathcal{B}_1(u_2, v_2)\Big)\Big\langle \nabla u_2(\cdot, t), \nabla\varphi \Big\rangle \\ &\quad + \Big\langle \widetilde{\mathbf{P}}^{\alpha}_{\varepsilon}(\mathcal{U})(\cdot, t), \varphi \Big\rangle + \Big\langle F_{R^\varepsilon}(\cdot, t, u_1, v_1) - F_{R^\varepsilon}(\cdot, t, u_2, v_2), \varphi \Big\rangle. \end{aligned} \tag{58}$$

Taking $\varphi = \mathcal{U}(x, t)$, and then integrating from t to T, we obtain

$$\|\mathcal{U}(\cdot, t)\|^2_{L^2(\Omega)} + 2\int_t^T \mathcal{B}_1(u_1, v_1)(\eta)\|\nabla\mathcal{U}(\cdot, \eta)\|^2_{L^2(\Omega)}\mathrm{d}\eta = -(I_1 + I_2 + I_3), \tag{59}$$

$$I_1 = 2\int_t^T \Big(\mathcal{B}_1(u_1, v_1) - \mathcal{B}_1(u_2, v_2)\Big)(\eta)\Big\langle \nabla u_2(\cdot, \eta), \nabla\mathcal{U}(\cdot, \eta)\Big\rangle \mathrm{d}\eta,$$

$$I_2 = 2\int_t^T \Big\langle \widetilde{\mathbf{P}}^{\alpha}_{\varepsilon}(\mathcal{U})(\cdot, \eta), \mathcal{U}(\cdot, \eta)\Big\rangle \mathrm{d}\eta,$$

$$I_3 = 2\int_t^T \Big\langle F_{R^\varepsilon}(\cdot, \eta, u_1, v_1) - F_{R^\varepsilon}(\cdot, \eta, u_2, v_2), \mathcal{U}(\cdot, \eta)\Big\rangle \mathrm{d}\eta.$$

We first estimate I_1 by using Hölder's inequality, (A_3), (A_4), the inequality $\|y\|\|z\| \le c\|y\|^2 + \frac{1}{c}\|z\|^2$, it yields

$$\begin{aligned}|I_1| &\le 2\int_t^T \left|\mathcal{B}_1(u_1, v_1) - \mathcal{B}_1(u_2, v_2)\right| \left\|\nabla u_2(\cdot, \eta)\right\|_{L^2(\Omega)} \left\|\nabla\mathcal{U}(\cdot, \eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\le \frac{1}{2(M_0 - M)}\int_t^T \left|\mathcal{B}_1(u_1, v_1) - \mathcal{B}_1(u_2, v_2)\right|^2 \left\|\nabla u_2(\cdot, \eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta \\ &\quad + 2(M_0 - M)\int_t^T \left\|\nabla\mathcal{U}(\cdot, \eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta \\ &\le \frac{\mathcal{L}^2_{\max}}{2(M_0 - M)}\int_t^T \left\|(\mathcal{U}, \mathcal{V})(\cdot, \eta)\right\|^2_{[L^2(\Omega)]^2}\left\|\nabla u_2(\cdot, \eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta + 2(M_0 - M)\int_t^T \left\|\nabla\mathcal{U}(\cdot, \eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta.\end{aligned} \tag{60}$$

For I_2, using Lemma 4.3, we have

$$|I_2| \le \frac{2}{T}\log\left(\frac{1}{\alpha}\right)\int_t^T \left\|\mathcal{U}(\cdot, \eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta. \tag{61}$$

Next, we deduce estimates for I_3

$$\begin{aligned}|I_3| &\le 2\int_t^T \left\|F_{R^\varepsilon}(\cdot, \eta, u_1, v_1) - F_{R^\varepsilon}(\cdot, \eta, u_2, v_2)\right\|_{L^2(\Omega)}\left\|\mathcal{U}(\cdot, \eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\le 2K_{R^\varepsilon}\int_t^T \left\|(\mathcal{U}, \mathcal{V})(\cdot, \eta)\right\|_{[L^2(\Omega)]^2}\left\|\mathcal{U}(\cdot, \eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta.\end{aligned} \tag{62}$$

Combining (59)–(62), we obtain

$$\left\|\mathcal{U}(\cdot,t)\right\|^2_{L^2(\Omega)} \le \frac{\mathcal{L}^2_{\max}}{2(M_0-M)}\int_t^T \left\|(\mathcal{U},\mathcal{V})(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\left\|\nabla u_2(\cdot,\eta)\right\|^2_{L^2(\Omega)}d\eta$$
$$+\frac{2\log(\frac{1}{\alpha})}{T}\int_t^T \|\mathcal{U}(\cdot,\eta)\|^2_{L^2(\Omega)}d\eta + 2K_{R^\varepsilon}\int_t^T \left\|(\mathcal{U},\mathcal{V})(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\left\|\mathcal{U}(\cdot,\eta)\right\|_{L^2(\Omega)}d\eta.$$

In a similar way with $\mathcal{V}$, we arrive at

$$\left\|(\mathcal{U},\mathcal{V})(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \le 2\left\|\mathcal{U}(\cdot,t)\right\|^2_{L^2(\Omega)} + 2\left\|\mathcal{V}(\cdot,t)\right\|^2_{L^2(\Omega)} \le C_4\int_t^T \left\|(\mathcal{U},\mathcal{V})(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}d\eta.$$

Using Gronwall's inequality, we have

$$\left\|(\mathcal{U},\mathcal{V})(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \le 0,$$

which implies that $\mathcal{U}=\mathcal{V}=0$, or $(u_1,v_1)=(u_2,v_2)$. The proof of the theorem is completed. □

5. Error analysis

So far, in the previous section, we have constructed a modified QR problem and proved the existence and uniqueness of weak regularized solution. Hence, we are now in position to establish some error estimations in L^2 and H^1 norms. To complete the theoretical part, it is essential to show that these errors reach 0, as the noise level tends to 0.

5.1. L^2-estimate

Theorem 5.1. Suppose that $(A_1)-(A_6)$ hold. The solution of the system (1) satisfies

$$(u_{\text{ex}},v_{\text{ex}}) \in \left[\left(L^2(0,T;G_\Upsilon(\Omega))\cap L^\infty(0,T;H^1_0(\Omega))\cap C^1(0,T;L^2(\Omega))\right)\cap L^\infty(Q_T)\right]^2,$$

with $\Upsilon \ge 2M_0T$. Denote

$$E=\max\left\{\|(u_{\text{ex}},v_{\text{ex}})\|_{[L^2(0,T;G_\Upsilon(\Omega))]^2},\ \|(u_{\text{ex}},v_{\text{ex}})\|_{[C^1(0,T;L^2(\Omega))]^2},\ \|(u_{\text{ex}},v_{\text{ex}})\|_{[L^\infty(0,T;H^1_0(\Omega))]^2}\right\}.$$

Let us choose R^ε such that

$$K_{R^\varepsilon} \le \frac{1}{4T}\log\left(\log^\gamma\left(\frac{1}{\alpha}\right)\right), \tag{63}$$

for some $\gamma>0$. Then there exist $A_0=A_0(u_{\text{ex}},v_{\text{ex}})$, $B_0=B_0(u_{\text{ex}},v_{\text{ex}})$, for which the following estimation holds

$$\left\|\left((U^\varepsilon_{\text{reg}},V^\varepsilon_{\text{reg}})-(u_{\text{ex}},v_{\text{ex}})\right)(\cdot,t)\right\|_{[L^2(\Omega)]^2} \le \sqrt{A_0\frac{\varepsilon^2}{\alpha^2}+B_0}\,\alpha^{\frac{t}{T}}\sqrt{\log^\gamma\left(\frac{1}{\alpha}\right)}. \tag{64}$$

Remark 5.1. Let us choose the regularization parameter $\alpha(\varepsilon)=\varepsilon$. From (64) we imply the stability for $t\in(0,T]$. Moreover, there exists $t_\varepsilon\in(0,T):\lim_{\varepsilon\to0}t_\varepsilon=0$, such that

$$\left\|(U^\varepsilon_{\text{reg}},V^\varepsilon_{\text{reg}})(\cdot,t_\varepsilon)-(u_{\text{ex}},v_{\text{ex}})(\cdot,0)\right\|_{[L^2(\Omega)]^2} \le \left(C_0\sqrt{\log^\gamma\left(\frac{1}{\varepsilon}\right)}+2E\right)\sqrt{\frac{T}{\log(\frac{1}{\varepsilon})}}, \tag{65}$$

where $C_0=\sqrt{A_0+B_0}$. Notice that if we take $0<\gamma<1$, then the right hand side of (65) tends to 0, we have the stability at $t=0$.

Remark 5.2. In the previous theorem, it is assumed that u and v belong to $L^2(0,T;G_\Upsilon(\Omega))$, where the Gevrey space of functions $G_\Upsilon(\Omega)$ has been defined in (11). At this stage, there are unknown to us sufficient conditions on the data entering the problem given by Eqs. (4), (7) and (9) to ensure that the solution $(u,v)\in\left[L^2(0,T;G_\Upsilon(\Omega))\right]^2$, but we point out to Refs. [31,32] for some useful results on Gevrey regularity for parabolic equations.

Proof. Let us define

$$\mathcal{X}^\varepsilon(x,t)=e^{q_\alpha(t-T)}(U^\varepsilon_{\text{reg}}-u_{\text{ex}})(x,t),\quad \mathcal{Y}^\varepsilon(x,t)=e^{q_\alpha(t-T)}(V^\varepsilon_{\text{reg}}-v_{\text{ex}})(x,t),$$

where $q_\alpha > 0$ is a positive constant, which will be chosen later. From (7), (28), with some computations, we have

$$\begin{aligned}
&\left\langle \mathcal{X}_t^\varepsilon(\cdot,t),\varphi\right\rangle - \mathcal{B}_1\left(U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon\right)(t)\left\langle \nabla\mathcal{X}^\varepsilon(\cdot,t),\nabla\varphi\right\rangle - e^{q_\alpha(t-T)}\left\langle \widetilde{\mathbf{Q}_\varepsilon^\alpha}(u_{\text{ex}})(\cdot,t),\varphi\right\rangle \\
&+ e^{q_\alpha(t-T)}\left(\mathcal{D}_1\left(U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon\right)-\mathcal{D}_1\left(u_{\text{ex}},v_{\text{ex}}\right)\right)\left\langle \nabla u_{\text{ex}}(\cdot,t),\nabla\varphi\right\rangle - q_\alpha\left\langle \mathcal{X}^\varepsilon(\cdot,t),\varphi\right\rangle \\
&= \left\langle \widetilde{\mathbf{P}_\varepsilon^\alpha}(\mathcal{X}^\varepsilon)(\cdot,t),\varphi\right\rangle + e^{q_\alpha(t-T)}\left\langle F_{R^\varepsilon}(\cdot,t,U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon)-F(\cdot,t,u_{\text{ex}},v_{\text{ex}}),\varphi\right\rangle.
\end{aligned} \tag{66}$$

Taking $\varphi = \mathcal{X}^\varepsilon$, and integrating from t to T, it yields

$$\begin{aligned}
&\|\mathcal{X}^\varepsilon(\cdot,T)\|_{L^2(\Omega)}^2 - \|\mathcal{X}^\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2 - 2\int_t^T q_\alpha\|\mathcal{X}^\varepsilon(\cdot,\eta)\|_{L^2(\Omega)}^2 \mathrm{d}\eta \\
&- 2\int_t^T \mathcal{B}_1\left(U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon\right)\|\nabla\mathcal{X}^\varepsilon(\cdot,\eta)\|_{L^2(\Omega)}^2 \mathrm{d}\eta \\
&= \underbrace{2\int_t^T e^{q_\alpha(\eta-T)}\left\langle \widetilde{\mathbf{Q}_\varepsilon^\alpha}(u_{\text{ex}})(\cdot,\eta),\mathcal{X}^\varepsilon(\cdot,\eta)\right\rangle \mathrm{d}\eta}_{K_1} + \underbrace{2\int_t^T \left\langle \widetilde{\mathbf{P}_\varepsilon^\alpha}(\mathcal{X}^\varepsilon)(\cdot,\eta),\mathcal{X}^\varepsilon(\cdot,\eta)\right\rangle \mathrm{d}\eta}_{K_2} \\
&+ \underbrace{2\int_t^T e^{q_\alpha(\eta-T)}\left\langle F_{R^\varepsilon}(\cdot,\eta,U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon)-F(\cdot,\eta,u_{\text{ex}},v_{\text{ex}}),\mathcal{X}^\varepsilon(\cdot,\eta)\right\rangle \mathrm{d}\eta}_{K_3} \\
&- \underbrace{2\int_t^T e^{q_\alpha(\eta-T)}\left(\mathcal{D}_1\left(U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon\right)-\mathcal{D}_1\left(u_{\text{ex}},v_{\text{ex}}\right)\right)\left\langle \nabla u_{\text{ex}}(\cdot,\eta),\nabla\mathcal{X}^\varepsilon(\cdot,\eta)\right\rangle \mathrm{d}\eta}_{K_4}.
\end{aligned} \tag{67}$$

Applying Hölder's inequality and Lemma 4.3

$$\begin{aligned}
|K_1| &\le 2\int_t^T \left\|\widetilde{\mathbf{Q}_\varepsilon^\alpha}(u_{\text{ex}})(\cdot,\eta)\right\|_{L^2(\Omega)}\left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\
&\le \frac{2\alpha}{T}\int_t^T \left\|u_{\text{ex}}(\cdot,\eta)\right\|_{G_\Upsilon(\Omega)}\left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\
&\le \frac{\alpha^2}{T^2}\int_t^T \left\|u_{\text{ex}}(\cdot,\eta)\right\|_{G_\Upsilon(\Omega)}^2\mathrm{d}\eta + \int_t^T \left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}^2\mathrm{d}\eta.
\end{aligned} \tag{68}$$

For K_2, using Lemma 4.3

$$\begin{aligned}
|K_2| &\le 2\int_t^T \left\|\widetilde{\mathbf{P}_\varepsilon^\alpha}(\mathcal{X}^\varepsilon)(\cdot,t)\right\|_{L^2(\Omega)}\left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\
&\le \frac{2}{T}\log\left(\frac{1}{\alpha}\right)\int_t^T \left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}^2\mathrm{d}\eta.
\end{aligned} \tag{69}$$

Next, we estimate K_3. Notice that $R^\varepsilon \to \infty$, when $\varepsilon \to 0$, since $u_{\text{ex}}, v_{\text{ex}} \in L^\infty(Q_T)$, we can choose a sufficiently small ε, such that for a.e. $(x,t) \in \overline{Q}_T : |u_{\text{ex}}(x,t)| + |v_{\text{ex}}(x,t)| < R^\varepsilon$, or $F(x,t,u_{\text{ex}},v_{\text{ex}}) = F_{R^\varepsilon}(x,t,u_{\text{ex}},v_{\text{ex}})$ a.e. $(x,t) \in Q_T$. Thus, we obtain

$$\begin{aligned}
|K_3| &\le 2\int_t^T e^{q_\alpha(\eta-T)}\left|\left\langle F_{R^\varepsilon}(\cdot,\eta,U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon)-F_{R^\varepsilon}(\cdot,\eta,u_{\text{ex}},v_{\text{ex}}),\mathcal{X}^\varepsilon(\cdot,\eta)\right\rangle\right|\mathrm{d}\eta \\
&\le 2K_{R^\varepsilon}\int_t^T e^{q_\alpha(\eta-T)}\left\|\left((U_{\text{reg}}^\varepsilon,V_{\text{reg}}^\varepsilon)-(u_{\text{ex}},v_{\text{ex}})\right)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\
&= 2K_{R^\varepsilon}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta.
\end{aligned} \tag{70}$$

Using (A_3), (A_4), Hölder's inequality and Cauchy's inequality

$$\begin{aligned}
|K_4| &\le 2\mathcal{L}_{\max}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\left\|\nabla u_{\text{ex}}(\cdot,\eta)\right\|_{L^2(\Omega)}\left\|\nabla\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\
&\le \frac{\mathcal{L}_{\max}^2E^2}{2(M_0-M)}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}^2\mathrm{d}\eta + 2(M_0-M)\int_t^T \left\|\nabla\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}^2\mathrm{d}\eta.
\end{aligned} \tag{71}$$

Combining (67)–(71), and choosing $q_\alpha = \frac{1}{T}\log(\frac{1}{\alpha})$, we have

$$\begin{aligned}\left\|\mathcal{X}^\varepsilon(\cdot,t)\right\|^2_{L^2(\Omega)} \le \varepsilon^2 + \frac{\alpha^2}{T^2}\int_t^T \left\|u_{\rm ex}(\cdot,\eta)\right\|^2_{G_\Upsilon(\Omega)}d\eta + \int_t^T \left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|^2_{L^2(\Omega)}d\eta \\ + 2K_{R^\varepsilon}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\left\|\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}d\eta \\ + \frac{\mathcal{L}^2_{\max}E^2}{2(M_0-M)}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}d\eta.\end{aligned} \tag{72}$$

In a similar manner, we obtain the estimate for $\mathcal{Y}^\varepsilon$, summing with (72), we deduce

$$\begin{aligned}\left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} &\le 2\left\|\mathcal{X}^\varepsilon(\cdot,t)\right\|^2_{L^2(\Omega)} + 2\left\|\mathcal{Y}^\varepsilon(\cdot,t)\right\|^2_{L^2(\Omega)} \\ &\le 4\varepsilon^2 + C_5\alpha^2 + (C_6+4K_{R^\varepsilon})\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}d\eta.\end{aligned} \tag{73}$$

Applying Gronwall's inequality, we arrive at

$$\left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \le \left(4\varepsilon^2 + C_5\alpha^2\right)\exp\left[(C_6+4K_{R^\varepsilon})(T-t)\right],$$

which leads to

$$\begin{aligned}\left\|\left((U^\varepsilon_{\rm reg},V^\varepsilon_{\rm reg})-(u_{\rm ex},v_{\rm ex})\right)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} &= e^{2q_\alpha(T-t)}\left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \\ &\le C_7\left(4\frac{\varepsilon^2}{\alpha^2}+C_5\right)\alpha^{\frac{2t}{T}}\log^\gamma\left(\frac{1}{\varepsilon}\right),\end{aligned}$$

where we recalled $q_\alpha = \frac{1}{T}\log\left(\frac{1}{\alpha}\right)$, $\quad K_{R^\varepsilon} \le \frac{1}{4T}\log\left(\log^\gamma\left(\frac{1}{\varepsilon}\right)\right)$. From this, we can easily imply (64).

Now, for every small $\varepsilon > 0$, let us take the unique solution t_ε in $(0,T)$ of the equation $t = \varepsilon^{\frac{t}{T}}$. Notice that $\lim_{\varepsilon\to 0} t_\varepsilon = 0$ and $t_\varepsilon \le \sqrt{\frac{T}{\log(\frac{1}{\varepsilon})}}$. Thus, from (64), we obtain

$$\begin{aligned}&\left\|(U^\varepsilon_{\rm reg},V^\varepsilon_{\rm reg})(\cdot,t_\varepsilon)-(u_{\rm ex},v_{\rm ex})(\cdot,0)\right\|_{[L^2(\Omega)]^2} \\ &\le \left\|(U^\varepsilon_{\rm reg},V^\varepsilon_{\rm reg})(\cdot,t_\varepsilon)-(u_{\rm ex},v_{\rm ex})(\cdot,t_\varepsilon)\right\|_{[L^2(\Omega)]^2} + \left\|(u_{\rm ex},v_{\rm ex})(\cdot,t_\varepsilon)-(u_{\rm ex},v_{\rm ex})(\cdot,0)\right\|_{[L^2(\Omega)]^2} \\ &\le C_0\varepsilon^{\frac{t_\varepsilon}{T}}\sqrt{\log^\gamma\left(\frac{1}{\varepsilon}\right)} + t_\varepsilon\left(\|\partial_t u_{\rm ex}\|_{C([0,T];L^2(\Omega))} + \|\partial_t v_{\rm ex}\|_{C([0,T];L^2(\Omega))}\right) \\ &\le \sqrt{\frac{T}{\log(\frac{1}{\varepsilon})}}\left(C_0\sqrt{\log^\gamma\left(\frac{1}{\varepsilon}\right)}+2E\right).\end{aligned}$$

The proof of the theorem is completed. □

5.2. $H^1(\Omega)$-estimate

Theorem 5.2. Suppose that $(A_1)-(A_4)$*and*(A_6) hold. The exact solution of the system (1) satisfies

$$(u_{\rm ex},v_{\rm ex}) \in \left[\left(L^2(0,T;G_\Upsilon(\Omega))\cap L^\infty(0,T;H^2(\Omega))\cap C^1(0,T;H^1_0(\Omega))\right)\cap L^\infty(Q_T)\right]^2,$$

with $\Upsilon \ge 2M_0T$. Denote

$$E^* = \max\left\{\|(u_{\rm ex},v_{\rm ex})\|_{[L^2(0,T;G_\Upsilon(\Omega))]^2}, \|(u_{\rm ex},v_{\rm ex})\|_{[C^1(0,T;H^1_0(\Omega))]^2}, \|(u_{\rm ex},v_{\rm ex})\|_{[L^\infty(0,T;H^2(\Omega))]^2}\right\}.$$

Let us assume that

$$\begin{aligned}&\Psi^\varepsilon,\Phi^\varepsilon,\Psi,\Phi \in H^1(\Omega),\\ &\|\Psi^\varepsilon-\Psi\|_{H^1(\Omega)} + \|\Phi^\varepsilon-\Phi\|_{H^1(\Omega)} \le \varepsilon.\end{aligned}$$

Choose R^ε such that

$$8K_{R^\varepsilon} + \frac{12K^2_{R^\varepsilon}}{M_0-M} \le \frac{1}{T}\log\left(\log^\gamma\left(\frac{1}{\alpha}\right)\right), \tag{74}$$

for $0 < \gamma < 1$. Then there exist $A_0^* = A_0^*(u_{\text{ex}}, v_{\text{ex}})$, $B_0^* = B_0^*(u_{\text{ex}}, v_{\text{ex}})$, for which the following estimation holds

$$\left\| \left((U^\varepsilon_{\text{reg}}, V^\varepsilon_{\text{reg}}) - (u_{\text{ex}}, v_{\text{ex}})\right)(\cdot, t)\right\|_{[H^1(\Omega)]^2} \le \sqrt{A_0^* \frac{\varepsilon^2}{\alpha^2} + B_0^* \alpha^{\frac{t}{T}}} \sqrt{\log^\gamma\left(\frac{1}{\alpha}\right)}. \tag{75}$$

We choose the regularization parameter $\alpha(\varepsilon) = \varepsilon$. From this we imply the stability for $t \in (0, T]$. Furthermore, there exists $t_\varepsilon \in (0, T) : \lim_{\varepsilon \to 0} t_\varepsilon = 0$, $C_0^* = \sqrt{A_0^* + B_0^*}$, such that

$$\left\|(U^\varepsilon_{\text{reg}}, V^\varepsilon_{\text{reg}})(\cdot, t_\varepsilon) - (u_{\text{ex}}, v_{\text{ex}})(\cdot, 0)\right\|_{[H^1(\Omega)]^2} \le \left(C_0^* \sqrt{\log^\gamma\left(\frac{1}{\varepsilon}\right)} + 2E^* \right) \sqrt{\frac{T}{\log\left(\frac{1}{\varepsilon}\right)}}, \tag{76}$$

or we have the stability at $t = 0$.

Proof. We first prove that the solution $\Delta U^\varepsilon_{\text{reg}}$ and $\Delta V^\varepsilon_{\text{reg}}$ to the problem (30)–(31) belong to $\in L^2([0, T]; L^2(\Omega))$. Since the basis $\{\mu_p\}_{p=0}^\infty \subset \mathbb{U}$, we have that

$$\Delta U^n = -\sum_{p=0}^n \lambda_p U_{np}(t)\mu_p(x) \text{and} \quad \Delta V^n = -\sum_{p=0}^n \lambda_p V_{np}(t)\mu_p(x)$$

also lie in $\mathbb{U}$ for a.e. $t \in [0, T]$, where (U^n, V^n) is the Galerkin approximate solution. In (32), taking $\varphi = \Delta U^n$, and integrating from t to T, we obtain

$$\begin{aligned}
&\|\nabla U^n(\cdot, t)\|^2_{L^2(\Omega)} + 2\int_t^T \mathcal{B}_1(U^n, V^n)(\eta)\|\Delta U^n(\cdot, \eta)\|^2_{L^2(\Omega)} \mathrm{d}\eta \\
&= \|\nabla \Psi^n\|^2_{L^2(\Omega)} + 2\int_t^T \left\langle \widetilde{\mathbf{P}}^\alpha_\varepsilon(U^n)(\cdot, \eta), \Delta U^n(\cdot, \eta)\right\rangle \mathrm{d}\eta + 2\int_t^T \langle F_{R^\varepsilon}(\cdot, \eta, U^n, V^n), \Delta U^n(\cdot, \eta)\rangle \mathrm{d}\eta \\
&\le \|\Psi^n\|^2_{H_0^1(\Omega)} + 2\int_t^T \left(\|\widetilde{\mathbf{P}}^\alpha_\varepsilon(U^n)(\cdot, \eta)\|_{L^2(\Omega)} + \|F_{R^\varepsilon}(\cdot, \eta, U^n, V^n)\|_{L^2(\Omega)}\right) \|\Delta U^n(\cdot, \eta)\|_{L^2(\Omega)} \mathrm{d}\eta \\
&\le C_8 + C_9 \int_t^T \|(U^n, V^n)(\cdot, \eta)\|^2_{[L^2(\Omega)]^2} \mathrm{d}\eta + (M_0 - M)\int_t^T \|\Delta U^n(\cdot, \eta)\|^2_{L^2(\Omega)} \mathrm{d}\eta,
\end{aligned}$$

where we have used Hölder's inequality, Cauchy's inequality, Lemma 4.3 and Lipschitz property of F_{R^ε}.

Hence, using (43), we arrive at $\|\Delta U^n\|_{L^2(0,T;L^2(\Omega))} \le C_{10}$. Then the limit function $U^\varepsilon_{\text{reg}}$ also satisfies this estimate. Using the same arguments for $V^\varepsilon_{\text{reg}}$, we have $\Delta U^\varepsilon_{\text{reg}}, \Delta V^\varepsilon_{\text{reg}} \in L^2(0, T; L^2(\Omega))$. As in the previous section, we define

$$\mathcal{X}^\varepsilon(x, t) = e^{q_\alpha(t-T)}(U^\varepsilon_{\text{reg}} - u_{\text{ex}})(x, t), \quad \mathcal{Y}^\varepsilon(x, t) = e^{q_\alpha(t-T)}(V^\varepsilon_{\text{reg}} - v_{\text{ex}})(x, t).$$

Since the hypothesis $u_{\text{ex}}, v_{\text{ex}} \in L^\infty(0, T; H^2(\Omega))$, it yields $\Delta\mathcal{X}^\varepsilon, \Delta\mathcal{Y}^\varepsilon \in L^2(0, T; L^2(\Omega))$.

From (66), taking $\varphi = \lambda_p\langle \mathcal{X}^\varepsilon(\cdot, t), \mu_p\rangle\mu_p(x)$, summing from $p = 0$ to ∞, and then integrating from t to T. By some simple calculations, we have that

$$\begin{aligned}
&\left\|\nabla \mathcal{X}^\varepsilon(\cdot, T)\right\|^2_{L^2(\Omega)} - \left\|\nabla \mathcal{X}^\varepsilon(\cdot, t)\right\|^2_{L^2(\Omega)} - 2q_\alpha \int_t^T \left\|\nabla \mathcal{X}^\varepsilon(\cdot, \eta)\right\|^2_{L^2(\Omega)} \mathrm{d}\eta \\
&\quad - 2\int_t^T \mathcal{B}_1\left(U^\varepsilon_{\text{reg}}, V^\varepsilon_{\text{reg}}\right)(\eta)\left\|\Delta \mathcal{X}^\varepsilon(\cdot, \eta)\right\|^2_{L^2(\Omega)} \mathrm{d}\eta \\
&= \underbrace{-2\int_t^T e^{q_\alpha(\eta-T)}\Big(\mathcal{D}_1\left(U^\varepsilon_{\text{reg}}, V^\varepsilon_{\text{reg}}\right) - \mathcal{D}_1\left(u_{\text{ex}}, v_{\text{ex}}\right)\Big)(\eta)\Big\langle \Delta u_{\text{ex}}(\cdot, \eta), \Delta\mathcal{X}^\varepsilon(\cdot, \eta)\Big\rangle \mathrm{d}\eta}_{G_1} \\
&\quad - \underbrace{2\int_t^T e^{q_\alpha(t-T)}\Big\langle \widetilde{\mathbf{Q}}^\alpha_\varepsilon(u_{\text{ex}})(\cdot, \eta), \Delta\mathcal{X}^\varepsilon(\cdot, \eta)\Big\rangle \mathrm{d}\eta}_{G_2} + \underbrace{2\int_t^T \Big\langle \widetilde{\mathbf{P}}^\alpha_\varepsilon(\nabla\mathcal{X}^\varepsilon)(\cdot, \eta), \nabla\mathcal{X}^\varepsilon(\cdot, \eta)\Big\rangle \mathrm{d}\eta}_{G_3} \\
&\quad - \underbrace{2\int_t^T e^{q_\alpha(t-T)}\Big\langle F_{R^\varepsilon}(\cdot, \eta, U^\varepsilon_{\text{reg}}, V^\varepsilon_{\text{reg}}) - F(\cdot, \eta, u_{\text{ex}}, v_{\text{ex}}), \Delta\mathcal{X}^\varepsilon(\cdot, \eta)\Big\rangle \mathrm{d}\eta}_{G_4}.
\end{aligned} \tag{77}$$

The above terms make sense since the linearity of $\widetilde{\mathbf{P}}^\alpha_\varepsilon$, $\widetilde{\mathbf{Q}}^\alpha_\varepsilon$, Lipschitz property of F_{R^ε}, and the fact that $\Delta\mathcal{X}^\varepsilon$, $\Delta\mathcal{Y}^\varepsilon$, Δu_{ex}, Δv_{ex} $\in L^2(0, T; L^2(\Omega))$.

Using Hölder's inequality and hypotheses (A_3), (A_4)

$$\begin{aligned}|G_1| &\le 2\mathcal{L}_{\max}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\left\|\Delta u_{\mathrm{ex}}(\cdot,\eta)\right\|_{L^2(\Omega)}\left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\le \frac{3\mathcal{L}_{\max}^2(E^*)^2}{2(M_0-M)}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta \\ &\quad+\frac{2(M_0-M)}{3}\int_t^T \left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta.\end{aligned} \tag{78}$$

Using Hölder's inequality, Cauchy's inequality and Lemma 4.3, it gives

$$\begin{aligned}|G_2| &\le 2\int_t^T \left\|\widetilde{\mathbf{Q}}^\alpha_\varepsilon(u_{\mathrm{ex}})(\cdot,\eta)\right\|_{L^2(\Omega)}\left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\le \frac{3}{2(M_0-M)}\int_t^T \left\|\widetilde{\mathbf{Q}}^\alpha_\varepsilon(u_{\mathrm{ex}})(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta+\frac{2(M_0-M)}{3}\int_t^T \left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta \\ &\le \frac{3\alpha^2}{2(M_0-M)T^2}\int_t^T \left\|u_{\mathrm{ex}}(\cdot,\eta)\right\|^2_{G_\Upsilon(\Omega)}\mathrm{d}\eta+\frac{2(M_0-M)}{3}\int_t^T \left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta \\ &\le \frac{3\alpha^2(E^*)^2}{2(M_0-M)T^2}+\frac{2(M_0-M)}{3}\int_t^T \left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta.\end{aligned} \tag{79}$$

Thanks to Lemma 4.3, we have

$$\begin{aligned}|G_3| &\le 2\int_t^T \left\|\widetilde{\mathbf{P}}^\alpha_\varepsilon(\nabla\mathcal{X}^\varepsilon)(\cdot,\eta)\right\|_{L^2(\Omega)}\left\|\nabla\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\le \frac{2}{T}\log\left(\frac{1}{\alpha}\right)\int_t^T \left\|\nabla\mathcal{X}^\varepsilon(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta.\end{aligned} \tag{80}$$

With an analogous argument as in Section 5.1, we can choose a sufficiently small ε, such that $F(x,t,u_{\mathrm{ex}},v_{\mathrm{ex}}) = F_{R^\varepsilon}(x,t,u_{\mathrm{ex}},v_{\mathrm{ex}})$ for a.e. $(x,t)\in Q_T$, where K_{R^ε} satisfies (74). Therefore

$$\begin{aligned}|G_4| &= 2\int_t^T e^{q_\alpha(\eta-T)}\left|\left\langle F_{R^\varepsilon}(\cdot,\eta,U^\varepsilon_{\mathrm{reg}},V^\varepsilon_{\mathrm{reg}})-F_{R^\varepsilon}(\cdot,\eta,u_{\mathrm{ex}},v_{\mathrm{ex}}),\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\rangle\right|\mathrm{d}\eta \\ &\le 2\int_t^T e^{q_\alpha(\eta-T)}\left\|F_{R^\varepsilon}(\cdot,\eta,U^\varepsilon_{\mathrm{reg}},V^\varepsilon_{\mathrm{reg}})-F_{R^\varepsilon}(\cdot,\eta,u_{\mathrm{ex}},v_{\mathrm{ex}})\right\|_{L^2(\Omega)}\left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\le 2K_{R_\varepsilon}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|_{[L^2(\Omega)]^2}\left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|_{L^2(\Omega)}\mathrm{d}\eta \\ &\le \frac{3K^2_{R_\varepsilon}}{2(M_0-M)}\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta+\frac{2(M_0-M)}{3}\int_t^T \left\|\Delta\mathcal{X}^\varepsilon(\cdot,\eta)\right\|^2_{L^2(\Omega)}\mathrm{d}\eta.\end{aligned} \tag{81}$$

Choosing $q_\alpha = \frac{1}{T}\log\left(\frac{1}{\alpha}\right)$. From (77)–(81), we deduce

$$\left\|\nabla\mathcal{X}^\varepsilon(\cdot,t)\right\|^2_{L^2(\Omega)} \le \varepsilon^2 + C_{10}\alpha^2 + \left(C_{11}+\frac{3K^2_{R^\varepsilon}}{2(M_0-M)}\right)\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta.$$

In a similar manner, we obtain estimate for $\mathcal{Y}^\varepsilon$. Therefore

$$\begin{aligned}&\left\|(\nabla\mathcal{X}^\varepsilon,\nabla\mathcal{Y}^\varepsilon)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \\ &\qquad\le 4\varepsilon^2+4C_{10}\alpha^2+\left(4C_{11}+\frac{6K^2_{R^\varepsilon}}{(M_0-M)}\right)\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|^2_{[L^2(\Omega)]^2}\mathrm{d}\eta.\end{aligned} \tag{82}$$

Combining (73) and (82), yields

$$\begin{aligned}\left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,t)\right\|^2_{[H^1(\Omega)]^2} &\le 2\left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2}+2\left\|(\nabla\mathcal{X}^\varepsilon,\nabla\mathcal{Y}^\varepsilon)(\cdot,t)\right\|^2_{[L^2(\Omega)]^2} \\ &\le 16\varepsilon^2+C_{12}\alpha^2+\left(C_{13}+8K_{R^\varepsilon}+\frac{12K^2_{R^\varepsilon}}{(M_0-M)}\right)\int_t^T \left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,\eta)\right\|^2_{[H^1(\Omega)]^2}\mathrm{d}\eta\end{aligned}$$

Applying Gronwall's inequality, we arrive at

$$\left\|(\mathcal{X}^\varepsilon,\mathcal{Y}^\varepsilon)(\cdot,t)\right\|^2_{[H^1(\Omega)]^2} \le \left(16\varepsilon^2+C_{12}\alpha^2\right)\exp\left[\left(C_{13}+8K_{R^\varepsilon}+\frac{12K^2_{R^\varepsilon}}{(M_0-M)}\right)(T-t)\right].$$

Thus, by choosing K_{R^ε} which satisfies (74), we can deduce (75).

Moreover, with $\alpha = \varepsilon$, as in Section 5.1, we choose $t = t_\varepsilon$ the unique solution of the equation $\varepsilon^{\frac{t}{T}} = t$, yields

$$\begin{aligned}
&\left\|(U^\varepsilon_{\text{reg}}, V^\varepsilon_{\text{reg}})(\cdot, t_\varepsilon) - (u_{\text{ex}}, v_{\text{ex}})(\cdot, 0)\right\|_{[H^1(\Omega)]^2} \\
&\le \left\|(U^\varepsilon_{\text{reg}}, V^\varepsilon_{\text{reg}})(\cdot, t_\varepsilon) - (u_{\text{ex}}, v_{\text{ex}})(\cdot, t_\varepsilon)\right\|_{[H^1(\Omega)]^2} + \left\|(u_{\text{ex}}, v_{\text{ex}})(\cdot, t_\varepsilon) - (u_{\text{ex}}, v_{\text{ex}})(\cdot, 0)\right\|_{[H^1(\Omega)]^2} \\
&\le C_0^* \varepsilon^{\frac{t_\varepsilon}{T}} \sqrt{\log^\gamma\left(\frac{1}{\varepsilon}\right)} + t_\varepsilon \left(\|\partial_t u_{\text{ex}}\|_{C([0,T];H^1(\Omega))} + \|\partial_t v_{\text{ex}}\|_{C([0,T];H^1(\Omega))}\right) \\
&\le t_\varepsilon \left(C_0^* \sqrt{\log^\gamma\left(\frac{1}{\varepsilon}\right)} + 2E^*\right),
\end{aligned}$$

which implies (76). This completes the proof of the theorem. □

Remark 5.3. In Theorems 5.1 and 5.2, to ensure the convergence of regularized solution, it is sufficient to choose α such that $\lim_{\varepsilon\to 0} \alpha = 0$, $\lim_{\varepsilon\to 0} \frac{\varepsilon}{\alpha} = const$. In order to obtain the optimal rate of convergence, we chose $\alpha = \varepsilon$. With this choice, the convergence speed of regularized solution in both L^2 and H^1-norms is of order $\mathcal{O}(\varepsilon^{\frac{t}{T}} \sqrt{\log^\gamma \frac{1}{\varepsilon}})$ for $t > 0$.

Remark 5.4. In Section 4, instead of the locally Lipschitz condition (A_6) of F and G, we can impose the locally Lipschitz property in $L^2 - norm$

$\forall R > 0, \exists K^R_F, K^R_G : 0 < K^R_F, K^R_G < \infty, \forall (u_i, v_i) \in \mathbb{B}_R, i = 1, 2, \forall t \in [0, T]$, yield

$$\begin{aligned}
&\|F(\cdot, t, u_1, v_1) - F(\cdot, t, u_2, v_2)\|_{L^2(\Omega)} \le K^R_F \|\left((u_1, v_1) - (u_2, v_2)\right)(\cdot, t)\|_{[L^2(\Omega)]^2}, \\
&\|G(\cdot, t, u_1, v_1) - G(\cdot, t, u_2, v_2)\|_{L^2(\Omega)} \le K^R_G \|\left((u_1, v_1) - (u_2, v_2)\right)(\cdot, t)\|_{[L^2(\Omega)]^2},
\end{aligned}$$

where $\mathbb{B}_R$ is the closed ball in $[L^2(\Omega)]^2$ of centre 0 and radius R. Then we can use the following sequences of globally Lipschitz functions

$$F_{R^\varepsilon}(x, t, u, v) = \begin{cases} F(x, t, u, v), & \text{if } \|(u, v)(\cdot, t)\|_{[L^2(\Omega)]^2} \le R^\varepsilon, \\ F\left(x, t, \dfrac{R^\varepsilon u}{\|(u, v)(\cdot, t)\|_{[L^2(\Omega)]^2}}, \dfrac{R^\varepsilon v}{\|(u, v)(\cdot, t)\|_{[L^2(\Omega)]^2}}\right), & \\ \qquad \text{if } \|(u, v)(\cdot, t)\|_{[L^2(\Omega)]^2} > R^\varepsilon, & \end{cases}$$

$$G_{R^\varepsilon}(x, t, u, v) = \begin{cases} G(x, t, u, v), & \text{if } \|(u, v)(\cdot, t)\|_{[L^2(\Omega)]^2} \le R^\varepsilon, \\ G\left(x, t, \dfrac{R^\varepsilon u}{\|(u, v)(\cdot, t)\|_{[L^2(\Omega)]^2}}, \dfrac{R^\varepsilon v}{\|(u, v)(\cdot, t)\|_{[L^2(\Omega)]^2}}\right), & \\ \qquad \text{if } \|(u, v)(\cdot, t)\|_{[L^2(\Omega)]^2} > R^\varepsilon & \end{cases}$$

to approximate F, G and our QR method can be straightforwardly applied. We can easily prove that the reaction function considered in Section 3 satisfies the globally Lipschitz condition in L^2- norm and the regularized solution for this unstable problem can be found.

6. Numerical results

This section concentrates on establishing some numerical tests in 1-D and 2-D regions Ω to illustrate our numerical strategy and verify the error estimates given in the theoretical parts. Let us start with the 1-D case.

6.1. Generalized Fisher–Kolmogorov model for the response of low-grade gliomas to radiotherapy

We first generalize the model expressing the response of the tumour cells to radiation [3]

$$\begin{aligned}
&\frac{\partial}{\partial t} u(x, t) = \mathcal{D}_1(u, v_d)(t) \Delta u + \rho(1 - u - v_d)u + F_1(x, t), \\
&\frac{\partial}{\partial t} v_d(x, t) = \mathcal{D}_2(u, v_d)(t) \Delta v_d - \frac{\rho}{k}(1 - u - v_d)v_d + F_2(x, t),
\end{aligned}$$

where u is the density of functionally alive tumour cells, v_d is the density of irreversibly damaged cells after irradiation. $\tau = 1/\rho$ is the tumour population doubling time. The parameter k has the meaning of the average number of mitosis cycles that damaged cells are able to complete before dying.

By simple computations, one can easily check that reaction terms F, G are locally Lipschitz functions w.r.t u, v_d, and with the Lipschitz constant

$$K_{R^\varepsilon} = \sqrt{8} \max\{\rho, \frac{\rho}{k}\}(2R^\varepsilon + 1).$$

We implement the model with the domain $\Omega = [0, \pi]$. The Laplacian operator has eigenfunctions, satisfying Neumann boundary condition: $\mu_p(x) = \sqrt{\frac{2}{\pi}}\cos(px)$, with corresponding eigenvalues, $\lambda_p = p^2$, $p \in \mathbb{N}$. This sequence $\{\mu_p(x)\}_0^\infty$ forms an orthonormal basis of $L^2(\Omega)$.

A uniform grid of mesh-points (x_i, t_k) is used to discretize the space and time intervals

$$\Delta x = \frac{\pi}{N_x}, x_i = (i-1)\Delta x, i = \overline{1, N_x+1},$$
$$\Delta t = \frac{T}{N_t}, t_k = (k-1)\Delta t, k = \overline{1, N_t+1}.$$

The inner product in $L^2(0,\pi)$ can be approximated by 1-D composite Simpson rule of numerical integration

$$\int_0^\pi f(x)\mathrm{d}x \approx \frac{\Delta x}{3}\sum_{i=1}^{N_x+1} c_i f(x_i),$$

where $c_i = \begin{cases} 1, & \text{if } (i=1) \vee (i = N_x+1), \\ 2, & \text{if } i = 2l+1, \\ 4, & \text{if } i = 2l. \end{cases}$

As a consequence, discrete norm in $L^2(\Omega)$: ℓ^2- norm can be defined by

$$\|w\|_{\ell^2(\Omega)} = \sqrt{\frac{\Delta x}{3}\sum_{i=1}^{N_x+1} c_i w^2(x_i)}.$$

The input data is perturbed as:

$$\Psi^\varepsilon(x) = \Psi(x)\left(1 + \frac{\varepsilon}{\|\Psi\|_{\ell^2(\Omega)} + \|\Phi_d\|_{\ell^2(\Omega)}}\left(2\,\mathrm{rand}(\mathrm{size}(x)) - 1\right)\right),$$
$$\Phi^\varepsilon(x) = \Phi(x)\left(1 + \frac{\varepsilon}{\|\Psi\|_{\ell^2(\Omega)} + \|\Phi_d\|_{\ell^2(\Omega)}}\left(2\,\mathrm{rand}(\mathrm{size}(x)) - 1\right)\right),$$

where rand(size(x)) is a random array with the same size with x, having values in [0, 1].

Denote by $(\widetilde{u}, \widetilde{v})$ our numerical regularized solution, and by $(u_{\mathrm{ex}}, v_{\mathrm{ex}})$ the exact solution.

The absolute errors are evaluated by

$$\epsilon_u = \|\widetilde{u} - u_{\mathrm{ex}}\|_{\ell^2(\Omega)}, \quad r\epsilon_u = \frac{\epsilon_u}{\|u_{\mathrm{ex}}\|_{\ell^2(\Omega)}},$$
$$\epsilon_v = \|\widetilde{v} - v_{\mathrm{ex}}\|_{\ell^2(\Omega)}, \quad r\epsilon_v = \frac{\epsilon_v}{\|v_{\mathrm{ex}}\|_{\ell^2(\Omega)}}.$$

6.1.1. Case 1

We investigate the model in the first case with

$$T = \frac{1}{3}, \rho = 0.01, k = 1, \mathcal{D}1 = 0.2 + 0.1\sin(\mathbf{b}(v_d)), \mathcal{D}_2 = 0.2 + 0.1\sin(\mathbf{c}(u)),$$
$$\mathbf{a}(u)(t) = 0, \mathbf{b}(v_d)(t) = \int_\Omega x\cdot v_d(x,t)\mathrm{d}x, \quad \mathbf{c}(u)(t) = \int_\Omega x\cdot u(x,t)\mathrm{d}x, \quad \mathbf{d}(v_d)(t) = 0.$$

One can easily see that $0.1 \le \mathcal{D}(\vartheta, \nu) \le 0.3$. Thus, we can choose $M_0 = 0.31$.

Taking the external sources

$$F_1 = e^{-5t}\cos 2x(0.01\cos x - 0.4\sin(2e^{-2t})) + 0.01e^{-6t}\cos^2 2x - 2.21e^{-3t}\cos 2x,$$
$$F_2 = -1.79e^{-2t}\cos x - 0.01(e^{-4t}\cos^2 x + e^{-5t}\cos 2x\cos x),$$

and the final data

$$\Psi(x) = e^{-1}\cos 2x, \quad \Phi(x) = e^{-2/3}\cos x$$

the problem admits $u_{\mathrm{ex}}(x,t) = e^{-3t}\cos 2x$, $v_{\mathrm{ex}}(x,t) = e^{-2t}\cos x$ as the exact solution. Then the radius chosen here $R^\varepsilon \ge 2$. Consequently, we have $K_{R^\varepsilon} \ge 0.05\sqrt{8}$. In order to guarantee the condition (63) with $\gamma = \frac{1}{2}$, we must have $\varepsilon \le 0.2327$.

Finite difference numerical solution.

In this example, we first try to use the traditional backward Euler method to take into account the numerical regularized solution to the system (24)–(25) with the regularization parameter chosen here $\alpha(\varepsilon) = \varepsilon$ as follows

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} - \mathcal{D}_1(U^{k+1}, V^{k+1})\frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{\Delta x^2} = F_i^{k+1} + \widetilde{\mathbf{Q}}_\varepsilon^\alpha(U_i^{k+1}), \tag{83}$$

Table 1
The errors of finite difference solution at $t \in \{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\}$, for various amounts of noise $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, \}$ for Ex. 1, case 1.

ε	$\epsilon_u(\frac{1}{4})$	$\epsilon_v(\frac{1}{4})$	$\epsilon_u(\frac{1}{6})$	$\epsilon_v(\frac{1}{6})$	$\epsilon_u(\frac{1}{12})$	$\epsilon_v(\frac{1}{12})$
10^{-1}	0.2539	0.5119	1.6491	1.7270	37.6326	8.3029
10^{-2}	0.2070	0.0801	0.5151	0.2093	1.1453	0.1852
10^{-3}	0.0685	0.0326	0.1046	0.0607	0.1616	0.1038
10^{-4}	0.0664	0.0221	0.0831	0.0258	0.1064	0.0317
10^{-5}	0.0674	0.0219	0.0841	0.0250	0.1062	0.0301

$$\frac{V_i^{k+1} - V_i^k}{\Delta t} - \mathcal{D}_2(U^{k+1}, V^{k+1})\frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\Delta x^2} = G_i^{k+1} + \widetilde{\mathbf{Q}_\varepsilon^\alpha}(V_i^{k+1}), \tag{84}$$

where $U_i^k = U_{\text{reg}}^\varepsilon(x_i, t_k)$, $V_i^k = V_{\text{reg}}^\varepsilon(x_i, t_k)$, $F_i^{k+1} = F_{R^\varepsilon}\left(x_{k+1}, t_i, U_{\text{reg}}^\varepsilon(x_{k+1}, t_i), V_{\text{reg}}^\varepsilon(x_{k+1}, t_i)\right)$ and $G_i^{k+1} = G_{R^\varepsilon}\left(x_{k+1}, t_i, U_{\text{reg}}^\varepsilon(x_{k+1}, t_i), V_{\text{reg}}^\varepsilon(x_{k+1}, t_i)\right)$.

The term $\widetilde{\mathbf{Q}_\varepsilon^\alpha}(V_i^{k+1})$ is defined as

$$\widetilde{\mathbf{Q}_\varepsilon^\alpha}\left(V_i^{k+1}\right) = \sum_{p=0}^{P} \frac{\log(1+\varepsilon e^{TM_0\lambda_p})}{T}\Big\langle V^{k+1}, \mu_p\Big\rangle \mu_p(x_i).$$

Furthermore, to find the Fourier coefficient $\Big\langle V^{k+1}, \mu_p\Big\rangle$ we imply the Filon numerical integration method (see [32]). The Neumann boundary condition (3) is implemented at $x = 0$ and $x = \pi$ as follows:

$$U_1^k = U_2^k, \quad V_1^k = V_2^k, \quad U_{N_x}^k = U_{N_x+1}^k, \quad V_{N_x}^k = V_{N_x+1}^k, \quad \text{for } k = \overline{1, N_t+1}.$$

Then, we will find the solution vector $\mathbf{X}^k = [U_2^k\ U_3^k\ \dots\ U_{N_x}^k\ V_2^k\ V_3^k\ \dots\ V_{N_x}^k]^{\mathrm{T}}$ by solving the linear system

$$\mathbf{A}(U^{k+1}, V^{k+1})\mathbf{X}^k = \mathbf{B}(U^{k+1}, V^{k+1}),$$

where

$$\mathbf{B}\left(U^{k+1}, V^{k+1}\right) = \begin{bmatrix} U_2^{k+1} - \Delta t F_2^{k+1} - \Delta t\widetilde{\mathbf{Q}_\varepsilon^\alpha}\left(U_2^{k+1}\right) \\ \dots \\ U_{N_x}^{k+1} - \Delta t F_{N_x}^{k+1} - \Delta t\widetilde{\mathbf{Q}_\varepsilon^\alpha}\left(U_{N_x}^{k+1}\right) \\ V_2^{k+1} - \Delta t G_2^{k+1} - \Delta t\widetilde{\mathbf{Q}_\varepsilon^\alpha}\left(V_2^{k+1}\right) \\ \dots \\ V_{N_x}^{k+1} - \Delta t G_{N_x}^{k+1} - \Delta t\widetilde{\mathbf{Q}_\varepsilon^\alpha}\left(V_{N_x}^{k+1}\right) \end{bmatrix},$$

$$\mathbf{A}\left(U^{k+1}, V^{k+1}\right) = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix},$$

with

$$\mathbf{A}_i = \begin{bmatrix} 1-\mathbf{D}_i h & \mathbf{D}_i h & 0 & 0 & 0 & \cdots & 0 \\ \mathbf{D}_i h & 1-2\mathbf{D}_i h & \mathbf{D}_i h & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{D}_i h & 1-2\mathbf{D}_i h & \mathbf{D}_i h & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \mathbf{D}_i h & 1-2\mathbf{D}_i h & \mathbf{D}_i h \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{D}_i h & 1-\mathbf{D}_i h \end{bmatrix},$$

where $h = \frac{\Delta t}{\Delta x^2}$, $\mathbf{D}_i = \mathcal{D}_i\left(U^{k+1}, V^{k+1}\right)$.

Choosing $N_x = 8$, $N_t = 100$, $P = 10$, we present the relative errors between the regularized and the true solutions at $t \in \{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\}$ for various amounts of noise $\varepsilon \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$ in Table 1. .

From them, we can observe that the errors at $t = \frac{1}{6}$ are greater than those at $t = \frac{1}{4}$ and smaller than those at $t = \frac{1}{12}$. Furthermore, with the smaller errors of input data, the results obtained are more accurate, which verifies the theoretical result in Theorem 5.1.

One more thing to remark is that using this method, if we choose the number of intervals for x : $N_x > 10$, then the numerical solution quickly looses its stability and tends to infinity. Hence, we choose $N_x = 8$ only.

Table 2
The errors of Fourier-mode solution at $t \in \{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\}$, for various amounts of noise $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$, with $N_x = 8$ for Ex. 1, case 1.

ε	$\epsilon_u(\frac{1}{4})$	$\epsilon_v(\frac{1}{4})$	$\epsilon_u(\frac{1}{6})$	$\epsilon_v(\frac{1}{6})$	$\epsilon_u(\frac{1}{12})$	$\epsilon_v(\frac{1}{12})$
10^{-1}	0.1291	0.0979	0.2033	0.1754	0.3873	0.3787
10^{-2}	0.1159	0.0843	0.2184	0.1952	0.4066	0.4022
10^{-3}	0.1175	0.0924	0.2233	0.2134	0.4170	0.4360
10^{-4}	0.1176	0.0962	0.2250	0.2231	0.4258	0.4530
10^{-5}	0.1178	0.0973	0.2271	0.2259	0.4322	0.4573

Table 3
The errors at $t = \frac{1}{4}, t = \frac{1}{6}, t = \frac{1}{12}$, for various amounts of noise $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$ with $N_x = 100$ for Ex. 1, case 1.

ε	$\epsilon_u(\frac{1}{4})$	$\epsilon_v(\frac{1}{4})$	$\epsilon_u(\frac{1}{6})$	$\epsilon_v(\frac{1}{6})$	$\epsilon_u(\frac{1}{12})$	$\epsilon_v(\frac{1}{12})$
10^{-1}	0.0515	0.1984	0.2977	1.2888	4.2858	157.2620
10^{-2}	0.0037	0.0039	0.0156	0.0700	0.2743	4.6714
10^{-3}	0.0012	0.0005	0.0066	0.0042	0.0176	0.0156
10^{-4}	0.0012	0.0004	0.0065	0.0027	0.0174	0.0072
10^{-5}	0.0012	0.0004	0.0066	0.0027	0.0173	0.0072

Fourier-mode numerical solution

Next, we construct a new numerical regularized solution to the system (24)–(25), which is of the following form

$$U^{\varepsilon}_{\text{reg}} = \sum_{p=0}^{\infty} \left[\exp\left(\lambda_p \int_t^T \mathcal{D}_1(U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}) \mathrm{d}\eta \right) \left(1 + \varepsilon e^{M_0 T \lambda_p}\right)^{\frac{t-T}{T}} \Psi^{\varepsilon}_p \right] \mu_p(x) \tag{85}$$
$$- \sum_{p=0}^{\infty} \left[\int_t^T \exp\left(\lambda_p \int_t^{\eta} \mathcal{D}_1(U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}) \mathrm{d}\xi \right) \left(1 + \varepsilon e^{M_0 T \lambda_p}\right)^{\frac{t-\eta}{T}} \langle F(\cdot, \eta, U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}), \mu_p \rangle \mathrm{d}\eta \right] \mu_p(x),$$

$$V^{\varepsilon}_{\text{reg}} = \sum_{p=0}^{\infty} \left[\exp\left(\lambda_p \int_t^T \mathcal{D}_2(U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}) \mathrm{d}\eta \right) \left(1 + \varepsilon e^{M_0 T \lambda_p}\right)^{\frac{t-T}{T}} \Phi^{\varepsilon}_p \right] \mu_p(x) \tag{86}$$
$$- \sum_{p=0}^{\infty} \left[\int_t^T \exp\left(\lambda_p \int_t^{\eta} \mathcal{D}_2(U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}) \mathrm{d}\xi \right) \left(1 + \varepsilon e^{M_0 T \lambda_p}\right)^{\frac{t-\eta}{T}} \langle G(\cdot, \eta, U^{\varepsilon}_{\text{reg}}, V^{\varepsilon}_{\text{reg}}), \mu_p \rangle \mathrm{d}\eta \right] \mu_p(x),$$

where we have used some similar calculations as in the proof of Lemma 3.2.

We herein use the Picard-like procedure [32] to approximate the Volterra-type integral Eq. (85) as follows (the same way is applied for v)

$$U^{k}_{\text{reg},p} = \langle U^{\varepsilon}_{\text{reg}}(\cdot, t_k), \mu_p \rangle = e^{\lambda_p \Delta t \sum_{l=k}^{N_t} \mathcal{D}_1(U^{l+1}_{\text{reg}}, V^{l+1}_{\text{reg}})} \left(1 + \varepsilon e^{M_0 T \lambda_p}\right)^{\frac{t_k - T}{T}} \Psi^{\varepsilon}_p$$
$$- \sum_{l=k}^{Nt} \int_{t_l}^{t_{l+1}} e^{\lambda_p (\eta - t_l) \mathcal{D}_1(U^{l+1}_{\text{reg}}, V^{l+1}_{\text{reg}})} \left(1 + \varepsilon e^{M_0 T \lambda_p}\right)^{\frac{t_l - \eta}{T}} \langle F(\cdot, \eta, U^{l+1}_{\text{reg}}, V^{l+1}_{\text{reg}}), \mu_p \rangle \mathrm{d}\eta, \tag{87}$$

with $U^l_{\text{reg}} = U^{\varepsilon}_{\text{reg}}(x, t_l), V^l_{\text{reg}} = V^{\varepsilon}_{\text{reg}}(x, t_l)$. We will find these Fourier coefficients up to $p = P$.

With the choice that $N_x = 8, N_t = 100$ and $P = 10$, the errors at $t \in \{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\}$, for various amounts of noise $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$ are presented in Table 2. We can easily see that the numerical solution is not stable, and the errors are greater than those of the finite difference numerical solution.

However, when we increase the parameter N_x to 100, the stability of the numerical solution has been much improved. Table 3 and Fig. 1 show the ℓ_2-errors between $(\widetilde{u}, \widetilde{v})$ and $(u_{\text{ex}}, v_{\text{ex}})$ at various times $t \in \{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\}$ (with $N_x = N_t = 100, P = 10$) for various amounts of noise $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, \}$, and the graphs of the approximate solutions at $t = \frac{1}{6}$ for $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, \}$ (with $N_x = N_t = 50, P = 10$).

6.1.2. Case 2

We now consider the model with

$$T = 1, \rho = 0.001, k = 1, \quad \mathcal{D}1 = \mathcal{D}2 = 0.1 + \frac{0.4}{1 + \mathbf{a}^2(u)} + \frac{0.5}{1 + \mathbf{b}^2(v_d)},$$
$$\mathbf{a}(u)(t) = \mathbf{c}(u)(t) = \int_{\Omega} u(x, t) \mathrm{d}x, \quad \mathbf{b}(v_d)(t) = \mathbf{d}(v_d)(t) = \int_{\Omega} v_d(x, t) \mathrm{d}x.$$

Notice that $0.1 \le \mathcal{D}_1, \mathcal{D}_2 \le 1$. We then choose $M_0 = 1 + 2.10^{-13}$.

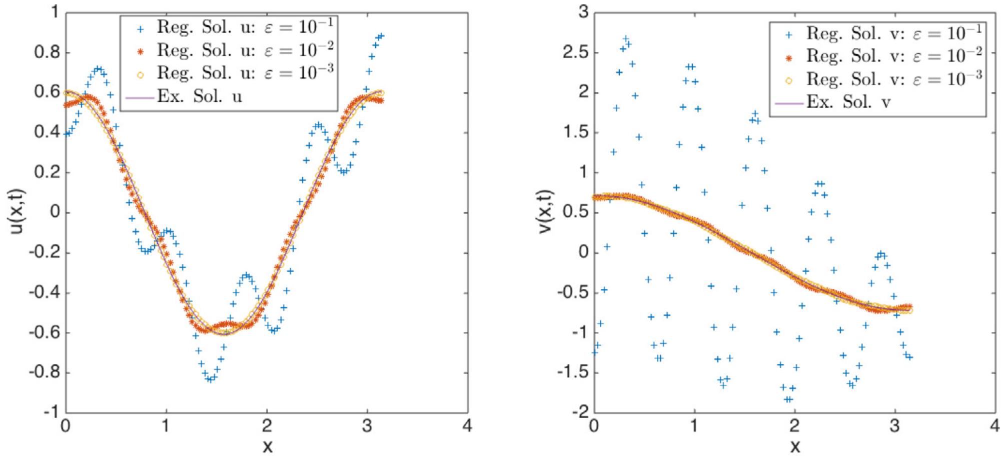

Fig. 1. The regularized solutions in Ex. 1, case 1, at $t = \frac{1}{6}$, with $\varepsilon = 10^{-1}, \varepsilon = 10^{-2}, \varepsilon = 10^{-3}$.

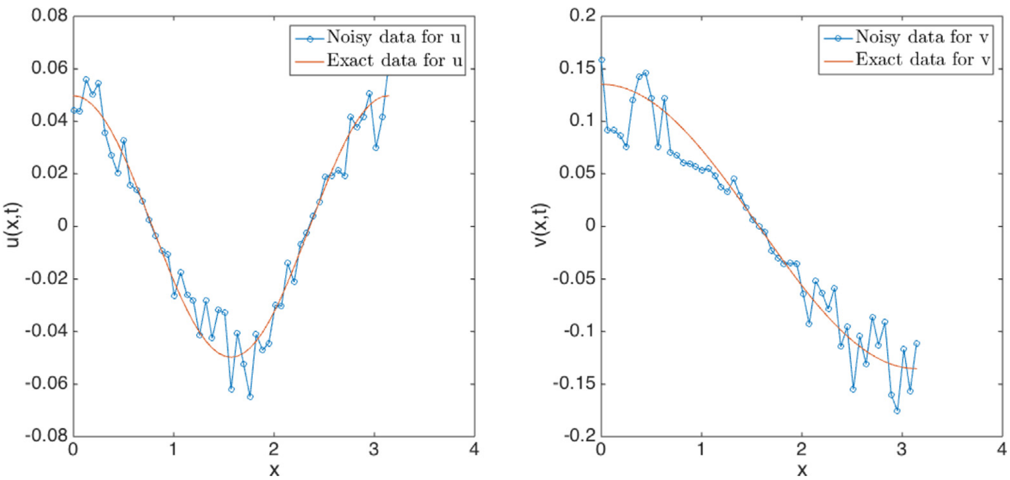

Fig. 2. The noisy data in Ex. 1, case 2, with $\varepsilon = 0.1$.

Let the external sources

$$F_1(x,t) = 0.999e^{-3t}\cos 2x + 0.001(e^{-5t}\cos 2x\cos x + e^{-6t}\cos^2 2x),$$
$$F_2(x,t) = -0.999e^{-2t}\cos x - 0.001(e^{-4t}\cos^2 x + e^{-5t}\cos x\cos 2x),$$

and the final data

$$\Psi(x) = e^{-3}\cos 2x, \quad \Phi(x) = e^{-2}\cos x$$

such that $u_{\text{ex}}(x,t) = e^{-3t}\cos 2x$, $v_{\text{ex}}(x,t) = e^{-2t}\cos x$ are also the exact solution. Here $R^{\varepsilon} \geq 2, K_{R^{\varepsilon}} \geq 0.005\sqrt{8}$, ε must satisfy $\varepsilon \leq 0.3263$.

Choosing $N_x = N_t = 50, P = 10$. Fig. 2 gives us a picture of noise of level $\varepsilon = 0.1$ in input data. Figs. 3 and 4 compare the numerical exact solution with noisy data and the regularized one at $t = 0.8$, with $\varepsilon = 0.1$. The error shown in the case of unregularized solution is extremely large, compared with the exact solution. This reflects the instability of the exact solution, which has been proved in Lemma 3.2 for an example.

Table 4 shows us that as t reaches 0, the numerical regularized solution tends to lose its stability.

6.2. Turing pattern

We now consider an example of pattern formation in a Turing-type reaction–diffusion system [6]

$$u_t(x,y,t) = D_u\Delta u + k_1u^2v - k_2u + F_1(x,y,t),$$

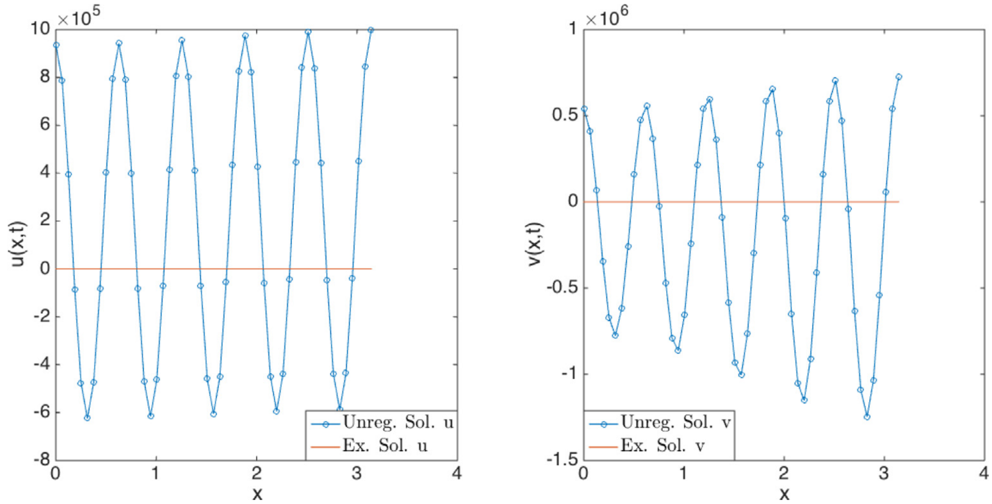

Fig. 3. The unregularized solution in Ex. 1, case 2, at $t = 0.8$, with $\varepsilon = 0.1$.

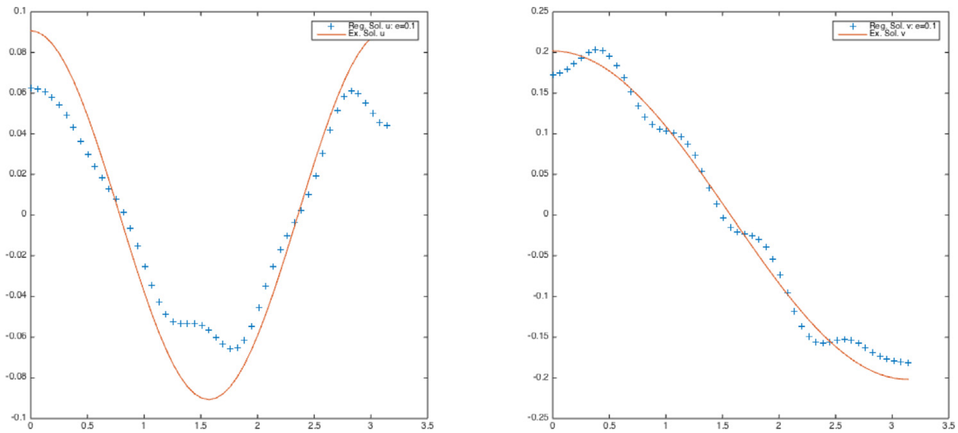

Fig. 4. The regularized solution in Ex. 1, case 2, at $t = 0.8$, with $\varepsilon = 0.1$.

Table 4
The relative errors at various times $t \in \{0.8, 0.6, 0.4, 0.2\}$, for various amounts of noise $\varepsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-5}, 10^{-5}, 10^{-6}, 10^{-7}\}$ for Ex. 1, case 2.

ε	$r\epsilon_u(0.8)$	$r\epsilon_v(0.8)$	$r\epsilon_u(0.6)$	$r\epsilon_v(0.6)$	$r\epsilon_u(0.4)$	$r\epsilon_v(0.4)$	$r\epsilon_u(0.2)$	$r\epsilon_v(0.2)$
10^{-1}	0.3508	0.0860	0.5472	0.1704	0.6854	0.1971	0.8374	0.2763
10^{-2}	0.0448	0.0207	0.0196	0.0587	0.0941	0.1113	0.2337	0.1770
10^{-3}	0.0413	0.0102	0.2014	0.0443	0.4542	0.0884	0.7961	0.1458
10^{-4}	0.0522	0.0098	0.2285	0.0432	0.5057	0.0879	0.8776	0.1359
10^{-5}	0.0534	0.0097	0.2315	0.0432	0.5109	0.0876	0.8862	0.1361
10^{-6}	0.0535	0.0097	0.2318	0.0431	0.5115	0.0876	0.8870	0.1363
10^{-7}	0.0535	0.0097	0.2318	0.0431	0.5115	0.0876	0.8870	0.1363

$$v_t(x, y, t) = D_v \Delta v + k_2 u^2 v + F_2(x, y, t),$$

where u, v are concentrations of activator, and substrate, respectively. D_u, D_v are the diffusion coefficients of u, v and D_v is significantly faster than that of the activating species, i.e $D_v \gg D_u$. k_1, k_2 are reaction rate constants. This model is observed in several biological systems including zebra (stripes), minor worker termites (concentric circles), aggregation of slime moulds (spirals) and leopards (randomly distributed dots).

The reaction terms $F = k_1 u^2 v - k_2 u + F_1(x, y, t)$ and $G = k_2 u^2 v + F_2(x, y, t)$ are locally Lipschitz functions with Lipschitz constant $K_{R^\varepsilon} = \sqrt{8} \max\{k_1, k_2\}(2(R^\varepsilon)^2 + 1)$. We shall implement the model in 2-D region $\Omega = [0, \pi]^2$ with the parameters $k_1 = k_2 = 0.01$, $T = 1$, $\mathcal{D}_u = 0.01$, $\mathcal{D}_v = 0.2$, and choose $M_0 = 0.2 + 10^{-10}$.

Table 5
The relative errors at various times $t \in \{0.9, 0.5, 0.3, 0.1\}$, for various amounts of noise $\varepsilon \in \{10^{-2}, 10^{-3}, 10^{-4}, \}$ for Ex. 2.

ε	$r\epsilon_u(0.9)$	$r\epsilon_v(0.9)$	$r\epsilon_u(0.5)$	$r\epsilon_v(0.5)$	$r\epsilon_u(0.3)$	$r\epsilon_v(0.3)$	$r\epsilon_u(0.1)$	$r\epsilon_v(0.1)$
10^{-2}	0.0358	0.0158	0.0346	0.0221	0.0358	0.0536	0.0403	0.1002
10^{-3}	0.0221	0.0095	0.0106	0.0384	0.0185	0.0981	0.0328	0.2557
10^{-4}	0.0095	0.0072	0.0531	0.1136	0.0679	0.4506	0.0997	1.8405

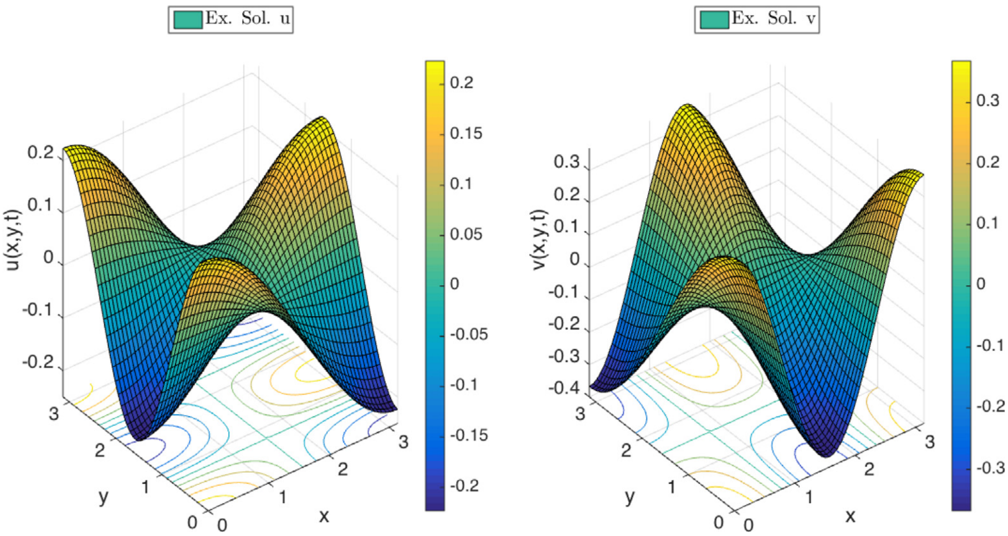

Fig. 5. The exact solution in Ex. 2 at $t = 0.5$.

At the discretization level, a uniform grid of mesh-points (x_i, y_j, t_k) is used

$$\Delta x = \frac{\pi}{N_x}, x_i = (i-1)\Delta x, i = \overline{1, N_x+1},$$
$$\Delta y = \frac{\pi}{N_y}, y_j = (j-1)\Delta x, j = \overline{1, N_y+1},$$
$$\Delta t = \frac{T}{N_t}, t_k = (k-1)\Delta t, k = \overline{1, N_t+1}.$$

An orthonormal basis in $[L^2(\Omega)]^2$, consisting eigenfunctions of Laplacian, satisfying Neumann boundary condition, with the corresponding eigenvalues

$$\mu_{mn}(x,y) = \frac{2}{\pi}\cos(mx)\cos(ny), \quad \lambda_{mn} = m^2 + n^2, \quad m, n \in \mathbb{N}.$$

Denote the number of terms of truncated Fourier series by $(M+1)(N+1) : 0 \le m \le M, 0 \le n \le N$. We use aforementioned Picard iteration method, as in (87) to find the Fourier coefficients

$$\widetilde{u}_{mn} = \langle \widetilde{u}, \mu_{mn} \rangle, \quad \widetilde{v}_{mn} = \langle \widetilde{v}, \mu_{mn} \rangle.$$

The composite Simpson rule for 2-D integration is represented as follows

$$\int_0^\pi \int_0^\pi f(x,y)\mathrm{d}x\mathrm{d}y \approx \frac{\Delta x \Delta y}{9} \sum_{i=1}^{N_x+1} \sum_{j=1}^{Ny+1} c_{ij} f(x_i, y_j),$$

where $c_{ij} = \begin{cases} 4, & \text{if } ((i=1 \vee i = N_x+1) \wedge j = 2l) \vee ((j=1 \vee j = N_y+1) \wedge i = 2l), \\ 2, & \text{if } ((i=1 \vee i = N_x+1) \wedge j = 2l+1) \vee ((j=1 \vee j = N_y+1) \wedge i = 2l+1), \\ 8, & \text{if } (i \neq 1, N_x+1) \wedge (j \neq 1, N_y+1) \wedge (i+j = 2l+1), \\ 16, & \text{if } (i = 2l, j = 2k), \\ 4, & \text{if } (i \neq 1, N_x+1) \wedge (j \neq 1, N_y+1) \wedge (i = 2l+1, j = 2k+1), \end{cases}$ for some integers k, l.

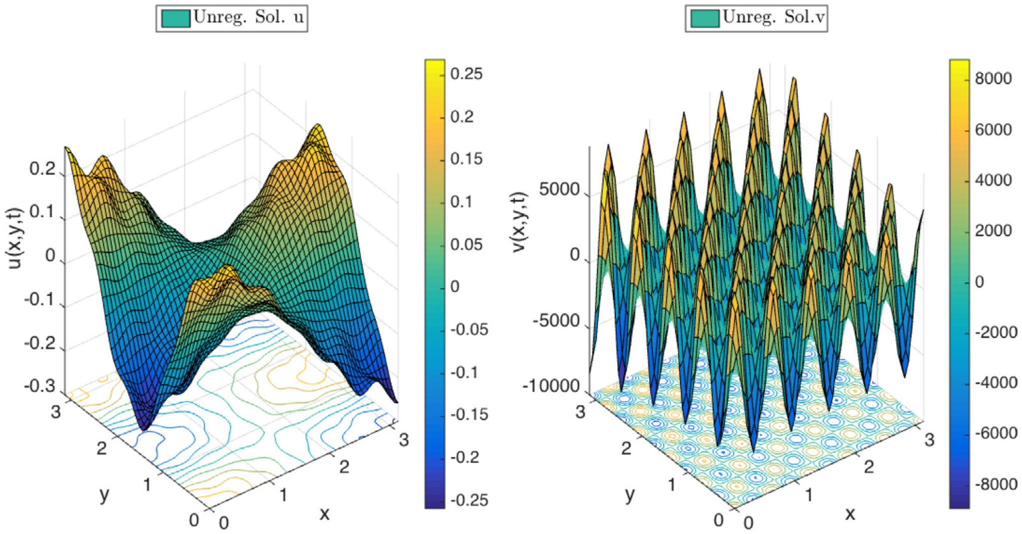

Fig. 6. The unregularized solution in Ex. 2 at $t = 0.5$, with $\varepsilon = 10^{-3}$.

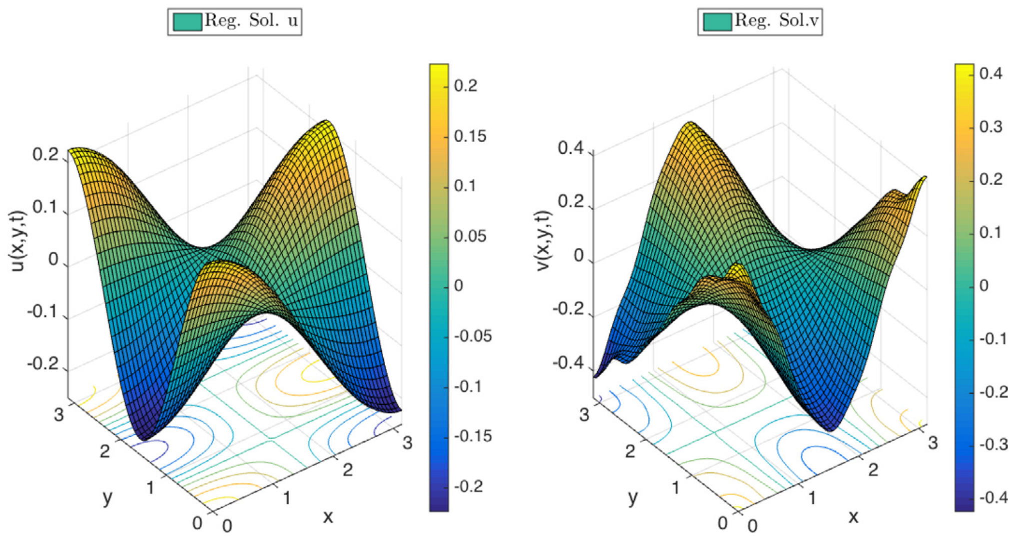

Fig. 7. The regularized solution in Ex. 2 at $t = 0.5$, with $\varepsilon = 10^{-3}$.

Taking the external sources

$$F_1(x, y, t) = -2.98e^{-3t} \cos 2y \cos x - 0.01e^{-8t} \cos 2x \cos^2 2y \cos^2 x \cos y,$$
$$F_2(x, y, t) = -1.2e^{-2t} \cos 2x \cos y - 0.01e^{-8t} \cos 2x \cos^2 2y \cos^2 x \cos y,$$

such that our exact solution is

$$u_{\text{ex}}(x, y, t) = e^{-3t} \cos x \cos 2y, \quad v_{\text{ex}}(x, y, t) = e^{-2t} \cos 2x \cos y.$$

The final data are given by

$$\Psi(x, y) = e^{-3} \cos x \cos 2y, \quad \Phi(x, y) = e^{-2} \cos 2x \cos y.$$

The radius chosen here $R^{\varepsilon} \geq 2$. Consequently, we have $K_{R^{\varepsilon}} \geq 0.2546$. The condition (63) with $\gamma = \frac{2}{3}$ holds true when $\varepsilon \leq 0.01$ Let $N_x = N_y = N_t = 50$, and the truncation levels: $M = N = 10$. The exact solution, numerical unregularized solution and the solution after regularization are presented in Figs. 5, 6, 7. Table 5 gives the relative error comparison at various times $t \in \{0.9, 0.5, 0.3, 0.1\}$ for various noise levels $\varepsilon \in \{10^{-2}, 10^{-3}, 10^{-4}\}$, and shows the similar phenomenon as in the previous subsection: when t reaches 0, the relative errors increases rapidly and the numerical regularized solution tends to lose its stability, the smaller the noise level, the greater the magnitude of approximation errors.

Remark 6.1. In case that we cannot find the analytical true solution, first, we can solve the forward problem to numerically simulate the terminal data, perturb them, and then use the perturbed final data for the backward problem.

7. Conclusion

In this study, we solved the unstable backward problem (1) with nonlocal diffusions and locally Lipschitz nonlinear reactions by suggesting a modified QR method. In the theoretical results, we obtained the error estimates in both L^2- and H^1-norms. We implemented 2 biological models in simple cases to verify the result in L^2-norm only. From the numerical tests, it shows that the regularized solutions are convergent to the true solutions and the convergence speed decreases rapidly as t tends to 0.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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