Mathematical Circus Clown Tricks

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Resumo

The Mathematical Circus project was created in 2011 by the LUDUS association with the main goal of promoting the interest and motivation for learning mathematics. The Mathematical Circus team performs mathematical magic shows where complementary skills are merged to produce a high intervention capacity within a wide geographical range in Portugal. Some tricks are performed by a mathematical clown, a unique character in the circus world, who brings together the usual foolish characteristics of a clown and the rigorous mathematical knowledge. In this paper we will describe and explain some of the tricks of the Mathematical Circus repertoire involving the clown.

Keywords: mathematical magic, recreational mathematics, mathematical circus.

1 Introduction

2 Who wants to be a millionaire trick

The mathematical circus's trick "Who wants to be a millionaire"starts with the problem of finding "which object is the odd one out"and is connected mathematical logic. This kind of problems is one of the important topics in competitive exams or employment tests in which mathematics logical questions are being asked and the solution is expected to be unique. Nevertheless, while the criteria one uses to get the correct solution is the most obvious one, many times we could use different criteria and obtain another answer that is still based on logical thought.

For example, if one asks

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Which one of the numbers $28, 49, 63, 23$ and 56 is the odd one out

The most obvious answer would be 23 because all the other numbers are divisible by 7, but any of the following answers would also be logically correct.

Only 63 has less units than tens.
Only 23 is prime
Only 49 has a O in its spelling
Only 28 has an H
Only 56 has two Fs (or two Is) in its spelling
Only 49 has two Ns in its spelling
Only 49 is a square
Only 23 has no digit with a closed shape $(8, 6, 9, 0, 4)$ in it
Only 49 has 2 digits with closed shapes
Only 49 has no digit in common with any of the other numbers.

In this trick, each volunteer is convinced that his is the correct answer, but after each of them to explains his reasoning they all (and all the members of the audience) become aware of the other possible logical explanations.

2.1 Clown's Divisions

The second part of the trick (clown's elementary arithmetic operations) serves to explain why writing an algorithm in the correct form is so important. This is one of our most successful tricks and everyone at the end smiles, even the students of ages between 7 and 10 who start by shouting "that is wrong!", "you don't know how to do it!", "No! The correct answer is ...".

When the clown "proves"that 24 divided by four is 15, he uses the following reasoning

> "four doesn't go into two, but four goes into four, once. One times four is four, subtracted from four is zero. Next, we pull down the two. Now, four doesn?t go into two neither does it go into zero. But four goes into twenty, five times. Finally, subtracting twenty from twenty leaves us with a zero remainder"

So, one question that comes to mind is: For which numbers can we apply a similar reasoning and obtain a zero remainder? Any such example we will call a *clown division*.

Suppose we want to divide a two-digits number n, i.e. $n = n_1 \times 10 + n_0$, with $n_1 > 0$ by a one-digit number $d < 10$. If we follow the clown reasoning, we obtain the following conditions:

- $d > n_1$;
- $d \leq n_0$ and $n_0 = dq_1 + r$, where q_1 and r are positive integers and $r < d$;
- after pulling down n_1 one realizes that d can't go into n_1 or into r, but it can go into $n_1 \times 10 + r$, q_0 times, that is $n_1 \times 10 + r = dq_0$.

Therefore, in order to perform the clown's "division algorithm", we must have

$$\begin{aligned} n_1 \times 10 + n_0 &= n_1 \times 10 + d \times q_1 + r \\ &= d \times q_0 + d \times q_1 \\ &= d \times (q_1 + q_0) \end{aligned}$$

with the conditions

$$10 > n_0 \geq d > n1 > 0 \quad \text{and} \quad n_0 = d \times q_1 + r \quad \text{where} \quad r < d.$$

The only numbers that satisfy all these conditions and so, for which we can perform the clown's division, are the following:

$$\begin{array}{rcl} 12 \div 2 &=& 15 \\ 14 \div 2 &=& 25 \\ 16 \div 2 &=& 35 \\ 18 \div 2 &=& 45 \\ 15 \div 3 &=& 14 \\ 18 \div 3 &=& 24 \\ 24 \div 3 &=& 17 \\ 27 \div 3 &=& 27 \end{array} \quad \begin{array}{rcl} 16 \div 4 &=& 13 \\ 24 \div 4 &=& 15 \\ 28 \div 4 &=& 25 \\ 36 \div 4 &=& 18 \\ 15 \div 5 &=& 12 \\ 25 \div 5 &=& 14 \\ 35 \div 5 &=& 16 \\ 45 \div 5 &=& 18 \end{array} \quad \text{and} \quad \begin{array}{rcl} 18 \div 6 &=& 12 \\ 36 \div 6 &=& 15 \\ 48 \div 6 &=& 17 \\ 28 \div 7 &=& 13 \\ 49 \div 7 &=& 16 \\ 48 \div 8 &=& 15 \end{array}$$

2.2 Number of clown's division in a basis b

Suppose we want to perform this trick using a basis b different from 10, i.e. we want to "divide" $n_1 \times b + n_0$ by d and obtain $q_1 q_0$ such that

$$b > n_0 \geq d > n1 > 0, \qquad n_0 = d \times q_1 + r \quad \text{where} \quad r < d$$

and

$$n_1 \times b + n_0 = d(q_1 + q_0).$$

For example we get the clown divisions:

Basis 12: $14 \div 4 = 13$ and $13 \div 3 = 14$
Basis 4: $12 \div 2 = 12$
Basis 5: $13 \div 2 = 13.$

So, given a basis b, how many clown's divisions do we have?

Let $b > d \geq 1$ be integers and denote by $\Omega_{b,d}$ the number of clown's divisions one can perform in basis b, when the quotient is d and denote by Γ_b the total number of clown's divisions that are possible in basis b. From above, we can see that $\Omega_{10,i} = 4$ for $i = 2, 3, 4, 5$ and for $i = 6$. Also $\Gamma_1 0 = 22$.

If $b = 2$ there exists no integers that satisfy $2 > n_0 > n_1 > 0$, If $b = 3$ then we would need to have $n_1 = 1, d = 2$ and $n_0 = 2$, but $1 \times 3 + 2 = 5$ which is not a multiple of 2. So, it is not possible to make clown divisions in basis two or three, that is, $\Gamma_2 = Gamma_3 = 0$.

Let x be a real number and denote by $\lfloor x \rfloor$ hhe greatest integer that is smaller or equal to x and by $\lceil x \rceil$ the smallest integer that is greater or equal to x.

Now, suppose $b \geq 4$ and take d such that $2 \leq d < b$. We know that there are b multiples of d in the sequence $1, 2, 3, \ldots bd$ and there are $\lfloor \frac{b}{d} \rfloor$ multiples of d in the sequence $1, 2, 3, \ldots, b$. Therefore, we have $b - \lfloor \frac{b}{d} \rfloor$ multiples of d between $b + 1$ and db. Below, we will see how many of these multiples, when written in basis b, have the units-digit greater or equal to d but before that we need to recall the following elementary result about congruences.

Proposition 2.1 ([3][Theorem 57)**.** *] Let a, m and n be integers and consider $\delta = \gcd(a, n)$. Then the congruence $ax \equiv m \mod n$ is soluble if and only if δ divides m, in which case it has δ solutions.*

Let's proceed in our quest to find how many clown divisions exists in basis b. Take $\delta = \gcd(d, b)$. In the sequence $0, 1, \ldots, d - 1$ there are $\frac{d}{\delta}$ multiples of δ. Let v be a multiple of δ in this sequence, then the congruence $dx \equiv v \mod b$ has δ solutions. Since $\frac{d}{\delta} \times \delta = d$ there are d multiples of d between 1 and db that have a units-digit (when written in basis b) smaller than d.

Now, if d is not a divisor of b then all the multiples of d smaller or equal to b have a units-digit greater or equal to d, so the d multiples of d that have a units-digit smaller than d are all greater than b. So

Therefore, if b is not a multiple of d we have

$$b - \left\lfloor \frac{b}{d} \right\rfloor - d$$

clown divisions for the divisor d.

If d divides b then we have exactly one multiple of d which has its units-digit smaller than d, namely $1 \times b + 0$. So the number of clown divisions in this case, is

$$b - \frac{b}{d} - (d - 1).$$

Note that, if b is not a multiple of d then

$$b - \left\lceil \frac{b}{d} \right\rceil - (d - 1) = b - \left\lfloor \frac{b}{d} \right\rfloor - d,$$

and if d divides b then

$$b - \left\lceil \frac{b}{d} \right\rceil - (d - 1) = b - \frac{b}{d} - (d - 1).$$

We just proved the following result

Theorem 2.2. *Let $b > 1$ be a positive integer and $1 \leq d \leq b$. Then there are exactly*

$$\Omega_{b,d} = b - \left\lceil \frac{b}{d} \right\rceil - (d-1)$$

multiples of d for which it is possible to do a clown division. Or in another words, there are exactly $\Omega b, d$ quadruples (n_0, n_1, q_0, q_1) such that $n_1 \times b + n_0 = d(q_0 + q_1)$,

$$10 > n_0 \geq d > n1 > 0 \quad \text{and} \quad n_0 = d \times q_1 + r \quad \text{where} \quad r < d.$$

For example, we obtain $\Omega_{10,4} = 10-3-3 = 4$ or $\Omega_{b,b-1} = b-2-(b-2) = 0$, for any $b > 1$.

In order to obtain an approximation for the number of clown divisions in a basis b we recall a result connected with the divisors summatory funtion and Dirichlet divisor problem.

Theorem 2.3 ([4][section 3.2)**.** *] Let x be a real number and γ be the Euler-Mascheroni constant. Let $\tau(n)$ be the number of divisors of n. Then*

$$\sum_{1 \leq n \leq x} \tau(n) = \sum_{1 \leq n \leq x} \left\lfloor \frac{x}{n} \right\rfloor = x \log(x) + (2\gamma - 1)x + \Delta(x),$$

where $\Delta(x) = O(x^{\frac{1}{3}})$.

Theorem 2.4. *Let $b > 1$ be a positive integer. Then there are exactly*

$$\Gamma_b = \frac{(b-2)(b+1)}{2} - \sum_{2 \leq d \leq b-1} \left\lceil \frac{b}{d} \right\rceil.$$

Moreover, Γ_b is approximately

$$\frac{(b-2)(b+1)}{2} - b\log(b) + (2\gamma - 1)b + \tau(b),$$

where γ is the Euler-Mascheroni constant.

Demonstração. We have

$$\begin{aligned}
\Gamma_b &= \sum_{2 \leq d \leq b-1} \Omega_{b,d} \\
&= \sum_{2 \leq d \leq b-1} \left(b - \left\lceil \frac{b}{d} \right\rceil - (d-1) \right) \\
&= b(b-2) - \frac{(b-1)(b-2)}{2} - \sum_{2 \leq d \leq b-1} \left\lceil \frac{b}{d} \right\rceil \\
&= \frac{(b\ 2)(b+1)}{2} + \sum_{2 \leq d \leq b-1} \left\lceil \frac{b}{d} \right\rceil
\end{aligned}$$

The second statement follows immediately from theorem?? and the fact that

$$\sum_{2\leq d\leq b-1}\left\lceil\frac{b}{d}\right\rceil=[\sum_{1\leq d\leq b-1}\left\lfloor\frac{b}{d}\right\rfloor-\tau(b)$$

□

Foe example $\Gamma_{10}=44-22=22$ and, for hexadecimal numbers, $\Gamma_{16}=119-44=75$. For the first values of b we obtain the following values of Γ_b:

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Γ_b	0	0	1	2	5	7	12	16	22	28	37	43	54	64	75

Note that $\Gamma_b=a(b-1)$, where $a(n)$ is the sequence A161664 in Sloane's The Online Encyclopedia of Integer Sequences (https://oeis.org/A161664).

2.3 Clown's greedy divisions

Suppose that in the "Who wants to be a millionaire"trick, the clown has 27 sweets and there are three winners, if the clown uses his "division"algorithm, he would be really greedy and would get all the 27 sweets. If we have a clown's division for which the dividend is equal to the quotient we will call that "division"a *clown's greedy division*. We saw above that we can also have clown's greedy divisions in other basis, e.g. in basis 4, $12\div 2=12$ and in basis 5, $13\div 3=13$. So, now we will focus on what are the necessary conditions for a basis b to have a clown's greedy division.

Let b,d,n^0,n_1 and r be integers such that $b>n_0\geq d>n1>0$, $n_0=d\times n_1+r$, $0\leq r<d$ and $n_1\times b+n_0=d(n_1+n_0)$. From the last condition we obtain

$$(b-d)n_1=(d-1)n_0.$$

Since $n_0=d\times n_1+r$ and $r\geq 0$, then $n_0\geq dn_1$. So, $(b-d)n_1\geq d(d-1)n_1$. Hence,

$$b\geq d^2.$$

Let $v=b-d^2$. Then $n_1\times b+n_0=d(n_1+n_0)$ and $n_0=d\times n_1+r$ implies $n_1(d^2+v)+dn_1+r=dn_1+d(dn_1+r)$. Therefore,

$$n_1v=(d-1)r.$$

Let $\delta=\gcd(v,d-1)$. Then $\gcd\left(\frac{v}{\delta},\frac{d-1}{\delta}\right)=1$. Now, since $\frac{d-1}{\delta}$ divides $n_1\frac{v}{\delta}$ and $\frac{d-1}{\delta}$ is coprime with $\frac{v}{\delta}$, then $\frac{d-1}{\delta}$ must divide n_1. Similarly, $\frac{v}{\delta}$ divides r.

Moreover, since $d>r$, $\delta\leq d-1$ and $\frac{v}{\delta}\leq r$, we must have

$$v\leq(d-1)^2.$$

We can now state and prove a result that identifies all the clown's greedy divisions for a basis b.

Theorem 2.5. *Let $b > 1$ be an integer. If there is no positive integer d such that $d^2 \leq b \leq d^2 + (d-1)^2$, then there is no clown's greedy division for the basis b. Otherwise take d such that $d^2 \leq b$ and $d^2 + (d-1)^2 \geq b$, take $v = b - d^2$ and take $\delta = \gcd(v, d-1)$. If there exists a multiple, of $\frac{v}{\delta}$, say r, such that*

a) $0 \leq r \leq d - 1$,

b) $\frac{(d-1)r}{v} \leq d - 1$,

then, taking $n_1 = \frac{(d-1)r}{v}$ and $n_0 = dn_1 + r$, we obtain the clown's greedy division $n_1 \times b + n_0 \div d = n_1 n_0$.

Demonstração. Clearly, we just need to see if $dn_0 = n_1 \times b + r$. But

$$\begin{aligned} n_1 \times b + r &= n_1 d^2 + n_1 v + r \\ &= n_1 d^2 + (d-1)r + r \\ &= dn_0. \end{aligned}$$

□

If $b = 10$ then d must be 3, $v = 1$ and $\delta = 1$. Therefore $r = 2$ would imply $n_1 = 4 > d - 1$, so $r = 1$, $n_1 = 2$ and $n_0 = 7$. Hence we get the clown division $2 \times 10 + 7 \div 3 = 27$.

Suppose $b = 18$, for example. Then $d = 4$, $v = 2$ and $\delta = 1$. Therefore $r = 2$, $n_1 = 3$ and $n_0 = 14$. Let's denote n_0 by E. Then $3 \times b + E \div 4 = 3E$.

There are many basis b for which we have no clown greedy division. For example if $b = 14$ or $b = 15$ we have no clown greedy divisions because there exists no d for which $d^2 \leq b \leq d^2 + (d-1)^2$. Also, if $b = 20$, $b = 21$, $b = 23$ or $b = 24$, then d would have to be 4, and v would be $4, 5, 7$ or 8 respectively. But in each case we would get $\delta = 1$ which would imply $r \geq d$.

One question remains "Is there a value b_0, such that if $b \geq b_0$ there is always one clown's greedy division for the basis b?".

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