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Minimal surfaces with arbitrary topology in $\mathbb{H}^2 \times \mathbb{R}$

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We show that any open orientable surface S can be properly embedded in $\mathbb{H}^2 \times \mathbb{R}$ as an area-minimizing surface.

53A10; 53C42

1 Introduction

Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ has been an attractive topic for the last two decades. After Nelli and Rosenberg's seminal results [15] on minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, the theory has flourished very quickly with substantial results on the existence, regularity and other properties of minimal and CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$, from eg Collin and Rosenberg [1], Coskunuzer, Meeks and Tinaglia [5], Hauswirth, Nelli, Sa Earp and Toubiana [8], Kloeckner and Mazzeo [10], Martín, Mazzeo and Rodríguez [11], Mazet, Rodríguez and Rosenberg [14], Nelli, Sa Earp, Santos and Toubiana [16], Rodríguez and Tinaglia [17] and Sa Earp and Toubiana [18].

Here we are interested in the following question: *hat type of surfaces can be embedded into* $\mathbb{H}^2 \times \mathbb{R}$ *as a complete minimal surface*? Ros conjectured that any open orientable surface can be properly embedded in $\mathbb{H}^2 \times \mathbb{R}$ as a minimal surface (see Martín and Rodríguez [12]). We prove this conjecture.

Theorem 1.1 *Any open orientable surface* S *can be properly embedded in* $\mathbb{H}^2 \times \mathbb{R}$ *as a complete area-minimizing surface.*

In particular, this implies that any open orientable surface S can be realized as an complete, embedded, minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. The key step is to show a vertical bridge principle for tall curves in $S^1_\infty \times \mathbb{R}$ (Section 3). Then, by using the positive solutions of the asymptotic plateau problem, we give a general construction to obtain complete, properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with arbitrary topology, ie any (finite or infinite) number of genus and ends.

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The outline of the method is as follows: We start with a simple exhaustion of the open orientable surface S, ie $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$, where $S = \bigcup_{n=1}^{\infty} S_n$. In particular, the surface S is constructed by starting with a disk $D = S_1$, and by adding 1–handles iteratively, ie $S_{n+1} - \mathrm{int}(S_n)$ is either a pair of pants or a cylinder with a handle (see Figure 4). Hence, after proving a bridge principle at infinity for $\mathbb{H}^2 \times \mathbb{R}$ for vertical bridges in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ (Theorem 3.2), we started the construction with an area-minimizing plane Σ_1 in $\mathbb{H}^2 \times \mathbb{R}$. Then, by following the iterative process dictated by the simple exhaustion, if S_{n+1} is a pair of pants attached to S_n, then we attach one vertical bridge in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ to the corresponding component of $\partial_\infty \Sigma_n$. Similarly, if S_{n+1} is a cylinder with a handle attached to S_n, then we attach two vertical bridges successively to $\partial_\infty \Sigma_n$ (see Figure 5) so that the number of boundary components of $\partial \Sigma_n$ and $\partial \Sigma_{n+1}$ are the same. By iterating this process, we inductively construct a properly embedded minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with the same topological type as S.

The organization of the paper is as follows. In the next section, we give some definitions, and introduce the basic tools which we use in our construction. In Section 3, we show the bridge principle at infinity in $\mathbb{H}^2 \times \mathbb{R}$ for sufficiently long vertical bridges. In Section 4, we prove the main result above. In Section 5, we discuss generalization of our result to H–surfaces and the finite total curvature case. We postpone some technical steps to the appendix, where we also prove a generic uniqueness result for tall curves.

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2 Preliminaries

In this section, we introduce our setup, and the basic tools which we use in our construction.

Throughout, $\overline{\mathbb{H}^2 \times \mathbb{R}} = \overline{\mathbb{H}^2} \times \overline{\mathbb{R}} = \mathbb{H}^2 \times \mathbb{R} \cup \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ represents the natural product compactification of $\mathbb{H}^2 \times \mathbb{R}$. In particular, $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) = (S^1_\infty \times \mathbb{R}) \cup (\overline{\mathbb{H}^2} \times \{\pm\infty\})$

represents the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$. Also, we call $S^1_\infty \times \mathbb{R}$ as the asymptotic cylinder, and $\overline{\mathbb{H}^2} \times \{\pm\infty\}$ as the caps at infinity.

Convention Throughout the paper, by *curve* we mean a finite collection of smooth Jordan curves unless otherwise stated.

A curve Γ in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ is *finite* if $\Gamma \subset S^1_\infty \times \mathbb{R}$. If $\Gamma \cap \overline{\mathbb{H}^2} \times \{\pm\infty\} \neq \varnothing$, we say Γ is *infinite*. Throughout the paper, all the curves in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ will be finite curves unless stated otherwise. For the asymptotic plateau problem for infinite curves, see [10; 3].

Definition 2.1 A compact surface with boundary Σ is called an *area-minimizing surface* if Σ has the smallest area among surfaces with the same boundary. A noncompact surface is called an *area-minimizing surface* if any compact subsurface is an area-minimizing surface.

For our construction, one of our key ingredients is the solutions of the following problem:

The asymptotic plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ *Let Γ be a collection of Jordan curves in $S^1_\infty \times \mathbb{R}$. Does there exist a complete, embedded minimal surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$?*

Here, Σ is an open, complete surface in $\mathbb{H}^2 \times \mathbb{R}$, and $\partial_\infty \Sigma$ represents the asymptotic boundary of Σ in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$. Then $\overline{\Sigma}$ is the closure of Σ in $\overline{\mathbb{H}^2 \times \mathbb{R}}$, then $\partial_\infty \Sigma = \overline{\Sigma} \cap \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$. Here, we stated the most general version of this problem. There are various results on this problem in the literature. For our construction, we need the positive solutions in a special case: tall curves (Lemma 2.4).

Definition 2.2 (tall curves) Consider the asymptotic cylinder $S^1_\infty \times \mathbb{R}$ with the coordinates (θ, t) where $\theta \in [0, 2\pi)$ and $t \in \mathbb{R}$. We call a rectangle $R = [\theta_1, \theta_2] \times [t_1, t_2] \subset S^1_\infty \times \mathbb{R}$ *tall* if $t_2 - t_1 > \pi$.

We call a finite collection of disjoint simple closed curves Γ in $S^1_\infty \times \mathbb{R}$ *tall* if the region $\Gamma^c = S^1_\infty \times \mathbb{R} - \Gamma$ can be written as a union of open tall rectangles $R_i = (\theta^i_1, \theta^i_2) \times (t^i_1, t^i_2)$, ie $\Gamma^c = \bigcup_i R_i$.

We call a region Ω in $S^1_\infty \times \mathbb{R}$ a *tall region* if Ω can be written as a union of tall rectangles, ie $\Omega = \bigcup_i R_i$, where R_i is a tall rectangle.

Note that tall rectangles in $S^1_\infty \times \mathbb{R}$ are very special. In a way, they behave like round circles in $S^2_\infty(\mathbb{H}^3)$.

Lemma 2.3 [4, Lemma 3.2] *Let R be a tall rectangle in $S^1_\infty \times \mathbb{R}$. Then there exists a unique minimal surface $\mathcal{P}$ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \mathcal{P} = \partial R$.*

Furthermore, Sa Earp and Toubiana [18] gave an explicit description of the disk type minimal surface $\mathcal{P}$ [4, Section 3].

The key component of our construction is the positive solution of following special case of the asymptotic plateau problem:

Lemma 2.4 (tall curves are strongly fillable [4, Theorem 4.1]) *Let Γ be a finite collection of disjoint, smooth Jordan curves in $S^1_\infty \times \mathbb{R}$ with $h(\Gamma) \neq \pi$. Then there exists a complete, embedded, area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma = \Gamma$ if and only if Γ is a tall curve.*

The next lemma is an asymptotic regularity result for complete, embedded, area-minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Lemma 2.5 [4, Lemma 7.6] *Let Σ be a complete area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$. Let $\overline{\Sigma}$ be the closure of Σ in $\overline{\mathbb{H}^2 \times \mathbb{R}}$, and let $\Gamma = \partial_\infty \Sigma$. If Γ is a tall curve, then $\overline{\Sigma}$ is a surface with boundary.*

Remark 2.6 In the lemma above, everything is in the $\mathcal{C}^0$ category. In [10, Section 3], Kloeckner and Mazzeo proved a stronger asymptotic regularity result for complete, embedded, minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ bounding $\mathcal{C}^{k,\alpha}$ smooth curves in $S^1_\infty \times \mathbb{R}$.

The following classical result of geometric measure theory will be very useful for our construction:

Lemma 2.7 existence and regularity of area-minimizing surfaces [6, Theorems 5.1.6 and 5.4.7] *Let M be a homogeneously regular, closed (or mean convex) 3–manifold. Let γ be a nullhomologous smooth curve in γ. Then γ bounds an area-minimizing surface Σ in M. Furthermore, any such area-minimizing surface is smoothly embedded.*

Now we state the convergence theorem for area-minimizing surfaces, which will be used throughout the paper. Note that we use convergence in the sense of geometric measure theory, ie the convergence of rectifiable currents in the flat metric.

Lemma 2.8 (convergence) *Let $\{\Sigma_i\}$ be a sequence of complete area-minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$, where $\Gamma_i = \partial_\infty \Sigma_i$ is a finite collection of closed curves in $S^1_\infty \times \mathbb{R}$. If Γ_i converges to a finite collection of closed curves $\widehat{\Gamma}$ in $S^1_\infty \times \mathbb{R}$, then there* *exists a subsequence* $\{\Sigma_{n_j}\}$ *such that* Σ_{n_j} *converges to an area-minimizing surface* $\widehat{\Sigma}$ *(possibly empty) with* $\partial_\infty \widehat{\Sigma} \subset \widehat{\Gamma}$. *In particular, the convergence is smooth on compact subsets of* $\mathbb{H}^2 \times \mathbb{R}$.

Proof Let $\Delta_n = \boldsymbol{B}_n(0) \times [-C, C]$ be convex domains in $\mathbb{H}^2 \times \mathbb{R}$, where $\boldsymbol{B}_n(0)$ is the closed disk of radius n in $\mathbb{H}^2$ with center 0 and $\widehat{\Gamma} \subset S^1_\infty \times (-C, C)$. For n sufficiently large, consider the surfaces $S_i^n = \Sigma_i \cap \Delta_n$. Since the areas of the surfaces $\{S_i^n \subset \Delta_n\}$ is uniformly bounded by $|\partial \Delta_n|$, and ∂S_i^n can be bounded by using standard techniques. Hence, if $\{S_i^n\}$ is an infinite sequence, then we get a convergent subsequence of $\{S_i^n\}$ in Δ_n with *nonempty limit* S^n. This S^n is an area-minimizing surface in Δ_n by the compactness theorem for rectifiable currents (codimension-1) with the flat metric of geometric measure theory (Lemma 2.7). By the regularity theory, the limit S^n is a smoothly embedded area-minimizing surface in Δ^n.

If the sequence $\{S_i^n\}$ is an infinite sequence for infinitely many n, we get an infinite sequence of compact area-minimizing surfaces $\{S^n\}$. Then, by using the diagonal sequence argument, we can find a subsequence of $\{\Sigma_i\}$ converging to an area-minimizing surface $\widehat{\Sigma}$ with $\partial_\infty \widehat{\Sigma} \subset \widehat{\Gamma}$ as $\Gamma_i \to \widehat{\Gamma}$. Note also that for fixed n, the curvatures of $\{S_i^n\}$ are uniformly bounded by curvature estimates for area-minimizing surfaces. Hence, with the uniform area bound, we get smooth convergence on compact subsets of $\mathbb{H}^2 \times \mathbb{R}$. For further details, see [13, Theorem 3.3]. □

Remark 2.9 In the lemma above, we can allow $\Gamma_i \subset S^1_\infty \times \mathbb{R}$ to be a collection of closed curves which may not be simple. Let Σ_i be an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma_i = \Gamma_i$. As Σ_i is an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$, it must be embedded by the regularity of area-minimizing surfaces. Hence, in such a case, $\overline{\Sigma}_i$ may not be an embedded surface with boundary in $\overline{\mathbb{H}^2 \times \mathbb{R}}$, even though Σ_i is an open embedded surface in $\mathbb{H}^2 \times \mathbb{R}$. Similarly, the limit $\widehat{\Sigma}$ (if nonempty) is an open embedded surface in $\mathbb{H}^2 \times \mathbb{R}$, even if $\partial_\infty \widehat{\Sigma} \subset \widehat{\Gamma}$ is not embedded. For the case when $\Gamma \subset S^1_\infty \times \mathbb{R}$ is tall and embedded, see also Lemma 2.5.

3 Vertical bridges at infinity

In this section, we prove a bridge principle at infinity for sufficiently long vertical bridges. Then, by using these bridges, we construct area-minimizing surfaces of arbitrary topology in $\mathbb{H}^2 \times \mathbb{R}$ in the next section.

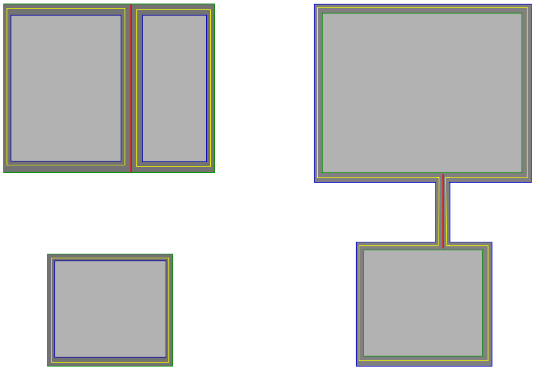

Figure 1: Here $\Gamma = \partial\Omega^\pm$ consists of the green curves with two components. Light shaded regions (in the right) represent Ω^+. Left: the case when the bridge α (red vertical line segment) is in Ω^+. Right: the case when α is in Ω^-. The family $\{\Gamma_t\}$ (yellow curves) foliate $\widehat{N}$ (dark shaded region). Here, $\Gamma_\epsilon \subset \partial\widehat{N}$ consists of the blue curves.

Definition 3.1 Let Γ be a collection of disjoint Jordan curves in $S^1_\infty \times \mathbb{R}$. If Γ bounds a unique area-minimizing surface Σ in $\mathbb{H}^2 \times \mathbb{R}$, we call Σ *a uniquely minimizing surface*, and we call Γ *a uni-curve*.

Notation and setup Let L_{θ_0} be a vertical line in $S^1_\infty \times \mathbb{R}$, ie $L_{\theta_0} = \{\theta_0\} \times \mathbb{R}$. Let $K_0 > \pi$ be as in Lemma A.7. Let Γ be a smooth tall uni-curve in $S^1_\infty \times \mathbb{R}$ with $\Gamma \cap L_{\theta_0} = \varnothing$ and $h(\Gamma) > K_0$. Let $\Omega^\pm$ be the tall regions in $S^1_\infty \times \mathbb{R}$ with $\Gamma^c = \Omega^+ \cup \Omega^-$ and $\partial\overline{\Omega^\pm} = \Gamma$.

Let $\alpha = \{\theta_1\} \times [c_1, c_2]$ be a vertical line segment in $S^1_\infty \times \mathbb{R}$ such that $\alpha \cap \Gamma = \partial\alpha$ and $\alpha \perp \Gamma$. Notice that $h(\Gamma) > K_0$ implies $c_2 - c_1 > K_0$, and $\alpha \cap \Gamma = \partial\alpha$ implies $\alpha \subset \overline{\Omega^+}$ or $\alpha \subset \overline{\Omega^-}$.

Consider a small open neighborhood $N(\Gamma \cup \alpha)$ of $\Gamma \cup \alpha$ in $S^1_\infty \times \mathbb{R}$. If $\alpha \subset \Omega^+$, let $\widehat{N} = N(\Gamma \cup \alpha) \cap \Omega^+$. If $\alpha \subset \Omega^-$, let $\widehat{N} = N(\Gamma \cup \alpha) \cap \Omega^-$. In other words, we only take one side $\widehat{N}$ of the open neighborhood $N(\Gamma \cup \alpha)$. Foliate $\widehat{N}$ by the smooth curves $\{\Gamma_t \mid t \in (0, \epsilon)\}$ with $\Gamma_\epsilon \subset \partial\widehat{N}$, and $\Gamma_0 = \Gamma \cup \alpha$ (see Figure 1). By taking a smaller neighborhood $N(\Gamma \cup \alpha)$ to start if necessary, we can assume that Γ_t is a smooth tall curve for any t.

Let S_α be a thin strip along α in $S^1_\infty \times \mathbb{R}$. In particular, if $N(\alpha)$ is a small neighborhood of α in $S^1_\infty \times \mathbb{R}$, then S_α is the component of $N(\alpha) - \Gamma$ containing α, ie $S_\alpha \sim [\theta_1 - \delta, \theta_1 + \delta] \times [c_1, c_2]$. In Figure 1, a tall curve Γ with two components is pictured.

In the left figure, the bridge α is in Ω^+, while in the right, α is in Ω^-. Notice that if $\partial\alpha$ is in the same component of Γ, then $\sharp(\Gamma_t) = \sharp(\Gamma) + 1$, where $\sharp(\cdot)$ represents the number of components (Figure 1, left). Similarly, if $\partial\alpha$ is in the different components of Γ, then $\sharp(\Gamma_t) = \sharp(\Gamma) - 1$ (Figure 1, right).

Now, consider the upper half-plane model for $\mathbb{H}^2 \simeq \{(x, y) \mid y > 0\}$. Without loss of generality, let $\theta_0 \in S^1_\infty(\mathbb{H}^2)$ correspond to the point at infinity in the upper half-plane model. We use the upper half-space model for $\mathbb{H}^2 \times \mathbb{R}$ with the identification $\mathbb{H}^2 \times \mathbb{R} = \{(x, y, z) \mid y > 0\}$, where $\mathbb{H}^2$ corresponds the xy–half-plane and $\mathbb{R}$ corresponds to the z–coordinate. Hence, the xz–plane will correspond to $S^1_\infty \times \mathbb{R}$. By using the isometries of the hyperbolic plane and the translation along the $\mathbb{R}$–direction, we assume that $\theta_1 \in S^1_\infty(\mathbb{H}^2)$ will correspond to 0, and the vertical line segment $\alpha \subset S^1_\infty \times \mathbb{R}$ above will have $\alpha = \{(0, 0)\} \times [c_1, c_2]$ and $S_\alpha \sim [-\delta, \delta] \times \{0\} \times [c_1, c_2]$ in (x, y, z)–coordinates.

With this notation, we can state the bridge principle at infinity for vertical bridges in $S^1_\infty \times \mathbb{R}$ as follows.

Theorem 3.2 (vertical bridges at infinity) *Let Γ be a tall uni-curve with $h(\Gamma) \geq K_0$ as above. Define α, Γ_t and S_α accordingly, as described above. Let Σ be the uniquely minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$, where $\partial_\infty \Sigma = \Gamma$. Assume also that $\overline{\Sigma}$ has finite genus. Then there exists a sufficiently small $t > 0$ such that Γ_t bounds a unique area-minimizing surface Σ_t, where Σ_t is homeomorphic to $\Sigma \cup S_\alpha$, ie $\Sigma_t \simeq \Sigma \cup S_\alpha$.*

Outline of the proof Let $\Gamma_t \to (\Gamma \cup \alpha)$ as above. Let Σ_t be the area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma_t = \Gamma_t$. Intuitively, for sufficiently large $n > 0$, we want to show that Σ_{t_n} is just Σ with a thin strip along α, where the thin strip vanishes as $n \to \infty$. We split the proof into four steps. In Step 1, we blow up the sequence $\{\Sigma_{t_n}\}$ and show that the limit $T = \lim \Sigma_{t_n}$ cannot contain the vertical segment α. In Step 2, we show that Σ_t does not develop any genus near the asymptotic boundary. In Step 3, we show that $\Sigma_t \simeq \Sigma \cup S_\alpha$ for t sufficiently close to 0. Finally, in Step 4, by using generic uniqueness, we show that we can choose $t > 0$ such that Γ_t bounds a unique area-minimizing surface Σ_t.

Proof First, by Lemma 2.4, for any $\Gamma_t \subset S^1_\infty \times \mathbb{R}$, there exists an area-minimizing surface Σ_t with $\partial_\infty \Sigma_t = \Gamma_t$.

As $t_n \searrow 0$, $\Gamma_{t_n} \to \Gamma \cup \alpha$. Since Γ_{t_n} is a tall curve, there exists an area-minimizing surface Σ_{t_n} in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma_{t_n} = \Gamma_{t_n}$ by Lemma 2.4. By Lemma 2.8, there exists a convergent subsequence, say Σ_n, converging to an area-minimizing surface T with $\partial_\infty T \subset \Gamma \cup \alpha$. Since $\Gamma \cup \alpha$ is a tall curve, the limit T is nonempty by the proof of Lemma 2.4.

Now, we claim that $\partial_\infty T = \Gamma$. In other words, the limit area-minimizing surface T with $\partial_\infty T \subset \Gamma \cup \alpha$ cannot have the vertical segment α in its asymptotic boundary. Then, since Γ bounds a unique area-minimizing surface Σ, $\partial_\infty T = \Gamma$ would imply $T = \Sigma$.

Step 1 $\partial_\infty T = \Gamma$.

Proof By the above, we know that $\partial_\infty T \subset \Gamma \cup \alpha$. By Lemma 2.5, $\overline{T} = T \cup \partial_\infty T$ is a surface with boundary in $\overline{\mathbb{H}^2 \times \mathbb{R}}$. As $\overline{\mathbb{H}^2 \times \mathbb{R}}$ is topologically a closed ball, $\overline{T}$ is separating in $\overline{\mathbb{H}^2 \times \mathbb{R}}$.

Assume that there is a point $p \in \alpha - \partial\alpha$ such that $p \in \partial_\infty T$. By using the notation and the upper half-space model described before the theorem, recall that $\alpha = \{(0,0)\} \times [c_1, c_2]$, and, without loss of generality, assume $p = (0,0,0) \in \alpha \subset S^1_\infty \times \mathbb{R}$. Consider the hyperbolic plane $P = \mathbb{H}^2 \times \{0\} = \{(x,y,0) \mid y > 0\}$ in $\mathbb{H}^2 \times \mathbb{R}$. Let γ_i be the geodesic arc in P with $\partial_\infty \gamma_i = \{(-r_i,0,0),(+r_i,0,0)\}$, where $r_i \searrow 0$. Let $\mathcal{U}_i = \gamma_i \times [-\epsilon_0, +\epsilon_0]$ for some fixed ϵ_0. Then, since $p \in \partial_\infty T$, $T \cap \mathcal{U}_i \neq \varnothing$ for $i > N_0$ for some N_0. Let $q_i \in T \cap \mathcal{U}_i$ for any $i > N_0$.

Now, let φ_i be the isometry of $\mathbb{H}^2 \times \mathbb{R}$ with $\varphi_i(x,y,z) = ((1/r_i)x, (1/r_i)y, z)$. Define a sequence of area-minimizing surfaces $T_i = \varphi_i(T)$. Let $\hat{\gamma}$ be the geodesic in P with $\partial_\infty \hat{\gamma} = \{(-1,0,0),(1,0,0)\}$. Hence, by construction, $\varphi_i(\gamma_i) = \hat{\gamma}$ and $\varphi_i(\mathcal{U}_i) = \hat{\mathcal{U}} = \hat{\gamma} \times [-\epsilon_0, +\epsilon_0]$ for any $i > 0$. Let $\hat{q}_i = \varphi_i(q_i)$ for any $i > N_0$. Then $\hat{q}_i \subset T_i \cap \hat{\mathcal{U}}$ for any $i > N_0$. Again by using Lemma 2.8, we get a subsequence of $\{T_i\}$ which converges to an area-minimizing surface $\hat{T}$. Let R^+ and R^- be two tall rectangles in opposite sides of α disjoint from $\Gamma \cup \alpha$, and let $P^\pm$ be the unique area-minimizing surfaces with $\partial_\infty P^\pm = \partial R^\pm$. By Lemma A.1 and Remark A.2, $T_i \cap P^\pm = \varnothing$ for any i. Let η be the finite segment in $\hat{\gamma}$ with $\partial\eta \subset P^+ \cup P^-$. Let $\hat{\mathcal{V}} = \eta \times [-\epsilon_0, +\epsilon_0]$. Then $\{\hat{q}_i\} \subset \hat{\mathcal{V}} \subset \hat{\mathcal{U}}$. As $\hat{\mathcal{V}}$ is compact, $\{\hat{q}_i\}$ has a convergent subsequence. This implies $\hat{T} \cap \hat{\mathcal{V}} \neq \varnothing$. This proves that the limit area-minimizing surface $\hat{T}$ does not escape to infinity. Furthermore, in the above construction, we can choose $\hat{\mathcal{U}}$ as close as we want to infinity $\{(0,0)\} \times [-\epsilon_0, \epsilon_0]$, and we can choose $\epsilon_0 > 0$ as small as we want, so we conclude that $p \in \partial_\infty \hat{T}$, too.

Now, by the construction of the sequence $\{T_i\}$, $\hat{T}$ and hence $\partial_\infty \hat{T}$ are invariant by the isometry $\varphi_\lambda(x,y,z) = (\lambda x, \lambda y, z)$. Notice that the isometry φ_λ fixes only the points

$(0,0)$ and ∞ in S^1_∞ and the horizontal lines $L_i=\{(t,0,c_i)\mid t\in\mathbb{R}\}$ in $S^1_\infty\times\mathbb{R}$. This implies $\partial_\infty\widehat{T}\subseteq\widehat{\Gamma}$, where $\widehat{\Gamma}\subset\alpha\cup L_1\cup\cdots\cup L_{m_1}\cup\beta_1\cup\cdots\cup\beta_{m_2}$, where β_j is a vertical line segment with x–coordinate 0. In particular, in the cylindrical model for $\mathbb{H}^2\times\mathbb{R}$, $\widehat{\Gamma}\subset\alpha\cup\bigcup_{i=1}^{m_1}\gamma_{c_i}\cup\bigcup_{j=1}^{m_2}\beta_j\cup\bigcup_{k=1}^{m_3}\widehat{\beta}_k$, where $\gamma_{c_i}=S^1_\infty\times\{c_i\}$ is the horizontal circle corresponding to L_i in $S^1_\infty\times\mathbb{R}$. Then $\beta_j=\{\theta_1\}\times[c_j^-,c_j^+]$ and $\widehat{\beta}_k=\{\theta_0\}\times[d_k^-,d_k^+]$, where $\theta_0\sim\infty$ and $\theta_1\sim(0,0)$ in the upper half-space model. Since $h(\Gamma)>K_0$, then $c_j^+-c_j^->K_0>\pi$ and $d_k^+-d_k^->K_0>\pi$ by construction. This implies the area-minimizing surface $\widehat{T}$ satisfies the conditions of Lemma A.7. By the lemma, we conclude that $\partial_\infty\widehat{T}\subset\bigcup_{i=1}^{m_1}\gamma_{c_i}$, ie $\partial_\infty\widehat{T}$ is a collection of horizontal circles in $S^1_\infty\times\mathbb{R}$, and cannot have any vertical line segments like α. However, this gives a contradiction as $p\in\partial_\infty\widehat{T}$. Step 1 follows. □

Now, we show that Σ_t does not develop genus near the asymptotic boundary.

Step 2 *There exists* $a_\Gamma>0$ *such that for sufficiently large* n, $\Sigma_n\cap\mathcal{R}_{a_\Gamma}$ *has no genus, ie* $\Sigma_n\cap\mathcal{R}_{a_\Gamma}\simeq\Gamma_n\times(0,a_\Gamma)$.

Proof Assume on the contrary that for $a_n\searrow 0$, there exists a subsequence $\Sigma_n\cap\mathcal{R}_{a_n}$ with positive genus. Recall that by Lemma 2.5, $\overline{\Sigma}_n=\Sigma_n\cup\Gamma_n$ is a surface with boundary in $\overline{\mathbb{H}^2\times\mathbb{R}}$ and separating in $\overline{\mathbb{H}^2\times\mathbb{R}}$. Let Δ_n be the component of $\overline{\mathbb{H}^2\times\mathbb{R}}-\overline{\Sigma}_n$ which contains the bridge α. Since $\Sigma_n\cap\mathcal{R}_{a_n}$ has positive genus, then $\Delta_n\cap\mathcal{R}_{a_n}$ must be a nontrivial handlebody, ie not a 3–ball. Hence, there must be a point p_n in $\Sigma_n\cap\mathcal{R}_{a_n}$, where the normal vector $v_{p_n}=\langle 0,1,0\rangle$ is pointing *inside* Δ_n by Morse theory. By genericity of Morse functions, we can modify the ∞ point in $\partial_\infty\mathbb{H}^2$ if necessary to get y–coordinate as a Morse function.

Let $p_n=(x_n,y_n,z_n)$. By construction, $y_n\to 0$ as $y_n<a_n$. Consider the isometry $\psi_n(x,y,z)=((x-x_n)/y_n,y/y_n,z-z_n)$, which is a translation by $-(x_n,0,0)$ first by a parabolic isometry of $\mathbb{H}^2$, and translation by $-(0,0,z_n)$ in the $\mathbb{R}$–direction. Then, by composing with the hyperbolic isometry $(x,y,z)\to(x/y_n,y/y_n,z)$, we get the isometry ψ_n of $\mathbb{H}^2\times\mathbb{R}$. Then consider the sequence of area-minimizing surfaces $\Sigma_n'=\psi_n(\Sigma_n)$ and $p_n'=\psi_n(p_n)=(0,1,0)$. Let $\Gamma_n'=\psi_n(\Gamma_n)=\partial_\infty\Sigma_n'$. After passing to a subsequence, we get the limits $\Sigma_n'\to\Sigma'$, $p_n'\to p'=(0,1,0)\in\Sigma'$ and $\Gamma_n'\to\Gamma'$. Note also that by construction the normal vector to the area-minimizing surface Σ' at p' is $v_{p_n}\to v_p'=<0,1,0>$ pointing inside Δ'.

Consider $\Gamma'=\lim\Gamma_n'$. Let l_z be the z–axis in $S^1_\infty\times\mathbb{R}$, ie $l_z=\{(0,0,t)\mid t\in\mathbb{R}\}$. Let $\Gamma'\cap l_z=\{(0,0,c_1),(0,0,c_2),\ldots,(0,0,c_k)\}$. Notice that as $h(\Gamma)>K_0$, $|c_i-c_j|>K_0$

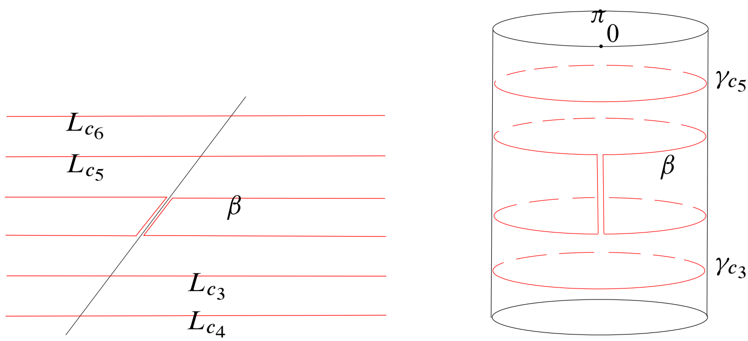

Figure 2: $\Gamma' \subset S^1_\infty \times \mathbb{R}$ is pictured in the upper half-space model and cylinder model for $\mathbb{H}^2 \times \mathbb{R}$.

for any $i \neq j$. Recall that $\partial\alpha = \{(0,0,c_1),(0,0,c_2)\}$. Note that by Lemma A.7, Γ' cannot have a vertical line segment $\alpha_j = \{(0,0)\} \times [c_j^-, c_j^+]$. Hence, by the construction of Γ'_n, we get $\Gamma' = \beta \cup L_{c_3} \cup \cdots \cup L_{c_k}$ where L_{c_i} is the horizontal line in $S^1_\infty \times \mathbb{R}$ with L_{c_i}, and β is the component of Γ' near α (see Figure 2, left). In particular, in cylinder model for $\mathbb{H}^2 \times \mathbb{R}$, L_{c_i} is the horizontal circle $\gamma_{c_i} = S^1_\infty \times \{c_i\}$ in $S^1_\infty \times \mathbb{R}$, and β is a tall rectangle $\beta = \partial R$, where $R = [\delta, 2\pi - \delta] \times [c_1, c_2]$ assuming $\alpha = \{0\} \times [c_1', c_2']$ (see Figure 2, right). Note that as ψ_n is only translating in the z–direction, $c_1 - c_2 = c_1' - c_2'$. Here, the limit area-minimizing surface Σ' is nonempty, as $(0,0,1) \in \Sigma'$ by construction. Also, δ depends on the comparison between $y_n \searrow 0$ and $d(\Gamma_n, \alpha) \searrow 0$. As Σ' does not escape infinity, we make sure that such a $\delta < \pi$ exists. Indeed, $\delta > 0$ can be explicitly computed by using the fact that there is a unique minimal surface P_β in $\mathbb{H}^2 \times \mathbb{R}$ containing (0,0,1) with $\partial_\infty P_\beta = \beta = \partial R$ by Lemma 2.3 as R is a tall rectangle.

Σ' bounds a unique area-minimizing surface with $\Sigma' = P_\beta \cup P_{c_1} \cup \cdots \cup P_{c_k}$, where P_β is the unique area-minimizing surface with $\partial_\infty P_\beta = \beta$ by Lemma 2.3, and P_{c_i} is the horizontal plane $\mathbb{H}^2 \times \{c_i\}$ in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty P_{c_i} = \gamma_{c_i}$. This is because $|c_i - c_j| > \pi$, so there is no connected minimal surface with asymptotic boundary containing more than one component of Γ'. In particular, if there were a connected area-minimizing surface Y with $\partial_\infty Y \supset \gamma_{c_1} \cup \gamma_{c_2}$ with $c_2 - c_1 > \pi$, one could place a minimal catenoid $\mathcal{C}$ with $\partial_\infty \mathcal{C} = \gamma_{c_1'} \cup \gamma_{c_2'}$, where $c_1' > c_1$ and $c_2' < c_2$ with $c_2' - c_1' = \pi - \epsilon$, so that $\mathcal{C} \cap S = \varnothing$ [4, Section 7.1]. Then, by using an hyperbolic isometry φ_t, one could push $\mathcal{C}$ towards S horizontally. As S is connected, there must be a first point of contact, which gives a contradiction with the maximum principle. This implies each component of Γ' bounds a component of Σ'. Since each component is uniquely minimizing, Σ' is a uniquely minimizing surface with $\partial_\infty \Sigma' = \Gamma'$.

Hence, by construction, $p' = (0,0,1)$ is on the P_β component of Σ'. Recall that the normal vector $v'_p = \langle 0,1,0 \rangle$ points inside of Δ', which is the component of $\mathbb{H}^2 \times \mathbb{R} - \Sigma'$ containing α. However, P_β is a plane, and the normal vector v'_p points outside of Δ', not inside. This is a contradiction. Step 2 follows. □

Step 3 *For sufficiently small* $t > 0$, Σ_t *is homeomorphic to* $\Sigma \cup S_\alpha$.

Proof Assume that for $\epsilon_n \searrow 0$ there exists $0 < t_n < \epsilon_n$ such that Σ_{t_n}, say Σ_n, is not homeomorphic to $\widehat{\Sigma} = \Sigma \cup S_\alpha$. Since the number of ends are same, this means Σ_n and $\widehat{\Sigma}$ have different genus.

Let $\mathcal{R}_a = \{0 \le y \le a\}$ in $\overline{\mathbb{H}^2 \times \mathbb{R}}$ be as in Step 2. Let $\mathcal{K}_a = \{y \ge a\}$ and let $\Sigma^a = \Sigma \cap \mathcal{K}_a$. Then, since $\Sigma_n \to \Sigma$ converges smoothly on compact sets, $\Sigma_n^a \to \Sigma^a$ smoothly. Hence, by Gauss–Bonnet, Σ_n^a and Σ^a must have the same genus. By Step 2, this implies for sufficiently large n, Σ_n and Σ must have the same genus. However, this contradicts with our assumption that Σ_n and Σ have different genus for any n. Therefore, this implies that for sufficiently small $\epsilon' > 0$, Σ_t is homeomorphic to $\Sigma \cup S_\alpha$ for $0 < t < \epsilon'$. Step 3 follows. □

Step 4 *For all but countably many* $0 < t < \epsilon'$, Γ_t *bounds a unique area-minimizing surface in* $\mathbb{H}^2 \times \mathbb{R}$.

Proof We adapt the proof of Theorem A.5 to this case. The family of tall curves $\{\Gamma_t \mid t \in (0,\epsilon)\}$ foliates $\widehat{N}$, where $\partial \widehat{N} = \Gamma_\epsilon \cup \Gamma$ and $\Gamma_0 = \Gamma \cup \alpha$. In particular, for any $0 < t_1 < t_2 < \epsilon$, $\Gamma_{t_1} \cap \Gamma_{t_2} = \varnothing$. If Σ_t is an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$, then $\Sigma_{t_1} \cap \Sigma_{t_2} = \varnothing$ too, by Lemma A.1. By Lemma A.3, if Γ_s does not bound a unique area-minimizing surface Σ_s, then we can define two disjoint canonical minimizing Σ_s^+ and Σ_s^- with $\partial_\infty \Sigma_s^\pm = \Gamma_s$. Then, by the proof of Theorem A.5, for all but countably many $s \in [0,\epsilon']$, Γ_s bounds a unique area-minimizing surface. Step 4 follows. □

Steps 1–3 imply the existence of $\epsilon' > 0$ such that any Σ_t with $\partial_\infty \Sigma_t = \Gamma_t$ for $t \in (0,\epsilon')$ is homeomorphic to $\Sigma \cup S_\alpha$. Step 4 implies the generic uniqueness for the family $\{\Gamma_t \mid t \in (0,\epsilon')\}$. Hence, Steps 1–4 together imply the existence of a smooth curve Γ_t with $t \in (0,\epsilon')$, where Γ_t bounds a *unique* area-minimizing surface Σ_t, and Σ_t has the desired topology, ie $\Sigma_t \simeq \Sigma \cup S_\alpha$. The proof of the theorem follows. □

4 Minimal surfaces of arbitrary topology in $\mathbb{H}^2 \times \mathbb{R}$

In this section, we prove that any open orientable surface can be embedded in $\mathbb{H}^2 \times \mathbb{R}$ as an area-minimizing surface. First, we show a simple construction for the finite topology case. Then we finish the proof by giving a very general construction for the infinite topology case.

4.1 Surfaces with finite topology

While our main result later applies to both finite and infinite topology orientable surfaces, we start with a very simple construction for surfaces with finite topology as a warm-up. In particular, by using vertical bridges as 1–handles, we give a construction of an area-minimizing surface Σ_k^g of genus g with k ends.

Euler characteristics Recall that if T_k^g is an orientable surface of genus g, and k boundary components, then $\chi(T_k^g) = 2-2g-k$. Adding a bridge (a 1–handle in topological terms) to a surface decreases the Euler characteristics by one. On the other hand, if you add a bridge to a surface where the endpoints of the bridge are in the same boundary component, then the number of boundary components increases by one. If you add a bridge whose endpoints are in the different boundary components, then the number of boundary components decreases by one (see Figure 1).

Now, adding a bridge to the same boundary component of a surface would increase the number of ends. In other words, let S_{n+1} obtained from S_n by attaching a bridge (1–handle) to S_n whose endpoints are in the same component of ∂S_n. Then $\chi(S_{n+1}) = \chi(S_n)-1$, $g(S_n) = g(S_{n+1})$ and $\sharp(\partial S_{n+1}) = \sharp(\partial S_n)+1$, where $\sharp$ is the number of components.

If we want to increase the genus, first add a bridge α_n whose endpoints are in the same component of ∂S_n, and get $S_n' \simeq S_n \natural S_{\alpha_n}$, where $S_n \natural S_{\alpha_n}$ represents the surface obtained by adding a bridge (thin strip) to S_n along α_n. Then, by adding another bridge α_n' whose endpoints are in different components of S_n', one gets $S_{n+1} \simeq S_n' \natural S_{\alpha_n'}$. Hence, $\chi(S_{n+1}) = \chi(S_n)-2$, and the number of boundary components are same. This implies that if $S_n \simeq T_k^g$, then $S_{n+1} \simeq T_k^{g+1}$. This shows that S_{n+1} is obtained by attaching a cylinder with handle to S_n, ie $S_{n+1} - S_n$ is a cylinder with handle.

Construction for finite topology surfaces There is a very elementary construction for open orientable surfaces of finite topology as follows: Let S be an open orientable

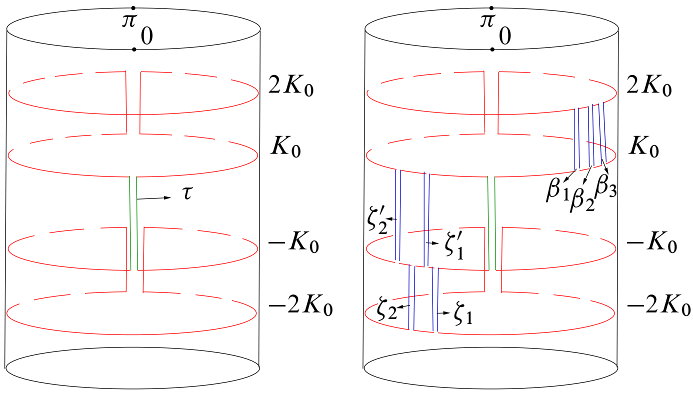

Figure 3: Left: the tall curve Γ_1 which bounds the area-minimizing surface $\Sigma_1 \sim P^+ \cup P^- \natural S_\tau$. Right: We first add bridges $\beta_1, \dots, \beta_k$ to Σ to increase the number of ends by k (here for $k = 3$). Then we add g pairs of bridges $\zeta_1, \zeta'_1, \dots, \zeta_g, \zeta'_g$ to increase the genus (here $g = 2$). Hence, Σ is a genus 2 surface with four ends.

surface of genus g and k ends. Construct the area-minimizing surface Σ_1 which is topologically a disk as in Figure 3, left. For $k+1$ ends, add k vertical bridges $\beta_1, \beta_2, \dots, \beta_k$ to Σ_1 as in Figure 3, right. Then, for genus g, add g pairs of vertical bridges ζ_i and ζ'_i successively as in Figure 3, right. Hence, the final surface Σ is an area-minimizing surface of genus g and $k+1$ ends. Furthermore, $\overline{\Sigma}$ is a compact embedded surface with boundary in $\overline{\mathbb{H}^2\times\mathbb{R}}$ by Lemma 2.5.

4.2 Surfaces with infinite topology

Now we prove any open orientable surface (finite or infinite topology) can be embedded in $\mathbb{H}^2\times\mathbb{R}$ as an area-minimizing surface. In this part, we mainly follow the techniques in [13; 2]. In particular, for a given surface S, we start with a compact exhaustion of S, $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$, and by using the bridge principle proved in the previous section, we inductively construct the area-minimizing surface with the desired topology.

In particular, by [7], for any open orientable surface S, there exists a simple exhaustion. A simple exhaustion $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$ is a compact exhaustion with the following properties: S_1 is a disk, and $S_{n+1} - S_n$ contains a *unique nonannular piece*, which is either a cylinder with a handle or a pair of pants, by [7] (see Figure 4).

First, we need a lemma which will be used in the construction.

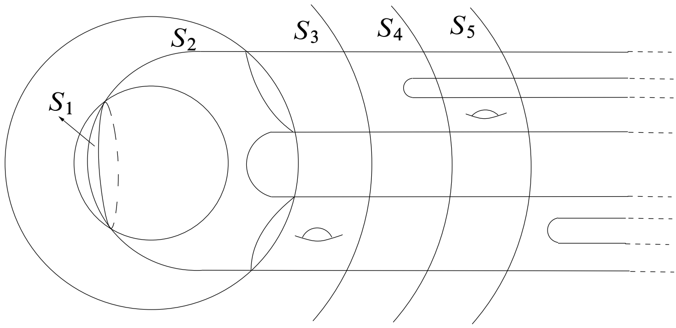

Figure 4: In the simple exhaustion of S, S_1 is a disk, and $S_{n+1}-S_n$ contains a unique nonannular part, which is a pair of pants (eg S_4-S_3) or a cylinder with a handle (eg S_3-S_2).

Lemma 4.1 *Let $R=[-1,1]\times[-4\pi,4\pi]$ and $R_c=[-c,c]\times[-2\pi,2\pi]$ be rectangles in $S^1_\infty\times\mathbb{R}$ with $0<c<1$. Let $\gamma=\partial R$, $\gamma_c=\partial R_c$ and $\Gamma_c=\gamma\cup\gamma_c$. Then there exists $\rho>0$ such that, for any $0<c\le\rho$, the area-minimizing surface Σ_c with $\partial_\infty\Sigma_c=\Gamma_c$ is $P\cup P_c$, where P and P_c are the unique area-minimizing surfaces with $\partial_\infty P=\gamma$ and $\partial_\infty P_c=\gamma_c$.*

Proof If the area-minimizing surface Σ_c is not connected, then it must be $P\cup P_c$ because the rectangles γ and γ_c bound unique area-minimizing surfaces P and P_c, respectively, by Lemma 2.3. Hence, we assume on the contrary that the area-minimizing surface Σ_c is connected for any $0<c<1$. We abuse notation and say $\Sigma_n=\Sigma_{1/n}$. Consider the sequence $\{\Sigma_n\}$. By Lemma 2.8, we get a convergent subsequence and limiting area-minimizing surface Σ with $\partial_\infty\Sigma\subset\gamma\cup\beta$, where β is the vertical line segment $\{0\}\times[-2\pi,2\pi]$.

Let $Q=\left[-\frac{1}{2},\frac{1}{2}\right]\times[-3\pi,3\pi]$ be another rectangle in $S^1_\infty\times\mathbb{R}$, and let T be the unique area-minimizing surface in $\mathbb{H}^2\times\mathbb{R}$ with $\partial_\infty T=\partial Q$. Since, by assumption, Σ_n is connected, and $\overline{T}$ separates the boundary components γ_n and γ of Σ_n, then $T\cap\Sigma_n\neq\varnothing$ for any $n>2$. By construction, this implies $\Sigma\cap T\neq\varnothing$.

As $\partial_\infty\Sigma\subset\gamma\cup\beta$, we have two cases. Either $\partial_\infty\Sigma=\gamma$ or $\partial_\infty\Sigma=\gamma\cup\beta$. If $\partial_\infty\Sigma=\gamma$, then γ bounds a unique area-minimizing surface P. In other words, Σ must be P and $P\cap T=\varnothing$. This is a contradiction.

If $\partial_\infty\Sigma=\gamma\cup\beta$, we get a contradiction as follows. Let's go back to cylinder model of $\mathbb{H}^2\times\mathbb{R}$. Then we can represent $\gamma=\partial R$, where $R=[-\theta_1,\theta_1]\times[-4\pi,4\pi]$ for some $\theta_1\in(0,\pi)$, and $\beta=\{0\}\times[-2\pi,2\pi]$ in $S^1_\infty\times\mathbb{R}$. Let φ_t be the isometry of $\mathbb{H}^2\times\mathbb{R}$

corresponding to $\varphi_t(x,y,z)=(tx,ty,z)$ in the upper half-space model of $\mathbb{H}^2\times\mathbb{R}$. In particular, $\{\pi\}\times\mathbb{R}$ represents the point at infinity, and φ_t pushes every point in $\mathbb{H}^2\times\mathbb{R}$ from $\{0\}\times\mathbb{R}$ to $\{\pi\}\times\mathbb{R}$ in the Poincaré disk model. Let $\Sigma^n=\varphi_n(\Sigma)$. Again, by Lemma 2.8, we get a limit area-minimizing surface $\widehat{\Sigma}$ with $\partial_\infty\widehat{\Sigma}\subset\Gamma^+\cup\Gamma^-\cup\beta\cup\alpha$, where $\Gamma^\pm=S^1_\infty\times\{\pm 4\pi\}$ and $\alpha=\{\pi\}\times[-4\pi,4\pi]$.

We claim that $\widehat{\Sigma}$ is nonempty and, furthermore, $\partial_\infty\widehat{\Sigma}=\Gamma^+\cup\Gamma^-\cup\beta\cup\alpha$. Since the original Σ is connected by assumption, $\Sigma\cap\mathbb{H}^2\times\{c\}$ contains an infinite curve l_c with $\partial_\infty l_c=\{(0,c),(\theta_1,c)\}$, where $c\in(-2\pi,2\pi)$. Then $\varphi_n(l_c)=l^n_c\subset\Sigma^n\cap\mathbb{H}^2\times\{c\}$, and l^n_c converges to a line $\hat{l}_c\subset\widehat{\Sigma}\cap\mathbb{H}^2\times\{c\}$ with $\partial_\infty\hat{l}_c=\{(0,c),(\pi,c)\}$. This shows $\partial_\infty\Sigma=\Gamma^+\cup\Gamma^-\cup\beta\cup\alpha$.

Finally, let $\mathcal{C}$ be Daniel's parabolic catenoid with $\partial_\infty\mathcal{C}=\lambda^+\cup\lambda^-\cup\tau$, where $\lambda^+=S^1_\infty\times\left\{\frac{7\pi}{2}\right\}$, $\lambda^-=S^1_\infty\times\left\{\frac{5\pi}{2}\right\}$ and $\tau=\{\pi\}\times\left[\frac{5\pi}{2},\frac{7\pi}{2}\right]$. As $\partial_\infty\mathcal{C}$ is invariant by φ_t, $\mathcal{C}_t\varphi_t(\mathcal{C})$ is also a parabolic catenoid with $\partial_\infty\mathcal{C}_t=\partial_\infty\mathcal{C}$. Furthermore, for sufficiently small $\epsilon>0$, $\mathcal{C}_\epsilon$ is very close to asymptotic cylinder $S^1_\infty\times\mathbb{R}$. Hence, we can choose sufficiently small $\epsilon>0$ with $\mathcal{C}_\epsilon\cap\widehat{\Sigma}=\varnothing$. Then, by pushing $\mathcal{C}_\epsilon$ towards $\widehat{\Sigma}$ via isometries φ_t, we get a first point of contact $\mathcal{C}_{t_0}$ with $\widehat{\Sigma}$ which contradicts to the maximum principle. The proof follows. □

Now we are ready to prove the existence result for properly embedded area-minimizing surfaces in $\mathbb{H}^2\times\mathbb{R}$ with arbitrary topology.

Theorem 4.2 *Any open orientable surface S can be embedded in $\mathbb{H}^2\times\mathbb{R}$ as an area-minimizing surface Σ.*

Proof Let S be an open orientable surface. Now, we inductively construct an area-minimizing surface Σ in $\mathbb{H}^2\times\mathbb{R}$ which is homeomorphic to S. Let $S_1\subset S_2\subset\cdots\subset S_n\subset\cdots$ be a simple exhaustion of S, ie $S_{n+1}-S_n$ contains a unique nonannular piece, which is either a cylinder with a handle or a pair of pants.

By following the simple exhaustion, we define a sequence of area-minimizing surfaces Σ_n so that Σ_n is homeomorphic to S_n, ie $\Sigma_n\simeq S_n$. Furthermore, the sequence Σ_n induces the same simple exhaustion for the limiting surface Σ. Hence, we get an area-minimizing surface Σ which is homeomorphic to the given surface S.

Now we follow the idea described in Section 4.1. Note that we are allowed to use only vertical bridges.

Let $R=\left[-\frac{\pi}{2},+\frac{\pi}{2}\right]\times[0,K_0]$ be a tall rectangle in $S^1_\infty\times\mathbb{R}$, where K_0 is as in Theorem 3.2. Let Σ_1 be the unique area-minimizing surface with $\partial_\infty\Sigma_1=\partial R$.

Clearly, $\Sigma_1 \simeq S_1$. Note that throughout our construction, our curves need to be tall curves $\{\Gamma_n\}$ of height $\geq K_0$ in order to apply the vertical bridge principle (Theorem 3.2) successively.

We define Σ_n inductively as follows. We will add only vertical bridges to Γ_n so that the resulting curve Γ_{n+1} bounds a unique area-minimizing surface Σ_{n+1} by Theorem 3.2.

By Section 4.1, adding one bridge β_{n+1} to Σ_n, where the endpoints of β_{n+1} are in the same component of $\Gamma_n = \partial_\infty \Sigma_n$, would suffice to increase the number of ends of Σ_n by one. This operation corresponds to *adding a pair of pants to the surface*. Similarly, by Section 4.1, adding two bridges successively so that the endpoints of the first bridge are in the same component and the endpoints of the second bridge are in different components (components containing the opposite sides of the first bridge) increases the genus, and keeps the number of the ends same. This operation corresponds to *adding a cylinder with handle to the surface*.

Now we continue inductively to construct the sequence $\{\Sigma_n\}$ dictated by the simple exhaustion (see Figure 4). There are two cases: $S_{n+1} - S_n$ contains either a pair of pants or a cylinder with handle.

Pair of pants case Assume that $S_{n+1} - S_n$ contains a pair of pants. Let the pair of pants be attached to the component γ in ∂S_n. Let γ' be the corresponding component of $\Gamma_n = \partial_\infty \Sigma_n$. By construction, γ' bounds a disk D in $S^1_\infty \times \mathbb{R}$ with $D \cap \Gamma_n = \gamma'$. Let $\beta_n = \{c_n\} \times [0, K_0]$ be a vertical segment with $\beta_n \subset D$. Since Σ_n bounds a unique area-minimizing surface by construction, and $\beta_n \perp \Gamma_n$, we can apply Theorem 3.2, and get an area-minimizing surface Σ_{n+1}, where Σ_{n+1} is homeomorphic to S_{n+1}. □

Cylinder with handle case Assume that $S_{n+1} - S_n$ contains a cylinder with handle. Again, let the pair of pants be attached to the component γ in ∂S_n. Let γ' be the corresponding component of $\Gamma_n = \partial_\infty \Sigma_n$. By construction, γ' bounds a disk D in $S^1_\infty \times \mathbb{R}$ with $D \cap \Gamma_n = \gamma'$. Let β_n be a vertical segment $\{c_n\} \times [0, K_0]$ such that $(c_n - \epsilon_n, c_n + \epsilon_n) \times \mathbb{R} \cap \Gamma_n \subset D$ for some $\epsilon_n > 0$. Again, we apply Theorem 3.2 for β_n and Σ_n, and get an area-minimizing surface Σ'_{n+1}. Say $\Gamma'_{n+1} = \partial_\infty \Sigma'_{n+1}$. We can choose the thickness of the bridge along β_n as small as we want. So, we can assume that the thickness of the bridge along β_n is smaller than $\frac{1}{4}\rho.\epsilon_n$, where $\rho > 0$ is the constant in Lemma 4.1.

Now consider the rectangle $Q_n = \left[c_n - \frac{1}{2}\rho.\epsilon_n, c_n + \frac{1}{2}\rho.\epsilon_n\right] \times [-6\pi - K_0, -4\pi - K_0]$ (see Figure 5). Let T_n be the unique area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty T_n = \partial Q_n$

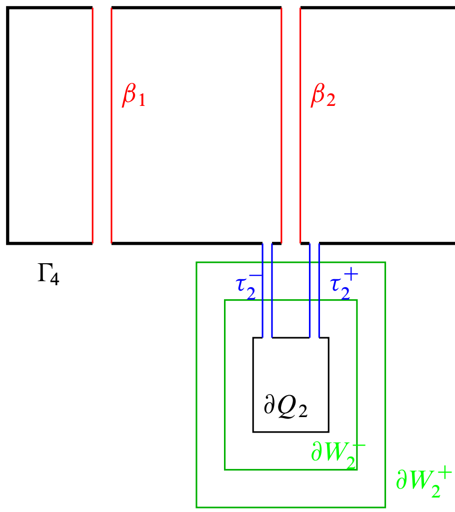

Figure 5: Here $S_2 - S_1$ is a pair of pants and $S_3 - S_2$ is a cylinder with handle. When constructing Σ_3, β_2 is attached to the corresponding component in Γ_2, then a hanger, the pair of vertical bridges $\tau_2^\pm$ and a thin rectangle Q_n are added to obtain the cylinder with handle. Here $\partial W_2^\pm$ is needed to show that $\Sigma_2' \cup T_2$ is a uniquely area-minimizing surface to apply Theorem 3.2.

by Lemma 2.3. Let $\widehat{\Gamma}_{n+1} = \Gamma'_{n+1} \cup \partial Q_n$. We claim that $\widehat{\Gamma}_{n+1}$ bounds a unique area-minimizing surface $\widehat{\Sigma}_{n+1}$ in $\mathbb{H}^2\times\mathbb{R}$ and $\widehat{\Sigma}_{n+1} = \Sigma'_{n+1} \cup T_n$. Notice that Σ'_{n+1} and T_n are uniquely minimizing surfaces. Hence, if we show that $\widehat{\Gamma}_{n+1}$ cannot bound any connected area-minimizing surface, then we are done.

Assume that $\widehat{\Gamma}_{n+1}$ bounds a connected area-minimizing surface $\widehat{\Sigma}_{n+1}$. Consider the pair of rectangles

$$W_n^+ = [c_n - \epsilon_n, c_n + \epsilon_n] \times [-9\pi - K_0, -\pi - K_0],$$
$$W_n^- = [c_n - \rho.\epsilon_n, c_n + \rho.\epsilon_n] \times [-7\pi - K_0, -3\pi - K_0].$$

Let $\Upsilon_n = \partial W_n^+ \cup \partial W_n^-$. Then, by Lemma 4.1, the uniquely minimizing surface F_n with $\partial_\infty F_n = \Upsilon_n$ must be $P_n^+ \cup P_n^-$, where $P_n^\pm$ is the unique area-minimizing surface with $\partial_\infty P_n^\pm = \partial W_n^\pm$. As $\widehat{\Gamma}_{n+1} \cap \Upsilon_n = \varnothing$, the area-minimizing surfaces $\widehat{\Gamma}_{n+1}$ and F_n must be disjoint by Lemma A.1 (see Figure 5). On the other hand, the area-minimizing surface $F_n = P_n^+ \cup P_n^-$ separates the components, Γ'_{n+1} and ∂Q_n, of $\widehat{\Gamma}_{n+1}$. Since $\widehat{\Gamma}_{n+1} \cap F_n = \varnothing$, this implies $\widehat{\Sigma}_{n+1}$ is disconnected. This proves that $\widehat{\Sigma}_{n+1} = \Sigma'_{n+1} \cup T_n$ is the unique area-minimizing surface with $\partial_\infty \widehat{\Sigma}_{n+1} = \widehat{\Gamma}_{n+1}$.

Now let $\tau_n^+ = \{c_n + \frac{1}{4}\rho.\epsilon_n\} \times [-4\pi - K_0, 0]$ be the vertical arc segment in $S^1_\infty \times \mathbb{R}$. When we apply Theorem 3.2 to the uniquely minimizing surface $\widehat{\Sigma}_{n+1}$ and the arc τ_n^+, we obtain a new uniquely minimizing surface $\widehat{\Sigma}'_{n+1}$. Similarly, let $\tau_n^- = \{c_n - \frac{1}{4}\rho.\epsilon_n\} \times [-4\pi - K_0, 0]$. Again, we apply Theorem 3.2 for $\widehat{\Sigma}'_{n+1}$ and τ_n^-; we obtain another uniquely minimizing surface Σ_{n+1}. Furthermore, we assume that both the bridges along τ_n^+ and τ_n^- have thickness less than $\frac{1}{4}\rho.\epsilon_n$. The pair of vertical bridges along $\tau_n^\pm$ with the thin rectangle Q_n looks like a hanging picture frame (see Figure 5).

By construction, Σ_{n+1} is homeomorphic to S_{n+1}. In particular, we have added a cylinder with handle to Σ_n along the corresponding component γ' in Γ_n. This finishes the description of the inductive step, when $S_{n+1} - S_n$ contains a cylinder with handle. □

The limit and the properly embeddedness Notice that in the bridge principle at infinity (Theorem 3.2), as the thickness of the bridge α goes to 0, the height of the strip S_α goes to 0, too. In particular, let Γ, Σ, α, Γ_t and Σ_t be as in the statement of Theorem 3.2. Let $S^t_\alpha = \Sigma_t \cap N_\epsilon(\alpha)$, where $N_\epsilon(\alpha)$ is the sufficiently small neighborhood of α in the compactification $\overline{\mathbb{H}^2 \times \mathbb{R}}$. Then, as $t \searrow 0$, $d(L_z, S^t_\alpha) \to \infty$, where L_z is the vertical line through the origin in $\mathbb{H}^2 \times \mathbb{R}$, ie $L_z = \{0\} \times \mathbb{R}$. This is because, as $t \searrow 0$, $\Sigma_t \to \Sigma$.

Let $\widehat{B}_r = B_r(0) \times [-2K_0, 2K_0]$ be a compact region in $\mathbb{H}^2 \times \mathbb{R}$, where $B_r(0)$ is the r–ball around the origin in $\mathbb{H}^2$. As $t_n \searrow 0$; then the thickness of the bridge in Σ_n near β_n (or $\tau_n^\pm$) goes to 0. Hence, by choosing $t_n < 1/10n^2$ sufficiently small, we can make sure that $d(L_z, S^{t_n}_{\beta_n}) > r_n$ and $d(L_z, S^{t_n}_{\tau_n^\pm}) > r_n$ for a sequence $r_n \nearrow \infty$. This implies that, for $m \geq n$, $\widehat{B}_{r_n} \cap \Sigma_m \simeq S_n$ as the thickness (and hence height) of the bridges β_n and ζ_n goes to 0.

Now, Σ_n is a sequence of absolutely area-minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Let $\Sigma'_n = \widehat{B}_{r_n} \cap \Sigma_n$. By Lemma 2.8, by using a diagonal sequence argument, we get a limiting surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ where the convergence is smooth on compact sets. Σ is an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ as it is the limit of area-minimizing surfaces. Notice that for $m \geq n$, $\widehat{B}_{r_n} \cap \Sigma_m \simeq S_n$ and the convergence is smooth on compact sets. This implies $\Sigma \cap \widehat{B}_{r_n} \simeq S_n$ for any n, and hence $\Sigma \simeq S$.

We also note that the bridges do not collapse in the limit as, for every bridge along β_n and $\tau_n^\pm$, we can place a thin, tall rectangle R_n "under" the bridge *disjoint* from the minimizing sequence. In other words, the area-minimizing plane P_n with $\partial_\infty P_n = R_n$

(Lemma 2.3) will be a barrier for bridges to collapse, as, for any $m > n$, $P_n \cap \Sigma_m = \varnothing$ since $\Gamma_m \cap R_n = \varnothing$ by Lemma A.1.

Finally, Σ is properly embedded in $\mathbb{H}^2 \times \mathbb{R}$ as, for any compact set $K \subset \mathbb{H}^2 \times \mathbb{R}$, there exists $r_n > 0$ with $K \subset \widehat{B}_{r_n}$ and $\widehat{B}_{r_n} \cap \Sigma \simeq S_n$, which is compact. The proof of the theorem follows. □

5 Final remarks

5.1 H–surfaces

The constant mean curvature surfaces could be considered as a natural candidate to generalize our results. Hence, consider the following question:

Question *What kind of surfaces can be embedded in* $\mathbb{H}^2 \times \mathbb{R}$ *as a complete* H*–surface for* $0 < H < \frac{1}{2}$*?*

In other words, is it possible to embed any open orientable surface S in $\mathbb{H}^2 \times \mathbb{R}$ as a complete H–surface for $0 < H < \frac{1}{2}$. A positive answer to this question would be a generalization of Theorem 4.2 to H–surfaces.

Unfortunately, it is hardly possible to generalize our methods to this problem. By [16], for $H > 0$, if Σ is an H–surface with $\partial_\infty \Sigma \neq \varnothing$ and $\Sigma \cup \partial_\infty \Sigma$ is a C^1 surface up to the boundary, then $\partial_\infty \Sigma$ must be a collection of vertical line segments in $S^1_\infty \times \mathbb{R}$. In particular, this implies the asymptotic plateau problem practically has no solution for H–surfaces in $\mathbb{H}^2 \times \mathbb{R}$, since, if Γ is a C^1 simple closed curve in $S^1_\infty \times \mathbb{R}$, there is no H–surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ where $\Sigma \cup \Gamma$ is a C^1 surface up to the boundary. Hence, because of this result, our methods for Theorem 4.2 cannot be generalized to this case. However, it might be possible to construct a complete H–surface Σ of any finite topology with only vertical ends, ie $\partial_\infty \Sigma$ consists of only vertical lines in $S^1_\infty \times \mathbb{R}$.

5.2 Finite total curvature

Our construction of area-minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$ produces surfaces of infinite total curvature. In [11], Martín, Mazzeo and Rodríguez recently showed that for any $g \geq 0$, there exists a complete, finite total curvature, embedded minimal surface Σ_{g,k_g} in $\mathbb{H}^2 \times \mathbb{R}$ with genus g and k_g ends for sufficiently large k_g. Even though this result is great progress to construct examples of minimal surfaces of finite total curvature, the question of existence (or nonexistence) of minimal surfaces of finite total curvature with any finite topology is still a very interesting open problem.

It is well known that a complete, properly embedded, minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature also has finite topology [9]. On the other hand, there are surfaces with finite topology which cannot be embedded in $\mathbb{H}^2 \times \mathbb{R}$ as a complete minimal surface with finite total curvature. For example, by [8], a twice-punctured torus cannot be embedded as a complete minimal surface with finite total curvature into $\mathbb{H}^2 \times \mathbb{R}$. Hence, the following question becomes very interesting:

Question *For which $g \geq 0$, and $k \geq 0$, does there exist a complete embedded minimal surface S_k^g in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature, where S_k^g is an orientable surface of genus g with k ends?*

Appendix

In this section, we prove some technical steps used in our construction. Basically, we prove two technical result in this section. The first one is the generic uniqueness result (Theorem A.5), which is needed in our construction in the main theorem (Theorem 4.2, Step 4). In particular, to apply the bridge principle, we needed unique curves, and this generic uniqueness result give us the desired curve with a slight modification of the original curve. The second result is an important technical step (Lemma A.7), which is needed to prove the vertical bridge principle (Theorem 3.2).

A.1 Generic uniqueness of area-minimizing surfaces

In this part, we prove a generic uniqueness result for tall curves in $S^1_\infty \times \mathbb{R}$. We start with a lemma which roughly says that disjoint curves in $S^1_\infty \times \mathbb{R}$ bounds disjoint area-minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Lemma A.1 (disjointness) *Let Ω_1 and Ω_2 be two closed regions in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$, where $\partial\Omega_i = \Gamma_i$ is a finite collection of disjoint simple closed curves. Further assume that $\Omega_1 \cap \Omega_2 = \varnothing$ or $\Omega_1 \subset \operatorname{int}(\Omega_2)$. If Σ_1 and Σ_2 are area-minimizing surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma_i = \Gamma_i$, then $\Sigma_1 \cap \Sigma_2 = \varnothing$.*

Proof Assume that $\Sigma_1 \cap \Sigma_2 \neq \varnothing$. As both surfaces are minimal, by the maximum principle, the intersection cannot contain isolated points. As $\Gamma_1 \cap \Gamma_2 = \varnothing$, then $\Sigma_1 \cap \Sigma_2 = \alpha$, which is a collection of closed curves.

Since $\mathbb{H}^2 \times \mathbb{R}$ is topologically a ball, any surface would be separating. Let Δ_i be the components of $\mathbb{H}^2 \times \mathbb{R} - \Sigma_i$ with $\partial_\infty \overline{\Delta}_i = \Omega_i$. In other words, as $\Sigma_i \cup \Omega_i$ is a closed surface in the contractible space $\overline{\mathbb{H}^2 \times \mathbb{R}}$, it bounds a region Δ_i in $\overline{\mathbb{H}^2 \times \mathbb{R}}$.

If $\Omega_1 \subset \operatorname{int}(\Omega_2)$, let $S_1 = \Sigma_1 - \Delta_2$ and let $S_2 = \Sigma_2 \cap \overline{\Delta}_1$. Then, as $\Omega_1 \subset \operatorname{int}(\Omega_2)$, with this operation, we cut the surfaces S_i from the noncompact parts in Σ_i. Therefore, $\partial_\infty S_1 = \partial_\infty S_2 = \varnothing$ and both S_1 and S_2 are compact surfaces with $\partial S_1 = \partial S_2 = \alpha$.

If $\Omega_1 \cap \Omega_2 = \varnothing$, let $S_1 = \Sigma_1 \cap \overline{\Delta}_2$ and let $S_2 = \Sigma_2 \cap \overline{\Delta}_1$. Again, as $\Omega_1 \cap \Omega_2 = \varnothing$, $\partial_\infty S_1 = \partial_\infty S_2 = \varnothing$ and both S_1 and S_2 are compact surfaces with $\partial S_1 = \partial S_2 = \alpha$.

As Σ_1 and Σ_2 are area-minimizing surfaces, so are $S_1 \subset \Sigma_1$ and $S_2 \subset \Sigma_2$. Hence, as $\partial S_1 = \partial S_2$, $|S_1| = |S_2|$, where $|\cdot|$ represents the area. Let T_1 be a compact subsurface in Σ_1 containing S_1, ie $S_1 \subset T_1 \subset \Sigma_1$. Consider $T_1' = (T_1 - S_1) \cup S_2$. Since T_1 is area-minimizing and $|T_1'| = |T_1|$, so is T_1'. However, T_1' is not smooth along α, which contradicts the regularity of area-minimizing surfaces (Lemma 2.7). The proof follows. □

Remark A.2 In the lemma above, $\Omega_1 \cap \Omega_2 = \varnothing$ or $\Omega_1 \subset \operatorname{int}(\Omega_2)$ are indeed equivalent conditions. This is because we can always replace Ω_2 with $\overline{\Omega_2^c}$. Notice also that the proof above is simply a swapping argument (S_1 and S_2) for area-minimizing surfaces, and the proof actually works for the more general case. In particular, we do not need Γ_i to be a collection of *simple* closed curves, but only to be $\Gamma_i = \partial \Omega_i$, where $\Omega_1 \subset \operatorname{int}(\Omega_2)$ for the swapping argument. So, with the same proof, the lemma above can also be stated as follows: *Let* Ω_1 *and* Ω_2 *be two open regions in* $\partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ *with* $\overline{\Omega}_1 \subset \Omega_2$. *If* Σ_1 *and* Σ_2 *are area-minimizing surfaces in* $\mathbb{H}^2 \times \mathbb{R}$ *with* $\partial_\infty \Sigma_i = \partial \overline{\Omega}_i$, *then* $\Sigma_1 \cap \Sigma_2 = \varnothing$.

Now we show that if a tall curve $\Gamma \subset S^1_\infty \times \mathbb{R}$ does not bound a unique area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$, it bounds two canonical area-minimizing surfaces $\Sigma^\pm$, where any other area-minimizing surface Σ' with $\partial_\infty \Sigma' = \Gamma$ must be "between" Σ^+ and Σ^-.

Lemma A.3 (canonical surfaces) *Let* Γ *be a tall curve in* $S^1_\infty \times \mathbb{R}$. *Then either there exists a unique area-minimizing surface* Σ *in* $\mathbb{H}^2 \times \mathbb{R}$ *with* $\partial_\infty \Sigma = \Gamma$, *or there are two canonical disjoint extremal area-minimizing surfaces* Σ^+ *and* Σ^- *in* $\mathbb{H}^2 \times \mathbb{R}$ *with* $\partial_\infty \Sigma^\pm = \Gamma$.

Proof We mainly adapt the techniques of [2, Lemma 4.3] (a similar result for $\mathbb{H}^3$) to the $\mathbb{H}^2 \times \mathbb{R}$ context. Let Γ be a tall curve in $S^1_\infty \times \mathbb{R}$, and let $\Gamma^c = \Omega^+ \cup \Omega^-$, where

$\Omega^{\pm}$ are two tall regions in $S^1_\infty \times \mathbb{R}$ with $\partial\overline{\Omega^+} = \partial\overline{\Omega^-} = \Gamma$. Let $N_\epsilon(\Gamma)$ be a small open neighborhood of Γ in $S^1_\infty \times \mathbb{R}$. Let $N^+ = N_\epsilon(\Gamma) \cap \Omega^+$ and let $N^- = N_\epsilon(\Gamma) \cap \Omega^-$. Let the family of curves $\{\Gamma^{\pm}_t \mid t \in [0,\epsilon)\}$ foliate the region $N^{\pm}$ with $\Gamma_0 = \Gamma$. Let $\Gamma^{\pm}_n = \Gamma^{\pm}_{t_n}$ for $t_n \searrow 0$. By choosing $\epsilon > 0$ sufficiently small, we can assume $\Gamma^{\pm}_n$ is tall for any $n > 0$. Let $\Sigma^{\pm}_n$ be an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma^{\pm}_n = \Gamma^{\pm}_n$ by Lemma 2.4.

By replacing the sequence Σ_n with $\widehat{B}_n \cap \Sigma^{\pm}_n$ in the proof of Lemma 2.4, we can show that Σ^+_n converges (up to a subsequence) to an area-minimizing surface Σ^+ with $\partial_\infty \Sigma^+ = \Gamma$. Similarly, Σ^-_n converges to an area-minimizing surface Σ^- with $\partial_\infty \Sigma^- = \Gamma$.

Assume that $\Sigma^+ \neq \Sigma^-$ and they are not disjoint. By the maximum principle, they cannot have isolated points in the intersection. Therefore, nontrivial intersection implies some part of Σ^- lies *above* Σ^+, ie some part of Σ^- separated by $\Sigma^=$. Then, since $\Sigma^+ = \lim \Sigma^+_n$, Σ^- must also intersect some Σ^+_n for sufficiently large n. However, by Lemma A.1 (swapping argument), Σ^+_n is disjoint from Σ^- as $\partial_\infty \Sigma^+_n = \Gamma^+_n$ is disjoint from $\partial_\infty \Sigma^- = \Gamma$. This is a contradiction. This shows Σ^+ and Σ^- are disjoint. By using similar techniques to [2, Lemma 4.3], it can be shown that $\Sigma^{\pm}$ are canonical, ie independent of the sequences $\{\Sigma^{\pm}_n\}$.

Similar arguments show that $\Sigma^{\pm}$ are disjoint from any area-minimizing hypersurface Σ' with $\partial_\infty \Sigma' = \Gamma$. As the sequences of Σ^+_n and Σ^-_n form a barrier for other area-minimizing hypersurfaces asymptotic to Γ, any such area-minimizing hypersurface must lie in the region bounded by Σ^+ and Σ^- in $\mathbb{H}^2 \times \mathbb{R}$. This shows that if $\Sigma^+ = \Sigma^-$, then there exists a unique area-minimizing hypersurface asymptotic to Γ. The proof follows. □

Remark A.4 If a finite collection of simple closed curves Γ is not assumed to be tall in the lemma above, the same proof is still valid. Hence, for any such Γ, either there is no solution ($\nexists\Sigma$), or a unique solution ($\exists!\Sigma$), or two canonical solutions ($\exists\Sigma^{\pm}$) for the asymptotic plateau problem for Γ ($\partial_\infty \Sigma = \Gamma$).

Now, by using the lemma above, we show a generic uniqueness result for tall curves.

Theorem A.5 (generic uniqueness) *A generic tall curve in $S^1_\infty \times \mathbb{R}$ bounds a unique area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$.*

Proof Let Γ_0 be a tall curve in $S^1_\infty \times \mathbb{R}$. Let $N(\Gamma_0)$ be a small open neighborhood of Γ_0 in $S^1_\infty \times \mathbb{R}$ which is a finite collection of annuli. Let $\{\Gamma_t \mid t \in (-\epsilon, \epsilon)\}$ be a foliation of $N(\Gamma_0)$. In particular, for any $-\epsilon < t_1 < t_2 < \epsilon$, $\Gamma_{t_1}\cap\Gamma_{t_2}=\varnothing$. We can assume $N(\Gamma_0)$ is sufficiently thin that Γ_t is a tall curve for any $t\in(-\epsilon,\epsilon)$. Let Σ_t be an area-minimizing surface in $\mathbb{H}^2\times\mathbb{R}$ with $\partial_\infty\Sigma_t=\Gamma_t$.

As in the proof of the lemma above, let $\Gamma_t^c=\Omega_t^+\cup\Omega_t^-$ with $\partial\overline{\Omega_t^+}=\partial\overline{\Omega_t^-}=\Gamma_t$. Then $\Omega_t^+\subset\Omega_s^+$ for $t<s$. Hence, by Lemma A.1, $\Sigma_t\cap\Sigma_s=\varnothing$ for $t<s$. Furthermore, by Lemma A.3, if Γ_s does not bound a unique area-minimizing surface Σ_s, then we can define two disjoint canonical minimizing surfaces Σ_s^+ and Σ_s^- with $\partial_\infty\Sigma_s^\pm=\Gamma_s$. Hence, $\Sigma_s^+\cup\Sigma_s^-$ separates a region V_s in $\mathbb{H}^2\times\mathbb{R}$. If Γ_s bounds a unique area-minimizing surface Σ_s, then let $V_s=\Sigma_s$ (say V_s is a degenerate neighborhood). Notice that, by Lemma A.1, $\Sigma_t\cap\Sigma_s=\varnothing$ for $t\neq s$, and hence $V_t\cap V_s=\varnothing$ for $t\neq s$.

Now consider a short arc segment η in $\mathbb{H}^2\times\mathbb{R}$ with one endpoint in Σ_{t_1} and the other endpoint in Σ_{t_2}, with $-\epsilon<t_1<0<t_2<-\epsilon$. Hence, η intersects all area-minimizing surfaces Σ_t with $\partial_\infty\Sigma_t=\Gamma_t$, where $t_1\le t\le t_2$. Now, for $t_1<s<t_2$, define the *thickness* λ_s of V_s as $\lambda_s=|\eta\cap V_s|$, ie λ_s is the length of the piece of η in V_s. Hence, if Γ_s bounds more than one area-minimizing surface, then the thickness is $\lambda_s>0$. In other words, if $\lambda_s=0$, then Γ_s bounds a unique area-minimizing surface in $\mathbb{H}^2\times\mathbb{R}$.

As $V_t\cap V_s=\varnothing$ for $t\neq s$, we have $\sum_{t_1}^{t_2}\lambda_s<|\eta|$. Hence, as $|\eta|$ is finite, $\lambda_s>0$ for only countably many $s\in[t_1,t_2]$. This implies that, for all but countably many $s\in[t_1,t_2]$, $\lambda_s=0$, and hence Γ_s bounds a unique area-minimizing surface. Similarly, this implies that, for all but countably many $s\in(-\epsilon,\epsilon)$, Γ_s bounds a unique area-minimizing surface. Then, by using the techniques in [2, Lemma 3.2], generic uniqueness in the Baire sense follows. □

Remark A.6 (generalization to minimal surfaces) The results in this section are mostly for area-minimizing surfaces, and may not be true for minimal surfaces in general. This is mostly because the disjointness lemma (Lemma A.1) uses the surgery argument, which only holds for area-minimizing surfaces. Because of this, the generic uniqueness result above may not be true for minimal surfaces in general. On the other hand, for *short curves*, there is a trivial counterexample as follows: Let $\Gamma_t=S^1_\infty\times\{c_0,c_0+t\}$ be a pair of round circles in $S^1_\infty\times\mathbb{R}$ for $t\in(0,\pi)$. Then $\{C_t\}$ gives a continuous family of nonuniqueness curves in $S^1_\infty\times\mathbb{R}$ as follows: Let $\mathcal{C}_t$ be the minimal horizontal catenoid of height t with $\partial_\infty\mathcal{C}_t=\Gamma_t$ [16]. Let $\Sigma_t=\mathbb{H}^2\times\{c_0,c_0+t\}$ be the pair of horizontal geodesic planes with $\partial_\infty\Sigma_t=\Gamma_t$. Therefore, any curve in the continuous family $\{\Gamma_t\}$ bounds at least two minimal surfaces. As this is a continuous family, it gives a counterexample to the generic uniqueness result above.

A.2 Nonexistence results for the vertical bridge principle

The following lemma rules out some special cases the for asymptotic plateau problem, and used in the proof of the bridge principle.

Lemma A.7 *Let $\gamma_c = S^1_\infty \times \{c\}$ represent the round circle in $S^1_\infty \times \mathbb{R}$ with $\{z = c\}$. Let $\Gamma = \bigcup_{i=1}^N \gamma_{c_i} \cup \bigcup_{j=1}^M \alpha_j$, where $\alpha_j = \{\theta_j\} \times [c_{j_1}, c_{j_2}]$ for some $\theta_j \in S^1_\infty$, and $c_i < c_{i+1}$. Then there exists a $K_0 > \pi$ such that the following holds: if $c_{i+1} - c_i > K_0$ for any i and Σ is an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma \subset \Gamma$, then Σ is a collection of horizontal planes, ie $M = 0$ and $\Sigma = \mathbb{H}^2 \times \{c_{i_1}, c_{i_2}, \dots, c_{i_k}\}$.*

Notice that the statement implies that, for such $K_0 > 0$, the asymptotic boundary of such an area-minimizing surface cannot contain any vertical line segment.

Proof Without loss of generality, we assume $N = 2$ as the other cases are similar. We divide the proof into two cases: $M = 1$ and $M > 1$.

Case 1 Assume $M = 1$, ie $\Gamma = \gamma_{c_1} \cup \gamma_{c_2} \cup \alpha_1$, where $\alpha_1 = \{\theta_1\} \times [c_1, c_2]$ for some $\theta_1 \in S^1_\infty$. Let Σ be the area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty \Sigma \subset \Gamma$. Recall that $c_2 - c_1 > K_0 > \pi$. Let R_i be a sequence of tall rectangles exhausting the region bounded by Γ, ie $R_i = \partial([\theta_1 + \epsilon_i, \theta_1 - \epsilon_i + 2\pi] \times [c_1 + \rho_i, c_2 - \rho_i])$ in $S^1_\infty \times \mathbb{R}$, where $\epsilon_i \searrow 0$ and $\rho_i \searrow 0$. Clearly, R_i is disjoint from Γ for any i, and $R_i \to \Gamma$ as $i \to \infty$.

Let P_i be the unique area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty P_i = R_i$ (Lemma 2.3). By Lemma A.1 and Remark A.2, $\Sigma \cap P_i = \varnothing$ for any i. On the other hand, the explicit description of P_i in [18] shows that P_i is foliated by horizontal equidistant curves $\beta_i^t = P_i \cap \mathbb{H}^2 \times \{t\}$ to the geodesic τ_i with $\partial_\infty \tau_i = \{\theta_1 + \epsilon_i, \theta_1 - \epsilon_i + 2\pi\}$. In particular, for $d_i(t) = d(\beta_i^t, \tau_i)$, $d_i(t) \to \infty$ as $t \to c_1$ or $t \to c_2$, while $d_i(c*) < C_0$, where $c* = \frac{1}{2}(c_1 + c_2)$ (see the discussion before Lemma 2.3). Hence, as $i \to \infty$, τ_i and hence β_i^{c*} escape to infinity. This shows P_i converges to two horizontal geodesic planes $\mathbb{H}^2 \times \{c_1, c_2\}$. However, this implies $\Sigma \cap P_i \neq \varnothing$ for sufficiently large i unless $\partial_\infty \Sigma \subset \gamma_{c_1} \cup \gamma_{c_2}$. Hence, the $M = 1$ case follows.

Case 2 Now assume $M > 1$. By using a simple trick, we reduce this case to $M = 2$. Let $\theta_0 \in S^1_\infty - \{\theta_1, \theta_2, \dots, \theta_M\}$. Let τ be the geodesic in $\mathbb{H}^2$ with $\partial_\infty \tau = \{\theta_0, \theta_1\}$. Let φ be the hyperbolic isometry fixing τ pushing from θ_1 to θ_0 with translation length $l > 0$. Let $\widehat{\varphi}$ be the isometry of $\mathbb{H}^2 \times \mathbb{R}$ with $\widehat{\varphi}(x, t) = (\varphi(x), t)$. Then define the sequence of area-minimizing surfaces $\Sigma_n = \widehat{\varphi}^n(\Sigma)$. Then, by Lemma 2.8, there exists a subsequence of $\{\Sigma_n\}$ converging to an area-minimizing surface $\widehat{\Sigma}$ in $\mathbb{H}^2\times\mathbb{R}$. Let $\widehat{\Gamma}=\partial_\infty\widehat{\Sigma}$. By construction, $\widehat{\Sigma}$ is invariant under $\widehat{\varphi}$, then so is $\widehat{\Gamma}$. As $\{\theta_0,\theta_1\}$ are the fixed points of φ, this implies $\widehat{\Gamma}\subset\gamma_{c_1}\cup\gamma_{c_2}\cup\alpha_0\cup\alpha_1$, where $\alpha_i=\{\theta_i\}\times[c_1,c_2]$.

We claim that $\widehat{\Gamma}=\gamma_{c_1}\cup\gamma_{c_2}\cup\alpha_0\cup\alpha_1$. Clearly, $\widehat{\Gamma}\supset\gamma_{c_1}\cup\gamma_{c_2}$ by construction. Now consider a component S of Σ with $\partial_\infty S\supset\alpha_1$ (possibly $\Sigma=S$). Since we assumed $M>1$, $\partial_\infty S$ must contain another α_{j_0} for some $j_0>1$. By Lemma 2.5, $\overline{S}=S\cup\partial_\infty S$ is a surface with boundary in $\overline{\mathbb{H}^2\times\mathbb{R}}$. Consider the collection of curves $\lambda_c=S\cap\mathbb{H}^2\times\{c\}$ for $c\in(c_1,c_2)$. As S is connected, there exists a $c\in(c_1,c_2)$ such that λ_c contains a infinite line l_c in $\mathbb{H}^2\times\{c\}$ with $\partial_\infty l_c=\{\theta_1,\theta_j\}$. Let $l_c^n=\widehat{\varphi}^n(l_c)\subset\Sigma_n\cap\mathbb{H}^2\times\{c\}$. Then, by construction, l_c^n converges to $\hat{l}_c\subset\widehat{\Sigma}\cap\mathbb{H}^2\times\{c\}$, where $\partial_\infty\hat{l}_c=\{\theta_1,\theta_0\}$. This proves that $\alpha_0\cup\alpha_1\supset\widehat{\Gamma}$. Hence, we reduce the $M>1$ case to the $M=2$ case.

Now we finish this case. Recall that, by construction $\widehat{\Sigma}$ is invariant by $\widehat{\Sigma}$, ie $\widehat{\varphi}(\widehat{\Sigma})=\widehat{\Sigma}$. Because of this invariance, we first claim that $\widehat{\Sigma}=\mathcal{P}_0\cup\mathcal{P}_1$, where $\mathcal{P}_i$ is the unique area-minimizing plane with asymptotic boundary a rectangle R_i, ie $\partial_\infty\mathcal{P}_0=R_0=\partial([\theta_0,\theta_1]\times[c_1,c_2])$ and $\partial_\infty\mathcal{P}_1=R_1=\partial([\theta_1,\theta_0+2\pi]\times[c_1,c_2])$. In order to see this, let $\theta_2=\frac{1}{2}(\theta_0+\theta_1)$ and $\theta_3=\theta_2+\pi$ in S^1_∞. Let η be the geodesic in $\mathbb{H}^2$ with $\partial_\infty\eta=\{\theta_2,\theta_3\}$. Let $\mathcal{W}=\eta\times\mathbb{R}$ be the vertical plane in $\mathbb{H}^2\times\mathbb{R}$. Consider $Z=\mathcal{W}\cap\widehat{\Sigma}$. By construction, Z is a collection of curves with $\partial_\infty Z$ is the four points, (θ_2,c_1), (θ_2,c_2), (θ_3,c_1) and (θ_3,c_2). Invariance of $\widehat{\Sigma}$ by $\widehat{\varphi}$ implies that Z is the generating curves for $\widehat{\Sigma}$. Assuming $\widehat{\Sigma}\neq\mathbb{H}^2\times\{c_1,c_2\}$, by [18], we conclude that $Z=\mu_0\cup\mu_1$, where μ_0 is the generating curve for $\mathcal{P}_0$ and μ_1 is the generating curve for $\mathcal{P}_1$ such that $\partial_\infty\mu_0=\{(\theta_2,c_1),(\theta_2,c_2)\}$ and $\partial_\infty\mu_1=\{(\theta_3,c_1),(\theta_3,c_2)\}$. Now, even though the union $\mathcal{P}_0\cup\mathcal{P}_1$ is a minimal surface in $\mathbb{H}^2\times\mathbb{R}$, we show that it is not an area-minimizing surface, and finish the proof of Case 2.

Claim $\mathcal{P}_0\cup\mathcal{P}_1$ *is not an area-minimizing surface.*

We show that a sufficiently long annulus $\mathcal{A}$ between $\mathcal{P}_0$ and $\mathcal{P}_1$ has less area than the sum of the areas of the corresponding disks D_0 in $\mathcal{P}_0$ and D_1 in $\mathcal{P}_1$, ie $\partial\mathcal{A}=\partial D_0\cup\partial D_1$ (see Figure 6).

Without loss of generality, let $c_1=-K$ and $c_2=K$, and $\theta_0=0$ and $\theta_1=\pi$ in S^1_∞. By [18, Proposition 2.1(1)] and Lemma 2.3, we have a very good understanding of the area-minimizing planes $\mathcal{P}_0$ and $\mathcal{P}_1$. By the symmetry, we work with only $\mathcal{P}_0$. Let υ be the geodesic in $\mathbb{H}^2$ with $\partial_\infty\upsilon=\{\frac{\pi}{2},\frac{3\pi}{2}\}$. Recall that $\mathcal{P}_0$ has the generating curve c_0

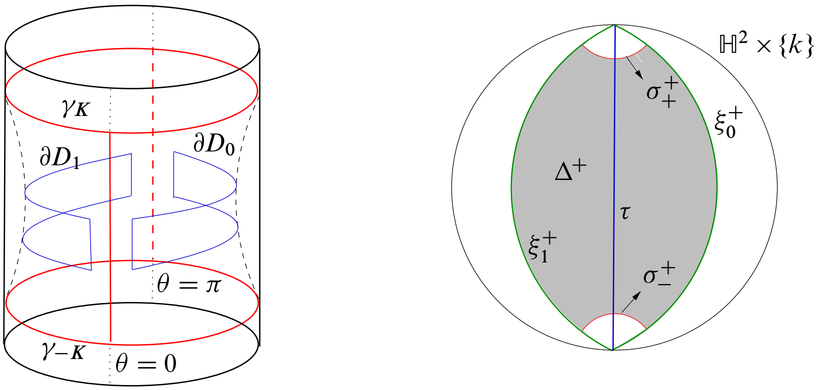

Figure 6: Left: the red curve represents $\widehat{\Gamma}$ in $S^1_\infty \times \mathbb{R}$; blue curves represent ∂D_i in $\mathcal{P}_i$. Right: the domain Δ^+ is depicted in the banana region between equidistant lines to τ.

in the vertical plane $\upsilon \times \mathbb{R}$, where $\partial_\infty c_0 = \{(\frac{\pi}{2}, -K), (\frac{\pi}{2}, K)\}$. The parametrization of the generating curve c_0 has explicitly been given in the proof of [18, Proposition 2.1] as $\lambda(\rho)$ for $d > 1$ (Case 1). Now recall that $\partial_\infty \mathcal{P}_0 = R_0 = \partial([0, \pi] \times [-K, K])$. Let $t \in [-K, K]$ represent the height in $\mathbb{H}^2 \times \mathbb{R}$. Parametrize c_0 as $\lambda(t) = (\rho(t), t)$ in $\upsilon \times \mathbb{R}$, where $\rho(t)$ is the distance of $(0, t)$ to $c_0 \cap \mathbb{H}^2 \times \{t\}$.

Recall that τ is the geodesic in $\mathbb{H}^2$ with $\partial_\infty \tau = \{0, \pi\}$. Parametrize τ so that $\tau(s)$ is the signed distance from the origin for $s \in (-\infty, +\infty)$. In particular, $\tau(+\infty) = \pi$ and $\tau(+\infty) = \pi$ in S^1_∞. Let φ_t be the hyperbolic isometry fixing τ with translation length $t \in \mathbb{R}$. Then, by [18], $\varphi_t(\mathcal{P}_0) = \mathcal{P}_0$ for any $t \in \mathbb{R}$. Let $\mathcal{P}_o \cap \mathbb{H}^2 \times \{t\} = \eta_t$. Then, by construction, η_t is the equidistant line to τ with distance $\rho(t)$. Parametrize η_t so that the closest point to $\tau(s)$ in η_t is $\eta_t(s)$ for $s \in (-\infty, +\infty)$.

Now we describe D_i in $\mathcal{P}_i$ by defining its boundary ∂D_i. Like $\mathcal{P}_0$ and $\mathcal{P}_1$, D_0 and D_1 will be symmetric with respect to $\mathcal{T} = \tau \times \mathbb{R}$, so let's only consider D_0. The boundary ∂D_0 is a rectangle in $\mathcal{P}_0$ with the following four edges. Fix $k > \frac{\pi}{2}$ be half the height of the rectangle with $k \ll K$. Let the upper edge ξ_0^+ be the segment in η_k between the points $\eta_k(-l)$ and $\eta_k(l)$, where $l \gg 0$ will be determined later. Similarly, let the lower edge ξ_0^- be the segment in η_{-k} between the points $\eta_{-k}(-l)$ and $\eta_{-k}(l)$. Let the short edges be the vertical paths ν_0^+ and ν_0^- in $\mathcal{P}_0$ with endpoints $\{\eta_k(l), \eta_{-k}(l)\}$ and $\{\eta_k(-l), \eta_{-k}(-l)\}$, respectively. Hence, D_0 is the rectangle in $\mathcal{P}_0$ with $\partial D_0 = \xi_0^+ \cup \nu_0^+ \cup \xi_0^- \cup \nu_0^-$. Similarly, define D_1 in $\mathcal{P}_1$ as $\partial D_1 = \xi_1^+ \cup \nu_1^+ \cup \xi_1^- \cup \nu_1^-$ as the symmetric rectangle with respect to the vertical plane $\mathcal{T}$ (see Figure 6, left).

Now we define the *competitor* annulus $\mathcal{A}$ with $\partial\mathcal{A} = \partial D_0 \cup \partial D_1$. Let σ_+^+ be the geodesic between $\eta_k(l)$ and its reflection with respect to $\mathcal{T}$. Let σ_-^+ be the reflection of σ_+^+ with respect to $\upsilon \times \mathbb{R}$. Let σ_+^- be the reflection of σ_+^+ with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$. Similarly, let σ_-^- be the reflection of σ_-^+ with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$.

Now let Δ^+ be the region in the horizontal plane $\mathbb{H}^2 \times \{+k\}$ such that $\partial\Delta^+ = \xi_0^+ \cup \sigma_+^+ \cup \xi_1^+ \cup \sigma_-^+$ (see Figure 6, right). Let Δ^- be the reflection of Δ^+ with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$. Let Ω^+ be the region in the vertical plane containing σ_+^+ and σ_+^- such that $\partial\Omega^+ = \sigma_+^+ \cup \nu_0^+ \cup \sigma_+^- \cup \nu_1^+$. Similarly, define Ω^- in the opposite side. Hence, $\mathcal{A} = \Delta^+ \cup \Delta^- \cup \Omega^+ \cup \Omega^-$. Then we have $\partial\mathcal{A} = \partial D_0 \cup \partial D_1$.

Let $|\cdot|$ represent the area. We claim that $|\mathcal{A}| < |D_0| + |D_1|$ for sufficiently large $l > 0$ and $K > 0$. First note that $|D_i| > 4kl$ as $2k$ is the height of the rectangle D_i, and any horizontal segment $\eta_t \cap D_i$ has length greater than $2l$ by construction.

Consider $|\mathcal{A}|$. Note Δ^+ belongs to the banana region in $\mathbb{H}^2 \times \{k\}$ bounded by η_k and its reflection. Let $\beta(t)$ be the asymptotic angle between the geodesic τ and the equidistant line η_t. Note that there is a one-to-one correspondence between the equidistance $\rho(t)$ and the angle $\beta(t)$. Let $\beta_0 = \beta(k)$. In this setting, if $t \to K$, then $\rho(t) \to \infty$ and $\beta(t) \to \frac{\pi}{2}$. Then a simple computation shows that $|\Delta^+| = 4l \tan\beta_0$. Furthermore, $|\Omega^\pm| < 2k\|\sigma_+^+\|$ as Ω^+ is a rectangle in the vertical plane with height $2k$ and all horizontal segments have length $2\rho(t)$ for $t \in [0,k]$. As $\|\sigma_+^+\| = 2\rho(k)$, we have $|\Omega^\pm| < 4k\rho(k)$.

Hence, we have $|\mathcal{A}| = 2|\Delta| + 2|\Omega| < 4l \tan\beta_0 + 8k\rho(k)$.

Since $|D_i| > 4kl$, $|\mathcal{A}| < |D_0| + |D_1|$ is equivalent to

$$8k\rho(k) < 4l.(2k - \tan\beta_0).$$

Now fix $k > \frac{\pi}{2}$. Notice that, by the explicit description of $\mathcal{P}_i$ in [18], if the height of $\mathcal{P}_i$ is $K \to \infty$ then $\beta_0 \to 0$ and $\rho(k) \to 0$. Hence, by choosing K sufficiently large, we can make sure that $\tan\beta_0 < 2k$. Then, for sufficiently large $l > 0$, we have the desired inequality. The proof of the claim and Case 2 follows.

Now we finish the proof of the lemma. So far, we have shown that if Σ is an area-minimizing surface in $\mathbb{H}^2 \times \mathbb{R}$ with $\partial_\infty\Sigma \subset \bigcup_{i=1}^N \gamma_{c_i} \cup \bigcup_{j=1}^M \alpha_j$, then $\partial_\infty\Sigma \subset \bigcup_{i=1}^N \gamma_{c_i}$. In other words, we prove that the asymptotic boundary of such an area-minimizing surface cannot have any vertical segments. Now we show that every component of Σ is a horizontal plane. In particular, assume that a component S of Σ

contains more than one horizontal circle, say $\gamma_{c_1} \cup \gamma_{c_2}$. By assumption, $|c_1 - c_2| > K_0 > \pi$. Let $[d_1, d_2] \subset (c_1, c_2)$ with $d_2 - d_1 = \pi$. Then consider the parabolic catenoid $\mathfrak{C}$ with $\partial_\infty \mathcal{C} = \gamma_{d_1} \cup \gamma_{d_2} \cup \alpha$, where α is the vertical segment corresponding to $\{0\} \times [d_1, d_2]$ in upper half-space model. In particular, in the upper half-space model, $\mathfrak{C} = \sigma \times \mathbb{R}$, where σ is the generating curve in xy–plane $\mathbb{H}^2$ with $\partial_\infty \sigma = \{(d_1, 0), (d_2, 0)\}$. Let $\varphi_\lambda(x, y, z) = (\lambda x, \lambda y, z)$ be the isometry of $\mathbb{H}^2 \times \mathbb{R}$ in the upper half-space model. Then $\mathfrak{C}_\lambda = \varphi_\lambda(\mathfrak{C})$ is another parabolic catenoid with generating curve $\lambda \cdot \sigma$. Now, for sufficiently small $\lambda > 0$, $\mathfrak{C} \cap S = \varnothing$. On the other hand, when $\lambda \to \infty$, $\mathfrak{C}$ converges to $\mathbb{H}^2 \times \{d_1, d_2\}$. This means that if $\partial_\infty S \supset \gamma_{c_1} \cup \gamma_{c_2}$, by increasing λ, for some $\lambda_0 > 0$, we can find the first point of contact between S and $\mathfrak{C}_{\lambda_0}$. However, this contradicts the maximum principle.

Finally, we show that if $\Gamma = \bigcup_{i=1}^N \gamma_{c_i}$ ($M = 0$), then Σ is indeed a collection of horizontal planes. Assume that there is a component S in Σ with $\partial_\infty S = \gamma_{c_j} \cup \gamma_{c_k}$. Since $h(\Gamma) = K_0 > \pi$, let $[e^-, e^+] \subset (c_j, c_k)$ with $e^+ - e^- = \pi$. Let $\mathcal{C}$ be Daniel's parabolic catenoid with $\partial_\infty \mathcal{C} = \gamma_{e^+} \cup \gamma_{e^-}$. We can push $\mathcal{C}$ towards $S^1_\infty \times \mathbb{R}$ as much as we want by using isometries, so that we can assume $\mathcal{C} \cap S = \varnothing$. Then, by pushing $\mathcal{C}$ towards S by using the isometries, we get a first point of contact, which contradicts the maximum principle. This proves that Σ must be a collection of horizontal planes, ie $\Sigma = \bigcup_{i=1}^N \mathbb{H}^2 \times \{c_i\}$. The proof follows. □

Remark A.8 (bridge height K_0) The above lemma is the only reason we need large K_0 for the vertical bridge principle. However, the constant K_0 in the lemma above might be highly improved (conjecturally $K_0 = \pi$) by using similar ideas. In particular, the estimates we use in Lemma A.7 are very rough, and by using the explicit description of the generating curve for $\mathcal{P}_i$ in [18], one can choose $k \in (\pi, h(\Gamma))$ more elegantly. Then, by choosing l sufficiently large, one might get a vertical bridge principle for all tall curves ($h(\Gamma) > \pi$), not just curves with $h(\Gamma) > K_0$. Furthermore, it might also be possible to prove a similar result for any collection of arcs $\{\alpha_i\}$ without the verticality condition on α_i.

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