

# IRREDUCIBLE COMPONENTS OF THE MODULI SPACE OF RANK 2 SHEAVES OF ODD DETERMINANT ON THE PROJECTIVE SPACE

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**ABSTRACT.** We describe new irreducible components of the moduli space of rank 2 semistable torsion free sheaves on the three-dimensional projective space whose generic point corresponds to non-locally free sheaves whose singular locus is either 0-dimensional or consists of a line plus disjoint points. In particular, we prove that the moduli spaces of semistable sheaves with Chern classes  $(c_1, c_2, c_3) = (-1, 2n, 0)$  and  $(c_1, c_2, c_3) = (0, n, 0)$  always contain at least one rational irreducible component. As an application, we prove that the number of such components grows as the second Chern class grows, and compute the exact number of irreducible components of the moduli spaces of rank 2 semistable torsion free sheaves with Chern classes  $(c_1, c_2, c_3) = (-1, 2, m)$  for all possible values for  $m$ ; all components turn out to be rational. Furthermore, we also prove that these moduli spaces are connected, showing that some of sheaves here considered are smoothable.

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## 1. INTRODUCTION

Following the proof of existence of a projective moduli scheme parametrizing S-equivalence classes of semistable sheaves on a projective variety by Maruyama [25], the study of the geometry of such moduli spaces has been a central topic of research within algebraic geometry. Although a lot is known for curves and surfaces, general results for three dimensional varieties are still lacking. In fact, moduli spaces of sheaves on 3-folds turn out to be quite complicated spaces (as it is illustrated by Vakil's Murphy's law [37]), particularly with several irreducible components of various dimensions.

The goal of this paper is to advance on the study of the moduli space of semistable rank 2 sheaves on  $\mathbb{P}^3$  with fixed Chern classes  $(c_1, c_2, c_3)$ , which we will denote by  $\mathcal{M}(c_1, c_2, c_3)$ . Additionally, we will also consider the open subset consisting of stable reflexive sheaves, denoted by  $\mathcal{R}(c_1, c_2, c_3)$ ; when  $c_3 = 0$ , this is actually the moduli space of stable locally free sheaves, and this will be denoted by  $\mathcal{B}(c_1, c_2)$ . Questions on the geometry of such spaces, such as

connectedness, or the number of irreducible components, seem to be less explored if compared to the study of the geometry of the Hilbert schemes of curves in the projective 3-space for instance; some results for Hilbert schemes of curves can be found in [12, 20, 21, 30, 31].

A rich literature on these moduli spaces was produced, especially in the 1980's and 1990's, studying  $\mathcal{R}(c_1, c_2, c_3)$  and  $\mathcal{B}(c_1, c_2)$  for specific values of the Chern classes. For instance, the geometry of  $\mathcal{B}(0, c_2)$  and  $\mathcal{B}(-1, c_2)$  is completely understood for  $c_2$  up to 5, see [1, 3, 5, 9, 14] for  $c_1 = 0$ , and [2, 15] for  $c_1 = -1$ . In addition, Ein characterized an infinite series of irreducible components of  $\mathcal{B}(c_1, c_2)$  and proved that the number of irreducible components of  $\mathcal{B}(c_1, c_2)$  goes to infinity as the  $c_2$  goes to infinity [8].

Regarding reflexive sheaves,  $\mathcal{R}(c_1, c_2, c_3)$  is known for  $c_2 \leq 3$  and all possible values for  $c_3$ , see [6] and the references therein. Some extremal values are also known, namely,  $\mathcal{R}(-1, c_2, c_2^2)$  was studied by Hartshorne in [13], Chang described  $\mathcal{R}(0, c_2, c_2^2 - 2c_2 + 4)$  in [7], while Miró-Roig studied  $\mathcal{R}(-1; c_2; c_2^2 - 2c_2 + 4)$  in [28], and the moduli spaces  $\mathcal{R}(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$  for  $1 \leq r \leq (-1 + \sqrt{4c_2 - 7})/2$ , and  $c_2$  greater than 5, and  $\mathcal{R}(-1, c_2, c_2^2 - 2(r-1)c_2)$  for  $c_2$  greater than 8 in [27].

Even less is known for torsion free sheaves. Okonek and Spindler proved in [32] that  $\mathcal{M}(0, c_2, c_2^2 - c_2 + 2)$  and  $\mathcal{M}(-1, c_2, c_2^2)$  are irreducible for  $c_2 \geq 6$ . For small values of  $c_2$ , Miró-Roig and Trautmann proved in [29] that  $\mathcal{M}(0, 2, 4)$  is irreducible, while Le Potier showed in [24, Chapter 7] that  $\mathcal{M}(0, 2, 0)$  has exactly 3 irreducible rational components; more recently, it was shown in [19] that  $\mathcal{M}(0, 2, 0)$  is connected. Trautmann has also argued that  $\mathcal{M}(0, 2, 2)$  has exactly 2 irreducible components [36]. More recently, Schmidt proved in [34], that  $\mathcal{M}(0, c_2, c_2^2 - c_2 + 2)$  and  $\mathcal{M}(-1, c_2, c_2^2)$  are irreducible for any  $c_2 \geq 0$ , using methods different from the ones employed by Okonek and Spindler and by Miró-Roig and Trautmann.

$\mathcal{M}(0, c_2, 0)$  for  $c_2 \geq 2$  was studied in [19], where new infinite series of irreducible components are described. The starting point is the identification of three different types of torsion free sheaves; more precisely, let  $E$  be a torsion free sheaf on  $\mathbb{P}^3$ , and set  $Q_E := E^{\vee\vee}/E$ , which we assume to be nontrivial; we have the following fundamental sequence

$$(1) \quad 0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow Q_E \rightarrow 0$$

and say that  $E$  has

- *0-dimensional singularities* if  $\dim Q_E = 0$ ;
- *1-dimensional singularities* if  $Q_E$  has pure dimension 1;
- *mixed singularities* if  $\dim Q_E = 1$ , but  $Q_E$  is not pure.

With this definition in mind, a systematic way of producing examples of irreducible components of  $\mathcal{M}(0, c_2, 0)$  whose generic point corresponds to a torsion free sheaf with 0-dimensional and 1-dimensional singularities is given in [19]. Furthermore, the third author and Ivanov [18] constructed irreducible components of  $\mathcal{M}(0, 3, 0)$  whose generic point corresponds to a torsion free sheaf with mixed singularities. Additionally, in a recent paper [17], Ivanov proved that  $\mathcal{M}(0, 3, 0)$  has at least 11 irreducible components.

Our first goal in this paper is to generalize the results presented in [18, 19], and show how to produce irreducible components of  $\mathcal{M}(c_1, c_2, c_3)$ , for values of  $c_1, c_2$  and  $c_3$  also including cases with  $c_1 = -1$  and  $c_3 \neq 0$ , for sheaves with 0-dimensional, 1-dimensional, and mixed singularities. More precisely, we prove the following two statements.

**Main Theorem 1.** *For each  $e \in \{-1, 0\}$ , let  $n$  and  $m$  be positive integers such that  $en \equiv m \pmod{2}$ . Let  $\mathcal{R}^*$  be a nonsingular, irreducible component of  $\mathcal{R}(e, n, m)$  of expected dimension  $8n - 3 + 2e$ .*

- (i) *For each  $l \geq 1$ , there exists an irreducible component*

$$T(e, n, m, l) \subset \mathcal{M}(e, n, m - 2l)$$

of dimension  $8n - 3 + 2e + 4l$  whose generic sheaf  $[E]$  satisfies  $[E^{\vee\vee}] \in \mathcal{R}^*$  and  $\text{length}(Q_E) = l$ .

- (ii) For each  $r \geq 2$  and  $s \geq 1$  such that  $2r + 2s \leq m + e + 2$ , or  $r = 1$  and  $s = 0$  when  $-e = n = m = 1$ , there exists an irreducible component

$$X(e, n, m, r, s) \subset \mathcal{M}(e, n + 1, m + 2 + c_1 - 2r - 2s)$$

of dimension of dimension  $8n + 4s + 2r + 2 + e$ , whose generic sheaf  $[E]$  satisfies  $[E^{\vee\vee}] \in \mathcal{R}^*$  and  $Q_E$  is supported on a line plus  $s$  points.

The case  $e = 0$  of the first part of the previous theorem is just [19, Theorem 7]; we prove here the case  $e = -1$ . The second part is a generalization of [18, Theorem 3], which covers the cases  $e = 0, n = 2, m = 2, 4$ .

Our second goal in this paper concerns the problem of rationality of irreducible components of the moduli spaces  $\mathcal{M}(e, n, m)$ . The study of this problem for the moduli components of locally free sheaves, which are contained in  $\mathcal{M}(-1, 2n, 0)$  and  $\mathcal{M}(0, n, 0)$ ,  $n \geq 1$ , dates back to late 70-ies and early 1980-ies. The rationality of these moduli components was proved for  $n \leq 3$  in case  $e = 0$  [3, 9, 14, 15]. The first infinite series of rational moduli components were constructed and studied in [4, 9, 38, 39]. Recently, A. Kytmanov, A. Tikhomirov and S. Tikhomirov [22] showed that there is a large infinite series of rational moduli components of locally free sheaves from  $\mathcal{M}(-1, 2n, 0)$  and  $\mathcal{M}(0, n, 0)$  which includes the above mentioned series. These are the so-called Ein components which were first found and studied by A. P. Rao [33] and L. Ein [8]. However, it is still an open question whether these components exist for every  $n$  sufficiently (there are gaps for some small values of  $n$ , see [22] for details). One of the central results of our paper states that, for any  $n \geq 1$  there exist rational irreducible components of  $\mathcal{M}(-1, 2n, 0)$  and of  $\mathcal{M}(0, n, 0)$ . The precise statement is given by the following theorem.

**Main Theorem 2.**

- (i) For any  $n \geq 1$ , the scheme  $\mathcal{M}(-1, 2n, 0)$  contains at least one rational, generically reduced, irreducible component with generic sheaf having 0-dimensional singularities. For any  $n \geq 3$ ,  $\mathcal{M}(-1, 2n, 0)$  contains at least one rational, generically reduced, irreducible component with generic sheaf having purely 1-dimensional singularities, respectively, at least  $2(n^2 - n - 1)$  rational, generically reduced, irreducible components with generic sheaves having singularities of mixed dimension.
- (ii) For any  $n \geq 2$ , the scheme  $\mathcal{M}(0, n, 0)$  contains at least one rational, generically reduced, irreducible component with generic sheaf having 0-dimensional singularities. For any  $n \geq 3$ , the scheme  $\mathcal{M}(0, n, 0)$  contains at least one rational, generically reduced, irreducible component with generic sheaf having purely 1-dimensional singularities. For any  $n \geq 4$ , the scheme  $\mathcal{M}(0, n, 0)$  contains at least  $\frac{n(n-3)}{2}$  rational, generically reduced, irreducible components with generic sheaves having singularities of mixed dimension.

In addition, we also show that  $\mathcal{M}(e, n, m)$  has rational irreducible components for  $e = -1, 0$  and  $n, m$  varying in a wide range (see Theorem 13 for the precise statement).

The proof of this theorem is based on the above mentioned results of Chang [7], Miró-Roig [27], Okonek–Spindler [32] and Schmidt [34] on reflexive sheaves, and uses elementary transformations of reflexive sheaves along finite sets of points.

We give two applications of our constructions. First, we prove that the number of irreducible components of  $\mathcal{M}(-1, n, 0)$  whose generic point corresponds to a sheaf with mixed singularities grows as  $n$  grows, see Theorem 11 below. Second, we provide a full description of the irreducible components of  $\mathcal{M}(-1, 2, m)$ .

**Main Theorem 3.** *The moduli spaces  $\mathcal{M}(-1, 2, m)$  are connected and*

- (i)  $\mathcal{M}(-1, 2, 4)$  *is irreducible and rational of dimension 11;*
- (ii)  $\mathcal{M}(-1, 2, 2)$  *is connected and has exactly 2 irreducible rational components of dimensions 11, and 15;*
- (iii)  $\mathcal{M}(-1, 2, 0)$  *is connected and has exactly 4 irreducible rational components of dimensions 11, 11, 15, and 19.*

Note that the rationality of all of these components follows directly from Main Theorem 2.

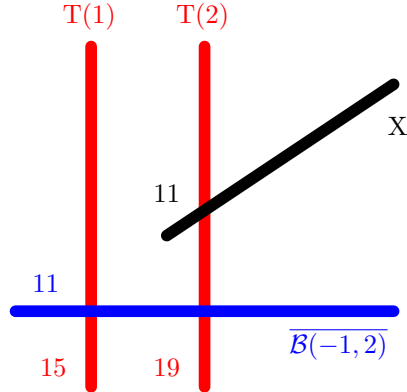


FIGURE 1. This is a representation of the geography of the moduli space  $\mathcal{M}(-1, 2, 0)$ . Each segment represents one of the irreducible components of  $\mathcal{M}(-1, 2, 0)$  and it is accompanied by its dimension. The blue component is the closure of  $\mathcal{B}(-1, 2)$ . The red components are of the type described in the first part of Main Theorem 1, namely  $T(l) = T(-1, 2, 2l, l)$ . The black component is of the type described in the second part of Main Theorem 1, namely  $X = X(-1, 1, 1, 1, 0)$ . The intersection of lines indicate when the corresponding components intersect.

We emphasize that proving that these moduli spaces are connected is quite relevant, since it is not known whether moduli spaces of rank 2 sheaves are in general connected, as it is the case for Hilbert schemes. In addition, we also provide very concrete descriptions of the generic points in each irreducible component; for a more detailed statement, see Theorems 25 and 26 for the cases  $c_3 = 2$  and  $c_3 = 0$ , respectively. A representation of the geography of  $\mathcal{M}(-1, 2, 0)$  is presented in Figure 1, showing how the various irreducible components intersect one another.

Another important aspect of the proof of the connectedness part of Main Theorem 3 is that we are implicitly showing that some of the sheaves presented in Main Theorem 1 are smoothable. To be more precise, a semistable non locally free sheaf with  $c_3 = 0$  is said to be *smoothable* if it can be deformed into a stable locally free sheaf, that is, if it lies in the closure of an irreducible component of  $\mathcal{B}(c_1, c_2)$  within  $\mathcal{M}(c_1, c_2, 0)$ . In the observations following the proof of Theorem 28 we provide certain sufficient conditions for smoothability of sheaves in  $\mathcal{M}(-1, 2, 0)$ .

The paper is organized as follows. In Section 2 we build up some basic techniques, and preliminary results. We compute the dimensions of the Ext groups of torsion free sheaves in terms of their Chern classes, and use it in Section 3 in order to produce the examples of

irreducible components of the moduli space of torsion free sheaves, and to prove Main Theorem 1. These results are then explored in end of Section 3 to prove that the number of irreducible components of  $\mathcal{M}(c_1, c_2, 0)$  whose generic point correspond to a sheaf with mixed singularities goes to infinity as  $c_2$  goes to infinity, thus providing our first application. Main Theorem 2 is proved in Section 4.

The remainder of the paper is occupied with the proof of Main Theorem 3. The irreducibility of  $\mathcal{M}(-1, 2, 4)$  is established in Section 5. After further technical results in Sections 6 and 7 regarding the families of sheaves introduced in Main Theorem 1, we dedicate Sections 8 and 9 to describing all irreducible components of  $\mathcal{M}(-1, 2, m)$  for  $c_3 = 2$  and  $c_3 = 0$ , respectively. The connectedness of  $\mathcal{M}(-1, 2, m)$  is finally established in Section 10.

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## 2. FIRST COMPUTATIONS

In order to study the moduli spaces of torsion free sheaves on  $\mathbb{P}^3$  we will need an explicit method to compute  $\dim \text{Ext}^1(E, E)$ , which gives us the dimension of the tangent space of the isomorphism class of a stable torsion free sheaf  $E$  as a point the moduli space. Our main goal in this section is to prove the following theorem.

**Theorem 1.** *Let  $E$  be a stable rank 2 torsion free sheaf on  $\mathbb{P}^3$  with  $e := c_1(E) \in \{-1, 0\}$ . Then*

$$\dim \text{Ext}^1(E, E) - \dim \text{Ext}^2(E, E) = 8c_2(E) - 3 - 2c_1(E)^2 = 8c_2(E) - 3 - 2e.$$

Note that this result generalizes [19, Lemma 5d)] and [19, Lemma 10], which establish the formula above for stable rank 2 torsion free sheaves with 0- and 1-dimensional singularities, respectively, in the case  $c_1(E) = 0$ . The proofs for sheaves with 0- and 1-dimensional singularities with arbitrary  $c_1$  are quite similar to the one in [19]; therefore, we only include here the proof for sheaves with mixed singularities.

Theorem 1 together with the deformation theory yields

**Corollary 2.** *Any irreducible component of the moduli space  $\mathcal{M}(e, c_2, c_3)$  has dimension at least  $8c_2 - 3 + 2e$ .*

**Lemma 3.** *If  $E$  is a torsion free sheaf on  $\mathbb{P}^3$ , then:*

- (i)  $\text{Ext}^1(E, E) = H^1(\mathcal{H}om(E, E)) \oplus \ker d_2^{01}$ ;
- (ii)  $\text{Ext}^2(E, E) = \ker d_3^{02} \oplus \ker d_2^{11} \oplus \text{coker } d_2^{01}$ ;
- (iii)  $\text{Ext}^3(E, E) = \text{coker } d_3^{02}$ .

Here,  $d_j^{pq}$  are the differentials in the  $j$ -th page of the spectral sequence for local to global ext's  $E_2^{pq} := \mathbb{H}^p(\mathcal{E}xt^q(E, E))$ . In particular, we have

$$\sum_{j=0}^3 (-1)^j \dim \text{Ext}^j(E, E) = \chi(\mathcal{H}om(E, E)) - \chi(\mathcal{E}xt^1(E, E)) + h^0(\mathcal{E}xt^2(E, E)).$$

*Proof.* The first part is a standard calculation with the spectral sequence  $E_2^{pq} := \mathbb{H}^p(\mathcal{E}xt^q(E, E))$ , which converges in its fourth page, because the spectral maps vanish. Note that  $\mathbb{H}^p(\mathcal{E}xt^q(E, E)) = 0$  for  $p \geq 2$  and  $q \geq 1$ , since  $\dim \mathcal{E}xt^q(E, E) \leq 1$  for  $q \geq 1$ . Furthermore, applying the functor  $\mathcal{H}om(\cdot, E)$  to the fundamental sequence (1), we get an epimorphism  $\mathcal{E}xt^3(E^{\vee\vee}, E) \rightarrow \mathcal{E}xt^3(E, E)$  and the isomorphism  $\mathcal{E}xt^2(E, E) \simeq \mathcal{E}xt^3(Q_E, E)$ ; however, the sheaf on the left vanishes because  $E^{\vee\vee}$  is reflexive, so  $\mathcal{E}xt^3(E, E) = 0$  as well. Finally, we also check that  $\dim \mathcal{E}xt^2(E, E) = 0$ ; indeed,  $E$  admits a resolution of the form

$$(2) \quad 0 \rightarrow L_2 \rightarrow L_1 \rightarrow L_0 \rightarrow E \rightarrow 0,$$

where  $L_k$  are locally free sheaves; we then get an epimorphism

$$\mathcal{E}xt^3(Q_E, \mathcal{O}_{\mathbb{P}^3}) \otimes L_0 \rightarrow \mathcal{E}xt^3(Q_E, E),$$

which implies that  $\dim \mathcal{E}xt^3(Q_E, E) = 0$  since  $\dim \mathcal{E}xt^3(Q_E, \mathcal{O}_{\mathbb{P}^3}) = 0$ .

The second claim is an immediate consequence of the first, since  $\dim \mathcal{E}xt^2(E, E) = 0$ .  $\square$

Assuming that  $E$  is  $\mu$ -semistable provides a useful simplification of the previous general result.

**Lemma 4.** *If  $E$  be a  $\mu$ -semistable torsion free sheaf on  $\mathbb{P}^3$ , then:*

- (i)  $\text{Ext}^1(E, E) = \mathbb{H}^1(\mathcal{H}om(E, E)) \oplus \ker d_2^{01}$ ;
- (ii)  $\text{Ext}^2(E, E) = \mathbb{H}^0(\mathcal{E}xt^2(E, E)) \oplus \mathbb{H}^1(\mathcal{E}xt^1(E, E)) \oplus \text{coker } d_2^{01}$ ;
- (iii)  $\text{Ext}^3(E, E) = 0$ .

Here,  $d_2^{01}$  is the spectral sequence differential  $d_2^{01} : \mathbb{H}^0(\mathcal{E}xt^1(E, E)) \rightarrow \mathbb{H}^2(\mathcal{H}om(E, E))$ .

*Proof.* The last item follows from Serre duality, we have

$$\text{Ext}^3(E, E) \simeq \text{Hom}(E, E(-4))^* = 0,$$

with the vanishing given by  $\mu$ -semistability. In addition, we argue that  $\mu$ -semistability also implies that  $\mathbb{H}^3(\mathcal{H}om(E, E)) = 0$ . Indeed, applying the functors  $\mathcal{H}om(\cdot, E)$  and  $\mathcal{H}om(E^{\vee\vee}, \cdot)$  to the fundamental sequences (1) we obtain, respectively,

$$0 \rightarrow \mathcal{H}om(E^{\vee\vee}, E) \rightarrow \mathcal{H}om(E, E) \rightarrow \mathcal{E}xt^1(Q_E, E) \rightarrow \dots$$

and

$$0 \rightarrow \mathcal{H}om(E^{\vee\vee}, E) \rightarrow \mathcal{H}om(E^{\vee\vee}, E^{\vee\vee}) \rightarrow \mathcal{H}om(E^{\vee\vee}, Q_E) \rightarrow \dots$$

In both sequences, the rightmost sheaf has dimension at most 1, hence so does the cokernel of the leftmost monomorphism, and it follows that

$$\mathbb{H}^3(\mathcal{H}om(E, E)) \simeq \mathbb{H}^3(\mathcal{H}om(E^{\vee\vee}, E)) \simeq \mathbb{H}^3(\mathcal{H}om(E^{\vee\vee}, E^{\vee\vee})).$$

However

$$\mathbb{H}^3(\mathcal{H}om(E^{\vee\vee}, E^{\vee\vee})) = \text{Ext}^3(E^{\vee\vee}, E^{\vee\vee}) \simeq \text{Hom}(E^{\vee\vee}, E^{\vee\vee}(-4))^* = 0;$$

the first equality follows from the spectral sequence for local to global ext's for  $E^{\vee\vee}$ , the isomorphism in the middle is given by Serre duality, and the vanishing is a consequence of the  $\mu$ -semistability of  $E^{\vee\vee}$ .

It follows that  $d_2^{pq} = 0$  except for  $d_2^{01}$ , while  $d_3^{pq} = 0$  for every  $p$  and  $q$ . This means that  $E_2^{pq}$  converges in its third page, providing the desired result.  $\square$

The following technical lemma will be helpful in our next argument.

**Lemma 5.** *Let  $F$  be a torsion free sheaf. If  $E$  is a subsheaf of  $F$  for which the quotient sheaf  $Z := F/E$  is 0-dimensional, then*

$$(3) \quad \sum_{j=0}^3 (-1)^j \chi(\mathcal{E}xt^j(Z, E)) + \sum_{j=0}^3 (-1)^j \chi(\mathcal{E}xt^j(F, Z)) = 0.$$

*Proof.* Break a locally free resolution of  $E$  as in (2) into two short exact sequences

$$0 \rightarrow L_2 \rightarrow L_1 \rightarrow K \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K \rightarrow L_0 \rightarrow E \rightarrow 0.$$

Applying functor  $\mathcal{H}om(Z, -)$  and passing to Euler characteristic on the first sequence, we have:

$$(4) \quad \begin{aligned} \chi(\mathcal{E}xt^2(Z, K)) - \chi(\mathcal{E}xt^3(Z, K)) &= \chi(\mathcal{E}xt^3(Z, L_2)) - \chi(\mathcal{E}xt^3(Z, L_1)) = \\ &= (\text{rk}(L_2) - \text{rk}(L_1))\chi(Z), \end{aligned}$$

since  $\chi(\mathcal{E}xt^3(Z, L_k)) = \chi(\mathcal{E}xt^3(Z, \mathcal{O}_{\mathbb{P}^3}) \otimes L_k) = \text{rk}(L_k) \cdot \chi(Z)$ . Now, applying the functor  $\mathcal{H}om(Z, -)$  to the second exact sequence we obtain the isomorphism  $\mathcal{E}xt^1(Z, E) \simeq \mathcal{E}xt^2(Z, K)$  and passing to the Euler characteristic we have

$$\chi(\mathcal{E}xt^2(Z, E)) - \chi(\mathcal{E}xt^3(Z, E)) = \chi(\mathcal{E}xt^3(Z, K)) - \chi(\mathcal{E}xt^3(Z, L_0)).$$

Subtracting  $\chi(\mathcal{E}xt^1(Z, E))$  from the left hand side and  $\chi(\mathcal{E}xt^2(Z, K))$  from the right hand side, and then substituting for (4) we have:

$$(5) \quad \sum_{j=0}^3 (-1)^j \chi(\mathcal{E}xt^j(Z, E)) = (\text{rk}(L_1) - \text{rk}(L_2) - \text{rk}(L_0)) \cdot \chi(Z) = -\text{rk}(E)\chi(Z).$$

Since  $\dim \mathcal{E}xt^j(Z, E) = 0$ , we have

$$\begin{aligned} \chi(\mathcal{E}xt^j(Z, E)) &= h^0(\mathcal{E}xt^j(Z, E)) = \\ &= \dim \text{Ext}^j(Z, E) \stackrel{\text{SD}}{=} \dim \text{Ext}^{3-j}(E, Z) = \chi(\mathcal{E}xt^{3-j}(E, Z)), \end{aligned}$$

where the superscript SD indicates the use of Serre duality. The formula (5) applied to the sheaf  $F$  then yields

$$\sum_{j=0}^3 (-1)^j \chi(\mathcal{E}xt^j(F, Z)) = \text{rk}(F)\chi(Z).$$

The fact that  $\text{rk}(F) = \text{rk}(E)$  provides the desired identity.  $\square$

**Lemma 6.** *Let  $E$  be a rank 2 torsion free sheaf with mixed singularities. Then:*

$$\sum_{j=0}^3 (-1)^j \dim \text{Ext}^j(E, E) = -8c_2(E) + 4 + 2c_1(E)^2.$$

*Proof.* Let  $Z_E \hookrightarrow Q_E$  the maximal 0-dimensional subsheaf of  $Q_E$ , and set  $T_E := Q_E/Z_E$  to be the pure 1-dimensional quotient; we assume that both  $Z_E$  and  $T_E$  are nontrivial. Let  $E'$  be the kernel of the composed epimorphism  $E^{\vee\vee} \rightarrow Q_E \rightarrow T_E$ ; note that it also fits into the following short exact sequence

$$(6) \quad 0 \rightarrow E \rightarrow E' \rightarrow Z_E \rightarrow 0.$$

Note that  $c_1(E') = c_1(E)$  and  $c_2(E') = c_2(E)$ . In addition,  $(E')^{\vee\vee} \simeq E^{\vee\vee}$ , and  $Q_{E'} \simeq T_E$ , thus  $E'$  is a torsion free sheaf with 1-dimensional singularities. It follows that  $E'$  has homological

dimension 1 (that is  $\mathcal{E}xt^p(E', G) = 0$  for  $p \geq 2$  and every coherent sheaf  $G$ ), so the proof of [13, Proposition 3.4] also applies for  $E'$ , and we conclude that

$$\sum_{j=0}^3 (-1)^j \dim \operatorname{Ext}^j(E', E') = -8c_2(E) + 4 + 2c_1(E)^2.$$

Therefore, it is enough to prove that

$$\sum_{j=0}^3 (-1)^j \dim \operatorname{Ext}^j(E, E) = \sum_{j=0}^3 (-1)^j \dim \operatorname{Ext}^j(E', E'),$$

which, by Lemma 3 is equivalent to show that

$$\chi(\mathcal{H}om(E, E)) - \chi(\mathcal{E}xt^1(E, E)) + h^0(\mathcal{E}xt^2(E, E)) = \chi(\mathcal{H}om(E', E')) - \chi(\mathcal{E}xt^1(E', E')).$$

To see this, note that applying the functor  $\mathcal{H}om(E', -)$  to the sequence (6) we obtain:

$$\begin{aligned} & \chi(\mathcal{H}om(E', E)) - \chi(\mathcal{H}om(E', E')) + \chi(\mathcal{H}om(E', Z_E)) - \\ & \chi(\mathcal{E}xt^1(E', E)) + \chi(\mathcal{E}xt^1(E', E')) - \chi(\mathcal{E}xt^1(E', Z_E)) = 0. \end{aligned}$$

Next, applying the functor  $\mathcal{H}om(-, E)$  to the sequence (6) we have

$$\begin{aligned} & \chi(\mathcal{H}om(E', E)) - \chi(\mathcal{H}om(E, E)) + \chi(\mathcal{E}xt^1(Z_E, E)) - \\ & \chi(\mathcal{E}xt^1(E', E)) + \chi(\mathcal{E}xt^1(E, E)) - \chi(\mathcal{E}xt^2(Z_E, E)) = 0. \end{aligned}$$

Taking the difference between these last two equations we obtain

$$\begin{aligned} & \chi(\mathcal{H}om(E', E')) - \chi(\mathcal{E}xt^1(E', E')) = \chi(\mathcal{H}om(E, E)) - \chi(\mathcal{E}xt^1(E, E)) + \\ & -\chi(\mathcal{E}xt^1(Z_E, E)) + \chi(\mathcal{E}xt^2(Z_E, E)) + \chi(\mathcal{H}om(E', Z_E)) - \chi(\mathcal{E}xt^1(E', Z_E)) = \\ & \chi(\mathcal{H}om(E, E)) - \chi(\mathcal{E}xt^1(E, E)) + \chi(\mathcal{E}xt^3(Z_E, E)), \end{aligned}$$

with the second equality following from applying the formula established in Lemma 5 to the sheaves  $E$  and  $E'$ . Applying the functor  $\mathcal{H}om(-, E)$  to the sequences

$$0 \rightarrow E' \rightarrow E^{\vee\vee} \rightarrow T_E \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Z_E \rightarrow Q_E \rightarrow T_E \rightarrow 0$$

we conclude that  $\mathcal{E}xt^3(T_E, E) = 0$  and  $\mathcal{E}xt^3(Q_E, E) \simeq \mathcal{E}xt^3(Z_E, E)$ . We already noticed in the proof of Lemma 3 that  $\mathcal{E}xt^3(Q_E, E) \simeq \mathcal{E}xt^2(E, E)$ , thus

$$\chi(\mathcal{E}xt^3(Z_E, E)) = \chi(\mathcal{E}xt^2(E, E)) = h^0(\mathcal{E}xt^2(E, E)),$$

as desired.  $\square$

Gathering the above results we are in position to prove the Theorem 1.

*Proof of Theorem 1.* By Lemma 6, it is enough to show that  $\dim \operatorname{Hom}(E, E) = 1$  and  $\operatorname{Ext}^3(E, E) = 0$ , but these follow easily from the stability of  $E$ .  $\square$

The following proposition will be a technical tool that will help us to compute explicitly the dimension of  $\operatorname{Ext}^1(E, E)$  for certain torsion free sheaves.

**Proposition 7.** *Let  $F$  be a stable rank 2 reflexive sheaf on  $\mathbb{P}^3$ , with  $\dim \operatorname{Ext}^2(F, F) = 0$ . Let  $Z$  be an artinian sheaf, and  $T$  be a sheaf of pure dimension 1 such that  $H^1(\mathcal{H}om(F, T)) = 0$ ; set  $Q := Z \oplus T$  and assume also that  $\operatorname{Sing}(F) \cap \operatorname{Supp}(Q) = \emptyset$ . If  $\varphi : F \rightarrow Q$  is an epimorphism, then, for  $E := \ker \varphi$ ,*

- (i)  $E$  is a stable rank 2 torsion free sheaf;
- (ii)  $c_1(E) = c_1(F)$  and  $c_2(E) = c_2(F) + \operatorname{mult}(T)$ , where  $\operatorname{mult}(T)$  denotes the multiplicity of the sheaf  $T$ ;



$$(iii) \text{Ext}^2(E, E) = \text{H}^0(\mathcal{E}xt^3(Z, E)) \oplus \text{Ext}^3(T, E).$$

*Proof.* The items (i) and (ii) are straightforward calculations; we will prove (iii). First we will show that the spectral sequence map

$$d_2^{01} : \text{H}^0(\mathcal{E}xt^1(E, E)) \rightarrow \text{H}^2(\mathcal{H}om(E, E))$$

is an epimorphism. Consider the exact sequence:

$$(7) \quad 0 \rightarrow E \rightarrow F \rightarrow Q \rightarrow 0.$$

Applying the functor  $\mathcal{H}om(F, -)$  to (7), once  $\text{coker}\{\mathcal{H}om(F, E) \rightarrow \mathcal{H}om(E, E)\}$  is supported in dimension 1, we have

$$(8) \quad \text{H}^2(\mathcal{H}om(F, E)) \rightarrow \text{H}^2(\mathcal{H}om(F, F)) \rightarrow 0.$$

Next apply  $\text{Hom}(F, -)$  in the sequence (7), by hypothesis,  $\text{Ext}^2(E, E) = 0$ , then we have

$$\text{Ext}^1(F, Q) \rightarrow \text{Ext}^2(F, E) \rightarrow 0.$$

To see that  $\text{Ext}^1(F, Q)$  vanishes, note that  $\text{Ext}^p(F, Q) = 0$  for  $p = 2, 3$  because  $F$  is reflexive.  $\mathcal{E}xt^1(F, Q) = 0$  because  $\text{Sing}(F) \cap \text{Supp}(Q) = \emptyset$ . In addition,  $\text{H}^p(\mathcal{H}om(F, Q)) = 0$  for  $p = 2, 3$  because  $\dim Q = 1$ . From the spectral sequence,  $\text{Ext}^1(F, Q) = \text{H}^1(\mathcal{H}om(F, Q))$  which vanishes by hypothesis. Therefore  $d_2^{01} : \text{H}^0(\mathcal{E}xt^1(F, E)) \rightarrow \text{H}^2(\mathcal{H}om(F, E))$  is surjective. Then we have

$$(9) \quad \begin{array}{ccc} \text{H}^0(\mathcal{E}xt^1(F, E)) & \xrightarrow{d_2^{01}} & \text{H}^2(\mathcal{H}om(F, E)) \\ \downarrow & & \downarrow \\ \text{H}^0(\mathcal{E}xt^1(E, E)) & \xrightarrow{d_2^{01}} & \text{H}^2(\mathcal{H}om(E, E)), \end{array}$$

where the vertical arrow in the left is the natural map coming from the exact sequence (7), and horizontal maps came from the spectral sequence. Since the top row map, and the right vertical map are surjective, we have that the bottom map is surjective as we wanted. Now, applying  $\mathcal{H}om(-, E)$  to the sequence (7) we have

$$\mathcal{E}xt^2(E, E) \simeq \mathcal{E}xt^3(Q, E) \simeq \mathcal{E}xt^3(Z, E) \oplus \mathcal{E}xt^3(T, E).$$

Furthermore, there is an exact sequence

$$\mathcal{E}xt^1(F, E) \longrightarrow \mathcal{E}xt^1(E, E) \xrightarrow{f} \mathcal{E}xt^2(Q, E) \longrightarrow 0,$$

where  $\dim \ker f = 0$ , since  $\dim \mathcal{E}xt^1(F, E) = 0$ . Thus

$$(10) \quad \text{Ext}^2(E, E) = \text{H}^0(\mathcal{E}xt^3(Z, E)) \oplus \text{H}^0(\mathcal{E}xt^3(T, E)) \oplus \text{H}^1(\mathcal{E}xt^2(T, E)).$$

Since  $F$  is reflexive, from [16, Proposition 1.1.6], we have  $\mathcal{E}xt^p(T, F) = 0$  for  $p = 0, 1$  and  $\text{codim } \mathcal{E}xt^p(T, F) \geq p$  for  $p = 2, 3$ . Clearly,  $\dim \mathcal{E}xt^p(T, E) \leq 1$  for  $p > 0$ , while  $\mathcal{H}om(T, E) = 0$ ; using these facts, we obtain from the spectral sequence for  $\text{Ext}(T, E)$  that

$$(11) \quad \text{Ext}^3(T, E) = \text{H}^0(\mathcal{E}xt^3(T, E)) \oplus \text{H}^1(\mathcal{E}xt^2(T, E)).$$

Putting together the equations (10) and (11) we obtain item (iii).  $\square$

**Remark 1.** *Item (iii) of Proposition 7 also holds when  $T = 0$  without assuming that  $\text{Ext}^2(F, F) = 0$ , see the proof of Main Theorem 2, starting in page 23 below.*

An important ingredient of the Proposition 7 is a family of stable reflexive sheaves, that fills out an irreducible component of the moduli space, with the expected dimension. A priori, it is not clear why such family should exist. In [19] the authors proved that, indeed, such families exists for infinitely many values of the second Chern class, provided that the first Chern class is even. Below we state a theorem that shows that this happens also for sheaves with odd first Chern class. For simplicity of notation, we define

$$G_{(a,b,c)} := a \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c \cdot \mathcal{O}_{\mathbb{P}^3}(-1).$$

**Theorem 8.** *For each triple  $(a, b, c)$  of positive integers such that  $3a + 2b + c$  is odd, the family of rank 2 reflexive sheaves  $F$  obtained as the cokernel of the maps  $\alpha$  below*

$$(12) \quad 0 \rightarrow a \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (a + b + c + 2) \cdot \mathcal{O}_{\mathbb{P}^3} \rightarrow F(k) \rightarrow 0,$$

where  $k := (3a + 2b + c + 1)/2$ , fills out a nonsingular irreducible component  $\mathcal{S}(a, b, c)$  of  $\mathcal{R}(-1; n; m)$  of expected dimension  $8n - 5$ , with  $n$  and  $m$  are given by the expressions:

$$(13) \quad \begin{aligned} n &= \frac{1}{4}(3a + 2b + c + 1)^2 + 3a + b, \\ m(a, b, c) &= 27 \binom{a+2}{3} + 8 \binom{b+2}{3} + \binom{c+2}{3} + 3(3a + 2b + 5)ab + \\ &+ \frac{3}{2}(3a + c + 4)ac + (2b + 3c + 3)bc + 6abc. \end{aligned}$$

More precisely, let  $\tilde{\mathcal{S}}(a, b, c) \subset \text{Hom}(G_{(a,b,c)}, (a + b + c + 2) \cdot \mathcal{O}_{\mathbb{P}^3})$  be the open subset consisting of monomorphisms with 0-dimensional degeneracy loci; then

$$\mathcal{S}(a, b, c) = \tilde{\mathcal{S}}(a, b, c) / ((\text{Aut}(G_{(a,b,c)}) \times GL(a + b + c + 2)) / \mathbb{C}^*).$$

*Proof.* Let  $a, b, c \in \mathbb{Z}$ , such that  $3a + 2b + c$  is odd and non zero, and consider morphisms of the form

$$\alpha : G_{(a,b,c)} \rightarrow (a + b + c + 2) \cdot \mathcal{O}_{\mathbb{P}^3}.$$

If the degeneracy locus

$$\Delta(\alpha) := \{x \in \mathbb{P}^3 \mid \alpha(x) \text{ is not injective}\}$$

is 0-dimensional, then the cokernel of  $\alpha$  is a rank 2 reflexive sheaf  $F$  on  $\mathbb{P}^3$ , which we twist by  $k := (3a + 2b + c + 1)/2$ , so that  $c_1(F) = -1$ , and the exact sequence in display (12) is satisfied. The dimension of this family of rank 2 reflexive sheaves is given by

$$\begin{aligned} \dim \text{Hom}(G_{(a,b,c)}, (a + b + c + 2) \cdot \mathcal{O}_{\mathbb{P}^3}) - \dim \text{Aut}(G_{(a,b,c)}) - (a + b + c + 2)^2 + 1 = \\ 8k^2 + 24a + 8b - 5 = 8c_2(F) - 5. \end{aligned}$$

Note that for such sheaf  $F$  satisfies  $h^0(F(-1)) = 0$ , thus  $F$  is always stable. It only remains for us to check that  $\dim \text{Ext}^2(F, F) = 0$ . This follows from applying the functor  $\text{Hom}(\cdot, F(k))$  to the exact sequence in display (12), and observing that  $H^1(F(t)) = 0$  for every  $t \in \mathbb{Z}$  and that  $H^2(F(k)) = 0$ . Therefore the family of sheaves given by (12) provides a component of the moduli space of stable rank 2 reflexive sheaves on  $\mathbb{P}^3$ .  $\square$

The case that deserves special attention is the case  $a = b = 0$  and  $c = 1$ , that give us  $c_1(F) = -1$ ,  $c_2(F) = c_3(F) = 1$ . In [13, Lemma 9.4] is shown that every reflexive sheaf in  $\mathcal{R}(-1, 1, 1)$  admits a resolution of the form:

$$(14) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\alpha} 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow F \rightarrow 0.$$

From this sequence we can easily deduce the splitting behaviour of a sheaf  $F$  in  $\mathcal{R}(-1, 1, 1)$ . Indeed, each one of the 3 rows of the map  $\alpha$  can be viewed as the equation of a hyperplane

in  $\mathbb{P}^3$ , since  $\alpha$  is injective, the hyperplane must intersect in exactly one point  $p$ , that coincides with the singularity of the sheaf  $F$ . Thus, if  $l \subset \mathbb{P}^3$  is a line, if  $p \notin l$ , then the restriction of  $F$  on  $l$ ,  $F|_l$ , is isomorphic to  $\mathcal{O}_l(-1) \oplus \mathcal{O}_l$ . On the other hand, if  $p \in l$ , from sequence (14), we have that  $F|_l \simeq \mathcal{O}_p \oplus 2\mathcal{O}_l(-1)$ . Summarizing, we have:

$$(15) \quad F|_l = \begin{cases} \mathcal{O}_l(-1) \oplus \mathcal{O}_l, & \text{if } p \notin l, \\ \mathcal{O}_p \oplus 2 \cdot \mathcal{O}_l(-1), & \text{if } p \in l. \end{cases}$$

**Remark 2.** From [13, Example 4.2.1] it follows that  $\mathcal{R}(-1, 1, 1)$  is irreducible non-singular and rational of dimension 3. Moreover, there is an isomorphism  $\mathcal{R}(-1, 1, 1) \xrightarrow{\sim} \mathbb{P}^3$ ,  $[F] \mapsto \text{Sing}(F)$ , and every sheaf  $[F] \in \mathcal{R}(-1, 1, 1)$  fits in an exact triple  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow F \rightarrow 0$ . This yields that  $\text{Ext}^2(F, F) = 0$ . Also Theorem 8 implies that  $\mathcal{S}(0, 0, 1) = \mathcal{R}(-1, 1, 1)$ . Besides, under the isomorphism  $\mathcal{R}(-1, 1, 1) \simeq \mathbb{P}^3$ , the above exact triple globalizes to the exact triple over  $\mathbb{P}^3 \times \mathbb{P}^3$ :

$$(16) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes T_{\mathbb{P}^3}(-1) \rightarrow \mathbf{F} \rightarrow 0,$$

where  $\mathbf{F}$  is the universal family of reflexive sheaves over  $\mathcal{R}(-1, 1, 1)$ , the morphism  $\alpha$  is the composition  $\mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{i(-1) \boxtimes \text{id}} 4 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\text{id} \boxtimes \epsilon} \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes T_{\mathbb{P}^3}(-1)$ , and  $i, \epsilon$  are the morphisms in the Euler exact triple  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{i} 4 \cdot \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} T_{\mathbb{P}^3}(-1) \rightarrow 0$ .

### 3. SHEAVES WITH 0-DIMENSIONAL AND MIXED SINGULARITIES

In [19] the authors produced examples of irreducible components with 0-dimensional singularities and pure 1-dimensional singularities, in the moduli space of rank 2 stable torsion free sheaves with first Chern class equals to 0, and in [18] the authors proved the existence of irreducible components in  $\mathcal{M}(0, 3, 0)$  whose generic point is a sheaf with mixed singularities, with first Chern class equals to 0. The first natural question that arises is that if similar constructions can be made for sheaves with odd first Chern class, and if it is possible similar irreducible components for non zero third Chern class.

We will explicit construct examples of irreducible components of the moduli space of torsion free sheaves with mixed singularities. We refer the reader to [18] for some examples in  $\mathcal{M}(0, 3, 0)$ .

For the rest of this work, let  $e \in \{-1, 0\}$ , and  $n, m$  be two integers such that  $en \equiv m \pmod{2}$ . Let

$$(17) \quad \mathcal{R}^*(e, n, m) := \{[F] \in \mathcal{R}(e, n, m) \mid \text{Ext}^2(F, F) = 0\}.$$

By semicontinuity,  $\mathcal{R}^*(e, n, m)$  is an open smooth subset of  $\mathcal{R}(e, n, m)$  such that, in view of Theorem 1 and Corollary 2,

$$(18) \quad \dim_{[F]} \mathcal{R}^*(e, n, m) = \dim \text{Ext}^1(F, F) = 8n - 3 + 2e, \quad [F] \in \mathcal{R}^*(e, n, m).$$

(Here and below by the dimension  $\dim_x X$  of a given scheme  $X$  locally of finite type at a point  $x \in X$  we mean the maximum of dimensions of irreducible components of  $X$  passing through the point  $x$ .)

Let  $(\mathbb{P}^3)_0^s$  be the open dense subset of  $(\mathbb{P}^3)^s$  consisting of disjoint unions of  $s$  distinct points in  $\mathbb{P}^3$ . For any closed point  $[F] \in \mathcal{R}^*(e, n, m)$ , define the sets

$$(19) \quad \Pi_{[F]} := \{S \in (\mathbb{P}^3)_0^s \mid S \cap \text{Sing}(F) = \emptyset\},$$

$$(20) \quad \mathcal{X}_{[F]} := \{(l, S) \in \text{G}(2, 4) \times (\mathbb{P}^3)_0^s \mid l \cap S = \emptyset, (l \cup S) \cap \text{Sing}(F) = \emptyset, \text{ and } F|_l = \mathcal{O}_l(e) \oplus \mathcal{O}_l\}.$$

Note that, since any reflexive sheaf  $F$  from  $\mathcal{R}^*(e, n, m)$  has 0-dimensional singularities, the set

$$(21) \quad (\mathcal{R}^*(e, n, m) \times (\mathbb{P}^3)_0^s)_0 := \{([F], S) \in \mathcal{R}^*(e, n, m) \times (\mathbb{P}^3)_0^s \mid S \in \Pi_{[F]}\}$$

is a dense open subset in  $\mathcal{R}^*(e, n, m) \times (\mathbb{P}^3)^s$ , hence by (18) it is smooth equidimensional of dimension

$$(22) \quad \dim (\mathcal{R}^*(e, n, m) \times (\mathbb{P}^3)_0^s)_0 = 8n + 3s + 2e - 3.$$

Respectively, by the Grauert–Müllich Theorem, the set

$$(23) \quad (\mathcal{R}^*(e, n, m) \times G(2, 4) \times (\mathbb{P}^3)_0^s)_0 := \{([F], (l, S)) \in \mathcal{R}^*(e, n, m) \times G(2, 4) \times (\mathbb{P}^3)_0^s \mid (l, S) \in \mathcal{X}_{[F]}\}$$

is a dense open subset in  $\mathcal{R}^*(e, n, m) \times G(2, 4) \times (\mathbb{P}^3)_0^s$ , hence by (18) it is smooth equidimensional of dimension

$$(24) \quad \dim (\mathcal{R}^*(e, n, m) \times G(2, 4) \times (\mathbb{P}^3)_0^s)_0 = 8n + 3s + 2e + 1.$$

For a pair  $([F], S) \in (\mathcal{R}^*(e, n, m) \times (\mathbb{P}^3)_0^s)_0$ , consider the  $2s$ -dimensional vector space  $\text{Hom}(F, \mathcal{O}_S)$  and its open dense subset  $\text{Hom}(F, \mathcal{O}_S)_e$  of epimorphisms  $F \rightarrow \mathcal{O}_S$ . By construction, the group  $\text{Aut}(\mathcal{O}_S)$  acts on  $\text{Hom}(F, \mathcal{O}_S)_e$ , and it follows that the quotient space  $\text{Hom}(F, \mathcal{O}_S)_e / \text{Aut}(\mathcal{O}_S)$  is a smooth irreducible scheme isomorphic to a product of projective spaces, where  $S = (q_1, \dots, q_s)$ :

$$(25) \quad \begin{aligned} \text{Hom}(F, \mathcal{O}_S)_e / \text{Aut}(\mathcal{O}_S) &\simeq \prod_{i=1}^s \mathbb{P}_{q_i}^1, \\ \dim \text{Hom}(F, \mathcal{O}_S)_e / \text{Aut}(\mathcal{O}_S) &= s, \end{aligned}$$

and where  $\mathbb{P}_{q_i}^1 = \text{Hom}(F, \mathcal{O}_{q_i})_e / \text{Aut}(\mathcal{O}_{q_i})$ ,  $i = 1, \dots, s$ .

Now, for any element  $\phi \in \text{Hom}(F, \mathcal{O}_S)_e$  the torsion free sheaf  $E_\phi := \ker(\phi : F \rightarrow \mathcal{O}_S)$  is stable, and defines a closed point in  $\mathcal{M}(e, n, m - 2s)$ . Furthermore,  $E_\phi \simeq E_{\phi'}$  if, and only if, there is a  $g \in \text{Aut}(\mathcal{O}_S)$  such that  $\phi = g \circ \phi'$ . Denote by  $[\phi]$  the equivalence class of  $\phi$  modulo  $\text{Aut}(\mathcal{O}_S)$  and consider the set

$$(26) \quad \tilde{\mathcal{T}}(e, n, m, s) := \{x = ([F], S, [\phi_x]) \mid ([F], S) \in (\mathcal{R}^*(e, n, m) \times (\mathbb{P}^3)_0^s)_0, \phi_x \in \text{Hom}(F, \mathcal{O}_S)_e\}.$$

By definition,  $\tilde{\mathcal{T}}(e, n, m, s)$  is fibered over  $(\mathcal{R}^*(e, n, m) \times (\mathbb{P}^3)_0^s)_0$  with fiber  $\text{Hom}(F, \mathcal{O}_S)_e / \text{Aut}(\mathcal{O}_S)$  over a given point  $([F], S)$ . Thus by (18) and (30) we conclude that  $\tilde{\mathcal{T}}(e, n, m, s)$  is naturally endowed with a structure of smooth equidimensional scheme of dimension

$$\dim \tilde{\mathcal{T}}(e, n, m, s) = 8n + 4s + 2e - 3,$$

and the number of irreducible components of  $\tilde{\mathcal{T}}(e, n, m, s)$  is equals to the number of those of  $\mathcal{R}^*(e, n, m)$ . Furthermore, for any point  $t = ([F], S, [\phi_x]) \in \tilde{\mathcal{T}}(e, n, m, s)$ , the sheaf  $E(t) := \ker\{\phi : F \rightarrow \mathcal{O}_S\}$  is a stable sheaf from  $\mathcal{M}(e, n, m - 2s)$ . Hence there is a well-defined modular morphism

$$\Phi : \tilde{\mathcal{T}}(e, n, m, s) \hookrightarrow \mathcal{M}(e, n, m - 2s), \quad t \mapsto [E(t)].$$

$\Phi$  is clearly an embedding, since the data  $x = ([F], (l, S), [\phi_x])$  are recovered uniquely from  $[E(t)] : F \simeq E(t)^{\vee\vee}$ ,  $S = \text{Supp}(E(t)^{\vee\vee}/E(t))$  and  $\phi$  is the canonical quotient morphism  $E(t)^{\vee\vee} \rightarrow \mathcal{O}_S \simeq E(t)^{\vee\vee}/E(t)$ . We thus set

$$(27) \quad \mathcal{T}(e, n, m, s) := \Phi(\tilde{\mathcal{T}}(e, n, m, s)) \simeq \tilde{\mathcal{T}}(e, n, m, s).$$

Let  $\mathbb{T}(e, n, m, s)$  be the closure  $\overline{\mathcal{T}(e, n, m, s)}$  of  $\mathcal{T}(e, n, m, s)$  in  $\mathcal{M}(e, n, m - 2s)$ . Formula (18) yields:

$$(28) \quad \dim \mathbb{T}(e, n, m, s) = \dim \mathcal{T}(e, n, m, s) = 8n + 4s + 2e - 3.$$

Respectively, let  $r \geq e$ . For a triple  $([F], (l, S)) \in (\mathcal{R}^*(e, n, m) \times G(2, 4) \times (\mathbb{P}^3)_0^s)_0$ , set

$$(29) \quad Q_{(l,S),r} := \mathcal{O}_S \oplus i_* \mathcal{O}_l(r),$$

where  $i : l \hookrightarrow \mathbb{P}^3$  is a closed immersion. Consider the  $(2r + 2s + 2 - e)$ -dimensional vector space  $\text{Hom}(F, Q_{(l,S),r})$  and its open dense subset  $\text{Hom}(F, Q_{(l,S),r})_e$  of epimorphisms  $F \rightarrow Q_{(l,S),r}$ . By construction, the group  $\text{Aut}(Q_{(l,S),r})$  acts on  $\text{Hom}(F, Q_{(l,S),r})_e$ , and it follows that the quotient space

$$\text{Hom}(F, Q_{(l,S),r})_e / \text{Aut}(Q_{(l,S),r})$$

is a smooth irreducible scheme isomorphic to a product of projective spaces:

$$(30) \quad \begin{aligned} \text{Hom}(F, Q_{(l,S),r})_e / \text{Aut}(Q_{(l,S),r}) &\simeq \mathbf{P}_l^{2r+1-e} \times \prod_{i=1}^s \mathbb{P}_{q_i}^1, \\ \dim \text{Hom}(F, Q_{(l,S),r})_e / \text{Aut}(Q_{(l,S),r}) &= 2r + s + 1 - e, \end{aligned}$$

and where

$$(31) \quad \mathbf{P}_l^{2r+1-e} = \text{Hom}(F, i_* \mathcal{O}_l(r))_e / \text{Aut}(i_* \mathcal{O}_l(r))$$

and  $\mathbb{P}_{q_i}^1$  are the same as in (25).

For any element  $\phi \in \text{Hom}(F, Q_{(l,S),r})_e$  the torsion free sheaf  $E_\phi := \ker \phi$  is stable, and defines a closed point in  $\mathcal{M}(e, n + 1, m - 2r - 2s - 2 - e)$ . Furthermore,  $E_\phi \simeq E_{\phi'}$  if, and only if, there is a  $g \in \text{Aut}(Q_{(l,S),r})$  such that  $\phi = g \circ \phi'$ . Denote by  $[\phi]$  the equivalence class of  $\phi$  modulo  $\text{Aut}(Q_{(l,S),r})$  and consider the set

$$(32) \quad \begin{aligned} \tilde{\mathcal{X}}(e, n, m, r, s) &:= \{x = ([F], (l, S), [\phi_x]) \mid ([F], (l, S)) \in (\mathcal{R}^*(e, n, m) \times G(2, 4) \times (\mathbb{P}^3)_0^s)_0, \\ &[\phi_x] \in \text{Hom}(F, Q_{(l,S),r})_e / \text{Aut}(Q_{(l,S),r})\}. \end{aligned}$$

By definition,  $\tilde{\mathcal{X}}(e, n, m, r, s)$  is fibered over  $(\mathcal{R}^*(e, n, m) \times G(2, 4) \times (\mathbb{P}^3)_0^s)_0$  with fiber  $\text{Hom}(F, Q_{(l,S),r})_e / \text{Aut}(Q_{(l,S),r})$  over a given point  $([F], (l, S))$ . Thus by (18) and (30)  $\tilde{\mathcal{X}}(e, n, m, r, s)$  is naturally endowed with a structure of smooth equidimensional scheme of dimension

$$(33) \quad \dim \tilde{\mathcal{X}}(e, n, m, r, s) = 8n + 4s + 2r + e + 2,$$

and the number of irreducible components of  $\tilde{\mathcal{X}}(e, n, m, r, s)$  equals the number of those of  $\mathcal{R}^*(e, n, m)$ . Furthermore, for any point

$$x = ([F], (l, S), [\phi_x]) \in \tilde{\mathcal{X}}(e, n, m, r, s),$$

the sheaf  $E(x) := \ker\{\phi : F \rightarrow Q_{(l,S),r}\}$  is a stable sheaf from  $\mathcal{M}(e, n + 1, m + 2 + e - 2r - 2s)$ . Hence there is a well-defined modular morphism

$$\Psi : \tilde{\mathcal{X}}(e, n, m, r, s) \hookrightarrow \mathcal{M}(e, n + 1, m + 2 + e - 2r - 2s), \quad x \mapsto [E(x)].$$

$\Psi$  is clearly an embedding, since the data  $x = ([F], (l, S), [\phi_x])$  are recovered uniquely from  $[E(x)] : F \simeq E(x)^{\vee\vee}$ ,  $l \sqcup S = \text{Supp}(E(x)^{\vee\vee}/E(x))$  and  $\phi$  is the canonical quotient morphism  $E(x)^{\vee\vee} \rightarrow Q_{(l,S),r} \simeq E(x)^{\vee\vee}/E(x)$ . We thus set

$$(34) \quad \mathcal{X}(e, n, m, r, s) := \overline{\Psi(\tilde{\mathcal{X}}(e, n, m, r, s))} \simeq \tilde{\mathcal{X}}(e, n, m, r, s).$$

Let  $X(e, n, m, r, s)$  be the closure  $\overline{\mathcal{X}(e, n, m, r, s)}$  of  $\mathcal{X}(e, n, m, r, s)$  in  $\mathcal{M}(e, n + 1, m + 2 + e - 2r - 2s)$ . Formula (33) yields:

$$(35) \quad \dim X(e, n, m, r, s) = \dim \mathcal{X}(e, n, m, r, s) = 8n + 4s + 2r + 2 + e.$$

**Remark 3.** By Remark 2,  $\mathcal{R}^*(-1, 1, 1) = \mathcal{R}(-1, 1, 1)$  is smooth irreducible of the expected dimension 3. Thus (35) yields that  $X(-1, 1, 1, -1, 0)$  is an irreducible scheme of dimension 7.

We now prove the following general result about the schemes  $T(e, n, m, s)$ .

**Theorem 9.** *Given  $s > 0$ , we have:*

- (i) *For any nonsingular irreducible component  $\mathcal{R}^*$  of  $\mathcal{R}(e, n, m)$  there corresponds an irreducible component of  $T(e, n, m, s)$  of dimension  $8n - 3 + 2e + 4s$  which is also an irreducible component of  $\mathcal{M}(e, n, m - 2s)$ . In particular, if  $\mathcal{R}(e, n, m)$  is irreducible, then  $T(e, n, m, s)$  is also irreducible.*
- (ii) *The generic sheaf  $[E]$  of any irreducible component of  $T(e, n, m, s)$  satisfies the conditions that  $[E^{\vee\vee}] \in \mathcal{R}^*(e, n, m)$  and  $Q_E = E^{\vee\vee}/E$  is an artinian sheaf of length  $s$ .*

*Proof.* For the statement (i) of Theorem, it is enough to prove that, for each  $[E(t)] \in \mathcal{T}(e, n, m, s)$ ,  $\text{Ext}^2(E(t), E(t)) = 4s$ . Indeed, in this case Theorem 1 yields that  $\dim \text{Ext}^1(E(t), E(t)) = 8n - 3 + 2e + 4s =$ , and this dimension coincides with the dimension of  $T(e, n, m, s)$  by (28), and therefore by the deformation theory any irreducible component of  $T(e, n, m, s)$  is an irreducible component of  $\mathcal{M}(e, n, m - 2s)$ .

From Proposition 7 we have

$$\dim \text{Ext}^2(E(t), E(t)) = h^0(\mathcal{E}xt^3(Q, E(t))),$$

where  $Q = E(t)^{\vee\vee}/E(t)$ . To compute this group, note that, since, by the definition of  $\mathcal{T}(e, n, m, s)$ ,  $Q = \mathcal{O}_S$ , where  $S = \{q_1, \dots, q_s\} \in (\mathbb{P}^3)_0^s$  and  $S \cap \text{Sing}(E(t)^{\vee\vee}) = \emptyset$ , we have

$$(36) \quad \text{Ext}^2(E(t), E(t)) \simeq H^0(\mathcal{E}xt^3(Q, E(t))) \simeq \bigoplus_{q_i \in S} \text{Ext}_{\mathbb{P}^3, q_i}^3(\mathcal{O}_{q_i}, E(t)_{q_i}).$$

Take a point  $q = q_j$  for some  $1 \leq j \leq s$ , and an open subset  $U$  in  $\mathbb{P}^3$  not containing other points of  $\text{Sing}(E(t))$ . Consider the exact sequence  $0 \rightarrow E(t) \rightarrow E(t)^{\vee\vee} \rightarrow Q \rightarrow 0$  and restrict it onto  $U$ . We then obtain the following exact sequence of sheaves on  $U$ :

$$0 \rightarrow \mathcal{O}_U \oplus \mathcal{I}_{q,U} \rightarrow 2 \cdot \mathcal{O}_U \rightarrow \mathcal{O}_q \rightarrow 0,$$

where  $\mathcal{I}_{q,U}$  denotes the ideal sheaf of the point  $p \in U$  and  $\mathcal{O}_q$  denotes the structure sheaf of the point  $q$  as a subscheme of  $U$ . In particular,  $E(t)|_U \simeq \mathcal{O}_U \oplus \mathcal{I}_{q,U}$ , so that

$$(37) \quad \text{Ext}_{\mathbb{P}^3, q}^3(\mathcal{O}_q, E(t)_q) \simeq H^0(\mathcal{E}xt_{\mathcal{O}_U}^3(\mathcal{O}_q, \mathcal{I}_{q,U})) \oplus H^0(\mathcal{E}xt_{\mathcal{O}_U}^3(\mathcal{O}_q, \mathcal{O}_U)).$$

Applying the functor  $\mathcal{H}om(-, \mathcal{I}_{q,U})$  to the sequence  $0 \rightarrow \mathcal{I}_q \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_q \rightarrow 0$ , we obtain:  $\mathcal{E}xt_{\mathcal{O}_U}^3(\mathcal{O}_q, \mathcal{I}_{q,U}) \simeq \mathcal{E}xt_{\mathcal{O}_U}^2(\mathcal{I}_{q,U}, \mathcal{I}_{q,U})$ . The last sheaf is an artinian sheaf of length 3 by the proof of [19, Proposition 6]. Thus, since  $\mathcal{E}xt_{\mathcal{O}_U}^3(\mathcal{O}_q, \mathcal{O}_U) \simeq \mathcal{O}_q$ , it follows from (37) that each point in  $S$  contributes with 4 to the dimension of  $\text{Ext}^2(E(t), E(t))$ , hence, by (36),  $\dim \text{Ext}^2(E(t), E(t)) = 4s$ . The other claims in the statement of Theorem are clear from the definition of  $T(e, n, m, s)$ .  $\square$

We next proceed to a general result about the schemes  $X(e, n, m, r, s)$ .

**Theorem 10.** *Let  $e, n, m, r, s$  be integers such that  $e \in \{-1, 0\}$ ,  $n, m > 0$ ,  $r \geq e$  and  $s \geq 0$ . Then the scheme  $X(e, n, m, r, s)$  is equidimensional of dimension  $8n + 4s + 2r + 2 + e$ , and the number of irreducible components of  $X(e, n, m, r, s)$  is the same as that of  $\mathcal{R}^*(e, n, m)$ . Furthermore,  $X(e, n, m, r, s)$  contains a dense open subset  $\mathcal{X}(e, n, m, r, s)$  such that, for  $[E] \in \mathcal{X}(e, n, m, r, s)$ , the following statements hold.*

- (i) *If  $r \geq 1$ , then  $\dim \text{Ext}^1(E, E) = 8n + 4s + 2r + 2 + e = \dim X(e, n, m, r, s)$ . Hence, if  $\mathcal{R}^*(e, n, m)$  is irreducible, then  $X(e, n, m, r, s)$  is an irreducible  $(8n + 4s + 2r + 2 + e)$ -dimensional component of  $\mathcal{M}(e, n + 1, m + 2 + e - 2r - 2s)$ .*

(ii) If  $0 \geq r \geq e$ , then  $\dim \text{Ext}^1(E, E) = 8n + 4s + 5 + 2e > \dim X(e, n, m, r, s)$ .

*Proof.* The first claim follows from (35) and the above considerations. For the statements (i) and (ii), consider any sheaf  $[E] \in \mathcal{X}(e, n, m, r, s)$ . By definition  $[E]$  defines a line  $l$  and a set of points  $S$  considered as a reduced scheme as:  $l \sqcup S = \text{Supp}(E^{\vee\vee}/E)$ . Note that, by Proposition 7.(iii) in which we set  $T = i_*\mathcal{O}_l(r)$ ,  $Z = \mathcal{O}_S$ , where  $i : Z \hookrightarrow \mathbb{P}^3$  is the embedding, one has

$$(38) \quad \dim \text{Ext}^2(E, E) = h^0(\mathcal{E}xt^3(\mathcal{O}_S, E)) + \dim \text{Ext}^3(i_*\mathcal{O}_l(r), E).$$

First, one has

$$(39) \quad h^0(\mathcal{E}xt^3(\mathcal{O}_S, E)) = 4s.$$

The proof of this equality is given in [19, Proof of Prop. 6] in the case  $e = 0$ . However, since  $\mathcal{E}xt^2(E, E)$  is 0-dimensional, the computation of  $h^0(\mathcal{E}xt^2(E, E))$  is purely local, and gives the same result for  $e = -1$ . Next,  $\text{Ext}^3(i_*\mathcal{O}_l(r), E) \simeq \text{Hom}(E, i_*\mathcal{O}_l(r-4))^\vee$  by Serre duality, and

$$\text{Hom}(E, i_*\mathcal{O}_l(r-4)) \simeq H^0(\mathcal{H}om(E, i_*\mathcal{O}_l(r-4))).$$

To compute  $h^0(\mathcal{H}om(E, i_*\mathcal{O}_l(r-4)))$ , apply the functor  $i^*(- \otimes i_*\mathcal{O}_l)$  to the triple

$$0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow \mathcal{O}_S \oplus i_*\mathcal{O}_l(r) \rightarrow 0.$$

Using the fact that  $E^{\vee\vee}|_l \simeq \mathcal{O}_l \oplus \mathcal{O}_l(e)$  we obtain the exact sequence

$$0 \rightarrow i^*Tor_1(i_*\mathcal{O}_l(r), i_*\mathcal{O}_l) \rightarrow E|_l \xrightarrow{f} \mathcal{O}_l(e) \oplus \mathcal{O}_l \xrightarrow{g} \mathcal{O}_l(r) \rightarrow 0.$$

Whence  $\ker g \simeq \mathcal{O}_l(-r+e)$ , and since  $i^*Tor_1(i_*\mathcal{O}_l(r), i_*\mathcal{O}_l) \simeq N_{l/\mathbb{P}^3}^\vee \otimes \mathcal{O}_l(r) \simeq 2 \cdot \mathcal{O}_l(r-1)$ , we obtain an exact sequence  $0 \rightarrow 2 \cdot \mathcal{O}_l(r-1) \rightarrow E|_l \rightarrow \mathcal{O}_l(-r+e) \rightarrow 0$ . This triple easily implies that  $h^0(\mathcal{H}om(E, i_*\mathcal{O}_l(r-4)))$  equals  $2r-3-e$  for  $r \geq 1$  and, respectively, equals 0 for  $0 \geq r \geq e$ . This together with (38), (39) and Theorem 1 with  $c_2(E) = n+1$  yields the statements (i) and (ii) of Theorem.  $\square$

We conclude this section with our first application of Theorem 10.

The case of moduli spaces  $\mathcal{M}(-1, n, 0)$  is interesting from the point of view that it contains, among others, all those irreducible components that have locally free sheaves (i. e., vector bundles) as their generic points. Ein had shown in [8] that the number of these components for given number  $n$  is unbounded as  $n$  grows infinitely. Therefore, it is important to understand whether the components of  $\mathcal{M}(-1, n, 0)$  with non-locally free sheaves as their generic points satisfy the similar property. In this section we give an affirmative answer to this question in Theorem 11 below. Namely, the components  $X(-1, n, m, r, s)$  described in Theorem 10 will serve for this purpose, with the numbers  $n, m, r, s$  chosen appropriately.

**Theorem 11.** *Let  $\eta_n$  and  $\xi_n$  denote the number of irreducible components of  $\mathcal{M}(-1, n, 0)$  whose generic point corresponds to a non-locally free sheaf with mixed singularities and with 1-dimensional singularities, respectively. Then*

$$\limsup_{n \rightarrow \infty} \eta_n = \limsup_{n \rightarrow \infty} \xi_n = \infty.$$

*Proof.* For any odd integer  $q \geq 1$  set  $n_q = 9q^2 - 9q + 1$ , and for any integer  $i$  such that  $0 \leq i \leq q-1$ , set  $a_{q,i} = i$ ,  $b_{q,i} = 3q - 3i - 3$ ,  $c_{q,i} = 3i + 1$ . Then, according to Theorem 8 of [19], for an odd integer  $m_{q,i} = m(a_{q,i}, b_{q,i}, c_{q,i})$  given by (13) the scheme  $\mathcal{R}^*(-1; n_q, m_{q,i})$  defined in (17) is nonempty. Thus, by Theorem 10, to any integers  $s$  and  $r$  such that  $0 \leq s \leq n_q - 1$ ,  $r = \frac{1}{2}(m_{q,i} + 1 - 2s)$ , there corresponds an equidimensional union  $X(-1, n_q - 1, m_{q,i}, r, s)$  of irreducible components of  $\mathcal{M}(-1, n_q, 0)$  of dimension  $8n_q + 2s + m_{q,i} + 3$ , where the number of irreducible components of  $X(-1, n_q - 1, m_{q,i}, r, s)$  is the same as that of  $\mathcal{R}^*(-1; n_q, m_{q,i})$ . Therefore, since  $0 \leq i \leq q-1$ , for each odd  $q$  we obtain at least  $q$  different irreducible

components of  $X(-1, n_q - 1, m_{q,i}, r, s)$ ,  $i = 0, \dots, q - 1$ , which are irreducible components of  $\mathcal{M}(-1, n_q, 0)$ , generic points of which are sheaves with mixed singularities. Taking  $s = 0$  we obtain the claim about sheaves with 1-dimensional singularities.  $\square$

A similar result also holds for sheaves with 0-dimensional singularities. The proof is very similar to [19, Theorem 9], using the series of components of  $\mathcal{R}(-1, n, m)$  produced in Theorem 8. More precisely:

**Theorem 12.** *Let  $\zeta_n$  denote the number of irreducible components of  $\mathcal{M}(-1, n, 0)$  whose generic point corresponds to a non-locally free sheaf with 0-dimensional singularities. Then*

$$\limsup_{n \rightarrow \infty} \zeta_n = \infty.$$

#### 4. INFINITE COLLECTIONS OF RATIONAL MODULI COMPONENTS

In this section we will construct an infinite collection of rational moduli components of the spaces  $\mathcal{M}(-1, n, m)$  and  $\mathcal{M}(0, n, m)$  with generic sheaves  $E$  satisfying  $\dim \text{Sing}(E) = 0$ . This collection will include all previously known rational moduli components of  $\mathcal{M}(-1, n, m)$  and  $\mathcal{M}(0, n, m)$  whose generic sheaf has the property above. As a consequence, we can conclude that the moduli schemes  $\mathcal{M}(-1, n, m)$  and  $\mathcal{M}(0, n, m)$  contain at least one rational irreducible components for all  $n \geq 1$  and all admissible  $m$ .

The desired collection will be constructed via elementary transformations at sets of points from certain moduli components of stable reflexive rank 2 sheaves on  $\mathbb{P}^3$ . For this, we invoke results of Chang [6, 7], Miró-Roig [27], Okonek–Spindler [32] and Schmidt [34] on the moduli spaces of reflexive sheaves  $\mathcal{R}(e, n, m)$ ,  $e \in \{0, 1\}$ . These are the results concerning the moduli spaces of reflexive sheaves with Chern classes belonging to the set of triples of integer numbers

$$(40) \quad \Sigma := \Sigma_{-1} \cup \Sigma_0,$$

where  $\Sigma_{-1}$  and  $\Sigma_0$  being respectively given by

$$(41) \quad \{(-1, n, n^2) \mid n \geq 1\} \cup \{(-1, n, n^2 - 2rn + 2r(r+1)) \mid n \geq 5, 1 \leq r \leq (\sqrt{4n-7}-1)/2\}$$

and

$$(42) \quad \{(0, n, n^2 - n + 2) \mid n \geq 3\} \cup \{(0, n, n^2 - n) \mid n \geq 4\} \cup \{(0, n, n^2 - 3n + 8) \mid n \geq 5\}.$$

According to [7, 27, 32, 34], for each triple  $(e, n, m) \in \Sigma$ , the moduli space of stable rank 2 reflexive sheaves  $\mathcal{R}(e, n, m)$  satisfies the following properties:

- (I) Each  $R = \mathcal{R}(e, n, m)$  is an irreducible, nonsingular and rational scheme, and it is a dense open subset of an irreducible component of  $\mathcal{M}(e, n, m)$ ;
- (II)  $R$  is a fine moduli space, i. e., there exists a universal family of reflexive sheaves  $\mathcal{F}$  on  $R \times \mathbb{P}^3$ . (In the case  $e = -1$  this a well-known property the moduli spaces of rank 2 sheaves on  $\mathbb{P}^3$  with odd determinant - see for instance [16, Thm. 4.6.5]. In the case  $e = 0$  this follows from the explicit constructions of reflexive sheaves from  $R$  as extensions of standard sheaves. These constructions are provided in [7, 27, 32, 34].)
- (III) The dimension of each  $R$  is given by:

$$(43) \quad \begin{aligned} \dim \mathcal{R}(-1, n, n^2) &= n^2 + 3n + 1 \quad \text{if } n \geq 2, \quad \text{resp., } 3 \quad \text{if } n = 1, \\ \dim \mathcal{R}(-1, n, n^2 - 2rn + 2r(r+1)) &= n^2 + (3 - 2r)n + 2r^2 + 5, \quad \text{if } r \geq 2, \quad n \geq 5, \\ \dim \mathcal{R}(-1, n, n^2 - 2rn + 2r(r+1)) &= n^2 + n + 6, \quad \text{if } r = 1, \quad n \geq 5, \end{aligned}$$



$$\begin{aligned}
 (44) \quad & \dim \mathcal{R}(0, n, n^2 - n + 2) = n^2 + 2n + 5, \quad n \geq 4, \quad \text{resp.}, \quad 21 \quad \text{if} \quad n = 3, \\
 & \dim \mathcal{R}(0, n, n^2 - n) = n^2 + 2n + 5, \quad n \geq 4, \\
 & \dim \mathcal{R}(0, n, n^2 - 3n + 8) = n^2 + 11 \quad \text{if} \quad n \geq 6, \quad \text{resp.}, \quad 37 \quad \text{if} \quad n = 5.
 \end{aligned}$$

(IV) For  $e = -1$  and arbitrary integer  $n \geq 1$ , the maximal possible  $m$  such that  $\mathcal{M}(-1, n, m) \neq \emptyset$  equals  $n^2$ ; note that  $(-1, n, n^2) \in \Sigma_{-1}$  for  $n \geq 1$ . For  $e = 0$  and arbitrary integer  $n \geq 1$ , the maximal possible  $m$  such that  $\mathcal{M}(-1, n, m) \neq \emptyset$  equals  $n^2 - n + 2$ ; note that  $(-1, n, n^2 - n + 2) \in \Sigma_0$ .

(V) The dimensions of the components  $\mathcal{R}(e, n, m)$  satisfy the relations:

$$(45) \quad \dim \mathcal{R}(e, n, m) = \dim \text{Ext}^1(F, F), \quad [F] \in \mathcal{R}(e, n, m), \quad (e, n, m) \in \Sigma_{-1} \cup \Sigma_0.$$

Now take an arbitrary scheme  $R = \mathcal{R}(e, n, m)$  for  $(e, n, m) \in \Sigma$  and, similarly to (21), set

$$(46) \quad (R \times (\mathbb{P}^3)_0^s)_0 := \{([F], S) \in R \times (\mathbb{P}^3)_0^s \mid S \in \Pi_{[F]}\},$$

$$(47) \quad (R \times G(2, 4) \times (\mathbb{P}^3)_0^s)_0 := \{([F], (l, S)) \in R \times G(2, 4) \times (\mathbb{P}^3)_0^s \mid (l, S) \in \mathcal{X}_{[F]}\},$$

where  $\Pi_{[F]}$  is defined in (19) for any reflexive sheaf  $[F] \in R$ . In particular, for  $s = 1$  we have

$$(48) \quad (R \times \mathbb{P}^3)_0 = \{([F], x) \in R \times \mathbb{P}^3 \mid x \notin \text{Sing}(F)\}.$$

By property (III) above there is a universal sheaf  $\mathcal{F}$  on  $R \times \mathbb{P}^3$ , and the definition (48) yields that

$$\mathbf{F} := \mathcal{F}|_{(R \times \mathbb{P}^3)_0}$$

is a locally free rank 2 sheaf. Hence, there exists an open subset  $U$  of  $(R \times \mathbb{P}^3)_0$  such that

$$(49) \quad \mathbf{P}(\mathbf{F}|_U) \simeq U \times \mathbb{P}^1,$$

and we have dense open inclusions

$$(50) \quad \mathbf{P}(\mathbf{F}) \xrightarrow{\text{open}} U \times \mathbb{P}^1 \xrightarrow{\text{open}} R \times \mathbb{P}^3 \times \mathbb{P}^1.$$

Now introduce a piece of notation. Let  $s$  be a positive integer and let  $f : X \rightarrow Y$  be an arbitrary morphism of schemes. The symmetric group  $G = \mathcal{S}_s$  acts on  $W = \prod_1^s X$  by permutations of factors, and the  $s$ -fold fibered product  $X \times_Y \cdots \times_Y X$  naturally embeds in  $W$  as a  $G$ -invariant subscheme. We will denote by  $\text{Sym}^s(X/Y)$  the quotient scheme  $(X \times_Y \cdots \times_Y X)/G$  and call this quotient the fibered symmetric product of  $X$  over  $Y$ .

Fix an integer  $s \geq 1$ . The composition of projections

$$f : \mathbf{P}(\mathbf{F}) \xrightarrow{\pi} (R \times \mathbb{P}^3)_0 \hookrightarrow R \times \mathbb{P}^3 \xrightarrow{pr_1} R,$$

where  $\pi$  is the structure morphism, defines the fibered symmetric product  $\text{Sym}^s(\mathbf{P}(\mathbf{F})/R)$  together with the projection  $f_s : \text{Sym}^s(\mathbf{P}(\mathbf{F})/R) \rightarrow R$  which factorizes as

$$f_s : \text{Sym}^s(\mathbf{P}(\mathbf{F})/R) \xrightarrow{\pi_s} R \times \text{Sym}^s(\mathbb{P}^3) \xrightarrow{pr_1} R.$$

The open embedding  $(\mathbb{P}^3)_0^s \hookrightarrow \text{Sym}^s(\mathbb{P}^3)$  together with the above projection  $\pi_s$  defines an open dense embedding of the fibered product

$$(51) \quad Y_R := \text{Sym}^s(\mathbf{P}(\mathbf{F})/R) \times_{\text{Sym}^s(\mathbb{P}^3)} (\mathbb{P}^3)_0^s \xrightarrow{\text{open}} \text{Sym}^s(\mathbf{P}(\mathbf{F})/R).$$

By the definition of  $Y_R$ , its set-theoretic description is the same as that of  $\tilde{\mathcal{T}}(e, n, m, s)$ , with  $\mathcal{R}^*(e, n, m)$  substituted by  $R$ :

$$(52) \quad Y_R := \{y = ([F], S, [\varphi_y]) \mid ([F], S) \in (R \times (\mathbb{P}^3)_0^s)_0, \varphi_y \in \text{Hom}(F, \mathcal{O}_S)_e\},$$

where  $(R \times (\mathbb{P}^3)_0^s)_0$  is defined in (46).

Now define a new set  $X_R$  by the formula similar to (32), with  $\mathcal{R}^*(e, n, m)$  substituted by  $R$ :

$$(53) \quad X_R := \{x = ([F], (l, S), [\varphi_x]) \mid ([F], (l, S)) \in (R \times G(2, 4) \times (\mathbb{P}^3)_0^s)_0, \\ [\varphi_x] \in \text{Hom}(F, Q_x)_e / \text{Aut}(Q_x), Q_x := Q_{(l, S), r}\},$$

where  $(R \times G(2, 4) \times (\mathbb{P}^3)_0^s)_0$  and  $Q_{(l, S), r}$  are defined in (47) and (29), respectively. There is a well-defined projection

$$(54) \quad \rho : X_R \rightarrow Y_R \times G(2, 4), \quad ([F], (l, S), [\varphi_x]) \mapsto (([F], S, [\varphi_x|_{\mathcal{O}_S}], l) ,$$

such that

$$\mathcal{V} = \rho(X_R)$$

is a dense open subset of  $Y_R \times G(2, 4)$ , and

$$(55) \quad \rho^{-1}(y, l) = \mathbf{P}_l^{2r+1-e}, \quad (y, l) \in \mathcal{V},$$

where  $\mathbf{P}_l^{2r+1-e}$  is described in (31) and (29). Let  $\Gamma \subset G(2, 4) \times \mathbb{P}^3$  be the graph of incidence with projections  $G(2, 4) \leftarrow \Gamma \rightarrow \mathbb{P}^3$ . Consider the natural projections

$$\mathcal{V} \xleftarrow{v} \mathcal{V} \times_G \Gamma \times_{\mathbb{P}^3} (R \times \mathbb{P}^3)_0 \xrightarrow{g} (R \times \mathbb{P}^3)_0 \xrightarrow{h} \mathbb{P}^3$$

and a locally free sheaf  $\mathbf{E}$  defined as

$$\mathbf{E} = (v_*(g^*\mathbf{F} \otimes h^*\mathcal{O}_{\mathbb{P}^3}(r))), \quad \text{rk } \mathbf{E} = 2r + 2 - e.$$

Then

$$(56) \quad \rho : X_R = \mathbf{P}(\mathbf{E}) \rightarrow \mathcal{V} \hookrightarrow Y_R \times G(2, 4),$$

is a locally trivial  $\mathbf{P}_l^{2r+1-e}$ -fibration with fibre  $\mathbf{P}_l^{2r+1-e}$  described in (55).

From the definition (46) of  $(R \times (\mathbb{P}^3)_0^s)_0$  it follows that  $\pi_s(Y_R) \subset (R \times (\mathbb{P}^3)_0^s)_0$ . We thus consider the composition

$$(57) \quad f_Y : Y_R \xrightarrow{\pi_s} (R \times (\mathbb{P}^3)_0^s)_0 \hookrightarrow R \times (\mathbb{P}^3)_0^s \xrightarrow{pr_1} R.$$

Note that the open embeddings in (50) commute with the natural projections  $f : \mathbf{P}(\mathbf{F}) \rightarrow R$ ,  $f' : U \times \mathbb{P}^1 \rightarrow R$  and  $f'' : R \times \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow R$  and therefore define the induced dense open embeddings

$$(58) \quad \text{Sym}^s(\mathbf{P}(\mathbf{F})/R) \xleftarrow{\text{open}} \text{Sym}^s(U \times \mathbb{P}^1/R) \xrightarrow{\text{open}} \text{Sym}^s(R \times \mathbb{P}^3 \times \mathbb{P}^1/R)$$

which commute with the induced projections  $f_s : \text{Sym}^s(\mathbf{P}(\mathbf{F})/R) \rightarrow R$ ,  $f'_s : \text{Sym}^s(U \times \mathbb{P}^1/R) \rightarrow R$  and  $f''_s : \text{Sym}^s(R \times \mathbb{P}^3 \times \mathbb{P}^1/R) \rightarrow R$ . The diagram (58) yields a birational isomorphism  $\text{Sym}^s(\mathbf{P}(\mathbf{F})/R) \xleftarrow{\text{bir}} \text{Sym}^s(R \times \mathbb{P}^3 \times \mathbb{P}^1/R)$ , hence the dense open embedding (51) leads to a birational isomorphism

$$(59) \quad Y_R \xleftarrow{\text{bir}} \text{Sym}^s(R \times \mathbb{P}^3 \times \mathbb{P}^1/R).$$

On the other hand, from the definition of  $\text{Sym}^s(R \times \mathbb{P}^3 \times \mathbb{P}^1/R)$  follows an isomorphism

$$(60) \quad \text{Sym}^s(R \times \mathbb{P}^3 \times \mathbb{P}^1/R) \simeq R \times \text{Sym}^s(\mathbb{P}^3 \times \mathbb{P}^1/R).$$

Since any symmetric product of a rational variety is also rational (see for instance [10, Ch. 4, Thm. 2.8]), it follows from (60) and the property (I) that  $\text{Sym}^s(R \times \mathbb{P}^3 \times \mathbb{P}^1/R)$  is a rational irreducible scheme of dimension  $4s + \dim R$ . Hence by (59)

$$(61) \quad Y_R \text{ is a rational irreducible scheme of dimension } 4s + \dim R.$$

This together with the the description (56) yields:

$$(62) \quad X_R \text{ is a rational irreducible scheme of dimension } 4s + 2r + 5 - e + \dim R.$$

**Theorem 13.** *For any  $(e, n, m) \in \Sigma_{-1} \cup \Sigma_0$  and  $R$  irreducible component of  $\mathcal{R}(e, n, m)$ , we have:*

- (i) *for any integer  $s$  such that  $0 \leq 2s \leq m$  there exists a rational, generically reduced, irreducible component  $\bar{Y}_R$  of the moduli space  $\mathcal{M}(e, n, m - 2s)$  having the dimension  $4s + \dim \mathcal{R}(e, n, m)$ , where  $\dim \mathcal{R}(e, n, m)$  is given by one of the corresponding formulas (43)-(44). A generic sheaf from  $\bar{Y}_R$  has 0-dimensional singularities;*
- (ii) *for any integers  $r, s$  such that  $s \geq 0, r \geq 3$  and  $2r + 2s \leq m + 2 + e$  there exists a rational, generically reduced, irreducible component  $\bar{X}_R$  of the moduli space  $\mathcal{M}(e, n + 1, m + 2 + e - 2r - 2s)$  having the dimension  $4s + 2r + 5 - e + \dim \mathcal{R}(e, n, m)$ , with  $\dim \mathcal{R}(e, n, m)$  given by the formulas mentioned above. A generic sheaf from  $\bar{X}_R$  has singularities of pure dimension 1 for  $s = 0$ , and, respectively, of mixed dimension for  $s \geq 1$ .*

*Proof. Item (i).* We are going to show that  $Y_R$  is an open dense subset of an irreducible component  $\bar{Y}_R$  of  $\mathcal{M}(e, n, m - 2s)$ , where  $R = \mathcal{R}(e, n, m)$ . To this aim, we first construct a family of sheaves on  $\mathbb{P}^3$  with base  $Y_R$  which are obtained from reflexive sheaves  $[F] \in R$  via elementary transformations along sets of  $s$  points. Let  $H = \text{Hilb}^s(\mathbb{P}^3)$  be the Hilbert scheme of 0-dimensional subschemes of length  $s$  in  $\mathbb{P}^3$ , together with the universal family of 0-dimensional schemes  $Z_H \hookrightarrow H \times \mathbb{P}^3$ . We have an open embedding  $(\mathbb{P}^3)_0^s \hookrightarrow H$  and the induced family  $Z = Z_H \times_H (\mathbb{P}^3)_0^s \hookrightarrow (\mathbb{P}^3)_0^s \times \mathbb{P}^3$ . Given a point  $\{S\} \in (\mathbb{P}^3)_0^s$ , we will denote also by  $S$  the corresponding 0-dimensional subscheme  $Z \times_{H(\mathbb{P}^3)_0^s} \{S\}$  in  $\mathbb{P}^3$ . (This will not cause an ambiguity since  $S$  is a reduced scheme by the definition of  $(\mathbb{P}^3)_0^s$ .) According to [11, Ch. II, Prop. 7.12], for a given sheaf  $[F] \in R$  and the above point  $\{S\}$  such that  $S \cap \text{Sing}(F) = \emptyset$ , choosing a class  $[\varphi]$  modulo  $\text{Aut}(\mathcal{O}_S)$  of an epimorphism

$$\varphi : F \rightarrow \mathcal{O}_S$$

is equivalent to choosing a section  $[\varphi]$  of the structure morphism  $\pi : \mathbf{P}(F|_S) \rightarrow S$ ,

$$[\varphi] : S \hookrightarrow \mathbf{P}(F|_S).$$

By the construction of  $Y_R$  (see (51)), the section  $[\varphi]$  is just a point of  $Y_R$  lying in the fiber  $\pi_s^{-1}([F], \xi)$  of the projection  $\pi_s : Y_R \rightarrow (R \times (\mathbb{P}^3)_0^s)_0$  defined in (57). Using this description of points of  $Y_R$ , define a family

$$(63) \quad \{[E_y] \in \mathcal{M}(e, n, m - 2s) \mid E_y = \ker(\varphi_y : F \rightarrow \mathcal{O}_S), y = ([F], S, [\varphi_y]) \in Y_R\}.$$

Here, by definition, for each  $y = ([F], S, [\varphi_y]) \in Y_R$ , the sheaf  $E = E_y$  satisfies the exact triple

$$(64) \quad 0 \rightarrow E \rightarrow F \xrightarrow{\varphi_y} \mathcal{O}_S \rightarrow 0, \quad F = E^{\vee\vee}.$$

In particular, this triple, together with the stability of  $F$ , yields by usual argument the stability of  $E$ , i. e., the definition of the family (63) is consistent. The family  $\{E_y\}_{y \in Y_R}$  globalizes in a standard way to a sheaf  $\mathbf{E}$  on  $Y_R \times \mathbb{P}^3$  such that, for any  $y \in Y_R$ ,  $\mathbf{E}|_{\{y\} \times \mathbb{P}^3} \simeq E_y$ . We thus have a natural modular morphism

$$(65) \quad \Psi : Y_R \rightarrow \mathcal{M}(e, n, m - 2s), \quad y \mapsto [\mathbf{E}|_{\{y\} \times \mathbb{P}^3}].$$

The morphism  $\Psi$  is clearly an embedding, since a point  $\{y\}$  is recovered from  $E = \ker(\varphi_y)$  as the (class of the) quotient map  $F = E^{\vee\vee} \twoheadrightarrow E^{\vee\vee}/E$ . We therefore identify  $Y_R$  with its image in  $\mathcal{M}(e, n, m - 2s)$ . Let  $\bar{Y}_R$  be the closure of  $Y_R$  in  $\mathcal{M}(e, n, m - 2s)$ .

We have to show that  $\bar{Y}_R$  is an irreducible rational component of  $\mathcal{M}(e, n, m - 2s)$ , where  $R = \mathcal{R}(e, n, m)$ . Here the rationality and the dimension of  $\bar{Y}_R$  are given in display (61). Since  $\bar{Y}_R$  is irreducible, to prove that  $\bar{Y}_R$  is an irreducible and generically reduced component of

$\mathcal{M}(e, n, m - 2s)$ , it is enough to show that, for an arbitrary point  $y \in Y_R$  the sheaf  $E = E_y$  satisfies the equality

$$(66) \quad \dim \operatorname{Ext}^1(E, E) = \dim Y_R = 4s + \dim \mathcal{R}(e, n, m).$$

(Note that the equality (66) is beyond the scope of Theorem 9, since we cannot assume that  $\operatorname{Ext}^2(F, F) = 0$  here).

Indeed, let  $E$  satisfy the triple (64). Then, since  $S$  is 0-dimensional and  $F$  is reflexive, it follows that  $\dim \operatorname{Sing}(E) = \dim \operatorname{Sing}(F) = 0$ , and therefore  $\dim \mathcal{E}xt^1(E, E) = \dim \mathcal{E}xt^1(F, E) = 0$ . Thus,

$$(67) \quad \mathbf{H}^1(\mathcal{E}xt^1(E, E)) = 0, \quad \mathbf{H}^1(\mathcal{E}xt^1(F, E)) = 0,$$

$$(68) \quad \mathbf{H}^2(\mathcal{E}xt^1(E, E)) = 0.$$

The first equality in (67) and Lemma 4.(ii) yield the equality  $\operatorname{Ext}^2(E, E) = \mathbf{H}^0(\mathcal{E}xt^2(E, E)) \oplus \operatorname{coker} d_2^{01}$  where  $d_2^{01}$  is the differential  $d_2^{01} : \mathbf{H}^0(\mathcal{E}xt^1(E, E)) \rightarrow \mathbf{H}^2(\mathcal{H}om(E, E))$  in the spectral sequence of local-to-global Exts. Moreover, this spectral sequence together with (68) yields an exact sequence

$$(69) \quad \mathbf{H}^0(\mathcal{E}xt^1(E, E)) \xrightarrow{d_2^{01}} \mathbf{H}^2(\mathcal{H}om(E, E)) \rightarrow \operatorname{Ext}^2(E, E) \rightarrow \mathbf{H}^0(\mathcal{E}xt^2(E, E)) \rightarrow 0.$$

Note that, since the reflexive sheaf  $F$  has homological dimension 1, it follows that

$$(70) \quad \mathcal{E}xt^2(F, E) = 0 = \mathcal{E}xt^3(F, E).$$

Therefore, applying to the triple (64) the functor  $\mathcal{E}xt^2(-, E)$  we obtain

$$(71) \quad \mathcal{E}xt^2(E, E) = \mathcal{E}xt^3(\mathcal{O}_S, E),$$

Here, as in (39), one has  $h^0(\mathcal{E}xt^3(\mathcal{O}_S, E)) = 4s$ , so that

$$(72) \quad h^0(\mathcal{E}xt^2(E, E)) = 4s.$$

Next, the equality  $\mathcal{E}xt^2(F, E) = 0$  (see (70)) together with the second equality in (67) yields an exact sequence similar to (69):

$$(73) \quad \mathbf{H}^0(\mathcal{E}xt^1(F, E)) \xrightarrow{d_2^{01}} \mathbf{H}^2(\mathcal{H}om(F, E)) \rightarrow \operatorname{Ext}^2(F, E) \rightarrow 0.$$

The exact sequences (69) and (73) fit in a commutative diagram extending (9)

$$(74) \quad \begin{array}{ccccccc} \mathbf{H}^0(\mathcal{E}xt^1(F, E)) & \xrightarrow{d_2^{01}} & \mathbf{H}^2(\mathcal{H}om(F, E)) & \longrightarrow & \operatorname{Ext}^2(F, E) & \longrightarrow & 0 \\ \downarrow & & \downarrow \simeq & & \downarrow & & \downarrow \\ \mathbf{H}^0(\mathcal{E}xt^1(E, E)) & \xrightarrow{d_2^{01}} & \mathbf{H}^2(\mathcal{H}om(E, E)) & \longrightarrow & \operatorname{Ext}^2(E, E) & \longrightarrow & \mathbf{H}^0(\mathcal{E}xt^2(E, E)) \longrightarrow 0. \end{array}$$

Here the second vertical map is an isomorphism. Indeed, applying to the exact triple (64) the functor  $\mathcal{H}om(-, E)$  we obtain an exact sequence  $0 \rightarrow \mathcal{H}om(F, E) \rightarrow \mathcal{H}om(E, E) \rightarrow \mathcal{A} \rightarrow 0$ , where  $\dim \mathcal{A} \leq 0$  since  $\mathcal{A}$  is a subsheaf of the sheaf  $\mathcal{E}xt^1(\mathcal{O}_S, E)$  of dimension  $\leq 0$ . Now passing to cohomology of the last exact triple we obtain the desired isomorphism. The diagram (74) together with (72) yields the relation

$$(75) \quad \dim \operatorname{Ext}^2(E, E) = \dim \operatorname{Ext}^2(F, E) + 4s.$$

Now applying to (64) the functor  $\operatorname{Hom}(F, -)$  we obtain the exact sequence

$$(76) \quad \operatorname{Ext}^1(F, \mathcal{O}_S) \rightarrow \operatorname{Ext}^2(F, E) \rightarrow \operatorname{Ext}^2(F, F) \rightarrow \operatorname{Ext}^2(F, \mathcal{O}_S).$$

Since  $\text{Supp}(\mathcal{O}_S) \cap \text{Sing}(F) = \emptyset$  and  $F$  is reflexive, it is easy to show that

$$(77) \quad \text{Ext}^j(F, \mathcal{O}_S) = 0, \quad j > 0,$$

- see, e. g., [19, Proof of Prop. 6] for the case  $e = 0$ ; in case  $e = -1$  this argument goes on without changing. Thus, from (76) it follows that  $\dim \text{Ext}^2(F, E) = \dim \text{Ext}^2(F, F)$ , and the relation (75) yields

$$(78) \quad \dim \text{Ext}^2(E, E) = \dim \text{Ext}^2(F, F) + 4s.$$

Next, since in (64)  $\dim \mathcal{O}_S = 0$ , it follows that  $c_i(E) = c_i(F)$ ,  $i = 1, 2$ . Thus, since both  $E$  and  $F$  are stable, Theorem 1 implies that  $\dim \text{Ext}^1(E, E) - \dim \text{Ext}^2(E, E) = \dim \text{Ext}^1(F, F) - \dim \text{Ext}^2(F, F)$ . Hence, by (78)  $\dim \text{Ext}^1(E, E) = \dim \text{Ext}^1(F, F) + 4s$ . This together with (45) implies (66).

**Item (ii).** We have to show that  $X_R$ , where  $R = \mathcal{R}(e, n, m)$ , is an open dense subset of an irreducible component  $\overline{X}_R$  of  $\mathcal{M}(e, n+1, m+2+e-2r-2s)$ . Using the pointwise description (53) of  $X_R$ , we consider similarly to (63) a family

$$(79) \quad \{[E_x] \in \mathcal{M}(e, n+1, m+2+e-2r-2s) \mid E_x = \ker(\varphi : F \rightarrow Q_x), x = ([F], (l, S), [\varphi_x]) \in X_R\}.$$

The rest of the argument below is parallel to that in the proof of statement (i) above. The difference is due to the fact that the triple (64) is modified as

$$(80) \quad 0 \rightarrow E \rightarrow F \xrightarrow{\varphi_x} Q_x \rightarrow 0, \quad Q_x = \mathcal{O}_S \oplus i_* \mathcal{O}_l(r), \quad F = E^{\vee\vee}, \quad E = E_x.$$

As for the sheaf  $E$  in (64), a standard argument for the sheaf  $E$  in the last triple in view of the stability of  $F$  yields the stability of  $E$ , i. e., the definition of the family (79) is consistent. This family  $\{E_x\}_{x \in X_R}$  globalizes in a standard way to a sheaf  $\mathbf{E}$  on  $X_R \times \mathbb{P}^3$  such that, for any  $x \in X_R$ ,  $\mathbf{E}|_{\{x\} \times \mathbb{P}^3} \simeq E_x$ . We thus have a natural modular morphism similar to (65):

$$(81) \quad \Psi : X_R \rightarrow \mathcal{M}(e, n+1, m+2+e-2r-2s), \quad x \mapsto [\mathbf{E}|_{\{x\} \times \mathbb{P}^3}].$$

The morphism  $\Psi$  is clearly an embedding, since a point  $\{x\}$  is recovered from  $E = \ker(\varphi_x)$  as the (class of the) quotient map  $F = E^{\vee\vee} \twoheadrightarrow E^{\vee\vee}/E$ . We therefore identify  $X_R$  with its image in  $\mathcal{M}(e, n+1, m+2+e-2r-2s)$ . Let  $\overline{X}_R$  be the closure of  $X_R$  in  $\mathcal{M}(e, n, m-2s)$ .

We have to prove that, for  $R = \mathcal{R}(e, n, m)$ , the scheme  $\overline{Y}_R$  is an irreducible rational component of  $\mathcal{M}(e, n+1, m+2+e-2r-2s)$ . Here the rationality and the dimension of  $\overline{X}_R$  are given in display (62). Since  $\overline{X}_R$  is irreducible, to prove that  $\overline{X}_R$  is an irreducible and generically reduced component of  $\mathcal{M}(e, n+1, m+2+e-2r-2s)$ , it is enough to show that, for an arbitrary point  $x \in X_R$  the sheaf  $E = E_x$  satisfies the equality

$$(82) \quad \dim \text{Ext}^1(E, E) = \dim \overline{X}_R = 4s + 2r + 5 - e + \dim \mathcal{R}(e, n, m).$$

(Remark that the equality (82) is beyond the scope of Theorem 9, since we cannot assume that  $\text{Ext}^2(F, F) = 0$  here).

Indeed, let  $E$  satisfy the triple (80). This triple and the definition (53) of  $X_R$  yield that

$$(83) \quad \text{Supp}(Q_x) = S \sqcup l, \quad \text{Sing}(E) = \text{Supp}(Q_x) \sqcup \text{Sing}(F), \quad \text{i. e.} \quad \text{Supp}(Q_x) \cap \text{Sing}(F) = \emptyset.$$

Hence, since  $F$  is reflexive,

$$(84) \quad \dim \mathcal{E}xt^1(F, E) = 0, \quad \text{Supp}(\mathcal{E}xt^1(F, E)) = \text{Sing}(F),$$

$$(85) \quad \mathcal{E}xt^i(F, E)|_{\mathbb{P}^3 \setminus \text{Sing}(F)} = 0, \quad i \geq 1.$$

Since  $F$  is locally free along  $l$  (see (83)) the isomorphisms  $F|_l \simeq \mathcal{O}_l(e) \oplus \mathcal{O}_l$  and  $\mathcal{E}xt^2(i_*\mathcal{O}_l, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{O}_l(2)$  imply that

$$(86) \quad \mathcal{E}xt^2(i_*\mathcal{O}_l(r), F) \simeq \mathcal{O}_l(2-r+e) \oplus \mathcal{O}_l(2-r),$$

$$(87) \quad \mathcal{E}xt^1(i_*\mathcal{O}_l(r), F) = \mathcal{E}xt^3(i_*\mathcal{O}_l(r), F) = 0.$$

By the same reason,  $\mathcal{E}xt^j(F, i_*\mathcal{O}_l(r)) = 0$ ,  $j > 0$ , so that, since  $r \geq 3$ , we have

$$(88) \quad \mathcal{E}xt^j(F, i_*\mathcal{O}_l(r)) = H^j(\text{Hom}(F, i_*\mathcal{O}_l(r))) \simeq H^j(\mathcal{O}_l(r-e) \oplus \mathcal{O}_l(r)) = 0, \quad j > 0.$$

Applying to the triple (80) the functor  $\mathcal{H}om(i_*\mathcal{O}_l(r), -)$  and using (86), (87) and the isomorphisms

$$\mathcal{E}xt^1(i_*\mathcal{O}_l(r), i_*\mathcal{O}_l(r)) = N_{l/\mathbb{P}^3} \simeq 2\mathcal{O}_l(1) \quad \text{and} \quad \mathcal{E}xt^2(i_*\mathcal{O}_l(r), i_*\mathcal{O}_l(r)) \simeq \det N_{l/\mathbb{P}^3} \simeq \mathcal{O}_l(2),$$

we obtain an exact sequence

$$(89)$$

$$0 \rightarrow 2\mathcal{O}_l(1) \rightarrow \mathcal{E}xt^2(i_*\mathcal{O}_l(r), E) \rightarrow \mathcal{O}_l(2-r+e) \oplus \mathcal{O}_l(2-r) \rightarrow \mathcal{O}_l(2) \xrightarrow{\gamma} \mathcal{E}xt^3(i_*\mathcal{O}_l(r), E) \rightarrow 0.$$

Here one easily sees that  $\text{Supp}(\mathcal{E}xt^3(i_*\mathcal{O}_l(r), E)) = l$ , hence  $\gamma$  is an isomorphism

$$(90) \quad \mathcal{E}xt^3(i_*\mathcal{O}_l(r), E) \simeq \mathcal{O}_l(2),$$

and (89) yields an exact triple  $0 \rightarrow 2\mathcal{O}_l(1) \rightarrow \mathcal{E}xt^2(i_*\mathcal{O}_l(r), E) \rightarrow \mathcal{O}_l(2-r+e) \oplus \mathcal{O}_l(2-r) \rightarrow 0$ . Passing to cohomology of this triple and using the condition  $r \geq 3$  we get

$$(91) \quad h^1(\mathcal{E}xt^2(i_*\mathcal{O}_l(r), E)) = 2r - e - 6.$$

Next, applying to the triple (80) the functor  $\mathcal{H}om(-, E)$  we obtain a long exact sequence

$$(92) \quad 0 \rightarrow \mathcal{H}om(F, E) \rightarrow \mathcal{H}om(F, E) \xrightarrow{\partial} \mathcal{E}xt^1(Q_x, E) \rightarrow \dots \rightarrow \mathcal{E}xt^3(F, E) \rightarrow \mathcal{E}xt^3(E, E) \rightarrow 0.$$

Denoting  $\mathcal{A} = \text{im}(\partial)$  we obtain an exact triple

$$(93) \quad 0 \rightarrow \mathcal{H}om(F, E) \rightarrow \mathcal{H}om(F, E) \rightarrow \mathcal{A} \rightarrow 0,$$

Since  $\mathcal{A}$  is a subsheaf of  $\mathcal{E}xt^1(Q_x, E)$  and by (83)  $\dim \mathcal{E}xt^1(Q_x, E) \leq 1$ , it follows that  $H^2(\mathcal{A}) = 0$ , and the triple (93) yields an epimorphism

$$(94) \quad H^2(\mathcal{H}om(F, E)) \twoheadrightarrow H^2(\mathcal{H}om(E, E)).$$

Next, restricting the sequence (92) onto  $\mathbb{P}^3 \setminus \text{Sing}(F)$  and using (85) we obtain the isomorphism

$$(95) \quad \mathcal{E}xt^1(E, E)|_{\mathbb{P}^3 \setminus \text{Sing}(F)} \simeq \mathcal{E}xt^1(i_*\mathcal{O}_l(r), E).$$

Since by (80) the sheaves  $E$  and  $F$  coincide outside  $\text{Supp}(Q_x)$ , it follows from (95) and the reflexivity of  $F$  that

$$(96) \quad \mathcal{E}xt^1(E, E) \simeq \mathcal{E}xt^2(i_*\mathcal{O}_l(r), E) \oplus \mathcal{E}xt^1(F, F), \quad \dim \mathcal{E}xt^1(F, F) = 0.$$

These equalities together with (91) imply

$$(97) \quad h^1(\mathcal{E}xt^1(E, E)) = h^1(\mathcal{E}xt^2(i_*\mathcal{O}_l(r), E)) = 2r - e - 6.$$

From (84) and (96) we find

$$(98) \quad H^1(\mathcal{E}xt^1(F, E)) = H^2(\mathcal{E}xt^1(F, E)) = 0,$$

$$(99) \quad H^2(\mathcal{E}xt^1(E, E)) = 0.$$

The spectral sequence of local-to-global Exts for the pair  $(E, E)$  together with (99) yields the exact sequences

$$(100) \quad H^0(\mathcal{E}xt^1(E, E)) \xrightarrow{d_2^{01}} H^2(\mathcal{H}om(E, E)) \rightarrow \text{coker } d_2^{01} \rightarrow 0,$$

$$(101) \quad 0 \rightarrow \ker \varepsilon \rightarrow \text{Ext}^2(E, E) \xrightarrow{\varepsilon} \text{H}^0(\mathcal{E}xt^2(E, E)) \rightarrow 0.$$

$$(102) \quad 0 \rightarrow \text{coker } d_2^{01} \rightarrow \ker \varepsilon \rightarrow \text{H}^1(\mathcal{E}xt^1(E, E)) \rightarrow 0.$$

Note that, since the sheaf  $F$  is reflexive, the equalities (70) are still true, so that the rightmost part of the long exact sequence (92) yields the isomorphisms

$$(103) \quad \mathcal{E}xt^2(E, E) \simeq \mathcal{E}xt^3(Q_x, E) = \mathcal{E}xt^3(\mathcal{O}_S, E) \oplus \mathcal{E}xt^3(i_*\mathcal{O}_l(r), E).$$

Here, as in (39), one has  $h^0(\mathcal{E}xt^3(\mathcal{O}_S, E)) = 4s$ , and by (90) we have  $h^0(\mathcal{E}xt^3(\mathcal{O}_S, E)) = 3$ , so that (103) implies

$$(104) \quad h^0(\mathcal{E}xt^2(E, E)) = 4s + 3.$$

Besides, similar to (100)-(102), the spectral sequence of local-to-global Exts for the pair  $(F, E)$  together with (98) and the first equality (70) yields the exact sequence

$$(105) \quad \text{H}^0(\mathcal{E}xt^1(F, E)) \xrightarrow{d_2^{01}} \text{H}^2(\mathcal{H}om(F, E)) \rightarrow \text{Ext}^2(F, E) \rightarrow 0,$$

The exact sequences (100) and (105) in view of (94) fit in a commutative diagram

$$(106) \quad \begin{array}{ccccccc} \text{H}^0(\mathcal{E}xt^1(F, E)) & \xrightarrow{d_2^{01}} & \text{H}^2(\mathcal{H}om(F, E)) & \longrightarrow & \text{Ext}^2(F, E) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{H}^0(\mathcal{E}xt^1(E, E)) & \xrightarrow{d_2^{01}} & \text{H}^2(\mathcal{H}om(E, E)) & \longrightarrow & \text{coker } d_2^{01} & \longrightarrow & 0. \end{array}$$

Now applying to (80) the functor  $\text{Hom}(F, -)$  we obtain the exact sequence

$$(107) \quad \text{Ext}^1(F, Q_x) \rightarrow \text{Ext}^2(F, E) \rightarrow \text{Ext}^2(F, F) \rightarrow \text{Ext}^2(F, Q_x).$$

Recall that  $Q_x = \mathcal{O}_S \oplus i_*\mathcal{O}_l(r)$ , and the equalities (77) are still true. This together with (88) yields  $\text{Ext}^i(F, Q_x) = 0$ ,  $i = 1, 2$ , and (107) implies the isomorphism.

$$(108) \quad \text{Ext}^2(F, E) \simeq \text{Ext}^2(F, F).$$

Now (101), (102), diagram (106) and (108) imply the inequality

$$\dim \text{Ext}^2(E, E) \leq \dim \text{Ext}^2(F, F) + h^0(\mathcal{E}xt^2(E, E)) + h^1(\mathcal{E}xt^1(E, E))$$

which in view of (97) and (104) can be rewritten as

$$(109) \quad \dim \text{Ext}^2(E, E) \leq \dim \text{Ext}^2(F, F) + 4s + 2r - e - 3.$$

Next, since  $c_1(Q_x) = 0$ ,  $c_2(Q_x) = -1$ , it follows from (80) that  $c_1(E) = c_1(F)$ ,  $i = 1, 2$ . Thus, since both  $E$  and  $F$  are stable, Theorem 1 implies that  $\dim \text{Ext}^1(E, E) - \dim \text{Ext}^2(E, E) = \dim \text{Ext}^1(F, F) - \dim \text{Ext}^2(F, F) + 8$ . Hence, by (109)  $\dim \text{Ext}^1(E, E) \leq \dim \text{Ext}^1(F, F) + 4s + 2r + 5 - e$ . This inequality in view of (62) and (45) can be rewritten as

$$\dim \text{Ext}^1(E, E) \leq 4s + 2r + 5 - e + \dim R = \dim \overline{X}_R.$$

On the other hand, since in (81) the modular morphism  $\Psi : X_R \rightarrow \mathcal{M}(e, n+1, m+2+e-2r-2s)$  is an embedding, it follows that  $\dim \text{Ext}^1(E, E) \geq \dim \overline{X}_R$ . Thus, the last inequality is a strict equality, and we obtain (82).  $\square$

We are finally in position to give the proof of Main Theorem 2.

*Proof of Main Theorem 2. Item (i).* It follows from Theorem 13 (i) and the property (IV). Namely, take  $R = \mathcal{R}(-1, 2n, (2n)^2)$ . If  $n \geq 1$ , then take  $s = 2n^2$ , so that, for this  $s$ ,  $Y_R$  is a rational generically reduced component of  $\mathcal{M}(-1, 2n, 0)$ , with generic sheaf having 0-dimensional singularities. If  $n \geq 3$ , then for  $(s, r) = (0, 2(n^2 - n - 1))$ , the scheme  $X_R$  is

a rational generically reduced component of  $\mathcal{M}(-1, 2n, 0)$ , with generic sheaf having purely 1-dimensional singularities; b) for each pair  $(s, r)$  such that  $1 \leq s \leq 2(n^2 - n - 1)$  and  $r = 2(n^2 - n) + 1 - s$ , the scheme  $X_R$  is a rational generically reduced component of  $\mathcal{M}(-1, 2n, 0)$ , with generic sheaf having singularities of mixed dimension.

**Item (ii).** It follows from Theorem 13 (i) and the property (IV). Take  $R = \mathcal{R}(0, n, n^2 - n + 2)$ . If  $n \geq 1$ , then for  $s = \frac{n^2 - n + 2}{2}$ , the scheme  $Y_R$  is a rational generically reduced component of  $\mathcal{M}(-0, n, 0)$ , with generic sheaf having 0-dimensional singularities. If  $n \geq 3$ , then for  $(s, r) = (0, \frac{n(n-3)}{2} + 3)$ , the scheme  $X_R$  is a rational generically reduced component of  $\mathcal{M}(0, n, 0)$ , with generic sheaf having purely 1-dimensional singularities. If  $n \geq 4$ , then for each pair  $(s, r)$  such that  $1 \leq s \leq \frac{n(n-3)}{2}$  and  $r = \frac{n(n-3)}{2} + 3 - s$ , the scheme  $X_R$  is a rational generically reduced component of  $\mathcal{M}(0, n, 0)$ , with generic sheaf having singularities of mixed dimension. Main Theorem 2 is proved.  $\square$

**Remark 4.** *As it was shown by Le Potier in [24], the moduli scheme  $\mathcal{M}(0, 2, 0)$  consists of three irreducible components: one is the closure of locally free sheaves, while the other two have, as a general point, sheaves with 0-dimensional singular locus obtained via elementary transformations of reflexive sheaves in  $\mathcal{R}(0, 2, 2)$  and  $\mathcal{R}(0, 2, 4)$  at points. While the results of this section do not cover these irreducible components, Le Potier has shown that they are also rational, via a different method.*

## 5. IRREDUCIBILITY OF $\mathcal{M}(-1, 2, 4)$

In the previous sections our results ensured the existence of irreducible components of the moduli spaces of torsion free sheaves with prescribed singularities, without focusing on the description of all irreducible components of the moduli space for given Chern classes. The aim of this and subsequent sections is to consider this problem for smallest value  $c_2 = 2$  of the second Chern class. Namely, in Sections 5-9 we will obtain the complete characterization of the moduli spaces  $\mathcal{M}(-1, 2, c_3)$  for all possible values of  $c_3$ . These results will illustrate why this study becomes too complicated for large values of  $c_2$ .

More precisely, in this section we will describe the irreducible components of the moduli spaces  $\mathcal{M}(-1, 2, c_3)$  for possible values  $c_3 = 0, 2, 4$  of the third Chern class. For the convenience of the reader, in the following proposition we will fix some numerical invariants of torsion free sheaves that we will use in this section.

**Proposition 14.** *Let  $E$  be a torsion free sheaf,  $E^{\vee\vee}$  its double dual and  $Q_E := E^{\vee\vee}/E$ . The following holds:*

- (i)  $\dim Q_E \leq 1$  and  $c_1(E^{\vee\vee}) = c_1(E)$ ;
- (ii) if  $\dim Q_E = 1$  then  $c_2(E^{\vee\vee}) = c_2(E) - \text{mult}(Q_E)$ ,  $c_3(E^{\vee\vee}) = c_3(E) + c_3(Q_E) - c_1(E) \cdot \text{mult}(Q_E)$ , where  $\text{mult}(Q_E)$  is the multiplicity of the sheaf  $Q_E$ ;
- (iii) if  $\dim Q_E = 0$  then  $c_2(E^{\vee\vee}) = c_2(E)$ ,  $c_3(E^{\vee\vee}) = c_3(E) + 2 \cdot \text{length}(Q_E)$ .

If, in addition  $E$  is stable with  $c_1(E) = -1$ , then

- (iv)  $E^{\vee\vee}$  is stable;
- (v) If  $\dim Q_E = 1$  then  $c_2(E) \geq \text{mult}(Q_E) \geq 1$ .

*Proof.* Since  $E$  is torsion free, it fits in the following exact sequence:

$$(110) \quad 0 \rightarrow E \rightarrow E^{\vee\vee} \xrightarrow{\varepsilon} Q_E \rightarrow 0.$$

The statement (i) is clear, since  $E$  is torsion free. Therefore, computing the Chern classes we have the items (ii) and (iii). To show (iv), it is enough to consider the triple  $0 \rightarrow A \xrightarrow{i} E^{\vee\vee} \rightarrow B \rightarrow 0$  where both  $A$  and  $B$  are rank-1 torsion free sheaves with  $c_1(A) + c_1(B) = c_1(E^{\vee\vee}) = -1$ .



Since  $\dim Q \leq 1$ , it follows that  $\dim \operatorname{im}(\varepsilon \circ i) \leq 1$ , where  $\varepsilon$  is the epimorphism in (110). Therefore, the rank 1 sheaf  $A' = \ker(\varepsilon \circ i)$  satisfies the equality  $c_1(A') = c_1(A)$ . On the other hand, since  $A'$  is a subsheaf of the stable sheaf  $E$ , it follows that  $c_1(A') \leq c_1(E) \leq -1$ . Hence,  $c_1(A) \leq c_1(E^{\vee\vee}) = -1$  and  $c_1(B) \geq 0$ , which implies that the reduced Hilbert polynomial of  $A$  is less than that of  $E^{\vee\vee}$ , that is  $E^{\vee\vee}$  is stable. In particular,  $c_2(E^{\vee\vee}) \geq 1$ , see [13, Cor. 3.3]. Thus, if  $\dim Q_E = 1$  then, by (iv),  $c_2(E) \geq \operatorname{mult} Q_E \geq 1$ .  $\square$

The next Lemma is an easy technical result that we use later in this section.

**Lemma 15.** *For each  $F \in \mathcal{R}(-1, 1, 1)$ , consider the set  $Y_F := \{l \in G(2, 4); \operatorname{Sing}(F) \subset l\}$ , and the set*

$$(111) \quad Y(r) := \{(F, l, \varphi) \mid (F, l) \in \mathcal{R}(-1, 1, 1) \times Y_F, \varphi \in \operatorname{Hom}(F, i_* \mathcal{O}_l(r))_e / \operatorname{Aut}(i_* \mathcal{O}_l(r))\}.$$

*Then, for each  $r \in \{-1, 0, 1\}$ , the set  $Y(r)$  is an irreducible scheme of dimension  $8 + 2r$ . In addition, the closure in  $\mathcal{M}(-1, 2, 2 - 2r)$  of the image of the morphism  $Y(r) \rightarrow \mathcal{M}(-1, 2, 2 - 2r)$ ,  $(F, l, \varphi) \mapsto [\ker \varphi]$  is never an irreducible component of  $\mathcal{M}(-1, 2, 2 - 2r)$ .*

*Proof.* For each  $[F] \in \mathcal{R}(-1, 1, 1)$ ,  $\operatorname{Sing}(F)$  is a unique point, so that the set  $Y_F$  is a surface in the Grassmannian  $G(2, 4)$  isomorphic to  $\mathbb{P}^2$ . Therefore it is irreducible of dimension 2. To see that the dimension of  $\operatorname{Hom}(F, i_* \mathcal{O}_l(r))_e / \operatorname{Aut}(i_* \mathcal{O}_l(r))$  is  $3 + 2r$ , apply the functor  $\mathcal{H}om(-, i_* \mathcal{O}_l(r))$  to the sequence (14), and recall that  $\dim H^0(\mathcal{E}xt^1(F, i_* \mathcal{O}_l(r))) = 1$ . Putting all these data together, we define the set  $Y(r)$  by (111). By construction it is an irreducible scheme of dimension  $8 + 2r$ . Indeed one has the surjective projection

$$Y(r) \rightarrow \mathcal{R}(-1, 1, 1) \times Y_F, \quad ([F], l, \varphi) \mapsto ([F], l),$$

onto an irreducible scheme  $\mathcal{R}(-1, 1, 1) \times Y_F$  of dimension 5 (see Remark 2), with fibers

$$\operatorname{Hom}(F, i_* \mathcal{O}_l(r))_e / \operatorname{Aut}(i_* \mathcal{O}_l(r)) \xrightarrow{\text{open}} \operatorname{Hom}(F, i_* \mathcal{O}_l(r)) / \operatorname{Aut}(i_* \mathcal{O}_l(r))$$

which have dimension  $3 + 2r$ .  $\square$

With the previous Lemma, we are already in position to prove the first main result of this section.

**Theorem 16.** *The moduli space  $\mathcal{M}(-1, 2, 4)$  of rank 2 stable sheaves on  $\mathbb{P}^3$  with Chern classes  $c_1 = -1$ ,  $c_2 = 2$ ,  $c_3 = 4$  is the closure  $\overline{\mathcal{R}(-1, 2, 4)}$  of the moduli space  $\mathcal{R}(-1, 2, 4)$  of the rank 2 reflexive sheaves with Chern classes  $c_1 = -1$ ,  $c_2 = 2$ ,  $c_3 = 4$ . Hence  $\mathcal{M}(-1, 2, 4)$  is irreducible, rational, generically smooth, and of dimension 11. Moreover,*

$$(112) \quad \{[F] \in \mathcal{M}(-1, 2, 4) \mid \dim \operatorname{Sing}(F) = 0\} = \mathcal{R}(-1, 2, 4).$$

*Proof.* By [13, Thm 9.2],  $\mathcal{R}(-1, 2, 4)$  is irreducible of dimension 11, and  $\overline{\mathcal{R}(-1, 2, 4)}$  is an irreducible component of  $\mathcal{M}(-1, 2, 4)$ . Consider  $[E] \in \mathcal{M}(-1, 2, 4) \setminus \overline{\mathcal{R}(-1, 2, 4)}$ . By Proposition 14, since  $E^{\vee\vee}$  is stable, and either  $\dim Q_E = 1$  and  $1 \leq \operatorname{mult} Q_E \leq c_2(E) = 2$ , or  $\dim Q_E = 0$ . We will study the possibilities for  $\dim Q_E$  and  $\operatorname{mult} Q_E$ .

i) If  $\dim Q_E = 1$ ,  $\operatorname{mult} Q_E = 2$ , then by Proposition 14.b)  $c_2(E^{\vee\vee}) = 0$ , and by [13, Thm 8.2],

$$(113) \quad c_3(E^{\vee\vee}) \leq c_2(E^{\vee\vee})^2,$$

since  $E^{\vee\vee}$  is stable. Therefore  $c_3(E^{\vee\vee}) = 0$ , that is  $E^{\vee\vee}$  is a stable locally free rank 2 sheaf. Since  $c_1(E^{\vee\vee}) = -1$ ,  $c_2(E^{\vee\vee}) = 0$ , this contradicts to [14, Cor. 3.5].

ii) If  $\dim Q_E = \text{mult } Q_E = 1$ , then  $c_2(E^{\vee\vee}) = 1$  and, as above,  $0 \leq c_3(E^{\vee\vee}) \leq c_2^2(E^{\vee\vee}) = 1$ . Moreover, the equality  $\text{mult } Q_E = 1$  implies that  $Q_E$  is supported on a line, say,  $l$  and it fits in an exact sequence of the form:

$$(114) \quad 0 \rightarrow Z_E \rightarrow Q_E \rightarrow i_*\mathcal{O}_l(r) \rightarrow 0,$$

where  $Z_E$  is the maximal 0-dimensional subsheaf of  $Q_E$  of length  $s \geq 0$ , and  $\mathcal{O}_l$  is the structure sheaf of the line  $l$ . This sequence and Proposition 14.b) yield

$$(115) \quad \chi(Q_E(t)) = t + r + s + 1, \quad c_3(Q_E) = 2(r + s - 1),$$

and  $c_2(E^{\vee\vee}) = 1$ ,  $c_3(E^{\vee\vee}) = 2r + 2s + 3 \geq 0$ . (Here the inequality  $c_3(E^{\vee\vee}) \geq 0$  follows from [13, Prop. 2.6].) Thus from (113) we obtain  $r + s = -1$ , i. e.

$$(116) \quad c_2(E^{\vee\vee}) = c_3(E^{\vee\vee}) = 1.$$

As  $E^{\vee\vee}$  is stable, this means that  $[E^{\vee\vee}] \in \mathcal{M}(-1, 1, 1)$ . Note that, by (114), there is an epimorphism  $E^{\vee\vee} \rightarrow Q_E$ , so that from (114) and the formula (15) in which we set  $F = E^{\vee\vee}$  it follows that  $r \geq -1$ . This together with the relation  $r + s = -1$  and the inequality  $s \geq 0$  shows that the only possible values for  $r$  and  $s$  are  $r = -1$ ,  $s = 0$ . We thus have  $Q_E/Z_E = i_*\mathcal{O}_l(-1)$ . This together with (116) yields that, if  $l \cap \text{Sing}(E^{\vee\vee}) = \emptyset$ , then, since  $[E^{\vee\vee}] \in \mathcal{M}(-1, 1, 1)$ , it follows that  $[E]$  belongs to the scheme  $\mathcal{X}(-1, 1, 1, -1, 0)$  defined in display (34). Note that  $\dim \mathcal{X}(-1, 1, 1, -1, 0) = 7$  by Remark 3. Since by the deformation theory (see Theorem 1) any irreducible component of  $\mathcal{M}(-1, 2, 4)$  has dimension at least 11, the last equality shows that the dimension of  $\mathcal{X}(-1, 1, 1, -1, 0)$  is too small to fill an irreducible component of  $\mathcal{M}(-1, 2, 4)$ .

iii) If  $\dim Q_E = 0$ , then  $s = \text{length}(Q_E) > 0$  and, by Proposition 14.c),  $c_2(E) = c_2(E^{\vee\vee}) = 2$ ,  $c_3(E^{\vee\vee}) = c_3(E) + 2s = 4 + 2s \geq 6$ . Therefore,  $4 = c_2^2(E^{\vee\vee}) < c_3(E^{\vee\vee})$ . But this inequality contradicts the stability of  $E^{\vee\vee}$  by [13, Thm. 8.2(d)].

In conclusion, we have proved that  $\mathcal{M}(-1, 2, 4) = \overline{\mathcal{R}(-1, 2, 4)}$ , and the equality in display (112) follows from iii) above. The rationality of  $\mathcal{R}(-1, 2, 4)$  is known from [7]. Hence,  $\mathcal{M}(-1, 2, 4)$  is rational.  $\square$

As a by product of the previous proof, we obtain the following interesting result.

**Corollary 17.** *The complement of  $\mathcal{R}(-1, 2, 4)$  in  $\mathcal{M}(-1, 2, 4)$  is precisely  $\mathcal{X}(-1, 1, 1, -1, 0)$ .*

## 6. DESCRIPTION OF FAMILIES WITH 0-DIMENSIONAL SINGULARITIES

In this section we describe explicitly the sheaves in the families  $\text{T}(-1, 2, 2, 1)$ ,  $\text{T}(-1, 2, 4, 1)$  and  $\text{T}(-1, 2, 4, 2)$ . This description will be used later in the study of irreducible components of the moduli spaces  $\mathcal{M}(-1, 2, c_3)$  for  $c_3 = 2$  and  $c_3 = 4$ . Everywhere below for a coherent sheaf  $F$  on a given scheme  $X$  we denote by  $\mathbf{P}(F)$  the projective spectrum of the symmetric algebra  $\text{Sym}_{\mathcal{O}_X}(F)$ . Besides, as before, for any point  $p \in \mathbb{P}^3$  we denote  $A_p = \text{Aut}(\mathcal{O}_p) \simeq \mathbf{k}$ .

We start with the following theorem describing, for  $i = 1$  and  $i = 2$ , the irreducible families  $\text{T}(-1, 2, 2i, 1)$  defined in Section 3 as the closures in  $\mathcal{M}(-1, 2, 2i - 2)$  of their open subsets  $\mathcal{T}(-1, 2, 2i, 1)$ . For these  $i$ , consider the moduli spaces  $R_i := \mathcal{R}(-1, 2, 2i)$  and the universal  $\mathcal{O}_{\mathbb{P}^3 \times R_i}$ -sheaves  $\mathbf{F}_i$ , respectively.

**Theorem 18.** *The scheme  $\text{T}(-1, 2, 2i, 1)$ , for  $i \in \{1, 2\}$ , is an irreducible 15-dimensional component of  $\mathcal{M}(-1, 2, 2i - 2)$ . This component contains an open subset of  $\text{T}(-1, 2, 2i, 1)$ , isomorphic to  $\mathbf{P}(\mathbf{F}_i)$ , which consists of all the points  $[E] \in \mathcal{M}(-1, 2, 2i - 2)$  such that  $E^{\vee\vee}/E$  is a 0-dimensional scheme of length 1. This subset  $\mathbf{P}(\mathbf{F}_i)$  contains the open subset  $\mathcal{T}(-1, 2, 2i, 1)$ .*

*Proof.* Let  $i \in \{1, 2\}$ . For any point  $y \in R_i$  we denote  $F_{i,y} = \mathbf{F}_i|_{\mathbb{P}^3 \times \{y\}}$ . By [35, Lemma 4.5]  $\mathbf{P}(F_{i,y})$  is an irreducible 4-dimensional scheme for any  $y \in R_i$ . Hence, since  $\dim R_i = 11$ ,  $i = 1, 2$ , it follows that  $\mathbf{P}(\mathbf{F}_i)$  is an irreducible 15-dimensional scheme. Consider the structure morphisms  $\pi_i : \mathbf{P}(\mathbf{F}_i) \rightarrow \mathbb{P}^3 \times R_i$  and the compositions  $\theta_i = pr_1 \circ \pi_i : \mathbf{P}(\mathbf{F}_i) \rightarrow \mathbb{P}^3$ . By the functorial property of projective spectra [11, Ch. II, Prop. 7.12] we have for  $i = 1, 2$ :

$$(117) \quad \mathbf{P}(\mathbf{F}_i) = \{z = (p, [F_i], [\psi] = \psi \bmod A_p) \mid (p, [F_i]) = \pi_i(z), \psi : F_i \rightarrow \mathcal{O}_p \text{ is an epimorphism}\}.$$

Hence, each point  $z = (p, [F_i], [\psi]) \in \mathbf{P}(\mathbf{F}_i)$  defines an exact triple

$$(118) \quad 0 \rightarrow E_i \rightarrow F_i \xrightarrow{\psi} \mathcal{O}_p \rightarrow 0, \quad [E_i = E_{i,z} := \ker \psi] \in \mathcal{M}(-1, 2, 2i - 2), \quad F_i = E_{i,z}^{\vee\vee}, \quad i = 1, 2.$$

This triple is globalized to an  $\mathcal{O}_{\mathbb{P}^3 \times R_i}$ -triple in the following way. Namely, let  $\tilde{\mathbf{F}}_i = \mathbf{F}_i \otimes_{\mathcal{O}_{R_i}} \mathcal{O}_{\mathbf{P}(\mathbf{F}_i)}$  and consider the "diagonal" embedding  $j : \mathbf{P}(\mathbf{F}_i) \hookrightarrow \mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_i)$ ,  $z \mapsto (\theta_i(z), z)$ . By construction,  $j^* \tilde{\mathbf{F}}_i = \pi_i^* \mathbf{F}_i$  and we obtain the composition of surjections  $\psi : \tilde{\mathbf{F}}_i \rightarrow j_* j^* \tilde{\mathbf{F}}_i = j_* \pi_i^* \mathbf{F}_i \rightarrow j_* \mathcal{O}_{\mathbf{P}(\mathbf{F}_i)}(1)$  which yields an exact  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_i)}$ -triple, where  $\mathbf{E}_i := \ker \psi$ :

$$(119) \quad 0 \rightarrow \mathbf{E}_i \rightarrow \tilde{\mathbf{F}}_i \xrightarrow{\psi} j_* \mathcal{O}_{\mathbf{P}(\mathbf{F}_i)}(1) \rightarrow 0, \quad i = 1, 2.$$

By construction, the sheaves in this triple are flat over  $R_i$ , hence its restriction onto  $\mathbb{P}^3 \times \{z\}$  for any  $z = (p, [F_i], [\psi]) \in \mathbf{P}(\mathbf{F}_i)$  yields the triple (118) with  $E_{i,z} = \mathbf{E}_i|_{\mathbb{P}^3 \times \{z\}}$ . Thus we obtain the modular morphism

$$(120) \quad f_i : \mathbf{P}(\mathbf{F}_i) \rightarrow \mathcal{M}(-1, 2, 2i - 2), \quad z \mapsto E_{i,z}, \quad i = 1, 2.$$

This morphism is clearly an embedding, since the data  $([F_i], p, [\psi])$  in the triple (118) are uniquely recovered from the point  $[E = E_{i,z}] \in \mathcal{M}(-1, 2, 2i - 2)$ ; namely,  $F_i := E^{\vee\vee}$ ,  $p := \text{Supp}(Q_E)$ , where  $Q_E := E^{\vee\vee}/E \simeq \mathcal{O}_p$  since  $\text{length } Q_E = 1$  and  $\psi : F_i \rightarrow \mathcal{O}_p$  is the quotient epimorphism. We therefore identify  $\mathbf{P}(\mathbf{F}_i)$  with its image under the morphism  $f_i$ .

Last, under the description (117) of  $\mathbf{P}(\mathbf{F}_i)$  we have, by the definition of  $\mathcal{T}(-1, 2, 2i, 1)$ , that  $\mathcal{T}(-1, 2, 2i, 1) = \{z = (p, [F_i], [\psi]) \in \mathbf{P}(\mathbf{F}_i) \mid p \notin \text{Sing}(F_i)\}$  is an open subset of  $\mathbf{P}(\mathbf{F}_i)$  which is dense since  $\mathbf{P}(\mathbf{F}_i)$  is irreducible. Hence, by definition, its closure in  $\mathcal{M}(-1, 2, 2i - 2)$  coincides with  $\mathcal{T}(-1, 2, 2i, 1)$ . In addition, it is an irreducible component of  $\mathcal{M}(-1, 2, 2i - 2)$  by Theorem 9.  $\square$

Let us introduce one more piece of notation. For any  $y \in R_2$ , let  $F_y := \mathbf{F}_2|_{\mathbb{P}^3 \times \{y\}}$  and let  $pr_2 : \mathbb{P}^3 \times R_2 \rightarrow R_2$  be the projection. Besides, for an arbitrary  $\mathcal{O}_{\mathbb{P}^3 \times R_2}$ -sheaf  $A$  and an integer  $m \in \mathbb{Z}$  let  $A(m) := A \otimes (\mathcal{O}_{\mathbb{P}^3}(m) \boxtimes \mathcal{O}_{R_2})$ . The following remark will be important below in the study of the scheme  $\mathbf{P}(\mathbf{E}_2)$  for the  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_2)}$ -sheaf  $\mathbf{E}_2$  defined in (119) for  $i = 2$ .

**Remark 5.** *From [13, Lemma 9.6 and Proof of Lemma 9.3] it follows that, for any  $y \in R_2$ , the sheaf  $F_y$  fits in an exact triple*

$$(121) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3) \rightarrow 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\phi} F_y \rightarrow 0.$$

*This triple clearly globalizes to a locally free  $\mathcal{O}_{\mathbb{P}^3 \times R_2}$ -resolution of the universal sheaf  $\mathbf{F}_2$ :*

$$(122) \quad 0 \rightarrow \mathbf{L}_2 \rightarrow \mathbf{L}_1 \xrightarrow{\Phi} \mathbf{F} \rightarrow 0, \quad \text{rk } \mathbf{L}_1 = 3, \quad \text{rk } \mathbf{L}_2 = 1.$$

*explicitly,  $\mathbf{L}_1$  fits in the exact triple  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes M_0 \rightarrow \mathbf{L}_1 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes M_1 \rightarrow 0$  and  $\mathbf{L}_2 = \mathcal{O}_{\mathbb{P}^3}(-3) \boxtimes M_2$ , where  $M_0, M_1, M_2$  are locally free  $\mathcal{O}_{R_2}$ -sheaves of ranks 2, 1, 1, respectively, which are determined by  $\mathbf{F}_2$  as:  $M_0 = pr_{2*}(\mathbf{F}_2(1))$ ,  $M_1 = pr_{2*}(\mathbf{F}_2(2))/pr_{2*}(\text{im}(ev))$ , where  $ev : \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes M_0 \rightarrow \mathbf{F}_2(2)$  is the evaluation morphism, and  $M_2 = \ker(pr_{2*} \mathbf{L}_1(3) \xrightarrow{pr_{2*} \Phi} pr_{2*} \mathbf{F}_2(3))$ .*

Consider the structure morphism  $\pi_2 : \mathbf{P}(\mathbf{F}_2) \rightarrow \mathbb{P}^3 \times R_2$ . Note that the triple (121) immediately yields that  $\pi_2^{-1}(p, y)$  equals to  $\mathbb{P}^1$  if  $p \notin \text{Sing}(F_y)$ , respectively, equals  $\mathbb{P}^2$  if  $p \in \text{Sing}(F_y)$ . As  $\text{codim}(\text{Sing}(F_y), \mathbb{P}^3) = 3$ , it follows by the definition of  $\mathcal{T}(-1, 2, 2i, 1)$  that

$$(123) \quad \text{codim}_{\mathbf{P}(\mathbf{F}_2)}(\mathbf{P}(\mathbf{F}_2) \setminus \mathcal{T}(-1, 2, 4, 1)) = 2.$$

Now proceed to the study of the scheme  $\mathbf{P}(\mathbf{E}_2)$  endowed with the structure morphism  $\pi : \mathbf{P}(\mathbf{E}_2) \rightarrow \mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_2)$  and consider the composition  $\tau = pr_1 \circ \pi : \mathbf{P}(\mathbf{E}_2) \rightarrow \mathbb{P}^3$ . Similarly to (117), in view of the functorial property of projective spectra [11, Ch. II, Prop. 7.12] we obtain the following description of the scheme  $\mathbf{P}(\mathbf{E}_2)$ :

$$(124) \quad \mathbf{P}(\mathbf{E}_2) = \{w = (q, [E_2], [\varphi] = \varphi \bmod A_q) \mid (q, [E_2]) = \pi(w), \varphi : E_2 \rightarrow \mathcal{O}_q \text{ is an epimorphism}\}.$$

It follows now that each point  $w = (q, [E_2], [\varphi]) \in \mathbf{P}(\mathbf{E}_2)$  defines an exact triple

$$(125) \quad 0 \rightarrow E_w \rightarrow E_2 \xrightarrow{\varphi} \mathcal{O}_q \rightarrow 0, \quad [E_w := \ker \varphi] \in \mathcal{M}(-1, 2, 0).$$

This triple is globalized to an  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_2)}$ -triple which is constructed completely similar to the triples (119). Namely, let  $\tilde{\mathbf{E}}_2 = \mathbf{E}_2 \otimes_{\mathcal{O}_{\mathbf{P}(\mathbf{F}_2)}} \mathcal{O}_{\mathbf{P}(\mathbf{E}_2)}$  and consider the "diagonal" embedding  $j : \mathbf{P}(\mathbf{E}_2) \hookrightarrow \mathbb{P}^3 \times \mathbf{P}(\mathbf{E}_2)$ ,  $w \mapsto (\tau(w), w)$ . Then  $j^* \tilde{\mathbf{E}}_2 = \pi^* \mathbf{E}_2$  and we obtain the composition of surjections  $\varphi : \tilde{\mathbf{E}}_2 \twoheadrightarrow j_* j^* \tilde{\mathbf{E}}_2 = j_* \pi^* \mathbf{E}_2 \twoheadrightarrow j_* \mathcal{O}_{\mathbf{P}(\mathbf{E}_2)}(1)$  which yields an exact  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathbf{E}_2)}$ -triple, where  $\mathbf{E} := \ker \varphi$ :

$$(126) \quad 0 \rightarrow \mathbf{E} \rightarrow \tilde{\mathbf{E}}_2 \xrightarrow{\varphi} j_* \mathcal{O}_{\mathbf{P}(\mathbf{E}_2)}(1) \rightarrow 0.$$

By construction, the restriction of this triple onto  $\mathbb{P}^3 \times \{w\}$  for any  $w = (p, [E_2], [\varphi]) \in \mathbf{P}(\mathbf{E}_2)$  yields the triple (125), where  $E_w = \mathbf{E}|_{\mathbb{P}^3 \times \{w\}}$  and where  $[E_2] \in \mathbf{P}(\mathbf{F}_2)$  by Theorem 18,(ii) fits in the triple (118) for  $i = 2$ :  $0 \rightarrow E_2 \rightarrow F_2 \xrightarrow{\psi} \mathcal{O}_p \rightarrow 0$ ,  $F_2 = E_2^{\vee\vee}$ . Combining this triple with (125), we obtain the equality  $F_2 = E_w^{\vee\vee}$  and two exact triples, where  $E = E_w$ :

$$(127) \quad 0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow Q_E \rightarrow 0, \quad 0 \rightarrow \mathcal{O}_q \rightarrow Q_E \rightarrow \mathcal{O}_p \rightarrow 0.$$

Besides, we have a modular morphism  $f : \mathbf{P}(\mathbf{E}_2) \rightarrow \mathcal{M}(-1, 2, 0)$ ,  $w \mapsto [E_w]$ . From (127) and the definition of the family  $\mathcal{T}(-1, 2, 4, 2)$  given in Theorem 9 it follows that

$$(128) \quad \mathcal{T}(-1, 2, 4, 2) = \{[E] \in f(\mathbf{P}(\mathbf{E}_2)) \mid \text{Supp}(Q_E) = p \sqcup q, \text{Supp}(Q_E) \cap \text{Sing}(E^{\vee\vee}) = \emptyset\}.$$

**Theorem 19.** *The scheme  $\mathcal{T}(-1, 2, 4, 2)$  is an irreducible 19-dimensional component of  $\mathcal{M}(-1, 2, 0)$ . This component contains a dense subset, isomorphic to  $f(\mathbf{P}(\mathbf{E}_2))$ , which consists of all the points  $[E] \in \mathcal{M}(-1, 2, 2)$  such that  $E^{\vee\vee}/E$  is a 0-dimensional scheme of length 2. This subset  $f(\mathbf{P}(\mathbf{E}_2))$  contains  $\mathcal{T}(-1, 2, 4, 2)$  as the dense open subset described in (128).*

*Proof.* We have to prove the irreducibility of  $\mathbf{P}(\mathbf{E}_2)$ . Since  $\mathbf{P}(\mathbf{F}_2)$  is irreducible, it is enough to prove that, for an arbitrary point  $z = (p, [F_2], [\psi]) \in \mathbf{P}(\mathbf{F}_2)$ , the fiber  $p_E^{-1}(z)$  of the composition  $p_E : \mathbf{P}(\mathbf{E}_2) \xrightarrow{\pi} \mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_2) \xrightarrow{pr_2} \mathbf{P}(\mathbf{F}_2)$  is irreducible of dimension 4. Note that the sheaves  $F_2$  and  $E_2 = \mathbf{E}_2|_{\mathbb{P}^3 \times \{z\}}$  fit in the exact triple (118) for  $i = 2$ . Besides,  $F_2$  fits in the exact triple

(121) in which we set  $F_y = F_2$ . These two triples are included in a commutative diagram

(129)

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-3) & \longrightarrow & \mathcal{G} & \longrightarrow & E_2 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-3) & \longrightarrow & 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) & \xrightarrow{\phi} & F_2 \longrightarrow 0 \\
 & & & & \downarrow \lambda & & \downarrow \psi \\
 & & & & \mathcal{O}_p & \xlongequal{\quad} & \mathcal{O}_p \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0,
 \end{array}$$

where  $\lambda := \psi \circ \phi$  and  $\mathcal{G} = \ker(\lambda)$ . Here, the surjection  $\lambda$  induces an embedding of a point

$$\sharp\lambda : w = \mathbf{P}(\mathcal{O}_p) \hookrightarrow W := \mathbf{P}(2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)),$$

and from standard properties of projective spectra it follows that  $\mathbf{P}(\mathcal{G})$  is a small birational modification of  $W$ . More precisely, this modification is the composition of the blowing up  $\sigma_w$  of  $W$  at the point  $w$  and the contraction of the proper preimage of the fiber  $\pi_W^{-1}(p, z)$  under  $\sigma_w$ , where  $\pi_W : W \rightarrow \mathbb{P}^3$  is the structure morphism. In particular,  $\mathbf{P}(\mathcal{G})$  is an irreducible projective scheme of dimension  $\dim \mathbf{P}(\mathcal{G}) = 5$ . By the same reason, from the rightmost vertical triple of (129) it follows that, if  $p \notin \text{Sing}(F_2)$ , the scheme  $\mathbf{P}(E_2)$  is a small birational modification of  $\mathbf{P}(F_2)$ . Namely, this modification is the composition of the blowing up  $\sigma_p$  of  $\mathbf{P}(F_2)$  at its smooth point  $\mathbf{P}(\mathcal{O}_p)$ , and the contraction of the proper preimage of the fiber  $\pi^{-1}(p, z)$  under  $\sigma_p$ . Therefore, since by [35, Lemma 4.5]  $\mathbf{P}(F_2)$  is irreducible,  $\mathbf{P}(E_2)$  is an irreducible scheme of dimension  $\dim \mathbf{P}(E_2) = 4$ , if  $p \notin \text{Sing}(F_2)$ , i. e. when  $z \in \mathcal{T}(-1, 2, 4, 1)$ . This implies that the scheme  $\mathbf{P}(E_2)_0 := p_E^{-1}(\mathcal{T}(-1, 2, 4, 1))$  is irreducible of dimension 19, since by Theorem 9  $\mathcal{T}(-1, 2, 4, 1)$  is irreducible of dimension 15.

Next, an easy computation with the diagram (129) yields:  $(E_2 \otimes \mathcal{O}_p)^\vee \subset (\mathcal{G} \otimes \mathcal{O}_p)^\vee = \mathbf{k}^5$ , hence

$$(130) \quad \pi_E^{-1}(z, p) = \mathbb{P}((E_2 \otimes \mathcal{O}_p)^\vee) \subset \mathbb{P}((\mathcal{G} \otimes \mathcal{O}_p)^\vee) = \mathbb{P}^4.$$

Now, according to Remark 5, the middle horizontal triple in (129) globalizes to the exact  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_2)}$ -triple  $0 \rightarrow \tilde{\mathbf{L}}_2 \rightarrow \tilde{\mathbf{L}}_1 \rightarrow \tilde{\mathbf{F}}_2 \rightarrow 0$  obtained by lifting the exact triple (122) from  $\mathbb{P}^3 \times R_2$  onto  $\mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_2)$ . Similarly, the rightmost vertical, the middle vertical and the upper horizontal triples in (129) globalize, respectively, to the triple (119) for  $i = 2$ , the triple  $0 \rightarrow \mathbf{G} \rightarrow \tilde{\mathbf{L}}_1 \rightarrow j_* \mathcal{O}_{\mathbf{P}(\mathbf{F})}(1) \rightarrow 0$  and the triple

$$(131) \quad 0 \rightarrow \tilde{\mathbf{L}}_2 \rightarrow \mathbf{G} \rightarrow \mathbf{E} \rightarrow 0,$$

where  $\mathbf{G}$  is an  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathbf{F}_2)}$ -sheaf such that  $\mathbf{G}|_{\{z\} \times \mathbb{P}^3} = \mathcal{G}$ . Consider the composition

$$p_G : \mathbf{P}(\mathbf{G}) \xrightarrow{\pi_G} \mathbf{P}(\mathbf{F}) \times \mathbb{P}^3 \xrightarrow{pr_1} \mathbf{P}(\mathbf{F})$$

where  $\pi_G$  is the structure morphism of  $\mathbf{P}(\mathbf{G})$ . Note that the sheaf  $\tilde{\mathbf{L}}_2$  in the triple (131) is invertible, hence this triple shows that  $\mathbf{P}(\mathbf{E})$  is a Cartier divisor in  $\mathbf{P}(\mathbf{G})$  defined as the zero-set of some section  $0 \neq s \in H^0(\mathcal{O}_{\mathbf{P}(\mathbf{G})}(1) \otimes p_G^* \tilde{\mathbf{L}}_2^\vee)$ . On the other hand, the fibers  $p_G^{-1}(z) = \mathbf{P}(\mathcal{G})$  of

$p_G$  are irreducible projective schemes, that is  $p_G$  is a projective morphism with irreducible 5-dimensional fibers over the irreducible 15-dimensional scheme  $\mathbf{P}(\mathbf{F}_2)$ . It follows that, if  $\mathbf{P}(\mathbf{E}_2)$  is reducible, then any its irreducible component  $U$  has dimension

$$(132) \quad \dim U = \dim \mathbf{P}(\mathbf{G}) - 1 = 19.$$

Note that, for any  $z = (p, [F_2], \psi \bmod A_p) \in \mathbf{P}(\mathbf{F}_2)$ , we have  $F_2|_{\mathbb{P}^3 \setminus \{p\}} = E_2|_{\mathbb{P}^3 \setminus \{p\}}$ , hence, since  $\dim \mathbf{P}(F_2) = 4$ ,  $\mathbf{P}(E_2|_{\mathbb{P}^3 \setminus \{p\}})$  is a 4-dimensional scheme. On the other hand, by definition,

$$\mathbf{P}(E_2) = p_E^{-1}(z) = \pi_E^{-1}(\{z\} \times \mathbb{P}^3) = \mathbf{P}(E_2|_{\mathbb{P}^3 \setminus \{p\}}) \cup \pi_E^{-1}(z, p).$$

Hence, by (130), for any  $z \in \mathbf{P}(\mathbf{F}_2)$ , we have

$$(133) \quad \dim \mathbf{P}(E_2) = 4.$$

This together with (123) implies that  $\mathbf{P}(\mathbf{E}_2) \setminus \mathbf{P}(\mathbf{E}_2)_0$  has codimension 2 in  $\mathbf{P}(\mathbf{E}_2)$ . Therefore, by (132),  $\mathbf{P}(\mathbf{E}_2)$  is irreducible and contains  $\mathbf{P}(\mathbf{E}_2)_0$  as a dense open subset.

Finally, remark that, by the description given in display (128), the set  $\mathcal{T}(-1, 2, 4, 2)$  is a nonempty open subset of  $\mathbf{P}(\mathbf{E}_2)_0$ , hence it is dense in  $\mathbf{P}(\mathbf{E}_2)$ .  $\square$

## 7. DESCRIPTION OF FAMILIES WITH MIXED SINGULARITIES

We now proceed to the description of the sets  $X(-1, 1, 1, -1, 1)$  and  $X(-1, 1, 1, 0, 1)$ . Our aim is to construct explicitly certain open dense subsets of them, together with a universal family of sheaves over these subsets, which will be used in our further results. We start with the following lemma.

**Lemma 20.** *Let  $G := G(2, 4)$  be the Grassmannian of lines in  $\mathbb{P}^3$  and  $M = \mathcal{M}(-1, 1, 1) \simeq \mathbb{P}^3$  (see Remark 2). Consider  $[F] \in M$ , and let  $\mathcal{E}$  and  $E$  be the sheaves on  $\mathbb{P}^3$  fitting in the exact triples*

$$(134) \quad 0 \rightarrow \mathcal{E} \rightarrow F \xrightarrow{\varepsilon} \mathcal{O}_l(-1) \rightarrow 0,$$

$$(135) \quad 0 \rightarrow E \rightarrow \mathcal{E} \xrightarrow{\gamma} \mathcal{O}_p \rightarrow 0,$$

for some line  $l \in G$  and some point  $p \in \mathbb{P}^3$ . Then  $\mathbf{P}(\mathcal{E})$  and  $\mathbf{P}(E)$  are irreducible generically smooth schemes of dimension 4.

*Proof.* We first show that  $\mathcal{E}$  fits in the exact triple:

$$(136) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\varepsilon} \mathcal{E} \rightarrow 0.$$

Let  $x_0 := \text{Sing}(F)$  (see Remark 2). Consider the two possible cases (a)  $x_0 \in l$  and (b)  $x_0 \notin l$ . Case (a):  $x_0 \in l$ . Note that from the definition of the sheaf  $F$  it follows easily that  $F$  fits in the exact triple  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow F \xrightarrow{\delta} \mathcal{I}_{l, \mathbb{P}^3} \rightarrow 0$ . Since  $\mathcal{I}_{l, \mathbb{P}^3} \otimes \mathcal{O}_l = N_{l/\mathbb{P}^3}^\vee \simeq 2 \cdot \mathcal{O}_l(-1)$ , it follows that there exists an epimorphism  $\beta : \mathcal{I}_{l, \mathbb{P}^3} \twoheadrightarrow \mathcal{O}_l(-1)$  such that  $\beta \circ \delta = \varepsilon$ , where  $\varepsilon$  is the epimorphism in (134). Besides,  $\ker \beta \simeq \mathcal{I}_{C, \mathbb{P}^3}$ , where  $C$  is a nonreduced conic supported on  $l$ , and we obtain exact triples

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{C, \mathbb{P}^3} \rightarrow 0, \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{I}_{C, \mathbb{P}^3} \rightarrow 0.$$

These two triples yield the resolution (136) by push-out, since  $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2), \mathcal{O}_{\mathbb{P}^3}(-1)) = 0$ .

Case(b):  $x_0 \notin l$ . Note that, by Remark (2),  $F$  fits in the exact triple  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} F \rightarrow 0$ . Since clearly  $\ker(\varepsilon \circ \alpha : 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \twoheadrightarrow \mathcal{O}_l(-1)) \cong 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{I}_{l, \mathbb{P}^3}(-1)$ , the last triple together with the triple (134) yields the exact triple

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{i} 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{I}_{l, \mathbb{P}^3}(-1) \rightarrow \mathcal{E} \rightarrow 0.$$

Let  $c : 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{I}_{l, \mathbb{P}^3}(-1) \rightarrow \mathcal{I}_{l, \mathbb{P}^3}(-1)$  be the canonical epimorphism and consider the composition  $c \circ i : \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{I}_{l, \mathbb{P}^3}(-1)$ . If this composition is the zero map, then  $\text{im}(i) \subset 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1)$  and  $\text{coker}(i) \subset \mathcal{E}$ . Since  $\mathcal{E}$  is torsion free, it follows that  $\text{coker}(i) = \mathcal{I}_{m, \mathbb{P}^3}$  for some line  $m$  distinct from  $l$ , and  $\mathcal{E}$  fits in the exact triple  $0 \rightarrow \mathcal{I}_{m, \mathbb{P}^3} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{l, \mathbb{P}^3}(-1) \rightarrow 0$ . This triple implies that  $m \subset \text{Sing}(\mathcal{E})$ , contrary to the evident equality  $\text{Sing}(\mathcal{E}) = x_0 \sqcup l$ . Hence the composition  $c \circ i$  is a nonzero morphism, so that  $\text{coker}(c \circ i) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$  for some projective plane  $\mathbb{P}^2$  in  $\mathbb{P}^3$ . We thus obtain an exact triple  $0 \rightarrow 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow 0$ . This triple and the exact triple  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-3) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow 0$  by push-out yield (136), since  $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-2), 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1)) = 0$ .

Now from (136) it follows that  $\mathbf{P}(\mathcal{E})$  is a Cartier divisor in  $W := \mathbf{P}(\mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1))$ , and the same argument as in the proof of Theorem 19 shows that  $\mathbf{P}(\mathcal{E})$  is irreducible. Next, the triples (135) and (136) yield exact triples

$$0 \rightarrow \cdot \mathcal{O}_{\mathbb{P}^3}(-3) \rightarrow \mathcal{G} \rightarrow E \rightarrow 0, \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\gamma \circ \epsilon} \mathcal{O}_p \rightarrow 0.$$

The second triple here shows that  $\mathcal{G}$  is irreducible as a small birational modification of the scheme  $W$  defined above, hence it is irreducible. On the other hand, the first triple shows that  $\mathbf{P}(E)$  is a Cartier divisor in  $\mathcal{G}$ , and again the same argument as in the proof of Theorem 19 yields the irreducibility of  $\mathbf{P}(E)$ .  $\square$

Now, let and  $\Gamma = \{(x, l) \in \mathbb{P}^3 \times G \mid x \in l\}$  the graph of incidence, and  $\mathcal{O}_{\mathbb{P}^3 \times M}$ -sheaf (see Remark 2). for  $l \in G$  denote  $A_l := \text{Aut}(\mathcal{O}_l(-1))$ ,  $A'_l := \text{Aut}(\mathcal{O}_l)$ ,  $A_l \simeq \mathbf{k}^* \simeq A'_l$ . Define the sets

$$(137) \quad B := \{(l, [F], \epsilon \bmod A_l) \mid (l, [F]) \in G \times M, \epsilon : F \rightarrow \mathcal{O}_l(-1) \text{ is an epimorphism}\}.$$

$$(138) \quad B' := \{(l, [F], \epsilon' \bmod A'_l) \mid (l, [F]) \in G \times M, \epsilon' : F \rightarrow \mathcal{O}_l \text{ is an epimorphism}\}.$$

We have the following proposition.

**Proposition 21.** *The following claims are true.*

- (i)  $B$ , respectively,  $B'$  is the set of closed points of an irreducible scheme of dimension 7, respectively, of dimension 9.
- (ii) There is an  $\mathcal{O}_{\mathbb{P}^3 \times B}$ -sheaf  $\mathcal{E}$  and an invertible  $\mathcal{O}_\Gamma$ -sheaf  $\mathbf{L}$  fitting in the exact triple  $0 \rightarrow \mathcal{E} \rightarrow \mathbf{F}_B \xrightarrow{\epsilon} \mathbf{L} \rightarrow 0$ , where  $\mathbf{F}_B = \mathbf{F} \otimes_{\mathcal{O}_M} \mathcal{O}_B$  and  $\Gamma = \Gamma \times_M B$ . Respectively, there is an  $\mathcal{O}_{\mathbb{P}^3 \times B'}$ -sheaf  $\mathcal{E}'$  and an invertible  $\mathcal{O}_{\Gamma'}$ -sheaf  $\mathbf{L}'$  fitting in the exact triple  $0 \rightarrow \mathcal{E}' \rightarrow \mathbf{F}_{B'} \xrightarrow{\epsilon'} \mathbf{L}' \rightarrow 0$ , where  $\mathbf{F}_{B'} = \mathbf{F} \otimes_{\mathcal{O}_M} \mathcal{O}_{B'}$  and  $\Gamma' = \Gamma \times_M B'$ . These triples, being restricted onto  $\mathbb{P}^3 \times \{b\}$ , respectively, onto  $\mathbb{P}^3 \times \{b'\}$  for any points  $b = (l, [F], \epsilon \bmod A_l) \in B$ ,  $b' = (l, [F], \epsilon' \bmod A'_l) \in B'$ , yield:

$$(139) \quad 0 \rightarrow \mathcal{E}_b \xrightarrow{\iota} F \xrightarrow{\epsilon} \mathcal{O}_l(-1) \rightarrow 0, \quad \mathcal{E}_b \cong \mathcal{E}|_{\mathbb{P}^3 \times \{b\}}.$$

$$(140) \quad 0 \rightarrow \mathcal{E}'_{b'} \xrightarrow{\iota'} F \xrightarrow{\epsilon'} \mathcal{O}_l \rightarrow 0, \quad \mathcal{E}'_{b'} \cong \mathcal{E}'|_{\mathbb{P}^3 \times \{b'\}}.$$

- (iii)  $\mathbf{P}(\mathcal{E})$ , respectively,  $\mathbf{P}(\mathcal{E}')$  is an irreducible generically smooth scheme of dimension 11, respectively, of dimension 13.

*Proof.* It is enough to argue fiberwise over  $M$ , i.e. for a fixed sheaf  $[F] \in M$ . Let  $y = \text{Sing}(F)$  and consider the sets  $B_y = B \times_M \{y\}$  and  $B'_y = B' \times_M \{y\}$ . Any points  $b = (l, [F], \epsilon) \in B_y$  and  $b' = (l, [F], \epsilon') \in B'_y$  define the exact triples (139) and (140) with  $\mathcal{E}_b = \ker(\epsilon)$  and  $\mathcal{E}'_{b'} = \ker(\epsilon')$ ,

respectively. These triples, together with the exact triple  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\delta} F \rightarrow 0$  from Remark 2, yield commutative diagrams with  $\mathcal{G} = \ker(\epsilon \circ \delta)$  and  $\mathcal{G}' = \ker(\epsilon' \circ \delta)$ , respectively:

$$(141) \quad \begin{array}{ccc} \mathcal{O}_{\mathbb{P}^3}(-2) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E}_b \\ \parallel & & \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{P}^3}(-2)[r] & \longrightarrow & 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\delta} & F \\ & & \downarrow \epsilon \circ \delta & & \downarrow \epsilon \\ & & \mathcal{O}_l(-1) & \xlongequal{\quad} & \mathcal{O}_l(-1), \end{array} \quad \begin{array}{ccc} \mathcal{O}_{\mathbb{P}^3}(-2) & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{E}'_{b'} \\ \parallel & & \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{P}^3}(-2)[r] & \longrightarrow & 3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\delta} & F \\ & & \downarrow \epsilon' \circ \delta & & \downarrow \epsilon' \\ & & \mathcal{O}_l & \xlongequal{\quad} & \mathcal{O}_l. \end{array}$$

Consider the scheme  $\Pi := \mathbf{P}(3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1)) \cong \mathbb{P}^3 \times \mathbb{P}^2$ . To the epimorphism  $\epsilon \circ \delta$  in the left diagram (141) there corresponds an injective morphism  $i : \mathbf{P}(\mathcal{O}_l(-1)) \hookrightarrow \Pi$  which defines a point  $x \in \mathbb{P}^2$  such that  $\text{im}(i) = l_x := \{x\} \times l$ . Respectively, to the epimorphism  $\epsilon' \circ \delta$  in the right diagram (141) there corresponds an injective morphism  $i' : \mathbf{P}(\mathcal{O}_l) \hookrightarrow \Pi$  which defines a point  $x' \in (\mathbb{P}_l)_e$ , where  $(\mathbb{P}_l)_e$  is the set of epimorphisms  $3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1) \twoheadrightarrow \mathcal{O}_l \bmod A'_l$  considered as a dense open subset of the projective 5-space  $\mathbb{P}(\text{Hom}(3 \cdot \mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{O}_l))$ . For this point  $x'$ , we denote  $l_{x'} := \text{im}(i')$ . Besides, to the epimorphism  $\delta$  in both diagrams there corresponds an injective morphism  $i_\delta : \mathbf{P}(F) \hookrightarrow \Pi$ . From now on we will identify  $\mathbf{P}(F)$  with its image under  $i_\delta$ . Now by (141) the condition  $b \in B_y$  and the condition  $b' \in B'_y$ , yield the inclusions

$$(142) \quad l_x \subset \mathbf{P}(F), \quad \text{respectively,} \quad l_{x'} \subset \mathbf{P}(F).$$

Next, by the middle horizontal triple in diagrams (141),  $\mathbf{P}(F)$  is a Cartier divisor on  $\Pi$  such that  $\mathcal{O}_\Pi(\mathbf{P}(F)) \cong \mathcal{O}_{\mathbb{P}^3}(2) \boxtimes \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_\Pi(1) \cong \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$ . Hence

$$(143) \quad \mathbf{P}(F) = (s)_0, \quad 0 \neq s \in H^0(\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)),$$

and the conditions (142) mean that  $s|_{l_x} = 0$ , respectively,  $s|_{l_{x'}} = 0$ .

Consider the first of these conditions  $s|_{l_x} = 0$ . Let  $\Pi = \mathbb{P}^3 \times \mathbb{P}^2 \xleftarrow{p} \Gamma \times \mathbb{P}^2 \xrightarrow{q} G \times \mathbb{P}^2$  be the projections. Then by construction the sheaf  $q_* p^*(\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1))$  is isomorphic to the sheaf  $\mathcal{A} = \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{Q}$ , where  $\mathcal{Q}$  is the universal quotient rank 2 bundle on  $G$ . In addition, under the natural isomorphism of spaces of sections  $H^0(\mathcal{O}_{\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbb{P}^2}(1)) \cong H^0(\mathcal{A})$ , the section  $s$  from (143) corresponds to the section  $\tilde{s} \in H^0(\mathcal{A})$ . The above condition  $s|_{l_x} = 0$  then means that the section  $\tilde{s}$  vanishes at the point  $(l, x) \in G \times \mathbb{P}^2$ . On the other hand, by the universal property of  $\mathbf{P}(F)$  (see [11, Ch. II, Prop. 7.12]) it follows that to give an epimorphism  $\epsilon : F \twoheadrightarrow \mathcal{O}_l(-1) \bmod A_l$  is equivalent to give an embedding  $l_x \hookrightarrow \mathbf{P}(F)$  in (142). This together with the condition  $(l, x) \in (\tilde{s})_0$  yields a natural isomorphism of schemes

$$(144) \quad B_y \simeq (\tilde{s})_0.$$

Under this isomorphism the fiber of the projection  $B_y \simeq (\tilde{s})_0 \rightarrow G$ ,  $(l, x) \mapsto l$  is naturally identified with  $\mathbb{P}(\text{Hom}(F, \mathcal{O}_l(-1)))$ . By (15) this projective space is a point if  $l \notin Z_y := \{l \in G \mid y \in l\}$ , respectively, is  $\mathbb{P}^1$  if  $l \in Z_y$ . This together with the universal property of blowing ups [11, Ch. II, Prop. 7.14] implies that  $B_y$  is isomorphic to the blow-up of  $G$  along the smooth center  $Z_y \simeq \mathbb{P}^2$ . In particular,  $B_y$  is irreducible, of dimension 4. Hence  $B$  is irreducible of dimension 7.

Now proceed to the second condition  $s|_{l_{x'}} = 0$ . For this, consider the scheme  $G' = \{(l, x') \mid l \in G, x' \in (\mathbb{P}_l)_e\}$  with the projection  $\psi : G' \rightarrow G$ ,  $(l, x') \mapsto l$ , and the graph of incidence  $\Gamma' = \{(z, l, x') \in \Pi \times G' \mid z \in l_{x'}\}$  with the projections  $\Pi \xleftarrow{p'} \Gamma' \xrightarrow{q'} G'$ . One checks that  $\mathcal{O}_\Pi(\mathbf{P}(F))|_{l_x} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ . This implies that, applying the functor  $q'_* p'^*$  to the section  $s$  from



(143) we obtain the section  $\tilde{s}' \in H^0(\psi^* S^2 Q \otimes \mathcal{D})$  for some invertible  $\mathcal{O}_{G'}$ -sheaf  $\mathcal{D}$  such that the condition  $s|_{l_{x'}} = 0$  is equivalent to the condition  $(l, x') \in (\tilde{s}')_0$ . This similarly to (144) yields  $B'_y \simeq (\tilde{s}')_0$ .

Under this isomorphism the fiber of the projection  $\psi|_{B'_y} : B'_y \simeq (\tilde{s}')_0 \rightarrow G$ ,  $(l, x') \mapsto l$  is naturally identified with  $\mathbb{P}(\text{Hom}(F, \mathcal{O}_l))$ . By (15) this projective space is  $\mathbb{P}^2$  if  $l \notin Z_y$ , respectively, is  $\mathbb{P}^3$  if  $l \in Z_y$ . This implies that  $\tilde{s}'$  as a section of a rank 3 vector bundle is regular, and its zero locus  $B'_y$  is irreducible. Hence  $B'$  is irreducible of dimension 9. We thus have proved the statement (i) of Lemma.

The statement (ii) is clear. To prove the statement (iii), it is also enough to argue fiberwise over  $M$ . For the above point  $b \in B_y$ , we have to prove the irreducibility and generic smoothness of the scheme  $\mathbf{P}(\mathcal{E}_b)$ . This is just the statement of Lemma (20) in which we set  $\mathcal{E} = \mathcal{E}_b$ . The irreducibility and generic smoothness of  $\mathbf{P}(\mathcal{E}'_{b'})$  for  $b' \in B'_y$  is completely similar.  $\square$

Let  $\rho : \mathbf{P}(\mathcal{E}) \rightarrow \mathbb{P}^3 \times B$  be the structure morphism, and consider the compositions  $\theta = pr_1 \circ \rho : \mathbf{P}(\mathcal{E}) \rightarrow \mathbb{P}^3$  and  $\tau = pr_2 \circ \rho : \mathbf{P}(\mathcal{E}) \rightarrow B$ . Set  $\tilde{\mathcal{E}} := (\text{id}_{\mathbb{P}^3} \times \tau)^* \mathcal{E} = \mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{O}_{\mathbf{P}(\mathcal{E})}$  and consider the "diagonal" embedding  $j : \mathbf{P}(\mathcal{E}) \hookrightarrow \mathbb{P}^3 \times \mathbf{P}(\mathcal{E})$ ,  $z \mapsto (\theta(z), z)$ . By construction,  $j^* \tilde{\mathcal{E}} = \rho^* \mathcal{E}$  and we obtain the composition of surjections  $\mathbf{e} : \tilde{\mathcal{E}} \rightarrow j_* j^* \tilde{\mathcal{E}} = j_* \rho^* \mathcal{E} \rightarrow j_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$  which yields an exact  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathcal{E})}$ -triple, where  $\mathbf{E} := \ker \mathbf{e}$ :

$$(145) \quad 0 \rightarrow \mathbf{E} \rightarrow \tilde{\mathcal{E}} \xrightarrow{\mathbf{e}} j_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \rightarrow 0.$$

In a similar way we define the morphisms  $\rho' : \mathbf{P}(\mathcal{E}') \rightarrow \mathbb{P}^3 \times B'$ ,  $\theta' = pr_1 \circ \rho' : \mathbf{P}(\mathcal{E}') \rightarrow \mathbb{P}^3$ ,  $j' : \mathbf{P}(\mathcal{E}') \hookrightarrow \mathbb{P}^3 \times \mathbf{P}(\mathcal{E}')$ ,  $z \mapsto (\theta'(z), z)$ , the sheaf  $\tilde{\mathcal{E}}' := \mathcal{E}' \otimes_{\mathcal{O}_{B'}} \mathcal{O}_{\mathbf{P}(\mathcal{E}')}$ , and the surjection  $\mathbf{e}' : \tilde{\mathcal{E}}' \rightarrow j'_* \mathcal{O}_{\mathbf{P}(\mathcal{E}')} (1)$  which yields an exact  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathcal{E}')}$ -triple, where  $\mathbf{E}' := \ker \mathbf{e}'$ :

$$(146) \quad 0 \rightarrow \mathbf{E}' \rightarrow \tilde{\mathcal{E}}' \xrightarrow{\mathbf{e}'} j'_* \mathcal{O}_{\mathbf{P}(\mathcal{E}')} (1) \rightarrow 0.$$

Below we will also consider extensions of  $\mathcal{O}_{\mathbb{P}^3}$ -sheaves of the form

$$(147) \quad 0 \rightarrow \mathcal{O}_q \rightarrow Q \xrightarrow{\gamma} i_* \mathcal{O}_l(-1) \rightarrow 0,$$

$$(148) \quad 0 \rightarrow \mathcal{O}_q \rightarrow Q \xrightarrow{\gamma} i_* \mathcal{O}_l \rightarrow 0,$$

where  $(q, l) \in \mathbb{P}^3 \times G$  and  $i : l \hookrightarrow \mathbb{P}^3$  is the embedding. Below we also set  $A_Q := \text{Aut}(Q)$  for  $Q$  in (147) and (148).

**Proposition 22.** *The following are true.*

(i) *There are isomorphisms of schemes  $\Phi : \mathbf{P}(\mathcal{E}) \xrightarrow{\simeq} X$  and  $\Phi' : \mathbf{P}(\mathcal{E}') \xrightarrow{\simeq} X'$ , where*

$$(149) \quad X = \{([F], Q, \delta \bmod A_Q) \mid [F] \in M, Q \text{ fits in (147), } \delta : F \rightarrow Q \text{ is surjective}\}.$$

$$(150) \quad X' = \{([F], Q, \delta \bmod A_Q) \mid [F] \in M, Q \text{ fits in (148), } \delta : F \rightarrow Q \text{ is surjective}\}.$$

(ii) *There are inclusions of dense open subschemes*

$$\mathcal{X}(-1, 1, 1, -1, 1) \hookrightarrow \mathbf{P}(\mathcal{E}) \quad \text{and} \quad \mathcal{X}(-1, 1, 1, , 1) \hookrightarrow \mathbf{P}(\mathcal{E}').$$

*The modular morphisms*

$$f : \mathbf{P}(\mathcal{E}) \rightarrow \mathcal{M}(-1, 2, 2), \quad z \mapsto [\mathbf{E}|_{\mathbb{P}^3 \times \{z}}] \quad \text{and} \quad f' : \mathbf{P}(\mathcal{E}') \rightarrow \mathcal{M}(-1, 2, 0), \quad z \mapsto [\mathbf{E}'|_{\mathbb{P}^3 \times \{z}}]$$

*are injective, and the closures of their images are  $X(-1, 1, 1, -1, 1)$  and  $X(-1, 1, 1, 0, 1)$ , respectively.*

*Proof.* (i) It is enough to consider  $\mathbf{P}(\mathcal{E})$ , since the argument with  $\mathbf{P}(\mathcal{E}')$  is similar. For any point  $z \in \mathbf{P}(\mathcal{E})$  let  $(q, b) = \rho(z)$ . By definition the triple (145) restricted onto  $\mathbb{P}^3 \times \{z\}$  is the triple

$$(151) \quad 0 \rightarrow E_z \rightarrow \mathcal{E}_b \xrightarrow{e_z} \mathcal{O}_q \rightarrow 0.$$

On the other hand, by Proposition 21.(ii),  $b = (l, [F], \epsilon \bmod A_l)$  and  $\mathcal{E}_b$  fits in the triple (139) in which  $F = \mathcal{E}_b^{\vee\vee}$ ,  $\iota : \mathcal{E}_b \rightarrow \mathcal{E}_b^{\vee\vee}$  is the canonical morphism and  $\epsilon : \mathcal{E}_b^{\vee\vee} \rightarrow \mathcal{O}_l(-1)$  is the quotient morphism. Since  $\mathcal{E}_b^{\vee\vee} = E_z^{\vee\vee}$ , the composition  $\tau : E_z \rightarrow \mathcal{E}_b \xrightarrow{\iota} E_z^{\vee\vee}$  is the canonical morphism of the sheaf  $E_z$  into its reflexive hull, and  $Q := \text{coker}(\iota)$  fits in the triple (147). We thus have an exact triple

$$(152) \quad 0 \rightarrow E_z \xrightarrow{\tau} E_z^{\vee\vee} \xrightarrow{\delta} Q \rightarrow 0,$$

where  $\delta$  is the quotient morphism. This defines a morphism

$$\Phi : \mathbf{P}(\mathcal{E}) \xrightarrow{\sim} X, \quad z \mapsto ([E_z^{\vee\vee}], Q = E_z^{\vee\vee}/E_z, \delta \bmod A_Q).$$

To construct the inverse morphism  $\Phi^{-1}$ , take a point  $x = ([F], Q, \delta \bmod A_Q)$  and set  $\mathcal{E} = \ker(\gamma \circ \delta : F \rightarrow \mathcal{O}_l(-1))$ , where  $\gamma$  in (147) is the morphism of factorization of  $Q$  by its maximal artinian subsheaf  $\mathcal{O}_q$ . We thus obtain the induced epimorphism  $e : \mathcal{E} \rightarrow \ker(\gamma) = \mathcal{O}_q$ , hence a point  $[e] = e \bmod A_q \in \mathbf{P}(\mathcal{E})$ . This yields the desired morphism  $\Phi^{-1} : X \xrightarrow{\sim} \mathbf{P}(\mathcal{E})$ ,  $x \mapsto [e]$ .

(ii) The injectivity of the modular morphism  $f$  is clear from the above. In addition, under the description (149), the scheme  $\mathcal{X}(-1, 1, 1, -1, 1)$  is the set of those points  $z = ([F], Q, \delta \bmod A_Q) \in \mathbf{P}(\mathcal{E})$ , with  $(q, l) = \rho(z)$ , for which  $q \notin l$ ,  $q \neq \text{Sing}(F) \notin l$ . This is clearly a nonempty open subset of the scheme  $\mathbf{P}(\mathcal{E})$  which is dense since  $\mathbf{P}(\mathcal{E})$  is irreducible by Proposition 21.(ii).  $\square$

Consider the sheaf  $\mathbf{E}$  defined in (145). Let  $\mathbf{r} : \mathbf{P}(\mathbf{E}) \rightarrow \mathbb{P}^3 \times \mathbf{P}(\mathcal{E})$  be the structure morphism, and consider the composition  $\mathbf{t} = pr_1 \circ \mathbf{r} : \mathbf{P}(\mathbf{E}) \rightarrow \mathbb{P}^3$  and the "diagonal" embedding  $\mathbf{j} : \mathbf{P}(\mathbf{E}) \hookrightarrow \mathbb{P}^3 \times \mathbf{P}(\mathcal{E})$ ,  $w \mapsto (\mathbf{t}(w), w)$ . Set  $\mathbf{P}(\hat{\mathbf{E}}) := \mathbf{E} \otimes_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}} \mathcal{O}_{\mathbf{P}(\mathbf{E})}$ . By construction,  $\mathbf{j}^* \hat{\mathbf{E}} = \mathbf{r}^* \mathbf{E}$  and we obtain the composition of surjections  $\tilde{\mathbf{e}} : \hat{\mathbf{E}} \twoheadrightarrow \mathbf{j}_* \mathbf{j}^* \hat{\mathbf{E}} = \mathbf{j}_* \mathbf{r}^* \mathbf{E} \twoheadrightarrow \mathbf{j}_* \mathcal{O}_{\mathbf{P}(\mathbf{E})}(1)$  which yields an exact  $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{P}(\mathbf{E})}$ -triple, where  $\hat{\mathbf{E}} := \ker \tilde{\mathbf{e}}$ :

$$(153) \quad 0 \rightarrow \hat{\mathbf{E}} \rightarrow \tilde{\mathbf{E}} \xrightarrow{\tilde{\mathbf{e}}} \mathbf{j}_* \mathcal{O}_{\mathbf{P}(\mathbf{E})}(1) \rightarrow 0.$$

We will also consider the exact triples of the form

$$(154) \quad 0 \rightarrow Z \rightarrow Q \rightarrow i_* \mathcal{O}_l(-1) \rightarrow 0, \quad \dim Z = 0, \quad \text{length}(Z) = 2,$$

where  $i : l \hookrightarrow \mathbb{P}^3$  is the embedding of a line  $l \in G$ .

**Proposition 23.** *The sheaf  $\hat{\mathbf{E}}$  defined in (153) determines the modular morphism*

$$\hat{\Phi} : \mathbf{P}(\mathbf{E}) \rightarrow \mathcal{M}(-1, 2, 0), \quad w \mapsto \left[ \hat{\mathbf{E}}|_{\mathbb{P}^3 \times \{w\}} \right],$$

and the closure  $\overline{\hat{\Phi}(\mathbf{P}(\mathbf{E}))}$  of its image in  $\mathcal{M}(-1, 2, 0)$  coincides with the scheme  $\mathbf{X}(-1, 1, 1, -1, 2)$ . In particular,  $\mathbf{X}(-1, 1, 1, -1, 2)$  contains all the points  $[E]$  such that  $Q = E^{\vee\vee}/E$  fits in the triple of the form (154).

*Proof.* First note that, by the definition of the sheaf  $\mathbf{E}$ , the scheme  $\mathbf{P}(\mathbf{E})$  is fibered over the scheme  $\mathbf{P}(\mathcal{E})$  with fibers of the form  $\mathbf{P}(E)$  described in Lemma 20. Hence by that Lemma, these fibers are irreducible of dimension 4. Besides, the scheme  $\mathbf{P}(\mathcal{E})$  is also irreducible of dimension 11 by Proposition 21.(iii). Hence the scheme  $\mathbf{P}(\mathbf{E})$  is irreducible of dimension 15. The fact that it contains the scheme  $\mathcal{X}(-1, 1, 1, -1, 2)$  as a dense open subset is proved in the same way as the statement of item (ii) of Proposition 22, based on the universal property of  $\mathbf{P}(\mathbf{E})$  from [11, Ch. II, Prop. 7.12].  $\square$

**Proposition 24.** *The following claims hold.*

- (i) *The scheme  $X(-1, 1, 1, -1, 1)$  is contained in  $T(-1, 2, 4, 1)$ .*
- (ii) *The scheme  $X(-1, 1, 1, -1, 2)$  is contained in  $T(-1, 2, 4, 2)$ .*
- (iii) *The scheme  $X(-1, 1, 1, 0, 1)$  is contained in  $T(-1, 2, 2, 1)$ .*

*Proof.* (i) We will construct a flat family  $\mathbb{E} = \{E_t\}_{t \in \mathbb{P}^1}$  of sheaves from  $T(-1, 2, 4, 1)$  such that, for a certain point  $t_0 \in \mathbb{P}^1$ ,  $E_{t_0}$  is a smooth point of  $\mathcal{M}(-1, 2, 2)$  lying in  $X(-1, 1, 1, -1, 1) \cap T(-1, 2, 4, 1)$ . From this the statement (i) will follow. Fix a plane  $\mathbb{P}^2$  in  $\mathbb{P}^3$  and choose a pencil of conics in  $\mathbb{P}^2$  considered as a divisor  $D$  in  $\mathbb{P}^2 \times \mathbb{P}^1$ , with the projection  $D \xrightarrow{p} \mathbb{P}^1$ , and, for  $t \in \mathbb{P}^1$ , denote  $C_t = p^{-1}(t)$ . We choose the pencil  $D$  in such a way that, for two distinct marked points  $t_0, t_1 \in \mathbb{P}^1$ ,  $C_{t_0} = l_1 \cup l_2$  is a union of two distinct lines and  $C_{t_1}$  is a smooth conic intersecting  $C_{t_0}$  at 4 distinct points. Fix a point  $q \in \mathbb{P}^3 \setminus \mathbb{P}^2$ , and on  $\Sigma = \mathbb{P}^3 \times \mathbb{P}^1$  consider the line  $L = \{q\} \times \mathbb{P}^1$  and the extension of sheaves, where  $\mathcal{A} = \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)$ :

$$(155) \quad 0 \rightarrow \mathcal{A} \otimes \mathcal{I}_{L, \Sigma} \rightarrow \mathbb{E} \rightarrow \mathcal{I}_{D, \Sigma} \rightarrow 0.$$

The extension group corresponding to (155) is  $V := \text{Ext}^1(\mathcal{I}_{D, \Sigma}, \mathcal{A} \otimes \mathcal{I}_{L, \Sigma}) \simeq \text{Ext}^1(\mathcal{I}_{D, \Sigma}, \mathcal{A}) \simeq \text{H}^0(\mathcal{E}xt^1(\mathcal{I}_{D, \Sigma}, \mathcal{A})) \simeq \text{H}^0(\mathcal{E}xt^2(\mathcal{O}_D, \mathcal{A})) \simeq \text{H}^0(\mathcal{O}_{\mathbb{P}^3}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}|_D) \simeq \text{H}^0(\mathcal{O}_{\mathbb{P}^2}(2))$ . Thus the element of  $V$  defining the extension (155) is understood as a section  $0 \neq s \in \text{H}^0(\mathcal{O}_{\mathbb{P}^2}(2))$ . Now pick the section  $s$  such that it vanishes on the line  $l_1$  and doesn't vanish on the line  $l_2$ ; hence it also doesn't vanish on the conic  $C_{t_1}$ . Then by the Serre construction we obtain the following properties of sheaves  $E_t = \mathbb{E}|_{\mathbb{P}^3 \times \{t\}}$ .

(i.a) Under the generic choice of the conic  $C_{t_1}$ , for generic  $t \in \mathbb{P}^1$ , the sheaf  $[E_t]$  is a generic sheaf from  $T(-1, 2, 4, 1)$ . In other words,  $\mathbb{P}^1 \subset T(-1, 2, 4, 1)$ . In particular,  $[E_{t_0}] \in T(-1, 2, 4, 1)$ .

(i.b) The sheaf  $E_{t_0}$  fits in the exact triple  $0 \rightarrow E_{t_0} \xrightarrow{\text{can}} E_{t_0}^{\vee\vee} \rightarrow \mathcal{O}_l(-1) \oplus \mathcal{O}_q \rightarrow 0$ , where  $[E_{t_0}^{\vee\vee}] \in M$  and, by the construction of  $E_{t_0}$ ,  $q \notin \text{Sing}(E_{t_0}^{\vee\vee}) \cup l$ . This means that  $[E_{t_0}] \in \mathcal{X}(-1, 1, 1, -1, 1)$ , and Theorem 10.(iii) implies that  $\dim \text{Ext}^1(E_{t_0}, E_{t_0}) = 15$ . Hence, since  $\dim T(-1, 2, 4, 1) = 15$  (see Theorem 9), it follows that  $E_{t_0}$  is a smooth point of  $T(-1, 2, 4, 1)$  and of  $\mathcal{M}(-1, 2, 2)$  as well.

(ii) This is completely similar to the statement (i) above. The only difference is that, instead of fixing a point  $q \in \mathbb{P}^3 \setminus \mathbb{P}^2$ , we fix two distinct points  $q_1, q_2 \in \mathbb{P}^3 \setminus \mathbb{P}^2$ , and on  $\Sigma = \mathbb{P}^3 \times \mathbb{P}^1$  consider the two corresponding lines  $L_i = \{q_i\} \times \mathbb{P}^1$  and the extension of sheaves similar to (155):  $0 \rightarrow \mathcal{A} \otimes \mathcal{I}_{L_1 \sqcup L_2, \Sigma} \rightarrow \mathbb{E} \rightarrow \mathcal{I}_{D, \Sigma} \rightarrow 0$ . Respectively, for  $V$  we take the group  $\text{Ext}^1(\mathcal{I}_{D, \Sigma}, \mathcal{A} \otimes \mathcal{I}_{L_1 \sqcup L_2, \Sigma}) \simeq \text{H}^0(\mathcal{O}_{\mathbb{P}^2}(2))$ . The rest of the argument is literally the same as in (i).

(iii) Similar to the above we construct a flat family  $\mathbb{E} = \{E_t\}_{t \in \mathbb{A}^1}$  of sheaves from  $T(-1, 2, 2, 1)$  such that, for a certain point  $t_0 \in \mathbb{A}^1$ ,  $E_{t_0}$  is a smooth point of  $\mathcal{M}(-1, 2, 0)$  lying in  $X(-1, 1, 1, 0, 1) \cap T(-1, 2, 2, 1)$ . From this the statement (ii) will follow. We will use the description of sheaves from  $\mathcal{M}(-1, 2, 2)$  given in [6, Lemma 2.4]. Thus, instead of the above family of conics  $D = \{C_t\}_{t \in \mathbb{P}^1}$  we take for  $C_t$ ,  $t \in \mathbb{A}^1$ , a fixed union  $Y = l_1 \sqcup l_2$  of two disjoint lines in  $\mathbb{P}^3$ , fix a point  $q \in \mathbb{P}^3 \setminus Y$ , set  $D = Y \times \mathbb{A}^1$ ,  $L = \{q\} \times \mathbb{A}^1$ . For these data consider the extension (155), where we set  $\mathcal{A} = \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbb{A}^1}$ , and then d the extension group  $V = \text{Ext}^1(\mathcal{I}_{D, \Sigma}, \mathcal{A} \otimes \mathcal{I}_{L, \Sigma})$  as above. One easily see that  $V \cong \text{H}^0(\mathcal{O}_{l_1}) \oplus \text{H}^0(\mathcal{O}_{l_2})$ . For  $i = 1, 2$  pick a nonzero vector  $v_i \in \text{H}^0(\mathcal{O}_{l_i})$  and identify the base  $\mathbb{A}^1$  of the family  $\{C_t\}$ ,  $t \in \mathbb{A}^1$ , with the subset  $\{(v_1, tv_2) | t \in \mathbf{k}\}$ . By the Serre construction we obtain the following properties of sheaves  $E_t = \mathbb{E}|_{\mathbb{P}^3 \times \{t\}}$ .

(a) For  $t \in \mathbb{A}^1 \setminus \{0\}$ , the sheaf  $[E_t]$  by definition belongs to  $\mathcal{T}(-1, 2, 2, 1)$ . It it follows that  $\mathbb{A}^1 \subset T(-1, 2, 2, 1)$ . In particular,  $[E_0] \in T(-1, 2, 2, 1)$ .

(b) The sheaf  $E_0$  fits in the exact triple  $0 \rightarrow E_0 \xrightarrow{\text{can}} E_0^{\vee\vee} \rightarrow \mathcal{O}_2 \oplus \mathcal{O}_q \rightarrow 0$ , where  $[E_0^{\vee\vee}] \in M$

and, by the construction of  $E_0$ ,  $q \notin \text{Sing}(E_0^{\vee\vee}) \cup l$ . This means that  $[E_0] \in \mathcal{X}(-1, 1, 1, 0, 1)$ , and Theorem 10.(iii) implies that  $\dim \text{Ext}^1(E_0, E_0) = 15$ . Hence, since  $\dim \text{T}(-1, 2, 2, 1) = 15$  (see Theorem 9), it follows that  $E_0$  is a smooth point of  $\text{T}(-1, 2, 2, 1)$  and of  $\mathcal{M}(-1, 2, 0)$  as well.  $\square$

## 8. IRREDUCIBLE COMPONENTS OF $\mathcal{M}(-1, 2, 2)$

Now, we are in position to prove the next main result of this paper.

**Theorem 25.** *The moduli space  $\mathcal{M}(-1, 2, 2)$  of rank 2 stable sheaves on  $\mathbb{P}^3$  with Chern classes  $c_1 = -1, c_2 = 2, c_3 = 2$ , has exactly 2 irreducible rational components, namely:*

- (i) *the closure  $\overline{\mathcal{R}(-1, 2, 2)}$  of the family of reflexive sheaves  $\mathcal{R}(-1, 2, 2)$ , of dimension 11;*
- (ii) *the irreducible component  $\text{T}(-1, 2, 4, 1)$  given by Theorem 9, of dimension 15, whose generic element is a torsion free sheaf  $E$  such that  $E^{\vee\vee} \in \mathcal{R}(-1, 2, 4)$  and  $Q_E$  is a sheaf of length 1;*
- (iii) *in addition,  $\mathcal{M}(-1, 2, 2) \setminus \overline{\mathcal{R}(-1, 2, 2)} = \text{T}(-1, 2, 4, 1)$ .*

*Proof.* By [6, Thm 2.5],  $\mathcal{R}(-1, 2, 2)$  is irreducible, nonsingular of dimension 11, and its closure  $\overline{\mathcal{R}(-1, 2, 2)}$  in  $\mathcal{M}(-1, 2, 2)$  is an irreducible component of  $\mathcal{M}(-1, 2, 2)$  of dimension 11. Consider  $E \in \mathcal{M}(-1, 2, 2) \setminus \overline{\mathcal{R}(-1, 2, 2)}$ . By Proposition 14, either  $\dim Q_E = 1$  and  $1 \leq \text{mult } Q_E \leq 2$ , or  $\dim Q_E = 0$ . Consider all the possibilities for  $\dim Q_E$  and  $\text{mult } Q_E$ .

i) If  $\dim Q_E = 1$  and  $\text{mult } Q_E = 2$ , then  $c_2(E^{\vee\vee}) = 0$ , and, as in the case i) of the proof of Theorem 16, we are led to a contradiction.

ii) If  $\dim Q_E = 1$  and  $\text{mult } Q_E = 1$ , then  $c_2(E^{\vee\vee}) = 1$  and  $c_3(E^{\vee\vee}) = 1$  and  $Q_E$  is supported on a line. Then  $Q_E$  fit in an exact sequence of the form (114) where  $Z_E$  is the maximal artinian subsheaf of  $Q_E$ . Then the Euler characteristic of  $Q_E(t)$  is given by formula (115) which together with (110) yields:  $-1 = \chi(E) = \chi(E^{\vee\vee}) - \chi(Q_E) = -1 - r - s$ . Hence  $-r - s = 0$  and, since we have an epimorphism  $\delta : E^{\vee\vee} \rightarrow Q_E$ , from equation (15) it follows that  $r \geq -1$ . This implies that the possible values for  $r$  and  $s$  are  $r = s = 0$  or  $r = -1, s = 1$ .

Case ii.1) Assume that  $r = s = 0$ . In this case,  $Q_E \simeq i_* \mathcal{O}_l$ , for some line  $l$ , where  $i : l \hookrightarrow \mathbb{P}^3$  is the embedding. If  $l \cap \text{Sing } E^{\vee\vee} = \emptyset$ , then  $E$  is a sheaf in  $\mathcal{X}(-1, 1, 1, 0, 0)$ , i. e. a generic sheaf in  $\text{X}(-1, 1, 1, 0, 0)$ , where  $\dim \text{X}(-1, 1, 1, 0, 0) = 9$  by (35). However, this dimension is too small for  $\text{X}(-1, 1, 1, 0, 0)$  to be an irreducible component of  $\mathcal{M}(-1, 2, 2)$ . Next, if  $l \cap \text{Sing } E^{\vee\vee} \neq \emptyset$ , then  $E \in Y(0)$  and by Lemma 15 also  $Y(0)$  does not fill an irreducible component of  $\mathcal{M}(-1, 2, 2)$ .

Case ii.2) Assume that  $r = -1$  and  $s = 1$ . In this case,  $Q_E$  fits into the exact triple (147) for some pair  $(q, l) \in \mathbb{P}^3 \times G$ , where  $i : l \hookrightarrow \mathbb{P}^3$  is the embedding and  $Q = Q_E$ . Since  $[E^{\vee\vee}] \in M = \mathcal{M}(-1, 1, 1)$ , Proposition 22.(i),(ii) yields that  $[\delta : E^{\vee\vee} \twoheadrightarrow Q] = \delta \bmod A_Q$  is the point in  $\mathbf{P}(\mathcal{E})$ , i. e.,  $[E] \in \text{X}(-1, 1, 1, -1, 1)$ . This together with Proposition 24 implies that  $[E] \in \text{T}(-1, 2, 4, 1)$ .

iii) If  $\dim Q_E = 0$ , then  $s = \text{length}(Q_E) > 0$ . By Proposition 14,  $c_2(E) = c_2(E^{\vee\vee}) = 2$  and  $c_3(E^{\vee\vee}) = c_3(E) + 2s = 2 + 2s \geq 4$ . On the other hand,  $c_3(E^{\vee\vee}) \leq c_2(E^{\vee\vee})^2 = 4$  by (113) since  $E^{\vee\vee}$  is stable. Hence  $s = 1, c_3(E^{\vee\vee}) = 4$ , i. e.  $Q_E \simeq \mathcal{O}_p$  for some point  $p \in \mathbb{P}^3$ . Then by Theorem 18.(ii)  $[E]$  belongs to the irreducible component  $\text{T}(-1, 2, 4, 1)$  of  $\mathcal{M}(-1, 2, 2)$ .

In conclusion, we have proved that  $\mathcal{M}(-1, 2, 2) = \overline{\mathcal{R}(-1, 2, 2)} \cup \text{T}(-1, 2, 4, 1)$ . Finally, remark that the rationality of  $\overline{\mathcal{R}(-1, 2, 2)}$  is known from [7], and the rationality of  $\text{T}(-1, 2, 4, 1)$  follows from Main Theorem 2.  $\square$

## 9. IRREDUCIBLE COMPONENTS OF $\mathcal{M}(-1, 2, 0)$

We are now ready to describe all the irreducible components of  $\mathcal{M}(-1, 2, 0)$ .

**Theorem 26.** *The moduli space  $\mathcal{M}(-1, 2, 0)$  of rank 2 stable sheaves on  $\mathbb{P}^3$  with Chern classes  $c_1 = -1, c_2 = 2, c_3 = 0$ , has exactly 4 irreducible rational components, namely:*

- (i) *The closure of the family of stable rank 2 locally free sheaves  $\mathcal{B}(-1, 2)$ , of dimension 11;*
- (ii) *The irreducible component  $X(-1, 1, 1, 1, 0)$  of dimension 11, described by Theorem 10, whose generic element is a torsion free sheaf  $E$  such that  $E^{\vee\vee} \in \mathcal{R}(-1, 1, 1)$  and  $Q_E = i_*\mathcal{O}_l(1)$  for some line  $i : l \hookrightarrow \mathbb{P}^3$ .*
- (iii) *The irreducible component  $T(-1, 2, 2, 1)$  of dimension 15 described in Theorem 9, whose generic sheaf is a torsion free sheaf  $E$  such that  $E^{\vee\vee} \in \mathcal{R}(-1, 2, 2)$  and  $Q_E$  is a sheaf of length 1.*
- (iv) *The irreducible component  $T(-1, 2, 4, 2)$  of dimension 19 described by the Theorem 9, whose generic sheaf is a torsion free sheaf  $E$  such that  $E^{\vee\vee} \in \mathcal{R}(-1, 2, 4)$  and  $Q_E$  is a sheaf of length 2, supported at two distinct points.*

*Proof.* By [14, Proposition 4.1],  $\overline{\mathcal{B}(-1, 2)}$  is an irreducible component of  $\mathcal{M}(-1, 2, 0)$  of dimension 11. Consider  $[E] \in \mathcal{M}(-1, 2, 0) \setminus \overline{\mathcal{B}(-1, 2)}$ ; again, By Proposition 14, either  $\dim Q_E = 1$  and  $\leq \text{mult } Q_E \leq 2$ , or  $\dim Q_E = 0$ . We will study the possibilities for  $\dim Q_E$  and  $\text{mult } Q_E$ .  
 i) If  $\dim Q_E = 1$ ,  $\text{mult } Q_E = 2$ , then  $c_2(E^{\vee\vee}) = 0$ , and by (113)  $c_3(E^{\vee\vee}) = 0$ . Thus  $E^{\vee\vee}$  is a stable rank 2 vector bundle with  $c_1(E^{\vee\vee}) = -1$ ,  $c_2(E^{\vee\vee}) = 0$ , contrary to [14, Cor. 3.5].  
 ii) If  $\dim Q_E = \text{mult } Q_E = 1$ , then  $c_2(E^{\vee\vee}) = 1$ . Hence  $Q_E$  is supported on a line  $i : l \hookrightarrow \mathbb{P}^3$  and, possibly, isolated points and fits in the exact sequence 114, where  $\dim Z_E \leq 0$ ,  $\text{length } Z_E = s$ . Then the second formula (115) and Proposition 14.b) yield  $c_3(E^{\vee\vee}) = 2(r + s) - 1$ . This together with (113) implies that  $0 \leq c_3(E^{\vee\vee}) = 2(r + s) - 1 \leq 1$ . Hence,  $c_3(E^{\vee\vee}) = r + s = 1$ , i. e.  $[E^{\vee\vee}] \in \mathcal{M}(-1, 1, 1)$ . Since we have an epimorphism  $E^{\vee\vee} \xrightarrow{\delta} Q_E$ , it follows from (15) that  $r \geq -1$ . This together with the inequality  $s \geq 0$  implies that the possible values for  $r$  and  $s$  are:  $r = 1$  and  $s = 0$ , or  $r = 0$  and  $s = 1$ , or  $r = -1$  and  $s = 2$ . Consider these three cases.  
 Case ii.1):  $r = 1$  and  $s = 0$ . In this case,  $Q_E \simeq i_*\mathcal{O}_l(1)$ . If  $l \cap \text{Sing}(E^{\vee\vee}) = \emptyset$ , then by definition  $[E] \in \mathcal{X}(-1, 1, 1, 1, 0)$ , that is  $E$  is a generic sheaf in  $X(-1, 1, 1, 1, 0)$ . Here by Theorem 10.(i)  $X(-1, 1, 1, 1, 0)$  is the irreducible component of dimension 11 in  $\mathcal{M}(-1, 2, 0)$ . If  $l \cap \text{Sing}(E) \neq \emptyset$ , then by the Lemma 15 the family of all such  $E$  cannot constitute an irreducible component of  $\mathcal{M}(-1, 2, 0)$ .

Case ii.2):  $r = 0$  and  $s = 1$ . In this case, the triple (114) is:  $0 \rightarrow \mathcal{O}_p \rightarrow Q_E \xrightarrow{\text{can}} i_*\mathcal{O}_l \rightarrow 0$ , where  $p$  is some point in  $\mathbb{P}^3$ . This together with the Snake Lemma implies that the sheaf  $\mathcal{E}$  defined as the kernel of the composition  $E^{\vee\vee} \xrightarrow{\text{can}} Q_E \xrightarrow{\delta} i_*\mathcal{O}_l$  fits in the exact triple  $0 \rightarrow E \rightarrow \mathcal{E} \rightarrow \mathcal{O}_p \rightarrow 0$ . This triple and the stability of  $E$  implies the stability of  $\mathcal{E}$ , hence  $[\mathcal{E}] \in \mathcal{M}(-1, 2, 2)$ . Since  $\dim \text{Sing}(E) = 1$ , we have by definition that  $[E] \in X(-1, 1, 1, 0, 1)$ . From Theorem 24.(iii) it follows now that  $[E] \in T(-1, 2, 2, 1)$ .

Case ii.3):  $r = -1$  and  $s = 2$ . In this case,  $Q_E$  fits into an exact sequence of the form:  $0 \rightarrow Z_E \rightarrow Q_E \rightarrow i_*\mathcal{O}_l(-1) \rightarrow 0$ , where  $Z_E$  has length 2 and where  $i : l \hookrightarrow \mathbb{P}^3$  is the embedding of some line  $l$ . Therefore, by Proposition 23,  $[E] \in X(-1, 1, 1, -1, 2)$ . Then from Proposition 24.(ii) it follows that  $[E] \in T(-1, 2, 4, 2)$ .

iii) If  $\dim Q_E = 0$ , let  $s = \text{length}(Q_E)$ , since we are assuming that  $E$  is properly torsion free, it follows that  $s > 0$ . By Proposition 14,  $c_2(E) = c_2(E^{\vee\vee})$  and  $c_3(E^{\vee\vee}) = c_3(E) + 2s$ . Therefore, either  $s = 1$ ,  $c_3(E^{\vee\vee}) = 2$ , or  $s = 2$ , and  $c_3(E^{\vee\vee}) = 4$ . Consider both these cases.  
 Case 1:  $s = 1$ , then  $c_3(E^{\vee\vee}) = 2$ . In this case  $Q_E \simeq \mathcal{O}_p$  for some point  $p \in \mathbb{P}^3$ , so that  $[E] \in T(-1, 2, 2, 1)$  by Theorem Theorem 18.(i).

Case 2:  $s = 2$ , then  $Q_E$  has length 2, and  $[E] \in T(-1, 2, 4, 2)$  by Theorem 19.

In conclusion, we have proved that  $\mathcal{M}(-1, 2, 0) = \overline{\mathcal{B}(-1, 2)} \cup \mathbb{T}(-1, 2, 2, 1) \cup \mathbb{T}(-1, 2, 4, 2) \cup \mathbb{X}(-1, 1, 1, 1, 0)$ . Remark also that the rationality of  $\overline{\mathcal{B}(-1, 2)}$  is proved in [15], the rationality of  $\mathbb{T}(-1, 2, 2, 1)$  and  $\mathbb{T}(-1, 2, 4, 2)$  follows from Main Theorem 2, and the rationality of  $\mathbb{X}(-1, 1, 1, 1, 0)$  also follows from Main Theorem 2 with small additional argument due to the elementary transformations of sheaves along the line  $l$ .  $\square$

**Remark 6.** *Meseguer, Sols and Strømme proved in [26] that  $\mathcal{M}(-1, 2, 0)$  contains, besides  $\mathcal{B}(-1, 2)$ , at least two families of non locally free sheaves containing sheaves that are not limits of locally free sheaves. Zavodchikov then proved in [40] that these families of sheaves form irreducible components of dimension 15 and 19; they coincide with the components we denoted by  $\mathbb{T}(-1, 2, 2, 1)$  and  $\mathbb{T}(-1, 2, 4, 2)$ , respectively. Later, Zavodchikov proved in [41] that  $\mathcal{M}(-1, 2, 0)$  consists of exactly 4 irreducible components; this article, however, is only available in russian.*

*We emphasize that our proof of Theorem 26 is completely independent from the results in [26, 40, 41], treating  $\mathcal{M}(-1, 2, c_3)$  in a uniform manner for all 3 possible values of  $c_3$ . Additionally, it also provides further information on the generic element of each component.*

## 10. CONNECTEDNESS OF THE SPACES $\mathcal{M}(-1, 2, c_3)$

Since the space  $\mathcal{M}(-1, 2, 4)$  is irreducible, it is obviously connected. In this section we will prove that the spaces  $\mathcal{M}(-1, 2, 2)$  and  $\mathcal{M}(-1, 2, 0)$  are also connected.

**Theorem 27.** *The moduli space  $\mathcal{M}(-1, 2, 2)$  is connected.*

*Proof.* First, remark that one easily constructs a flat family of curves  $\mathcal{Z}$  in  $\mathbb{P}^3$  with base  $\mathbb{A}^1$ , i. e., a subscheme  $\mathcal{Z}$  of  $\mathbb{P}^3 \times \mathbb{A}^1$  with the projection  $\pi : \mathcal{Z} \rightarrow \mathbb{P}^3 \times \mathbb{A}^1 \xrightarrow{pr_2} \mathbb{A}^1$  satisfying the properties:

- a) for  $t \in \mathbb{A}^1 \setminus \{0\}$ , the fiber  $Z_t := \pi^{-1}(t)$  is a disjoint union  $l_{1t} \sqcup l_{2t}$  of two lines;
- b) the zero fiber  $Z_0 := \pi^{-1}(0)$  as a set is the union of two distinct lines  $l_{10}$  and  $l_{20}$  meeting at a point, say,  $p$  such that  $(Z_0)_{red} = l_{10} \cup l_{20}$  and  $Z_0$  as a scheme has  $p$  as an embedded point; more precisely, there is an exact triple:

$$(156) \quad 0 \rightarrow \mathcal{O}_p \rightarrow \mathcal{O}_{Z_0} \rightarrow \mathcal{O}_{(Z_0)_{red}} \rightarrow 0.$$

Indeed, to construct the family  $\mathcal{Z}$ , consider the projective space  $\mathbb{P}^4$  with coordinates  $(u : x : y : z : w)$  and the affine line  $\mathbb{A}^1$  with coordinate  $t$ . In  $\mathbb{P}^4$  consider a reduced subscheme  $W$  given by the equations  $\{xz = xw = yz = yw = 0\}$  (This  $W$  is just a union of two projective planes in  $\mathbb{P}^4$  intersecting at the point  $\tilde{p} = (1 : 0 : 0 : 0 : 0)$ .) Next, in  $\mathbb{P}^4 \times \mathbb{A}^1$  take a divisor  $D = \{tu = x + y + z + w\}$ , and let  $\tilde{\mathcal{Z}} = D \cap W \times \mathbb{A}^1$ . Furthermore, in  $\mathbb{P}^4$  take a hyperplane  $\mathbb{P}_0^3 = \{x + y + z + w = 0\}$ , fix some isomorphism  $f : D \xrightarrow{\sim} \mathbb{P}_0^3 \times \mathbb{A}^1 \simeq \mathbb{P}^3 \times \mathbb{A}^1$  and set  $p = f(\tilde{p})$ . (For instance, one can take for  $f$  a morphism  $((u : x : y : z : w), t) \mapsto ((u : x : y : z : -(x + y + z)), t)$ . Then the subscheme  $\mathcal{Z} = f(\tilde{\mathcal{Z}})$  satisfies the above properties a) and b).

Let  $p_2 : \mathbb{P}^3 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$  be the projection. One checks that, for  $t \in \mathbb{A}^1$ ,  $\text{Ext}^1(\mathcal{I}_{Z_t}, \mathcal{O}_{\mathbb{P}^3}(-1))$  has fixed dimension 4 while the higher Ext-groups of this pair vanish, hence the base change for relative Ext-sheaves (see, e. g., [23, Thm. 1.4]) shows that the sheaf  $\mathcal{A} = \mathcal{E}xt_{p_2}^1(\mathcal{I}_{\mathcal{Z}, \mathbb{P}^3 \times \mathbb{A}^1}, \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbb{A}^1})$  is a locally free  $\mathcal{O}_{\mathbb{A}^1}$ -sheaf and there exists a nowhere vanishing section  $s \in H^0(\mathcal{A})$ . Furthermore, by the spectral sequence of global-to-relative Ext we may consider  $s$  as an element of the group  $\text{Ext}_{p_2}^1(\mathcal{I}_{\mathcal{Z}, \mathbb{P}^3 \times \mathbb{A}^1}, \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbb{A}^1})$ . This element defines an extension of  $\mathcal{O}_{\mathbb{P}^3 \times \mathbb{A}^1}$ -sheaves

$$(157) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbb{A}^1} \rightarrow \mathbf{E} \rightarrow \mathcal{I}_{\mathcal{Z}, \mathbb{P}^3 \times \mathbb{A}^1} \rightarrow 0.$$

The sheaf  $\mathbf{E}$  is flat over  $\mathbb{A}^1$  and, by construction, for  $t \in \mathbb{A}^1$ , the restriction of (157) is a nonsplitting extension of  $\mathcal{O}_{\mathbb{P}^3}$ -sheaves  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E_t \rightarrow \mathcal{I}_{Z_t, \mathbb{P}^3} \rightarrow 0$ , where  $E_t := \mathbf{E}|_{\mathbb{P}^3 \times \{t\}}$ . Hence,  $[E_t] \in \mathcal{M}(-1, 2, 2)$ , i. e., we obtain a modular morphism  $\Phi : \mathbb{A}^1 \rightarrow \mathcal{M}(-1, 2, 2)$ ,  $t \mapsto [E_t]$ .

Note that, for  $t \neq 0$ ,  $[E_t] \in \mathcal{R}(-1, 2, 2)$  by [6, Lemma 2.4], i. e.  $\Phi(\mathbb{A}^1 \setminus \{0\}) \subset \mathcal{R}(-1, 2, 2)$ . It follows that  $[E_0] \in \overline{\mathcal{R}(-1, 2, 2)}$ . Besides,  $E_0$  fits the exact sequence

$$(158) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{r} E_0 \rightarrow \mathcal{I}_{Z_0, \mathbb{P}^3} \rightarrow 0.$$

From (156) and (158) we deduce the following exact sequences:

$$(159) \quad 0 \rightarrow E_0 \rightarrow E_0^{\vee\vee} \rightarrow \mathcal{O}_p \rightarrow 0,$$

$$(160) \quad 0 \rightarrow \mathcal{O}(-1) \xrightarrow{s} E_0^{\vee\vee} \rightarrow \mathcal{I}_{(Z_0)_{red}, \mathbb{P}^3} \rightarrow 0.$$

where  $s$  is the composition morphism  $r$  in the sequence (158) with the canonical monomorphism  $E_0 \rightarrow E_0^{\vee\vee}$ . From the sequence (160) and [13, Proposition 4.2] we conclude that  $E_0^{\vee\vee}$  is stable. Moreover, by (159) and Theorem 18.(ii),  $[E_0] \in \mathcal{T}(-1, 2, 4, 1)$ . This yields the proof since, by Theorem 25,  $\mathcal{M}(-1, 2, 2) = \overline{\mathcal{R}(-1, 2, 2)} \cup \mathcal{T}(-1, 2, 4, 1)$ .  $\square$

**Theorem 28.** *The moduli space  $\mathcal{M}(-1, 2, 0)$  is connected.*

*Proof.* By Theorem 26,

$$\mathcal{M}(-1, 2, 0) = \overline{\mathcal{B}(-1, 2)} \cup \mathcal{T}(-1, 2, 2, 1) \cup \mathcal{T}(-1, 2, 4, 2) \cup \mathcal{X}(-1, 1, 1, 1, 0).$$

We are going to prove that: (i) the component  $\overline{\mathcal{B}(-1, 2)}$  intersects the components  $\mathcal{T}(-1, 2, 2, 1)$  and  $\mathcal{T}(-1, 2, 4, 2)$ ; (ii) the component  $\mathcal{X}(-1, 1, 1, 1, 0)$  intersects the component  $\mathcal{T}(-1, 2, 4, 2)$ . This will imply the connectedness of  $\mathcal{M}(-1, 2, 0)$ .

(i) By [13, Example 3.1.2], the generic sheaf  $[E] \in \mathcal{B}(-1, 2)$  fits into an exact triple of the form

$$(161) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow E \rightarrow \mathcal{I}_Z(1) \rightarrow 0,$$

where  $\mathcal{I}_Z$  is the ideal sheaf of a disjoint union of two conics in  $\mathbb{P}^3$ . The proof is similar to the proof of Theorem 27, with minor changes, that we will include here for completeness.

Consider the following two 1-dimensional flat families of curves  $\mathcal{Z}^1$  and  $\mathcal{Z}^2$  in  $\mathbb{P}^3$ , with base  $U$  open subset in  $\mathbb{A}^1$  containing the point 0, and with projections  $\pi_i : \mathcal{Z}^i \hookrightarrow \mathbb{P}^3 \times U \xrightarrow{pr_2} U$ ,  $i = 1, 2$ , such that  $\mathcal{Z}^i$  satisfies the conditions (a) and (b)<sub>*i*</sub>,  $i = 1, 2$ , where:

(a) for  $t \neq 0$ , the fiber  $Z_t^i := \pi_i^{-1}(t)$  is a disjoint union of two conics;

(b)<sub>1</sub> the fiber  $Z_0^1$  at 0 as a set is the union of two smooth conics,  $C_1$  and  $C_2$  meeting in a unique point, say  $p$ , i. e.,  $(Z_0^1)_{red} = C_1 \cup C_2$  and  $p = C_1 \cap C_2$ , and as a scheme  $Z_0^1$  has an embedded point  $p$  such that the following triple is exact:

$$(162) \quad 0 \rightarrow \mathcal{O}_p \rightarrow \mathcal{O}_{Z_0^1} \rightarrow \mathcal{O}_{(Z_0^1)_{red}} \rightarrow 0.$$

(b)<sub>2</sub> the fiber  $Z_0^2$  at 0 is a union of two conics,  $C_1$  and  $C_2$  meeting in two distinct fat points of multiplicity 2. That is,  $(Z_0^2)_{red} = C_1 \cup C_2$ , and  $\{p_1, p_2\} = C_1 \cap C_2$ , and as scheme  $Z_0^2$  has two embedded points  $p_1$  and  $p_2$ :

$$(163) \quad 0 \rightarrow \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \rightarrow \mathcal{O}_{Z_0^2} \rightarrow \mathcal{O}_{(Z_0^2)_{red}} \rightarrow 0.$$

Now similarly to (157) we obtain the exact triples:

$$(164) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes \mathcal{O}_U \rightarrow \mathbf{E}^i \rightarrow \mathcal{I}_{Z^i, \mathbb{P}^3 \times U} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_U \rightarrow 0, \quad i = 1, 2,$$

such that, for  $t \in U$ , the restriction of (164) is a nonsplitting extension of  $\mathcal{O}_{\mathbb{P}^3}$ -sheaves  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow E_t^i \rightarrow \mathcal{I}_{Z_t^i, \mathbb{P}^3}(1) \rightarrow 0$ , where  $E_t^i := \mathbf{E}^i|_{\mathbb{P}^3 \times \{t\}}$ . Hence,  $[E_t^i] \in \mathcal{M}(-1, 2, 0)$ , i. e.,

we obtain modular morphisms  $\Phi_i : U \rightarrow \mathcal{M}(-1, 2, 0)$ ,  $t \mapsto [E_t^i]$ . Note that, for  $t \neq 0$ , each  $[E_t^i] \in \mathcal{B}(-1, 2)$  by [14, Example 3.1.2]. Hence, also  $[E_0^i] \in \overline{\mathcal{B}(-1, 2)}$ ,  $i = 1, 2$ . Besides,  $E_0^i$  fit in the following exact triples:

$$(165) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{r_i} E_0^i \rightarrow \mathcal{I}_{Z_0^i, \mathbb{P}^3}(1) \rightarrow 0, \quad i = 1, 2.$$

The triples (162), (163) and (165), yield the following exact sequences:

$$(166) \quad 0 \rightarrow E_0^1 \rightarrow (E_0^1)^{\vee\vee} \rightarrow \mathcal{O}_p \rightarrow 0,$$

$$(167) \quad 0 \rightarrow E_0^2 \rightarrow (E_0^2)^{\vee\vee} \rightarrow \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \rightarrow 0,$$

$$(168) \quad 0 \rightarrow \mathcal{O}(-2) \xrightarrow{s_i} (E_0^i)^{\vee\vee} \rightarrow \mathcal{I}_{(Z_0^i)_{red}, \mathbb{P}^3}(1) \rightarrow 0, \quad i = 1, 2,$$

where  $s_i$  is the composition morphism  $r_i$  from (165) with the canonical monomorphism  $E_0^i \rightarrow (E_0^i)^{\vee\vee}$ . From sequence (168) and [13, Proposition 4.2] we conclude that  $(E_0^i)^{\vee\vee}$  is stable, i. e.  $[(E_0^i)^{\vee\vee}] \in \mathcal{M}(-1, 2, 2i)$ ,  $i = 1, 2$ . Thus, (166) and Theorem 26.(c) yield  $[E_0^1] \in \mathbb{T}(-1, 2, 2, 1)$ ; respectively, (167) and Theorem 26.(d) yield  $[E_0^2] \in \mathbb{T}(-1, 2, 4, 2)$ . Since, by the above,  $[E_0^1], [E_0^2] \in \overline{\mathcal{B}(-1, 2)}$ ,  $i = 1, 2$ , it follows, that  $\overline{\mathcal{B}(-1, 2)} \cap \mathbb{T}(-1, 2, 2, 1) \neq \emptyset$  and  $\overline{\mathcal{B}(-1, 2)} \cap \mathbb{T}(-1, 2, 4, 2) \neq \emptyset$ , as stated.

(ii) Fix a sheaf  $[F] \in \mathcal{R}^*(-1, 1, 1)$ , a line  $l$  in  $\mathbb{P}^3$  such that  $l \cap \text{Sing}(F) = \emptyset$ , and two distinct points  $p_1, p_2 \in l$ . Consider the surface  $S = l \times \mathbb{A}^1$  with the projection  $pr_2 : S \rightarrow \mathbb{A}^1$ . The points  $p_1, p_2$  define two points  $\tilde{p}_i = (p_i, 0) \in pr_2^{-1}(0)$ ,  $i = 1, 2$ . Since  $F|_l \cong \mathcal{O}_l \oplus \mathcal{O}_l(-1)$  (see (15)), it follows that there exists an epimorphism  $\mathbf{e} : F \boxtimes \mathcal{O}_{\mathbb{A}^1} \rightarrow \mathcal{B} := \mathcal{I}_{\tilde{p}_1 \sqcup \tilde{p}_2, S} \otimes \mathcal{O}_l(1) \boxtimes \mathcal{O}_{\mathbb{A}^1}$ . Consider an  $\mathcal{O}_{\mathbb{P}^3 \times \mathbb{A}^1}$ -sheaf  $\mathbf{E} = \ker \mathbf{e}$ , flat over  $\mathbb{A}^1$  and, for  $t \in \mathbb{A}^1$ , set  $E_t := \mathbf{E}|_{\mathbb{P}^3 \times \{t\}}$ . By construction, the restriction of the exact triple  $0 \rightarrow \mathbf{E} \rightarrow F \boxtimes \mathcal{O}_{\mathbb{A}^1} \rightarrow \mathcal{B} \rightarrow 0$  onto  $\mathbb{P}^3 \times \{t\}$  yields the exact sequences

$$(169) \quad 0 \rightarrow E_t \rightarrow F \rightarrow \mathcal{O}_l(1) \rightarrow 0, \quad t \in \mathbb{A}^1 \setminus \{0\},$$

$$(170) \quad 0 \rightarrow E_0 \rightarrow F \rightarrow \mathcal{O}_l(-1) \oplus \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \rightarrow 0,$$

and there is a modular morphism  $\Psi : \mathbb{A}^1 \rightarrow \mathcal{M}(-1, 2, 0)$ ,  $t \mapsto [E_t]$ . By the definition of  $\mathcal{X}(-1, 1, 1, 1, 0)$  and  $\mathbb{X}(-1, 1, 1, 1, 0)$  (see (32) and (34)), it follows from (169) that  $[E_t] \in \mathcal{X}(-1, 1, 1, 1, 0)$  for  $t \in \mathbb{A}^1 \setminus \{0\}$ , i. e.  $\Psi(\mathbb{A}^1 \setminus \{0\}) \subset \mathcal{X}(-1, 1, 1, 1, 0) \subset \mathbb{X}(-1, 1, 1, 1, 0)$ . Hence,  $[E_0] \in \mathbb{X}(-1, 1, 1, 1, 0)$ . On the other hand, by the definition of  $\mathcal{X}(-1, 1, 1, -1, 2)$ ,  $[E_0] \in \mathcal{X}(-1, 1, 1, -1, 2) \subset \mathbb{X}(-1, 1, 1, -1, 2)$ . Since by Theorem 24.(ii)  $\mathbb{X}(-1, 1, 1, -1, 2)$  lies in  $\mathbb{T}(-1, 2, 4, 2)$ , it follows that  $\mathbb{X}(-1, 1, 1, 1, 0) \cap \mathbb{T}(-1, 2, 4, 2) \neq \emptyset$ .  $\square$

The first part of the previous proof can also be regarded as a proof of the following claim. Let  $E$  be a stable torsion free sheaf with  $(c_1(E), c_2(E), c_3(E)) = (-1, 2, 0)$  such that

- (i) there exists a nontrivial section in  $H^0(E^{\vee\vee}(2))$  that vanishes along the union of two conics intersecting in a point  $p$ , and
- (ii)  $E^{\vee\vee}/E = \mathcal{O}_p$ .

Then  $E$  is smoothable. Indeed, these two hypotheses imply that  $E$  fits into the following exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow E \rightarrow I_Z(1) \rightarrow 0,$$

where  $Z$  coinciding the scheme  $Z_0^1$  described in item (b<sub>1</sub>) above. Deforming  $Z$  into a union of disjoint conics, we obtain a deformation of  $E$  into a locally free sheaf  $F$  with  $[F] \in \mathcal{B}(-1, 2)$ .

Similarly, if  $E$  satisfies the following two hypotheses

- (i') there exists a nontrivial section in  $H^0(E^{\vee\vee}(2))$  that vanishes along the union of two conics intersecting in two points  $p$  and  $q$ ;



$$(ii') \quad E^{\vee\vee}/E = \mathcal{O}_p \oplus \mathcal{O}_q,$$

then  $E$  is smoothable.

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