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(1) Introduction

The basic assumptions of psychological models of development are formulated, as a rule, on so general a theoretical level that their claims to validity go far beyond what can be verified or falsified by investigations of ontogenetic processes in developmental psychology. On one hand, it could be the case that psychologists have succumbed to the temptation of generalizing their theoretical insights obtained by examining ontogenetic development much further than is warranted. On the other hand there are also grounds for the generalization, which lie in the subject matter. Developmental psychology is forced to take up problems that can only be dealt with from a perspective embracing ontogeny and historical development; for, with the question about the ontogenetic development of fundamental cognitive structures of human thinking the question is always also posed as to the nature of these structures; and whatever the answer given to the question, it will include willy-nilly statements on the relation of such structures to cultural evolution.

This observation on the implications of the theoretical models of developmental psychology applies particularly to the psychological studies of the development of the concept of number in children which have come in the wake of Jean Piaget's epoch-making study (Piaget/Szeminska 1941). No one was more aware than Piaget himself of the fact that his investigation not only gave answers to narrowly formulated questions concerning ontogenetic development but also took a position in a controversy reaching back into antiquity over the nature of numbers--a controversy which was fought out primarily in mathematics and philosophy and much less so in psychology.

Piaget (e.g. Piaget, 1950, 1967; Piaget/Garcia 1983) took up the challenge of a subject not reducible to psychological states of affairs. He also advocated with initiative those consequences, too, which led him out of the narrower field of research in developmental psychology. His success there was, however, less conclusive than in psychology; for such a success it would have been necessary to conduct detailed investigations of the states of affairs in the history and theory of mathematics, to which Piaget appeals rather eclectically. His conclusions therefore evoke the critical question if it is permissible to generalize from the ontogenetic development of the concept of number to historical processes of development the way he does.

Doubts about the competence of psychology for dealing adequately with historical issues probably play a considerable role in psychological research; among the numerous, scarcely surveyable studies carried out in the wake of Piaget, one very rarely finds a study that has made this question posed by Piaget on the relation of ontogenetic and historical development the direct subject of a thorough investigation. It is, however, precisely the precarious status of the generalizing propositions of developmental psychology that provides the opportunity to submit the findings of developmental psychology to some control, to supplement or contrast them and possibly even to bring controversial questions of developmental psychology into a form in which they can be decided by indications from outside the field.

What will be said in this chapter is connected with just that purpose in mind. First, some theoretical arguments advanced in the investigation of the development of the child's conception of number will be discussed briefly. Attention will be focused on the following theoretical alternatives: The cultural evolution of arithmetical thinking can be interpreted either in terms of Piaget's epigenetic conception or in terms of a theory which asserts a substantial influence of representations on the development of cognitive structures as argued by Bruner and Aebli. It will be claimed that these alternatives do not lead to very different conclusions if only applied to ontogenetic development. Applied, however, to the process of cultural evolution there are strikingly different consequences which may be confirmed or rejected by findings of historical research. Second, some findings on arithmetical techniques documented by the archaic texts of the early Mesopotamian city of Uruk (ca. 3000 B.C.) will be presented. The basic claim here is that these techniques represent a missing link between proto-arithmetical techniques as known from ethnological studies about preliterate cultures and the abstract number concept. The attempt will be made to demonstrate that constructivist structuralism in the tradition of Piaget provides a good explanation of the cultural evolution of the number concept. To arrive at this explanation, however, Piaget's epigenetic conception of the number concept has to be discarded. Arithmetical techniques developed in connection with the invention of writing, namely, give evidence that in cultural evolution there is no synchronous appearance of the structural elements of the number concept which Piaget asserts to be universal.

(2) Piaget's Epigenetic Conception of the Development of the Concept of Number

According to Piaget's research on the child's intellectual development the concept of number is the result of the construction of a cognitive structure based on experience. Hence it is neither an inherited intellectual scheme nor a property directly abstracted from real objects. Piaget showed that the cognitive structure underlying the concept of number is not present in early childhood but rather arises as the result of ontogenetic developmental processes only at a particular stage of development.

At the preoperational stage a child still lacks a crucial prerequisite, without which a meaningful concept of number is inconceivable. It lacks the intellectual insight that in every set of objects there is something that remains identical independent of what is done with the objects and in spite of how their perceptual appearance changes, as long as nothing is added or taken away--that is, their quantity. At the preoperational stage there is in a child's thought no consciousness of the necessity that, when counting a number of objects, the same result must always be obtained however the objects are arranged and in whatever order they are counted. Only when this "conservation of quantity" has been mentally constructed is there a substratum available to the child, i.e. a logical carrier of the properties "more" and "less" to be expressed by numbers.

How is the number concept construed in the process of ontogenetic development? According to Piaget this cognitive structure is the result of reflection on actively dealing with objects. He explains the concept of number as the result of the coordination of actions such as the construction of one to one correspondences between sets or relations (cardinal or ordinal correspondence), the arranging of objects according to qualitative characteristics (seriation), the adding and taking away of objects and groups of objects (additive composition), and the combinatoric coordination of sets and relations (logical multiplication). In the fully developed cognitive structure of the concept of number these actions are mentally represented by reversible cognitive operations constituting a closed system of possible inferences. Additive and multiplicative connections between quantities can be inferred from class-logical and relation-logical relationships between sets of objects, and vice versa. The arithmetically structured substratum of the number concept is an ideal substratum. Numbers are ideal objects of thought whose existence is not bound to the existence of the material objects to which they are applied. They are independent of their particular representations by the various systems of numerical signs which differ from one culture to another. According to Piaget the concept of number is thus the result of experience and abstraction; however, not abstraction from the objects but rather a "reflective abstraction" from the actions carried out with the objects.

This constructivist view of the concept of number is basic also for the study presented here. A further characteristic of the reconstruction of the development of the concept of number as given by Piaget must now be taken up and critically examined.

According to Piaget the concept of number, though not inherited but acquired through the experience of actions, is in its substance not influenced by the content of these experiences. He interprets the coordination of actions as resulting from an endogenous unfolding of biologically predetermined possibilities in interaction with the environment, and hence sees the basic structures of cognition, to which the structure of the concept of number belongs, as a special human form of biological coordination of behavior (Piaget 1967). Its endogenous development, while not preformed by inheritance, is nonetheless an epigenetic process governed by inner necessity.

This conception results in a distinction between two different forms of knowledge, namely empirical knowledge, whose contents are determined by the object, and mathematical-logical knowledge, whose contents represent a cognitive universal of the human race. Each form has its own modality of knowledge. Empirical knowledge can lead only to empirical certainty, to which alternatives are conceivable; mathematical-logical knowledge, on the contrary, leads to knowledge with the status of logical necessity, which, once it has been acquired by constructing the appropriate cognitive structure, cannot be challanged by any empirical experience.

For the universalism of the mathematical-logical structures to be plausible, this concept must be restricted to very elementary structures. Thus, for instance, Piaget's concept of number is narrowed down to a few fundamental properties. Piaget does not accept the assumption that the concept of number can be reduced to logic, but he does emphasize its proximity to logic. For him the concept of number is nothing but the cognitive structure in which the operations of the logic of classes and relations are united to a closed system of thought forms. This closedness of the developed structure is the reason for the logical necessity of mathematical knowledge, for this is determined by the system and not by real objects, which are interpreted by this system.

(3) The Role of Representations

The basic epigenetic conception of Piaget's theory of the development of cognitive structures has not been accepted without reservations. Doubts have been raised especially when individual or cultural differences are investigated or when, as in the case of didactics, means are developed to overcome such differences. On the one hand, Piaget's constructivist structuralism makes available excellent theoretical instruments to understand better the structural transformations between different conceptual systems both in cultural development and in the individual learning process; on the other hand, it is precisely here that his theory should not be applicable, for an epigenetic explanation is only meaningful for cognitive universals. Moreover, the epigenetic conception excludes the view that material representations of cognitive structures in actions, pictures, and symbols, as well as the social interactions in which the meaning of such representations is mediated, have an influence on structural transformations.

Bruner (et al. 1966), for instance, fundamentally questions Piaget's epigenetic conception in a publication of comparative studies on the development of elementary cognitive structures in different cultures. There he argues for the substantial role of representations in the development of mental structures. In this influential publication he thereby generalizes the concept of cognitive structure in that he no longer deals only with the mathematical-logical basic skeleton of thought, but also applies it generally to systems of evaluation and interpretation of reality. Nevertheless, he retains the basic idea of Piaget's conception that there are essential foundations for such systems (for instance, the spatio-temporal frame of reference) which cannot be derived immediately from reality.

This insistence on a constructivist structuralism while at the same time generalizing the concept of cognitive structure to individual or culturally specific systems of thought suggests that the representations of such systems are to be credited with a key role in the development of the systems. In Piaget's epigenetic conception a universal endogenous process guides the construction. For a generalized conception of cognitive development something else has to be identified which can mediate between experience of reality and specific prerequisites determined by the cultural environment which guide the construction in a particular way: and representations can do just that.

Bruner conceives these representations as means for modelling reality which serve as "amplifiers" of the fundamental human abilities to act, perceive, and know and as such guide the development of structures that coordinate them. In his view, three media for the representation of cognitive structures are available: the medium of action for the "enactive representation", the medium of images for "iconic representation", and the medium of the conventially assignable signs for "symbolic representation".

This media theory of representation has often been taken up and has been put to extensive use, at least as an eclectically applied scheme of description for the material means of thought and of the development of thought structures. This revision of Piaget's conception has proved to be especially helpful for the discussion of problems of didactics. For instance, it reconciles constructivist structuralism with the fact that didactic embodiments of number structures were the necessary prerequisites for the elementary concept of number to become a general basic qualification for most of the population in the industrial countries.

Aebli (1963), too, has suggested a fundamental revision in his attempt to apply Piaget's constructivist structuralism to instructional questions. He points out that one can adequately grasp the structuring potential of thought in concrete processes of problem-solving only if one does not assume the cognitive structures to be simply given and universally applicable. Rather they must be "elaborated" anew in the process of application; i.e., reconstructed in connection with the concrete object. Later, in a broad-based attempt to ground the development of thought forms in actions, he places the concept of representation in a medium at the center of his theory (Aebli, 1980/1981). In doing this, he develops further the concept of medium, which for Bruner is a relatively unarticulated and neutral concept, to a differentiated concept for widely differing means of thought (Aebli, 1980/1981, Vol. II, Chap. VI). At least one distinction should be mentioned here as an example, namely the distinction between primary and secondary media of representation. Representation by primary media coordinates action and cognition directly whereas representation by secondary media serves data-processing in the process of thinking, thus coordinating action and cognition indirectly. This distinction makes it possible theoretically to grasp reflexive processes of thought on the level of the representation of cognitive structures.

This distinction is especially important with regard to the development of the concept of number. In Piaget's conception representations of numbers and number operations do not play a substantial role in the construction of the concept of number. In Bruner's conception such representations have a central function. All operations relating to numbers which are to be carried out in thought belong to the cognitive structure and this structure is embodied in the representations. His conception of the three media of representation, however, allows only the static representation of the structures in these media, so that their function can only be to "amplify" the thought operations. With Aebli's distinction between primary and secondary media, the essential difference that exists between representations of numbers or number operations of differing degrees of generality can be grasped theoretically, and thus one can determine the function that the representations take on in the process of reflection by which cognitive structures are reorganized and generalized.

(4) Consequences for a Theory of the Cultural Evolution of Arithmetical Techniques

When constructivist structuralism is applied only to the ontogenesis, in particular to such elementary cognitive structures as the number concept, Piaget's epigenetic conception scarcely differs in its consequences from these revisions, where representation is accorded a more important place. This is so because ontogenetic processes are usually directed towards a culturally present end, as if they were epigenetic.

But applied to the problem of how to interpret the cultural evolution of cognitive structures, Piaget's epigenetic conception differs fundamentally from the revisions of constructivist structuralism just discussed. In the following it will be attempted to make this clearer using the problem of how to interpret theoretically the cultural development of number representations and computing techniques with regard to the underlying cognitive processes.

The forms of representation of numbers and number operations, as is well known, are subject to great cultural variation, and comprehensive processes of historical development can be ascertained from the first number notations which have come down to us up to modern arithmetic. Piaget's epigenetic conception immediately poses the fundamental question which is crucial for any comprehension of the cognitive background of the historical development of arithmetic with psychological categories, whether the number concept which he described as a universal cognitive structure was already present as a result of ontogenetic processes of development before corresponding representations appeared in history. In this case the development of the forms of representation has at best only marginally to do with the emergence of the number concept. The alternative is that this structure arose only in the course of history and developed gradually to an universal structure.

In both cases Piaget's epigenetic conception drastically restricts the possible answers to the question. In the first case Piaget's constructivist structuralism needs a theory to explain the transformation of the preexisting universal cognitive structure of the number concept into the process structure of historically changing forms of representation. In the second case we ought to be able to identify, behind the manifold developments in the history of arithmetic, a psychological process which somehow corresponds to the synchronous emergence of logical and numerical operations in ontogeny. Whatever answer one gives to the question within Piaget's epigenetic conception, the growth of cognitive structures such as the number concept has to be conceived in this conception as an ontogenetic process that represents a result of biological development, whereas cultural evolution represents only a phenomenon derived from this ontogenetic universal.

To the question which of these two alternatives is true, Piaget (e.g. 1950, Vol. I, Chap. I 12; Vol. II, Chap. IV 7; Vol. III, Chap. XII 7) gives a complicated two-fold answer. First he parallels the thinking of "primitives" with that of children, ascribing to them both an insufficient store of experience in dealing with the elementary actions which underlie the concrete operations and especially the concept of number. However, the resulting parallel between ontogeny and cultural evolution is essentially completed in Greek antiquity, that is, at a stage of development with deductive thought forms and in particular a developed number concept. In the second part of his answer, Piaget reconstructs the history of arithmetic as a progressive process of "becoming aware" of the operations already present in the cognitive structure of the concept of number, from which ever more general forms of representation arise through reflective abstraction.

Let us turn now to the consequences that result if we reject Piaget's epigenetic conception and like Bruner and Aebli accord the representation of cognitive structures a substantial role in the development of such structures. Since such representations are obviously culture-dependent, the relations between ontogeny and cultural evolution are completely changed. Bruner in fact arrives at this conclusion, although inherent in his model is an essentially ahistorical conception of the media of representation. Culture, for Bruner, is the embodiment of cognitive structures in the means of thinking and acting, which he describes theoretically as representations in the three media of acting, imaging, and symbolizing. With the origin of humans, cultural evolution joins the biological domain as a new form of evolution, and even contributes the selection criteria for the change of body form and expansion of the brain in the last stage of the development to homo sapiens. In the form of "models" of reality, the culture prescribes to ontogeny instruments for interpreting experience. By using these instruments the inherent models are adapted. This happens either spontaneously or initiated by interaction, for instance in education. Bruner (et. al. 1966, p. 321) accuses psychology of not being able to adequately grasp this dependence of ontogeny on culture in spite of its "lip service to such ideas as 'cultural-and-personality'. We are, alas, wedded to the idea that human reality exists within the limiting boundary of the human skin."

With this change in the relation of ontogeny and cultural evolution, Piaget's constructivist structuralism is by no means abandoned. The interpretation of the process of cultural evolution, however, is fundamentally changed. If the ontogeny of cognitive structures is not an endogenous process, but rather the structures are prescribed by representations, then the representations embody stages of development from which the developmental results of ontogeny that are possible at any particular stage of development can be reconstructed. Cultural evolution consists then in the realization of such possibilities and consequently in the creation of new conditions for ontogeny in the form of representations of further developed cognitive structures.

This conception widens the possibilities of interpreting historical developments as processes of change in cognitive structures, in comparison with those permitted by the more rigid epigenetic assumptions of Piaget. Developments can be included and explained in detail which in Piaget's conception are declared to be only accidental consequences of the primary process of constructing and becoming reflectively aware of fundamental cognitive structures. At the same time, however, assumptions about the cultural presence of a cognitive structure are submitted to stringent criteria. Its presence must be shown in the representations and the possibility of its ontogenetic realization must be shown in the given cultural milieu, whereas in Piaget's conception this is simply guaranteed by biological endowments.

Such theoretical pretensions cannot, however, be backed up with the relatively simple concept of medium used by Bruner. The wealth of relationships humans produce among perceived objects can only be tapped if one takes into account two issues. The first is the "efficiency of information processing" of secondary means of representation, as Aebli (1980/1981, Vol. II, p. 321) has characterized them theoretically. In the second, one must determine the particular role of representation for the dynamics of change through reflection in cognitive structures.

If we apply to the historical development of arithmetic these implications which constructivist structuralism has for the explanation of processes of the cultural evolution of cognitive structures, then the following theoretical alternatives result.

According to Piaget's epigenetic conception, either stages of historical development corresponding to ontogenetic stages have to be identified, or it has to be demonstrated that there are no substantial changes in cognitive structure as far back as the evidence for this thinking reaches in history. In the first case the synchronous appearance of prelogical and proto-arithmetical processes and their integration to a closed structure defining the origin of the number concept has to be located in human history. Furthermore it has to be demonstrated that, after the developmental stage of the concept of number has been reached, as well as in the second case in general, no more substantial changes in the underlying cognitive structure can be registered. In this case it has to be shown that the observable cultural evolution of arithmetical techniques merely documents the becoming aware of this structure.

The alternative assumption, that the representations of cognitive structures have the function of marking off the horizon of possibilities for the ontogenetic realizations of cognitive structures, admits many more assumptions about cognitive structures that underlie the arithmetical techniques of a particular culture. The reconstruction of these structures includes demonstrating that they can in fact also be realized ontogenetically with the representations of numbers and number operations in the culture considered. Fundamental historical changes in these structures can be conceived as an outcome of the invention of new arithmetical means which implicitly represent the new structural elements of a more developed number concept. Thus representations of numbers and number operations play a crucial role in the evolution of the cognitive structures of arithmetical thinking. The theoretical reconstruction of a historical transition from one structure to another requires on one hand demonstrating that the representations of the more developed structure can be contructed under the given cognitive conditions. On the other hand it has to be shown that this structure may indeed be the outcome of handling the new representations and elaborating the implicit consequences of their use beyond the aims and the anticipated utility of their invention.

(5) Proto-arithmetic in preliterate cultures

Until very recently there existed so-called 'primitive peoples' who still had no contact with the arithmetical techniques of more developed cultures and whose life-world was to a certain extent comparable to the living conditions of prehistoric times. This opens up the possibility of obtaining indications about very early forms of arithmetical thinking. And precisely at the time Piaget was beginning his work, attempts were being made to use ethnological literature to obtain insights into such early forms of arithmetical thinking (Levy-Brühl, 1921; Wertheimer, 1925/1967).

Now there is a difficulty, already pointed out by Max Wertheimer (1925/1967, p. 106), that arises especially when one tries to reconstruct early forms of thought which are unfamiliar to us in their logic and form of appearance: "What has been presented about the thinking of primitive peoples in the reports of research expeditions is for the most part still insufficient for real psychological knowledge. ... It is not enough to ask which numbers and operations of our mathematics the peoples of other cultures have, especially the so-called primitive peoples. ... The question must be: What kind of thought structures (Denkgebilde) do they have in this area? ... One approaches the thinking of the so-called primitive peoples with our categories--number, cause, abstract concepts--as they have emerged in our biological, social relations, in our history, with the firm prejudice that their thinking is nothing but a preliminary stage, only more vague, rudimentary or even less capable; thus one blocks his own path to a real knowledge of the actually given."

With regard to the number concept, Wertheimer (1925/1967, p. 108) conjectured the cognitive structures which are connected with numerical techniques in preliterate cultures to be fundamentally different from the modern number concept: "The ideal of universal transferability (abstractness) of thought structures (Denkgebilde), of the unified constructive arrangement of a series, need not be the necessary direction of thought in this area. These are structures, which, less abstract than our numbers, serve analogous ends or function in their place. These structures do not completely abstract from what is given in nature and from natural relationships." Careful studies carried out recently have in fact proved this conjecture to be true (for an overview see Saxe and Posner, 1983). Two examples may illustrate the character of the findings.

A case study carried out by Gay and Cole (1967) among the Kpelle of Liberia demonstrates how beliefs and practices of a traditional preliterate culture affect learning readiness for the abstract mathematical concepts of the Western culture. The results can be summarized (p. 35): "The child has no occasion in village life to use mathematical skills learned by rote in school, and has no knowledge of how to use these skills, other than to please the teacher. The subject is isolated and irrelevant, a curious exercise in memory and sly guessing". Detailed analysis of linguistic behaviour and indigenous arithmetic-like techniques have revealed the reasons for this. In Kpelle culture, dealing with quantities has a different logical background and another meaning than in Western culture. It is described by Gay and Cole (p. 52) the following way: "There is a well-developed system of terminology for putting objects into sets and materials into containers. The classification system implied by this is not commonly used, however, and is certainly not part of a Kpelle's normal response to the world. Objects are counted, but there are no independent abstract numerals. Numerals must modify a noun or a pronoun. ... Addition and substraction are performed in concrete fashion only. Multiplication and addition exist only as repeated addition and substraction. Operations are not usually performed, and when performed normally invoke only numbers as high as about 30 or 40."

Saxe (1982a, 1982b) studied sociohistorical changes in indigenous arithmetical techniques and corresponding cognitive structures among the Oksapmin of Papua New Guinea, initiated by participation in Western-style economic exchange and the introduction of money. Before having contact with Western culture the Oksapmin used body parts as a notational system for quantities, but (1982b, p. 160) "in traditional life there is virtually no context in which Oksapmin use the system to do arithmetical computations." Arithmetical techniques were closely connected with practices meaningful in the Oksapmin culture. Even such a fundamental quantity as length only has a meaning in a concrete context. Saxe (1982b, p. 166) reports with respect to Piaget's conservation tasks that "only a little over one-third of the adult subjects passed the standard test (with sticks), whereas virtually all of them demonstrated an understanding of conservation when the task was embedded in the context of measuring string bags." The building of trade stores and the introduction of money brought an arithmetical context which was very new to Oksapmin life. To solve the problems involved, store owners invented modifications of the standard indigenous body-part counting system, making it a hybrid by sticking to the counting system but changing the character of its representing quantities. A new technique for carrying out additions resulted in a differentiation between body-parts as body-parts, and body-parts as numerical symbols. Thus there was introduced a functional equivalent of the Western abstract notion of quantity within an essentially context-dependent arithmetical system.

There are five characteristics which may be typical of the proto-arithmetical techniques of preliterate cultures in general:

(1) One of the most elementary characteristics is the existence of number words, which usually also can be arranged linearly in one or more counting series. These counting series possess such familiar structures as counting steps, constructive formation of number words for larger numbers out of the number words for smaller numbers. Often they possess base numbers such as 10 in the decimal system.

(2) A characteristic of many counting series of early cultural stages is their finite character. The counting series is concluded by certain words which express only plurality as such.

This needs an explanation, for in a formal sense all counting series are finite, insofar as the giving of names to numbers ceases at some point or other because it makes no practical sense to pass out names to even larger numbers. Only with certain systems of number signs, such as for instance our modern decimal place value system, do we find formation principles for the designations which no longer display this formally finite character. The counting series of early cultural stages often display a finite character in a more strict sense than that of such a formally finite character. They contain no law of formation which embodies the potential infinity of the counting series such as, for instance, the decimal formation law of higher number words in most modern languages. The finite character of many counting series of early cultural stages in this narrower sense of lacking a generative law of formation can be taken as evidence that these counting series are finite not merely for technical reasons, but also because the underlying cognitive structure is only capable of interpreting large quantities as unspecified pluralities.

(3) A further mark of early arithmetical techniques is the existence of variegated auxiliary means for solving the arithmetical problems arising in particular situations. There can even be such aids for the counting process itself, as the various body-part counting systems show, in which parts of the body are touched

in a particular succession. Besides, such means as fingers, toes, counters, shell money, knotted strings, knotched sticks, etc. serve to carry out additive operations. This can occur in close connection with a counting series, for instance, when the counting series is adapted to the structure of such means or when designations referring to these means function directly as number words. However, in these early arithmetical systems the function of number words to identify pluralities is scarcely connected with the function of such arithmetical means of representing additive actions in particular contexts of application.

(4) Perhaps the most important characteristic of the arithmetical techniques of early cultural stages is the subordinate importance and restriction of significance of these techniques to quite specific problem areas within the aggregate context of everyday life situations. One consequence of this subordinate importance of arithmetical techniques in early cultures is one of the most common observations about these techniques: the tying of the meaning of number words and symbolic representations in arithmetical means to particular, concrete areas of application and contexts of action. This has been supported by grammatical evidence (for instance, the adjectival use of number words including the assumption of the gender or other characters of the subject) and the etymological derivations of number words. Perhaps the strongest support is given by the coexistence of different counting series or arithmetical techniques for different specific purposes which cannot be generalized to every given application, so that there is no reason to assume their integration in a superordinated cognitive structure of an

arithmetical nature. This use of different techniques independent of one another, which seem from our point of view to be parallel techniques, is much better accounted for by the assumption that the coordination of the techniques occurs only within the patterns of interpretation of the connection of their purposes in the life situation. Finally, the aversion to questions referring to number words as such, which is very often observed, or even the incapacity to recognize such questions as meaningful or answerable, can be regarded as evidence for the same conclusion.

(5) Such a dependence can be more pointedly seen in computing operations, when their performability is tied to their substantial meaning in the context of application concerned. Thus, tying addition to the condition that what is to be added must make up a meaningful unity prevents the formulation of arithmetical rules like those characteristic of modern arithmetical techniques on the basis of a developed concept of number.

The exact nature of the cognitive structures connected with proto-arithmetical techniques must be left an open question here. It is sufficient to note that the ethnological sources give evidence of an early stage of development in arithmetical thinking characterized as follows: on the one hand, numerous arithmetical techniques exist and arithmetical problems can be solved practically; on the other hand, the characteristics of the developed number concept, by and large, are not yet characteristics of the cognitive structures through which these techniques are interpreted and through which the actions to be carried out are coordinated.

(6) The Role of Early Civilizations in the Development of the Concept of Number

If we assume that the cognitive structures connected with proto-arithmetical techniques of the so-called primitive cultures can be set parallel to the structures of thought in prehistoric times when similar conditions of life prevailed, then the early civilizations take on a key role in understanding the cultural evolution which led from proto-arithmetic to the concept of number. What I have in mind is period of the origin of writing and the period in which, alongside the small autonomous economic units and living spaces in which arithmetical problems played only a subordinate role, forms of more comprehensive economic relations arose which made quite new demands on arithmetic. This is the period of the rise of the city and of the state, of the growth of an administrative apparatus with officials and their various means for conducting and representing the the activities of planning and administration. And this is the period when the first educational institutions appear in which techniques, such as writing, were handed down. In the early civilizations the first highly developed arithmetical systems with complicated computing techniques also arose.

Another fact that makes this period so very interesting is that the development of arithmetical methods in each of the different early civilizations was apparently quite independent of the others, and the evolution of the various cultures also produced very different results. One might be able to undertake comparative studies of the cultural development which in the end led to a unified concept of number but nonetheless took different courses.

Unfortunately, detailed comparative studies encounter many limitations. Among them is the problem that historical sources which provide information about the early phase of the early civilizations do not contain nearly so much information about the development of arithmetical techniques as would be necessary for well-grounded inferences about the forms of thought that emerged with these techniques.

With regard to sources, there is, however, a remarkable exception, namely the Mesopotamian civilization. It is generally known that as early as the first half of the second millennium B.C., at the time of the First Dynasty of Babylon (Old-Babylonian Period), there existed an early form of mathematics, usually called "Babylonian Algebra", according to recent research (Høyrup, 1985), however, generated by geometrical cognition. It allowed the application of an arithmetical technique which was so highly developed that problems could be formulated and solved that have the same structure as modern quadratic equations of today. This Babylonian algebra is known to us through a few hundred cuneiform tablets containing for the most part problems and their numerical solutions. It is much less well known that the thousand-year prehistory of this Babylonian algebra is also documented by a wealth of source material (Friberg, 1982). From the emergence of writing around the end of the 4th millennium B.C. onward, economic texts bear witness to the development of computing technique (Damerow, 1981); there are not merely a few hundred, but several thousand such texts.

It is even possible to obtain information about the arithmetical techniques for the time before the invention of writing. Numerical tablets such as for instance those from Ǧebel Aruda (van Driel, 1982) have come down to us from the 4th millennnium. These tablets contain no writing, but along with seal markings they contain signs which are very similar to numerical signs which are to be found in later economic texts. Beside this, there was a system of clay tokens which was used in the administrative supervision of economic transactions. If the finds have been interpreted correctly, this system, whose origins have been dated as early as the 8th millennium, was used throughout the Near East.

Thus if a chance exists at all of using historical sources to reconstruct the cultural evolution of cognitive structures from proto-arithmetic to the number concept, then Mesopotamia probably provides the most promising opportunity. I have been working on this problem for a number of years and would like to point out with examples how the results of this work can be of significance for the examination of basic assumptions of developmental psychology. I shall concentrate on a turning point in cultural evolution which presumably had the most profound effects an arithmetical thinking, namely, the invention of writing.

(7) The Origin of Numerical Notations in Mesopotamia

The clay tokens of the preliterate period are small, geometrically shaped objects (spheres, pellets, cones, tetrahedrons, cylinders, ovoids, etc.), which in their later forms are further specified by incised markings. The interpretation of these objects as tokens or symbols for particular natural objects used as means in a mechanical system of accounting for the administrative supervision of economic transactions was in essence first suggested by Amiet (1966). He recognized in the finds from Susa early evidences of an arithmetical technique which Oppenheim (1959) had shrewdly construed from a much later single find from Nuzi. This find consisted of a clay ball filled with 48 small pellets and bearing a seal and an inscription on its surface. From the inscription it can be learned that the 48 pellets represent a herd of sheep and goats. Its composition of ewes, lambs and rams is listed in detail. If one adds the individual items together, the resulting sum of all animals is 48. Oppenheim interpreted the find as a kind of document concerning the herd which made use of an arithmetical technique that could be handled even by illiterate people. The interpretation of the clay tokens from the preliterate period as early evidence of such a technique was then expanded by Schmandt-Besserat (1977, 1978) into a theory of the origin of writing.

With regard to the use of the clay tokens, this much seems certain: they represented counting and measuring units for particular goods or produce; the symbols used in a particular context express in their form the kind, and in their number the quantity, of the goods represented. If all quantitatively significant economic transactions were simultaneously reiterated with the clay tokens, then these tokens could always give an account of the inventory and changes in inventory.

Insert Fig. 1 about here

The arithmetical function of the clay tokens has been reconstructed mainly from their often being deposited in closed and mostly sealed clay bullae (cf. Fig. 1). The use of such clay bullae--which are similar to the much later find analyzed by Oppenheim--in the 4th millennium even before the invention of writing has been substantiated by numerous finds in different places (Schmandt-Besserat, 1980). These clay bullae often bear marks made with a stylus or by pressing clay tokens into them, which give an account of their contents. It is furthermore important for the interpretation of the function of the clay tokens that some of them by their form and by the incised markings correspond in their appearance to the written signs of the archaic economic texts. In so far as the meaning of these signs is known, conjectures can be made as to the meaning of the corresponding clay tokens. Schmandt-Besserat has, moreover, proposed the theory that the numerical signs impressed with a stylus in the numerical tablets dating from the preliterate period are derived directly from the marks on the surface of the clay bullae and that the written signs of the archaic economic texts originated in pictures of the appropriate clay tokens of the preliterate period. It is, however, a controversial question whether the temporal sequence of this theory can really be supported by the available findings (Liebermann, 1980; Dittmann, 1986; Jasim/Oates, 1986; Shendge, 1983).

The clay tokens bear witness to an arithmetical technique that is older than writing. With respect to the question of the development of arithmetic the crucial problem consists in construing the kind of number representation and number use from the combinations of clay tokens substantiated in the clay bullae and from the "numerical signs" of the numerical tablets. Unfortunately only a few reports on the contents of clay bullae have been published so far (overview: Schmandt-Besserat, 1980), and there is no systematic study of the numerical tablets from which one could learn which of the sign combinations are from tablets definitely predating the invention of writing. Schmandt-Besserat at first proposed the thesis that besides the clay tokens which represented particular goods there were also tokens for abstract numbers (Schmandt-Besserat, 1977, 1978). Since then, however, she has revised her view in favor of a strict one-to-one correspondence between clay tokens and the appropriate counting and measuring units (Schmandt-Besserat, 1983). All indications are that numbers were represented using the clay tokens only in one-to-one correspondence between token and object and that this principle of representation was transferred mechanically to the marks on the surface of the clay bullae. Should this conjecture be confirmed, then the clay tokens testify to a very simple arithmetical technique which would be applied only to small amounts and to a limited number of different goods.

Let us compare this preliterate arithmetical technique with the arithmetic of the archaic numerical signs. In Mesopotamia writing material consisted primarily of clay tablets, on which the signs of cuneiform writing were presse with a stylus. The oldest group of these tablets are called archaic texts. The written characters of these texts are pictograms; they do not yet possess the characteristic cuneiform elements which gave the later writing its name. The writing of the archaic texts still is not adequately understood. However, since many of the pictograms of the writing can be identified as predecessors of later cuneiform characters, it is often possible to construe their meanings from the later writing, which has been substantially deciphered. Moreover, for many characters the pictorial sense can still be perceived.

The latest group of the archaic texts, a group of about 370 texts from Ur (Burrows, 1935), will not be taken into account in the following. The remainder consists for the most part of some 3900 texts from Uruk, which can be divided paleographically into two groups: the elder can be assigned to the level Uruk IVa, the younger to the level Uruk III. Only about 600 of these texts have been published up to now (Falkenstein, 1936). In addition to the texts from Uruk, there are also about 240 texts from Gemdet Nasr (Langdon, 1928), which according to paleographic evidence date from the same period as the texts from the level Uruk III. The results on the numerical signs in the whole corpus of these texts which I report here are based on a statistical analysis of these numerical signs, which I conducted using a computer, as well as on subsequent philological analyses done in cooperation with Robert Englund and Hans Nissen.(Damerow/Englund, 1986) who are preparing the bulk of the unpublished text for publication.

The archaic texts are for the most part economic records, the rest are lexical lists, which presumably were used in the training of scribes (Nissen, 1985, 1986). The written characters of the archaic texts are mostly word-signs with a limited number of semantic functions, among them in particular the following: the designation of the supervised goods, designation of the purpose of the record (for instance, the designation of a list of provisions by the sign for "food"), the designation of persons (by combinations of signs) and of titles, the designation of the function of individual entries (e.g., as the sum of the other entries). The tablets are divided in different ways into cases for the individual items. This format is often essential for the interpretation of the entries (cf. Fig. 2).

Insert Fig. 2 about here

Let us now turn to the numerical signs of these texts. As opposed to the written characters, which in these older texts were still drawn in the clay with a pointed stylus, these numerical signs were pressed deep into the clay with rounded styluses of various sizes as in the older numerical tablets. These signs represent, as did the clay tokens of the preliterate period, counting and measuring units, and, as was the case for the clay tokens, here too a particular quantity is represented by repeating the sign that represents the counting and measuring units in correspondence with the number of units to be represented. Now, however, these signs are quite systematically subsumed under higher valued signs whenever a particular number is reached. The numerical signs of these texts thus constitute coherent systems of signs with fixed size relation between each two signs of one and the same system.

Insert Table 1 and Table 2 about here

Table 1 gives an overview of the numerical signs of the archaic texts, table 2 gives the reconstructed numerical systems. The systems are noted as factor diagrams, whereby the numbers above the arrows tell us what multiple of the value of the right-hand sign the higher-valued sign on the left possesses.

There are five basic numerical systems, designated here as sexagesimal system (system S), bisexagesimal system (system B), SE system (system Š), GAN_2 system (system G), and EN system (system E).

The sexagesimal system is used for discrete objects of various kinds. This system with its strict sexagesimal structure corresponds to the series of number words of the Sumerian language (Powell, 1973), which is assumed to have been the language of the inhabitants of the southern part of Mesopotamia at the time of the invention of writing. From this it cannot, however, be concluded that this system of numerical signs arose as a representation of the series of number words. During the entire third millennium numerical signs were used in the texts almost exclusively; number words are for this period practically unattested. Our knowledge of the Sumerian series of number words is derived from lexical lists written down more than 1200 years after the archaic texts. Only the number words from one to ten are attested by two earlier sources, for which the time interval is only a few hundred years (Edzard, 1980; Civil, 1982), found recently in excavations of the archive at Ebla. It is therefore completely impossible to decide whether the Sumerian series of number words was as developed as the sexagesimal system of numerical signs before this system arose, for which, in the texts from the later level Uruk III, instances are to be found up to a sign with a numerical value of 36,000.

In particular, it cannot be decided whether the series of number words already had a sexagesimal structure before the invention of writing. Only with the number 3.600 does the sexagesimal system reach the second power of the base number 60. Etymologically, one can derive from the number word series only the counting step at 60 with the word g̃eš, a counting step of unknown etymology, and a counting limit at 3.600 through the word šar (= all/everything). The number word g̃eš is the last individual name of this series of number words. The word g̃eš-u (sixty-ten) for the number 600 only expresses the counting of units of sixty. In general, the series of number words follows the number writing so exactly that the suspicion of a much later, artificial formation of the series of number words is hard to avoid.

For the sexagesimal system of numerical signs in the archaic texts we can thus only ascertain a close connection with the formation of the series of number words and assume with a certain probability that the šar represented a counting limit which in the course of the change to written representation was identified with the second power of the counting unit sixty and was surpassed in the usual manner by counting higher counting units.

The bisexagesimal system of the archaic texts is a second system of numerical signs for discrete objects, which agrees with the sexagesimal system up to the sign with the value 60. There are not indications of this second system in the series of number words, unless one wants to take the break in the formation laws of the number words at 120 as evidence of a former counting limit. The numerical signs of the sexagesimal and bisexagesimal systems were used completely separately from each other--nowhere, for instance, is the sign for 600 in the sexagesimal system used together with the sign for 120 of the bisexagesimal system. Here we can see the problematic character of ascribing modern, abstract numbers to the numerical signs of the archaic texts.

The bisexagesimal system was used to count particular things, especially bread, cheese (?), and a certain kind of fish. It is difficult to give any reasonable explanation for the exclusive use of the bisexagesimal system in these contents. The only common feature of the goods to which the bisexagesimal system was applied seems to be that they were foodstuffs for mass consumption. Hence, they may have had a special function in the ration system. A few hundred years later, the bisexagesimal system disappears without trace from the economic texts in favor of the surviving sexagesimal system.

The ŠE system is a system of numerical signs, the signs of which in all probability designated measures of grain. Often, the sign ŠE for grain is added for closer characterization. The smaller units follow an untypical formation; they are formed as fractions which stand in no whole-number size relationships to one another. These small units served above all to designate the types of bread according to the amount of grain contained in them. The ŠE system is found only in the archaic texts; in its place in the later economic texts we find another system of numerical signs with a very similar function, whose arithmetical structure was changed with every reform of the measures of grain. This fact proves that the numerical signs of the ŠE system and the corresponding systems of later periods represented very concrete units of measurement and not abstract ideas of quantity.

The GAN$_2$ system is a system of numerical signs, in which the signs designate field measures. As a rule this was expressed by adding the sign GAN$_2$ (=field, field area) or the sign KI (=place). The GAN$_2$ system, like the sexagesimal system, remained in use for a long time without any change in the arithmetical structure. The reason for this should presumably be seen in its function for a particular procedure for calculating the area of fields from the results of measuring lengths, which can already be found in the archaic texts from Ǧemdet Nasr.

The EN system ist the least well known of the systems which we identified in the archaic texts. Except for one fragment, texts with notations in this system stem from the same find spot. All the texts are related to the same content. Unfortunately, however, up to now we have been unable to grasp the meaning of the sign designating this content.

There are several variants of the basic systems--we could identify five--which need not be discussed here. They generally have specific functions in connection with particular applications. The function of such a variant may be quite different in various contexts of application. For instance, the signs of system S' which is a variant of the sexagesimal system probably designate in animal husbandry those animals that have died (cf. Fig. 3). In connection with texts on beer production, on the other hand, these numerical signs are used to count jugs of a particular type of beer (Green, 1980; Damerow/Englund, 1986, p. 131).

Insert Fig. 3 about here

One particular pecularity of the systems of numerical signs in the archaic texts should be emphasized: the base signs are used in several systems at the same time but with completely different substantial and arithmetical meanings. Taken by themselves they are ambiguous, and acquire their precise sense only from the context of their application. This is perhaps the most remarkable property of the systems of numerical signs in this early phase of the development of writing. Even the most elementary signs are not exempt from such ambiguity: the signs for the "numbers" 1 and 10 of the sexagesimal system represent at the same time two units of the ŠE system; these, however, stand not in the size relation 1 to 10 but in the size relation 1 to 6 (cf. Fig. 4). On the usual interpretation of the signs as "number signs" in an absolute sense, such a state of affairs seemed so unbelievable that only in 1978 did the mathematician Jöran Friberg (1978, 1979)--coming from outside the field--point out and correct a decade-old mistake. It had been assumed against all evidence from the available texts that the ŠE system had a decimal structure.

Insert Fig. 4 about here

In conclusion we can characterize the numerical signs of the archaic texts as follows: these are organized in highly complex systems with which great arithmetical spans between individual counting or measuring units as well as large quantities can be mastered arithmetically. The signs, however, possessed scarcely any context-independent meaning. The concrete content determines how the signs are to be interpreted arithmetically and which arithmetical operations can be applied to them.

(8) Psychological Prerequisites and Consequences of the Written Representation of Proto-arithmetical Ideas

With the proto-arithmetical techniques of the preliterate period witnessed by the clay tokens, and the respective techniques of the archaic economic texts, we have encountered two techniques which stand for two very different stages of development in the cultural evolution of the concept of number, although the latter probably grew out of the former and the periods of the two techniques overlap. The preliterate representations of quantities by a corresponding number of tokens is an arithmetical technique tied to small amounts and to a limited area of application, which presumably has its origin in the village culture of the prehistoric settlements and settlement complexes in the Near East. The complicated systems of numerical signs in the archaic texts, on the other hand, are an invention made by administrators in the urban economic centers of the late Uruk period. They aided the economic supervision as well as the mastering of new kinds of problems of the early city-states which simply from their order of magnitude have no counterparts in the limited framework of a village economy.

If we compare these two techniques with the ethnological and psychological findings, the first thing we can ascertain is that the preliterate technique of representation using clay tokens is in no way different from the arithmetical techniques of the so-called primitive cultures. The technique of representation by clay tokens displays the same ties to particular, concrete contexts of application and action as do the arithmetical ideas of the primitive cultures, especially through the simultaneous representation of quality and quantity. This applies above all to the arithmetical operations, which in this technique are bound to particular substantial actions by the type of representation itself. At the present state of knowledge there is no indication which could give any reason to suspect a different type of cognitive structure behind this technique than that of the context-bound numerical structures analyzed by Wertheimer.

With the proto-arithmetical technique of representation by clay tokens, quantities are represented by one-to-one correspondences, and the additive operations are conducted exclusively enactively by reiterating the quantitatively relevant transactions in the medium of representation. The genesis of such techniques is not bound to any cognitive prerequisites of a specifically arithmetical nature, but only to the appropriate handling of the symbol function, which is attributed to the especially early developmental results both in ontogeny and historical development. The development and application of such a technique presupposes the ability to ascribe to objects a symbolic meaning--a clay token must be able to symbolize a sheep, which it might have to represent in this technique. Their appropriate handling, furthermore, calls for the ability to remember and anticipate the results of actions to a certain degree. Only these elementary abilities must be assumed as cognitive prerequisites, in order to be able to reconstruct psychologically the genesis of the proto-arithmetical technique of representation by clay tokens.

With this, it is by no means implied that the application of this techniques does in fact occur at such a low cognitive level. Rather, one would expect as a cognitive effect of the extensive use of such a technique that mental operations will be constructed intellectually that are appropriate to the correspondence actions, and thus that essential prerequisites for the concept of number will arise. Such correspondence operations completely match the system of mental operations described by Piaget as the basis of the conservation of quantity. One can scarcely avoid the compelling conclusion that such a technique of the intellectual construction of correspondence must have brought forth as a result of reflection an equivalent to the conservation of quantity defined ontogenetically by Piaget. There is, however, no reason to assume that at the same time all the other structural elements were abstracted which in the epigenetic conception of Piaget are conceived as universals of the number concept.

The study of the impact of the invention of writing upon the proto-arithmetical cognitive structure of the preliterate period offers a unique chance to examine whether this inference holds true. We are enabled to investigate the next step of cultural evolution which consisted in a reflective processing of proto-arithmetical operations in the medium of symbolic representation by numerical signs, i.e. in a secondary medium of representation (in Aebli's sense). We can suppose with some certainty that the use of sealed clay bullae already indicates applications of the technique of representing quantities by clay tokens, to new problem contexts. Advances such as differentiation, specialization, and normalization in the technique of representation attest the cognitive progress which is probably implied. The real transition from the proto-arithmetical technique of representation through clay tokens to the use of numerical signs in the archaic tablet texts is, however, not to be understood as an increase in complexity through the extensive use of already available possibilities. Rather, it involves a change of the medium of representation, which as such presupposes neither changed cognitive structures nor an expanded arithmetical technique. Apparently it did, however, have lasting cognitive effects among those who, as specialists in an administrative organization, operated extensively with the new medium.

These effects are indicated first of all by the elaborated systems of numerical signs in the archaic texts. The written fixing of size relationships among units of measure which were originally determined as natural measures detaches them from the real context of conditions determining them. Simply by representing them in the secondary medium of written symbols, it makes them the object of possible formal norming even beyond the area of intuitively comprehensible or imaginable size relationships. The results are the systems of numerical signs, still context bound but transcending individual contexts of application, which we find even in the oldest group of archaic texts.

The effects are also indicated by the arithmetical operations which can be found in the archaic texts. On the action level of pure enactive representation the relationship between part and whole can only be represented successively. On the level of symbolic representation by written signs, however, the sum does not eliminate the summanda. The relationship between summandum and sum is represented spatially and synchronously on the written tablets. The emergence of a cognitive structure that comprehends at least the operations of elementary additions is surely the necessary consequence. In fact even the somewhat formalized way of writing the combinations of repeated signs, which represent the various quantities in the texts seems to make evident that these combinations are not merely the results of enactively completed operations, but rather already at least partly represent the results of mental operations which refer primarily to the numerical signs and only indirectly to the objects represented. Moreover it seems to be impossible that the scribes of the archaic texts could have exhibited the extraordinary intellectual complexity which is now connected with many applications of the signs without a reflective use of the rules for handling the numerical sign systems.

As a result of this change of medium we obtain a situation to which there is no parallel either in the ethnological attested proto-arithmetical techniques or in the modern abstract concept of number. The systems of numerical signs of the archaic texts with their strong ties to the contexts of things and to external purposiveness of the situation of application still display all the traits with which the cognitive structure of proto-arithmetical techniques was characterized. But the semantic functions of the representations of quality and of quantity are now to a certain extent already separated and given to different groups of signs, namely the pictograms and the number signs. The systems of numerical signs are handled formally to an extent, and independently of the individual problem in a manner, that one would have expected within the framework of Piaget's epigenetic conception only on the basis of a developed concept of number.

Hence the arithmetical techniques documented by the archaic texts represent a missing link in the cultural evolution from proto-arithmetic to the number concept providing that this process does not display the synchronous character of the emergence of the various structural elements of the number concept, which we can ascertain in ontogenetic development and on which Piaget has based his epigenetic conception. The fact that the peculiarities of the context-bound arithmetic of the archaic text can be interpreted as consequence of the change of medium suggest a substantial influence of culturally transmitted representations on the emergence of cognitive structures in ontogenetic development.

It is not my intention to draw far reaching conclusions from a single example. I want only to demonstrate with this example what possibilities the development of the early civilizations offers for clarifying basic theoretical problems of a theory of cognitive structures embracing ontogeny and historical development.

The coherent interpretation of longer developmental processes seems to me to be necessary to carry out convincingly such a program. In Mesopotamia it took approxismately one thousand years before the invention of the place value system brought about a similarly fundamental turning point in the cultural evolution of the concept of number as the one I have attributed to the invention of writing. Elsewhere (Damerow, 1981) I have attempted to argue that only at this point we can assume the existence of an abstract idea of number in the domain of Mesopotamian civilization, an idea that due to its dependence on the specific form of representation still displays crucial differences to the developed concept of number. The thousand years between the invention of writing and the development of place value notation are characterized by an extreme progress in the capacity of computing techniques, which likewise needs an explanation.

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References

Aebli, H. (1963). Über die geistige Entwicklung des Kindes. Stuttgart.

Aebli, H. (1980, 1981). Denken: Das Ordnen des Tuns. 2 Vol. Stuttgart.

Amiet, P. (1966). Les Elamites inventaient l'ecriture. Archaelogia, 12, 17-23.

Bruner, J. S., et al. (1966). Studies in cognitive growth. New York/London/Sidney.

Burrows, E. (1935). Ur excavations texts II: Archaic texts. London.

Civil, M. (1982). Studies on early dynastic lexicography. I. Oriens Antiquus, 21, 1-26.

Damerow, P. (1981). Die Entstehung des arithmetischen Denkens. In: P. Damerow & W. Lefèvre (Eds.), Rechenstein, Experiment, Sprache. Stuttgart, 11-13.

Damerow, P., & Englund, R. K. (1986). Die Zahlzeichensysteme der Archaischen Texte aus Uruk. In: M. W. Green & H. J. Nissen (Eds.), Zeichenliste der Archaischen Texte aus Uruk. Berlin (Archaische Texte aus Uruk. Vol. 2), 117-156.

Dittmann, R. (1986). Seals, sealing and tablets. In: Beihefte zum Tübinger Atlas des Vorderen Orients. Reihe B (Geisteswissenschaften). Bd. 62. Wiesbaden.

van Driel, G. (1982). Tablets from Jebel Aruda. In: G. van Driel et al. (Eds.), Zikir Šumin. Leiden, 12-25.

Edzard, D. O. (1980). Sumerisch 1 bis 10 in Ebla. Studi Eblaiti, 3, 121-127.

Falkenstein, A. (1936). Archaische Texte aus Uruk. Leipzig.

Friberg, J. (1978). A method for the decipherment, through mathematical and metrological analysis, of Proto-Sumerian and Proto-Elamite semi-pictographic inscriptions. CTH-GU Göteborg.

Friberg, J. (1979). Metrological relations in a group of semi-pictographic tablets of the Jemdet Nasr type, probably from Uruk-Warka. CTH-GU Göteborg.

Friberg, J. (1982). A survey of publications on Sumero-Akkadian mathematics, metrology and related matters (1845-1982). CTH-CU Göteborg.

Gay, J., & Cole, M. (1967). The new mathematics and an old culture. A study of learning among the Kpelle of Liberia. New York.

Green, M. W. (1980). Animal husbandry at Uruk in the Archaic period. Journal of Near Eastern Studies, 39, 1-35.

Høyrup, J. (1985). Algebra and naive geometry. An investigation of some basic aspects of Old Babylonian mathematical thought. forthcoming (Prepr. Roskilde Univ. Center).

Jasim, S. A., & Oates, J. (1986). Early tokens and tablets in Mesopotamia: New informations from Tell Abada and Tell Brak. World Archaelogy, 17, 348-362.

Langdon, S. (1928). Pictographic inscriptions from Jemdet Nasr. Oxfort.

Levy-Brünl, L. (1921). Das Denken der Naturvölker. Wien/Leipzig.

Liebermann, S. J. (1980). Of clay pebbles, hollow clay balls, and writing: A Sumerian view. American Journal of Archaelogy, 84, 339-358.

Nissen, H. J. (1985). The emergence of writing in the Ancient Near East. Interdisciplinary Science Reviews, 10, 349-361.

Nissen, H. J. (1986). The Archaic texts from Uruk. World Archaelogy, 17, 317-334.

Oppenheim, A. L. (1959). On an operational device in Mesopotamian bureaucracy. Journal of Near Eastern Studies, 39, 121-128.

Piaget, J. (1950). Introduction à l'épistémologie génétique. 3 Vol. Paris.

Piaget, J. (1967). Biologie et connaissance. Paris.

Piaget, J., & Garcia, R. (1983). Psychogenèse et histoire des sciences. Paris.

Piaget, J., & Szeminska, A. (1941). La genèse du nombre chez l'enfant. Neuchatel/Paris.

Powell, M. A. (1973). Sumerian numeration and metrology. Ann Arbor (Univ. Microfilms).

Saxe, G. B. (1982a). Developing forms of arithmetical thought among the Oksapmin of Papua New Guinea. Developmental Psychology, 18, 583-594.

Saxe, G. B. (1982b). Culture and the development of numerical cognition: Studies among the Oksapmin of Papua New Guinea. In: C. J. Brainerd (Ed.), The development of logical and mathematical cognition. New York, 157-176.

Saxe, G. B., & Posner, J. (1983). The development of numerical cognition: Cross-cultural perspectives. In: H. P. Ginsburg (Ed.), The developmental of mathematical thinking. New York, 291-317.

Schmandt-Besserat, D. (1977). An Archaic recording system and the origin of writing. Syro-Mesopotamian Studies, 31-70.

Schmandt-Besserat, D. (1978). The earliest precursor of writing. Scientific American, 238, 38-47.

Schmandt-Besserat, D. (1977). An Archaic recording system and the origin of writing. *Syro-Mesopotamian Studies*.

Schmandt-Besserat, D. (1978). The earliest precursor of writing. *Scientific American*, *238*, 38-47.

Schmandt-Besserat, D. (1980). The envelopes that bear the first writing. *Technology and Culture*, *21*, 357-385.

Schmandt-Besserat, D. (1983). Tokens and counting. *Biblical Archaelogist*, *46*, 117-120.

Shendge, M. J. (1983). The use of seals and the invention of writing. *Journal of the Economic and Social History of the Orient*, *26*, 113-136.

Strommenger, E. (1980). *Habuba Kabira*. Mainz.

Wertheimer, M. (1925/1967). Über das Denken der Naturvölker, Zahlen und Zahlgebilde. In: M. Wertheimer, *Drei Abhandlungen zur Gestalttheorie*. Erlangen 1925 (Reprint Darmstadt 1967), 106-163.

Table 1: Numerical signs in the archaic texts from Uruk.
(Source: Damerow/Englund 1986)

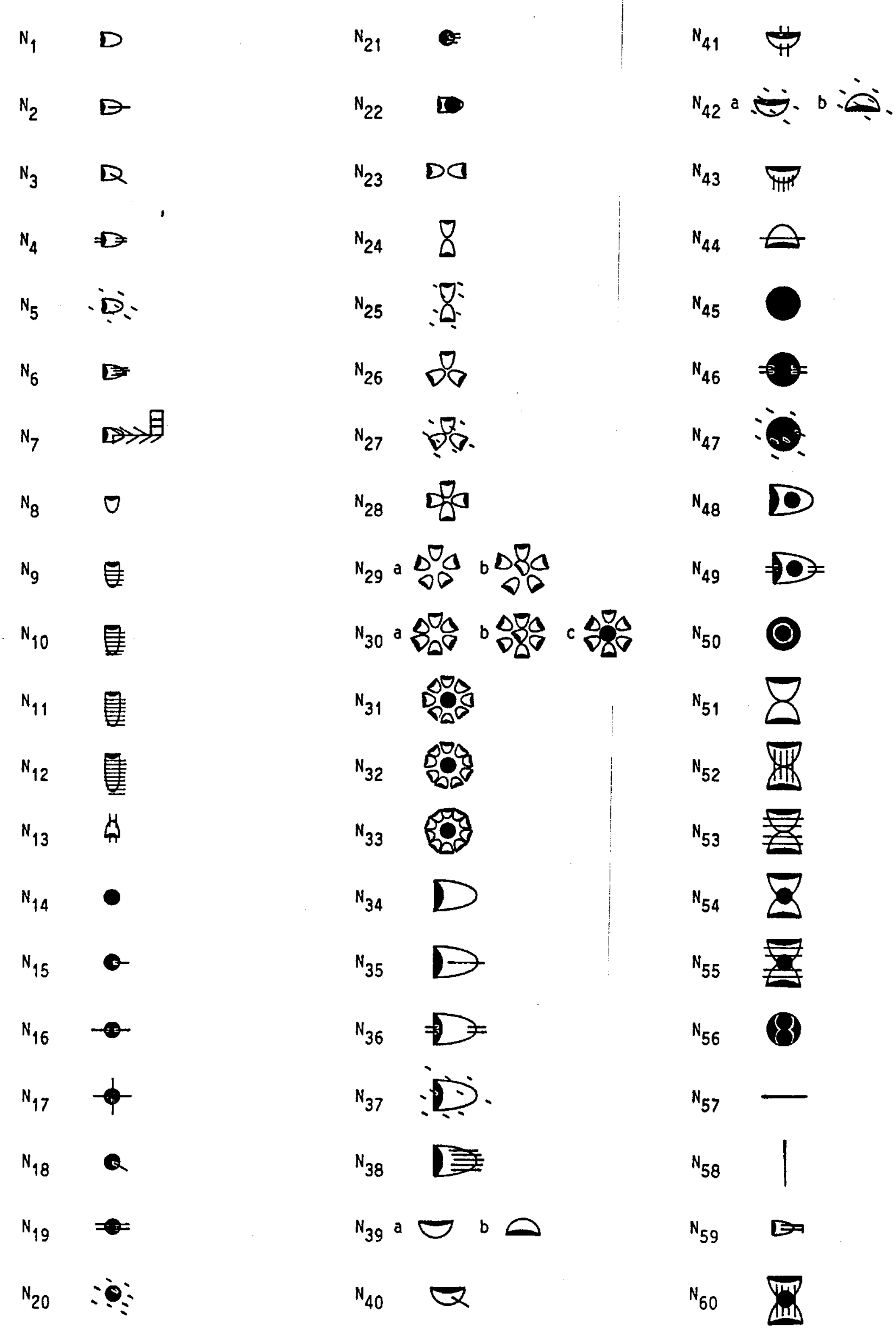


Table 2: Systems of numerical signs identified in the archaic texts from Uruk.

(Source: Damerow/Englund 1986)

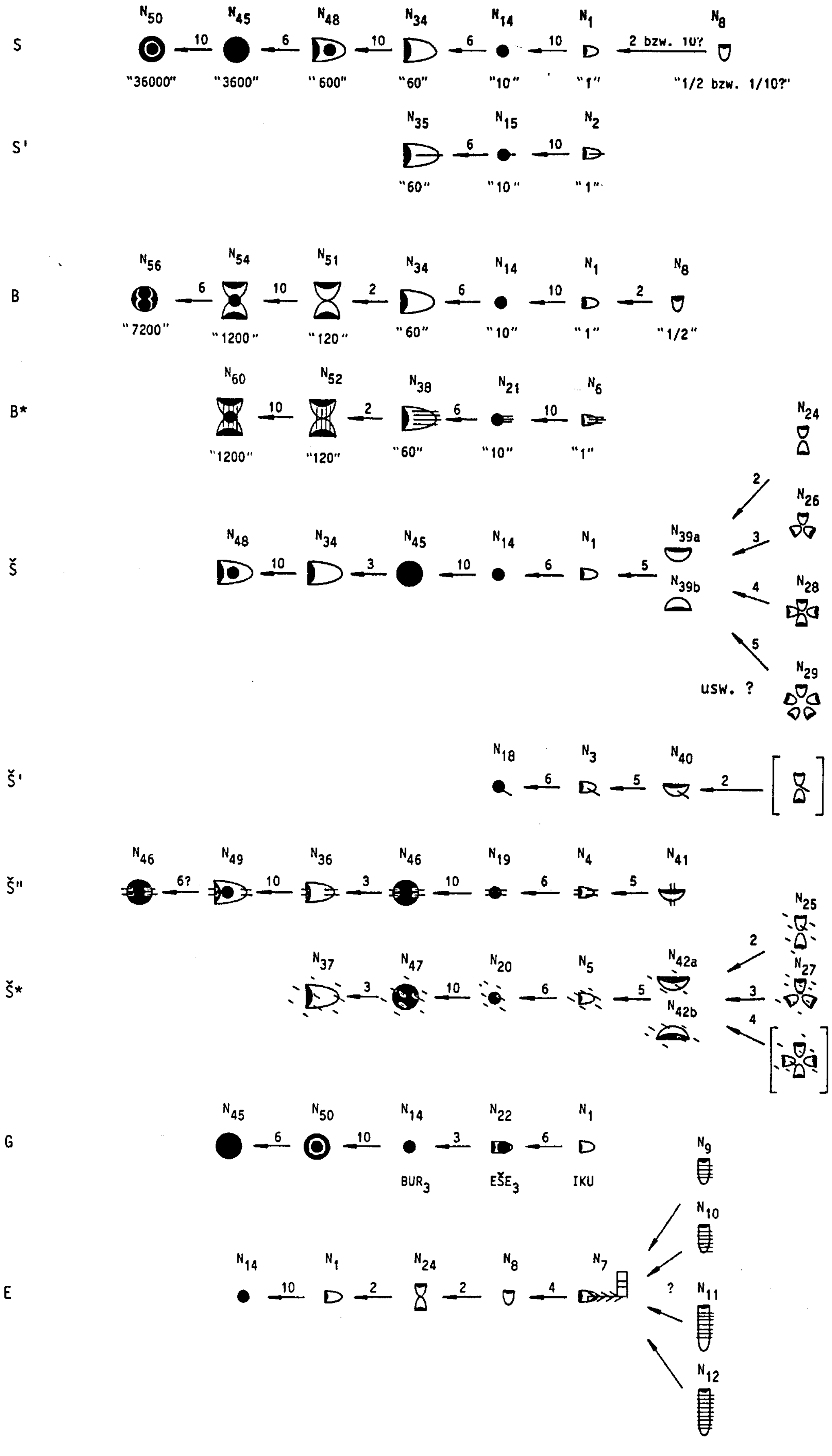


Figure 1: Clay bulla of the preliterate period from Habuba Kabira (see Strommenger, 1980). The bulla originally contained tokens probably corresponding to the impressions on the surface. The bulla was broken, however, when it was found and only two tokens have been preserved.

(Photo: Deutsche Orientgesellschaft e. V.)

Figure 2: Two archaic administrative tablets with similar content. The tablets are divided into cases, each case with a pictografic sign representing the nature of the counted or measured object and mostly one or more numerical signs. At the bottom of both tablets there is a signature. The tablets show the importance of the tablet format. They represent something like a business form. On the left tablet for some unknown reason some quantities are not filled in. The tablets probably deal with products of animal husbandry.

(Photo: Deutsches Archäologisches Institut, Abt. Bagdad)

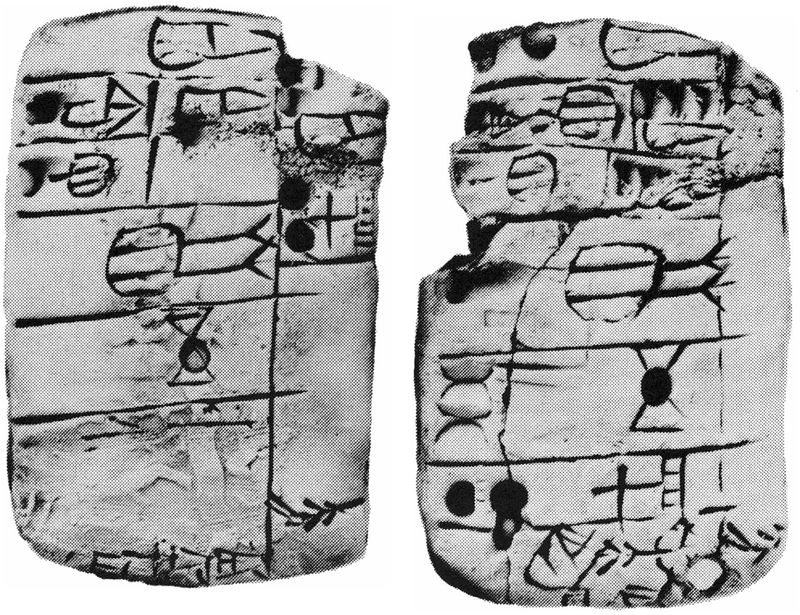


Figure 3: Reverse of an archaic administrative tablet containing information about a small herd of sheep. It probably is an annual account of a shepherd with details about the composition of the herd written on the obverse (which is destroyed) and totals and subtotals written on the partly preserved reverse of the tablet. According to a tentative interpretation the middle column gives the information that the herd at the end of the year consists of 33 sheep, that the number of offspring for the year is 12, that 2 sheep have been taken away for some reason, and that 4 sheep (written with signs of another system) died. The left column gives the total of 39 sheep, the offspring not included. Below this information the total quantity of a produce (probably a milk product) is broken away. The right column is less clear. The meaning may be that 10 sheep have been fattened.

(Source: Green, 1980)

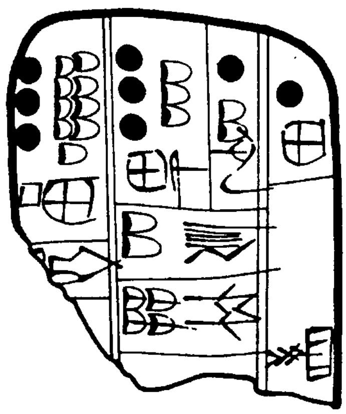
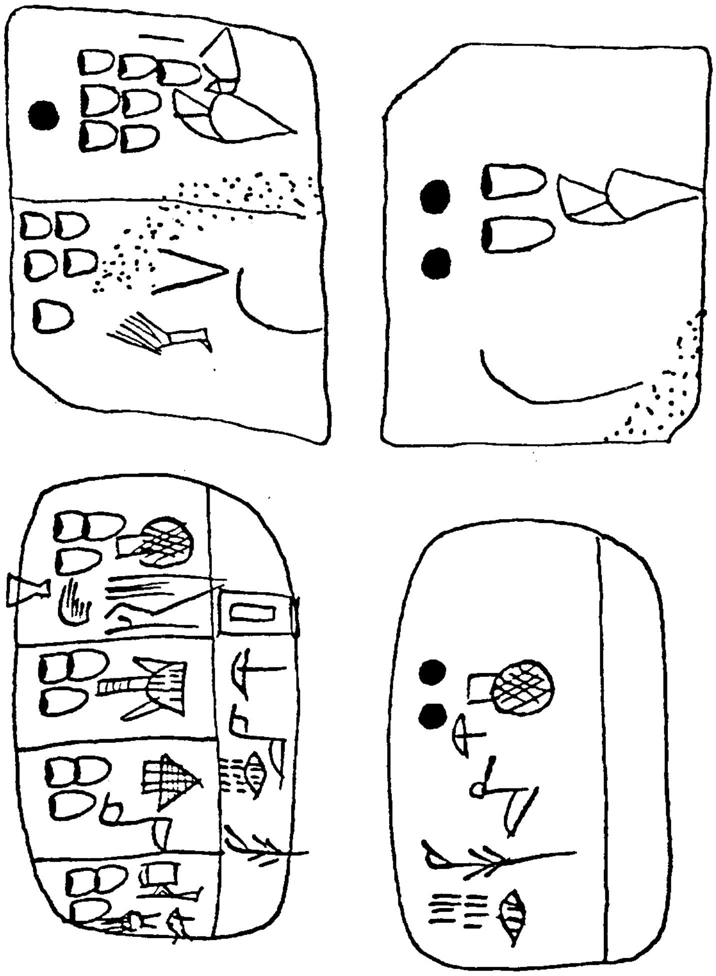


Figure 4: Two archaic tablets showing how the arithmetical meaning even of the most common numerical signs depends on context. Both tablets bear on the reverse the totals of the obverse entries. The upper tablet uses the sexagesimal system so that the total of 22 is written with two round and two half circle impressions. The lower tablet is written with the same signs but now they belong to the SE-system. So the total of 12 is given by two round marks.

(Copy H. Nissen)



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