Received: 31 July 2020 | Revised: 4 August 2021 | Accepted: 9 August 2021

DOI: 10.1112/mtk.12115

RESEARCH ARTICLE

Mathematika

Lower bounds for moments of zeta and *L*-functions revisited

Winston Heap[1] | **K. Soundararajan**[2]

[1] Max Planck Institute for Mathematics, Bonn, Germany

[2] Department of Mathematics, Stanford University, Stanford, California, USA

Correspondence
Winston Heap, Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany.
Email: winstonheap@gmail.com

Funding information
NSF, Grant/Award Number: DMS1854398; Simons Foundation

Abstract

This paper describes a method to compute lower bounds for moments of ζ and L-functions. The method is illustrated in the case of moments of $|\zeta(\frac{1}{2}+it)|$, where the results are new for small moments $0 < k < 1$.

MSC 2020
11M06 (primary)

1 | INTRODUCTION

This paper reexamines the problem of obtaining lower bounds of the correct order of magnitude for moments of the Riemann zeta function on the critical line, and related problems for central values in families of L-functions. Our work is motivated by recent work on the complementary problem of obtaining upper bounds for such moments. For example, [9] enunciates the principle that an upper bound for a particular moment (with a little flexibility) may be used to establish upper bounds of the correct order of magnitude for all smaller moments. Recent work of the authors with Radziwiłł [4] provides such upper bounds for all moments of the Riemann zeta-function below the fourth moment. In those papers, one key idea is to approximate Euler products that mimic suitable powers of the zeta-function using Dirichlet series of small length. The aim of this paper is to demonstrate how that idea may also be used to establish lower bounds of the right order of magnitude for all moments of the Riemann zeta-function.

Theorem 1. *Let T be large. Uniformly for $(\log T)^{-\frac{1}{2}} \leqslant k \leqslant (\log T)^{\frac{1}{2}-\delta}$ (for any fixed $\delta > 0$), we have*

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} dt \geqslant C_k T (\log T)^{k^2},$$

where we may take $C_k = C_1 k$ in the range $k \leqslant 1$, and $C_k = (C_2 k^2 \log(ek))^{-k^2}$ for some absolute positive constants C_1 and C_2.

There is a long history concerning such lower bounds for ζ and L-functions. To place our result briefly in context, we recall that in the range $k \geqslant 1$ such a lower bound was established by [8], although our quantification of C_k is better and the proof arguably simpler. While we have stated the result in a uniform range of k, the interest of the result is really in the case when k is a bounded real number. For example, in the range $k \geqslant \log\log T$, by modifying the constant C_2 and using Hölder's inequality, one can consider just integer values of k, where it is easy to obtain lower bounds (see, for example, [2]).

Theorem 1 is new in the range $0 < k \leqslant 1$. Previous work of Heath–Brown [5] had established such a bound for rational k in this range, and for real k such a bound was known to hold conditional on the Riemann hypothesis (see [5, 11, 12]). In the range $c(\log\log T)^{-\frac{1}{2}} \leqslant k = o(1)$, Laurinchikas [6] has shown that the $2k$th moment is $\sim T(\log T)^{k^2}$. The constant C_k in our result tends to zero as $k \to 0$; with more effort, our argument could be made to yield $C_k \gg 1$ for all $k \leqslant 1$, but we have not done so in the interest of keeping the exposition simple.

Combining the upper bound result of Heap, Radziwiłł, and Soundararajan [4] with the lower bound of Theorem 1, we obtain the following corollary.

Corollary 1. *For T large, uniformly for $(\log T)^{-\frac{1}{2}} \leqslant k \leqslant 2$ we have*

$$T(\log T)^{k^2} \gg \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} dt \gg kT(\log T)^{k^2}.$$

The moments of $\zeta(\frac{1}{2}+it)$ encode information on the distribution of large values of $|\zeta(\frac{1}{2}+it)|$. In [13] it was observed that the $2k$th moment of $|\zeta(\frac{1}{2}+it)|$ should be dominated by values of size $(\log T)^k$, which should occur on a set of measure about $T/(\log T)^{k^2}$. On RH, it was shown in [13] that the measure of $\{t \in [T, 2T] : |\zeta(\frac{1}{2}+it)| \geqslant (\log T)^k\}$ is $T(\log T)^{-k^2+o(1)}$ for any fixed positive k. From Corollary 1, we may obtain a sharper form of such a result unconditionally in the limited range $0 < k < 2$.

Corollary 2. *Uniformly in the range*

$$\sqrt{\log\log T}\log\log\log T \leqslant V \leqslant 2\log\log T - 2\sqrt{\log\log T}\log\log\log T,$$

we have

$$\text{meas}\{t \in [T, 2T] : |\zeta(\tfrac{1}{2}+it)| \geqslant e^V\} = T\exp\left(-\frac{V^2}{\log\log T} + O\left(\frac{V\log\log\log T}{\sqrt{\log\log T}}\right)\right).$$

Recall that Selberg's central limit theorem (see [10] for a proof related to ideas of this paper) states that for t chosen uniformly from $[T, 2T]$, $\log|\zeta(\frac{1}{2}+it)|$ has an approximately normal distribution with mean 0 and variance $\sim \frac{1}{2}\log\log T$. Radziwiłł [7] has established a uniform version of this result showing that for $V \leqslant (\log\log T)^{\frac{3}{5}-\epsilon}$ one has

$$\text{meas}\,\{t \in [T, 2T] : \log|\zeta(\tfrac{1}{2}+it)| \geqslant V = \Delta\sqrt{\tfrac{1}{2}\log\log T}\} \sim \frac{T}{\sqrt{2\pi}}\int_{\Delta}^{\infty} e^{-x^2/2}dx.$$

Corollary 2 gives a crude version of such a result but in a wider range for V.

2 | SETUP AND PLAN OF THE PROOF

Since Theorem 1 is really new only in the range $0 < k \leqslant 1$, we give a detailed proof in this range. In Section 6, we briefly indicate the modifications to the argument needed to establish Theorem 1 for $k \geqslant 1$, and also discuss lower bounds for moments of central values of L-functions in families.

Throughout, $\log_j$ will denote the j-fold iterated logarithm. Let T be large and assume that $1/\sqrt{\log T} \leqslant k \leqslant 1$. Let ℓ denote the largest integer such that $\log_\ell T \geqslant 10^4$. Define a sequence T_j by setting $T_1 = e^2$, and for $2 \leqslant j \leqslant \ell$ by

$$T_j := \exp\left(\frac{k\log T}{(\log_j T)^2}\right).$$

Note that T_2 is already large. Further, the sequence T_j is in ascending order, and lastly $k\log T \ll \log T_\ell \leqslant 10^{-8}k\log T$.

For each $2 \leqslant j \leqslant \ell$, set

$$\mathcal{P}_j(s) := \sum_{T_{j-1}\leqslant p<T_j} \frac{1}{p^s}, \qquad \text{and} \qquad P_j = \mathcal{P}_j(1) = \sum_{T_{j-1}\leqslant p<T_j} \frac{1}{p}.$$

Note that

$$P_j = \log\frac{\log T_j}{\log T_{j-1}} + O\left(\frac{1}{\log T_{j-1}}\right) \sim 2\log\left(\frac{\log_{j-1} T}{\log_j T}\right) = 2\log_j T - 2\log_{j+1} T,$$

so that $P_\ell \geqslant 10^4$, $P_{\ell-1} \geqslant \exp(10^4)$, and so on.

Let $\mathcal{N}$ denote the set of integers $n = n_2 \cdots n_\ell$ where each n_j is divisible only by primes in the interval T_{j-1} to T_j and such that $\Omega(n_j) \leqslant K_j := 500P_j$ for all $2 \leqslant j \leqslant \ell$. If $n \in \mathcal{N}$, then

$$n = n_2 \cdots n_\ell \leqslant T_2^{500P_2}T_3^{500P_3}\cdots T_\ell^{500P_\ell} \leqslant T^{k/9}. \tag{1}$$

Let $g(n)$ denote the multiplicative function given on prime powers by $g(p^r) = 1/r!$. Define, for any real number α and $2 \leqslant j \leqslant \ell$

$$\mathcal{N}_j(s,\alpha) = \sum_{r=0}^{K_j}\frac{1}{r!}(\alpha\mathcal{P}_j(s,\alpha))^r = \sum_{\substack{p|n\Rightarrow T_{j-1}\leqslant p\leqslant T_j \\ \Omega(n)\leqslant K_j}} \frac{\alpha^{\Omega(n)}g(n)}{n^s}, \tag{2}$$

and put

$$\mathcal{N}(s,\alpha) := \sum_{n\in\mathcal{N}} \frac{\alpha^{\Omega(n)}g(n)}{n^s} = \prod_{j=2}^{\ell} \mathcal{N}_j(s,\alpha). \tag{3}$$

In view of (1), $\mathcal{N}(s,\alpha)$ is a short Dirichlet polynomial. The idea is that $\mathcal{N}(s,\alpha)$ behaves in many ways like $\zeta(s)^\alpha$, but with the advantage that since $\mathcal{N}(s,\alpha)$ is a short Dirichlet polynomial, one can compute mean values involving it and $\zeta(s)$. The proof of our theorem rests on the following three propositions dealing with such mean values involving $\zeta(s)$ and $\mathcal{N}(s,\alpha)$ for suitable values of α.

Proposition 1. *Let T be large. Uniformly in the range $1 \geqslant k \geqslant 1/\sqrt{\log T}$ we have*

$$\int_T^{2T} \zeta(\tfrac12+it)\mathcal{N}(\tfrac12+it,k-1)\mathcal{N}(\tfrac12-it,k)dt \geqslant C_1 T(\log T)^{k^2},$$

for some positive constant C_1.

Proposition 2. *Let T be large. Uniformly in the range $1 \geqslant k \geqslant 1/\sqrt{\log T}$ we have*

$$\int_T^{2T} |\zeta(\tfrac12+it)\mathcal{N}(\tfrac12+it,k-1)|^2 dt \leqslant C_2 k^{-1} T(\log T)^{k^2},$$

for some positive constant C_2.

Proposition 3. *Let T be large. Uniformly in the range $1 \geqslant k \geqslant 1/\sqrt{\log T}$ we have*

$$\int_T^{2T} |\mathcal{N}(\tfrac12+it,k)|^{\frac{2}{k}}|\mathcal{N}(\tfrac12+it,k-1)|^2 dt \leqslant C_3 T(\log T)^{k^2},$$

for some positive constant C_3.

Two applications of Hölder's inequality give

$$\Big|\int_T^{2T} \zeta(\tfrac12+it)\mathcal{N}(\tfrac12+it,k-1)\mathcal{N}(\tfrac12-it,k)dt\Big|$$
$$\leqslant \Big(\int_T^{2T} |\zeta(\tfrac12+it)|^{2k}dt\Big)^{\frac12} \times \Big(\int_T^{2T} |\zeta(\tfrac12+it)\mathcal{N}(\tfrac12+it,k-1)|^2 dt\Big)^{\frac{1-k}{2}}$$
$$\times \Big(\int_T^{2T} |\mathcal{N}(\tfrac12+it,k)|^{\frac{2}{k}}|\mathcal{N}(\tfrac12+it,k-1)|^2 dt\Big)^{\frac{k}{2}},$$

so that the lower bound of the theorem follows at once from the three propositions.

Deducing Corollary 2 from Corollary 1. *Let V be in the range of the corollary, and put $k = V/\log\log T$ and $\delta = \log_3 T/\sqrt{\log\log T}$ so that $k+2\delta \leqslant 2$. The upper bound implicit in the* *corollary follows (in a stronger form) upon noting that*

$$\text{meas}\{t \in [T, 2T] : |\zeta(\tfrac{1}{2} + it)| \geqslant e^V\} \leqslant e^{-2kV} \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \ll T \exp\left(-\frac{V^2}{\log\log T}\right).$$

To prove the lower bound, consider

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2(k+\delta)} dt \gg (k+\delta) T (\log T)^{(k+\delta)^2} \gg \frac{T}{\sqrt{\log\log T}} (\log T)^{(k+\delta)^2}. \tag{4}$$

The contribution to the integral from t with $|\zeta(\frac{1}{2} + it)| \leqslant e^V$ *is*

$$\leqslant e^{2\delta V} \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \ll T (\log T)^{k^2 + 2k\delta} = o\left(\frac{T}{\sqrt{\log\log T}} (\log T)^{(k+\delta)^2}\right).$$

Similarly, the contribution to the integral from t with $|\zeta(\frac{1}{2} + it)| \geqslant e^V (\log T)^{2\delta}$ *is*

$$\leqslant (\log T)^{-2\delta(k+2\delta)} \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2(k+2\delta)} dt \ll T (\log T)^{k^2 + 2\delta k} = o\left(\frac{T}{\sqrt{\log\log T}} (\log T)^{(k+\delta)^2}\right).$$

Thus, the left side of (4) is dominated by values of $|\zeta(\frac{1}{2} + it)|$ *lying between* $e^V = (\log T)^k$ *and* $(\log T)^{k+2\delta}$ *and it follows that the measure of the set of such t is*

$$\gg (\log T)^{-2(k+\delta)(k+2\delta)} \int_{\substack{t \in [T, 2T] \\ (\log T)^{k+2\delta} \geqslant |\zeta(\frac{1}{2} + it)| \geqslant (\log T)^k}} |\zeta(\tfrac{1}{2} + it)|^{2(k+\delta)} dt$$

$$\gg \frac{T}{\sqrt{\log\log T}} (\log T)^{-k^2 - 4k\delta - 3\delta^2}.$$

The corollary follows.

3 | PROOF OF PROPOSITION 1

Expanding out, we have

$$\int_T^{2T} \zeta(\tfrac{1}{2} + it) \mathcal{N}(\tfrac{1}{2} + it, k-1) \mathcal{N}(\tfrac{1}{2} - it, k) dt$$

$$= \sum_{n, n \in \mathcal{N}} \frac{(k-1)^{\Omega(n)} k^{\Omega(m)} g(n) g(m)}{\sqrt{mn}} \int_T^{2T} \zeta(\tfrac{1}{2} + it) \left(\frac{m}{n}\right)^{it} dt. \tag{5}$$

Using the simple approximation

$$\zeta(1/2 + it) = \sum_{r \leqslant T} \frac{1}{r^{1/2+it}} + O(T^{-1/2}), \qquad t \in [T, 2T],$$

we find that

$$\int_T^{2T} \zeta(\tfrac{1}{2}+it)\Big(\frac{m}{n}\Big)^{it} dt = T\frac{\delta(rn=m)}{\sqrt{r}} + O\Bigg(T^{\frac{1}{2}} + \sum_{\substack{r\leqslant T\\ rn\neq m}} \frac{1}{\sqrt{r}|\log(rn/m)|}\Bigg).$$

Here $\delta(rn=m)$ equals 1 if $n|m$ and $r=m/n$, and there is no main term if $n\nmid m$. If $rn\neq m$, we may estimate $1/|\log(rn/m)|$ trivially by $\ll m$, and so the remainder term above is $O(mT^{\frac{1}{2}})$. From these remarks, it follows that the right side of (5) equals

$$T\sum_{\substack{m,n\in\mathcal{N}\\ n|m}} \frac{(k-1)^{\Omega(n)}k^{\Omega(m)}g(n)g(m)}{m} + O\Bigg(\sum_{m,n\in\mathcal{N}} \frac{1}{\sqrt{mn}} mT^{\frac{1}{2}}\Bigg). \tag{6}$$

Since the elements of $\mathcal{N}$ are all bounded by $T^{1/9}$, the error term above is seen to be $O(T^{7/9})$, which is negligible.

Now consider the main term in (6). Factor $n=n_2\cdots n_\ell$ and $m=m_2\cdots m_\ell$ where m_j and n_j are divisible only by the primes in the interval (T_{j-1},T_j) and $\Omega(m_j)$ and $\Omega(n_j)$ are bounded by K_j. Then the main term in (6) factors naturally as

$$T\prod_{j=2}^{\ell}\Bigg(\sum_{\substack{n_j,m_j\\ n_j|m_j\\ \Omega(m_j)\leqslant 500P_j}} \frac{(k-1)^{\Omega(n_j)}k^{\Omega(m_j)}g(n_j)g(m_j)}{m_j}\Bigg). \tag{7}$$

If we drop the condition that $\Omega(m_j)\leqslant K_j$, then the sums over n_j, m_j above may be replaced with (thinking of a as the power of p dividing m_j and b the power dividing n_j)

$$\prod_{T_{j-1}\leqslant p\leqslant T_j}\Bigg(1+\sum_{\substack{a\geqslant 1\\ a\geqslant b\geqslant 0}} \frac{k^a(k-1)^b}{p^a} g(p^a)g(p^b)\Bigg) \geqslant \prod_{T_{j-1}\leqslant p\leqslant T_j}\Big(1+\frac{k^2}{p}\Big).$$

The error incurred in dropping this condition is bounded in magnitude by

$$\begin{aligned}\sum_{\substack{n_j,m_j\\ n_j|m_j\\ \Omega(m_j)>K_j}} \frac{g(n_j)g(m_j)}{m_j} &\leqslant e^{-K_j}\sum_{\substack{n_j,m_j\\ n_j|m_j}} \frac{g(n_j)g(m_j)}{m_j}e^{\Omega(m_j)}\\ &= e^{-500P_j}\prod_{T_{j-1}\leqslant p\leqslant T_j}\Bigg(1+\sum_{a\geqslant 1}\frac{e^a}{a!p^a}\sum_{a\geqslant b\geqslant 0}\frac{1}{b!}\Bigg)\\ &\leqslant e^{-500P_j}\prod_{T_{j-1}\leqslant p\leqslant T_j}\Big(1+\frac{20}{p}\Big) \leqslant e^{-400P_j}.\end{aligned}$$

It follows that the main term (7) is

$$\geqslant T \prod_{j=2}^{\ell} \prod_{T_{j-1} \leqslant p \leqslant T_j} \left(1 + \frac{k^2}{p}\right)\left(1 - e^{-400P_j}\right) \geqslant CT(\log T_\ell)^{k^2},$$

for an absolute positive constant C. Since $\log T_\ell \gg k \log T$, and $k^{k^2} \gg 1$ for $0 < k \leqslant 1$, this proves Proposition 1.

4 | PROOF OF PROPOSITION 2

It is a simple matter to compute the mean square of the zeta function multiplied by a short Dirichlet polynomial. For example, from [1], we obtain

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)\mathcal{N}(\tfrac{1}{2}+it, k-1)|^2 dt = T \sum_{m,n \in \mathcal{N}} \frac{(k-1)^{\Omega(m)+\Omega(n)} g(m)g(n)}{[m,n]} \log\left(\frac{BT(m,n)^2}{mn}\right) + o(T), \tag{8}$$

for a constant B. We must now bound the main term above. While one can work out an asymptotic for this main term, we give a quick proof of an upper bound, which is all that is needed in Proposition 2.

Write

$$\log\left(\frac{BT(m,n)^2}{mn}\right) = \frac{1}{2\pi i} \int_{|z|=1/\log T} \left(\frac{BT(m,n)^2}{mn}\right)^z \frac{dz}{z^2},$$

so that the main term in (8) becomes

$$\frac{T}{2\pi i} \int_{|z|=1/\log T} \sum_{m,n \in \mathcal{N}} \frac{(k-1)^{\Omega(m)+\Omega(n)} g(m)g(n)}{[m,n]} \left(\frac{BT(m,n)^2}{mn}\right)^z \frac{dz}{z^2}.$$

By the triangle inequality, we may estimate the above by

$$\leqslant 3T \log T \max_{|z|=1/\log T} \left| \sum_{m,n \in \mathcal{N}} \frac{(k-1)^{\Omega(m)+\Omega(n)} g(m)g(n)}{[m,n]} \left(\frac{(m,n)^2}{mn}\right)^z \right|. \tag{9}$$

We can now analyze the sum over m and n in (9) by adapting the argument of the previous section. Thus, decompose $m = m_2 \cdots m_\ell$ and $n = n_2 \cdots n_\ell$ where m_j and n_j are composed only of the primes in (T_{j-1}, T_j) and $\Omega(m_j)$ and $\Omega(n_j)$ are both $\leqslant K_j$. By multiplicativity, the sum in (9) factors as

$$\prod_{j=2}^{\ell} \left(\sum_{\substack{m_j, n_j \\ \Omega(m_j), \Omega(n_j) \leqslant K_j}} \frac{(k-1)^{\Omega(m_j)+\Omega(n_j)} g(m_j)g(n_j)}{[m_j, n_j]} \left(\frac{(m,n)^2}{mn}\right)^z \right). \tag{10}$$

As before, we handle these terms by first dropping the condition on $\Omega(m_j)$ and $\Omega(n_j)$, and then bounding the error in doing so. If we drop the conditions on $\Omega(m_j)$ and $\Omega(n_j)$, the sums over m_j and n_j become

$$\prod_{T_{j-1}\leqslant p\leqslant T_j}\left(\sum_{a,b=0}^{\infty}\frac{(k-1)^{a+b}}{a!b!p^{\max(a,b)}}p^{-|b-a|z}\right)=\prod_{T_{j-1}\leqslant p\leqslant T_j}\left(1+\frac{(k-1)^2+2(k-1)p^{-z}}{p}+O\left(\frac{1}{p^2}\right)\right)$$
$$=\prod_{T_{j-1}\leqslant p\leqslant T_j}\left(1+\frac{k^2-1}{p}+O\left(\frac{\log p}{p\log T}+\frac{1}{p^2}\right)\right).$$

The error incurred in dropping the conditions on $\Omega(m_j)$ and $\Omega(n_j)$ is bounded in magnitude by

$$\leqslant e^{-K_j}\sum_{m_j,n_j}\frac{g(m)g(n)}{[m,n]}e^{\Omega(m_j)+\Omega(n_j)}\left(\frac{mn}{(m,n)^2}\right)^{1/\log T}$$
$$\leqslant e^{-K_j}\prod_{T_{j-1}\leqslant p\leqslant T_j}\left(1+2\sum_{a=1}^{\infty}\sum_{0\leqslant b\leqslant a}\frac{e^{a+b}}{a!b!p^a}p^{a/\log T}\right)$$
$$\leqslant e^{-500P_j}\prod_{T_{j-1}\leqslant p\leqslant T_j}\left(1+35\sum_{a=1}^{\infty}\frac{e^a}{a!p^a}\right)\leqslant e^{-500P_j}\exp\left(\sum_{T_{j-1}\leqslant p\leqslant T_j}\frac{35e}{p}\right)\leqslant e^{-400P_j}.$$

We conclude that the sum over m_j, n_j in (10) is

$$\prod_{T_{j-1}\leqslant p\leqslant T_j}\left(1+\frac{k^2-1}{p}+O\left(\frac{\log p}{p\log T}+\frac{1}{p^2}\right)\right)\left(1+O(e^{-300P_j})\right), \tag{11}$$

so that the quantity in (9) is

$$\ll T\log T\prod_{p\leqslant T_\ell}\left(1+\frac{k^2-1}{p}+O\left(\frac{\log p}{p\log T}+\frac{1}{p^2}\right)\right)\ll k^{-1}T(\log T)^{k^2}.$$

The proposition follows.

5 | PROOF OF PROPOSITION 3

Recall from (2) and (3) the definitions of $\mathcal{N}_j(s,\alpha)$ and $\mathcal{N}(s,\alpha)$. The following simple lemma is the key to establishing Proposition 3.

Lemma 1. *For* $2\leqslant j\leqslant\ell$

$$|\mathcal{N}_j(\tfrac12+it,k-1)\mathcal{N}_j(\tfrac12+it,k)^{\frac1k}|^2\leqslant|\mathcal{N}_j(\tfrac12,+it,k)|^2(1+O(e^{-K_j}/k))+O\Big(2^{2/k}Q_j(t)\Big),$$

where the implied constants are absolute, and

$$\mathcal{Q}_j(t) = \left(\frac{12|\mathcal{P}_j(\frac{1}{2}+it)|}{K_j}\right)^{2K_j} \sum_{r=0}^{K_j/k} \left(\frac{2e|\mathcal{P}_j(\frac{1}{2}+it)|}{r+1}\right)^{2r}.$$

Proof. We begin by observing that if $|z| \leqslant K/10$, then

$$\Big|\sum_{r=0}^{K} \frac{z^r}{r!} - e^z\Big| \leqslant \frac{|z|^K}{K!} \leqslant \Big(\frac{e}{10}\Big)^K,$$

so that

$$\sum_{r=0}^{K} \frac{z^r}{r!} = e^z\big(1 + O(e^{-K})\big). \tag{12}$$

Consider first the case $|\mathcal{P}_j(\frac{1}{2}+it)| \leqslant K_j/10$, where three applications of (12) show that

$$\begin{aligned}|\mathcal{N}_j(\tfrac{1}{2}+it, k-1)|^2 |\mathcal{N}_j(\tfrac{1}{2}+it, k)|^{\frac{2}{k}} &= \exp(2k\mathrm{Re}\mathcal{P}_j(\tfrac{1}{2}+it))\big(1 + O(e^{-K_j}/k)\big)\\ &= |\mathcal{N}_j(\tfrac{1}{2}+it, k)|^2\big(1 + O(e^{-K_j}/k)\big).\end{aligned}$$

The lemma follows in this case.

Suppose now that $|\mathcal{P}_j(\frac{1}{2}+it)| \geqslant K_j/10$. Here note that

$$\begin{aligned}|\mathcal{N}_j(\tfrac{1}{2}+it, k-1)| &\leqslant \sum_{r=0}^{K_j} \frac{|\mathcal{P}_j(\frac{1}{2}+it)|^r}{r!} \leqslant |\mathcal{P}_j(\tfrac{1}{2}+it)|^{K_j} \sum_{r=0}^{K_j} \Big(\frac{10}{K_j}\Big)^{K_j-r} \frac{1}{r!}\\ &\leqslant \left(\frac{12|\mathcal{P}_j(\frac{1}{2}+it)|}{K_j}\right)^{K_j}.\end{aligned} \tag{13}$$

Further, applying Hölder's inequality we find

$$\begin{aligned}|\mathcal{N}_j(\tfrac{1}{2}+it, k)|^{\frac{2}{k}} &\leqslant \left(\sum_{r=0}^{K_j} \frac{(k|\mathcal{P}_j(\frac{1}{2}+it)|)^r}{r!}\right)^{\frac{2}{k}} \leqslant \left(\sum_{r=0}^{K_j} \frac{(2k|\mathcal{P}_j(\frac{1}{2}+it)|)^{\frac{2r}{k}}}{r!^{2/k}}\right)\left(\sum_{r=0}^{K_j} 2^{-r}\right)^{\frac{2}{k}-1}\\ &\leqslant 2^{\frac{2}{k}} \sum_{r=0}^{K_j} (2k|\mathcal{P}_j(\tfrac{1}{2}+it)|)^{\frac{2r}{k}} \Big(\frac{e}{r+1}\Big)^{\frac{2r}{k}} \leqslant 2^{\frac{2}{k}} \sum_{r=0}^{K_j} \left(\frac{2e|\mathcal{P}_j(\frac{1}{2}+it)|}{r/k+1}\right)^{\frac{2r}{k}}.\end{aligned}$$

A little calculus allows us to bound the above by

$$\ll 2^{\frac{2}{k}} \sum_{r=0}^{K_j/k} \left(\frac{2e|\mathcal{P}_j(\frac{1}{2}+it)|}{r+1}\right)^{2r},$$

which when combined with (13) yields the lemma. ◻

We next show that $\mathcal{Q}_j(t)$ (which is always non-negative by definition) is small on average.

Lemma 2. *With the above notation*

$$\int_T^{2T} \mathcal{Q}_j(t)dt \ll Te^{-K_j}.$$

Proof. We begin by recalling a simple mean-value theorem for Dirichlet polynomials:

$$\int_T^{2T} \Big|\sum_{n\leqslant N} a(n)n^{-it}\Big|^2 dt = T\sum_{n\leqslant N}|a(n)|^2 + O\bigg(\sum_{m\neq n\leqslant N}\frac{|a(m)a(n)|}{|\log(m/n)|}\bigg),$$

and bounding $|a(m)a(n)|$ by $|a(m)|^2+|a(n)|^2$, it follows that

$$\int_T^{2T} \Big|\sum_{n\leqslant N} a(n)n^{-it}\Big|^2 dt = (T+O(N\log N))\sum_{n\leqslant N}|a(n)|^2. \tag{14}$$

Now, for $0\leqslant r\leqslant K_j/k$,

$$\mathcal{P}_j(\tfrac12+it)^{K_j+r} = \sum_{\substack{\Omega(n)=K_j+r\\ p|n\Rightarrow T_{j-1}\leqslant p\leqslant T_j}} \frac{(K_j+r)!g(n)}{n^{\frac12+it}},$$

is a short Dirichlet polynomial (since $T_j^{K_j(1+1/k)}\leqslant T^{1/10}$), and so by (14)

$$\begin{aligned}\int_T^{2T} |\mathcal{P}_j(\tfrac12+it)|^{2(K_j+r)}dt &= (T+O(T^{1/2}))\sum_{\substack{\Omega(n)=K_j+r\\ p|n\Rightarrow T_{j-1}\leqslant p\leqslant T_j}}\frac{(K_j+r)!^2g(n)^2}{n}\\ &\leqslant (K_j+r)!P_j^{K_j+r}(T+O(T^{1/2})),\end{aligned}$$

where the last bound follows upon noting that $g(n)^2\leqslant g(n)$. Using this bound in the definition of $\mathcal{Q}_j(t)$, we find

$$\int_T^{2T}\mathcal{Q}_j(t)dt \ll T\bigg(\frac{12}{K_j}\bigg)^{2K_j}\sum_{r=0}^{K_j/k}\Big(\frac{2e}{r+1}\Big)^{2r}(K_j+r)!P_j^{K_j+r}. \tag{15}$$

Stirling's formula and a little calculus show that the terms above attain a maximum for r around the solution to $r^2=4P_j(K_j+r)$, and since $K_j=500P_j$, such r satisfies $2\sqrt{P_jK_j}\leqslant r\leqslant 2.1\sqrt{P_jK_j}$. It follows that the right side of (15) is

$$\ll T\bigg(\frac{12}{K_j}\bigg)^{2K_j}\bigg(\frac{K_j}{k}\bigg)\bigg(\frac{2P_jK_j}{e}\bigg)^{K_j}e^{2.1\sqrt{P_jK_j}} \ll Te^{-K_j}.$$

□

We need one more observation for the proof of the proposition. Suppose that we are given R Dirichlet polynomials

$$A_j(s) = \sum_{n\in\mathcal{S}_j} a_j(n)n^{-s},$$

where the sets $\mathcal{S}_j$ satisfy the following two properties: (i) If $j_1 \neq j_2$, then the elements of $\mathcal{S}_{j_1}$ are all coprime to the elements of $\mathcal{S}_{j_2}$, and (ii) $\prod_{j=1}^R n_j \leqslant N$ for all $n_j \in \mathcal{S}_j$. The coprimality condition implies that there is at most one way to write $n = \prod_{j=1}^R n_j$ with $n_j \in \mathcal{S}_j$. Thus applications of (14) give

$$\begin{aligned}\frac{1}{T}\int_T^{2T}\prod_{j=1}^R |A_j(it)|^2 dt &= (1+O(NT^{-1}\log N))\sum_{n\leqslant N}\Big|\sum_{\substack{n=n_1\cdots n_R\\ n_j\in\mathcal{S}_j}}\prod_{j=1}^R a_j(n_j)\Big|^2\\ &= (1+O(NT^{-1}\log N))\prod_{j=1}^R\left(\sum_{n_j\in\mathcal{S}_j}|a_j(n_j)|^2\right)\\ &= (1+O(NT^{-1}\log N))\prod_{j=1}^R\left(\frac{1}{T}\int_T^{2T}|A_j(it)|^2dt\right). \qquad (16)\end{aligned}$$

We are now ready to combine the above observations to prove Proposition 3. Applying Lemma 1, we find

$$\begin{aligned}&\int_T^{2T}|\mathcal{N}(\tfrac{1}{2}+it,k-1)|^2|\mathcal{N}(\tfrac{1}{2}+it,k)|^{\frac{2}{k}}dt\\ &\qquad\leqslant \int_T^{2T}\prod_{j=2}^{\ell}\Big(|\mathcal{N}_j(\tfrac{1}{2}+it,k)|^2(1+O(e^{-K_j}/k))+O(2^{2/k}Q_j(t))\Big)dt.\end{aligned}$$

Appealing now to the observation (16), the above is

$$\ll T\prod_{j=2}^{\ell}\left(\frac{1}{T}\int_T^{2T}\Big(|\mathcal{N}_j(\tfrac{1}{2}+it,k)|^2(1+O(e^{-K_j}/k))+O(2^{2/k}Q_j(t))\Big)dt.\right. \qquad (17)$$

Applying the mean-value theorem for Dirichlet polynomials (14), we see that

$$\begin{aligned}\int_T^{2T}|\mathcal{N}_j(\tfrac{1}{2}+it,k)|^2dt &= (T+O(T^{1/2}))\sum_{\substack{p|n\Rightarrow T_{j-1}\leqslant p\leqslant T_j\\ \Omega(n)\leqslant K_j}}\frac{k^{2\Omega(n)}g(n)^2}{n}\\ &\leqslant (T+O(T^{1/2}))\prod_{T_{j-1}\leqslant p\leqslant T_j}\left(1+\frac{k^2}{p}+O\left(\frac{1}{p^2}\right)\right).\end{aligned}$$

Combining this with Lemma 2, we conclude that the quantity in (17) is

$$\ll T \prod_{p \leqslant T_\ell} \left(1 + \frac{k^2}{p} + O\left(\frac{1}{p^2}\right)\right),$$

which completes the proof of the proposition.

6 | EXTENSIONS OF THE RESULT

We first give the modifications needed to obtain Theorem 1 in the range $k \geqslant 1$. Once again let ℓ be the largest integer with $\log_\ell T \geqslant 10^4$, and now define T_j by $T_1 = k^4 e^2$ and for $2 \leqslant j \leqslant \ell$ by

$$T_j = \exp\left(\frac{\log T}{k^2 (\log_j T)^2}\right).$$

Define $\mathcal{P}_j(s)$, P_j exactly as before, and now put $K_j = 500 k^2 P_j$ with $\mathcal{N}(s, \alpha)$ defined accordingly. Analogously to Proposition 1, we may establish that

$$\int_T^{2T} \zeta(\tfrac{1}{2} + it) \mathcal{N}(\tfrac{1}{2} + it, k-1) \mathcal{N}(\tfrac{1}{2} - it, k) dt \gg T \prod_{T_1 \leqslant p \leqslant T_\ell} \left(1 + \frac{k^2}{p}\right).$$

Now Hölder's inequality gives that the left side above is

$$\leqslant \left(\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} dt\right)^{\frac{1}{2k}} \left(\int_T^{2T} |\mathcal{N}(\tfrac{1}{2} + it, k-1) \mathcal{N}(\tfrac{1}{2} + it, k)|^{\frac{2k}{2k-1}} dt\right)^{\frac{2k-1}{2k}}.$$

By modifying the argument of Proposition 3 (indeed the details are even a little simpler), the second term above may be bounded by

$$\ll \left(T \prod_{T_1 \leqslant p \leqslant T_\ell} \left(1 + \frac{k^2}{p} + O\left(\frac{k^4}{p^2}\right)\right)\right)^{\frac{2k-1}{2k}} \ll \left(T \prod_{T_1 \leqslant p \leqslant T_\ell} \left(1 + \frac{k^2}{p}\right)\right)^{\frac{2k-1}{2k}}.$$

The lower bound claimed in the theorem follows.

Examining our proof, we may extract the following principle. Given a family of L-functions, if one can compute the mean value of $L(\frac{1}{2})$ multiplied by suitable short Dirichlet polynomials, as well as the mean value of $|L(\frac{1}{2})|^2$ multiplied by suitable short Dirichlet polynomials, then one obtains a lower bound of the right order for the moments $|L(\frac{1}{2})|^k$ for all $k > 0$. If $k \geqslant 1$, then one needs only an understanding of the mean value of $L(\frac{1}{2})$ multiplied by short Dirichlet polynomials, and knowledge of the second moment of $L(\frac{1}{2})$ is not required. Thus, for example, one may establish that

$$\sum_{\chi \pmod q} |L(\tfrac{1}{2}, \chi)|^{2k} \gg_k q (\log q)^{k^2}, \tag{18}$$

where q is a large prime, and $k > 0$. Or, that for $k > 0$ and large X

$$\sum_{|d|\leqslant X}^{\flat} |L(\tfrac{1}{2}, \chi_d)|^k \gg_k X(\log X)^{\frac{k(k+1)}{2}}, \tag{19}$$

where the sum is over fundamental discriminants d. Previously, (18) and (19) were accessible for all $k \geqslant 1$ by [8], and (18) was known for rational $0 \leqslant k \leqslant 1$ by the work of Chandee and Li [3]. A third example is the family of quadratic twists of a new form f, where the second moment of the central L-values is not known. Here one can establish

$$\sum_{|d|\leqslant X}^{\flat} L(\tfrac{1}{2}, f \times \chi_d)^k \gg_k X(\log X)^{\frac{k(k-1)}{2}}, \tag{20}$$

for all $k \geqslant 1$. Such a result would be accessible also to the method of [8], but the problem of obtaining satisfactory lower bounds for the small moments $k < 1$ (which is connected to the delicate question of non-vanishing of L-values) remains open.

ACKNOWLEDGEMENTS

The second author is partially supported by grants from the NSF (including the FRG grant DMS1854398), and through a Simons Investigator grant from the Simons Foundation. We are grateful to Maksym Radziwiłł for many valuable discussions on these themes.

JOURNAL INFORMATION

Mathematika is owned by University College London and published by the London Mathematical Society. All surplus income from the publication of *Mathematika* is returned to mathematicians and mathematics research via the Society's research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. R. Balasubramanian, J. B. Conrey, and D. R. Heath-Brown, *Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial*, J. Reine Angew. Math. **357** (1985), 161–181.
2. R. Balasubramanian and K. Ramachandra, *On the frequency of Titchmarsh's phenomenon – III*, Proc. Indian Acad. Sci. **84A** (1977), no. 4, 341–351.
3. V. Chandee and X. Li, *Lower bounds for small fractional moments of Dirichlet L-functions*, Int. Math. Res. Not. **2013** (2013), no. 19, 4349–4381.
4. W. Heap, M. Radziwiłł, and K. Soundararajan, *Sharp upper bounds for fractional moments of the Riemann zeta function*, Q. J. Math. **70** (2019), no. 4, 1387–1396.
5. D. R. Heath-Brown, *Fractional moments of the Riemann zeta function*, J. Lond. Math. Soc. **24** (1981), no. 1, 65–78.
6. A. P. Laurinchikas, *Moments of the Riemann zeta-function on the critical line*, Math. Notes Acad. Sci. USSR **39** (1986), 267–272.
7. M. Radziwiłł, *Large deviations in Selberg's central limit theorem*, arXiv:1108.509v1 (2011).
8. M. Radziwiłł and K. Soundararajan, *Continuous lower bounds for moments of zeta and L-functions*, Mathematika **59** (2013), no. 1, 119–128.
9. M. Radziwiłł and K. Soundararajan, *Moments and distribution of central L-values of quadratic twists of elliptic curves*, Invent. Math. **202** (2015), no. 3, 1029–1068.

10. M. Radziwiłł and K. Soundararajan, *Selberg's central limit theorem for* $\log|\zeta(\frac{1}{2}+it)|$, Enseign. Math. **63** (2017), 1–19.
11. K. Ramachandra, *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series*, Ann. Acad. Sci. Fenn. Math. **5** (1980), 145–158.
12. K. Ramachandra, *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. II*, Hardy-Ramanujan J. **3** (1980), 1–24.
13. K. Soundararajan, *Moments of the Riemann zeta function*, Ann. Math. **170** (2009), 981–993.