CONSTRUCTING COPRODUCTS IN LOCALLY CARTESIAN CLOSED ∞-CATEGORIES

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Abstract. We prove that every locally Cartesian closed ∞-category with a subobject classifier has a *strict initial object* and *disjoint and universal binary coproducts*.

1. Introduction

Elementary Toposes and Finite Colimits. *Categorical logic* uses results and constructions from category theory to study *type theory*, *set theory* and other concepts in mathematical logic. One key concept in categorical logic is that of an *elementary topos*. Elementary toposes admit a natural interpretation of *higher-order logic* [Joh02b, Chapter D4], and also give rise to models of set theories [MLM94, JM95].

Elementary toposes were defined by Lawvere and Tierney as a generalization of *Grothendieck toposes*. The latter always admit small limits and colimits since they are defined as categories of sheaves and are therefore locally presentable [AGV72]. Hence, the first definitions of elementary topos assumed the existence of both finite limits and finite colimits [Law70, Tie72]. However, it was soon realized that the existence of finite colimits could in fact be deduced from the other axioms and concretely that we have the following result: every finitely complete Cartesian closed category with a subobject classifier has finite colimits [Mik72, Par74, Mik76].

The recent decades have witnessed significant advances in the study of *homotopy invariant* mathematics. In particular, there is now a well developed theory of homotopy invariant categories, known as $(\infty, 1)$-*categories* or simply ∞-*categories* [Ber10], which have been used extensively in many areas relevant to homotopy theory, such as *homotopy coherent algebraic structures* or *derived geometry* [Lur17].

The theory of Grothendieck toposes has successfully been generalized to the higher categorical setting – both in the context of model categories [Rez10] and ∞-categories [Lur09] – giving rise to the notion of (Grothendieck-)∞-*topos*.

At the same time, categorical logicians have devised a homotopy invariant interpretation of *Martin-Löf type theory* [MLS84], known as *homotopy type theory* [Uni13]. This interpretation was quickly conjectured to generalize from homotopy types to arbitrary ∞-toposes, and a complete proof of this fact has recently been given [Shu19].

Just as the interpretation of higher order logic in 1-toposes, the interpretation of type theory in ∞-toposes does not rely on the (co)completeness of the topos, which suggested to formulate a notion of ‘finitary’ or ‘elementary’ ∞-topos as natural target for the interpretation of type theory, analogous to Lawvere and Tierney’s elementary 1-toposes. Concrete proposals for a definition of elementary ∞-topos were given in [Shu17, Ras18], and similarly to the first definitions of elementary 1-topos, these definitions explicitly postulate the existence of finite colimits.

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This leaves us with the question whether we can recover finite colimits from the remaining axioms just as in the 1-dimensional case. In the present paper we give a partial answer, by proving the following main result.

Theorem 5.7. *Let $\{A_k\}_{k\in I}$ be a finite family of objects in a locally Cartesian closed ∞-category $\mathcal{C}$ with subobject classifier. Then the coproduct $\coprod_{k\in I} A_k$ exists, and pullback along the inclusion maps $i_k : A_k \to \coprod_{k\in I} A_k$ give rise to an equivalence of ∞-categories*

$$(i_k^*)_{k\in I} : \mathcal{C}_{/\coprod_{k\in I} A_k} \to \prod_{k\in I} \mathcal{C}_{/A_k}.$$

This result can be reformulated as saying that $\mathcal{C}$ admits a *strict* initial object and *disjoint and universal* binary coproducts. Of these properties, universality [Lur09, Definition 6.1.1.2] and strictness say that the respective colimits are preserved by pullback functors, which is a direct consequence of local Cartesian closure. *Disjointness* of binary coproducts says that the commutative squares

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A+B \end{array} \qquad\qquad \begin{array}{ccc} 0 & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A+B \end{array}$$

are pullbacks for all objects A, B, and the combination of universality and disjointness is the special case of Rezk's *descent condition* [Rez10, 6.5] for binary coproducts. In the context of 1-categories, descent for coproducts is also known as *extensivity* [CLW93].

What about pushouts? Having settled the issue of coproducts, the remaining question is that of pushouts and coequalizers.

However, it turns out that unlike the 1-categorical situation, assuming the existence of a subobject classifier in fact does not suffice to prove the existence of pushouts in locally Cartesian closed ∞-categories as we illustrate via the following example.

Example 6.3. *Let $\mathcal{S}^{\mathrm{tr}}$ be the full subcategory of the ∞-category $\mathcal{S}$ of spaces spanned by truncated spaces. Then $\mathcal{S}^{\mathrm{tr}}$ is locally Cartesian closed and the discrete space $1+1$ is a subobject classifier. However, the diagram*

$$1 \longleftarrow S^1 \longrightarrow 1$$

does not have a pushout.

We can in fact give a more conceptual argument why it is possible to recover coproducts from the subobject classifier but not pushouts: the universal property of coproducts in ∞-categories only depends on the homotopy types of the mapping spaces, since the diagram used for coproducts is *discrete* and so cannot involve any higher homotopies. On the other side the diagram used to construct pushouts $(\bullet \leftarrow \bullet \rightarrow \bullet)$ is not discrete which means that the universal property of pushouts necessarily involves the notion of *homotopy coherent diagram* [Lur09, Section 1.2.6].

Hence, it remains to determine what precise conditions we need to add to a locally Cartesian closed ∞-category with a subobject classifier to be able to construct all finite colimits. The current hope is that we can obtain this result by additionally assuming the existence of universes.

Structure of the paper. Section 2 recalls basic facts about locally Cartesian closed ∞-categories, including the Beck-Chevalley condition (Lemma 2.1), truncation levels (Subsection 2.1), and the *object of contractibility* (Subsection 2.2) – a technique which allows to reduce contractibility questions to contractibility of subterminals. In Section 3 we discuss subobject lattices and subobject classifiers, and show that if a locally Cartesian closed ∞-category has a subobject classifier, then its subobject lattices have finite joins (Theorem 3.5). Using this, we show in Section 4 that any locally Cartesian closed ∞-category with a subobject classifier has an initial object (Corollary 4.4), and in Section 5 that it has disjoint binary coproducts (Theorem 5.6). We conclude in Section 6 by discussing the relevance of our result to the notion of 'elementary ∞-topos'.

∞-Categorical Conventions. In this paper we use ∞-categorical language and results via the model of *quasi-categories* as developed in [Joy08] and [Lur09]. However, the results proven here only rely on 'model independent' properties of higher categories such as finite limits and locally Cartesian closure and so also hold analogously in any other ∞-cosmos [RV17].

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2. Some Facts about Locally Cartesian Closed ∞-Categories

Let $\mathcal{C}$ be an ∞-category with finite limits. Then for every morphism $f : A \to B$, the pullback functor $f^* : \mathcal{C}_{/B} \to \mathcal{C}_{/A}$ has a left adjoint $f_! : \mathcal{C}_{/A} \to \mathcal{C}_{/B}$ given by post-composition. If f^* furthermore has a *right* adjoint $f_* : \mathcal{C}_{/A} \to \mathcal{C}_{/B}$ for all f, then $\mathcal{C}$ is called *locally Cartesian closed*. If B is the terminal object, we informally identify $\mathcal{C}$ with $\mathcal{C}_{/1}$ (see [Lur09, 1.2.12.4]) and simply write $A_! \dashv A^* \dashv A_*$ for the adjoint string of functors along the terminal projection $A \to 1$.

Lemma 2.1 (Beck–Chevalley condition)**.** *Given a pullback square*

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

in an ∞-category $\mathcal{C}$ with pullbacks, the canonical transformation

$$h_! \circ k^* \to f^* \circ g_!$$

is an equivalence. If $\mathcal{C}$ is locally Cartesian closed, then the canonical natural transformation

$$f^* \circ g_* \to h_* \circ k^*$$

is an equivalence.

Proof. This is proven for the ∞-category $\mathcal{S}$ of spaces in [GHK21, Lemma 2.1.6], but the proof only relies on $\mathcal{C}$ being locally Cartesian closed. □

Recall that an ∞-category $\mathcal{C}$ is called *Cartesian closed* if it has finite products and for every $A \in \mathcal{C}$ the product functor $(- \times A) : \mathcal{C} \to \mathcal{C}$ has a right adjoint commonly written $(-)^A : \mathcal{C} \to \mathcal{C}$ and called 'exponentiation by A'. Every locally Cartesian closed ∞-category is Cartesian closed since $(- \times A)$ can be decomposed as $A_! \circ A^*$, and both $A_!$ and A^* have right adjoints – thus, exponentiation by A is given by $A_* \circ A^*$ in this case. Since slices of locally Cartesian closed ∞-categories are obviously locally Cartesian closed, we can conclude that all slices of locally Cartesian closed ∞-categories are Cartesian closed.[1]

Moreover, we can deduce from the Beck–Chevalley condition that exponentiation commutes with pullback functors:

Lemma 2.2. *Given morphisms $f : B \to A$, $g : C \to A$, and $h : D \to A$ in a locally Cartesian closed ∞-category $\mathcal{C}$ and $g, h \in \mathcal{C}_{/A}$, we have $f^*(h^g) \simeq (f^*h)^{f^*g}$.*

Proof. Form the pullback square

$$\begin{array}{ccc} P & \xrightarrow{\;\overline{f}\;} & C \\ {\scriptstyle f^*g}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{\;f\;} & A \end{array}$$

of g along f. We have

$$\begin{aligned} f^*(h^g) &\simeq (f^* \circ g_* \circ g^*)(h) \\ &\simeq ((f^*g)_* \circ \overline{f}^* \circ g^*)(h) && \text{by the Beck–Chevalley condition} \\ &\simeq ((f^*g)_* \circ (f^*g)^* \circ f^*)(h) && \text{since the square commutes} \\ &\simeq (f^*h)^{f^*g} \;. \end{aligned}$$

□

2.1. **Truncation and monomorphisms.** For $n \geq -2$, recall that an object A in an ∞-category $\mathcal{C}$ is called *n-truncated* if the mapping space $\mathsf{Map}_{\mathcal{C}}(X, A)$ is n-truncated for all objects $X \in \mathcal{C}$. The object is called *contractible* or *terminal* if it is (-2)-truncated, and *subterminal* if it is (-1)-truncated.

An *arrow* $f : A \to B$ in $\mathcal{C}$ is called n-truncated if for all $X \in \mathcal{C}$ the postcomposition operation $\mathsf{Map}_{\mathcal{C}}(X, f) : \mathsf{Map}_{\mathcal{C}}(X, A) \to \mathsf{Map}_{\mathcal{C}}(X, B)$ is an n-truncated map in $\mathcal{S}$, i.e. if its fibers are n-truncated spaces. If $\mathcal{C}$ has a terminal object 1 then an object A is n-truncated iff the morphism $A \to 1$ is n-truncated. Conversely, $f : A \to B$ is n-truncated as a morphism in $\mathcal{C}$ iff it is n-truncated as an object in $\mathcal{C}_{/B}$.

A morphism $f : A \to B$ is (-2)-truncated iff it is an equivalence. If $\mathcal{C}$ has pullbacks, then $f : A \to B$ is $(n+1)$-truncated iff its diagonal $\delta_f : A \to A \times_B A$ is n-truncated.

Maps that are (-1)-truncated are also called *monomorphisms*. Thus, $f : A \to B$ is an monomorphism iff its diagonal $\delta_f : A \to A \times_B A$ is an equivalence, i.e. the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\;\mathsf{id}\;} & A \\ {\scriptstyle \mathsf{id}}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\;f\;} & B \end{array}$$

is a pullback.

Lemma 2.3. *Let $m : U \rightarrowtail A$ be a monomorphism in an ∞-category $\mathcal{C}$ with finite limits.*

(1) *For every $f : B \to U$, the commutative square $\begin{array}{ccc} B & \xrightarrow{\mathsf{id}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle m \circ f} \\ U & \overset{m}{\rightarrowtail} & A \end{array}$ is a pullback.*

(2) *The adjunction $m_! \dashv m^*$ is a coreflection, i.e. its unit is an equivalence.*

(3) *If $\mathcal{C}$ is locally Cartesian closed then the adjunction $m^* \dashv m_*$ is a reflection, i.e. its counit is an equivalence.*

[1]Conversely, every ∞-category with finite limits and Cartesian closed slices is locally cartesian closed – the 1-categorical proof of this statement given in [Joh02a, Corollary A1.5.3] generalizes to ∞-categories in a straightforward manner.

Proof. The first claim follows from the pullback lemma since both small squares in the following diagram are pullbacks.

$$\begin{array}{ccc} B & \xrightarrow{\mathsf{id}} & B \\ f\downarrow & & \downarrow f \\ U & \xrightarrow{\mathsf{id}} & U \\ \mathsf{id}\downarrow & & \downarrow m \\ U & \xrightarrow{m} & A \end{array}$$

The second claim follows from the first since the unit of $m_! \dashv m^*$ at $f : U \to A$ is the canonical map from f to $m^*(m \circ f)$. The third claim follows from the second since the rightmost functor in an adjoint triple is fully faithful iff the leftmost is. □

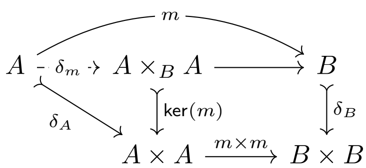
Lemma 2.4. *Two subterminal objects A, B in an ∞-category $\mathcal{C}$ are equivalent whenever there exist maps $f : A \to B$ and $g : B \to A$.*

Proof. This follows since all parallel maps into a subterminal are homotopic, in particular every endomorphism is homotopic to the identity. □

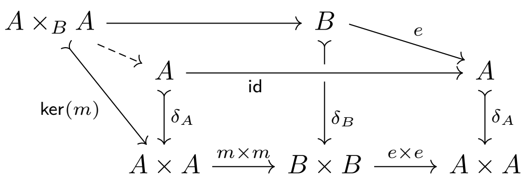
Lemma 2.5. *Let A and B be 0-truncated objects in an ∞-category $\mathcal{C}$, and let $m : A \to B$, $e : B \to A$ such that $e \circ m = \mathsf{id}_A$ in $\mathsf{Ho}(\mathcal{C})$. Then m is a monomorphism.*

Proof. We give a proof in $\mathcal{S}$ (or in any ∞-category with finite limits), the proof in general ∞-categories reduces to $\mathcal{S}$ by applying corepresentable functors $\mathsf{Map}_{\mathcal{C}}(X, -)$.

We have to show that $\delta_m : A \to A \times_B A$ is an equivalence. This map may be viewed as a map in the slice category over $A \times A$:



Since δ_A and $\mathsf{ker}(m)$ are monomorphisms it is sufficient by Lemma 2.4 to exhibit a map over $A \times A$ in the opposite direction of δ_m. Such a map is given by the mediating map in the following diagram



where the front rectangle is a pullback since $(e \times e) \circ (m \times m) \simeq \mathsf{id}$. □

2.2. **The object of contractibility.** Finally we will make use of the object of contractibility, motivated from homotopy type theory.

Definition 2.6. Given an object A in a locally Cartesian closed ∞-category $\mathcal{C}$, we define the object $\mathsf{isContr}(A)$ by $\mathsf{isContr}(A) = A_!(\pi_* \delta_A)$, where $\delta_A : A \to A \times A$ is the diagonal and $\pi : A \times A \to A$ is the first projection.

Proposition 2.7. *Let $\mathcal{C}$ be a locally Cartesian closed ∞-category and let $A \in \mathcal{C}$.*

1. *The object $\mathsf{isContr}(A)$ is always subterminal.*
2. *A is terminal iff $\mathsf{isContr}(A)$ is terminal.*
3. *Given a second object B, we have $B^*(\mathsf{isContr}(A)) \simeq \mathsf{isContr}(B^*A)$ in $\mathcal{C}_{/B}$.*

Proof. For 1,2 see [Ras21, Subsection 4.8].

The third claim follows from the Beck–Chevalley condition for the pullback squares

$$\begin{array}{ccccc} B\times A\times A & \longrightarrow & B\times A & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A\times A & \longrightarrow & A & \longrightarrow & 1 \end{array}$$

together with the equivalences

$$\mathcal{C}_{/1}\simeq\mathcal{C}\qquad \mathcal{C}_{/(B\times A)}\simeq(\mathcal{C}_{/B})_{/(B^*A)}\qquad \mathcal{C}_{/(B\times A\times A)}\simeq(\mathcal{C}_{/B})_{/(B^*A\times B^*A)}$$

where we already commented on the first one, and the other two two are special cases of the dual of [Lur09, 2.1.2.5]. □

For more details on the object of contractibility in locally Cartesian closed ∞-categories see [Ras21, Subsection 4.8].

3. Subobject Classifiers in ∞-Categories

3.1. **Subobject lattices.** Let $\mathcal{C}$ be an ∞-category with pullbacks. The *subobject lattice* $\mathsf{Sub}(A)$ of an object A in $\mathcal{C}$ is the full subcategory of $\mathcal{C}_{/A}$ spanned by monomorphisms. Then $\mathsf{Sub}(A)$ is closed under finite limits in $\mathcal{C}_{/A}$, and since parallel maps between subterminal objects are always homotopic it is (equivalent to the nerve of) a poset, whence the finite limits are actually finite 'meets' (infima), i.e. $\mathsf{Sub}(A)$ is a *meet-semilattice*.

If $\mathcal{C}$ is locally Cartesian closed then the Cartesian closure of its slices $\mathcal{C}_{/A}$ is inherited by the subobject lattices $\mathsf{Sub}(A)$ since exponentiation preserves truncatedness as a right adjoint. We shall refer to Cartesian closed posets as *Heyting semilattices*[2]. The Cartesian exponentiation operation is called *Heyting implication* in the posetal case, and denoted $(-\Rightarrow -)$.

For $f : B \to A$, the pullback functor $f^* : \mathcal{C}_{/A} \to \mathcal{C}_{/B}$ restricts to a monotone and finite-meet-preserving map between subobject lattices.

$$\begin{array}{ccc} \mathsf{Sub}(B) & \overset{f^*}{\dashleftarrow} & \mathsf{Sub}(A) \\ \downarrow & & \downarrow \\ \mathcal{C}_{/B} & \overset{f^*}{\longleftarrow} & \mathcal{C}_{/A} \end{array}$$

If $\mathcal{C}$ is locally Cartesian closed, then f^* furthermore preserves Heyting implication by Lemma 2.2, i.e. it is a *morphism of Heyting semilattices*.

Since homotopic maps in $\mathcal{C}$ induce equal maps between subobject lattices, the assignment $A \mapsto \mathsf{Sub}(A)$ is functorial *on the homotopy category*, i.e. it gives rise to a contravariant functor

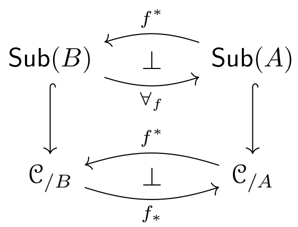
$$\mathsf{Sub}(-)\ :\ \mathsf{Ho}(\mathcal{C})^{\mathsf{op}} \to \mathsf{HSLat} \tag{3.1}$$

into the category HSLat of Heyting semilattices and monotone maps preserving finite meets and Heyting implication.

The postcomposition maps $f_! : \mathcal{C}_{/B} \to \mathcal{C}_{/A}$ do not generally restrict to subobject lattices (only if f itself is a monomorphism), but if $\mathcal{C}$ is locally Cartesian closed

[2]This is a back-formation from the common term *Heyting algebra*, which in our terminology is a Heyting semilattice with finite joins.

then the right adjoints f_* restrict to monomorphisms, so that for each $f : B \to A$ the adjunction between slices restricts to an adjunction between subobject lattices.



In other words, for each $f : B \to A$ in $\mathsf{Ho}(\mathcal{C})$, the monotone map $f^* : \mathsf{Sub}(A) \to \mathsf{Sub}(B)$ has a right adjoint which we denote $\forall_f : \mathsf{Sub}(B) \to \mathsf{Sub}(A)$.

By uniqueness of adjoints, this 'universal quantification' operation gives rise to a *covariant* functor of type $\mathsf{Ho}(\mathcal{C}) \to \mathsf{SLat}$ with the same object part as (3.1).

3.2. **Subobject classifiers.** Let $\mathcal{C}$ be again an ∞-category with pullbacks. We define $\mathsf{cMono}(\mathcal{C})$ to be the non-full subcategory of the arrow category $\mathsf{Fun}(\Delta^1, \mathcal{C})$ with monomorphisms as objects and pullback squares as morphisms. Then the codomain projection $p : \mathsf{cMono}(\mathcal{C}) \to \mathcal{C}$ is a right fibration [Lur09, 6.1.3.4]. Observe that for A in $\mathcal{C}$, the fiber of p over A is a Kan complex which is equivalent to the underlying set of $\mathsf{Sub}(A)$. We recall the following definition from [Lur09, 6.1.6.1].

Definition 3.2. A *subobject classifier* in $\mathcal{C}$ is a terminal object in $\mathsf{cMono}(\mathcal{C})$.

Thus, a subobject classifier is a monomorphism from which any other monomorphism can be obtained as a pullback in an essentially unique way.

Theorem 3.3. *Let $\mathtt{tt} : U \rightarrowtail \Omega$ be a subobject classifier in an ∞-category $\mathcal{C}$ with pullbacks. Then U is terminal and Ω is 0-truncated.*

Proof. The object Ω is 0-truncated because for every object A, the space $\mathsf{Map}_{\mathcal{C}}(A, \Omega)$ is equivalent to the fiber of p over A and therefore to the underlying set of $\mathsf{Sub}(A)$, i.e. $\mathsf{Map}(-, A)$ classifies the 0-presheaf of subobjects. The object U is terminal since it classifies *maximal* subobjects. □

Lemma 3.4. *Let $m : A \to B$, $e : B \to A$ be maps in a locally Cartesian closed ∞-category $\mathcal{C}$ such that $e \circ m = \mathsf{id}_A$ in $\mathsf{Ho}(\mathcal{C})$. Then given $U \in \mathsf{Sub}(B)$, we have $\forall_e U \leq m^* U$ in $\mathsf{Sub}(A)$.*

Proof. By adjunction we have $U \leq \forall_m m^* U$, and therefore we can argue

$$\forall_e U \leq \forall_e \forall_m m^* U \leq \forall_{e \circ m} m^* U \leq m^* U$$

by functoriality of $\forall$ on $\mathsf{Ho}(\mathcal{C})$. □

Theorem 3.5. *Let $\mathcal{C}$ be a locally Cartesian closed ∞-category with subobject classifier $\mathtt{tt} : U \rightarrowtail \Omega$. Then for every object $A \in \mathcal{C}$ the poset $\mathsf{Sub}(A)$ has finite joins.*

Proof. Given $A \in \mathcal{C}$ we claim that a least element of $\mathsf{Sub}(A)$ is given by

$$\perp = \forall_{\pi_1} \pi_2^* \, \mathtt{tt},$$

where $A \xleftarrow{\pi_1} A \times \Omega \xrightarrow{\pi_2} \Omega$ is a product span. Let $U \in \mathsf{Sub}(A)$, and let $f : A \to \Omega$ with $f^* \mathtt{tt} = U$. Then we have

$$\begin{aligned} \perp = \forall_{\pi_1} \pi_2^* \, \mathtt{tt} &\leq \langle \mathsf{id}_A, f \rangle^* \pi_2^* \, \mathtt{tt} && \text{by Lemma 3.4} \\ &\leq f^* \, \mathtt{tt} && \text{by functoriality of } (-)^* \\ &= U\,. \end{aligned}$$

Given $U, V \in \mathsf{Sub}(A)$ we claim that a binary join is given by

$$U \vee V = \forall_{\pi_1}\big((\pi_1^* U \Rightarrow \pi_2^* \mathtt{tt}) \wedge (\pi_1^* V \Rightarrow \pi_2^* \mathtt{tt}) \Rightarrow \pi_2^* \mathtt{tt}\big).$$

The derivation

$$\begin{array}{ll} & \pi_1^* U \wedge (\pi_1^* U \Rightarrow \pi_2^* \mathtt{tt}) \quad \leq \quad \pi_2^* \mathtt{tt} \\ \Rightarrow & \pi_1^* U \wedge (\pi_1^* U \Rightarrow \pi_2^* \mathtt{tt}) \wedge (\pi_1^* V \Rightarrow \pi_2^* \mathtt{tt}) \quad \leq \quad \pi_2^* \mathtt{tt} \\ \Leftrightarrow & \pi_1^* U \quad \leq \quad (\pi_1^* U \Rightarrow \pi_2^* \mathtt{tt}) \wedge (\pi_1^* V \Rightarrow \pi_2^* \mathtt{tt}) \Rightarrow \pi_2^* \mathtt{tt} \\ \Leftrightarrow & U \quad \leq \quad \forall_{\pi_1}\big((\pi_1^* U \Rightarrow \pi_2^* \mathtt{tt}) \wedge (\pi_1^* V \Rightarrow \pi_2^* \mathtt{tt}) \Rightarrow \pi_2^* \mathtt{tt}\big) \end{array}$$

shows that U is indeed smaller than $U \vee V$, and similarly for V. To show that $U \vee V$ is a least upper bound let $W \in \mathsf{Sub}(A)$ with $U \leq W$ and $V \leq W$, and let $g : A \to \Omega$ with $g^* \mathtt{tt} = W$. Then we have

$$\begin{aligned} & U \vee V \\ & = \forall_{\pi_1}\big((\pi_1^* U \Rightarrow \pi_2^* \mathtt{tt}) \wedge (\pi_1^* V \Rightarrow \pi_2^* \mathtt{tt}) \Rightarrow \pi_2^* \mathtt{tt}\big) \\ & \leq \langle \mathsf{id}_A, g\rangle^* \big((\pi_1^* U \Rightarrow \pi_2^* \mathtt{tt}) \wedge (\pi_1^* V \Rightarrow \pi_2^* \mathtt{tt}) \Rightarrow \pi_2^* \mathtt{tt}\big) && \text{by Lemma 3.4} \\ & = (U \Rightarrow W) \wedge (V \Rightarrow W) \Rightarrow W && \text{since } (-)^* \text{ preserves } \wedge, \Rightarrow \\ & = W \end{aligned}$$

□

Remark 3.6. The argument in the previous proof is well known from second order logic, and in its categorical incarnation from *tripos theory* [HJP80, Pit81] and elementary topos theory [BJ81]. It works in general whenever we have a presheaf $\mathcal{H} : \mathbb{C}^{\mathsf{op}} \to \mathsf{HSLat}$ of Heyting semilattices on a 1-category with finite products, such that

(1) reindexing maps along product projections have right adjoints, and
(2) $\mathcal{H}$ has a *generic predicate*, i.e. the category of elements of the underlying presheaf of sets of $\mathcal{H}$ has a weakly terminal object.

(Note that we do *not* require a Beck-Chevalley condition.)

From the point of view of locally Cartesian closed categories we point out that the construction applies exponentiation and pushforward functors f_* only to *subobjects* rather than general morphisms.

4. Initial Objects

In this section we prove that every locally Cartesian closed ∞-category with subobject classifier has a strict initial object.

Definition 4.1. An *initial* object in an ∞-category $\mathcal{C}$ is an object 0 such that $\mathsf{Map}_{\mathcal{C}}(0, A)$ is contractible for all $A \in \mathcal{C}$. The initial object is called *strict*, if $\mathcal{C}_{/0}$ is equivalent to the terminal ∞-category.

The following theorem gives a characterization of initial objects.

Theorem 4.2. *Let $\mathcal{C}$ be a locally Cartesian closed ∞-category and I an object of $\mathcal{C}$. Then the following are equivalent.*

(1) *I is initial in $\mathcal{C}$.*
(2) *$\mathcal{C}_{/I}$ is equivalent to the terminal ∞-category.*
(3) *$\mathsf{Sub}(I)$ is equivalent to the terminal preorder.*

Proof. Evidently (1) implies (3) since every subobject of an initial object has to be trivial.

Conversely, if $\mathsf{Sub}(I) \simeq 1$ then for any $X \to I$ the subobject $\mathsf{isContr}_I(X) \rightarrowtail I$ is maximal, meaning that $X \to I$ is an equivalence. This shows that (3) implies (2).

Finally, to show that I is initial we have to show that the mapping space $\mathsf{Map}_{\mathcal{C}}(I, X)$ is terminal for all $X \in \mathcal{C}$. Since $\mathsf{Map}_{\mathcal{C}}(I, X) \simeq \mathsf{Map}_{\mathcal{C}}(1, X^I)$ and $\mathsf{Map}_{\mathcal{C}}(1, -)$ preserves finite limits, it is enough to show that X^I is terminal in $\mathcal{C}$. Since $X^I = \Pi_I I^* X$ and $\Pi_I : \mathcal{C}_{/I} \to \mathcal{C}$ preserves limits, it is enough to show that $I^* X$ is terminal in $\mathcal{C}_{/I}$. This follows from (2). □

Remark 4.3. Implication (1) to (2) of the theorem tells us in particular that initial objects in locally Cartesian closed ∞-categories are always *strict* (Definition 4.1).

Corollary 4.4. *Let $\mathcal{C}$ be a locally Cartesian closed ∞-category with subobject classifier. Then $\mathcal{C}$ has a strict initial object.*

Proof. By Theorem 3.5, the terminal object of $\mathcal{C}$ has a least subobject $0 \rightarrowtail 1$. Since any subobject of a least subobject is trivial we have $\mathsf{Sub}(0) \simeq 1$, and Theorem 4.2 together with Remark 4.3 imply that 0 is a strict initial object. □

5. Binary Coproducts

In this section we prove that every locally Cartesian closed ∞-category with subobject classifier Ω has finite coproducts by using the fact that the subobject lattices have finite joins (Theorem 3.5). To motivate our proof, we start by discussing the 1-categorical case.

According to Johnstone [Joh02a, A2.2], the first proofs of the existence of finite colimits in elementary toposes were given by Mikkelsen and Paré [Mik72, Par74, Mik76]. Mikkelsen's proof does not seem to have been published. Paré proved – using Beck's theorem – that in any elementary 1-topos $\mathcal{E}$ the power object functor $\Omega^{(-)} : \mathcal{E}^{\mathsf{op}} \to \mathcal{E}$ is monadic, which implies that $\mathcal{E}^{\mathsf{op}}$ has finite limits as a category of Eilenberg-Moore algebras over a finite-limit category.

Although there is an ∞-categorical analogue of Beck's theorem [Lur17, Theorem 4.7.3.5], this proof cannot be generalized as the corresponding functor of ∞-categories $\Omega^{(-)} : \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$ is not monadic and in fact not even conservative for the most simple examples: if $\mathcal{C} = \mathcal{S}$ then $\Omega = \{0, 1\}$, the two element set, and the functor $\Omega^{(-)} : \mathcal{S}^{\mathsf{op}} \to \mathcal{S}$ takes every connected space to Ω, and every map between connected spaces to an equivalence.

Our proof of the existence of binary coproducts is based on an 'internal-language proof' in 1-toposes that avoids the monadicity theorem and was given as an Exercise in [LS86, Exercise II.5.]. The idea is to 'carve out' the coproduct $A+B$ as subobject of $\Omega^A \times \Omega^B$. In trying to adapt this proof to ∞-categories, we are met with two obstacles:

(1) While in a 1-topos every object A embeds into its power object Ω^A, this cannot work in higher toposes as, by Theorem 3.3, Ω – and therefore Ω^A and all its subobjects – are 0-truncated.
(2) To verify the universal property, the internal-language proof exhibits the unique arrow by first defining a (monic) binary relation, and then showing that it is single-valued and total. This kind of argument cannot work in the higher setting since it relies on the fact that the graph $\langle 1, f \rangle : A \to A \times B$ of a map $f : A \to B$ is always monic, which is not the case e.g. in $\mathcal{S}$.

To overcome the first hurdle, we replace the Ω^A in the construction with an object $\overline{A}$ known as *partial map classifier* or *partial map representer* [Joh02a, pg. 101] in 1-topos theory (Lemma 5.1). To address the second point, we replace the classical internal-logic proof by an argument which is inspired by homotopy type theory (Lemma 5.3), and which crucially relies on the technique of the object of contractibility, which we reviewed in Subsection 2.2.

Lemma 5.1. *Let A be an object in a locally Cartesian closed ∞-category $\mathcal{C}$ with subobject classifier $\mathsf{tt} : 1 \to \Omega$. Then there exists an object $\overline{A}$ admitting disjoint monomorphisms of A and 1, i.e. there exists a pullback square*

$$\begin{array}{ccc} 0 & \rightarrowtail & 1 \\ \downarrow & & \downarrow \\ A & \rightarrowtail & \overline{A} \end{array}$$

where all sides are monomorphisms and the upper left object is initial.

Proof. Let $a : A \to 1$ be the terminal projection, and define $(\overline{a} : \overline{A} \to \Omega) = \mathsf{tt}_* \, a$. Then by Lemma 2.3(3) we have $\mathsf{tt}^* \, \overline{a} \simeq a$, i.e. there is a pullback square

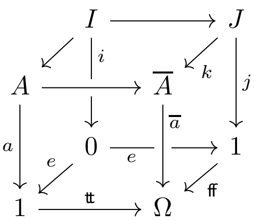
$$\begin{array}{ccc} A & \rightarrowtail & \overline{A} \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \overline{a}} \\ 1 & \xrightarrow{\mathsf{tt}} & \Omega \end{array} .$$

The lower map is a monomorphism by Lemma 2.5, and the upper map is a monomorphism by pullback stability. Now let

$$\begin{array}{ccc} 0 & \xrightarrow[e]{} & 1 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \mathsf{ff}} \\ 1 & \xrightarrow{\mathsf{tt}} & \Omega \end{array}$$

be the classifying pullback square of the least subobject $0 \rightarrowtail 1$ of 1, such that $\mathsf{ff} : 1 \rightarrowtail \Omega$ represents the truth value 'false'. Again, ff is a monomorphism by Lemma 2.5. The upper and left maps can be chosen to be equal since $\mathsf{Map}_{\mathcal{C}}(0, 1)$ is contractible.

Forming the pullback in the arrow category $\mathsf{Fun}(\Delta^1, \mathcal{C})$ we obtain a commutative cube



in which the left and right sides are pullbacks, since pullbacks are computed pointwise in functor categories. We already know that the front and bottom squares are pullbacks, and conclude that the remaining two are as well by the pullback lemma. The map i is an equivalence since $\mathcal{C}_{/0} \simeq 1$ by Theorem 4.2. Furthermore we have

$$j \simeq \mathsf{ff}^*(\mathsf{tt}_* \, a) \simeq e_*(e^* a) \simeq e_* i$$

by the Beck–Chevalley condition (Lemma 2.1), which means that j is an equivalence as well since terminal objects are preserved by right adjoints.

Finally, k is a monomorphism as a pullback of ff and the desired square is recovered on the top of the cube. □

Lemma 5.2. *Let $U, V \in \mathsf{Sub}(1)$ be subterminals in a locally Cartesian closed ∞-category $\mathcal{C}$, such that $U \vee V = \top$ in $\mathsf{Sub}(1)$. An object $A \in \mathcal{C}$ is contractible whenever $U^* A$ is contractible in $\mathcal{C}_{/U}$ and $V^* A$ is contractible in $\mathcal{C}_{/V}$.*

Proof. It is sufficient to show $\mathsf{isContr}(A) \geq U$ and $\mathsf{isContr}(A) \geq V$ in $\mathsf{Sub}(1)$, or equivalently that $U^* \, \mathsf{isContr}(A) \simeq 1$ and $V^* \, \mathsf{isContr}(A) \simeq 1$ in $\mathcal{C}_{/U}$ and $\mathcal{C}_{/V}$, respectively. This follows from the assumption together with Proposition 2.7(2) since we have $U^*(\mathsf{isContr}(A)) \simeq \mathsf{isContr}(U^*A)$ and $V^*(\mathsf{isContr}(A)) \simeq \mathsf{isContr}(V^*A)$ by Proposition 2.7(3). □

Lemma 5.3. *Let $U \overset{i}{\rightarrowtail} A \overset{j}{\leftarrowtail} V$ be a cospan of monomorphisms in a locally Cartesian closed ∞-category $\mathcal{C}$, such that $U \wedge V$ is a least subobject of A, and $\top$ is a least upper bound of U and V in $\mathsf{Sub}(A)$. Then i and j exhibit A as a disjoint coproduct of U and V.*

Proof. Since the forgetful functor $A_! : \mathcal{C}_{/A} \to \mathcal{C}$ preserves coproducts as a left adjoint we may w.l.o.g. work in the slice category and thus assume that $A = 1$.

To show that we have a coproduct, we have to check that for all objects $X \in \mathcal{C}$ and arrows $f : U \to X$, $g : V \to X$, the pullback of the cospan

$$\begin{array}{ccc} & & \mathsf{Map}_{\mathcal{C}}(1,X) \\ & & \big\downarrow {\scriptstyle \langle \mathsf{Map}_{\mathcal{C}}(i,X), \mathsf{Map}_{\mathcal{C}}(j,X)\rangle} \\ 1 & \xrightarrow{\langle f,g\rangle} & \mathsf{Map}_{\mathcal{C}}(U,X) \times \mathsf{Map}_{\mathcal{C}}(V,X) \end{array}$$

in $\mathcal{S}$ is contractible. This cospan is equivalent to the image of the cospan

$$\begin{array}{ccc} & & X \\ & & \big\downarrow {\scriptstyle \langle c,d\rangle} \\ 1 & \xrightarrow{\langle f,g\rangle} & X^U \times X^V \end{array} \tag{5.4}$$

under $\mathsf{Map}_{\mathcal{C}}(1,-)$, where c and d are exponential transposes of projection maps. Since $\mathsf{Map}_{\mathcal{C}}(1,-)$ preserves limits, it suffices to show that the pullback of the latter cospan is terminal in $\mathcal{C}$. By Lemma 5.2 and since pullback functors preserve limits, it suffices to show that the images of (5.4) under U^* and V^* are contractible in $\mathcal{C}_{/U}$ and $\mathcal{C}_{/V}$, respectively. By symmetry, it is enough to consider the first case. We have

$$U^*(X^U) = U^*(U_*(U^*X)) \simeq U^*X \tag{5.5}$$

since $U^* \dashv U_*$ is a reflection (Lemma 2.3), and by applying the Beck–Chevalley condition for the pullback square $\begin{smallmatrix} 0 & \xrightarrow{i} & V \\ {\scriptstyle i}\downarrow & & \downarrow \\ U & \to & 1 \end{smallmatrix}$ we get

$$U^*(X^V) = U^*(V_*(V^*X)) \simeq i_*(i^*(V^*X)) \simeq i_*1 \simeq 1,$$

since all objects over 0 are terminal (Theorem 4.2). Furthermore one can show that modulo the equivalence (5.5) we have $U^*(c) \simeq \mathsf{id}$, and since U^* preserves limits we conclude

$$U^*\left(\begin{array}{ccc} & & X \\ & & \downarrow {\scriptstyle \langle c,d\rangle} \\ 1 & \longrightarrow & X^U \times X^V \end{array}\right) \simeq \left(\begin{array}{ccc} & & U^*X \\ & & \downarrow {\scriptstyle \mathsf{id}} \\ U^*1 & \longrightarrow & U^*X \end{array}\right).$$

The pullback of the right hand cospan is contractible in $\mathcal{C}_{/U}$ since U^*1 is, and equivalences are stable under pullback.

Disjointness is clear since the injections are monic by assumption, and their pullback coincides with the meet $U \wedge V = \bot$ in $\mathsf{Sub}(A)$, which is initial by Theorem 4.2. □

Theorem 5.6. *Let $\mathcal{C}$ be a locally Cartesian closed ∞-category with a subobject classifier. Then $\mathcal{C}$ has disjoint binary coproducts.*

Proof. Let A and B be objects of $\mathcal{C}$. By Lemma 5.3 it is sufficient to find an object C admitting monomorphisms $A \rightarrowtail C$ and $B \rightarrowtail C$ such that $A \wedge B = \bot$ and $A \vee B = \top$ in $\mathsf{Sub}(C)$.

By Lemma 5.1 we have pullback squares

$$\begin{array}{ccc} 0 & \rightarrowtail & 1 \\ \downarrow & & \downarrow \\ A & \rightarrowtail & \overline{A} \end{array} \qquad\qquad \begin{array}{ccc} 0 & \rightarrowtail & 1 \\ \downarrow & & \downarrow \\ B & \rightarrowtail & \overline{B} \end{array}$$

Forming the 'transposed product'

$$\left(\begin{array}{ccc} 0 & \rightarrowtail & 1 \\ \downarrow & & \downarrow \\ A & \rightarrowtail & \overline{A} \end{array}\right) \times \left(\begin{array}{ccc} 0 & \rightarrowtail & B \\ \downarrow & & \downarrow \\ 1 & \rightarrowtail & \overline{B} \end{array}\right) = \left(\begin{array}{ccc} 0 & \rightarrowtail & B \\ \downarrow & & \downarrow \\ A & \rightarrowtail & \overline{A} \times \overline{B} \end{array}\right)$$

of these two pullbacks yields a pullback square exhibiting A and B as disjointly embedded in an object $\overline{A} \times \overline{B}$. The desired cospan $A \rightarrowtail C \leftarrowtail B$ is obtained by setting $C = A \vee B$ in $\mathsf{Sub}(\overline{A} \times \overline{B})$. □

The following summarizes all our results.

Theorem 5.7. *Let $\{A_k\}_{k \in I}$ be a finite family of objects in a locally Cartesian closed ∞-category $\mathcal{C}$ with subobject classifier. Then the coproduct $\coprod_{k \in I} A_k$ exists, and the inclusion maps $i_k : A_k \to \coprod_{k \in I} A_k$ give rise to an equivalence of ∞-categories*

$$(i_k^*)_{k \in I} : \mathcal{C}_{/\coprod_{k \in I} A_k} \to \prod_{k \in I} \mathcal{C}_{/A_k}.$$

Proof. If I is empty, then this is precisely the statement that the initial object exists and is strict (Corollary 4.4). For I non-empty, this is a direct consequence of the fact that coproducts exists and are disjoint (Theorem 5.6) and universal, as $\mathcal{C}$ is locally Cartesian closed and left adjoints preserve colimits [Lur09, Proposition 5.2.3.5]. □

6. Coproducts and Pushouts in an Elementary ∞-Topos

In this final section we apply our result to the theory of elementary ∞-toposes. Following [Ras18, Shu17] we consider the following definition.

Definition 6.1. An elementary ∞-topos is a finitely complete and cocomplete locally Cartesian closed ∞-category $\mathcal{E}$ with a subobject classifier and enough universes[3].

Theorem 5.7 immediately gives us the following.

Corollary 6.2. *An ∞-category $\mathcal{E}$ is an elementary ∞-topos if and only if it is locally Cartesian closed and has coequalizers, a subobject classifier, and enough universes.*

This result moves us closer to the modern definition of elementary toposes, with the main difference being that we still assume the existence of coequalizers. The final question is whether we can construct coequalizers from the remaining axioms.

The following example shows that a subobject classifier certainly does not suffice to construct pushouts.

[3]Here a *universe* is an arrow $p : \mathcal{U}_* \to \mathcal{U}$ such that for all objects A of $\mathcal{E}$ the induced map $\mathsf{Map}(A, \mathcal{U}) \to \mathsf{Core}(\mathcal{E}_{/A})$ is a monomorphism, and the class of pullbacks of p satisfies certain closure conditions. For details see [Ras18].

Example 6.3. Let $\mathcal{S}^{\mathsf{tr}}$ be the full subcategory of $\mathcal{S}$ spanned by truncated spaces. Note that $\mathcal{S}^{\mathsf{tr}}$ is locally Cartesian closed and the discrete space $1+1$ is a subobject classifier. We claim that the diagram

$$1 \longleftarrow S^1 \longrightarrow 1 \tag{6.4}$$

does not have a pushout in $\mathcal{S}^{\mathsf{tr}}$. First, note that the pushout in $\mathcal{S}$ is just the 2-sphere S^2. This implies that the n-truncation $\tau_{\leq n}S^2$ is the pushout of this diagram in the subcategory $\mathcal{S}^{\leq n}$ of n-truncated spaces. Now if (6.4) had a pushout C in $\mathcal{S}^{\mathsf{tr}}$ then the n-truncations of C would *also* be pushouts in $\mathcal{S}^{\leq n}$, which would imply that $\tau_{\leq n}C \simeq \tau_{\leq n}S^2$ for all $n \geq 0$. This is impossible since S^2 is not truncated [Gra69].

References

[AGV72] M. Artin, A. Grothendieck, and J. L. Verdier. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Tome 1: Théorie des topos. Exposés I à IV. 2e éd.*, volume 269. Springer, Cham, 1972. (On p. 1)

[Ber10] J.E. Bergner. A survey of $(\infty, 1)$-categories. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 69–83. Springer, New York, 2010. (On p. 1)

[BJ81] A. Boileau and A. Joyal. La logique des topos. *The Journal of Symbolic Logic*, 46(1):6–16, 1981. (On p. 8)

[CLW93] A. Carboni, S. Lack, and R.F.C. Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84(2):145–158, 1993. (On p. 2)

[GHK21] D. Gepner, R. Haugseng, and J. Kock. ∞-Operads as Analytic Monads. *International Mathematics Research Notices*, 04 2021. (On p. 3)

[Gra69] B. Gray. On the sphere of origin of infinite families in the homotopy groups of spheres. *Topology*, 8(3):219–232, 1969. (On p. 13)

[HJP80] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Cambridge Philos. Soc.*, 88(2):205–231, 1980. (On p. 8)

[JM95] A. Joyal and I. Moerdijk. *Algebraic set theory*, volume 220. Cambridge University Press, 1995. (On p. 1)

[Joh02a] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 1*, volume 43 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 2002. (On pp. 4 and 9)

[Joh02b] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2*, volume 44 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, Oxford, 2002. (On p. 1)

[Joy08] A. Joyal. Notes on quasi-categories. 2008. (On p. 3)

[Law70] F.W. Lawvere. Quantifiers and sheaves. In *Actes du congres international des mathematiciens, Nice*, volume 1, pages 329–334, 1970. (On p. 1)

[LS86] J. Lambek and P.J. Scott. *Introduction to higher order categorical logic*, volume 7 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. (On p. 9)

[Lur09] J. Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009. (On pp. 1, 2, 3, 6, 7, and 12)

[Lur17] J. Lurie. Higher algebra. Unpublished book (accessed 20.03.2021), September 2017. (On pp. 1 and 9)

[Mik72] C.J. Mikkelsen. Finite colimits in toposes. In *Talk at the conference on category theory at Oberwolfach*, 1972. (On pp. 1 and 9)

[Mik76] C.J. Mikkelsen. *Lattice theoretic and logical aspects of elementary topoi*. PhD thesis, Aarhus University, 1976. (On pp. 1 and 9)

[MLM94] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition. (On p. 1)

[MLS84] P. Martin-Löf and G. Sambin. *Intuitionistic type theory*, volume 9. Bibliopolis Napoli, 1984. (On p. 1)

[Par74] R. Paré. Colimits in topoi. *Bulletin of the American Mathematical Society*, 80(3):556–561, 1974. (On pp. 1 and 9)

[Pit81] A.M. Pitts. *The theory of triposes*. PhD thesis, Cambridge Univ., 1981. (On p. 8)

[Ras18] Nima Rasekh. A theory of elementary higher toposes. *arXiv preprint*, 2018. arXiv:1805.03805v3. (On pp. 1 and 12)

[Ras21] N. Rasekh. Every elementary higher topos has a natural number object. *Theory Appl. Categ.*, 37:Paper No. 13, pp 337–377, 2021. (On p. 6)

[Rez10] C. Rezk. Toposes and homotopy toposes (version 0.15). Unpublished notes (accessed 20.03.2021), 2010. (On pp. 1 and 2)

[RV17] E. Riehl and D. Verity. Fibrations and Yoneda's lemma in an ∞-cosmos. *Journal of Pure and Applied Algebra*, 221(3):499–564, 2017. (On p. 3)

[Shu17] M. Shulman. Elementary $(\infty, 1)$-topoi, 2017. *Blog post.* `https://golem.ph.utexas.edu/category/2017/04/elementary_1topoi.html`. (On pp. 1 and 12)

[Shu19] M. Shulman. All $(\infty, 1)$-toposes have strict univalent universes. *arXiv preprint arXiv:1904.07004*, 2019. (On p. 1)

[Tie72] M. Tierney. Sheaf theory and the continuum hypothesis. In *Toposes, algebraic geometry and logic*. Springer, 1972. (On p. 1)

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013. (On p. 1)

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