

Electron-phonon heat transfer in giant vortex states

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We examine energy relaxation of non-equilibrium quasiparticles in different vortex configurations in “dirty” s -wave superconductors. The heat flow from the electronic subsystem to phonons in a mesoscopic superconducting disk with a radius of the order of several coherence lengths is calculated both in the Meissner and giant vortex states using the Usadel approach. The recombination process is shown to be strongly affected by interplay of the subgap states, located in the vortex core and in the region at the sample edge where the spectral gap E_g is reduced by the Meissner currents. In order to uncover physical origin of the results, we develop a semiquantitative analytical approximation based on the combination of homogeneous solutions of Usadel equations in Meissner and vortex states of a mesoscopic superconducting disc and analytically calculate the corresponding spatially resolved electron-phonon heat rates. Our approach provides an important information about non-equilibrium quasiparticles cooling by the magnetic-field induced traps in various mesoscopic superconducting devices.

I. INTRODUCTION

The progress in cryogenic superconducting (SC) devices and technologies, such as electronic refrigerators and thermometers [1], radiation detectors [2, 3], and qubit systems for quantum information processing [4, 5] requires an improved understanding of the quasiparticle thermalization mechanisms. Such mesoscopic devices with nanoscale dimensions operating at subkelvin temperatures are easily driven out of equilibrium via processes involving electromagnetic field absorption, hot quasiparticle injection, or the operational drive. As a result, there is a significant concentration of non-equilibrium quasiparticles (QPs), present in a driven superconductor even for the temperatures far away from the drive location well below the temperature of superconducting transition [6]. Typically, the presence of excess QPs is characterized by an effective electron temperature T , which exceeds the temperature of the phonon bath T_{ph} . Excess QPs destroy coherence of qubit systems [7–9], decrease the quality factor of superconducting resonators [10, 11], reduce the efficiency of cooling in electronic refrigerators [12, 13], and result in the excess current in single electron [14–16] and heat quantum [17] turnstiles. The energy-relaxation rate of the non-equilibrium QPs is also known to affect the characteristics of detectors of electromagnetic radiation [6, 18]. Thus, the overheating of QPs or unwanted population in general appear to be major factors limiting the performance of low temperature SC devices. This is why the study of the mechanisms of non-equilibrium QPs relaxation in superconductors of mesoscopic dimensions seems to be rather important (see e.g., [19] for review).

In order to suppress quasiparticle poisoning and to prevent overheating, density of non-equilibrium QPs in a

device should be reduced. Simple lowering the phonon-bath temperature T_{ph} often is not sufficient, and to remove QPs in superconducting devices additional quasiparticle traps should be used: either normal-metal films connected to the superconductor [20–22], Andreev bound states in weak links [23], or regions with reduced or destroyed SC gap [24–27]. In this case QPs are trapped by the region with no energy gap (or suppressed gap) away from the active region. One of the possible types of QPs traps can be formed by regions with the suppressed SC gap, that appear in the Meissner and vortex states and can be successfully controlled by the external magnetic field [6, 28–36]. This method has a number of advantages due to using of the same material as the rest of the device, since perfect matching of different parts of the device without barriers or interface potentials is provided. Besides, magnetic field allows to tune the trap controllably. The controllable use of such traps in various applications mentioned above assumes, certainly, understanding of their cooling capacities.

The hot electrons dissipate the heat typically via the interaction with the phonons [37–39]. In three dimensional normal metals of volume \mathcal{V} (when the mean free path ℓ is small compared to all other length scales) the heat flow \dot{Q}_N related to the electron-phonon (e-ph) relaxation is determined by the well-known relation

$$\dot{Q}_N = \Sigma \mathcal{V} (T^5 - T_{\text{ph}}^5), \quad (1)$$

where Σ is the material constant for electron-phonon coupling [1]. Quasiparticle recombination in a bulk superconductor with an s -wave gap via coupling to the phonons has been widely studied [39–43]. The hard energy gap E_g in the quasiparticle spectrum of the superconductor is known to suppress the recombination of the hot QPs making the relaxation rate ν_{e-ph} to be expo-

nentially slow ($\sim \exp(-E_g/k_B T)$) because QPs need to possess an energy exceeding the gap E_g in order to recombine. The electron-phonon relaxation remains extremely slow even in the presence of Abrikosov vortices and subgap excitations localized in the vortex core in clean superconductors at ultra low temperatures [44]. The resulting relaxation rate enters the rate equations for the concentration of QPs and phonons which allow to get the full description of the non-equilibrium processes in superconducting devices [45]. The electron-phonon relaxation in mesoscopic SC samples and SC point contacts looks rather different compared to dynamics of bulk quasiparticles [46–49]. The simplified model of quasiparticle dynamics and electron cooling both in the Meissner and vortex states of a mesoscopic superconductor was proposed recently in Ref. 34 to explain experimental measurements of the characteristics of non-equilibrium quasiparticle distributions in a mesoscopic SC island in a single-electron transistor setup with normal metal leads, however, that model worked well only for the Meissner and single-quantum vortex states.

The main goal of our work is to overcome this limitation by developing a quantitative description of the electron-phonon heat transfer processes in a diffusive mesoscopic SC disk of the size comparable to several superconducting coherence lengths ξ_0 placed in external magnetic field oriented perpendicular to the plane of the disk. In such nanoscale samples theory predicts the existence of rather exotic vortex configurations – so-called “giant” (or multi-quantum) vortices in the disk center [50, 51]. These exotic vortex states are formed due to the confinement effect of the screening supercurrents and have been observed by a variety of experimental methods [50–55], including scanning tunneling microscopy/spectroscopy studies [56–61]. Electronic structure of mesoscopic superconductors is known to be sensitive to the applied magnetic field, related vorticity and vortex configurations [62–68]. The overall spectral characteristics and local density of states (LDOS) of the mesoscopic disk was shown to be strongly affected by an interplay of the subgap states, located in the vortex core and in the region at the sample edge where the spectral gap E_g is reduced by Meissner currents [69]. Keeping in mind the high sensitivity of e-ph relaxation to the local density of states [34], one can expect a strong dependence of the total QPs heat flow and the relaxation rate ν_{e-ph} in the mesoscopic disk on the applied magnetic field and related vorticity. Note that both the giant vortex cores and the region near the sample edge with the flowing Meissner screening currents can be clearly viewed as quasiparticle traps created by the applied magnetic field. It is the joint effect of contributions from the core and edge traps which determines the total QPs heat flow in a mesoscopic sample.

The paper is organized as follows. In Sec. II we briefly discuss the basic equations. In Sec. III we calculate the superconducting critical temperature T_c and study the switching between the states with different vorticity L

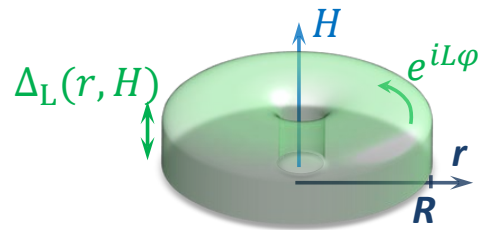


FIG. 1. (Color online) **Schematic picture of the spatial order parameter distribution $\Delta_L(H)$** (shown by semi-transparent green color) in the superconducting disk of the radius R with the giant L -quantum vortex, Eq. (2), in the applied perpendicular magnetic field H .

while sweeping the magnetic field. In Sec. IV we calculate numerically the electron-phonon heat flow in a mesoscopic SC disk with a giant vortex. In Sec. V we propose a simplified analytical model to describe the electron-phonon heat flow and check its applicability. We summarize our results in Sec. VI.

II. MODEL AND BASIC EQUATIONS

To set the stage, here we describe the considered system and main relevant equations and approximations. The section contains the standard notations and literature results, therefore a reader, acquainted with this formalism and interested in the original result, can directly jump to Sec. III.

The system in the focus is a thin mesoscopic superconducting (SC) disk of a radius R comparable to several superconducting coherence lengths ξ at a finite temperature T . We concentrate on more experimentally relevant s -wave superconductors in the dirty limit where the mean free path ℓ is the smallest length scale in the system. In terms of the elastic electron scattering rate τ^{-1} and the bare superconductor transition temperature T_{cs} the above dirty limit condition reads as follows $k_B T_{cs} \tau / \hbar \ll 1$. In order to generate vortices that trap quasiparticles in their cores, an external magnetic field $\mathbf{H} = H \mathbf{z}_0$, perpendicular to the SC disk plane, is applied (Fig. 1). We focus on the experimentally relevant situation of a small disk thickness d and disregard the magnetic field, induced by Meissner supercurrents in the disk. This approximation holds as soon as d is small compared to the London penetration depth λ in such a way that the effective magnetic field penetration depth $\Lambda = \lambda^2/d$ is large compared to the disk radius $\Lambda \gg R$. This allows us to take into account only the external field, i.e., $\text{rot} \mathbf{A} = \mathbf{B} \equiv \mathbf{H}$. In addition, due to smallness of the inelastic electron-electron scattering time τ_{ee} with respect to the electron-phonon one in experimentally relevant setups, we assume that the QP energy distribution in a superconducting disk is the quasiequilibrium Fermi distribution $f_T(E) = [e^{E/k_B T} + 1]^{-1}$, characterized by a certain (electronic) temperature $T \ll T_{cs}$ which may

differ from the bath temperature T_{ph} . In this regime, the normal (\mathcal{G}) and anomalous (\mathcal{F}) quasiclassical Green's functions obey the Usadel equations [70], which are valid for all temperatures and for distances exceeding the mean free path ℓ . We concentrate on the cylindrically symmetric case, introduce the coordinates (r, φ, z) as it is shown in Fig. 1, and look for the homogeneous solution along the z -axis solutions characterized by a certain integer angular momentum L (called a vorticity),

$$\Delta(\mathbf{r}) = \Delta_L(r) e^{iL\varphi}, \quad (2)$$

which describe the axisymmetric multiquantum vortex states with the vortex core located at the center of the disk, $r = 0$. The vorticity L coincides with the angular momentum of the anomalous Green's function \mathcal{F} . In the standard trigonometrical parametrization the quasiclassical Green's functions for L -th orbital mode can be encoded in the pairing angle $\theta_L(r)$ as

$$\mathcal{G} = \cos \theta_L, \quad \mathcal{F} = \sin \theta_L e^{iL\varphi}, \quad \mathcal{F}^\dagger = \sin \theta_L e^{-iL\varphi}.$$

The pairing angle θ_L obeys the following equation (see, e.g., Ref. [71])

$$-\frac{\hbar D}{2} \nabla_r^2 \theta_L + [\omega_n + \Gamma_L(r, H) \cos \theta_L] \sin \theta_L = \Delta_L(r) \cos \theta_L, \quad (3)$$

where the inhomogeneous depairing parameter

$$\Gamma_L(r, H) = \hbar v_L^2 / 2D \quad (4)$$

is expressed through the superfluid velocity $\mathbf{v}_s = (0, v_L, 0)$

$$v_L = D \left(\frac{L}{r} - \frac{\pi H}{\Phi_0} r \right) \quad (5)$$

and depends on the external magnetic field H . Here $\omega_n = \pi T(2n + 1)$ is the Matsubara frequency at the temperature T , $D = v_{\text{F}} \ell / 3$ is the diffusion coefficient, and $\Phi_0 = \pi \hbar c / e$ is the flux quantum. The relevant length scale in the Usadel equations is given by the superconducting coherence length $\xi_0 = \sqrt{\hbar D / 2\Delta_0}$, where Δ_0 is the SC gap at zero temperature. Further we treat only positive ω_n values, due to the symmetry of Usadel equations and \mathcal{F} being an even function of ω_n , $\mathcal{F}(\mathbf{r}, -\omega_n) = \mathcal{F}(\mathbf{r}, \omega_n)$. The singlet pairing potential $\Delta_L(r)$ in Usadel equations (3) is determined self-consistently by the equation

$$\frac{\Delta_L(r)}{g} - 2\pi T \sum_{n \geq 0} \sin \theta_L = 0, \quad (6)$$

where the pairing parameter g fixes the bare critical temperature T_{cs} as

$$\frac{1}{g} = \sum_{n=0}^{\Omega_D / (2\pi T_{\text{cs}})} \frac{1}{n + 1/2} \simeq \ln \left[\frac{\Omega_D}{2\pi T_{\text{cs}}} \right] + 2 \ln 2 + \gamma. \quad (7)$$

Here Ω_D is the Debye frequency and $\gamma \simeq 0.5772$ is the Euler–Mascheroni constant. The equations (3),(6) in the disk bulk should be accompanied by the boundary conditions at the disk edge $r = R$ for the order parameter Δ_L and the pairing angle θ_L :

$$\left. \frac{d\Delta_L}{dr} \right|_R = 0, \quad \left. \frac{d\theta_L}{dr} \right|_R = 0. \quad (8)$$

Both the Usadel (3) and the self-consistency (6) equations can be obtained by variation of the free energy functional

$$F_L = 2\pi N_0 d \left(\pi T \sum_{\omega_n < \Omega_D} \int_0^R r dr \left\{ \hbar D \left(\frac{\partial \theta_L}{\partial r} \right)^2 + 2\Gamma_L \sin^2 \theta_L - 4\omega_n \cos \theta_L - 4\Delta_L \sin \theta_L \right\} + \frac{1}{g} \int_0^R r dr \Delta_L^2 \right), \quad (9)$$

with N_0 representing the density of states at the Fermi level for a spin projection.

The power absorbed in the disk is associated with the heat transferred to phonons emitted by thermal QPs. In general inhomogeneous case, the electron-phonon heat flow $\dot{Q}_L(r)$ across the central part of the disk of the a radius $r \leq R$ into the phonon bath for the orbital mode L is given by the expressions [41, 42]

$$\dot{Q}_L(r) = 2\pi d \int_0^r dr' r' \mathcal{P}_L(r'), \quad \mathcal{P}_L(r) = \int_0^\infty \frac{\Sigma [n_T(\epsilon) - n_{T_{\text{ph}}}(\epsilon)]}{24\zeta(5)k_B^5} \epsilon^3 d\epsilon \int_{-\infty}^\infty M_L(r, E, \epsilon) [f_T(E) - f_T(E + \epsilon)] dE. \quad (10)$$

Here $\zeta(s)$ is the Riemann zeta function, k_B is the Boltzmann constant, $n_T(\epsilon) = [\exp(\epsilon/k_B T) - 1]^{-1}$ is the Bose function with a temperature T . The kernel of the integrals (10)

$$M_L(r, E, \epsilon) = N_L(r, E) N_L(r, E + \epsilon) - B_L(r, E) B_L(r, E + \epsilon), \quad (11)$$

with the local density of states (LDOS) $N_L(r, E)$ and superconducting correlations $B_L(r, E)$

$$N_L(r, E) = \text{Re}[\cos \theta_L(r)]|_{\omega_n = -iE}, \quad B_L(r, E) = \text{Im}[\sin \theta_L(r)]|_{\omega_n = -iE}, \quad (12)$$

depending on the radial coordinate r , due to the spatial dependence of the normal ($\sim \cos \theta_L$) and anomalous ($\sim \sin \theta_L$) Green's functions in the disk.

To calculate $M_L(r, E, \epsilon)$ we have to solve the Usadel equation (3) with the inhomogeneous depairing parameter $\Gamma_L(r, H)$ (4), (5) for $\omega_n = -iE$ in the presence of the L -quantum vortex in the disk center. The solution of the Usadel equation gives us the LDOS $N_L(r, E)$ and the spectral gap value E_g governing the contribution to the thermal relaxation mechanisms (see, e.g., [34, 35]). From the experimental point of view, LDOS $N_L(r, E)$ can be measured directly via the local differential conductance (e.g., in the scanning tunneling spectroscopy setting):

$$G_L(r, V) = \frac{dI/dV}{(dI/dV)_N} \quad (13)$$

$$= \int_{-\infty}^{\infty} dE \frac{N_L(r, E)}{N_0} \frac{\partial f_T(E - eV)}{\partial V}.$$

Here V is the applied bias voltage, $(dI/dV)_N$ is the conductance of the normal metal junction.

III. GIANT VORTEX STATES IN MESOSCOPIC DISK

The phase boundary $T_c(H)$ of the mesoscopic SC disk is known to exhibit an oscillatory behavior similar to the well-known Little-Parks oscillations [72, 73], caused by the transitions between the states with different angular momenta L . Here $T_c(0) = T_{cs}$. Figure 2 shows a typical dependence of the critical temperature

$$T_c(H) = \max_L \{ T_L(H) \},$$

on the external magnetic field H , affected by these transitions. The values of the normalized flux $\phi_L = \pi H_L R^2 / \Phi_0 = H_L / H_0$ through the disk of a radius R , where the switching of the orbital modes $L \rightleftharpoons L + 1$ takes place, do not depend on the disk radius R , and can be found using the of Eqs. (3), (6) linearized in the anomalous Green's function ($\cos \theta_L \simeq 1$, $\sin \theta_L \simeq \theta_L$) [69] or within the Ginzburg-Landau formalism [74] based on the free energy (9). Note that this energy-based approach (9) does not take into account possible hysteresis in the orbital-mode switching, caused by the barriers between different free-energy minima.

In order to analyze the transitions between different vortex states far from the phase transition line $T \ll T_c(H)$ we need to use the nonlinear Usadel theory (3 - 8). The Usadel equations have been solved numerically for different vorticities L which allowed us to calculate and compare the values of the free energy F_L (9). Figure 3 shows the magnetic field dependence of the free energy (9) and the zero bias conductance $G_L(R, 0)$ (13) at the Fermi level for a small disk radius $R = 4\xi_0$ and the temperature $T = 0.1T_{cs}$. The curves illustrate the

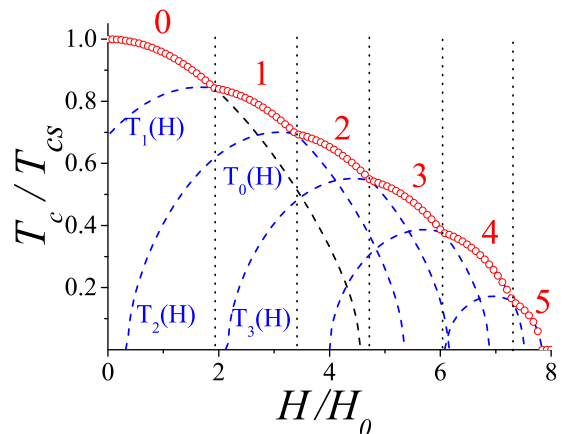


FIG. 2. (Color online) **Schematic dependence of the critical temperature T_L ($L = 1 \div 5$) (dashed blue lines) and T_c (red symbols) on the external magnetic field H .** The vorticity L is denoted by the numbers near the curves. The values of the magnetic field $H = H_L$ corresponding to the switching of the orbital modes between L and $L + 1$ are shown by the dotted vertical lines.

switching between the states with different vorticities $L = 0 \div 4$, which is similar to the Little-Parks-like switching of the critical temperature $T_c(H)$ shown in Fig. 2.

Sequential entries of vortices produce a set of branches

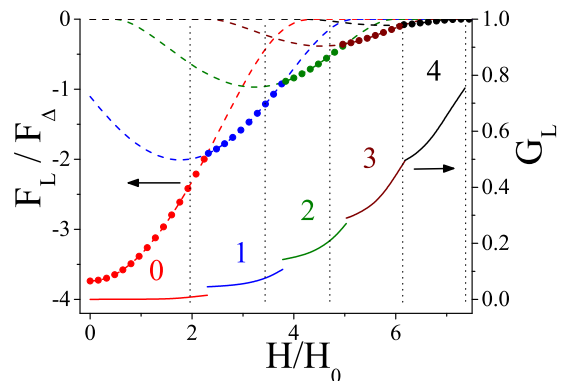


FIG. 3. (Color online) **The magnetic-flux $\phi = H/H_0$ -dependence of the normalized zero bias conductance (ZBC) $G_L(0, R)$ (13) at the edge of the SC disk (solid lines) and the free energy F_L (9) (symbol \bullet) for the temperatures $T = 0.1T_{cs}$.** The disk radius is $R = 4\xi_0$, the SC coupling constant is $g = 0.18$. The dependence $F_L(\phi)$ for fixed vorticity $L = 0 \div 4$ are shown by the dashed lines. Like in Fig. 2, the corresponding values of vorticity L are denoted by the numbers near the curves, while the values of the flux $H_L/H_0 = \phi_L$ corresponding to the switching of the orbital modes between L and $L + 1$, with different critical temperatures T_c , are shown by the dotted vertical lines. Here the free energy is normalized by $F_\Delta = \pi \hbar D N_0 d \Delta_0$.

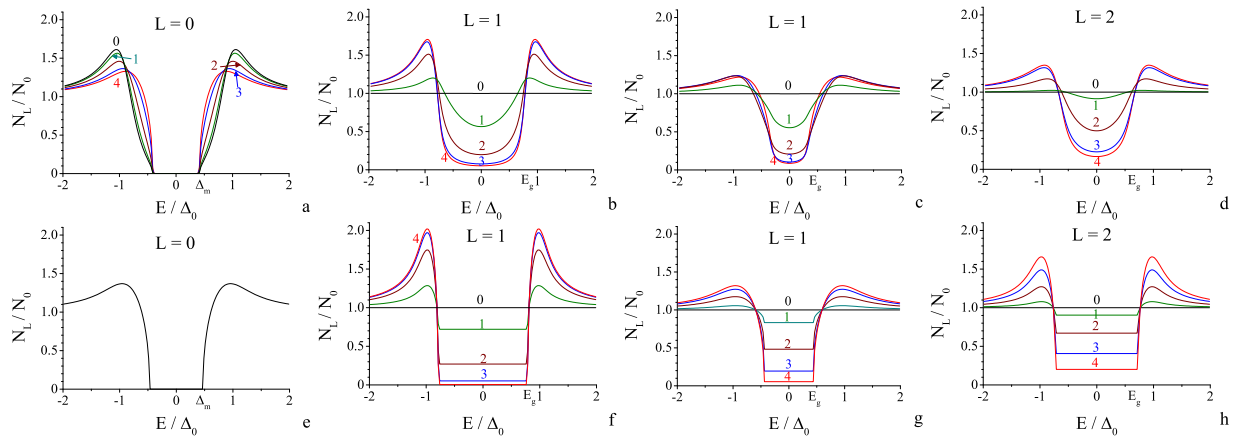


FIG. 4. (Color online) **Comparison of the exact (12) (panels a-d) and approximate (14) (panels e-h) LDOS $N_L(r, E)$ versus energy E for several distances r from the vortex core, vorticities L , and magnetic fields H .** The SC disk radius $R = 4\xi_0$ and the temperature $T = 0.1T_{cs}$ are fixed for all panels. The magnetic field values H are: (a, b, e, f) $H = H_{s1} = 2.24H_0$; (c, d, g, h) $H = H_{s2} = 3.84H_0$. The distances r (in the units of ξ_0) are shown by numbers of the corresponding colors near the curves. The soft minigap at the edge E_g in (b-d) is assumed to correspond to the maximal slope of the energy dependence of the LDOS $dN_L(R, E)/dE$.

F_L with different vorticity L on the $F(H)$ and $dI/dV(H)$ curves. The transitions between different vortex states are accompanied by an abrupt change in the zero bias conductance (ZBC) at the disk edge, which is attributed to the entry/exit of a vortex while sweeping the magnetic field. Note, that the field values H_{sL} at which the jumps in vorticity ($L - 1 \rightarrow L$) occur at low temperature are always larger than the values H_L found from the calculations of the critical temperature behavior $T_c(H)$. For a fixed disk radius R the direction of jumps (upward/downward) in the dependence of ZBC $G_L(R, 0)$ vs magnetic field depends on the temperature T and reflects a crossover between the edge-dominated and core-dominated regimes in the magnetic field dependence of the tunneling conductance (see Ref. 69 for details). Figure 4 illustrates evolution of the spatially resolved LDOS $N_L(r, E)$ vs energy E for values of the magnetic field $H = H_{s1}, H_{s2}$ corresponding to the vorticity switching. In the Meissner state ($L = 0$), the hard minigap Δ_m in the spectrum survives and $N_{L=0}(r, E < \Delta_m) = 0$ until the first vortex entry, Fig. 4(a). In vortex states ($L \geq 1$), Figs. 4(b-d), the density of states $N_L(0, E)$ is equal to the normal-metal electronic density of states at the Fermi level N_0 , indicating a full suppression of the spectral gap in the disk center due to the vortex entry. At the same time, at the edge of the disk the superconductivity survives though the gap becomes soft, $0 < N_L(R, E) < N_0$. The soft minigap at the edge E_g is assumed to correspond to the maximal slope of the energy dependence of the LDOS $dN_L(R, E)/dE$. Figures 4(a-b) illustrate the switching between the states with hard and soft gaps at $H = H_{s1} \simeq 2.24H_0$ when the Meissner state ($L = 0$) transforms to the single-vortex state ($L = 1$). At fixed vorticity $L = 1$ as the magnetic field increases, the LDOS $N_L(r, E)$ smoothly evolves with the simultaneous decrease of the soft minigap at the disk edge E_g , Figs. 4(b-

c). This smooth evolution of the LDOS is interrupted at $H = H_{s2} \simeq 3.84H_0$ by a next vortex entry which restores the superconductivity near the disk edge and results in increase of the soft minigap E_g , Figs. 4(c-d). Certainly, these drastic changes in the LDOS $N_L(r, E)$ and the spectral gap value E_g directly impact the thermal relaxation mechanisms and manifest themselves in peculiarities of the magnetic field dependence of the electron-phonon heat flow \dot{Q} , which we consider further.

IV. ELECTRON-PHONON HEAT FLOW IN GIANT VORTEX CONFIGURATIONS

As a next step we calculate the thermal relaxation rate in the superconducting disk assuming the electronic temperature T to be much larger than the bath (phonon) temperature T_{ph} , thus, neglecting for simplicity all the exponential terms like $e^{-E_g/k_B T_{ph}}$. Filled circles in Fig. 5 show the magnetic field dependence of the total electron-phonon heat flow \dot{Q} versus the applied magnetic field H for a small disk. Here and further we focus on the disk radius $R = 4\xi_0$ and the temperature $T = 0.1T_{cs}$ (if not mentioned otherwise). The total $\dot{Q}(R)$ curve consists of several separate branches \dot{Q}_L (10) corresponding to the states with different vorticity $L = 0 \div 4$. The transitions between different vortex states are visualized by abrupt changes (or jumps) $\Delta\dot{Q}$ in the heat flow at $H = H_{sL}$, where switching of the orbital modes $L - 1 \rightleftharpoons L$ takes place. The smooth growth of the electron-phonon heat flow \dot{Q}_L occurs while sweeping the magnetic field up within the branch L , $H_{sL} < H < H_{s(L+1)}$, due to decrease of the value of the spectral gap (soft or hard) and increase in subgap LDOS at the disk edge [69]. The especially strong growth of \dot{Q}_L appears in the Meissner state

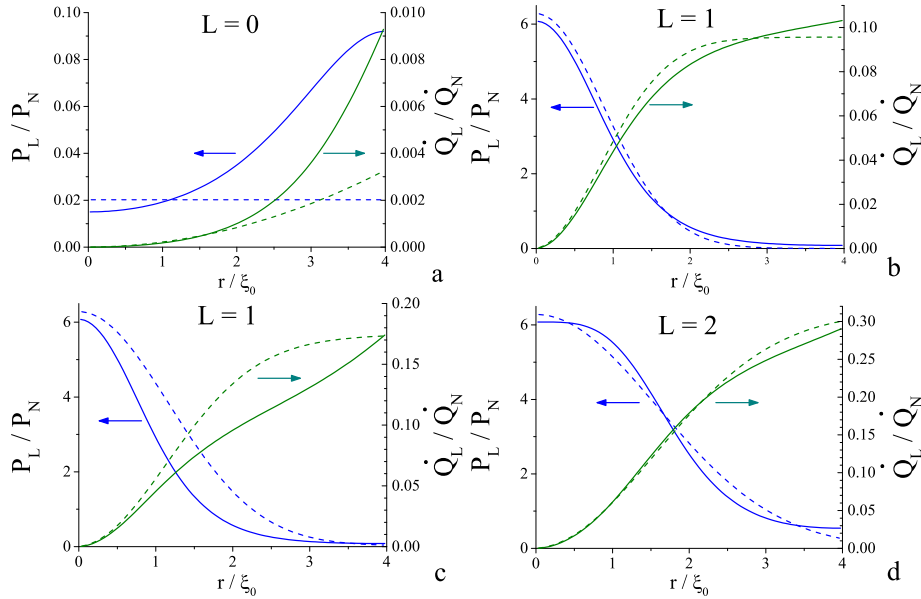


FIG. 6. (Color online) The spatially resolved heat flow $\dot{Q}_L(r)$ and the density of the electron-phonon heat flow $P_L(r)$ (10) (solid lines) for several values of magnetic field H : (a, b) $H = H_{s1} = 2.24 H_0$; (c, d) $H = H_{s2} = 3.84 H_0$. The dashed lines show the results of fitting $\dot{Q}_L(r)$ (26), (27) for $Z_{DL}(r) = e^{-r^2/D_L^2}$ ($L \geq 1$) described in section V: $D_1 = 1.6\xi_0$, $D_2 = 2.7\xi_0$. ($R = 4\xi_0$, $T = 0.1T_{cs}$, $\dot{Q}_N = \Sigma\mathcal{V}T^5$, $P_N = \Sigma T^5$, $\mathcal{V} = \pi R^2 d$).

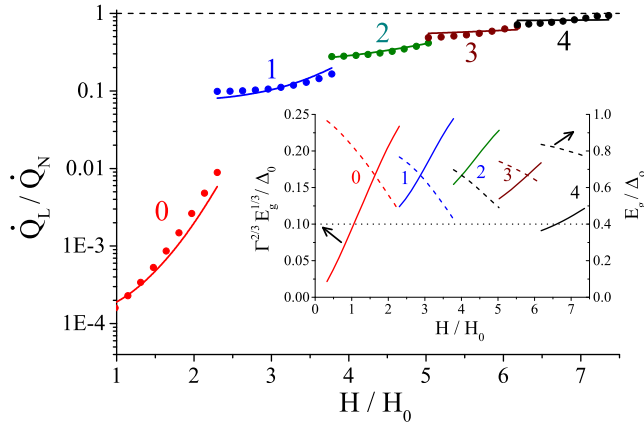


FIG. 5. (Color online) The dependence of the electron-phonon heat flow $\dot{Q}_L(R)$ (10) on the magnetic flux $\phi = H/H_0$ across the SC disk of the radius $R = 4\xi_0$ for the temperatures $T = 0.1T_{cs}$ (filled circles). The solid lines show the results of fitting \dot{Q}_L (26), (27) in homogeneous approximation described in Sec. V. The inset shows the dependence of the hard gap \bar{E}_g (solid lines) and the value $\bar{\Gamma}^{2/3}\bar{E}_g^{1/3}$ (dashed lines) determined by the relations (17) on the magnetic flux ϕ . The dotted line shows the value of the temperature T . The numbers near the curves denote the corresponding values of vorticity $L = 0 \div 4$. ($\dot{Q}_N = \Sigma\mathcal{V}T^5$, $\mathcal{V} = \pi R^2 d$).

($\dot{Q}_0(H_{s1})/\dot{Q}_0(0) \sim 10^2$) owing to essential suppression of the hard minigap Δ_m in the spectrum by the screening currents, Fig. 4(a).

To compare the contributions to the total electron-phonon heat flow $\dot{Q}(R)$ (10) from the vortex core and the region near the sample edge with the flowing Meissner current, we present in Fig. 6 radial distributions of the heat flow $\dot{Q}_L(r)$ and the flow density $\mathcal{P}_L(r)$ (10) for two values of the applied magnetic field $H = H_{s1}, H_{s2}$ corresponding to the switching between the states $L = 0 \rightarrow 1$ and $L = 1 \rightarrow 2$, respectively. In the Meissner state the spatially resolved heat flow density $\mathcal{P}_L(r)$ and the heat flow $\dot{Q}_L(r)$ are more pronounced near the disk edge where the screening superconducting currents have higher density and the SC order parameter is suppressed, Fig. 6(a). A vortex in the mesoscopic SC disk leads to QP redistribution so that the heat flow density in the vortex core is always higher than near the disk edge ($\mathcal{P}_{L \neq 0}(0) > \mathcal{P}_{L \neq 0}(R)$). Despite this, the contribution of the region outside the vortex core to the total electron-phonon heat flow appears to be comparable to the contribution of the vortex core itself because of the relatively small volume of the core region. Note, that the significant growth of $\dot{Q}_L(r)$ (10) in Figs. 6(b-d) outside the vortex core region illustrate the existence of a noticeable interplay between the contributions from the subgap states, located in the vortex cores and in the region with the reduced spectral gap E_g near the sample edge. The

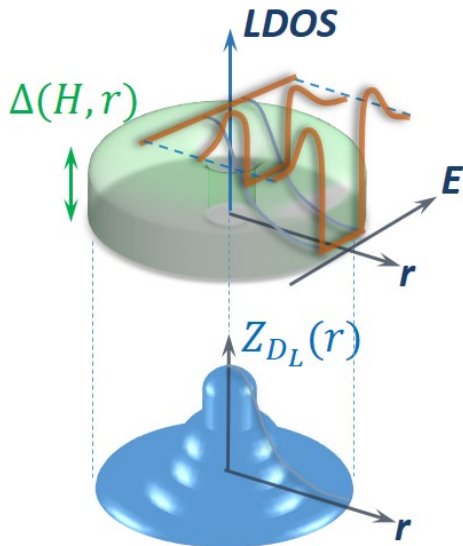


FIG. 7. (Color online) **Sketch of the homogeneous approximation.** (top) The soft superconducting gap (shown by semi-transparent green color) in the superconducting disk of the radius R with the giant L -quantum vortex is accompanied by the energy-dependence of the local density of states (orange lines), Eq. (14), being a superposition of the vortex $\bar{N}_V(E) = 1$ and homogeneous $\bar{N}_L(E)$ LDOS with the radial profile $Z_{\mathcal{D}_L}(r)$ (shown by two parallel blue curves). (bottom) The cylindrical-symmetric representation of the radial function $Z_{\mathcal{D}_L}(r)$ in the SC disk.

interplay increases when the field is swept from H_{sL} to $H_{s(L+1)}$ for a fixed vorticity L and with the increase in the number of vortices L trapped in the center of the sample.

The electronic properties of the vortex states will be surely modified if we further increase the radius of the disk R compared to the coherence length ξ_0 . In this case the core of a multiquantum vortex does not extend to the edge of the disk, and quasiparticles in the vortex core remain well localized near the disk center. Clearly, in this case the profiles of the heat flow density $\mathcal{P}_L(r)$ and the heat flow $\dot{Q}_L(r)$ (10) have to reveal a local minimum and a plateau, respectively.

V. SEMIQUANTITATIVE HOMOGENEOUS APPROXIMATION FOR THE ELECTRON-PHONON HEAT FLOW

For the analysis of the experimental data it is often useful and convenient to build some semiquantitative approximations for the measurable quantities which would allow to avoid the extensive use of the numerical simulations. To develop such a description we try to fit the above numerical simulations of the magnetic field dependence of the electron-phonon heat flow $\dot{Q}_L(H)$ (see Fig. 5) by a simplified model taking into account that the overall spectral characteristics and local density of

states of the mesoscopic sample result from the interplay of the subgap states, located in the vortex core and in the edge regions with the spectral gap reduced by Meissner currents [69]. We assume that both the order parameter $\Delta_L(r)$ and $\theta_L(r)$ function vanish inside the vortex core and separate the energy and the coordinate dependence in LDOS $N_L(r, E)$ and superconducting correlations $B_L(r, E)$ (12) by the following approximate expressions (see Fig. 7)

$$N_L(r, E) \simeq [1 - Z_{\mathcal{D}_L}(r)] \bar{N}_L(E) + Z_{\mathcal{D}_L}(r) \bar{N}_V(E) \quad (14)$$

$$B_L(r, E) \simeq [1 - Z_{\mathcal{D}_L}(r)] \bar{B}_L(E) + Z_{\mathcal{D}_L}(r) \bar{B}_V(E) \quad (15)$$

Here the functions $\bar{N}_V(E) = \text{Re}[\cos \theta_V] = 1$ and $\bar{B}_V(E) = \text{Im}[\sin \theta_V] = 0$ ($\theta_V = 0$), describe normal metal properties of the vortex core region. A radial profile $Z_{\mathcal{D}_L}(r)$ which describes a contribution of the subgap states in the vortex core ($Z_{\mathcal{D}_L}(0) = 1$, $Z_{\mathcal{D}_L}(R) \ll 1$) is assumed to be a monotonically decaying function with a certain L -dependent length scale \mathcal{D}_L , see Fig. 7 (bottom). A residual small contribution of the subgap states related to the vortex core at the disk edge ($\sim Z_{\mathcal{D}_L}(R)$) corresponds to a nonzero density of states at the Fermi level and is responsible for the destruction of the hard gap in the density of states. In the Meissner state we get $Z_0(r) = 0$ and, thus, $\mathcal{D}_0 = 0$.

The functions $\bar{N}_L(E)$ and $\bar{B}_L(E)$ describe the averaged characteristics of the superconducting state in the vortex free phase and outside the vortex core region. To get these functions we average the Eq. (3) over the radial coordinate neglecting the gradient terms, i.e. assume the $\theta_L(r)$ function to depend slowly on r . The spatially averaged value $\bar{\theta}_L$ satisfies to the Usadel equation for the normal ($\cos \bar{\theta}_L$) and anomalous ($-i \sin \bar{\theta}_L$) Green's functions [75]

$$(iE - \bar{\Gamma}_L \cos \bar{\theta}_L) \sin \bar{\theta}_L = \bar{\Delta}_L \cos \bar{\theta}_L, \quad (16)$$

which describes different depairing effects in dirty superconductors with the effective depairing energy $\bar{\Gamma}_L$ [71, 76–78]. The solution $\bar{\theta}_L$ of the algebraic equation (16) at $\omega_n = -iE$ gives us the energy dependence of the density of states $\bar{N}_L(E) = \text{Re}[\cos \bar{\theta}_L]$ and the function $\bar{B}_L(E) = \text{Im}[\sin \bar{\theta}_L]$. The hard gap \bar{E}_g in the density of states $\bar{N}_L(E)$ and the order parameter $\bar{\Delta}_L$ are determined by the relations

$$\bar{E}_g = \bar{\Delta}_L \left(1 - \gamma_L^{2/3}\right)^{3/2}, \quad \bar{\Delta}_L = \Delta_0 e^{-\pi \gamma_L/4}, \quad (17)$$

which in turn depend on the magnetic field H via parameter

$$\gamma_L = \bar{\Gamma}_L / \bar{\Delta}_L. \quad (18)$$

The effective depairing energy $\bar{\Gamma}_L$ takes into account the phase gradient created by the L -quantum vortex and the effect of the magnetic field H , and can be calculated by the averaging of the inhomogeneous depairing

parameter $\Gamma_L(r, H)$ (4), (5):

$$\bar{\Gamma}_L(H, \mathcal{D}_L) = \frac{\pi\hbar}{DS_Z} \int_0^R r [1 - Z_{\mathcal{D}_L}(r)]^2 v_L^2(r) dr, \quad (19)$$

$$S_Z = 2\pi \int_0^R r [1 - Z_{\mathcal{D}_L}(r)] dr.$$

The relation (19) accounts for the presence of the core of the L -quantum vortex via the factor $1 - Z_{\mathcal{D}_L}(r)$ in the superfluid velocity $[1 - Z_{\mathcal{D}_L}(r)]v_L(r)$ and the effective area of the disk S_Z . As a result, the effective depairing energy $\bar{\Gamma}_L$ (19) and parameter γ_L (18) depend on the magnetic field H and the length scale \mathcal{D}_L . The kernel (11) takes the form

$$M_L(r, E, \epsilon) \simeq Z_{\mathcal{D}_L}(r)^2 + [1 - Z_{\mathcal{D}_L}(r)]^2 \bar{M}_L(E, \epsilon) + Z_{\mathcal{D}_L}(r) [1 - Z_{\mathcal{D}_L}(r)] \bar{K}_L(E, \epsilon), \quad (20)$$

where the functions

$$\bar{M}_L(E, \epsilon) = \bar{N}_L(E)\bar{N}_L(E + \epsilon) - \bar{B}_L(E)\bar{B}_L(E + \epsilon) \quad (21)$$

$$\bar{K}_L(E, \epsilon) = \bar{N}_L(E) + \bar{N}_L(E + \epsilon) \quad (22)$$

do not depend on radius r .

In the Meissner state ($L = 0, Z_0 = 0$) the depairing energy $\bar{\Gamma}_0$ can be easily calculated by the averaging of the inhomogeneous depairing parameter $\Gamma_0(r, H)$ (4), (5) over the radial direction

$$\bar{\Gamma}_0(H) = \frac{2}{R^2} \int_0^R dr r \Gamma_0(r, H) = \frac{\hbar D}{4R^2} \left(\frac{H}{H_0} \right)^2. \quad (23)$$

The kernel $M_0(r, E, \epsilon)$ of the second integral in (10) does not depend on radial coordinate r ($M_0(r, E, \epsilon) \simeq \bar{M}_0(E, \epsilon)$) and can be found by replacing the functions $N_0(r, E)$ and $B_0(r, E)$ with $\bar{N}_0(E)$ and $\bar{B}_0(E)$, respectively, in the expression (11). The comparison of the above averaged description with the full numerical solution for the density of states can be seen in Fig. 4(e) where the homogeneous density of states $\bar{N}_0(E) = \text{Re}[\cos \bar{\theta}_0]$ determined by the solution of Eq. (16) is shown.

In the vortex state ($L \neq 0$) the unknown scale \mathcal{D}_L depends on the disk geometry via the spatial distributions

of the order parameter $\Delta_L(r)$ and the local density of states $N_L(r, E)$ in the disk, and in general \mathcal{D}_L is a function of temperature T , magnetic field H and vorticity L . In order to account for the reduction in the average order parameter $\bar{\Delta}_L$ and the minigap \bar{E}_g which occurs with increasing H for a fixed vorticity L , the cutoff radius \mathcal{D}_L is assumed to obey the following relation

$$\mathcal{D}_L(H) = a_L \xi_L(H), \quad (24)$$

where the coherence length

$$\xi_L(H) = \sqrt{\hbar D / 2 \bar{E}_g} \quad (25)$$

plays the role of the characteristic length scale of the Green's functions outside the vortex core, and a_L is a fixed fitting parameter for the L -th orbital mode.

Finally, the expressions (17)-(19), (24)-(25) define the implicit relation between the scale \mathcal{D}_L and magnetic field H , which after the substitution into the expression (19) for the homogeneous depairing energy $\bar{\Gamma}_L$ gives us the average normal ($\sim \cos \bar{\theta}_L$) and anomalous ($\sim \sin \bar{\theta}_L$) Green's functions from the solution of the algebraic equation (16).

The general expressions (10) for the electron-phonon heat flow can be significantly simplified in low temperature limit $T_{\text{ph}} \ll T \ll \bar{E}_g/k_B$, where we neglect the terms $e^{-\bar{E}_g/k_B T_{\text{ph}}}$ with respect to $e^{-\bar{E}_g/k_B T}$, using approximation of the kernel of the integrals $M(r, E, \epsilon)$ (20)-(22). We calculate the coordinate-resolved electron-phonon heat flux $\mathcal{P}_L(r)$ for any profile $Z_{\mathcal{D}_L}(r)$ combining the procedure, described in Refs. [39, 41], and the solution of Usadel equation (16) (see Appendix A for details).

As a result in the leading approximation in $\bar{E}_g/(k_B T)$ for $T_{\text{ph}} = 0$ and $T \ll \bar{\Gamma}^{2/3} \bar{E}_g^{1/3}/k_B$ the electron-phonon heat flow $\dot{Q}_L(r)$ within the central part of the disk of the a radius $r \leq R$ into the phonon bath for the orbital mode L is given by the expressions

$$\dot{Q}_L(r) = \frac{\dot{Q}_N}{\pi R^2} \int_0^r dr' r' \bar{\mathcal{P}}_L(r'), \quad (26)$$

$$\bar{\mathcal{P}}_L(r) \approx 2\pi \left\{ Z_{\mathcal{D}_L}^2(r) + [1 - Z_{\mathcal{D}_L}(r)]^2 \frac{128}{189\zeta(5)} \frac{k_B T}{\Gamma^{2/3} \bar{E}_g^{1/3}} \left[1 + \frac{21\pi}{256} \left(\frac{\bar{E}_g}{k_B T} \right)^3 e^{-\bar{E}_g/k_B T} \right] e^{-\bar{E}_g/k_B T} + Z_{\mathcal{D}_L}(r) [1 - Z_{\mathcal{D}_L}(r)] \frac{\sqrt{\pi/6}}{48\zeta(5)} \left(\frac{\bar{E}_g \bar{\Delta}}{\bar{\Gamma}^2} \right)^{1/3} \left(\frac{\bar{E}_g}{k_B T} \right)^{7/2} e^{-\bar{E}_g/k_B T} \right\}, \quad (27)$$

where the first term in curly brackets in (27) corresponds

to the normal core contribution, the second term — to

the superconducting part with the hard gap, while the last term provides the cross-contribution with the kernel (22). $Q_N = \mathcal{V}\Sigma T^5$ is the electron-phonon heat flow in the normal state of the disk of volume $\mathcal{V} = \pi R^2 d$.

The results of the fitting for the trial function $Z_{\mathcal{D}_L}(r) = e^{-r^2/\mathcal{D}_L^2}$ are shown in Fig. 5. The best fits of $Q_L(R)$ (26), (27) to the numerical simulations are obtained with $a_L \simeq 1.5; 2.5; 4.1; 8.0$ for $L = 1 \div 4$, respectively. The inset in Fig. 5 shows the dependence of the value $\bar{\Gamma}^{2/3}\bar{E}_g^{1/3}$ (solid lines) and the hard gap \bar{E}_g (dashed lines) determined by the relations (17) on the magnetic flux ϕ for the best fits. The condition $\bar{\Gamma}^{2/3}\bar{E}_g^{1/3} > k_B T$ determines the values of the applied magnetic field $H \gtrsim H_0$ for which the approximate expressions (26), (27) are correct. One can see from Fig. 5 that within the validity range $\bar{\Gamma}^{2/3}\bar{E}_g^{1/3} > k_B T$ the approximation agrees reasonably well with the exact result. In the range $H < H_0$ of the Meissner regime, where the above validity condition is violated, our semiquantitative model obviously fails as it does not take into account highly inhomogeneous distribution of Meissner currents (see, e.g., P_0 in Fig. 6(a)).

Figures 4(e-h) provide the comparison of the exact spatially resolved LDOS $N_L(r, E)$ vs energy E (12) shown in Figs. 4(a-d) to the approximate ones (14) corresponding to the best fits for values of the magnetic field $H = H_{s1}, H_{s2}$.

The corresponding spatially resolved heat flow density $\bar{P}_L(r)$ (27) and the heat flow $\bar{Q}_L(r)$ (26) are shown in Fig. 6 by the dashed lines. One should notice the semiquantitative agreement between exact numerically calculated and approximate curves, except for the cases Fig. 6(a, c), when the Meissner screening currents essentially suppress the SC order parameter $\Delta(r)$ near the disk edge, because the simple approximate model (14), (15) fails to account for this suppression.

Figure 8 shows the dependencies of the coherence length $\xi_L(H)$ and the corresponding length scale $\mathcal{D}_L(H)$ on the magnetic field H for different vorticities L . For a fixed vorticity L the coherence length $\xi_L(H)$ (25) grows while sweeping the magnetic field up, $H_{sL} \leq H \leq H_{s(L+1)}$, due to increase of the screening currents in a small disk and shrinking of the gap \bar{E}_g . As a result, the radius \mathcal{D}_L increases slightly within the branch L . The downward jumps in the coherence length $\xi_L(H)$ (25) at $H = H_{sL}$ are caused by the increase in the hard spectral gap value \bar{E}_g as the vortex enters the sample, see Fig. 4.

VI. CONCLUSIONS

In conclusion, on the basis of the Usadel theory we have calculated the electron-phonon heat transfer \bar{Q} in a diffusive mesoscopic SC disk of the size comparable to several superconducting coherence lengths ξ_0 , placed in the external magnetic field H oriented perpendicular to

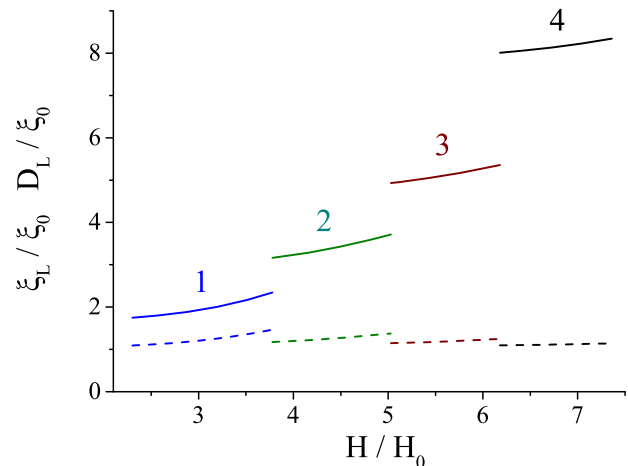


FIG. 8. (Color online) **The dependence of the scale \mathcal{D}_L (solid lines) and the coherence length ξ_L (dashed lines) on the magnetic field H in the SC disk of the radius $R = 4\xi_0$ for the temperature $T = 0.1T_{cs}$. The numbers near the curves denote the corresponding values of vorticity L .**

the plane of the disk. The strong confinement effects of the screening supercurrents are responsible for the formation of the giant (L -quantum) vortices in the disk center. The giant vortex core and the regions near the sample edge with the reduced spectral gap form potential wells (traps) for quasiparticles responsible for the heat transfer. We have demonstrated that the transitions between the superconducting states with different vorticities L provoke abrupt changes (jumps) in $\bar{Q}(H)$ attributed to the entry/exit of vortices while sweeping the magnetic field. The smooth growth of the electron-phonon heat flow $\bar{Q}_L(H)$ takes place for a fixed vorticity L while sweeping the magnetic field up, due to the increase of the screening currents and shrinking of the hard or soft spectral gap in the density of states. We have shown that the electron-phonon heat flow in the Meissner and vortex states of mesoscopic samples can be effectively controlled by the external magnetic field. We develop the semiquantitative approximate model the description of the quasiparticle trapping which works reasonably well beyond the Meissner state. Our numerical analysis of the electron-phonon relaxation confirms the validity and efficiency of the above semiquantitative model. We expect our results can stimulate further experimental work on the controllable QPs trapping in the vortex state of mesoscopic superconductors and superconducting hybrid devices.

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Appendix A: Homogeneous approximation in a mesoscopic disk

In this section we derive the expression for the electron-phonon heat flow \dot{Q}_{eph} (10)-(12) in a mesoscopic superconducting disk with a L -quantum vortex in the center within the approximation (14), (15). Using the kernel (20), we decompose the heat flow density (10) into three parts as follows

$$\mathcal{P}_L(r) = \Pi^{(1)} F^2(r) + \Pi^{(2)} [1 - F(r)]^2 + \Pi^{(3)} F(r) [1 - F(r)], \quad (\text{A1})$$

where the coefficients $\Pi^{(i)}$ are given by the unified expression

$$\Pi_L^{(i)} = \frac{\Sigma}{24\zeta(5)k_B^5} \int_0^\infty \epsilon^3 [n_T(\epsilon) - n_{T_{\text{ph}}}(\epsilon)] d\epsilon \int_{-\infty}^\infty \mathcal{M}_L^{(i)}(E, \epsilon) [f_T(E) - f_T(E + \epsilon)] dE, \quad (\text{A2})$$

with the kernels $\mathcal{M}_L^{(1)}(E, \epsilon) = 1$, $\mathcal{M}_L^{(2)}(E, \epsilon) = \bar{M}_L(E, \epsilon)$, and $\mathcal{M}_L^{(3)}(E, \epsilon) = \bar{K}_L(E, \epsilon)$, respectively. The coefficient

$$\Pi^{(1)} = \frac{\Sigma}{24\zeta(5)k_B^5} \int_0^\infty \epsilon^3 [n_T(\epsilon) - n_{T_{\text{ph}}}(\epsilon)] d\epsilon \int_{-\infty}^\infty [f_T(E) - f_T(E + \epsilon)] dE = \Sigma [T^5 - T_{\text{ph}}^5], \quad (\text{A3})$$

describes the heat flow density in the normal state and determines the contribution of the vortex core.

For simplicity the further calculations are done in low temperature limit $T_0 \ll T \ll \bar{E}_g/k_B$ in order to neglect the temperature dependence of the gap and the order parameter and the terms $e^{-\bar{E}_g/k_B T_{\text{ph}}}$ with respect to $e^{-\bar{E}_g/k_B T}$. In this case the values $\Pi^{(i)}$ (A2) do not depend on the phonon temperature T_{ph} and $\Pi^{(1)} \approx \Sigma T^5$.

In the low temperature limit the main contributions to the coefficients $\Pi^{(2)}$ and $\Pi^{(3)}$ arise from the vicinity $0 < |E| - \bar{E}_g \lesssim k_B T$ of the hard gap value $\pm \bar{E}_g$, leading to the smallness of the positive parameter $\delta E = |E| - \bar{E}_g \ll \bar{E}_g$. The expansion of $\bar{N}_L(E)$ and $\bar{B}_L(E)$ over δE takes the form

$$\bar{N}_L^2(E) \simeq \Theta(\delta E) \frac{2\delta E \bar{\Delta}_L^{2/3}}{3\bar{\Gamma}_L^{4/3} \bar{E}_g^{1/3}}, \quad \frac{\bar{B}_L(E)}{\bar{N}_L(E) \text{sign}(E)} \simeq \left(\frac{\bar{E}_g}{\bar{\Delta}_L} \right)^{1/3}. \quad (\text{A4})$$

Here $\Theta(x)$ is the Heaviside theta-function. Substituting (A4) into expressions (21, 22) we obtain the final expression for the kernels $\bar{M}_L(E, \epsilon)$ and $\bar{K}_L(E, \epsilon)$

$$\bar{M}_L(E, \epsilon) \simeq \frac{2\sqrt{\delta E \delta E'} \Theta(\delta E) \Theta(\delta E')}{3\bar{\Gamma}_L^{2/3} \bar{E}_g^{1/3}}, \quad \bar{K}_L(E, \epsilon) \simeq \sqrt{\frac{2\bar{\Delta}_L^{2/3}}{3\bar{\Gamma}_L^{4/3} \bar{E}_g^{1/3}}} \left[\sqrt{\delta E} \Theta(\delta E) + \sqrt{\delta E'} \Theta(\delta E') \right], \quad (\text{A5})$$

which are correct for $T \ll \bar{\Gamma}^{2/3} \bar{E}_g^{1/3}/k_B$, where $\delta E' = |E + \epsilon| - \bar{E}_g$. Splitting the integration in $\Pi^{(2)}$ (A2) into positive and negative parts in the electronic energy E and taking into account that $\bar{M}(E, \epsilon) = 0$ if any of two conditions $|E| < \bar{E}_g$ and $|E + \epsilon| < \bar{E}_g$ is valid, the coefficient $\Pi^{(2)}$ can be simplified in the low temperature limit as follows

$$\Pi^{(2)} \approx \frac{\Sigma}{24\zeta(5)k_B^5} \left[2 \int_0^\infty \epsilon^3 e^{-\epsilon/k_B T} d\epsilon \int_{\bar{E}_g}^\infty \bar{M}_L(E, \epsilon) e^{-E/k_B T} dE + \int_{2\bar{E}_g}^\infty \epsilon^3 e^{-\epsilon/k_B T} d\epsilon \int_{\bar{E}_g - \epsilon}^{-\bar{E}_g} \bar{M}_L(E, \epsilon) dE \right], \quad (\text{A6})$$

As a result taking into account only the leading terms in the small parameter $k_B T/\bar{E}_g$ we obtain

$$\Pi^{(2)} = \frac{128 \Sigma T^5}{189 \zeta(5)} \frac{k_B T}{\bar{\Gamma}_L^{2/3} \bar{E}_g^{1/3}} \left[1 + \frac{21\pi}{256} \left(\frac{\bar{E}_g}{k_B T} \right)^3 e^{-\bar{E}_g/k_B T} \right] e^{-\bar{E}_g/k_B T}. \quad (\text{A7})$$

In order to simplify the expression for $\Pi^{(3)}$ with $E \geq \bar{E}_g$, we neglect $n_{T_{\text{ph}}}(\epsilon)$ and use the following equalities

$$n_T(\epsilon) f_T(E - \epsilon) = f_T(E) [n_T(\epsilon) + f_T(E - \epsilon)], \quad n_T(\epsilon) f_T(E + \epsilon) = f_T(E) [n_T(\epsilon) - f_T(E + \epsilon) e^{E/k_B T}],$$

As a result we obtain

$$\Pi^{(3)} \approx \frac{\Sigma}{12\zeta(5)k_B^5} \int_0^\infty \epsilon^3 n_T(\epsilon) d\epsilon \int_{\bar{E}_g}^\infty \bar{N}_L(E) \left[f_T(E - \epsilon) + f_T(E + \epsilon) e^{E/k_B T} \right], \quad (\text{A8})$$

leading to

$$\Pi^{(3)} \approx \frac{\sqrt{\pi/6} \Sigma T^5}{48\zeta(5)} \left(\frac{\bar{E}_g \bar{\Delta}_L}{\bar{\Gamma}_L^2} \right)^{1/3} \left(\frac{\bar{E}_g}{k_B \bar{T}} \right)^{7/2} e^{-\bar{E}_g/k_B \bar{T}}. \quad (\text{A9})$$

Finally, substituting (A1), (A3), (A8), (A9) into the expressions (10), one obtains simplified expressions (26), (27) for the electron-phonon heat flow $\dot{Q}_L(r)$ into the phonon bath for the orbital mode L .

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