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The Odd Nadarajah-Haghighi Family of Distributions: Properties and Applications

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Abstract
We study some mathematical properties of a new generator of continuous distributions called the Odd Nadarajah-Haghighi (ONH) family. In particular, three special models in this family are investigated, namely the ONH gamma, beta and Weibull distributions. The family density function is given as a linear combination of exponentiated densities. Further, we propose a bivariate extension and various characterization results of the new family. We determine the maximum likelihood estimates of ONH parameters for complete and censored data. We provide a simulation study to verify the precision of these estimates. We illustrate the performance of the new family by means of a real data set.

1. Introduction
Statistical models are very useful in describing and predicting real world phenomena. Over the last decades, numerous extended distributions have been successfully used for modeling data obtained from applications in several areas. Recent developments aim to generate new families by extending well-known distributions and, at the same time, provide great flexibility in modeling data in practice. Thus, several classes distributions via generating new distributions by adding one or more parameters have been proposed in the statistical literature. Some well-known families are the exponentiated-G (exp-G) by Mudholkar and Srivastava ([14]), Marshall-Olkin-G by Marshall and Olkin ([13]), beta-G by Eugene et al. ([5]), gamma-G by Zografos and Balakrishnan ([21]), Kumaraswamy-G by Cordeiro and de Castro ([6]), McDonald-G by Alexander et al. ([1]), T-X family by Alzaatreh et al. ([3]), Weibull-G by Bourguignon et al. ([4]), beta odd log-logistic generalized by Cordeiro et al. ([8]), logistic-X family by Tahir et al. ([19]), Burr X generator by Yousof et al. ([20]) and odd-Burr generalized family by Alizadeh et al. ([2]).

Recently, Nadarajah and Haghighi ([17]) proposed the Nadarajah-Haghighi (NH) distribution, which has cumulative distribution function (cdf) and probability density function (pdf) given by (for $t > 0$)

$$F(t) = 1 - e^{1-(1+\lambda t)^{\alpha}} \text{ and } f(t) = \alpha\lambda(1+\lambda t)^{\alpha-1}e^{1-(1+\lambda t)^{\alpha}},$$

respectively, where $\alpha, \lambda > 0$. The NH quantile function (qf) is given by (Nadarajah and Haghighi, 2011)

(1)

$$Q(p) = \lambda^{-1}\left\{[1-\log(1-p)]^{\frac{1}{\alpha}} - 1\right\}.$$

Such distribution has proved to be a good alternative to the gamma, Weibull and log-normal models. We denote the NH random variable by $T \sim \mathrm{NH}(\alpha, \lambda)$.

In this paper, we consider the problem in which the odds ratio (associated with a baseline cdf $G(x;\ \boldsymbol{\xi})$ depending on a parameter vector $\boldsymbol{\xi}$) is represented by "$G(x;\ \boldsymbol{\xi})/\bar{G}(x;\ \boldsymbol{\xi})$", where $\bar{G}(x;\ \boldsymbol{\xi}) =\ 1\ G(x;\ \boldsymbol{\xi})$. Setting $p\ =\ F\ (x;\ \alpha, \lambda, \boldsymbol{\xi})$ in Equation (1), the cdf of the proposed family is given by

(2)

$$\frac{G(x;\ \boldsymbol{\xi})}{\bar{G}(x;\ \boldsymbol{\xi})}=\frac{\{1-\log[1-F\ (x;\ \alpha,\lambda,\boldsymbol{\xi})]\}^{1/\alpha}-1}{\lambda}\Leftrightarrow$$

$$F\ (x;\ \alpha,\lambda,\boldsymbol{\xi})=1-e^{1-\left[1+\lambda\frac{G(x;\ \xi)}{\bar{G}(x;\ \xi)}\right]^{\alpha}}$$

where $\alpha,\lambda\ >\ 0$ and $x\in\chi\subseteqq\mathbb{R}$. For each baseline cdf G, the *Odd NadarajahHaghighi-G* (ONH-G for short) family is defined by the cdf (2). Equation (2) can be understood as the cdf of the random variable X by solving the non-linear stochastic equation

$\frac{G(X;\,\xi)}{\bar{G}(X;\,\xi)}=T$, where $T\sim NH(\alpha\lambda)$ and $G(\cdot;\boldsymbol{\xi})$ is the baseline cdf.

Let *X* ONH-G$(\alpha,\lambda,\boldsymbol{\xi})$ be a random variable with cdf (2). The ONH-G distribution contains as special case the odd exponential-G family proposed by Bourguignon et al. ([4]) when $\alpha\ =\ 1$. Henceforth, $G(x)=\ G(x;\ \boldsymbol{\xi}),\bar{G}(x)=\ \bar{G}(x;\ \boldsymbol{\xi})$ and $g(x)=\ g(x;\ \boldsymbol{\xi})$ and we omit the dependence on the parameters.

This paper is organized as follows. In Section 2, we present the new family and its density, random number generator, three of its special cases and some of its asymptotes and shapes. Characterizations for the new family are given in Section 3. In Section 4, we demonstrate that the ONH-G pdf is given by a linear combination of exponentiated-G (exp-G) densities. In Section 5, we provide some mathematical and statistical properties of the ONH-G family. In Section 6, a new bivariate family is introduced. In Section 7, we discuss the maximum likelihood estimation procedure for the ONH-G parameters. Section 8 is devoted to a simulation study and an application. Section 9 offers some concluding remarks.

2. The new family and its motivation

The pdf corresponding to (2) is given by

(3)

$$f(x)=\frac{\alpha\lambda g(x)}{\bar{G}(x)^{2}}\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha-1}e^{1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}},x\in\mathbb{R},$$

where $g(x)$ is the baseline pdf. Further, the hazard rate function (hrf) of X is given by

(4)

$$h(x)=\frac{\alpha\lambda g(x)}{\bar{G}(x)^{2}}\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha-1},x\in\mathbb{R},$$

By inverting equation (2), the ONH-G random variable can be easily generated as follows:

1. Generate *u* as an outcome of *U* ~ *U* (0, 1);
2. Obtain

(5)

$$x = G^{-1}\left\{\frac{-1+[1-\log(1-u)]^{\frac{1}{\alpha}}}{\lambda-1+[1-\log(1-u)]^{\frac{1}{\alpha}}};\boldsymbol{\xi}\right\},$$

as an outcome of $X \sim$ ONH-G$(\alpha,\lambda,\boldsymbol{\xi})$.

Further, Theorem 1 provides some relations of some distributions with the ONH-G family.

THEOREM 1. *Let* $X \sim \mathrm{ONH-G}(\alpha,\lambda,\boldsymbol{\xi})$.

$(a)\ If\ Y = G(X;\boldsymbol{\xi}), then\ F_Y(y) = 1 - e^{1-\left(1+\frac{\lambda y}{1-y}\right)^{\alpha}}\ for\ 0<y<1.$

$(b) If\ Y = \dfrac{G(X)}{\bar{G}(X)}, then\ Y \sim NH(\alpha,\lambda).$

The proof of this theorem is obtained by direct application of (2) along with the basic properties of the cdf.

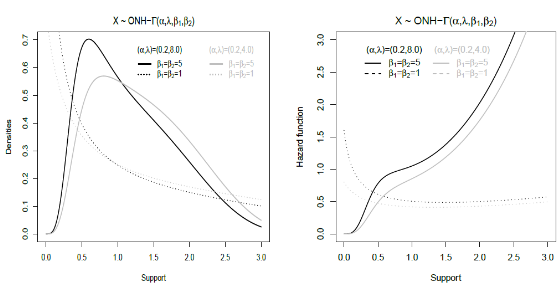
In what follows, we present three special cases of the ONH-G family.

2.1. The ONH-gamma (ONH-Γ) distribution

The ONHΓ density is given by ($\beta_1, \beta_2 > 0$)

$$f(x) = \frac{\alpha\lambda\beta_2^{\beta_1}}{\Gamma(\beta_1)}\frac{x^{\beta_1-1}e^{-\beta_2 x}}{[1-\gamma_1(\beta_1,\beta_2 x)]}\left[1+\lambda\frac{\gamma_1(\beta_1,\beta_2 x)}{1-\gamma_1(\beta_1,\beta_2 x)}\right]^{\alpha-1} \times \exp\left\{1-\left[1+\lambda\frac{\gamma_1(\beta_1,\beta_2 x)}{1-\gamma_1(\beta_1,\beta_2 x)}\right]^{\alpha}\right\}, x>0,$$

where $\gamma_1(\alpha,\beta x) = \Gamma(\alpha)^{-1}\int_0^{\beta x} t^{\alpha-1}\mathrm{d}t$ is the incomplete gamma function ratio. Figure 1 displays some plots of the pdf and hrf of the ONH-Γ distribution for selected parameter values.



(a) Pdfs. (b) Hrfs.

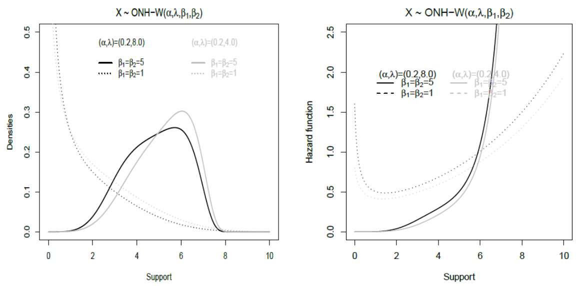
Fig. 1. Density and hazard rate plots for the ONHΓ model.

2.2. The ONH-Weibull (ONH-W) distribution

The ONH-W density is given by $(\beta_1, \beta_2 > 0)$

$$f(x) = \frac{\alpha\lambda\beta_1}{\beta_2^{\beta_1}}x^{\beta_1-1}\left[1+\lambda\left(e^{\left(\frac{x}{\beta_2}\right)^{\beta_1}}-1\right)\right]^{\alpha-1} \times \exp\left\{1-\left[1+\lambda\left(e^{\left(\frac{x}{\beta_2}\right)^{\beta_1}}-1\right)\right]^{\alpha}\right\}, x>0.$$

Figure 2 provides some plots of the pdf and hrf of the ONH-W distribution for selected parameter values.



(a) Pdfs. (b) Hrfs.

Fig. 2. Density and hazard rate plots for the ONH-W model.

2.3. The ONH-beta (ONH-β) distribution

The ONHβ density is given by $(\beta_1, \beta_2 > 0)$

$$f(x) = \frac{\alpha\lambda\beta_1}{\beta_2^{\beta_1}} x^{\beta_1 - 1} \left[1 + \lambda \frac{I_x(\beta_1, \beta 2)}{1 - I_x(\beta_1, \beta 2)}\right]^{\alpha - 1} \times \exp\left\{1 - \left[1 + \lambda \frac{I_x(\beta_1, \beta 2)}{1 - I_x(\beta_1, \beta 2)}\right]^{\alpha}\right\}, x > 0.$$

where $I_x(\beta_1, \beta 2) = B(\beta_1, \beta 2)^{-1} B(x; \beta_1, \beta 2)$ and $B(x; \beta_1, \beta 2) = \int_0^x t^{\beta_1 - 1}(1-t)^{\beta_2 - 1}\,\mathrm{d}t$ denotes the incomplete beta function. Figure 3 displays some plots of the ONHβ pdf for selected parameter values.

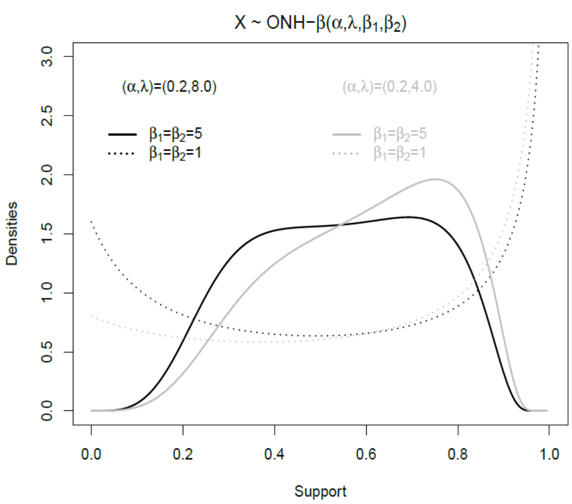


Fig. 3. Density plots for the ONHβ model.

3. Characterizations

In this section we present certain characterizations of ONH-G distribution. These characterizations are in terms of:

i. the truncated moment involving two functions;
ii. a simple relationship between two truncated moments and
iii. the hazard function.

One of the advantages of characterization (ii) is that the cdf is not required to have a closed form.

We present our characterizations (i)–(iii) in three subsections.

3.1. Characterizations based on truncated moment involving two functions

Our first characterization is based on the following Proposition.

PROPOSITION 1. *Let* $X : \Omega \rightarrow \mathbb{R}$ *be a continuous random variable with cdf* F. *Let* $\psi(x)$ *and* $\varphi(x)$ *be two differentiable functions on* $\mathbb{R}$ *such that* $\int_{-\infty}^{\infty} \frac{\varphi'(t)}{\varphi(t)-\psi(t)} dt = \infty$. *Then*

(6)

$$E[\psi(X)|X \geq x] = \varphi(x), x \in \mathbb{R}$$

implies

(7)

$$F(x) = 1 - \exp\left\{-\int_{-\infty}^{x} \frac{\varphi'(t)}{[\varphi(t)-\psi(t)]} dt\right\}, x \in \mathbb{R},$$

PROOF. If (6) holds, then

$$\int_{x}^{\infty} \psi(u) f(u) du = \big(1 - F(x)\big)\varphi(x).$$

Differentiating both sides of the above equation and rearranging terms, we arrive at

$$\frac{f(x)}{1-F(x)} = \frac{\varphi'(x)}{\varphi(x)-\psi(x)}, x \in \mathbb{R}.$$

Integrating the last equation with respect to *t* from −∞ to *x*, we have

$$-\ln[1-F(x)] = \int_{-\infty}^{x} \frac{\varphi'(t)}{[\varphi(t)-\psi(t)]} dt,$$

from which we obtain (7).

REMARK1. *For* $\psi(x) = 2\varphi(x), \varphi(x) = e^{1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}}$ *and the fact that* $\lim_{x\rightarrow-\infty}\varphi(x) = 1$, *we have*

$$F(x) = 1 - e^{1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}}, x \in \mathbb{R}.$$

which is cdf (2).

3.2. Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of ONH-G distribution in terms of the ratio of two truncated moments. This characterization result employs a theorem due to Glänzel ([10]), see Theorem 2 of Appendix B. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could also be applied when the cdf F does not have a closed form. As shown in (Glänzel, 1990), this characterization is stable in the sense of weak convergence.

PROPOSITION 2. *Let* $X : \Omega \rightarrow \mathbb{R}$ *be a continuous random variable and let* $q_1 \equiv$ *and* $q_2(x) = e^{1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}}$ *for* $x \in \mathbb{R}$*. The random variable* X *has pdf (3) if and only if the function* η *defined in Theorem 2 has the form*

$$\eta(x) = \frac{1}{2}e^{1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}}, x \in \mathbb{R}.$$

PROOF. Let *X* be a random variable with pdf (3), then

$$\big(1 - F(x)\big)E[q_1(X)|X \geq x] = e^{1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}}, x \in \mathbb{R}.$$

and

$$\big(1 - F(x)\big)E[q_2(X)|X \geq x] = \frac{1}{2}e^{2\left(1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}\right)}, x \in \mathbb{R}.$$

and finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{1}{2}e^{1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}} < 0 \text{ for } x \in \mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{\alpha\lambda g(x)}{\bar{G}(x)^2}\left[1 + \lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha-1} \; x \in \mathbb{R},$$

and hence

$$s(x) = \left[1 + \lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}, x \in \mathbb{R}.$$

Now, in view of Theorem 2, X has density (3). □

COROLLARY 1. *Let* $X : \Omega \rightarrow \mathbb{R}$ *be a continuous random variable and let* $q_1(x)$ *be as in Proposition 2. The pdf of* X *is* (3) *if and only if there exist functions* q_2 *and* η *defined in Theorem 2 satisfying the differential equation*

(8)

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{\alpha\lambda g(x)}{\bar{G}(x)^2}\left[1 + \lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha-1}, x \in \mathbb{R}.$$

The general solution of the differential equation (8) *is*

$$\eta(x) = e^{\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}} - 1\left[-\int \frac{\alpha\lambda g(x)}{\bar{G}(x)^2}\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha-1} e^{1-\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}} \left(q_1(x)\right)^{-1} q_2(x) + D\right],$$

where D is a constant.

Note that a set of functions satisfying the above differential equation is given in Proposition 2 with $D = 0$. However, it should be also noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 2.

3.3. Characterization based on hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization of ONH-G distribution, which is not of the above trivial form.

PROPOSITION 3. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The pdf of X is* (3)*, if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$h'_F(x) - \frac{g'(x)}{g(x)} h_F(x) = \alpha\lambda g(x) \frac{d}{dx}\left\{\frac{\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha-1}}{\bar{G}(x)^2}\right\}, x \in \mathbb{R}.$$

PROOF. If X has pdf (3), then clearly the above differential equation holds. Now, if this differential equation holds, then

$$\frac{d}{dx}\{g(x)^{-1}h_F(x)\} = \alpha\lambda \frac{d}{dx}\left\{\frac{\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha-1}}{\bar{G}(x)^2}\right\}, x \in \mathbb{R},$$

from which, we obtain

$$h_F(x) = \frac{\alpha\lambda g(x)\left[1+\lambda\frac{G(x)}{\bar{G}(x)}\right]^{\alpha-1}}{\bar{G}(x)^2}, x \in \mathbb{R},$$

which is the hazard function of ONH-G distribution.

REMARK 2. *For $\alpha = 1$, we have the following simple differential equation*

$$h'_F(x) - \frac{g'(x)}{g(x)} h_F(x) = \frac{2\lambda g(x)^2}{\bar{G}(x)^3}, x \in \mathbb{R}.$$

4. Linear representations

First, using Taylor expansion for every $z > 0$, we can write

$$z^{\alpha} = \sum_{j=0}^{\infty} \frac{(\alpha)j}{j!}(z-1)^{j},$$

where $(\alpha)_j = \alpha(\alpha-1)\cdots(\alpha-j+1)$ is the falling factorial. Thus, we obtain

$$F(x) = 1 - \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\left[1 + \frac{\lambda G(x)}{\bar{G}(x)}\right]^{i\alpha} = 1 - \sum_{i,j=0}^{\infty} \frac{(-1)^i (i\alpha)_j}{i!}\left[\frac{\lambda G(x)}{\bar{G}(x)}\right]^{j}$$

$$= 1 - \sum_{i,j,l=0}^{\infty} \frac{(-1)^{i+l}(i\alpha)_j \lambda^j}{i!}\binom{-j}{l} G(x)^{j+l}.$$

Second, we have

(9)

$$F(x) = 1 - \sum_{k=0}^{\infty} w_k G(x)^k = \sum_{k=0}^{\infty} v_k G(x)^k = \sum_{k=0}^{\infty} v_k H_k(x),$$

where $v_0 = 1 - w_0, v_k = w_k$ for $k \geq 1$, and (for $k \geq 0$)

$$w_k \sum_{j,l \in I_k}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+l}(i\alpha)_j \lambda^j}{i!}\binom{-j}{l}, I_k = \{(j,l) | k = j + l; j, l = 0,1,2,\ldots\}$$

and $H_{\gamma}(x)$ is the exp-G cdf with power parameter γ. By differentiating (9), we obtain

(10)

$$f(x) = \sum_{k=0}^{\infty} v_{k+1} h_{k+1}(x),$$

where $h_{\gamma}(x) = \gamma_g(x) G(x)^{\gamma-1}$ denotes the exp-G family density with power parameter $\gamma > 0$. Equation (10) reveals that the ONH-G family density is a linear combination of exp-G densities. So, some mathematical properties of this family can be determined from those of the exp-G distribution. The properties of exp-G distributions have been studied by many authors in recent years, see Mudholkar and Srivastava ([14]) for exponentiated Weibull and Nadarajah ([15]) for exponentiated-type distributions, among others. Equations (9) and (10) are the main results of this section.

5. Properties

The formulae derived throughout this section can be easily handled in most symbolic computation software platforms such as **Maple**, **Mathematica** and **Matlab**.

5.1. Moments and generating function

Henceforth, let Y_{k+1} denote the exp-G distribution with power parameter $(k + 1)$. The rth moment of X, say μ'_r, can be obtained from (10) as

(11)

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty} v_{k+1} E(Y_{k+1}^r).$$

For $\gamma > 0$, we have $E(Y_\gamma^r) = \alpha \int_{-\infty}^{\infty} x^r g(x;\boldsymbol{\xi}) G(x;\boldsymbol{\xi})^{\gamma-1} dx$, which can be computed numerically in terms of the baseline qf $Q_G(u;\boldsymbol{\xi}) = G^{-1}(u;\boldsymbol{\xi})$ as $E(Y_\gamma^n) = \gamma \int_0^1 Q_G(u;\boldsymbol{\xi})^n u^{\gamma-1} du$.

Next, we provide two formulae for the moment generating function (mgf) $M(t) = E(e^{tX})$ of X. Clearly, the first one can be derived from equation (10) as

$$M(t) = \sum_{k=0}^{\infty} v_{k+1} M_{k+1}(t),$$

where $M_{k+1}(t)$ is the mgf of Y_{k+1}. Hence, $M(t)$ can be determined from the exp-G generating function. A second formula for $M(t)$ follows from (10) as $M(t) = \sum_{k=0}^{\infty} v_{k+1} \varsigma(t,k)$, where $\varsigma(t,k) = \int_0^1 \exp[tQ_G(u;\boldsymbol{\xi})] u^k du$. For the ONH-W model discussed in Section 2.2, we obtain the results $(r > -\beta_1)$:

$$\mu'_r = \beta_2^r \Gamma\left(1 + \frac{r}{\beta_1}\right) \sum_{k,h=0}^{\infty} \frac{(-1)^h (k+1) v_{k+1}}{(h+1)^{(r+\beta_1)/\beta_1}} \binom{k}{h}$$

and

$$M(t) = \sum_{k,r,h=0}^{\infty} \frac{(-1)^h \beta_2^r (k+1) v_{k+1} t^r}{r!\,(h+1)^{(r+\beta_1)/\beta_1}} \binom{k}{h} \Gamma\left(1 + \frac{r}{\beta_1}\right).$$

5.2. Incomplete moments

The sth incomplete moment, say $\varphi_s(t)$, of X can be expressed from (10) as

(12)

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x) dx = \sum_{k=0}^{\infty} v_{k+1} \int_{-\infty}^{t} x^s h_{k+1}(x) dx.$$

For the ONH-W model, we obtain (for $s > -\beta_1$)

$$\varphi_s(t) = \beta_2^s \gamma\left(1 + \frac{s}{\beta_1}, \left(\frac{1}{\beta_2 t}\right)^{\beta_1}\right) \sum_{k,h=0}^{\infty} \frac{(-1)^h (k+1) v_{k+1}}{(h+1)^{(s+\beta_1)/\beta_1}} \binom{k}{h},$$

where $\gamma\ (\cdot,\cdot)$ is the lower incomplete gamma function.

The mean deviations about the mean $[\delta_1 = E(|X - \mu_1'|)]$ and about the median $[\delta_2 = E(|X - M|)]$ of X are given by

$$\delta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1') \text{ and } \delta_2 = \mu_1' - 2\varphi_1(M),$$

respectively, where $\mu_1' = E(X)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily calculated from (2) and $\varphi_1(t)$ is the first incomplete moment given by (12) with $s = 1$. Expressions for $\varphi_1(t)$ can be derived from $\varphi_s(t)$ as

$$\varphi_1(t) = \sum_{k=0}^{\infty} \upsilon_{k+1}\delta_{k+1}(x) = \sum_{k=0}^{\infty} \upsilon_{k+1}\eta_{k+1}(t),$$

where $\delta_{k+1}(x) = \int_{-\infty}^{t} x h_{k+1}(x) dx$ is the first incomplete moment of the exp-G distribution and $\eta_{k+1}(t) = (k + 1)\int_0^{G(t)} Q_G(u) u^k du$ can be computed numerically. These equations for $\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \varphi_1(q)/\pi\mu_1'$ and $L(\pi) = \varphi_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the qf of X at π.

5.3. Moment residual life and reversed residual life

The *n*th moment of the residual life is defined by (for $t > 0$)

$$z_n(t) = E[(X - t)^n | X > t] = \frac{1}{1 - F(t)} \int_t^{\infty} (x - t)^n dF(x), \qquad \text{for } n = 1, 2, \ldots,$$

and determines $F(x)$ uniquely. One can show that $z_n(t)$ is given by

$$z_n(t) = \frac{1}{1 - F(t)} \sum_{k=0}^{\infty} \upsilon_{k+1}^{\star} \int_t^{\infty} x^r \pi_{k+1}(x)\, dx,$$

where $\upsilon_{k+1}^{\star} = \upsilon_{k+1}(1 - t)^n$. In particular, the mean residual life function or the life expectation at age t is given by $z_1(t)$, which represents the expected additional life length for a unit which is alive at age t.

On the other hand, the *n*th moment of the reversed residual life, say $Z_1(t)$, is defined by (for $t > 0$)

$$Z_n(t) = E[(t - X)^n | X \le t] = \frac{1}{F(t)} \int_0^{t} (t - x)^n dF(x), \qquad \text{for } n = 1, 2, \ldots,$$

and also determines $F(x)$ uniquely. The last quantity can be expressed as

$$Z_n(t) = \frac{1}{F(t)} \sum_{k=0}^{\infty} \upsilon_{k+1}^{\star} \int_0^{t} x^r \pi_{k+1}(x)\, dx.$$

The mean reversed residual life function, also called the mean inactivity time given by $Z_1(t)$, represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

5.4. Order statistics

Suppose $X_1, \ldots, X_n$ is a random sample from the ONH-G model. Let $X_{i:n}$ denote the *i*th order statistic. The pdf of $X_{i:n}$ is given by

(13)

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F\,(x)^{j+i-1}.$$

Following similar algebraic developments of Nadarajah et al. ([18]), we can write the density function of $X_{i:n}$ as

(14)

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} b_{r,k}\, \pi_{r+k+1}(x),$$

where

$$b_{r,k} = \frac{n!\,(r\,+\,1)(i\,-\,1)!\,w_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!\,j!},$$

w_{r+1} and v_r are given in Section 3 $(r\, \geq\, 0)$ and the quantities $f_{j+i-1,k}$ can be determined (with $f_{j+i-1,0}\,=\,v_0^{j+i-1}$) recursively (for $k\, \geq\, 1$) by

$$f_{j+i-1,k} = (kv_0)^{-1} \sum_{m=1}^{k} [m(j\,+\,i) -\,k] v_m f_{j+i-1,k-m}\,.$$

Equation (14) is the main result of this section. It reveals that the pdf of the ONH-G order statistics is a linear combination of exp-G densities. Therefore, some mathematical quantities of the ONH-G order statistics such as the ordinary, incomplete and factorial moments, mean deviations and several others can be determined from those quantities of the exp-G distribution.

5.5. Quantile power series

In this section, we derive a power series for $Q(u) =\, F^{-1}(u)$ of X by expanding equation (5). If $Q_G(u)$ does not have a closed-form expression, it can be written as a power series

(15)

$$Q_G(u) = \sum_{i=0}^{\infty} a_i u^i,$$

whose coefficients $a_i's$ are suitably chosen real numbers. They depend on the parameters of the G distribution. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions but it can be expanded as in equation (15).

According to the discussion of Appendix A, the argument, say A, of $Q_G(\cdot)$

(16)

$$A = \sum_{k=0}^{\infty} \delta_k u^k,$$

where $\delta_0 = \frac{a_0}{b_0}$ and, for $k \geq 1$, we obtain

$$\delta_k = b_0^{-1}\left(a_k - b_0^{-1}\sum_{r=1}^{k} b_r \delta_{k-r}\right).$$

Then, the qf of X can be expressed as

(17)

$$Q(u) = Q_G\left(\sum_{k=0}^{\infty} \delta_k u^k\right).$$

For any baseline *G* distribution, we combine (15) and (17) and write

$$Q(u) = Q_G\left(\sum_{m=0}^{\infty} \delta_m u^m\right) = \sum_{i=0}^{\infty} a_i \left(\sum_{m=0}^{\infty} \delta_m u^m\right)^i,$$

and then, using (10) and (11), we have

(18)

$$Q(u) = \sum_{m=0}^{\infty} e_m u^m,$$

where $e_m = \sum_{i=0}^{\infty} a_i d_{i,m}$, and, for $i = 0, 1, \ldots, d_{i,0} = \delta_0^i$ and (for $m \geq 1$)

$$d_{i,m} = (m\delta_0)^{-1} \sum_{n=1}^{m} [n(i+1) - m]\delta_n d_{i,m-n}.$$

Equation (18) reveals that the qf of the ONH-G family can be expressed as a power series. Then, several mathematical quantities of *X* can be reduced to integrals over (0*,* 1) based on this power series.

Let $W(\cdot)$ be any integrable function on the real line. We can write

(19)

$$\int_{-\infty}^{\infty} W(x)f(x)\mathrm{d}x = \int_{0}^{1} W\left(\sum_{m=0}^{\infty} e_m u^m\right) du.$$

Equations (18) and (19) are the main results of this section since we can obtain from them various ONH-G mathematical properties. In fact, they can follow by using the integral on the right-hand side for special $W(\cdot)$ functions, which are usually simpler than if they were based on the left-hand integral. For the great majority of these quantities, we can adopt twenty terms in this power series.

6. Bivariate extension

In this section, we introduce a bivariate version of model (2), whose joint cdf is given by

$$F\,(x,y;\,\alpha,\lambda,\boldsymbol{\xi}) = 1 - e^{1-\left[1+\lambda\frac{G(x,y;\boldsymbol{\xi})}{1-G(x,y;\boldsymbol{\xi})}\right]^{\alpha}}, (x,y) \in \mathbb{R}^2,$$

where $G(x,y;\,\boldsymbol{\xi})$ is a bivariate continuous distribution with marginal cdf's $G_1(x;\,\boldsymbol{\xi})$ and $G_2(y;\,\boldsymbol{\xi})$. We denote this distribution by the *bivariate Odd Nadarajah-Haghighi* -G (BONH-G) distribution. The marginal cdf's are given by

$$F_X(x) = 1 - e^{1-\left[1+\lambda\frac{G_1(x;\boldsymbol{\xi})}{G_1(x;\boldsymbol{\xi})}\right]^{\alpha}} \text{ and } F_Y(y) = 1 - e^{1-\left[1+\lambda\frac{G_2(y;\boldsymbol{\xi})}{G_2(y;\boldsymbol{\xi})}\right]^{\alpha}}.$$

The joint pdf of (*X, Y*) is

$$fX,Y(x,y) = \frac{\alpha\lambda A(x,y;\,\alpha,\lambda,\boldsymbol{\xi})}{[1-G(x,y;\boldsymbol{\xi})]^4}\left[1+\lambda\frac{G(x,y;\boldsymbol{\xi})}{1-G(x,y;\boldsymbol{\xi})}\right]^{\alpha-1} e^{1-\left[1+\lambda\frac{G(x;\boldsymbol{\xi})}{1-G(x,y;\boldsymbol{\xi})}\right]^{\alpha}},$$

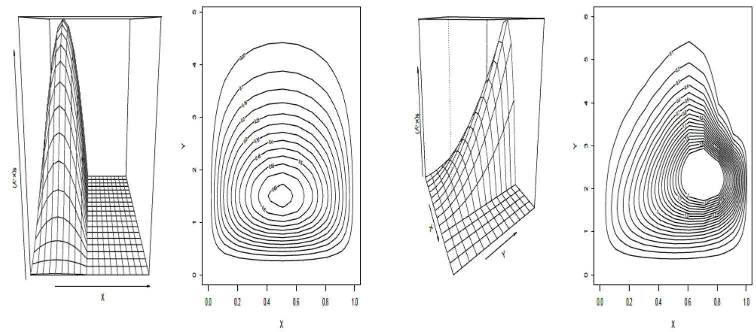
where

$$A(x,y;\alpha,\lambda,\boldsymbol{\xi}) = g(x,y;\boldsymbol{\xi}) + \frac{2}{1-G(x,y;\boldsymbol{\xi})}\frac{\partial G(x,y;\,\boldsymbol{\xi})}{\partial x}\,\frac{\partial G(x,y;\,\boldsymbol{\xi})}{\partial y}$$
$$+\frac{\lambda(\alpha-1)}{[1-G(x,y;\boldsymbol{\xi})]^2\left[1+\lambda\frac{G(x,y;\boldsymbol{\xi})}{1-G(x,y;\boldsymbol{\xi})}\right]}\frac{\partial G(x,y;\,\boldsymbol{\xi})}{\partial x}\,\frac{\partial G(x,y;\,\boldsymbol{\xi})}{\partial y}$$
$$+\frac{\lambda(\alpha-1)}{[1-G(x,y;\boldsymbol{\xi})]^2}\left[1+\lambda\frac{G(x,y;\boldsymbol{\xi})}{1-G(x,y;\boldsymbol{\xi})}\right]^{\alpha-1}\frac{\partial G(x,y;\,\boldsymbol{\xi})}{\partial x}\,\frac{\partial G(x,y;\,\boldsymbol{\xi})}{\partial y}.$$

In order to illustrate the proposed bivariate extension, we use as baseline the McKay's bivariate gamma distribution (see Nadarajah and Gupta, 2006), whose pdf is given by

$$g(u,v) = \frac{u^{a-1}(1-u)^{b-1}}{B(a,b)}\frac{v^{a+b-1}e^{-\frac{v}{\mu}}}{\mu^{a+b}\Gamma(a+b)}.$$

Figure 4 displays plots of the McKay's model (Figure 4(a)) and those of its BONH-McKay transformation (Figure 4(b)). It is observed from level curves that the extend bivariate model may produce more flexible distributions.



(a) McKay's model: $(a = b = \mu = 2)$ (b) BONH-McKay model: $(a=b=\mu=2)$ and $(\alpha, \lambda) = (1, 1/4)$

Fig. 4. Plots of the pdfs and level curves.

7. Estimation

This section addresses estimation procedures for the ONH parameters under two perspectives: uncensored and censored likelihoods. For both cases, the estimators cannot be expressed in closed-forms which imposes the use of numerical interactive methods (such as Newton-Raphson, BFGS and Nelder-Mead, among others). To this end, possible users of our proposal can employ standard routines implemented in softwares such as **R**, Ox and **SAS** to maximize the likelihoods directly.

7.1. Uncensored maximum likelihood estimation

In this section, we determine the maximum likelihood estimates (MLEs) of α and λ (additional parameters) and $\boldsymbol{\xi}$ (parameter vector of the baseline G) in the ONH-G family. Let $x_1, \ldots, x_n$ be an observed sample from $X \sim$ONH-G and $\boldsymbol{\theta} = \left(\alpha, \lambda, \boldsymbol{\xi}^\top\right)^\top$ be the parameter vector of dimension $r \times 1$. The total log-likelihood function for $\boldsymbol{\theta}$, say $\ell_n = \ell_n(\boldsymbol{\theta})$, is given by

$$\ell_n = n[\log(\alpha\lambda) + 1] + \sum_{i=1}^{n} \log[g(x_i; \boldsymbol{\xi})] - 2\sum_{i=1}^{n} \log[\bar{G}(x_i; \boldsymbol{\xi})]$$
$$+(\alpha - 1)\sum_{i=1}^{n} \log\left[1 + \lambda\frac{G(x_i; \boldsymbol{\xi})}{\bar{G}(x_i; \boldsymbol{\xi})}\right] + \sum_{i=1}^{n}\left\{1 - \left[1 + \lambda\frac{G(x_i; \boldsymbol{\xi})}{\bar{G}(x_i; \boldsymbol{\xi})}\right]^{\alpha}\right\}.$$

The MLE $\widehat{\boldsymbol{\theta}}$ can be defined as $\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}[\ell_n(\boldsymbol{\theta})]$, where $\boldsymbol{\Theta} \subseteq \mathbb{R}^r$ denotes the parametric space associated to the ONH-G family. The MLEs are obtained by solving the nonlinear system of equations

$$\boldsymbol{U}(\boldsymbol{\theta}) = \mathbf{0} \Leftrightarrow \left[U_\alpha, U_\lambda, \boldsymbol{U}_{\boldsymbol{\xi}}^\top\right] = \mathbf{0} \Leftrightarrow \left[\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \lambda}, \left(\frac{\partial \ell_n}{\partial \boldsymbol{\xi}}\right)^\top\right] = \mathbf{0},$$

where

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[1 + \lambda \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}\right]$$

$$-\sum_{i=1}^{n}\left[1 + \lambda \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}\right]^{\alpha} \log\left[1 + \lambda \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}\right],$$

$$U_\lambda = \frac{n}{\lambda} + (\alpha - 1)\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi}) + \lambda G(x_i;\boldsymbol{\xi})}$$

$$-\alpha \sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}\left[1 + \lambda \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}\right]^{\alpha-1},$$

$$\boldsymbol{U}_{\xi_r} = \sum_{i=1}^{n} \frac{g^{(r)}(x_i;\boldsymbol{\xi})}{g(x_i;\ \boldsymbol{\xi})} + 2\sum_{i=1}^{n} \frac{G^{(r)}(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}$$

$$+\lambda(\alpha - 1)\sum_{i=1}^{n} \frac{G^{(r)}(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})[\bar{G}(x_i;\ \boldsymbol{\xi}) + \lambda G(x_i;\ \boldsymbol{\xi})]}$$

$$-\alpha\lambda \sum_{i=1}^{n} \frac{G^{(r)}(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})^2}\left[1 + \lambda \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}\right]^{\alpha-1}$$

and $G^{(r)}(x_i;\boldsymbol{\xi})$ denotes the derivative of $G(x_i;\boldsymbol{\xi})$ with respect to ξ_r, etc.

7.2. Multi-censored maximum likelihood estimation

Often with lifetime data, we encounter censored observations. There are different forms of censoring: type I censoring, type II censoring, etc. Here, we consider the general case of multi-censored data: there are m subjects of which m_0 are known to have failed at the times $x_1, \ldots, x_{m_0}$; m_1 are known to have failed in the interval $[s_{j-1}, s_j], j = 1, \ldots, m_1$; m_2 survived to a time r_j $j = 1, \ldots, m_2$, *but not observed any longer. Note that* $m = m_0 + m_1 + m_2$ *and that type I censoring and type II censoring are special cases of multi-censoring. The log-likelihood function for* $\boldsymbol{\theta}$ *is given by*

$$\ell_m(\boldsymbol{\theta}) = m_0[\log(\alpha\lambda) + 1] + (\alpha - 1)\sum_{i=1}^{m_0} \log\left[1 + \lambda \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}\right]$$

$$+\sum_{i=1}^{m_0} \log[g(x_i;\boldsymbol{\xi})] + \sum_{i=1}^{m_0}\left\{1 - \left[1 + \lambda \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\ \boldsymbol{\xi})}\right]^{\alpha}\right\}$$

$$+\sum_{i=1}^{m_2}\left\{1 - \left[1 + \frac{\lambda G(r_i;\boldsymbol{\xi})}{\bar{G}(r_i;\ \boldsymbol{\xi})}\right]^{\alpha}\right\} - 2\sum_{i=1}^{m_0} \log\left[\bar{G}(x_i;\ \boldsymbol{\xi})\right]$$

$$+\sum_{i=1}^{m_1} \log\left[\left(1 - \exp\left\{1 - \left[1 + \frac{\lambda G(s_i;\boldsymbol{\xi})}{\bar{G}(s_i;\ \boldsymbol{\xi})}\right]^{\alpha}\right\}\right) - \left(1 - \exp\left\{1 - \left[1 + \frac{\lambda G(s_{i-1};\boldsymbol{\xi})}{\bar{G}(s_{i-1};\ \boldsymbol{\xi})}\right]^{\alpha}\right\}\right)\right].$$

The likelihood equations can be obtained from the authors upon request.

8. Numerical results

In this section, we fit some models under the proposed class for both empirical and real data. First, a Monte Carlo simulation study is performed in order to check the influence of the variation of the baseline parameters on the additional ones. Second, an application is performed in synthetic aperture radar data.

8.1. Monte Carlo simulation

We carry out a Monte Carlo simulation study (with 1,000 replications) to quantify some asymptotic properties of the MLEs of the model parameters. To that end, we consider the ONHβ, ONHΓ and ONH-W models presented in Section 2 under a parametric variation $k\,[\alpha, \lambda, \beta_1, \beta_2] = k \times [0.3, 0.5, 0.3, 0.5]$, where $k = 1, 5, 10$ and for sample sizes $n = 50, 100, 200$. For each parametric point, we obtain from the simulations three assessment measures: (i) the averages of the MLEs, (ii) the Kolmogorov-Smirnov distance and (iii) the Mean Square Errors (MSEs).

Table 1 lists the values obtained for the criteria (1)–(3). In general, as expected, one can note that both asymptotic biases and MSEs decrease when the sample size increases. In particular, with respect to the effects of parameter variation, larger MSEs are associated to larger parameters. This fact suggests the use of robust adjustment methods for the ONH-G model with high magnitude parameters.

Table 1. Simulation results

Factors	MLEs					MSEs			
	α	λ	β_1	β_1	K	α	λ	β_1	β_1
$\mathrm{ONH}-\ \Gamma$	0.459	0.436	0.354	0.329	0.076	0.023	0.081	0.019	0.083
$\mathrm{ONH}-\ \beta$	0.495	0.409	0.342	0.331	0.075	0.025	0.069	0.018	0.074
$\mathrm{ONH}-\mathrm{W}$	0.251	0.301	0.289	0.319	0.075	0.027	0.131	0.011	0.149
$(n,\alpha,\lambda,\ \beta_1,\beta_2) =\ (100, 0.3, 0.5, 0.3, 0.5)$									
	0.435	0.419	0.323	0.346	0.055	0.022	0.062	0.015	0.072
	0.423	0.435	0.326	0.386	0.053	0.023	0.052	0.015	0.067
	0.242	0.362	0.292	0.278	0.053	0.024	0.101	0.009	0.141
$(n,\alpha,\lambda,\ \beta_1,\beta_2) =\ (200, 0.3, 0.5, 0.3, 0.5)$									
	0.366	0.442	0.308	0.407	0.038	0.018	0.046	0.010	0.061
	0.387	0.462	0.316	0.407	0.037	0.020	0.033	0.010	0.052
	0.264	0.402	0.291	0.222	0.037	0.022	0.080	0.006	0.131
$(n,\alpha,\lambda,\ \beta_1,\beta_2) =\ (50, 1.5, 2.5, 1.5, 2.5)$									
	1.313	2.027	1.547	3.844	0.076	0.705	2.474	0.236	1.588
	1.210	2.287	1.607	4.124	0.076	0.686	2.386	0.266	1.567
	1.306	3.052	1.601	1.658	0.077	0.955	2.980	0.277	1.594
$(n,\alpha,\lambda,\ \beta_1,\beta_2) =\ (100, 1.5, 2.5, 1.5, 2.5)$									
	1.304	2.211	1.547	3.317	0.054	0.550	1.836	0.151	1.333
	1.228	2.301	1.578	3.602	0.054	0.571	1.829	0.170	1.513
	1.358	2.841	1.551	1.930	0.054	0.793	2.350	0.152	1.446
$(n,\alpha,\lambda,\ \beta_1,\beta_2) =\ (200, 1.5, 2.5, 1.5, 2.5)$									
	1.355	2.141	1.534	3.132	0.039	0.443	1.506	0.075	1.411
	1.395	2.228	1.507	3.139	0.039	0.438	1.434	0.080	1.480
	1.511	2.816	1.530	2.238	0.038	0.715	1.691	0.071	1.373
$(n,\alpha,\lambda,\ \beta_1,\beta_2) =\ (50, 3, 5, 3, 5)$									
	3.619	4.952	3.124	5.909	0.080	2.794	7.561	0.761	6.955
	3.570	4.672	3.114	6.064	0.079	2.699	7.913	0.741	7.675
	3.713	5.127	3.086	3.820	0.079	4.147	11.872	0.709	4.888
$(n,\alpha,\lambda,\ \beta_1,\beta_2) =\ (100, 3, 5, 3, 5)$									
	3.164	4.633	3.066	5.534	0.056	2.203	6.995	0.440	5.765
	3.411	4.556	3.043	5.417	0.056	2.172	7.148	0.454	6.524

	3.143	5.722	3.048	4.616	0.056	3.291	8.911	0.404	3.820
$(n, \alpha, \lambda, \beta_1, \beta_2) = (200, 3, 5, 3, 5)$									
	3.177	5.137	3.015	5.157	0.040	1.708	6.422	0.221	4.581
	3.066	5.020	3.017	5.278	0.040	1.721	6.015	0.210	5.371
	2.870	5.123	3.032	4.831	0.040	2.855	7.331	0.180	3.260

8.2. Intensities in (dB) of radar data

We carry out an application to a real data larger set. We use a database extracted from an image of Foulum (Denmark) obtained by the EMISAR sensor (Lee and Pottier, 2009) operated at Cand L-bands (though not simultaneously) with quad-polarizations.

The data are obtained at http://earth.eo.esa.int/polsarpro/datasets.html by means of the polSARpro software and, for each geographic position, each one of its element consists in norm squared of a complex number, which represents the information of the polarization channel resulting of a pulse both transmitted and recorded in horizontal direction. These informations are known as *intensities* and are given in Alizadeh et al. ([2]).

Table 2. MLEs and their estimated SEs for the current data

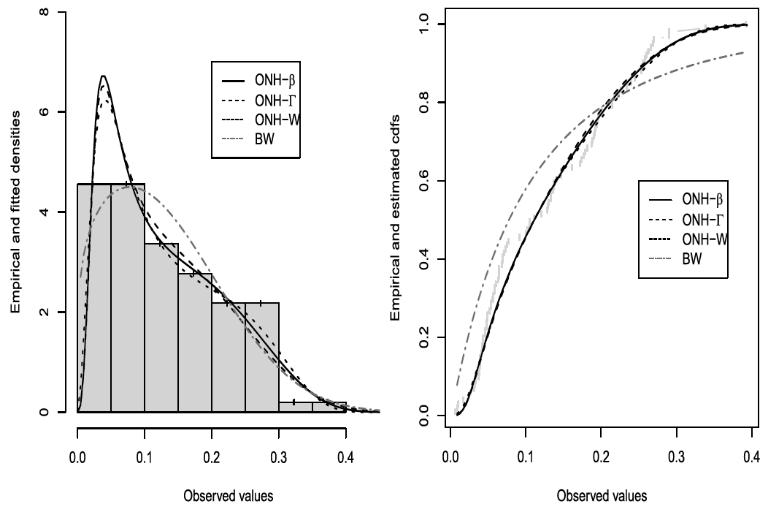
Model	$\widehat{\boldsymbol{\theta}}$[MLEs (SEs)]			
ONHβ	0.145 (0.010)	20.765(7.090)	3.695(0.365)	33.059(1.382)
ONHΓ	0.148(0.013)	12.490(6.890)	3.811(0.508)	42.647(1.533)
ONH-W	0.164(0.028)	54.255(14.823)	2.548(0.100)	0.165(0.021)
BW	0.561 (0.290)	0.735 (0.447)	2.162 (0.753)	5.951 (1.766)

Here, the special models presented in Section 2 are employed to describe the EMISAR intensities. They are compared with the four-parameter beta Weibull (BW) model discussed by Cordeiro et al. ([7]), whose pdf is given by (for $x > 0$)

$$f(x) = \frac{c\lambda^c}{B(a,b)} x^{c-1} e^{-b(\lambda x)^c} \left[1 - e^{-(\lambda x)^c}\right]^{a-1},$$

where $a, b, c, \lambda > 0$. Table 2 gives the MLEs of the parameters of the ONH$-\{\beta, \Gamma, W\}$ and BW distributions fitted to the current data. In order to perform a previous quality comparison, Figure 5 displays both empirical and estimated ONH$\{\beta, \Gamma, W\}$ and BW pdfs and their counterpart cdfs. In general, the estimated pdfs are close to the histogram for all cases. Further, with respect to the quality of to the empirical cdf, the ONH-G models outperform the BW model.

For presenting a quantitative base to analyze Figure 5, we use four goodness-of-fit (GoF) measures: (a) Akaike information criterion (AIC), (b) corrected AIC (AICc), (c) Bayesian information criterion (BIC) and (d) Kolmogorov Smirnov statistic (and its associated p-value). Table 3 presents their values for the radar intensities. We note that the ONHβ distribution yields the best fit according to all GoFs.



(a) For pdfs. (b) For cdfs.

Fig. 5. Application to Radar Intensities.

Table 3. Values of the GoF measures

Model	Depend on the pdf			Depend on the cdf	
	AIC	AICc	BIC	KS	p-value
ONHβ	-234.2379	-233.8212	-223.7774	0.0777	0.5753
ONHΓ	-232.9728	-232.5561	-222.5123	0.0837	0.4785
ONH-W	-233.6043	-233.1876	-223.1438	0.0796	0.5435
β-W	-222.7629	-222.3462	-212.3024	0.222	$< 9.5 \times 10^{-5}$

9. Conclusion

We propose a new class of continuous distributions, called the Odd Nadarajah-Haghighi (ONH) family, and investigate some of its mathematical properties. Three special ONH models are presented, namely: the ONHgamma, -beta and -Weibull distributions. We introduce a bivariate version of the ONH-G family and one special case called the McKay's bivariate gamma distribution is discussed. Further, we also developed a collection of characterization results for the proposed family. The maximum likelihood estimation for the ONH parameters is addressed. The performance of the maximum likelihood estimates is quantified for some parametric points. Finally, an application to SAR data is conducted and, according to four goodness-of-fit statistics, the results indicate that the ONHβ model provides a better fit to SAR intensities, when compared with the classical BW, ONHΓ and -W models.

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A. Mathematical support for representing quantile function

We use throughout the paper a result of Gradshteyn and Ryzhik ([11]) for a power series raised to a positive integer *n*

(20)

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i} u^i,$$

where $c_{n,0} = a_0^n$ and the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) can be determined from the recurrence equation

(21)

$$c_{n,i} = (i a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i] a_m c_{n,i-m}.$$

Next, we derive an expansion for the argument of $Q_G(\cdot)$ in (5):

$$A = \frac{-1 + [1 - \log(1-u)]^{1/\alpha}}{\lambda - 1 + [1 - \log(1-u)]^{1/\alpha}}.$$

Using **MATHEMATICA**, we have

(22)

$$-1 + [1 - \log(1-u)]^{1/\alpha} = \sum_{k=0}^{\infty} a_k u^k,$$

where

$$a_0 = 0, a_1 = \frac{1}{\alpha}, a_2 = \frac{1}{2\alpha^2}, a_3 = \frac{\alpha^2 + 1}{6\alpha^3}, a_4 = \frac{\alpha^3 + 4\alpha^2 + 1}{24\alpha^4}, \ldots$$

And then

(23)

$$\lambda - 1 + [1 - \log(1-u)]^{1/\alpha} = \sum_{k=0}^{\infty} b_k u^k,$$

where

$$b_0 = 0, b_1 = \frac{1}{\alpha}, b_2 = \frac{1}{2\alpha^2}, b_3 = \frac{\alpha^2 + 1}{6\alpha^3}, b_4 = \frac{\alpha^3 + 4\alpha^2 + 1}{24\alpha^4}, \ldots$$

By using the quotient of two power series, we can write

(24)

$$A = \frac{\sum_{k=0}^{\infty} a_k u^k}{\sum_{k=0}^{\infty} b_k u^k} = \sum_{k=0}^{\infty} b_k u^k,$$

where $\delta_0 = a_0/b_0$ and, for $k \geq 1$, we obtain

$$\delta_k = b_0^{-1}\left(a_k - b_0^{-1}\sum_{r=1}^{k} b_r \delta_{k-r}\right).$$

B. Mathematical support for Section 3

Theorem 2. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a given probability space and let* $H = [a, b]$ *be an interval for some* $d < b$ $(a = -\infty, b = \infty$ *might as well be allowed). Let* $X: \Omega \to H$ *be a continuous random variable with the distribution function* F *and let* q_1 *and* q_2 *be two real functions defined on* H *such that*

$$\mathbf{E}[q_2(X)|X \geq x] = \mathbf{E}[q_1(X)|X \geq x]\eta(x), \quad x \in H,$$

is defined with some real function η*. Assume that* $q_1, q_2 \in C^1(H), \eta \in C^2(H)$ *and* F *is twice continuously differentiable and strictly monotone function on the set* H*. Finally, assume that the equation* $\eta q_1 = q_2$ *has no real solution in the interior of* H*. Then* F *is uniquely determined by the functions* q_1*,* q_2 *and* η*, particularly*

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)}\right| \exp(-s(u))du,$$

where the function s *is a solution of the differential equation* $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ *and* C *is the normalization constant, such that* $\int_H dF = 1$*.*

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