WAVELETS ON IRREGULAR GRIDS WITH ARBITRARY DILATION MATRICES, AND FRAME ATOMS FOR $L^2(\mathbb{R}^d)$

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ABSTRACT. In this article, we develop a general method for constructing wavelets $\{|\det A_j|^{1/2}\psi(A_j x - x_{j,k}) : j \in J, k \in K\}$ on irregular lattices of the form $X = \{x_{j,k} \in \mathbb{R}^d : j \in J, k \in K\}$, and with an arbitrary countable family of invertible $d \times d$ matrices $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$ that do not necessarily have a group structure. This wavelet construction is a particular case of general atomic frame decompositions of $L^2(\mathbb{R}^d)$ developed in this article, that allow other time frequency decompositions such as non-harmonic Gabor frames with non-uniform covering of the Euclidean space $\mathbb{R}^d$. Possible applications include image and video compression, speech coding, image and digital data transmission, image analysis, estimations and detection, and seismology.

1. INTRODUCTION

Recently there has been a considerable interest in trying to obtain atomic decompositions of the space $L^2(\mathbb{R}^d)$. These decompositions are usually obtained in terms of frames generated by a family of functions translated on a regular grid, and dilated by powers of a dilation matrix. The uniformity of the grid and the structure of the dilations can be exploited to obtain very sharp results. For irregular grids and unstructured dilations or if dilations are replaced by other transformations the situation is more complex and requires different techniques. One method is to use the regular case and try to obtain perturbations of the grid that preserve the frame structure. Another possibility is to obtain irregular samples of the continuous transform, that have the required properties.

In this article we study frame decompositions of the space $L^2(\mathbb{R}^d)$ using translations of a family of functions on irregular grids, and arbitrary dilations, and we even replace dilations by other transformations.

Our approach is different and very general, allowing quite general constructions. We prove the existence of smooth time-frequency frame atoms in several variables. The setting includes as particular cases, wavelet frames on irregular lattices and with a set of dilations or transformations that do not have a group structure. Another particular case are non-harmonic Gabor frames with non-uniform covering of the Euclidean space. It also leads to new constructions of wavelet and Gabor frames

with regular lattice translates. One of the nice features of the proposed method is that it unifies different atomic decompositions.

For the case of regular lattices Guido Weiss and his group [HDW02, HDW03, Lab02] developed a very fundamental program to characterize a large class of decompositions of $L^2(\mathbb{R}^d)$ through certain equations that the generators must satisfy. This is an important attempt to unify Gabor and wavelets decompositions. Other fundamental construction of MRA wavelet frames on regular lattices can also be found in [CHS], [CHSS03], [CS00]. Our methods can be used to produce a substantial part of these systems.

A set $Q \subset \mathbb{R}^d$ is a wavelet set if the inverse Fourier transform of the characteristic function of the set is a wavelet. Wavelet sets, frame wavelet sets and methods for constructing such sets have been studied recently [BMM99], [BL99], [BL01], [BS03], [DLS97], [DLS98], [HL00] [Ola03], [OS03]. Our methods give constructions of wavelet sets with translations on irregular grids.

Let J and K be countable index sets. We consider families of functions $\{g_j\}_{j \in J} \subset L^2(\mathbb{R}^d)$ and discrete sets $X = \{x_{j,k} : j \in J, k \in K\} \subseteq \mathbb{R}^d$ such that the collection $\{g_j(x - x_{j,k}) : j \in J, k \in K\}$ form a frame for $L^2(\mathbb{R}^d)$. The wavelet case is obtained when $g_j = |\det(A)|^{j/2} g \circ A^j$ with A an expansive matrix and g a fixed atom. We want to stress here that our constructions are much more general, allowing for example a different invertible (not necessarily expansive) matrix A_j for each $j \in J$. For the case of orthogonal wavelets, Yang Wang [Wan02] has recently considered wavelet sets associated with arbitrary families of invertible matrices and irregular sets of translates. He gave conditions for the existence of such wavelet sets and related them to spectral pairs.

Irregular wavelet and Gabor frames also have been studied as perturbations of uniform (lattice translate) frames and also as sampling of the continuous wavelet/ Gabor transform. See [Bal97], [BCHL03], [Chr96], [Chr97], [CFZ01], [CH97], [CDH99], [FZ95], [FG89], [Grö91], [Grö93],[HK03], [OS92], [RS95], [SZ00], [SZ01], [SZ02], [SZ03], [SZ03].

The approach in this article can be considered in the spirit of the classic construction in 1 dimension of smooth regular tight frames done by Daubechies, Grossmann and Meyer in [DGM86]. They found, for the case of uniform lattices, general conditions on a compactly supported smooth function h, in order that it generates a tight Gabor frame of $L^2(\mathbb{R})$. In the affine case they found necessary and sufficient conditions for a band limited function in order that it generates a smooth wavelet frame. See also [HW89].

There were other related attempts to obtain atomic decompositions of functional spaces using very general systems. See for example [FG85], [Fei87] in the context of locally compact groups.

This paper is organized as follows: Section 2 introduces the notation and some preliminaries. Section 3 presents a Theorem on wavelet construction on arbitrary, sufficiently dense, but otherwise irregular grids and with arbitrary dilation or even invertible transformation matrices. Specific constructions of such wavelets are obtained in Section 4, first in the 1-D case and then in the multidimensional case. A general theory of frame atomic decomposition of $L^2(\mathbb{R}^d)$ is obtained in Section 5.

Using the concept of outer frame, reconstruction formulas for these atomic decompositions are obtained in Section 6.

2. Notation

Throughout the paper J and K will denote countable index sets, and e_x will stand for the function $e_x(\xi) = e^{-i2\pi x \cdot \xi}$. We will use $\mu(E)$ to denote the Lebesgue measure of a measurable set E.

A set $\mathcal{H} := \{h_j\}_{j \in J}$ of measurable functions on $\mathbb{R}^d$ is called a *Riesz partition of unity* (**RPU**), if there exist constants $0 < p \le P < +\infty$ such that

$$(2.1) \qquad p \le \sum_{j \in J} |h_j(x)|^2 \le P \quad \text{a.e.} \quad x \in \mathbb{R}^d.$$

Let $\mathcal{S} = \{S_j\}_{j \in J}$ be a family of measurable subsets of $\mathbb{R}^d$. A *Riesz partition of unity* **associated to** $\mathcal{S}$, is a set $\mathcal{H} := \{h_j\}_{j \in J}$ of measurable functions, such that

(1) $\text{Supp}\, h_j \subseteq S_j$
(2) There exist constants $0 < p \le P < +\infty$ such that

$$(2.2) \qquad p \le \sum_{j \in J} |h_j(x)|^2 \le P \quad \text{a.e.} \quad x \in \cup_j S_j.$$

Remarks.

- If $\mathcal{H} = \{h_j\}$ is a RPU, then $\overline{\mathcal{H}} = \{\overline{h}_j\}$ is also a RPU.
- If $p = P = 1$, we will say that $\mathcal{H} = \{h_j\}$ is a *regular partition of unity*.
- If the sets in $\mathcal{S}$ are essentially disjoint (i.e. $\mu(S_i \cap S_j) = 0, \forall i \ne j$), the family $\{h_j = \chi_{S_j}\}$ will yield a regular partition of unity associated to $\mathcal{S}$.
- Every RPU $\mathcal{H} = \{h_j\}$ can be normalized to obtain a regular partition of unity by considering

$$\tilde{h}_j = \frac{h_j}{(\sum_j |h_j|^2)^{1/2}}.$$

- Given a family $\mathcal{S} = \{S_j\}_{j \in J}$ of measurable sets on $\mathbb{R}^d$, define

$$(2.3) \qquad \rho_S(x) = \#(\{j \in J : x \in S_j\}) = \sum_{j \in J} \chi_{S_j}(x),$$

where $\#(B)$ is the cardinal of the set B. The value $\rho_S = \|\rho_S\|_\infty$ is called the *covering index* of $\mathcal{S}$.

We now recall the definition of frame for a given close subspace F of $L^2(\mathbb{R}^d)$.

Definition 2.1. A set of functions $\{g_j\}_{j \in J}$ is a *frame* for F if $g_j \in F$ and there exist constants $0 < m, M < +\infty$, such that

$$(2.4) \qquad m\|f\|^2 \le \sum_{j \in J} |<f, g_j>|^2 \le M\|f\|^2, \quad \forall \quad f \in F.$$

For a measurable set $Q \subseteq \mathbb{R}^d$ we will denote by $\mathcal{K}_Q$ the functions that have support in $\overline{Q}$, and by $\mathcal{B}_Q$ the functions whose Fourier transform has support in $\overline{Q}$, i.e.

$$(2.5) \qquad\qquad \mathcal{K}_Q := \{f \in L^2(\mathbb{R}^d) : \text{Supp}(f) \subset \overline{Q}\}$$

$$(2.6) \qquad\qquad \mathcal{B}_Q := \{f \in L^2(\mathbb{R}^d) : \text{Supp}(\hat{f}) \subset \overline{Q}\}.$$

We will need also the following definition.

Definition 2.2. Let H be a Hilbert space. A collection of vectors $\{g_j \in H\}_{j \in J}$ is an *outer frame* for a closed subspace F of H, if $\{P_F(g_j)\}_{j \in J}$ is a frame for F, where P_F is the orthogonal projection onto F, or equivalently, there exist constants $0 < m, M < +\infty$, such that

$$(2.7) \qquad\qquad m\|f\|^2 \le \sum_{j \in J} |<f, g_j>|^2 \le M\|f\|^2, \quad \forall \quad f \in F.$$

Related definitions to the concept of outer frames appear in [LO98], [FW01].

Remark. Throughout the paper, we will use the following immediate and very useful fact about frames:
If $\{g_j\}_{j \in J}$ is a frame for $\mathcal{K}_Q$, and $V \subset Q$, then $\{g_j\}_{j \in J}$ is an outer frame for $\mathcal{K}_V$.

3. Wavelets on arbitrary irregular grids and with arbitrary dilation matrices and other transformations

Our first results concerns the construction of wavelets $\{|\det A_j|^{1/2} \psi(A_j x - x_{j,k}) : j \in J, k \in K\}$ with translates on the arbitrary irregular grid $X = \{x_{j,k} \in \mathbb{R}^d : k \in K, j \in J\}$ and with an arbitrary countable family of invertible $d \times d$ matrices $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$.

Theorem 3.1 (Wavelets). *Let $Q \subset \mathbb{R}^d$ be a set of finite measure, h a function in $L^2(\mathbb{R}^d)$ and $\mathcal{A} = \{A_j \in GL_d(\mathbb{R}) : j \in J\}$ a family of invertible matrices.*

For each $j \in J$ set $B_j = (A_j^T)^{-1}$, $S_j = B_j^{-1}Q = A_j^T Q$, $h_j = h(B_j \cdot)$ and let $\mathcal{S} = \{S_j, j \in J\}$.

Assume that $\mathcal{S}$ is a covering of $\mathbb{R}^d$, $\mathcal{H}$ is a RPU with bounds p and P and that $\text{Supp}(h) \subset Q$.

Consider $X = \{x_{j,k} \in \mathbb{R}^d : j \in J, k \in K\}$ such that for each $j \in J$, the set $\{e_{x_{j,k}} \chi_Q : k \in K\}$ forms a frame for $\mathcal{K}_Q$ with lower and upper frame bounds m_j and M_j respectively. If $m := \inf_j m_j > 0$ and $M := \sup_j M_j < +\infty$, then the collection

$$\{|\det A_j|^{1/2} \psi(A_j x - x_{j,k}) : j \in J, k \in K\}$$

is a wavelet frame of $L^2(\mathbb{R}^d)$ with bounds mp and MP, generated by a single function ψ, where ψ is the inverse Fourier transform of h.

Proof. Since for each $j \in J$ we have that $\{e_{x_{j,k}} \chi_Q : k \in K\}$ forms a frame for $\mathcal{K}_Q$ with lower and upper frame bounds m_j and M_j respectively, an application of Part 2 of Corollary 5.2 for the matrix $B_j^{-1}*$ shows that $\{|B_j|^{1/2} e_{x_{j,k}}(B_j \omega) \chi_Q(B_j \omega) : k \in K\}$ forms a frame of $\mathcal{K}_{S_j}$ with the same bounds. From the definition of S_j,

$\{(\mu(S_j))^{-(1/2)}e_{(A_j^{-1}x_{j,k})}(\omega)\chi_{S_j}(\omega) : k \in K\}$ is then a frame for $\mathcal{K}_{S_j}$ with frame bounds $m_j\mu(Q)^{-1}$ and $M_j\mu(Q)^{-1}$.

On the other side, if $\mathrm{Supp}(h) \subset Q$ then $\mathcal{H}$ is associated to S. So, we can apply Proposition 5.7 to $S, \mathcal{H}$ and $\{\mu(S_j)^{-(1/2)}e_{(A_j^{-1}x_{j,k})}\chi_{S_j} : k \in K\}, j \in J$, to conclude that, $\{\mu(S_j)^{-1/2}h_je_{(A_j^{-1}x_{j,k})} : k \in K, j \in J\}$ forms a frame of $L^2(\mathbb{R}^d)$ with lower frame bound $mp\mu(Q)^{-1}$ and upper frame bound $MP\mu(Q)^{-1}$. This gives that,

$$\{|B_j|^{1/2}h(B_j\omega)e_{x_{j,k}}(B_j\omega) : k \in K, j \in J\}$$

forms a frame of $L^2(\mathbb{R}^d)$ with frame bounds mp (or mc) and MP. The theorem now follows from an application of the inverse Fourier transform.

$\square$

Remarks.

(1) The set of matrices $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$ can be arbitrary and need not have a group structure.
(2) The set $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$ can also be chosen to have a simple structure. For example, $J = \mathbb{Z}^2$, $A_{(i,j)} = R^iD^j$ where R is a rotation and D a dilation matrix, will be used to construct directional wavelets. An even simpler example is $J = \mathbb{Z}$, $A_j = A^j$, where A is an invertible matrix which gives a construction of wavelet frames on $\mathbb{R}^d$.
(3) Note that h does not need to be compactly supported.

We will use the theorem above to construct specific examples of wavelets, e.g., directional wavelets, isotropic wavelets, etc.

Interesting particular cases of Theorem 3.1.

(1) $x_{j,k} = x_k \, \forall \, j \in J$.
  Let $X = \{x_k \in \mathbb{R}^d : k \in K\}$ be such that $\{e_{x_k}\chi_Q, k \in K\}$ is a frame for $\mathcal{K}_Q$ with frame bounds m and M. Then

$$\{|\det A_j|^{1/2}\psi(A_jx - x_k) : j \in J, k \in K\}$$

  forms a wavelet frame of $L^2(\mathbb{R}^d)$ with bounds mp and MP.
(2) $A_j = A^j \, \forall \, j \in J$, with $A \in GL_d(\mathbb{R})$.
  Each of the following sets are wavelet frames of $L^2(\mathbb{R}^d)$ with bounds mp and MP:

$$\{|\det A|^{j/2}\psi(A^jx - x_{j,k}) : j \in J, k \in K\}, \text{ and}$$
$$\{|\det A|^{j/2}\psi(A^jx - x_k) : j \in J, k \in K\}.$$

Remarks.

(1) If the set $\mathcal{S}$ is a tiling of $\mathbb{R}^d$ then the wavelets constructed above are Shannon-like wavelets, thus not well localized in space. To obtain well localized space-frequency wavelets, $\mathcal{H}$ must be constructed to be a smooth partition of unity, e.g., at least $C^1(\mathbb{R}^d)$ as demonstrated in the examples in Section 4, below.

(2) Reconstruction formulas for such wavelet frames are developed in Section 6.

4. Examples of wavelet frames on irregular lattices and with arbitrary set of dilation matrices and other transformations

4.1. **Density and Separation.** To be able to use Theorem 3.1 to construct concrete examples of wavelet frames on irregular grids, we first need to construct exponential frames $\{e_{x_k}\chi_D\}$ (also called Fourier frames) for $\mathcal{K}_D$. Exponential frames play a central role in sampling theory for Paley-Wiener spaces (also known as spaces of band-limited functions).

The density of a set $X = \{x_k \subset \mathbb{R}^d : k \in K\}$ and separateness of the points in X play a fundamental role for finding exponential frames $\{e_{x_k}\chi_D\}$ for $\mathcal{K}_D$.

Definition 4.1. A sequence $X = \{x_k : k \in K\}$ is separated if

$$\inf_{k \neq l} \|x_k - x_l\| > 0.$$

There are many notions for the density of a set X. We start with three definitions that are due to Beurling.

Definition 4.2.

(1) A lower uniform density $D^-(X)$ of a separated sequence $X \subset \mathbb{R}^d$ is defined as

$$D^-(X) = \lim_{r \to \infty} \frac{\nu^-(r)}{(2r)^d}$$

where $\nu^-(r) := \min_{y \in \mathbb{R}^d} \# \left(X \cap (y + [-r, r]^d) \right)$, where $\#(Z)$ denotes the cardinal of the set Z.

(2) An upper uniform density $D^+(X)$ of a separated sequence X is defined as

$$D^+(X) = \lim_{r \to \infty} \frac{\nu^+(r)}{(2r)^d}$$

where $\nu^+(r) := \max_{y \in \mathbb{R}} \# \left(X \cap (y + [-r, r]^d) \right)$.

(3) If $D^-(X) = D^+(X) = D(X)$, then X is said to have uniform Beurling density $D(X)$.

Remark. The limits in the definitions of $D^-(X)$ and $D^+(X)$ exist (see [BW99]).

As an example, let $X \subset \mathbb{R}$ be separated and assume that there exists $L > 0$ such that $|x_k - \frac{k}{d}| \leq L$, for all $k \in \mathbb{Z}$. Then $D^-(X) = D^+(X) = d$. For the one dimensional case, Beurling proved the following Theorem.

Theorem 4.3. *(Beurling) Let $X \subset \mathbb{R}$ be separated, $a > 0$ and $\Omega = [-\frac{a}{2}, \frac{a}{2}]$. If $a < D^-(X)$ then $\{e_{x_k}\chi_\Omega\}$ is a frame for $\mathcal{K}_\Omega$.*

This previous result however is only valid in one dimension. For higher dimensions, Beurling introduced the following notion:

Definition 4.4. The gap ρ of the set $X = \{x_k : k \in K\}$ is defined as

$$\rho = \rho(X) = \inf \left\{ r > 0 : \bigcup_{k \in K} B_r(x_k) = \mathbb{R}^d \right\}$$

Equivalently, the gap ρ can be defined as

$$\rho = \rho(X) = \sup_{x \in \mathbb{R}^d} \inf_{x_k \in X} |x - x_k|.$$

It is not difficult to show that if X has gap ρ, then $D^-(X) \geq \frac{1}{2\rho}$. For a separated set X, and for the case where Ω is the ball $B_r(0)$ of radius r centered at the origin, Beurling [Beu66] proved the following result:

Theorem 4.5 (Beurling). *Let $X \subset \mathbb{R}^d$ be separated, and $\Omega = B_r(0)$. If $r\rho < 1/4$, then $\{e_{x_k}\chi_\Omega\}$ is a frame for $\mathcal{K}_\Omega$.*

For a very clear exposition of some of the Beurling density results see [BW99].

4.2. **Wavelet frames in 1-D.** The construction in the following theorem (which is a particular case of Theorem 3.1), generalizes a similar result of [DGM86] to the irregular case. See also [Grö93].

Theorem 4.6. *Let $Q = [-1, -1/2] \cup [1/2, 1]$, $0 \leq \epsilon < 1/2$, and let $\hat{h}_+$ be a real valued function such that $Q^1_\epsilon := \operatorname{Supp} \hat{h}_+ \subset [\frac{1}{2} - \epsilon, 1 + \epsilon]$, $|\hat{h}_+| \leq 1$, and $0 < c \leq |\hat{h}_+|$ on $[1/2, 1]$. Assume that for each $j \in \mathbb{Z}$, the sequence $X_j = \{x_{j,k}\}_{k \in \mathbb{Z}}$ is separated and that $D^-(X_j) > 2^{j+1}(1 + \epsilon)$. Then for each j, the set $\{e_{2^j x_{j,k}}\chi_{Q_\epsilon} : x_{j,k} \in X_j : k \in \mathbb{Z}\}$ is a frame of $\mathcal{K}_{Q_\epsilon}$, where $Q_\epsilon := Q^1_\epsilon \cup (-Q^1_\epsilon)$. If furthermore the sets $X_j = \{x_{j,k}\}_{k \in \mathbb{Z}}$ are chosen such that the frame bounds m_j and M_j satisfy $\inf_j m_j = m > 0$ and $\sup_j M_j = M < +\infty$, then the set $\{2^{j/2}\psi(2^j(\cdot - x_{j,k})) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ where $\psi(x) = 2Re(h_+(x))$ is a wavelet frame for $L^2(\mathbb{R})$.*

Remark. The wavelet frame constructed in the theorem above is of the form $\{2^{j/2}\psi(2^j(\cdot - x_{j,k})) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ which is slightly different form than the one constructed in Theorem 3.1. This discrepancy is due to a convenient choice of the irregular set $X_{j*} = \{x_{j,k}\}$ that we have adopted in the statement of the theorem above.

Note that the wavelets constructed in the theorem above are real and symmetric. Actually, if one wants ψ with good decay, $\hat{h}_+$ can be easily constructed to be C^r, $r \geq 1$, even C^∞.

As a corollary, if we choose the sampling sets X_j to be nested, i.e., $X_j \subset X_{j+1}$, we get

Corollary 4.7. *Let Q, ϵ, and $\hat{h}_+$ be as in Theorem 4.6. Assume that the sequences $X_j = \{x_{j,k}\}_{k \in \mathbb{Z}}$ are separated and such that $x_{j,k} = x_{j+1,2k}$. If $D^-(X_0) > 2(1 + \epsilon)$, then for each j, the set $\{e_{2^j x_{j,k}}\chi_{Q_\epsilon} : x_{j,k} \in X_j\}$ is a frame for $\mathcal{K}_{Q_\epsilon}$. If furthermore the frame bounds m_j and M_j satisfy $\inf_j m_j = m > 0$ and $\sup_j M_j = M < \infty$, then the set $\{2^{j/2}\psi(2^j(\cdot - x_{j,k})) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ where $\psi(x) = 2Re(h_+(x))$ is a wavelet frame for $L^2(\mathbb{R})$.*

Proof. Since $x_{j,k} = x_{j+1,2k}$ we have that $\#(X_{j+1} \cap [-r,r]) \geq 2\#(X_j \cap [-r,r] - 1)$. Thus $D^-(X_{j+1}) \geq 2D^-(X_j)$. But $D^-(X_0) > 2(1+\epsilon)$, therefore $D^-(X_j) > 2^{j+1}(1+\epsilon)$. The corollary then follows directly from Theorem 4.6. $\square$

From Theorems 4.6 and 3.1, we immediately get the following Corollary.

Corollary 4.8. *Let Q, ϵ, and h_+ be as in Theorem 4.6. Assume that the set $X = \{x_k\}_{k \in \mathbb{Z}}$ is separated and that $D^-(X) > 2(1+\epsilon)$. Then, the set $\{e_{x_k}\chi_{Q_\epsilon} : x_k \in X\}$ is a frame of $\mathcal{K}_{Q_\epsilon}$, and the set of functions $\{2^{j/2}\psi(2^j \cdot -x_k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ where $\psi(x) = 2Re(h_+(x))$ is a wavelet frame for $L^2(\mathbb{R})$.*

4.2.1. *Examples.*

(1) *Shannon-type wavelet frames* : We use Corollary 4.7, with $\epsilon = 0$ and $\hat{h}_+ = \chi_{[1/2,1]}$ to get wavelet frames of the form

$$\psi_{x_{j,k},j} = \{2^{-j/2}\cos(2^{-j-1}3\pi(x - x_{j,k}))\operatorname{sinc}(2^{-j-1}\pi(x - x_{j,k}))\}.$$

These wavelets are not well localized since the decay at ∞ is $O(|x|^{-1})$.

(2) *Shannon-type wavelet bases* : If we choose X_j such that $|x_{j,k} - 2^j k| \leq L_j < \frac{2^j}{4}$, $\forall\ k \in \mathbb{Z}$, then by Kadec's 1/4-Theorem, we immediately get that $\psi_{x_{j,k},j} = \{2^{-j/2}\cos(2^{-j-1}3\pi(x - x_{j,k}))\operatorname{sinc}(2^{-j-1}\pi(x - x_{j,k})) : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ constructed above form a wavelet Riesz basis for $L^2(\mathbb{R})$.

(3) *Well localized wavelet frames*: For faster decay of the wavelet frames, we choose $\hat{h}_+$ to be a smoother function. Let $\beta_n = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ be the B-spline of degree n (note that $\operatorname{Supp}\beta_n = [0, n+1]$). Let $\epsilon = 1/4$, and $\hat{h}_+(\xi) = n\beta_{n-1}((\xi - 1/4)n))$. Then we get a wavelet frame of the form

$$\psi_{x_{j,k},j} = \{2^{-j/2}\cos(2^{-j-1}3\pi(x - x_{j,k}))\operatorname{sinc}^n(2^{-j}\frac{\pi}{n}(x - x_{j,k}))\}.$$

For this case the wavelets decay as $O(|x|^{-n})$.

4.3. **Examples of wavelet frames in $\mathbb{R}^d$.**

(1) For $\mathbb{R}^2$, let $X = \{x_k : k \in \mathbb{Z}\}$, and let $Y = \{y_l : l \in \mathbb{Z}\}$. If $D^-(X) > 2$ and $D^-(Y) > 2$, then using Proposition 5.6 below for product frames, the set $\{e_{(x_k,y_l)} : (k,l) \in \mathbb{Z}^2\}$ form a frame for $\mathcal{K}_{[-1,1]^2}$. Let $Q := \{(x,y) \in \mathbb{R}^2 : 1/2 \leq x^2 + y^2 \leq 1\}$, and $A = 2I$, then $\mathbb{R}^2 = \cup_j A^j Q$. We can then use Theorem 3.1 to construct wavelet frames for $L^2(\mathbb{R}^2)$:

- *Shannon-type radial wavelets*: Let $h = \chi_Q$, then h is radial. Thus the function ψ defined as $\hat{\psi} = h$ satisfies $\psi(x,y) = g(r)$, where $r = (x^2 + y^2)^{1/2}$. We then construct the Shannon-like wavelet frame for $L^2(\mathbb{R}^2)$, as in Theorem 3.1. A related construction of non-separable radial Shannon-type frame wavelets and multiwavelets can be found in [PGKKH1], and [PGKKH2].

- *Well localized radial wavelets*: To construct wavelet frames with polynomial decay in space, we let $h(\xi_1, \xi_2) = n\beta_{n-1}\left((\xi_1^2 + \xi_2^2 - 1/4)n\right)$, and construct the wavelet frames using Theorem 3.1 (see Figure 1).
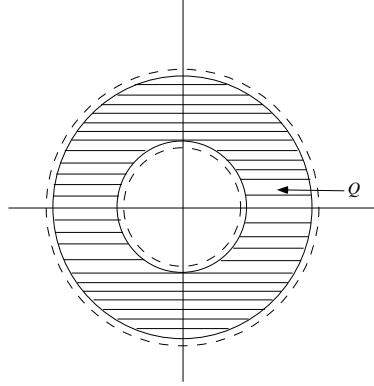
FIGURE 1. Radial wavelet frames that are well localized in space.

(2) The points $\{(x_k, y_l) : k, l \in \mathbb{Z}^2\}$ lie on an irregular grid of the form $X \times Y$. However, we may be interested in points $Z = \{(x_k, y_l) \subset \mathbb{R}^2 : (k, l) \in \mathbb{Z}^2\}$ that do not lie on irregular grids of the form $X \times Y$. For this case, the same constructions above can be used to form wavelets frame for $L^2(\mathbb{R}^2)$, as long as the gap $\rho(X) < \frac{1}{4(1+\epsilon)}$.

(3) As in the 1-D examples above, we can also use Corollary 4.7 to construct wavelets on irregular grids satisfying $X_j \subset X_{j+1}$.

(4) *Directional wavelet frames*: We can easily construct directional wavelet frames as follows: Let Q_1 be a region defined by $Q_1 = \{(x, y) \in \mathbb{R}^2 : x = r\cos(\theta), y = r\sin(\theta), 1/2 \leq r \leq 1, |\theta| \leq \frac{\pi}{8}\}$, and define $Q = (-Q_1) \cup Q_1$. Let $A = 2I$, and R be the matrix of a rotation by an angle $\pi/4$. Let ψ be such that $\hat{\psi} = \chi_Q$, then we obtain the wavelet frame for $L^2(\mathbb{R}^2)$ of the form $\{\psi_{j,k} = 4^{-j_1/2}\psi_j(2^{-j_1}R^{j_2} \cdot -x_{j,k}) : j = (j_1, j_2) \in \mathbb{Z} \times \{0, 1, 2, 3\}, k \in \mathbb{Z}\}$. The index j_1 codes for the resolution of the wavelet, while the index j_2 codes for four possible directions. Thus the wavelet frame coefficients encode time scale as well as directional information. Clearly one can choose any number of directions and adapt the previous construction. An obvious modification as shown in Figure 2, yields wavelet frames with polynomial decay. Very nice constructions of smooth directional wavelet frames on regular grids were obtained before in [AHNV01, ADH$^+$03].

(5) *Spiral* In this example we will define a dilation covering by spiral annulus sectors.

Let $a, b > 1$, and Γ the spiral curve defined by

$$\Gamma(t) = (a^t \cos(bt), a^t \sin(bt)) \quad t \in \mathbb{R}.$$

For $\alpha \in \mathbb{R}$ define R_α to be the rotation of angle α : $R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$. The curve Γ satisfies:

$$\Gamma(t + \alpha) = a^\alpha R_{b\alpha}\Gamma(t).$$

Note that for positive α the matrix $A = a^\alpha R_{b\alpha}$ is expansive.

Now we are ready to define the covering elements. Set $b = 2\pi$ and $\alpha = \frac{1}{m}$, for some integer $m \geq 2$ so that $A^m = aI_d$. Define the spiral annulus sector
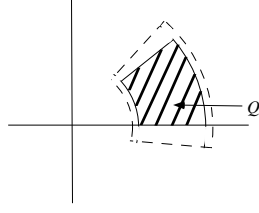
FIGURE 2. Well-localized directional wavelet: The regions $Q = Q^1 \cup (-Q^1)$ and $Q_\epsilon = Q^1_\epsilon \cup (-Q^1_\epsilon)$ that can be used to construct well localized directional wavelets.
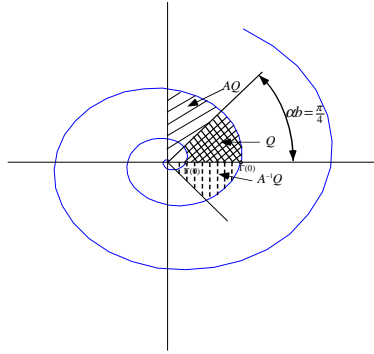

FIGURE 3. Spiral wavelet frames.

$Q = \{x \in \mathbb{R}^2 : x = \lambda\Gamma(\beta), 1 \le \lambda \le a, 0 \le \beta \le \alpha\}$ (see Figure 3). So Q is compact and $\{A^j Q : j \in \mathbb{Z}\}$ is a disjoint covering of $\mathbb{R}^d \setminus \{0\}$.

Choose $\varepsilon > 0$ sufficiently small and h a smooth function that does not vanish in Q and with support in Q_ε. Define $\hat{\psi} = h$. Select a separated set $X = \{x_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}^2$ such that $\rho(X) < \frac{1}{2\mathrm{diam}(Q_\varepsilon)}$.

The set $\{a^{j/m}\psi(a^{j/m}R_{-2\pi j/m}(x - x_k), k \in \mathbb{Z}, j \in \mathbb{Z}\}$ form a wavelet frame of $L^2(\mathbb{R}^2)$ generated by a single wavelet ψ that is band-limited, with good decay and directional in frequency.

(6) Obviously, all the constructions above can be generalized to $\mathbb{R}^d$ for any dimension $d > 2$.

Remark. Some of the wavelet frames may be associated with MRAs. For example, the so called Shannon wavelet frame constructed above is associated with the Shannon MRA $V_j = \{f \in L^2(\mathbb{R}) : \mathrm{Supp}\,\hat{f} \subset [-2^{-j-1}, 2^{-j-1}]\}$, $j \in \mathbb{Z}$. In general however, the precise relation needs further investigation.

5. General results on the construction of time-frequency atoms

In this section we will develop a general method for constructing time-frequency frame atoms in several variables. This construction allows us to construct the previously introduced wavelet frames on irregular grids and with arbitrary dilation matrices or other types of transformations. It also allows us to construct non-harmonic Gabor frames on non-uniform coverings of $\mathbb{R}^d$ as described in Section 5.1 below.

Let U and V be non-empty open subsets of $\mathbb{R}^d$, $T : U \to V$ is an invertible C^1 map, with C^1 inverse $S : V \to U$, i.e., T is a C^1 homeomorphism with C^1 inverse $S : V \to U$. Define $\alpha := \inf_{y \in U} |\det T'(y)|^{-1}$ and $\beta := \sup_{y \in U} |\det T'(y)|^{-1}$, where $T'(y)$ denotes the derivative of T at y, $T'(y) = \left[\frac{\partial T_i}{\partial y_j}(y) \right]$. We have the following Proposition :

Proposition 5.1. *With the above notation, assume that α is positive and β is finite, then if $\{g_j\}_{j \in J}$ is a frame for $\mathcal{K}_V$ with frame bounds m and M, then $\{g_j \circ T\}_{j \in J}$ is a frame for $\mathcal{K}_U$ with frame bounds αm and βM.*

Proof. Assume that $f \in \mathcal{K}_U$. Then for $j \in J$, by an application of the change of variables formula

$$(5.1) \quad < f, g_j \circ T > = \int_U f \overline{g_j \circ T} = \int_V (f \circ S) |\det S'| \overline{g_j} = < (f \circ S) |\det S'|, g_j > .$$

Since $|\det S'|$ is finite and bounded away from zero, $(f \circ S) |\det S'|$ is in $\mathcal{K}_V$.

Using that $\{g_j\}_{j \in J}$ is a frame of $\mathcal{K}_V$ with bounds m, M we have

$$(5.2) \qquad \sum_{j \in J} | < (f \circ S) |\det S'|, g_j > |^2 \leq M \|(f \circ S) |\det S'| \|^2.$$

Now, applying again the change of variables theorem, we obtain

$$\|(f \circ S) |\det S'| \|^2 = \int_V |f \circ S|^2 |\det S'|^2 = \int_U |f|^2 |\det(S' \circ T)|^2 |\det T'|.$$

Since $S = T^{-1}$ we have that $\det[(S' \circ T)(y)] = (\det[T'(y)])^{-1}$ for all $y \in U$. Thus

$$(5.3) \qquad \|(f \circ S) |\det S'| \|^2 = \int_U |f|^2 |\det T'|^{-1} \leq M \|f\|^2.$$

That is, from (5.1), (5.2) and (5.3) we have

$$\sum_{j \in J} | < f, g_j \circ T > |^2 \leq \beta M \|f\|^2.$$

This proves the upper inequality for $\{g_j \circ T\}_{j \in J}$. The lower inequality is obtained in the same way, with the obvious modifications. □

Proposition 5.1 establishes that T defines an isomorphism $\Pi_T : \mathcal{K}_U \to \mathcal{K}_V$ defined by $\Pi_T(g) = g \circ T$, thereby transforming frames into frames. By taking T to be a translation or a dilation, we get the well known result:

Corollary 5.2 (Translation and dilation of frames)**.** *Let $Q \subseteq \mathbb{R}^d$ be an open subset of $\mathbb{R}^d$. Let $y \in \mathbb{R}^d$ be any point, and $A \in GL_d(\mathbb{R})$ an invertible matrix. We have,*

(1) **Translation** $\{g_j\}_{j\in J}$ *is a frame for* $\mathcal{K}_Q$ *with bounds* m *and* M, *if and only if* $\{g_j(\cdot - y)\}_{j\in J}$ *is a frame for* $\mathcal{K}_{Q+y}$ *with the same bounds.*

(2) **Dilation** $\{g_j\}_{j\in J}$ *is a frame for* $\mathcal{K}_Q$ *with bounds* m *and* M, *if and only if* $\{|\det A|^{-1/2}g_j(A^{-1}\cdot)\}_{j\in J}$ *is a frame for* $\mathcal{K}_{AQ}$ *with the same bounds.*

Remark. In fact the corollary remains true if we only assume that Q is measurable.

Proof. Part 1 is a direct application of the Proposition, for the case that $T(x) = x - y$.

For Part 2, and the transformation $T(x) = A^{-1}x$, the Proposition tells us that $\{g_j(A^{-1}\cdot)\}_{j\in J}$ is a frame with frame bounds $|\det A|m$ and $|\det A|M$, therefore dividing each function by $|\det A|^{-1/2}$ we obtain the result. $\qquad\square$

For the next theorem, we need to introduce some definitions. Let $\mathcal{H} = \{h_j : j \in J\}$ be a RPU with bounds p and P. For each $j \in J$ set

$$(5.4) \qquad W_j = Closure_{L^2}\{\overline{h}_j f : f \in L^2(\mathbb{R}^d)\}.$$

Let $0 < c \le p$, and define

$$(5.5) \qquad Q_j = Q_j(c) = \{x \in \mathbb{R}^d : |h_j(x)|^2 > c\}, \quad \text{for each} \quad j \in J.$$

For a given c, we discard all those j such that Q_j has measure zero. Note that if $J_0 = \{j \in J : \mu(Q_j) > 0\}$, then we can only claim that $\mathcal{H}_0 = \{h_j\chi_{Q_j}, j \in J_0\}$ is a RPU associated to $\{Q_j\}_{j\in J_0}$ with constants c and P.

Theorem 5.3. *Let* $0 < c \le p$.

(1) *Assume that* $\{g_{j,k}\}_{k\in K}$ *is a frame for* W_j *with lower and upper frame bounds* m_j *and* M_j *respectively. If* $m := \inf_j m_j > 0$ *and* $M := \sup_j M_j < +\infty$, *then* $\{h_j g_{j,k} : j \in J, k \in K\}$ *is a frame for* $L^2(\mathbb{R}^d)$, *with frame bounds* pm *and* PM.

(2) *Assume that for each* $j \in J_0$, $\{g_{j,k}\}_{k\in K}$ *is a frame for* $\mathcal{K}_{Q_j}$ *with lower and upper frame bounds* m_j *and* M_j *respectively. If* $m := \inf_{j\in J_0} m_j > 0$ *and* $M := \sup_{j\in J_0} M_j < +\infty$, *then* $\{h_j g_{j,k} : j \in J_0, k \in K\}$ *is a frame for* $\mathcal{K}_{\cup Q_j}$, *with frame bounds* cm *and* PM.

Proof.

(1) Given $f \in L^2(\mathbb{R}^d)$ and denoting by $f_j = \overline{h}_j f \in W_j$ we will first show that

$$(5.6) \qquad p\|f\|^2 \le \sum_j \|f_j\|^2 \le P\|f\|^2,$$

for

$$p\|f\|^2 = \int p|f|^2 \le \int \sum_j |h_j|^2 |f|^2 = \sum_j \int |h_j f|^2 = \sum_j \|f_j\|^2,$$

where we used dominated convergence for the interchange of the integral with the sum. The other inequality is analogous.

For each $j \in J$ and $f \in L^2(\mathbb{R}^d)$, we use the fact that $\{g_{j,k}\}_k$ is a frame for W_j, and that $< f, h_j g_{j,k} > = < \overline{h}_j f, g_{j,k} >$, to obtain

$$m\|h_j f\|^2 \leq m_j \|h_j f\|^2 \leq \sum_k |<f, h_j g_{j,k}>|^2$$
$$\leq M_j \|h_j f\|^2 \leq M \|h_j f\|^2.$$

So summing over j

$$pm\|f\|^2 \leq m \sum_j \|f_j\|^2 \leq \sum_j \sum_k |<f, h_j g_{j,k}>|^2$$
$$\leq M \sum_j \|f_j\|^2 \leq PM\|f\|^2.$$

(2) For the second case we observe that the set $\lambda = \{\lambda_j = h_j \chi_{Q_j}, j \in J_0\}$ forms a RPU associated to $\{Q_j, j \in J_0\}$ with bounds c and P, for if $x \in \cup Q_j$,

$$c \leq c\,\#(\{j \in J_0 : x \in Q_j\}) \leq \sum_{j \in J_0 : x \in Q_j} |h_j(x)|^2$$
$$= \sum_{j \in J_0} |h_j(x)|^2 \chi_{Q_j}(x) \leq \sum_{j \in J_0} |h_j(x)|^2 \leq P.$$

Furthermore, by hypothesis,

$$\mathcal{K}_{Q_j} = \{\overline{\lambda}_j f : f \in L^2(\mathbb{R}^d)\}, \ j \in J_0.$$

These subspaces $\mathcal{K}_{Q_j}$ correspond to the subspaces W_j defined in (5.4) for the RPU λ, and therefore we can use the previous result (applied to $L^2(\cup Q_j)$ instead of $L^2(\mathbb{R}^d)$) to conclude that $\{\lambda_j g_{j,k} : j \in J, k \in K\}$ is a frame for $\mathcal{K}_{\cup Q_j} = L^2(\cup Q_j)$. The proof is complete by noting that $\lambda_j g_{j,k} = h_j g_{j,k}, j \in J_0$.

$\square$

Remarks.

- Note that in the previous theorem, instead of choosing a frame for the subspaces W_j, we could have chosen any collection of functions of $L^2(\mathbb{R}^d)$ that form an outer frame for W_j.
- If h is a bounded function and $Q = \operatorname{Supp} h$, then it is easy to see that closure$_{L^2}\{hf : f \in L^2(\mathbb{R}^d)\} = \mathcal{K}_Q$ if and only if $\mu(Q) = \mu(\{x : |h(x)| > 0\})$. So the spaces W_j defined in (5.4) will coincide in most of the cases with $\mathcal{K}_{\operatorname{Supp}(h_j)}$.

As a very important particular case of the previous theorem, we have the following Corollaries.

Corollary 5.4. *Let $\mathcal{S} = \{S_j \subset \mathbb{R}^d : j \in J\}$ be a family of subsets of $\mathbb{R}^d$, not necessarily disjoint, and let $\mathcal{H} = \{h_j\}$ be a RPU with constants p, and P not necessarily associated to $\mathcal{S}$. Assume that $\{g_{j,k}\}_{k \in K}$ is a frame for $\mathcal{K}_{S_j}$ with lower and upper frame bounds m_j and M_j respectively. If $m := \inf_j m_j > 0$ and $M := \sup_j M_j < +\infty$, then*

(1) If $\mathcal{H}$ is associated to $\mathcal{S}$ (i.e. $\mathrm{Supp}\, h_j \subseteq S_j, j \in J$), then $\big\{h_j g_{j,k} : j \in J, k \in K\big\}$ is a frame for $\mathcal{K}_{\cup S_j}$, with frame bounds pm and PM.

(2) If instead $|h_j(x)|^2 > c \; \forall x \in S_j, j \in J$, then also $\big\{h_j g_{j,k} : j \in J, k \in K\big\}$ is a frame for $\mathcal{K}_{\cup S_j}$, with frame bounds cm and PM.

Proof. Denote by P_{W_j} the orthogonal projection on the subspace W_j defined in (5.4). Since $\{g_{j,k} : k \in K\}$ forms a frame for $\mathcal{K}_{S_j}$ with bounds m_j and M_j, then $\{P_{W_j} g_{j,k} : k \in K\}$ forms a frame for W_j with the same bounds. So, by Theorem 5.3, $\{h_j P_{W_j} g_{j,k} : j \in J, k \in K\}$ forms a frame of $\mathcal{K}_{\cup S_j}$ with frame bounds mp and MP. We obtain part (1) of the Corollary observing that:

$$< f, h_j P_{W_j}(g_{j,k}) > = < \overline{h}_j f, P_{W_j}(g_{j,k}) > = < \overline{h}_j f, g_{j,k} > = < f, h_j g_{j,k} > .$$

The second part is a consequence of the fact that $\{h_j \chi_{S_j} : j \in J\}$ is a RPU associated to $\mathcal{S}$ with bounds c and P and that $Q_j(c) = S_j$. Therefore we can apply the second part of Theorem 5.3. $\qquad \square$

Remarks.

- For the first case, the set $\big\{h_j g_{j,k} : k \in K\big\}$ is not necessarily a frame for $\mathcal{K}_{S_j}$ or even for $W_j \subset \mathcal{K}_{S_j}$, even though $\big\{h_j g_{j,k} : j \in J, k \in K\big\}$ is a frame for $\mathcal{K}_{\cup S_j})$.
- In the second case, the subspace W_j is contained in $\mathcal{K}_{S_j}$.
- Note that if $\mathcal{H}$ is a regular partition of unity, then the frame bounds for the frame constructed in the Corollary, are m and M.

Corollary 5.5. *Let $\mathcal{H} = \{h_j : j \in J\}$ be a RPU with constants p, and P and $\{S_j \subset \mathbb{R}^d : j \in J\}$ and $\{Q_j \subset \mathbb{R}^d : j \in J\}$ be coverings of $\mathbb{R}^d$ such that $Q_j \subset S_j$ and $|h_j(x)|^2 \geq c$ a.e. $x \in Q_j$ for all $j \in J$ and some constant $c > 0$. Assume that $\{g_{j,k}\}_{k \in K}$ is a frame for $\mathcal{K}_{S_j}$ with lower and upper frame bounds m_j and M_j respectively. If $m := \inf_j m_j > 0$ and $M := \sup_j M_j < +\infty$, then $\big\{h_j g_{j,k} : j \in J, k \in K\big\}$ is a frame for $L^2(\mathbb{R}^d)$, with frame bounds cm and PM.*

Proof. As in the proof of the Theorem, we have the inequalities

$$(5.7) \qquad\qquad p\|f\|^2 \leq \sum_j \|\overline{h}_j f\|^2 \leq P\|f\|^2,$$

On the other side,

$$< f, h_j g_{j,k} > = < \overline{h}_j f, g_{j,k} \chi_{S_j} > = < \overline{h}_j f \chi_{S_j}, g_{j,k} >,$$

and then

$$\sum_j \sum_k |< f, h_j g_{j,k} >|^2 \leq M \sum_j \|\overline{h}_j f \chi_{S_j}\|_2^2 \leq M \sum_j \|\overline{h}_j f\|_2^2 \leq MP\|f\|_2^2,$$

and

$$\sum_j \sum_k |< f, h_j g_{j,k} >|^2 \geq m \sum_j \|\overline{h}_j f \chi_{S_j}\|_2^2 \geq mc \sum_j \|f \chi_{Q_j}\|_2^2 \geq mc\|f\|_2^2.$$

$\qquad \square$

The following proposition is a direct application of Fubini's theorem, and allows us to construct frames in a product space. By induction the Theorem can be generalized to hold for any finite number of factors.

Proposition 5.6 (Product of Frames). *Let E_1 and E_2 be measurable subsets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively, and let $\{h_j\}_{j \in J}$ and $\{g_k\}_{k \in K}$ be frames for $\mathcal{K}_{E_1}$ and $\mathcal{K}_{E_2}$ with frame bounds m_1, M_1 and m_2, M_2. Then $\{h_j g_k\}_{j,k \in J \times K}$ is a frame of $\mathcal{K}_{E_1 \times E_2}$ with frame bounds $m = m_1 m_2, M = M_1 M_2$.*

Proof. For any function f in $\mathcal{K}_{E_1 \times E_2}$,

$$\sum_{j \in J} \sum_{k \in K} \left| \int_{E_1 \times E_2} f(x_1, x_2) \overline{h_j(x_1) g_k(x_2)} dx_1 dx_2 \right|^2 =$$

$$\sum_{j \in J} \left(\sum_{k \in K} \left| \int_{E_2} \left(\int_{E_1} f(x_1, x_2) \overline{h_j(x_1)} dx_1 \right) \overline{g_k(x_2)} dx_2 \right|^2 \right)$$

$$\geq m_2 \int_{E_2} \sum_{j \in J} \left| \int_{E_1} f(x_1, x_2) \overline{h_j(x_1)} dx_1 \right|^2 dx_2$$

$$\geq m_1 m_2 \int_{E_2} \int_{E_1} |f(x_1, x_2)|^2 dx_1 dx_2,$$

which yields the lower frame bound $m = m_1 m_2$. The upper frame bound $M = M_1 M_2$ can be obtained in a similar fashion. $\square$

5.1. Construction of time-frequency atoms on arbitrary irregular grids and with arbitrary dilation matrices and other transformations. We now particularize our previous results to frames of the form $\{h e_{x_{j,k}}, j \in J, k \in K\}$, where h is a fixed function. Using the Fourier transform, these types of frames allow us to construct wavelets $\{|\det A_j|^{1/2} \psi(A_j x - x_{j,k}) : j \in J, k \in K\}$ with translates on the arbitrary irregular grid X and with an arbitrary countable family of invertible $d \times d$ matrices $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$ (cf. Theorem 3.1). First as a particular case of Corollary 5.4, we obtain the following Proposition.

Proposition 5.7. *Assume that $\mathcal{S} = \{S_j : j \in J\}$ forms a covering of $\mathbb{R}^d$, and let $\mathcal{H} = \{h_j : j \in J\}$ be a RPU with bounds p and P associated to $\mathcal{S}$. Assume also that $\{\mu(S_j)^{-1/2} e_{x_{j,k}} \chi_{S_j} : k \in K\}$ is a frame for $\mathcal{K}_{S_j}$ with lower and upper frame bounds m_j and M_j respectively. If $m := \inf_j m_j > 0$ and $M := \sup_j M_j < +\infty$, then*

$$\left\{ \mu(S_j)^{-1/2} h_j e_{x_{j,k}} : j \in J, k \in K \right\}$$

is a frame for $L^2(\mathbb{R}^d)$ with frame bounds mp and MP.

Remark. If $x_{j,k}$, S_j, and h_j are chosen such that $x_{j,k} = \alpha k$, $k \in \mathbb{Z}^d$, $\alpha \in \mathbb{R}$, $S_j = S + j$, $j \in \Gamma$ and $h_j = h(\cdot + j)$, $j \in \Gamma$, where Γ is a lattice in $\mathbb{R}^d$, then we obtain the standard Gabor or Weyl-Heisenberg frames. Thus in general, the construction above can be viewed as non-harmonic Gabor frames with variable windows h_j.

The following wavelet frame construction is a direct application of the previous proposition. Taking $\widehat{\psi}_j = h_j$, the set $\{\psi_{j,x_{j,k}} = \psi_j(\cdot - x_{j,k}) : j \in J, k \in K\}$ is a

wavelet frame for $L^2(\mathbb{R}^d)$ with frame bounds mp and MP. Therefore, there exists a dual frame $\tilde{\psi}_{j,x_{j,k}}$ such that

$$f = \sum_{j \in J} \sum_{x_{j,k}} < f, \tilde{\psi}_{j,x_{j,k}} > \psi_{j,x_{j,k}} \quad \forall f \in L^2(\mathbb{R}^d).$$

We can also use Proposition 5.7 and Beurling Theorem 4.3, to obtain a non-harmonic Gabor frame as the following example shows:

Example. [Non-harmonic Gabor frames] Let $Q = [1,3]$ and Let $\beta_3 = \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]}$ be the B-spline of degree 3. Clearly, $\mathbb{R} = \cup_{j \in \mathbb{Z}} Q + j$. Let $X_j = \{x_{j,k}\}_{k \in \mathbb{Z}}$ be sets in $\mathbb{R}$ chosen such that for each j, $D^-(X_j) > 2$ and such that the frame bounds m_j and M_j satisfy $\inf_j m_j = m > 0$ and $\sup_j M_j = M < +\infty$. Then using Proposition 5.7 and Beurling Theorem 4.3, we obtain a non-harmonic Gabor frame of the form $\{\beta_3(t-j)e^{-i2\pi x_{j,k}\omega} : j \in \mathbb{Z}, k \in \mathbb{Z}\}$. Obviously, we can also use a non-uniform partition of $\mathbb{R}$ and get generalized non-harmonic Gabor frames as discussed earlier.

Remark. If $Q \subset \mathbb{R}^d$ be a measurable subset of $\mathbb{R}^d$, and if the family of sets $\mathcal{S}$ is defined by means of expanding or contracting Q, then we obtain the theorems of Section 3 as particular cases of Proposition 5.7.

5.2. **Construction of exponential frames.** By Corollaries 5.2 and 5.4, we can construct a frame for $\mathcal{K}_Q \subset \mathbb{R}^d$ starting from a frame for $\mathcal{K}_D$ where D is *any* subset of $\mathbb{R}^d$ with nonempty interior. Hence to build a frame for $\mathcal{K}_Q$ it is enough to start with a frame for $\mathcal{K}_U$, where U is an open disk. Specifically, since any bounded measurable set can be covered by a finite number of translates of U, we can use Corollary 5.2(1) to find a frame for $\mathcal{K}_Q$. We can also expand U until it covers Q and use Corollary 5.2(2). This shows that there are many ways to construct frames for $\mathcal{K}_Q$ starting from a frame for $\mathcal{K}_U$. Obviously, the particular construction will depend on the application.

We will describe now two particular constructions.

(1) Given a frame for K_D where D is a measurable subset of $\mathbb{R}^d$, we will construct a frame for K_Q. Let Γ be a regular lattice in $\mathbb{R}^d$ (i.e. $\Gamma = R\mathbb{Z}^d$, where R is an invertible $d \times d$ matrix with real entries), and let D be a measurable subset of $\mathbb{R}^d$ such that $\mathbb{R}^d = \bigcup_{\gamma \in \Gamma} (D + \gamma)$ with a finite covering index. Let Q be a measurable subset of $\mathbb{R}^d$ and define $Q_\gamma := Q \cap (D + \gamma)$ for $\gamma \in \Gamma$. Let $\Delta := \{\gamma \in \Gamma : \mu(Q_\gamma) > 0\}$. By Corollary 5.2, if $\{e_{x_k}\chi_D : k \in K\}$ is a frame for $\mathcal{K}_D$, then $\{e_{x_k}\chi_{D+\gamma} : k \in K\}$ is a frame for $\mathcal{K}_{D+\gamma}$. Hence, $\{e_{x_k}\chi_{Q_\gamma} : k \in K\}$ is also a frame for $\mathcal{K}_{Q_\gamma}$. Therefore, by Corollary 5.4, $\{e_{x_k}\chi_{Q_\gamma} : \gamma \in \Delta, k \in K\}$ is a frame for $\mathcal{K}_Q$. As an example, when Q is a measurable subset of $\mathbb{R}^d$, $D = [0,1]^d$, and $\Gamma = \mathbb{Z}^d$, we have that if $\{e_{x_k}\chi_{[0,1]^d} : k \in K\}$ is a frame for $L^2([0,1]^d)$, then $\{e_{x_k}\chi_{Q_\gamma} : \gamma \in \Delta, k \in K\}$ is a frame for $\mathcal{K}_Q$ (recall that $Q_\gamma = Q \cap ([0,1]^d + \gamma)$).

It is easy to see that the covering requirement $\mathbb{R}^d = \bigcup_{\gamma \in \Gamma} (D + \gamma)$ in the previous construction is not restrictive. Specifically, we only need

$\mathbb{R}^d = \bigcup_{\gamma \in \Gamma} (\alpha D + \gamma)$, where α is any positive real number. Furthermore, the construction remains valid if for each Q_γ we choose a *different* set $\{x_{\gamma,k}\}$ such that $E_\gamma := \{e_{x_{\gamma,k}} \chi_D : \gamma \in \Delta, k \in K\}$ is a frame for $\mathcal{K}_D$, and $m := \inf_\gamma m_\gamma > 0$ and $M := \sup_\gamma M_\gamma < +\infty$, where m_γ and M_γ are the lower and upper frame bounds of E_γ. Finally, note that if $Q = \mathbb{R}^d$ we obtain a frame for $L^2(\mathbb{R}^d)$.

(2) If Q is bounded we can construct frames for K_Q using Theorem 4.5. Let $\delta = \mathrm{diam}(Q)$, and $x_0 \in \mathbb{R}^d$ such that $Q \subseteq B(x_0, \delta)$. Let $X = \{x_k, k \in K\}$ be such that $\rho(X) < \frac{1}{4\delta}$. Then using Beurlings Theorem (4.5) we obtain that $\{e_{x_k} \chi_{B(0,\delta)}, k \in K\}$ is a frame of $K_{B(0,\delta)}$. So, using Corollary 5.2(1), $\{e_{x_k} \chi_{B(x_0,\delta)}, k \in K\}$ is an outer frame of K_Q, and $\{e_{x_k} \chi_Q, k \in K\}$ is a frame for K_Q.

5.3. Existence and construction of Riesz partitions of unity.

In view of the previous results, we will be interested in constructing particular kinds of Riesz partitions of unity associated to special coverings of the space. The next results provide the necessary tools to accomplish this task.

If A is a $d \times d$ matrix, we will say that A is *expansive*, if $|\lambda| > 1$ for every eigenvalue λ of A.

We will use the following known result (see for example [HJ91],pg. 297).

Lemma 5.8. *Let B be in $\mathbb{C}^{d \times d}$ and $\varepsilon > 0$. There exists a matrix norm $\|\|\cdot\|\|$ such that*

$$s(B) \leq \|\|B\|\| \leq s(B) + \varepsilon,$$

where $s(B)$ is the spectral radius of the matrix B, and there exists a norm $\|\cdot\|$ in $\mathbb{C}^d$ such that

$$\|Bx\| \leq \|\|B\|\| \ \|x\|.$$

As a consequence of this lemma, if A is an expansive matrix, then there exists a norm $\|\cdot\|$ in $\mathbb{C}^d$ such that

$$\|A^{-1}x\| \leq c\|x\| \quad 0 < c < 1,$$

and therefore

$$\|Ax\| \geq c'\|x\| \quad c' > 1.$$

In particular for every $x \in \mathbb{C}^d$,

$$(5.8) \qquad \lim_{j \to \infty} \|A^{-j}x\| = 0 \quad \text{and} \quad \text{if } x \neq 0 \quad \lim_{j \to \infty} \|A^j x\| = +\infty.$$

Proposition 5.9. *Let A be a $d \times d$ expansive matrix and $V \subset \mathbb{R}^d$ a bounded set such that*

(i) *there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \cap V = \emptyset$.*
(ii) *$\bigcup_{j \in \mathbb{Z}} A^j V = \mathbb{R}^d \setminus \{0\}$.*

Then $\rho_{A,V}$, the covering index of the family $\{A^j V\}_{j \in \mathbb{Z}}$, is finite, i.e. there exists an integer $n \geq 1$ such that $1 \leq \rho_{A,V} \leq n$.

Proof. For $x \in \mathbb{R}^d$ define

$$z_x^+ = \{j \geq 0 : x \in A^j V\} \quad \text{and} \quad z_x^- = \{j < 0 : x \in A^j V\}.$$

Since $\rho_{A,V} \leq \operatorname{ess\,sup}_x(\#(z_x^+)) + \operatorname{ess\,sup}_x(\#(z_x^-))$, it is enough to prove that $\#(z_x^+)$ and $\#(z_x^-)$ are uniformly bounded in $\mathbb{R}^d$. We will see that $\#(z_x^+)$ is uniformly bounded. A similar argument proves the claim for $\#(z_x^-)$.

It is easy to see that

$$\#(z_x^+) = \#(z_{A^s x}^+) \quad \forall\, s \in \mathbb{Z}, \quad \forall\, x \in \mathbb{R}^d.$$

Thus, if $\#(z_x^+)$ is bounded on V, then, by (ii) $\#(z_x^+)$ is bounded in $\mathbb{R}^d \setminus \{0\}$ with the same bound. Now, using (i), we see that there exist $0 < c_1 < c_2 < +\infty$ such that $c_1 \leq \|x\| \leq c_2$, for all $x \in V$. Fix $x \in V$. If $j \in z_x^+$, then there exists $v \in V$ such that $x = A^j v$ and

$$c_2 \geq \|x\| = \|A^j v\| \geq \frac{\|v\|}{\|A^{-j}\|} \geq \frac{c_1}{\|A^{-j}\|},$$

and therefore $\|A^{-j}\| \geq \frac{c_1}{c_2}$. But since $\|A^{-j}\| \to 0$, necessarily there exists j_0 such that for every $s \geq j_0, \|A^{-s}\| \leq \frac{c_1}{c_2}$. Hence for every $x \in V$, $\#(z_x^+) \leq j_0$. $\qquad \square$

The next proposition shows the construction of a large class of Riesz partitions of unity, for families of sets obtained by dilation of a compact set.

Proposition 5.10. *Let $Q \subset \mathbb{R}^d$ be a compact set and A a $d \times d$ expansive matrix such that*

(i) $0 \notin Q$
(ii) $\bigcup_{j \in \mathbb{Z}} A^j Q = \mathbb{R}^d \setminus \{0\}$.

Let h be any measurable function, and $0 < c_1 \leq c_2 < +\infty$ some constants such that

(a) $0 \leq |h|^2 \leq c_2$
(b) $0 < c_1 \leq |h|^2$ on Q
(c) $h = 0$ on $\mathbb{R}^d \setminus Q_\varepsilon$, where $0 < \varepsilon < d(0,Q)$ and $Q_\varepsilon \equiv \{x \in \mathbb{R}^d : d(x,Q) \leq \varepsilon\}$.

Then the family of functions $\{h_j(\cdot) = h(A^{-j}\cdot)\}_{j \in \mathbb{Z}}$ is a RPU associated to $\{A^j Q_\varepsilon\}$.

Proof. If $x \neq 0$, by (ii) there exists $j \in \mathbb{Z}$ and $q \in Q$ such that $A^j q = x$, so by (b),

$$c_1 \leq |h(q)|^2 = |h(A^{-j}x)|^2. \quad \text{Thus} \quad \sum_{s \in \mathbb{Z}} |h(A^{-s}x)|^2 \geq c_1.$$

Now by Proposition 5.9, the covering index ρ_{A,Q_ε} of the family $\{A^j Q_\varepsilon\}_{j \in \mathbb{Z}}$ is finite. Using (i) and (ii), we see that $\operatorname{Supp}(h_j) \subset A^j Q_\varepsilon$ and since $0 \leq |h_j(x)|^2 \leq c_2 \,\forall\, x$, we obtain that

$$\sum_{j \in \mathbb{Z}} |h_j(x)|^2 \leq \rho_{A,Q_\varepsilon} c_2,$$

which proves the proposition. $\qquad \square$

Remark. This proposition generalizes easily to the case where we replace for each j, A^j by an invertible matrix A_j in such a way that $\{A_j Q\}_{j \in \mathbb{Z}}$ is a covering of $\mathbb{R}^d \setminus \{0\}$ with finite index.

The next Lemma shows, that the assumption of having a compact set that covers $\mathbb{R}^d$ by dilations is actually necessary, if one wants to construct a RPU associated to a covering of $\mathbb{R}^d$ of the form $\mathcal{S} = \{A^j V, j \in \mathbb{Z}\}$, where V is a bounded open set.

Lemma 5.11. *If A is invertible, and V is a bounded open set such that*

    (1) there exists $\varepsilon > 0$ with $B(0, \varepsilon) \cap V = \emptyset$
    (2) $\bigcup_{j \in \mathbb{Z}} A^j V = \mathbb{R}^d \setminus \{0\}$

then there exists a compact set $Q \subset V$ such that $\cup_{j \in \mathbb{Z}} A^j Q \supset \mathbb{R}^d \setminus \{0\}$.

Proof. It is clearly enough to find Q such that $\cup_j A^j Q \supset V$. Let

$$Q_n = \left\{ x \in V : d(x, \partial V) \geq \frac{1}{n} \right\}.$$

We will prove (by the contradiction) that for some $n \geq 1$, Q_n covers V by dilations by A.

Assume that for each $n \geq 1$, there exists $x_n \in V$ such that $x_n \notin \cup_j A^j Q_n$. Since $\overline{V}$ is compact, and $d(x_n, \partial V) < \frac{1}{n}$, there exists a subsequence $\{x_{n_k}\}$ and $x \in \partial V$ such that $x_{n_k} \to x$. By (1) and (2) there exists j_0 such that $x \in A^{j_0} V$, and since $A^{j_0} V$ is open, $x_{n_k} \in A^{j_0} V$ for $k \geq k_0$. Set $y = A^{-j_0} x$ and $y_k = A^{-j_0} x_{n_k}$, then $y, y_k \in V$ for $k \geq k_0$. Choose ε small enough such that $B(y, \varepsilon) \subset V$, and so $d(y, \partial V) \geq \varepsilon$. Let m_0 be such that if $m \geq m_0$, then $y_m \in B(y, \frac{\varepsilon}{2})$.

Then for $z \in B(y, \frac{\varepsilon}{2})$, and $v \in \partial V$ we have

$$\varepsilon - d(y, z) \leq d(y, v) - d(y, z) \leq d(z, v),$$

and thus

$$\frac{\varepsilon}{2} \leq d(z, \partial V).$$

So $B(y, \frac{\varepsilon}{2}) \subset Q_n$ for all n such that $\frac{1}{n} < \frac{\varepsilon}{2}$. This contradicts our assumption that $x_{n_m} = A^{j_0} y_{n_m} \notin Q_{n_m}$, since, if $m \geq m_0$ and $\frac{1}{n_m} < \frac{\varepsilon}{2}$, we have $y_m \in B(y, \frac{\varepsilon}{2}) \subset Q_{n_m}$, and so $x_{n_m} \in A^{j_0} Q_{n_m}$. $\qquad\square$

We show next that it is not difficult to construct bounded sets that cover $\mathbb{R}^d$ by dilations.

Lemma 5.12. *Let $V \subset \mathbb{R}^d$ be a bounded set such that $0 \in V^\circ$, and A an expansive $d \times d$ matrix. Set $Q = AV \setminus V$, then $\{A^j Q, j \in \mathbb{Z}\}$ is a covering of $\mathbb{R}^d \setminus \{0\}$ with finite covering index. Furthermore, if $V \subset AV$ then the sets $\{A^j Q\}$ are disjoint.*

Proof. Choose $\varepsilon > 0$ such that $B(0, \varepsilon) \subset V$. Let $x \in \mathbb{R}^d \setminus \{0\}$ be an arbitrary point. By equation (5.8) $\lim_{j \to \infty} \|A^{-j} x\| = 0$, for some norm $\| \cdot \|$ in $\mathbb{C}^d$.

So there exists a positive integer n such that $\forall j \geq n$, $y := A^{-j} x \in B(0, \varepsilon)$, i.e., $x \in A^j V, \forall j \geq n$.

Since V is bounded, and $\|x\| > 0$, there exists $j_0 \in \mathbb{Z}$ such that $x \in A^{j_0+1} V$ and $x \notin A^j V, \forall j \leq j_0$, i.e., $x \in A^{j_0+1} V \setminus A^{j_0} V = A^{j_0} Q$.

The finiteness of the covering index follows from Proposition 5.9, and if $V \subset AV$ the disjointness property follows immediately. $\qquad\square$

Remark.

- If we want to obtain coverings of $\mathbb{R}^d \setminus \{0\}$ with compact sets, we can choose the set $\overline{Q}$ where Q is the set constructed in the previous Lemma. Clearly the sets $\{A^j\overline{Q} : j \in J\}$ cover $\mathbb{R}^d \setminus \{0\}$, and if $\mu(\partial V) = 0$, then $\{A^j\overline{Q} : j \in J\}$ are almost disjoint, i.e. $\mu(A^j\overline{Q} \cap A^k\overline{Q}) = 0, j \neq k$.
- Even though for *any* set V it is true that $A^j(AV \setminus V) \cap A^{j+1}(AV \setminus V) = \emptyset$, in general it is not true that the family $\{A^j(AV \setminus V)\}$ is disjoint, as the following example shows: Let $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ and V the rectangle $[-3, 3] \times [-1, 1]$.

5.4. **Recipe for constructing smooth wavelet frames using a single dilation matrix and an irregular grid.** We can combine the results of this section with Theorem 3.1, to obtain the following recipe to construct smooth, wavelet frames of $\mathbb{R}^d$ associated to a single dilation matrix A and an irregular grid. The wavelets obtained by this method can be constructed to have polynomial decay of any degree, as exemplified in Section 4.2.

Recipe:

- Select a bounded set $V \subset \mathbb{R}^d$ such that $0 \in V^\circ$ and $\mu(\partial V) = 0$.
- Select a function h of class $C^r, r > 0$ such that $|h| \neq 0$ on $Q = \overline{A^T V \setminus V}$ and $\operatorname{Supp} h \subset Q_\varepsilon$ for some small $\varepsilon > 0$.
- Consider a set $X = \{x_k\}_{k \in K} \subset \mathbb{R}^d$, such that X is separated and $\rho(X) < \frac{1}{2\delta}$, where $\delta = \operatorname{diam}(Q_\varepsilon)$.

Then the following collection is a frame of $L^2(\mathbb{R}^d)$:

$$\{|\det A|^{j/2}\psi(A^j x - x_k) : k \in K, j \in \mathbb{Z}\},$$

where ψ is the inverse Fourier transform of h.

6. RECONSTRUCTION FORMULAS

We first note that although the set $\{|\det A_j|^{1/2}\psi(A_j x - x_{j,k}) : j \in J, k \in K\}$ in Theorem 3.1 is a wavelet frame for $L^2(\mathbb{R}^d)$, it is not in general true that for a fixed j the set $\{\psi_{j,x_{j,k}}(x) = |\det A_j|^{1/2}\psi(A_j x - x_{j,k}) : k \in K\}$ is a frame, unless the wavelet frame is of Shannon-type, hence, not well-localized. Thus, for well-localized wavelets, it appears that the reconstruction of a function $f \in L^2(\mathbb{R}^d)$ from the wavelet coefficients $\{< f, \psi_{j,x_{j,k}} >: j \in J, k \in K\}$ cannot be obtained in a stable way by first reconstructing at each level j and then obtaining f by summing over all levels j. However what follows shows that this is still possible:

For a matrix R, denote by D_R the dilation operator $D_R(f) = |\det R|^{1/2}f \circ R$. Using the notation of Theorem 3.1 we have that

$$< f, \psi_{j,x_{j,k}} >=< \hat{f}, h_j D_{B_j} e_{x_{j,k}} >=< \overline{h}_j\hat{f}, D_{B_j} e_{x_{j,k}} > .$$

This, together with the assumptions that $\{e_{x_{j,k}}\chi_Q : k \in K\}$ is a frame for $\mathcal{K}_Q$ for each j, and that $\operatorname{Supp} h_j \subset S_j$, permits to reconstruct $\hat{f}_j = \overline{h}_j\hat{f}$ using a dual

frame $\{\phi_{j,k} : k \in K\}$ of $\{e_{x_{j,k}}\chi_Q : k \in K\}$. Note that, since D_{B_j} is unitary then $\{D_{B_j}\phi_{j,k} : k \in K\}$ is a dual frame of $\{D_{B_j}(e_{x_{j,k}}\chi_Q) : k \in K\}$ for $\mathcal{K}_{S_j}$.

Thus, it is always possible to reconstruct each f_j in a stable way and then obtain f by summing up over all levels j. One drawback is that the dual frame may not be well-localized, since each $\phi_{j,k}$ may be discontinuous at the boundary of Q. To treat this problem we simply note that

$$\overline{h_j}\hat{f} = \sum_k < \hat{f}, h_j D_{B_j} e_{x_{j,k}} > D_{B_j}\phi_{j,k}.$$

Multiplying both sides by h_j we obtain

$$(6.1) \qquad |h_j|^2 \hat{f} = \sum_k < \hat{f}, h_j D_{B_j} e_{x_{j,k}} > D_{B_j}\theta_{j,k},$$

where $\theta_{j,k} = h\phi_{j,k}$. If we choose h to be in $C^r(\mathbb{R}^d)$, and if $\phi_{j,k}$ is in $C^r(Q)$, then $\theta_{j,k}$ will be in $C^r(\mathbb{R}^d)$ and therefore decays polynomially in space. Hence the partial sums of the series (6.1) will have good convergence properties. We can then sum equation (6.1) over $j \in J$ to obtain $\hat{f}\sum_j |h_j|^2$, and then divide by $\sum_j |h_j|^2$ to obtain $\hat{f}$.

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