

The ideal of p -compact operators: a tensor product approach

by

DANIEL GALICER, SILVIA LASSALLE and PABLO TURCO (Buenos Aires)

Abstract. We study the space of p -compact operators, \mathcal{K}_p , using the theory of tensor norms and operator ideals. We prove that \mathcal{K}_p is associated to $/d_p$, the left injective associate of the Chevet–Saphar tensor norm d_p (which is equal to g'_p). This allows us to relate the theory of p -summing operators to that of p -compact operators. Using the results known for the former class and appropriate hypotheses on E and F we prove that $\mathcal{K}_p(E; F)$ is equal to $\mathcal{K}_q(E; F)$ for a wide range of values of p and q , and show that our results are sharp. We also exhibit several structural properties of \mathcal{K}_p . For instance, we show that \mathcal{K}_p is regular, surjective, and totally accessible, and we characterize its maximal hull \mathcal{K}_p^{\max} as the dual ideal of p -summing operators, Π_p^{dual} . Furthermore, we prove that \mathcal{K}_p coincides isometrically with $\mathcal{QN}_p^{\text{dual}}$, the dual to the ideal of the quasi p -nuclear operators.

Introduction. In 1956, Grothendieck published his famous Résumé [9] in which he set out the basic theory of tensor products of Banach spaces. In the years following, the parallel theory of operator ideals was initiated by Pietsch [12]. Researchers in the field have generally preferred the language of operator ideals to the more abstruse language of tensor products, and so the former theory has received more attention in the succeeding decades. However, the monograph of Defant and Floret [3], in which the two fields are described in tandem, has initiated a period in which authors use indistinctly both languages.

In the recent years, Sinha and Karn [16] introduced the notion of (relatively) p -compact sets. The definition is inspired in Grothendieck's result which characterizes relatively compact sets as those contained in the convex hull of a norm null sequence of vectors of the space. In a similar form, p -compact sets are determined by norm p -summable sequences. Related to this concept, the ideal \mathcal{K}_p of p -compact operators and different approximation properties naturally appear (see definitions below). Since relatively p -compact sets are, in particular, relatively compact, p -compact operators

2010 *Mathematics Subject Classification*: Primary 47L20, 46A32; Secondary 47B07, 47B10.
Key words and phrases: tensor norms, p -compact operators, quasi p -nuclear operators, absolutely p -summing operators, approximation properties.

are compact. These concepts were first studied in [16] and thereafter in several other articles: see for instance [1, 2, 4, 5, 6, 17]. However, we believe that the benefits the space of p -compact operators inherits from the general theory of operator ideals and tensor products have not yet been fully exploited.

The main purpose of this article is to show that the principal properties of the class of p -compact operators can be easily obtained if we study this operator ideal within the theory of tensor products and tensor norms. This insight allows us to give new results, to recover many already known facts, and also to improve some of them.

The paper is organized as follows. In Section 1 we fix some notation and list the classical operator ideals, with their associated tensor norms, which we use thereafter. Section 2 is devoted to general results on p -compact sets and p -compact operators. We define a measure m_p to study the size of a p -compact set K in a Banach space E and show that this measure is independent of whether K is considered as a subset of E or as a subset of E'' , the bidual of E . This allows us to show that \mathcal{K}_p is regular (see definition below). In addition, we prove that \mathcal{K}_p coincides isometrically with $\mathcal{QN}_p^{\text{dual}}$, the dual of the ideal of quasi p -nuclear operators. Finally, we give a factorization for p -compact operators that may be compared with that of p -nuclear operators given in [8].

In Section 3 we use the Chevet–Saphar tensor norm d_p to find the appropriate tensor norm associated to the ideal of p -compact operators. We show that \mathcal{K}_p is associated to the left injective associate of d_p , denoted by $/d_p$, which is equal to g'_p . We use this to link the theory of p -summing operators with that of p -compact operators. Using the results known for the former class and natural hypotheses on E and F we show that $\mathcal{K}_p(E; F)$ and $\mathcal{K}_q(E; F)$ coincide for a wide range of values of p and q . We also use the limit orders of the ideals of p -summing operators [12] to show that our results are sharp. Furthermore, we prove that \mathcal{K}_p is surjective and totally accessible, and we characterize its maximal hull $\mathcal{K}_p^{\text{max}}$ as the dual to the ideal of p -summing operators, Π_p^{dual} .

For the sake of completeness, we list in the Appendix the limit orders of the ideal p -compact operators obtained by a simple transcription of those given in [12] for p -summing operators.

When the final version of this manuscript was being written, we got to know a preprint on the same subject authored by Albrecht Pietsch [13]. The main results in both articles coincide. However, the material in each paper was obtained independently. While A. Pietsch based his work on the classical theory of operator ideals following his monograph [12], we preferred the language of tensor products developed in the book by A. Defant and K. Floret [3].

1. Notation and preliminaries. Throughout this paper, E and F denote Banach spaces, E' and B_E denote respectively the topological dual and the closed unit ball of E . A sequence $(x_n)_n$ in E is said to be p -summable if $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$, and weakly p -summable if $\sum_{n=1}^{\infty} |x'(x_n)|^p < \infty$ for all $x' \in E'$. We denote by $\ell_p(E)$ and $\ell_p^w(E)$, respectively, the spaces of all p -summable and all weakly p -summable sequences in E , $1 \leq p < \infty$. Both spaces are Banach spaces, the first one endowed with the norm

$$\|(x_n)_n\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p}$$

and the second with the norm

$$\|(x_n)_n\|_p^w = \sup_{x' \in B_{E'}} \left\{ \left(\sum_{n=1}^{\infty} |x'(x_n)|^p \right)^{1/p} \right\}.$$

For $p = \infty$, we have the spaces $c_0(E)$ and $c_0^w(E)$ formed, respectively, by all null and all weakly null sequences of E , endowed with the natural norms. The p -convex hull of a sequence $(x_n)_n$ in $\ell_p(E)$ is defined as

$$p\text{-co}\{x_n\} = \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_{p'}} \right\}$$

where $1/p + 1/p' = 1$ ($\ell_{p'} = c_0$ if $p = 1$).

Following [16], we say that a subset $K \subset E$ is *relatively p -compact*, $1 \leq p \leq \infty$, if there exists a sequence $(x_n)_n \subset \ell_p(E)$ so that $K \subset p\text{-co}\{x_n\}$.

The space of bounded linear operators from E to F is denoted by $\mathcal{L}(E; F)$ and its subspace of finite rank operators by $\mathcal{F}(E; F)$. Often the finite rank operator $x \mapsto \sum_{j=1}^n x'_j(x)y_j$ is associated with the element $\sum_{j=1}^n x'_j \otimes y_j$ in $E' \otimes F$. In many cases, the completion of $E' \otimes F$ with a reasonable tensor norm produces a subspace of $\mathcal{L}(E; F)$. For instance the injective tensor product $E' \hat{\otimes}_{\varepsilon} F$ can be viewed as the approximable operators from E to F . The *Chevet–Saphar tensor norm* g_p defined on $E' \otimes F$ by

$$g_p(u) = \inf \left\{ \|(x'_n)_n\|_p \|(y_n)_n\|_{p'}^w : u = \sum_{j=1}^n x'_j \otimes y_j \right\}$$

gives the ideal $\mathcal{N}_p(E; F)$ of p -nuclear operators, $1 \leq p \leq \infty$. If we denote by $x' \underline{\otimes} y$ the 1-rank operator $x \mapsto x'(x)y$, then

$$\mathcal{N}_p(E; F) = \left\{ T = \sum_{n=1}^{\infty} x'_n \underline{\otimes} y_n : (x'_n)_n \in \ell_p(E') \text{ and } (y_n)_n \in \ell_{p'}^w(F) \right\}$$

is a Banach operator ideal endowed with the norm

$$v_p(T) = \inf \left\{ \|(x'_n)_n\|_p \|(y_n)_n\|_{p'}^w : T = \sum_{n=1}^{\infty} x'_n \underline{\otimes} y_n \right\}.$$

It is known that the space of p -nuclear operators is a quotient of $E' \hat{\otimes}_{g_p} F$, and the equality $\mathcal{N}_p(E; F) = E' \hat{\otimes}_{g_p} F$ holds if either E' or F has the approximation property (see [15, Chapter 6]). The definition of g_p is not symmetric, its transpose $d_p = g_p^t$ is associated with the operator ideal

$$\mathcal{N}^p(E; F) = \left\{ T = \sum_{n=1}^{\infty} x'_n \otimes y_n : (x'_n)_n \in \ell_{p'}^w(E') \text{ and } (y_n)_n \in \ell_p(F) \right\},$$

equipped with the norm

$$v^p(T) = \inf \left\{ \|(x'_n)_n\|_{\ell_{p'}^w(E')} \|(y_n)_n\|_{\ell_p(F)} : T = \sum_{n=1}^{\infty} x'_n \otimes y_n \right\}.$$

Here, $\mathcal{N}^p(E; F) = E' \hat{\otimes}_{d_p} F$ if either E' or F has the approximation property. Also, note that when $p = 1$, we obtain $\mathcal{N}_1 = \mathcal{N}^1 = \mathcal{N}$, the ideal of nuclear operators, and $d_1 = g_1 = \pi$, the projective tensor norm.

In this paper, we focus on the study of p -compact operators, introduced by Sinha and Karn [16] as those which map the closed unit ball into a p -compact set. The space of p -compact operators is denoted by $\mathcal{K}_p(E; F)$, $1 \leq p \leq \infty$; it is an operator Banach ideal endowed with the norm

$$\kappa_p(T) = \inf \{ \|(x_n)_n\|_p : T(B_E) \subset p\text{-co}\{x_n\} \}.$$

We want to understand this operator ideal in terms of tensor products and reasonable tensor norms. In order to do so we also make use of the ideal of quasi p -nuclear operators introduced and studied by Persson and Pietsch [14]. The space of quasi p -nuclear operators from E to F is denoted by $\mathcal{QN}_p(E; F)$. This ideal is associated by duality with the ideal of p -compact operators [6].

Recall that an operator T is *quasi p -nuclear* if and only if there exists a sequence $(x'_n)_n \subset \ell_p(E')$ such that

$$\|Tx\| \leq \left(\sum_n |x'_n(x)|^p \right)^{1/p}$$

for all $x \in E$, and the quasi p -nuclear norm of T is given by $v_p^Q(T) = \inf \{ \|(x'_n)_n\|_p \}$, where the infimum is taken over all sequences $(x'_n)_n \in \ell_p(E')$ satisfying the inequality above. It is known that $\mathcal{QN}_p = \mathcal{N}_p^{\text{inj}}$, where $\mathcal{N}_p^{\text{inj}}$ denotes the injective hull of \mathcal{N}_p .

The ideal of p -summing operators, denoted by Π_p , $1 \leq p < \infty$, will play an important role in Section 3. A full description of this operator ideal may be found, for instance, in [3, Section 11], [8, Chapter 2], [15, Section 6.3] and [12, Section 17.3.1].

For general background on tensor products and tensor norms we refer the reader to the monographs by Defant and Floret [3], by Diestel, Fourie and Swart [7], and by Ryan [15]. All the definitions and notation we use

regarding tensor norms and operator ideals can be found in [3]. For further reading on operator ideals we refer the reader to Pietsch's book [12].

2. On p -compact sets and p -compact operators. Given a relatively p -compact set K in a Banach space E there exists a sequence $(x_n)_n \subset E$ so that $K \subset p\text{-co}\{x_n\}$. Such a sequence is not unique, so we may consider the following definition.

DEFINITION 2.1. Let E be a Banach space and $K \subset E$ a p -compact set. For $1 \leq p \leq \infty$, we define

$$m_p(K; E) = \inf\{\|(x_n)_n\|_{\ell_p(E)} : K \subset p\text{-co}\{x_n\}\},$$

If $K \subset E$ is not a p -compact set, $m_p(K; E) = \infty$.

We say that $m_p(K; E)$ measures the size of K as a p -compact set of E .

There are some properties which derive directly from the definition of m_p . For instance, since $p\text{-co}\{x_n\}$ is absolutely convex, $m_p(K; E) = m_p(\overline{\text{co}}\{K\}; E)$. Also, by Hölder's inequality, we have $\|x\| \leq \|(x_n)_n\|_{\ell_p(E)}$, and as a consequence, $\|x\| \leq m_p(K)$ for all $x \in K$. Moreover, as compact sets can be considered p -compact sets for $p = \infty$ we see that any p -compact set is q -compact and $\sup_{x \in K} \|x\| = m_\infty(K; E) \leq m_q(K; E) \leq m_p(K; E)$ for $1 \leq p \leq q \leq \infty$.

Some other properties are less obvious. Suppose that E is a subspace of another Banach space F . It is clear that if $K \subset E$ is p -compact in E then K is p -compact in F and $m_p(K; F) \leq m_p(K; E)$. As we will see in Section 3, the definition of m_p depends on the space E . In other words, K may be p -compact in F but not in E . We show this in Corollary 3.5.

For the particular case when $F = E''$, the bidual of E , Delgado, Piñeiro and Serrano [6, Corollary 3.6] show that a set $K \subset E$ is p -compact if only if K is p -compact in E'' with $m_p(K; E'') \leq m_p(K; E)$. We want to prove that, in fact, the equality $m_p(K; E'') = m_p(K; E)$ holds. To do so we inspect various results concerning operators and their adjoints.

Recall that when E' has the approximation property, any operator $T \in \mathcal{L}(E; F)$ with nuclear adjoint T' is nuclear and both nuclear norms coincide, $v(T) = v(T')$ (see for instance [15, Proposition 4.10]). The analogous result for p -nuclear operators is due to Reinov [11, Theorem 1] and states that when E' has the approximation property and $T' \in \mathcal{N}_p(F'; E')$, then $T \in \mathcal{N}^p(E; F)$. However, the relationship between $v^p(T)$ and $v_p(T')$ is omitted. It is clear that whenever T is in $\mathcal{N}^p(E; F)$ its adjoint is p -nuclear and satisfies $v_p(T') \leq v^p(T)$. Proposition 2.3 below shows that the isometric result is also valid for p -nuclear operators. Before showing this, we need the following result.

PROPOSITION 2.2. Let E and F be Banach spaces, E' with the approximation property, and let $T \in \mathcal{L}(E; F)$. If $J_F T \in \mathcal{N}^p(E; F'')$ then $T \in \mathcal{N}^p(E; F)$ and $v^p(J_F T) = v^p(T)$.

Proof. We only need to show the equality of the norms, as the first part of the assertion corresponds to the first statement of [11, Theorem 1]. Note that since E' has the approximation property, we have $\mathcal{N}^p(E; F) = E' \hat{\otimes}_{d_p} F$ and $\mathcal{N}^p(E; F'') = E' \hat{\otimes}_{d_p} F''$. By the embedding lemma [3, 13.3], $E' \hat{\otimes}_{d_p} F$ is a subspace of $E' \hat{\otimes}_{d_p} F''$ via $\text{id}_{E'} \otimes J_F$. Therefore,

$$\nu^p(J_F T) = \nu^p(T),$$

and the proof is complete. ■

PROPOSITION 2.3. *Let E be a Banach space such that E' has the approximation property and let $1 \leq p < \infty$. If $T \in \mathcal{L}(E; F)$ has p -nuclear adjoint, then $T \in \mathcal{N}^p(E; F)$ and $v^p(T) = v_p(T')$.*

Proof. The first assertion is a direct consequence of [11, Theorem 1]. We only prove the isometric result. Take T as in the statement. Since $T' \in \mathcal{N}_p(F'; E')$, there exist sequences $(y''_n)_n \in \ell_p(F'')$ and $(x'_n)_n \in \ell_p^w(E')$ such that $T' = \sum_{n=1}^{\infty} y''_n \otimes x'_n$. Then $J_F T = T'' J_E = \sum_{n=1}^{\infty} x'_n \otimes y''_n$, which implies that $J_F T \in \mathcal{N}^p(E; F'')$. It is clear that $v_p(T') \geq v^p(J_F T)$. By Proposition 2.2 we have $T \in \mathcal{N}^p(E; F)$ and $v^p(J_F T) = v^p(T)$. The reverse inequality always holds. ■

Now we are ready to prove that the m_p -measure of a p -compact set $K \subset E$ does not change if K is considered as a subset of E'' .

THEOREM 2.4. *Let E be a Banach space and $K \subset E$. Then K is p -compact in E if and only if K is p -compact in E'' , and $m_p(K; E) = m_p(K; E'')$.*

Proof. We only need to show that $m_p(K; E) \leq m_p(K; E'')$ since the claim that K is p -compact in E if and only if K is p -compact in E'' is proved in [6, Corollary 3.6]. Also, in this case, the inequality $m_p(K; E'') \leq m_p(K; E)$ is obvious.

Suppose that $K \subset E$ is p -compact and define the operator $\Psi: \ell_1(K) \rightarrow E$ such that for $\alpha = (\alpha_x)_{x \in K}$,

$$\Psi(\alpha) = \sum_{x \in K} \alpha_x x.$$

Note that $K \subset \Psi(B_{\ell_1(K)}) \subset \overline{\text{co}}(K)$, thus Ψ and $J_E \Psi$ are p -compact operators. Also, $m_p(K; E) = \kappa_p(\Psi)$ and $m_p(K; E'') = \kappa_p(J_E \Psi)$. By [6, Proposition 3.1], $\Psi' J'_E$ belongs to $\mathcal{QN}_p(E'''; \ell_\infty(K))$ and $v_p^Q(\Psi' J'_E) \leq \kappa_p(J_E \Psi)$. Therefore $\Psi' \in \mathcal{QN}_p(E'; \ell_\infty(K))$ and $v_p^Q(\Psi') \leq v_p^Q(\Psi' J'_E)$.

Since $\ell_\infty(K)$ is injective, $\Psi' \in \mathcal{N}_p(E'; \ell_\infty(K))$ and $v_p(\Psi') = v_p^Q(\Psi')$ (see [14, Satz 38]). Now, an application of Proposition 2.3 shows that Ψ is in $\mathcal{N}^p(\ell_1(K); E)$ and $v^p(\Psi) = v_p(\Psi')$. In particular, $\Psi \in \mathcal{K}_p(\ell_1(K); E)$ and $\kappa_p(\Psi) \leq v^p(\Psi)$.

Thus, we have

$$m_p(K; E) = \kappa_p(\Psi) \leq v^p(\Psi) = v_p(\Psi') = v_p^Q(\Psi') \leq v_p^Q(\Psi' J'_E) \leq \kappa_p(J_E \Psi),$$

and the latter is equal to $m_p(K; E'')$, which completes the proof. ■

As an immediate consequence of the theorem above we show that the p -compact operators form a regular ideal. Recall that an operator ideal \mathcal{A} is said to be *regular* if given Banach spaces E, F , an operator T is in $\mathcal{A}(E; F)$ whenever $J_F T \in \mathcal{A}(E; F'')$.

THEOREM 2.5. *The ideal $(\mathcal{K}_p, \kappa_p)$ of p -compact operators is regular.*

Proof. Let E and F be Banach spaces and $T: E \rightarrow F$ be an operator such that $J_F T$ is p -compact. By Theorem 2.4, $m_p(J_F T(B_E); F'') = m_p(T(B_E); F)$ and T is p -compact. Hence, the result follows. ■

Also we obtain the isometric version of [6, Corollary 3.6] which is stated as follows.

COROLLARY 2.6. *Let E and F be Banach spaces. Then $T \in \mathcal{K}_p(E; F)$ if and only if $T'' \in \mathcal{K}_p(E''; F'')$ and $\kappa_p(T) = \kappa_p(T'')$.*

Proof. The statement that $T \in \mathcal{K}_p(E; F)$ if and only if $T'' \in \mathcal{K}_p(E''; F'')$ is part of [6, Corollary 3.6]. Let T be a p -compact operator. In particular, $T(B_E)$ is relatively compact and

$$J_F T(B_E) \subset T''(B_{E''}) \subset \overline{J_F T(B_E)}^{w*} = \overline{J_F T(B_E)}.$$

Applying Theorem 2.4 twice we get

$$\begin{aligned} m_p(T(B_E); F) &= m_p(T(B_E); F'') \leq m_p(T''(B_{E''}); F'') \\ &\leq m_p(\overline{J_F T(B_E)}; F'') = m_p(T(B_E); F). \end{aligned}$$

Since $\kappa_p(T) = m_p(T(B_E); F)$ and $\kappa_p(T'') = m_p(T''(B_{E''}); F'')$, the isometry is proved. ■

Now, we describe the duality between p -compact and quasi p -nuclear operators. On the one hand, an operator T is quasi p -nuclear if and only if T' is p -compact, and $\kappa_p(T') = v_p^Q(T)$ [6, Corollary 3.4]. On the other hand, T is p -compact if and only if its adjoint T' is quasi p -nuclear, and in this case $v_p^Q(T') \leq \kappa_p(T)$ [6, Proposition 3.8]. We improve this last result by showing the equality of the norms.

COROLLARY 2.7. *Let E and F be Banach spaces. Then $T \in \mathcal{K}_p(E; F)$ if and only if $T' \in \mathcal{QN}_p(F'; E')$, and $\kappa_p(T) = v_p^Q(T')$.*

Proof. The inequality $v_p^Q(T') \leq \kappa_p(T)$ and the equality $\kappa_p(T'') = v_p^Q(T')$ always hold. A direct application of Corollary 2.6 completes the proof. ■

Corollary 2.7 and the results mentioned above yield the following identities.

THEOREM 2.8. $\mathcal{K}_p^{\text{dual}} \stackrel{1}{=} \mathcal{QN}_p$ and $\mathcal{QN}_p^{\text{dual}} \stackrel{1}{=} \mathcal{K}_p$.

We finish this section with a factorization result for p -compact operators, which improves [16, Theorem 3.2] and [2, Theorem 3.1]. The characterization given below should be compared with [8, Proposition 5.23].

PROPOSITION 2.9. *Let E and F be Banach spaces. Then an operator $T \in \mathcal{L}(E; F)$ is p -compact if and only if T admits a factorization $T = ST_0R$ where T_0 is a p -compact operator, and R and S are compact.*

Moreover, $\kappa_p(T) = \inf\{\|S\|\kappa_p(T_0)\|R\|\}$ where the infimum is taken over all factorizations as above.

Proof. Suppose that $T \in \mathcal{K}_p(E; F)$. Then, given $\varepsilon > 0$, there exists $y = (y_n)_n \in \ell_p(F)$ such that $T(B_E) \subset p\text{-co}\{y_n\}$ with $\|(y_n)_n\|_p \leq \kappa_p(T)(1 + \varepsilon)$. We may choose $\beta = (\beta_n)_n \in B_{c_0}$ such that $z = (z_n)_n = (y_n/\beta_n)_n \in \ell_p(F)$ and $\|(z_n)_n\|_p \leq \|(y_n)_n\|_p(1 + \varepsilon)$. Now, $T(B_E) \subset \{\sum_{n=1}^{\infty} \alpha_n z_n : (\alpha_n)_n \in L\}$ where L is a compact set in $B_{\ell_{p'}}$. By the factorization in [16, Theorem 3.2], we have the commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{T} & F & \xleftarrow{\theta_z} & \ell_{p'} \\ & \searrow R & \uparrow \tilde{\theta}_z & \swarrow \pi & \\ & & \ell_{p'}/\ker \theta_z & & \end{array}$$

where π is the projection mapping, and θ_z and R are given by $\theta_z((\alpha_n)_n) = \sum_{n=1}^{\infty} \alpha_n z_n$ and $R(x) = [(\alpha_n)_n]$ where $(\alpha_n)_n \in L$ is a sequence satisfying $T(x) = \sum_{n=1}^{\infty} \alpha_n z_n$. Since $R(B_E) = \pi(L)$, we see that R is compact and $T = \tilde{\theta}_z R$.

Note also that $\tilde{\theta}_z$ is p -compact. Since $\|R\| \leq 1$, we have

$$\kappa_p(T) \leq \kappa_p(\tilde{\theta}_z) \leq \|(z_n)_n\|_p \leq \kappa_p(T)(1 + \varepsilon)^2.$$

Now, using [2, Theorem 3.1] we factorize $\tilde{\theta}_z$ via a p -compact operator T_0 and a compact operator S , as follows:

$$\begin{array}{ccc} \ell_{p'}/\ker \theta_z & \xrightarrow{\tilde{\theta}_z} & F \\ & \searrow T_0 & \nearrow S \\ & & \ell_1/M \end{array}$$

where M is a closed subspace of ℓ_1 . A close inspection of the proof in [2] allows us to choose the factorization with $\kappa_p(\tilde{\theta}_z) \leq \|S\|\kappa_p(T_0) \leq (1 + \varepsilon)\kappa_p(\tilde{\theta}_z)$ (just consider a sequence $(\beta_n)_n$ similar to that used above). Hence, the factorization satisfies the desired equality $\kappa_p(T) = \inf\{\|S\|\kappa_p(T_0)\|R\|\}$.

The converse is obvious. ■

Note that if both E' and F have the approximation property then $T \in \mathcal{K}_p(E; F)$ if and only if $T \in \mathcal{K}_p^{\min}(E; F)$. Moreover, $\kappa_p(T) = \kappa_p^{\min}(T)$. We show in the next section that the same result holds if only one of the spaces (E' or F) has the approximation property.

3. Tensor norms. Our purpose in this section is to bring together the theory of operator ideals and tensor products for the class of p -compact operators. To start with, we use the Chevet–Saphar tensor norm to find the appropriate tensor norm associated to the ideal of p -compact operators. The tensor norm obtained is $g'_{p'}$, which allows us to connect the theory of p -summing operators with that of p -compact operators. Using the results known for the former class, under additional hypotheses on E and F we show that $\mathcal{K}_p(E; F)$ and $\mathcal{K}_q(E; F)$ coincide for a wide range of p and q . We also use the limit orders of the ideal of p -summing operators [12] to show that the values considered for p and q cannot be improved. Some other properties describing the structure of the ideal of p -compact operators are given.

Recall that $d_p(u) = \inf\{\|(x_n)_n\|_{p'}^w \|(y_n)_n\|_p\}$ where the infimum is taken over all the possible representations of $u = \sum_{j=1}^n x_j \otimes y_j$. We denote by $/d_p$ the left injective associate of the tensor norm to d_p . Note that $/d_p = g'_{p'}$ [15, Theorem 7.20] and therefore $/d_p = (g_{p'}^*)^t$.

PROPOSITION 3.1. *The ideal $(\mathcal{K}_p, \kappa_p)$ of p -compact operators is surjective.*

Proof. Let $Q: G \xrightarrow{1} E$ be a quotient map. If TQ is p -compact, then $TQ(B_G) = T(B_E)$ is a p -compact set. Thus, T is p -compact and

$$\kappa_p(TQ) = m_p(TQ(B_G)) = m_p(T(B_E)) = \kappa_p(T).$$

This completes the proof. ■

In order to characterize the tensor norm associated to $(\mathcal{K}_p, \kappa_p)$ we need the following simple lemma. We sketch its proof for completeness. This result should be compared with [3, Theorem 20.11].

LEMMA 3.2. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be an operator ideal and let α be its associated tensor norm.*

- (a) *If \mathcal{A} is surjective, then α is left injective.*
- (b) *If \mathcal{A} is injective, then α is right injective.*

Proof. Suppose \mathcal{A} is surjective. Using the ‘left version’ of [3, Proposition 20.3(1)], we only need to see that α is left injective on FIN, the class of all finite-dimensional spaces.

Fix $N, M, W \in \text{FIN}$ such that $i: M \xrightarrow{1} W$. Then we have the commutative diagram

$$\begin{array}{ccc}
M \otimes_{\alpha} N & \xrightarrow{i \otimes \text{id}_N} & W \otimes_{\alpha} N \\
\parallel & & \parallel \\
\mathcal{A}(M'; N) & \xrightarrow{\phi} & \mathcal{A}(W'; N)
\end{array}$$

where ϕ is given by $T \mapsto Ti'$. As i is an isometry, i' is a metric surjection. Now, since \mathcal{A} is surjective, ϕ is an isometry, which proves (a).

The proof of (b) follows easily by a similar reasoning. ■

From [6, Proposition 3.11] we have $\mathcal{N}^p(\ell_1^n; N) \stackrel{1}{=} \mathcal{K}_p(\ell_1^n; N)$ for every n and every finite-dimensional space N . Since \mathcal{N}^p is associated to the tensor norm d_p , we have the following result.

THEOREM 3.3. *The ideal $(\mathcal{K}_p, \kappa_p)$ of p -compact operators is associated to the tensor norm $/d_p$ for every $1 \leq p < \infty$.*

Proof. Denote by α the tensor norm associated to \mathcal{K}_p . By Proposition 3.1 and the above lemma, α is left injective. Note that for every n and every finite-dimensional space N we have the isometric identities

$$\ell_{\infty}^n \otimes_{d_p} N = \mathcal{N}^p(\ell_1^n; N) = \mathcal{K}_p(\ell_1^n; N) = \ell_{\infty}^n \otimes_{\alpha} N.$$

Now, applying the ‘left version’ of [3, Proposition 20.9], we conclude that $\alpha = /d_p$. ■

PROPOSITION 3.4. *The ideal $(\mathcal{K}_p, \kappa_p)$ is not injective for any $1 \leq p < \infty$.*

Proof. Suppose that \mathcal{K}_p is injective. By Theorem 3.3 and Lemma 3.2 we see that $/d_p$, the associated tensor norm for \mathcal{K}_p , is right injective. Thus, its transpose g_p^* is left injective. Now, by [3, Theorem 20.11], Π_p is surjective, which is a contradiction. Note that, by Grothendieck’s theorem [3, Theorem 23.10], $\text{id}: \ell_2 \rightarrow \ell_2$ belongs to Π_p^{sur} and obviously is not p -summing. ■

As a consequence we show that the m_p -measure of a set depends on the space which contains the set.

COROLLARY 3.5. *Given $1 \leq p < \infty$, there exist a Banach space G , a subspace $F \subset G$ and a set $K \subset F$ such that K is p -compact in G but K fails to be p -compact in F .*

Proof. Since $(\mathcal{K}_p, \kappa_p)$ is not injective, there exist Banach spaces E , F and G with $F \xrightarrow{I_{F,G}} G$ and an operator $T \in \mathcal{L}(E; F)$ such that $I_{F,G}T$ is p -compact but T is not. Taking $K = T(B_E)$, we see that $m_p(K; G) < \infty$ while $m_p(K; F) = \infty$. ■

Now we characterize $\mathcal{K}_p^{\text{max}}$, the maximal hull of the operator ideal \mathcal{K}_p , in terms of the ideal Π_p of p -summing operators.

COROLLARY 3.6. *The operator ideal \mathcal{K}_p^{\max} coincides isometrically with Π_p^{dual} .*

Proof. The maximal hull of \mathcal{K}_p is also associated to the tensor norm $/d_p = (g_p^*)^t$. Since the ideal Π_p is associated to the tensor norm g_p^* , by Corollary 3 in [3, 17.8] the result follows. ■

Now we are in a position to show that $E' \widehat{\otimes}_{/d_p} F$ coincides with $\mathcal{K}_p^{\min}(E; F)$ for any Banach spaces E and F . For this we need the notion of totally accessible tensor norm and operator ideal.

Recall that a tensor norm is called *totally accessible* if it is finitely generated and cofinitely generated [3, 15.6]. An operator ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is *totally accessible* if for every finite rank operator $T \in \mathcal{L}(E; F)$ and $\varepsilon > 0$ there exist a finite-dimensional subspace $M \subset F$, a finite-codimensional subspace $L \subset E$ and $S \in \mathcal{L}(E/L; M)$ such that $T = I_F S Q_E$ and $\|S\|_{\mathcal{A}} \leq (1 + \varepsilon) \|T\|_{\mathcal{A}}$, where $Q_E: E \rightarrow E/L$ and $I_F: M \rightarrow F$ are the canonical quotient mapping and the inclusion, respectively [3, 21.2].

A finitely generated tensor norm α is totally accessible if and only if its associated maximal Banach ideal is [3, Proposition 21.3]. By [3, Proposition 21.1(3)] and the fact that $/(d_p/) = /d_p$ we see that the tensor norms $/d_p$ are totally accessible (see also [15, Corollary 7.15]). Therefore, we have the following two results. For the first one we use [3, Proposition 21.3] and for the second we use [3, Corollary 22.2].

REMARK 3.7. The operator ideal $\mathcal{K}_p^{\max} \stackrel{1}{=} \Pi_p^{\text{dual}}$ is totally accessible.

REMARK 3.8. For any Banach spaces E and F , $\mathcal{K}_p^{\min}(E; F) \stackrel{1}{=} E' \widehat{\otimes}_{/d_p} F$.

With the help of Corollary 3.6 we obtain an alternative way to compute the κ_p norm of a p -compact operator: just take the p -summing norm of its adjoint. Moreover, the same holds for the minimal norm. We also have the following isometric relations.

PROPOSITION 3.9. *There are isometric inclusions*

$$\mathcal{K}_p^{\min} \xrightarrow{1} \mathcal{K}_p \xrightarrow{1} \mathcal{K}_p^{\max} \stackrel{1}{=} \Pi_p^{\text{dual}}.$$

In particular, \mathcal{K}_p^{\min} and \mathcal{K}_p are totally accessible.

Proof. Let E and F be Banach spaces. We have

$$\mathcal{K}_p^{\min}(E; F) \stackrel{\leq 1}{\hookrightarrow} \mathcal{K}_p(E; F) \stackrel{\leq 1}{\hookrightarrow} \mathcal{K}_p^{\max}(E; F) \stackrel{1}{=} \Pi_p^{\text{dual}}(E; F).$$

Now, using the previous remark and [3, Corollary 22.5], we conclude that $\mathcal{K}_p^{\min}(E; F) \xrightarrow{1} \mathcal{K}_p^{\max}(E; F) \stackrel{1}{=} \Pi_p^{\text{dual}}(E; F)$, which implies that all the inclusions above are isometries. ■

The definition of the κ_p -approximation property in [5] was given in terms of operators: a Banach space F has the κ_p -approximation property if, for every Banach space E , $\mathcal{F}(E; F)$ is κ_p -dense in $\mathcal{K}_p(E; F)$. In other words,

$$\overline{\mathcal{F}(E; F)}^{\kappa_p} \stackrel{1}{=} \mathcal{K}_p(E; F).$$

On the other hand, by Remark 3.7, [3, Corollary 22.5] and the previous proposition we have

$$\mathcal{K}_p^{\min}(E; F) \stackrel{1}{=} \overline{\mathcal{F}(E; F)}^{\mathcal{K}_p^{\max}} \stackrel{1}{=} \overline{\mathcal{F}(E; F)}^{\kappa_p}.$$

Therefore, F has the κ_p -approximation property if and only if the equality $\mathcal{K}_p^{\min}(E; F) \stackrel{1}{=} \mathcal{K}_p(E; F)$ holds for every Banach space E .

Any Banach space with the approximation property enjoys the κ_p -approximation property for $1 \leq p < \infty$. This result can be deduced from [5, Theorem 3.1]. Below, we give a short proof using the language of operator ideals. It is worth mentioning that every Banach space has the κ_2 -approximation property (which can be deduced from [16, Theorem 6.4]) and for each $p \neq 2$ there exists a Banach space whose dual lacks the κ_p -approximation property [5, Theorem 2.4].

PROPOSITION 3.10. *If a Banach space has the approximation property then it has the κ_p -approximation property.*

Proof. We have shown that a Banach space F has the κ_p -approximation property if and only if $\mathcal{K}_p^{\min}(E; F) \stackrel{1}{=} \mathcal{K}_p(E; F)$ for every Banach space E . Suppose that F has the approximation property and let $T \in \mathcal{K}_p(E; F)$. Using [2, Theorem 3.1] we have a factorization

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ & \searrow T_0 & \nearrow S \\ & & G \end{array}$$

where T_0 is p -compact and S is compact (therefore approximable). Now, by [3, Proposition 25.2(1) b)], $T \in \mathcal{K}_p^{\min}(E; F)$, which concludes the proof. ■

Note that, in general, the converse of Proposition 3.10 is not true. For instance, if $1 \leq p < 2$, we may always find a subspace E of ℓ_q , $1 < q < 2$, without the approximation property. This subspace is reflexive and has cotype 2. Then, by the comment following [3, Proposition 21.7] one can apply [6, Corollary 2.5] to show that $F = E'$ has the κ_p -approximation property and fails to have the approximation property.

In this setting, the next theorem becomes quite natural. It states that the ideal of p -compact operators can be represented in terms of tensor products in the presence of the κ_p -approximation property.

THEOREM 3.11. *Let E and F be Banach spaces. Then*

$$E' \widehat{\otimes}_{/d_p} F \stackrel{1}{=} \mathcal{K}_p(E; F)$$

if and only if F has the κ_p -approximation property. Also, the isometry remains valid whenever E' has the approximation property, regardless of F .

Proof. Note that $/d_p$ is totally accessible (see the comments preceding Remark 3.7). Thus, the proof of the first claim is straightforward from Remark 3.8.

For the second statement, take $T \in \mathcal{K}_p(E; F)$. By Proposition 2.9, $T = T_0 R$ where R is a compact operator and T_0 is p -compact. Now, using the hypothesis that E' has the approximation property, R is approximable by finite rank operators and an application of [3, Proposition 25.2(2) b] gives that $T \in \mathcal{K}_p^{\min}(E; F)$. Again, the result follows by Remark 3.8. ■

Under natural conditions on E or F , we characterize the dual of $\mathcal{K}_p(E; F)$ in terms of the ideal \mathcal{I}_p of p -integral operators. The basic theory of p -integral operators may be found in [8, Chapter 5], [12, Section 19.2.1] and [15, Section 7.3]; see also [3, Sections 17.10–13] and [7, Section 1.4]. The next result improves [5, Proposition 3.3].

COROLLARY 3.12. *Let E and F be Banach spaces such that F has the κ_p -approximation property or E' has the approximation property. Then $\mathcal{K}_p(E; F)' \stackrel{1}{=} \mathcal{I}_{p'}(E'; F')$, $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$.*

Proof. The proof is straightforward from Theorem 3.11 and [15, p. 174]. See also [7, Section 1.4]. ■

In what follows, we compare p -compact and q -compact operators for certain classes of Banach spaces. We use some well known results stated for p -summing operators when the spaces involved are of finite cotype or $\mathcal{L}_{q,\lambda}$ -spaces for some q . Our results are stated in terms of $\mathcal{K}_p^{\min}(E; F)$, but if F has the κ_p -approximation property or E' has the approximation property, then by Theorem 3.11, they can be stated for $\mathcal{K}_p(E; F)$. First we need the following general result. As usual, for $s = \infty$, we consider $\mathcal{L}(X; Y)$ instead of $\Pi_s(X; Y)$, and $\overline{\mathcal{F}(Y; X)}$ instead of $\mathcal{K}_s^{\min}(Y; X)$.

THEOREM 3.13. *Let E and F be Banach spaces such that $\Pi_r(F'; E') = \Pi_s(F'; E')$ for some $1 \leq r < s \leq \infty$. Then $\mathcal{K}_s^{\min}(E; F) = \mathcal{K}_r^{\min}(E; F)$. Moreover, if $\pi_r(\cdot) \leq A\pi_s(\cdot)$ on $\Pi_s(F'; E')$ then $\kappa_r(\cdot) \leq A\kappa_s(\cdot)$ on $\mathcal{K}_s^{\min}(E; F)$.*

Proof. Since $\Pi_r(F'; E') = \Pi_s(F'; E')$ and Π_r is a maximal ideal, and since its associated tensor norm $g_{r'}^*$ is totally accessible [3, Corollary 21.1], by the embedding theorem [3, 17.6] we have $F'' \widehat{\otimes}_{g_{r'}^*} E' \xrightarrow{1} \Pi_r(F'; E')$. Now, using the embedding lemma [3, 13.3] we have the following commutative diagram:

$$\begin{array}{ccccc}
E' \widehat{\otimes}_{/d_s} F = F \widehat{\otimes}_{g_s^*} E' & \xrightarrow{1} & F'' \widehat{\otimes}_{g_s^*} E' & \xrightarrow{1} & \Pi_s(F'; E') \\
\downarrow \leq A & & & & \downarrow \leq A \\
E' \widehat{\otimes}_{/d_r} F = F \widehat{\otimes}_{g_r^*} E' & \xrightarrow{1} & F'' \widehat{\otimes}_{g_r^*} E' & \xrightarrow{1} & \Pi_r(F'; E')
\end{array}$$

Therefore, $/d_s \leq /d_r \leq A /d_s$ on $E' \otimes F$, which implies that $\mathcal{K}_s^{\min}(E; F) = \mathcal{K}_r^{\min}(E; F)$ and $\kappa_r(T) \leq A \kappa_s(T)$ for every $T \in \mathcal{K}_s^{\min}(E; F)$. ■

In order to compare the norm $\kappa_r(T)$ with $\|T\|$ or with $\kappa_s(T)$, we use the constants obtained in comparing summing operators, taken from [18]. Some of them involve the Grothendieck constant K_G , the constant B_r taken from Khintchine's inequality, and $C_q(E)$, the q -cotype constant of E . With this notation and the theorem above we have the following results.

COROLLARY 3.14. *Let E and F be Banach spaces such that E is an $\mathcal{L}_{2,\lambda'}$ -space and F is an $\mathcal{L}_{\infty,\lambda}$ -space. Then $\overline{\mathcal{F}(E; F)} = \mathcal{K}_1^{\min}(E; F)$ and $\kappa_1(T) \leq K_G \lambda \lambda' \|T\|$ for every $T \in \mathcal{F}(E; F)$.*

Proof. Note that E is an $\mathcal{L}_{2,\lambda'}$ -space if and only if E' is an $\mathcal{L}_{2,\lambda'}$ -space, and F is an $\mathcal{L}_{\infty,\lambda}$ -space if and only if F' is an $\mathcal{L}_{1,\lambda}$ -space (see [3, 23.2 Corollary 1] and [3, 23.3]). Now, use Theorem 3.13, with [3, Theorem 23.10] or [18, Theorem 10.11]. ■

COROLLARY 3.15. *Let E and F be Banach spaces such that F is an $\mathcal{L}_{1,\lambda}$ -space.*

- (a) *If E' has cotype 2, then $\overline{\mathcal{F}(E; F)} = \mathcal{K}_2(E; F) = \mathcal{K}_r^{\min}(E; F)$ for all $r \geq 2$, and*

$$\kappa_r(T) \leq \lambda [c C_2(E')^2 (1 + \log C_2(E'))]^{1/r} \|T\|$$

for all $T \in \mathcal{K}_r^{\min}(E; F)$.

- (b) *If E' has cotype q , $2 < q < \infty$, then $\overline{\mathcal{F}(E; F)} = \mathcal{K}_r^{\min}(E; F)$ for all $q < r < \infty$, and*

$$\kappa_r(T) \leq \lambda c q^{-1} (1/q - 1/r)^{-1/r'} C_q(E') \|T\|$$

for all $T \in \mathcal{K}_r^{\min}(E; F)$.

In each case, $c > 0$ is a universal constant.

Proof. Again, F is an $\mathcal{L}_{\infty,\lambda}$ -space if and only if F' is an $\mathcal{L}_{1,\lambda}$ -space. For the first statement, note that every space has the κ_2 -approximation property, $\mathcal{K}_2^{\min}(E; F) = \mathcal{K}_2(E; F)$, and use Theorem 3.13 combined with [18, Theorem 10.14 and Proposition 10.16]. For the second claim, use Theorem 3.13 and [18, Theorem 21.4(ii)]. ■

COROLLARY 3.16. *Let E and F be Banach spaces.*

- (a) *If E' has cotype 2, then $\mathcal{K}_r^{\min}(E; F) = \mathcal{K}_2(E; F)$ for all $2 \leq r < \infty$, and*

$$\kappa_2(T) \leq B_r C_2(E') \kappa_r(T)$$

for every $T \in \mathcal{K}_r^{\min}(E; F)$.

- (b) *If F' has cotype 2, then $\mathcal{K}_2^{\min}(E; F) = \mathcal{K}_1^{\min}(E; F)$ for all E , and*

$$\kappa_1(T) \leq c C_2(F') (1 + \log C_2(F'))^{1/2} \kappa_2(T)$$

for every $T \in \mathcal{K}_2^{\min}(E; F)$. In particular, for all $1 \leq r \leq 2$, $\mathcal{K}_r^{\min}(E; F) = \mathcal{K}_1^{\min}(E; F)$ for all E .

- (c) *If F' has cotype q , $2 < q < \infty$, then $\mathcal{K}_r^{\min}(E; F) = \mathcal{K}_1^{\min}(E; F)$ for all $1 \leq r < q'$ and E , and*

$$\kappa_1(T) \leq c q^{-1} (1/q - 1/r')^{-1/r} C_q(F') \kappa_r(T)$$

for every $T \in \mathcal{K}_r^{\min}(E; F)$.

In each statement, $c > 0$ is a universal constant. Note that if E' and F' have cotype 2, then $\mathcal{K}_r^{\min}(E; F) = \mathcal{K}_1^{\min}(E; F)$ for all $1 \leq r < \infty$.

Proof. Use Theorem 3.13 and [18, Theorem 10.15] for part (a). For (b) use [18, Corollary 10.18(a)]. Finally, use Theorem 3.13 together with [18, Corollary 21.5(i)] for the third claim. ■

We finish this section by showing that the conditions on r in the corollaries above are sharp. We make use of the notion of limit order [12, Chapter 14], which has proved useful, specially to compare different operator ideals. Recall that for an operator ideal \mathcal{A} , the limit order $\lambda(\mathcal{A}, u, v)$ is defined to be the infimum of all $\lambda \geq 0$ such that the diagonal operator D_λ belongs to $\mathcal{A}(\ell_u; \ell_v)$, where $D_\lambda: (a_n) \mapsto (n^{-\lambda} a_n)$ and $1 \leq u, v \leq \infty$.

LEMMA 3.17. *Let $1 \leq u, v, p \leq \infty$ and u', v', p' the respective conjugates. Then*

$$\lambda(\mathcal{K}_p, u, v) = \lambda(\Pi_p, v', u').$$

Proof. Denote by $\text{id}_{u,v}$ the identity map from ℓ_u^n to ℓ_v^n , for a fixed integer n . By Corollary 3.6 we have

$$\kappa_p(\text{id}_{u,v}: \ell_u^n \rightarrow \ell_v^n) = \pi_p(\text{id}_{v',u'}: \ell_{v'}^n \rightarrow \ell_{u'}^n).$$

Then a direct application of [12, Theorem 14.4.3] gives the result. ■

A direct transcription of the values of the limit orders $\lambda(\Pi_r, v', u')$, computed in Pietsch's monograph, gives the values of $\lambda(\mathcal{K}_r, u, v)$: just use a combination of Propositions 22.4.9, 22.4.12 and 22.4.13 in [12]. Now we have:

RESULT 3.18. *The conditions imposed on r in Corollaries 3.15 and 3.16 are sharp.*

Proof. (1) Let $E = \ell_u$ and $F = \ell_1$. Note that (see Appendix, (a) and (b))

$$\lambda(\mathcal{K}_r, u, 1) = \begin{cases} 1 - 1/u & \text{if } r' \leq u \leq \infty, \\ 1/r & \text{if } 1 \leq u \leq r'. \end{cases}$$

For fixed $1 \leq r < 2$ choose $2 < u < r'$. Then E' has cotype 2 and $\lambda(\mathcal{K}_r, u, 1) = 1/r \neq 1/u' = \lambda(\mathcal{K}_2, u, 1)$. Thus, $\mathcal{K}_r(\ell_u; \ell_1) \neq \mathcal{K}_2(\ell_u; \ell_1)$ and so r cannot be included in Corollary 3.15(a).

Now, fix $q > 2$ and let $E = \ell_{q'}$. Then E' has cotype q and given $r < q$, we see that $\lambda(\mathcal{K}_r, q', 1) = 1/r$. On the other hand, $\lambda(\mathcal{K}_s, q', 1) = 1/q$ for any $q < s$. This shows that $\mathcal{K}_r(\ell_{q'}; \ell_1) \neq \mathcal{K}_s(\ell_{q'}; \ell_1)$ for any $r < q < s$.

Note that we have also shown that if $r < \tilde{r} \leq q$, then $\lambda(\mathcal{K}_{\tilde{r}}, q', 1) \neq \lambda(\mathcal{K}_r, q', 1)$. Therefore, the inclusions $\mathcal{K}_{\tilde{r}}(\ell_{q'}, \ell_1) \subset \mathcal{K}_r(\ell_{q'}, \ell_1)$ are strict for any $r < \tilde{r} \leq q$.

For the case $r = q$, $2 < q < \infty$, take $E = L_{q'}[0, 1] = L_{q'}$ and $F = L_1[0, 1] = L_1$. Suppose that $\overline{\mathcal{F}(L_{q'}; L_1)} = \mathcal{K}_q(L_{q'}; L_1)$. By Theorem 3.11, $L_q \widehat{\otimes}_{/d_q} L_1 = L_q \widehat{\otimes}_\varepsilon L_1$ and $L_1 \widehat{\otimes}_{(/d_q)^t} L_q = L_1 \widehat{\otimes}_\varepsilon L_q$. Since $(/d_q)^t = (d_{q'})'$ and $\pi' = \varepsilon$, we have $L_1 \widehat{\otimes}_{d_{q'}} L_q = L_1 \widehat{\otimes}_{\pi'} L_q$ and so $(L_1 \widehat{\otimes}_{d_{q'}} L_q)' = (L_1 \widehat{\otimes}_{\pi'} L_q)'$. Since both L_∞ and $L_{q'}$ have the metric approximation property, by [3, 17.7] and [3, 12.4] we have the isomorphism $L_\infty \widehat{\otimes}_{d_{q'}} L_{q'} = L_\infty \widehat{\otimes}_\pi L_{q'}$. Therefore $(L_\infty \widehat{\otimes}_{d_{q'}} L_{q'})' = (L_\infty \widehat{\otimes}_\pi L_{q'})'$. In other words, $\Pi_q(L_\infty, L_q) = \mathcal{L}(L_\infty, L_q)$ (see [15, Section 6.3]), which contradicts [10, Theorem 7].

(2) For any $1 \leq p < \infty$, there exists a compact operator in $\mathcal{L}(\ell_p; \ell_p)$ (and therefore approximable) which is not p -compact [1, Example 3.1]. Thus, $\overline{\mathcal{F}(\ell_p; \ell_p)} \neq \mathcal{K}_p(\ell_p; \ell_p)$.

Fix $p \geq 2$ for all $2 \leq r < \infty$, we see that $\mathcal{K}_r^{\min}(\ell_p; \ell_p) = \mathcal{K}_p^{\min}(\ell_p; \ell_p) = \mathcal{K}_p(\ell_p; \ell_p) = \mathcal{K}_2(\ell_p; \ell_p)$, by Corollary 3.16(a). Thus, $r = \infty$ cannot be included in the first statement of this corollary.

Also, for $r < 2$, we may choose p and q such that $2 \leq p \leq r'$ and $1 \leq q \leq r$. Now, with $E = \ell_p$ and $F = \ell_q$ using the limit orders (see Appendix) we obtain $\lambda(\mathcal{K}_r, p, q) = 1/r$ and $\lambda(\mathcal{K}_2, p, q) = 1/2$, and conclude that the inclusion $\mathcal{K}_r(\ell_p; \ell_q) \subset \mathcal{K}_2(\ell_p; \ell_q)$ is strict.

(3) To see that the choice of r in Corollary 3.16(b) is sharp, fix $r > 2$. Take p and q such that $2 \leq q < r$ and $1 \leq p \leq r'$. Let $E = \ell_p$ and $F = \ell_{q'}$; the limit orders satisfy $\lambda(\mathcal{K}_2, p, q') = 1/2$ and $\lambda(\mathcal{K}_r, p, q') = 1/r$ (see Appendix, (b)). Thus, $\mathcal{K}_2(\ell_p; \ell_{q'}) \neq \mathcal{K}_r(\ell_p; \ell_{q'})$.

Here, we have also shown that for any r and \tilde{r} such that $2 \leq r < \tilde{r}$, the inclusion $\mathcal{K}_{\tilde{r}}(\ell_p; \ell_{q'}) \subset \mathcal{K}_r(\ell_p; \ell_{q'})$ is strict for suitable p and q .

(4) Now, we turn to Corollary 3.16(c). Fix $q > 2$ and let $E = \ell_1$ and $F = \ell_{q'}$. We claim that $\mathcal{K}_r(\ell_1, \ell_{q'}) \neq \mathcal{K}_1(\ell_1, \ell_{q'})$ for any $q' < r$. In fact, this follows by using the limit orders: $\lambda(\mathcal{K}_1, 1, q') = 1/q'$ and $\lambda(\mathcal{K}_r, 1, q') = 1/r$.

This also shows that $\mathcal{K}_{\tilde{r}}(\ell_1; \ell_{q'})$ is strictly contained in $\mathcal{K}_r(\ell_1; \ell_{q'})$ for any $q' \leq r < \tilde{r}$.

Finally, we deal with the remaining case, $r = q'$. Take $E = L_1[0, 1] = L_1$, $F = L_{q'}[0, 1] = L_{q'}$ and suppose that $\mathcal{K}_{q'}(L_1; L_{q'}) = \mathcal{K}_1(L_1; L_{q'})$, $2 < q < \infty$. Applying Theorem 3.11 we get $L_\infty \widehat{\otimes}_{g'_q} L_{q'} = L_\infty \widehat{\otimes}_{g'_\infty} L_{q'}$. Thus, the tensor spaces have isomorphic duals. By [3, 17.7 and 13.3] we obtain the isomorphism $L_1 \widehat{\otimes}_{g_q} L_q = L_1 \widehat{\otimes}_{g_\infty} L_q$. Since $g_\infty = \setminus \varepsilon$ and $g_q = \setminus g_{q'}^*$, by [3, Corollary 1 20.6] we have $L_1 \widehat{\otimes}_{g_{q'}^*} L_q = L_1 \widehat{\otimes}_\varepsilon L_q$. As shown in part (1), this cannot happen. ■

4. Appendix

(a) For $1 \leq r \leq 2$,

$$\lambda(\mathcal{K}_r, u, v) = \begin{cases} 1/r & \text{if } 1 \leq v \leq r, 1 \leq u \leq r', \\ 1 - 1/u & \text{if } 1 \leq v \leq r, r' \leq u \leq \infty, \\ 1/v & \text{if } r \leq v \leq 2, 1 \leq u \leq v', \\ 1 - 1/u & \text{if } r \leq v \leq 2, v' \leq u \leq \infty, \\ 1/v & \text{if } 2 \leq v \leq \infty, 1 \leq u \leq 2, \\ 1/2 - 1/u + 1/v & \text{if } 2 \leq v \leq \infty, 2 \leq u \leq \infty. \end{cases}$$

(b) For $2 < r < \infty$,

$$\lambda(\mathcal{K}_r, u, v) = \begin{cases} 1/r & \text{if } 1 \leq v \leq r, 1 \leq u \leq r', \\ 1 - 1/u & \text{if } 1 \leq v \leq 2, r' \leq u \leq \infty, \\ \rho & \text{if } 2 \leq v \leq r, r' \leq u \leq 2, \\ 1/v & \text{if } r \leq v \leq \infty, 1 \leq u \leq 2, \\ 1/2 - 1/u + 1/v & \text{if } 2 \leq v \leq \infty, 2 \leq u \leq \infty, \end{cases}$$

where

$$\rho = \frac{1}{r} + \frac{(\frac{1}{v} - \frac{1}{r})(\frac{1}{r'} - \frac{1}{u})}{\frac{1}{2} - \frac{1}{r}}.$$

Acknowledgements. The authors would like to thank Daniel Carando for helpful comments and suggestions. The authors also wish to thank the anonymous referee for the comments he/she made to improve this manuscript and also for providing them with additional references. This project was supported in part by UBACyT 1-746, UBACyT 1-218 and CONICET PIP 0624.

References

- [1] R. Aron, M. Maestre and P. Rueda, *p*-compact holomorphic mappings, Rev. R. Acad. Ciencias Exactas Fís. Nat. Ser. A. Mat. 104 (2010), 353–364.

- [2] Y. S. Choi and J. M. Kim, *The dual space of $(\mathcal{L}(X, Y); \tau_p)$ and the p -approximation property*, J. Funct. Anal. 259 (2010), 2437–2454.
- [3] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam, 1993.
- [4] J. M. Delgado, E. Oja, C. Piñeiro and E. Serrano, *The p -approximation property in terms of density of finite rank operators*, J. Math. Anal. Appl. 354 (2009), 159–164.
- [5] J. M. Delgado, C. Piñeiro and E. Serrano, *Density of finite rank operators in the Banach space of p -compact operators*, J. Math. Anal. Appl. 370 (2010), 498–505.
- [6] J. M. Delgado, C. Piñeiro and E. Serrano, *Operators whose adjoints are quasi p -nuclear*, Studia Math. 197 (2010), 291–304.
- [7] J. Diestel, J. H. Fourie and J. Swart, *The Metric Theory of Tensor Products, Grothendieck's Résumé Revisited*, Amer. Math. Soc., Providence, RI, 2008.
- [8] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43, Cambridge Univ. Press, Cambridge, 1995.
- [9] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo 8 (1956), 1–79.
- [10] S. Kwapien, *On a theorem of L. Schwartz and its applications to absolutely summing operators*, Studia Math. 38 (1970), 193–201.
- [11] O. Reinov, *On linear operators with p -nuclear adjoint*, Vestnik St. Petersburg Univ. Math. 33 (2000), no. 4, 19–21.
- [12] A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam, 1980.
- [13] A. Pietsch, *The ideal of p -compact operators and its maximal hull*, Proc. Amer. Math. Soc., to appear.
- [14] A. Persson und A. Pietsch, *p -nukleare und p -integrale Abbildungen in Banachräumen*, Studia Math. 33 (1969), 19–62.
- [15] R. Ryan, *Introduction to Tensor Products on Banach Spaces*, Springer, London, 2002.
- [16] D. P. Sinha and A. K. Karn, *Compact operators whose adjoints factor through subspaces of ℓ_p* , Studia Math. 150 (2002), 17–33.
- [17] D. P. Sinha and A. K. Karn, *Compact operators which factor through subspaces of ℓ_p* , Math. Nachr. 281 (2008), 412–423.
- [18] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite-Dimensional Operator Ideals*, Longman, Harlow, 1989.

Daniel Galicer, Silvia Lassalle, Pablo Turco
 Departamento de Matemática – Pab. I
 Facultad de Ciencias Exactas y Naturales
 Universidad de Buenos Aires
 (1428) Buenos Aires, Argentina
 E-mail: dgalicer@dm.uba.ar
 slassall@dm.uba.ar
 paturco@dm.uba.ar

Received August 29, 2012
Revised version October 10, 2012

(7609)