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SUPERCOOLED STEFAN PROBLEM WITH A NEUMANN TYPE BOUNDARY CONDITION

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ABSTRACT. We consider a supercooled one-dimensional Stefan problem with a Neumann boundary condition and a variable thermal diffusivity. We establish a necessary and sufficient condition for the heat flux at the fixed face $x = 0$, in order to obtain existence and uniqueness of a similarity type solution. Moreover we over-specified the fixed face $x = 0$ by a Dirichlet boundary condition aiming at the simultaneous determination of one or two thermal coefficients.

1. INTRODUCTION

We consider the following Stefan problem that describes the freezing of a supercooled liquid,

$$w_t = (D(w)w_x)_x\,, \quad 0 < x < s(t),\ t > 0 \tag{1.1}$$

$$D(w(0,t))w_x(0,t) = q^*(t) > 0\,, \quad t > 0 \tag{1.2}$$

$$D(w(s(t),t))w_x(s(t),t) = -\frac{l}{c}\dot{s}(t)\,, \quad t > 0 \tag{1.3}$$

$$w(s(t),t) = w_0\,, \quad t > 0 \tag{1.4}$$

$$w(x,0) = g(x) < w_0\,, \quad 0 < x < 1 \tag{1.5}$$

$$s(0) = 1\,, \tag{1.6}$$

where $w = w(x,t)$ represents the temperature of the material and $s = s(t)$ is the moving interface to be determined, c is the specific heat, l is the latent heat of fusion of the medium, w_0 is the phase change temperature and $g = g(x)$ is the initial temperature of the material. We impose a Neumann type condition at the fixed face $x = 0$ characterized by a heat flux $q^* = q^*(t) > 0$ [1], which corresponds to a supercooled liquid. We assume that the thermal diffusivity D is given by

$$D(w) = \frac{a}{(b+w)^2} \tag{1.7}$$

and we assume that the positive parameters a, b and the initial temperature g satisfy the following condition

$$b + w_0 \neq \frac{l}{c}, \quad -b < g(x) < w_0, \quad \text{for } 0 < x < 1. \tag{1.8}$$

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In [5] a supercooled one-phase Stefan problem with temperature-dependent thermal conductivity and a Dirichlet condition at fixed face was studied. Existence of a similarity type solution was obtained through the unique solution of an integral equation and it was showed that finite time blow-up occurs.

Free boundary problems with diffusion coefficient given by (1.7) or with temperature dependent conductivity were considered in [2, 4, 6, 7, 16, 19, 23, 30].

Free boundary problems which involves the freezing of a supercooled liquid can be seen in [12, 13, 14, 17, 18, 22, 24]. The problems with phase change occur in several and important biological, industrial and technological processes among others [8, 9, 10, 11, 15, 20, 21, 31]. A large bibliography on the subject is given in [29].

The organization of the paper is as follow: In Section 2 the Stefan problem is reduced through reciprocal transformations [25, 26, 27] to another one free boundary problem which admits a similarity type solution. We prove that there exists a similarity type solution if and only if a Neumann-type boundary condition given by (1.2) with $q^*(t) = Q_0/s(t)$ is considered, where $Q_0 > 0$ and $s(t)$ is the free boundary. In Section 3 we prove that the similarity solution to (1.1)-(1.6) can be obtained through the unique solution of an integral equation. In Section 4 we establish an over-condition of Dirichlet type at the fixed face $x = 0$ in order to determine one or two unknown thermal coefficients according to whether the free boundary (unknown) or the movil boundary (known) is considered.

2. Necessary and sufficient conditions for the existence and uniqueness of similarity type solution

To obtain an equivalent Stefan problem to (1.1)-(1.7) which admits a similarity type solution we give preliminaries results [5, 25, 26, 27, 28].

Lemma 2.1. *Under the reciprocal transformation*

$$u(x,t) = \frac{\sqrt{a}}{b + w(x,t)}, \tag{2.1}$$

problem (1.1)-(1.7) *is equivalent to the Stefan problem*

$$u_t = u^2 u_{xx}\,, \quad 0 < x < s(t)\,,\ t > 0\,, \tag{2.2}$$

$$u_x(0,t) = -r(t)\,, \quad t > 0\,, \tag{2.3}$$

$$u_x(s(t),t) = \frac{l}{c\sqrt{a}}\dot{s}(t)\,, \quad t > 0\,, \tag{2.4}$$

$$u(s(t),t) = \gamma\,, \quad t > 0\,, \tag{2.5}$$

$$u(x,0) = \frac{\sqrt{a}}{b + g(x)}, \quad 0 < x < 1\,, \tag{2.6}$$

$$s(0) = 1\,, \tag{2.7}$$

where

$$r(t) = \frac{q^*(t)}{\sqrt{a}}, \quad \gamma = \frac{\sqrt{a}}{b + w_0}. \tag{2.8}$$

The proof of the above lemma is standard; we omit it here. Now we introduce the transformation

$$v(y,t) = u(x,t), \quad y(x,t) = \int_0^x \frac{d\eta}{u(\eta,t)}\,, \quad 0 < x < s(t)\,,\ t > 0. \tag{2.9}$$

Lemma 2.2. *From* (2.9) *the free boundary problem* (2.2)–(2.7) *is equivalent to*

$$v_t = v_{yy} + r(t)v_y\,, \quad 0 < y < Y(t)\,,\ t > 0\,, \tag{2.10}$$

$$v_y(0,t) = -r(t)v(0,t), \quad t > 0\,, \tag{2.11}$$

$$v_y(Y(t),t) = \beta[\dot{Y}(t) + r(t)]\,, \quad t > 0\,, \tag{2.12}$$

$$v(Y(t),t) = \gamma\,, \quad t > 0 \tag{2.13}$$

$$v(y,0) = G(y)\,, \tag{2.14}$$

$$Y(0) = y_0\,, \tag{2.15}$$

where

$$Y(t) = y(s(t),t) = \int_0^{s(t)} \frac{d\eta}{u(\eta,t)}, \quad t > 0 \tag{2.16}$$

is the free boundary and

$$\beta = \frac{l\gamma}{c(b+w_0)-l}, \quad y_0 = \frac{1}{\sqrt{a}}(b + \int_0^1 g(\eta)d\eta), \tag{2.17}$$

$$G(y) = \frac{\sqrt{a}}{b + g(\phi(y))}\,, \quad \phi(y) = \int_0^y v(\eta,0)d\eta\,. \tag{2.18}$$

Proof. From (2.9), using (2.2)-(2.3) we have

$$u_x = \frac{v_y}{v}, \quad u_{xx} = \frac{v_{yy}}{v^2} - \frac{v_y^2}{v^3}, \quad u_t = -\frac{v_y^2}{v} - r(t)v_y + v_t\,.$$

Then, from (2.2) we obtain (2.10).

To prove (2.12), we differentiate $Y(t)$ in (2.16) and take into account the conditions (2.3)-(2.5), to obtain

$$\begin{aligned}
\dot{Y}(t) &= \frac{1}{u(s(t),t)}\dot{s}(t) + \int_0^{s(t)} \frac{-u_t(\eta,t)}{u^2(\eta,t)}d\eta \\
&= \frac{1}{\gamma}\dot{s}(t) - \int_0^{s(t)} u_{\eta\eta}(\eta,t)d\eta \\
&= \frac{1}{\gamma}\dot{s}(t) - u_x(s(t),t) + u_x(0,t) \\
&= \Big(\frac{1}{\gamma} - \frac{l}{c\sqrt{a}}\Big)\dot{s}(t) - r(t)
\end{aligned}$$

or equivalently

$$\dot{Y}(t) = \Big(\frac{c(b+w_0)-l}{c\sqrt{a}}\Big)\dot{s}(t) - r(t)\,. \tag{2.19}$$

Therefore, from (2.19) and (2.4) we obtain (2.12). From (2.3), (2.6), (2.7) and (2.17)-(2.18) follow immediately the conditions (2.11), (2.14) and (2.15). $\square$

Now, for the free boundary problem (2.10)-(2.15) we propose a similarity type solution given by

$$T(z) = v(y,t), \quad z = \frac{y}{Y(t)}. \tag{2.20}$$

We have the following results.

Theorem 2.3. *The free boundary problem* (2.10)-(2.15) *admits a solution of type* (2.20) *if and only if there exist constants* μ *and* R *such that*

$$Y(t)\dot{Y}(t) = \mu\,, \quad \forall t > 0\,, \tag{2.21}$$

$$Y(t)r(t) = R, \quad \forall t > 0. \tag{2.22}$$

Then the free boundary problem (2.10)-(2.15) *is equivalent to the problem*

$$T''(z) + T'(z)(\mu z + R) = 0\,, \quad 0 < z < 1\,, \tag{2.23}$$

$$T'(0) = -RT(0)\,, \tag{2.24}$$

$$T(1) = \gamma\,, \tag{2.25}$$

$$T'(1) = \beta(\mu + R)\,, \tag{2.26}$$

$$T(\frac{y}{y_0}) = G(y)\,, \quad 0 < y < y_0\,, \tag{2.27}$$

whose solution is given by

$$T(z) = \gamma \frac{-R\int_0^z \exp\big(-\frac{\mu}{2}(\eta + \frac{R}{\mu})^2\big)d\eta + \exp(-\frac{R^2}{2\mu})}{\exp(-\frac{R^2}{2\mu}) - R\int_0^1 \exp\big(-\frac{\mu}{2}(\eta + \frac{R}{\mu})^2\big)d\eta}, \quad 0 < z < 1\,, \tag{2.28}$$

and the coefficient μ *must be determined from the equation*

$$\frac{-R\gamma \exp\big(-\frac{\mu}{2}(1 + \frac{R}{\mu})^2\big)}{\exp(-\frac{R^2}{2\mu}) - R\int_0^1 \exp\big(-\frac{\mu}{2}(\eta + \frac{R}{\mu})^2\big)d\eta} = \beta(\mu + R). \tag{2.29}$$

Moreover, the function $\varphi(z) := G(zy_0)$ *is a solution of* (2.23).

Proof. Taking into account (2.20) we have

$$v_y = \frac{T'(z)}{Y(t)}, \quad v_{yy} = \frac{T''(z)}{Y^2(t)}, \quad v_t = -T'(z)z\frac{\dot{Y}(t)}{Y(t)}\,.$$

Then (2.10) is equivalent to

$$T''(z) + [r(t)Y(t) + Y(t)\dot{Y}(t)z]T'(z) = 0.$$

Moreover from (2.11) we have $T'(0) = -r(t)Y(t)T(0)$ and, by (2.12), we obtain

$$T'(1) = \beta\big(Y(t)\dot{Y}(t) + Y(t)r(t)\big)\,.$$

Therefore, we can obtain a similarity type solution $T = T(z)$ if and only if there exist constants μ and R such that (2.21) and (2.22) hold.

The conditions (2.26) and (2.27) follow from (2.13) and (2.14), respectively. It is easy to see that the solution to (2.23)-(2.25) is given by (2.28), and equation (2.29) for μ follows inmediatiely from (2.26).

Moreover, taking into account (2.27) we obtain that the function $\varphi(z) = G(zA)$, $0 < z < 1$ solves (2.23). □

Corollary 2.4. *The heat flux at fixed face* $x = 0$ *of problem* (1.1)-(1.6) *satisfies*

$$q^*(t) = \frac{R\sqrt{a}}{Y(t)}, \quad R > 0. \tag{2.30}$$

Remark 2.5. The solution to the supercooled Stefan problem with temperature condition at fixed face given by $w(0,t) = -B$, a phase-change temperature $w(s(t),t) = 0$ and a variable diffusivity given by $D(w) = \frac{k(w)}{\rho c} = \frac{1}{(a+bw)^2}$ obtained in [5] satisfies

$$D(w(0,t))w_x(0,t) = \frac{Q_0}{s(t)}\,, \tag{2.31}$$

where Q_0 is a positive constant which depends on the data.

In what follows for each $R > 0$ we will analyze the existence of the solution μ to (2.29). We have the following results.

Lemma 2.6. *There is no solution $\mu > 0$ for* (2.29).

Proof. If we suppose that there exists $\mu > 0$ which satisfies (2.21). Then

$$Y(t) = \sqrt{2\mu t + y_0^2} > 0,\ t > 0, \quad \dot{Y}(t) > 0. \tag{2.32}$$

Taking into account (2.19) and (2.21) we have

$$\Big(\frac{c(b+w_0)-l}{c\sqrt{a}}\Big)\dot{s}(t) = \frac{\mu+R}{Y(t)} \tag{2.33}$$

and from the boundary conditions of problem (2.2)-(2.7) we deduce $\dot{s}(t) < 0$ which implies $c(b+w_0) - l < 0$. Moreover we obtain

$$s(t) = 1 + \frac{c\sqrt{a}(\mu+R)}{\mu(c(b+w_0)-l)}\Big(\sqrt{y_0^2 + 2\mu t} - y_0\Big), \quad t > 0, \tag{2.34}$$

and there exists $t_0 > 0$ such that $s(t_0) = 0$ where

$$t_0 = \frac{m^2 - y_0^2}{2\mu} > 0, \quad m = y_0 - \frac{\mu(c(b+w_0)-l)}{(\mu+R)c\sqrt{a}}.$$

Then we have a contradiction: $s(t_0) = 0$ implies $Y(t_0) = 0$, but $Y(t) > 0$ for all $t > 0$. Therefore there is no solution $\mu > 0$ for (2.29). $\square$

Next, we study the existence of solution to equation (2.29) for $\mu < 0$. For this, it is suitable to define

$$\sigma = -\mu > 0, \quad f(\sigma) = \frac{R}{\sqrt{2\sigma}}, \quad j(\sigma) = f(\sigma) - \sqrt{\frac{\sigma}{2}}, \tag{2.35}$$

$$U(x) = \int_0^x \exp(z^2)dz. \tag{2.36}$$

Then equation (2.29) can be rewritten as

$$\frac{\exp(j^2(\sigma) - f^2(\sigma))}{1 + 2f(\sigma)\exp(-f^2(\sigma))[U(j(\sigma)) - U(f(\sigma))]} = p(R-\sigma) \tag{2.37}$$

in the unknown $\sigma > 0$, where

$$p = \frac{-\beta}{R\gamma} = \frac{l}{lR - cR(b+w_0)}. \tag{2.38}$$

Lemma 2.7. *The real function*

$$F_1(\sigma) := \frac{\exp(j^2(\sigma) - f^2(\sigma))}{1 + 2f(\sigma)\exp(-f^2(\sigma))[U(j(\sigma)) - U(f(\sigma))]} \tag{2.39}$$

satisfies

$$F_1(0) = 1, \quad F_1(+\infty) - \infty\,. \tag{2.40}$$

If $R \neq 2x_1^2$, then

$$F_1(R) = \frac{exp(-R/2)}{F'(\sqrt{\frac{R}{2}})}\,. \tag{2.41}$$

If $R = 2x_1^2$, then

$$F_1(R^-) = +\infty, \quad F_1(R^+) = -\infty\,, \tag{2.42}$$

where

$$F(x) = exp(-x^2)\int_0^x exp(z^2)dz \tag{2.43}$$

is the Dawson's function and x_1 is such that $F'(x_1) = 0$.

Moreover:

(a) *If $R < 2x_1^2$ there exists $\sigma_0 > R$ such that*

$$F_1(\sigma)\begin{cases} > 0, & \sigma < \sigma_0 \\ < 0, & \sigma > \sigma_0 \end{cases}$$

and $F_1(\sigma_0^-) = +\infty$, $F_1(\sigma_0^+) = -\infty$.

(b) *If $R > 2x_1^2$ then there exists $\sigma_0 < R$ such that*

$$F_1(\sigma)\begin{cases} > 0, & 0 < \sigma < \sigma_0 \\ < 0, & \sigma_0 < \sigma \end{cases}$$

and in both cases $F_1(\sigma_0^-) = +\infty$, $F_1(\sigma_0^+) = -\infty$

(c) *If $R = 2x_1^2$, then*

$$F_1(\sigma)\begin{cases} > 0, & \sigma < R \\ < 0, & \sigma > R \end{cases}$$

Proof. We have

$$\begin{aligned} F_1(\sigma) &= \frac{\exp(j^2(\sigma) - f^2(\sigma))}{1 - 2f(\sigma)F(f(\sigma)) + 2f(\sigma)\exp(-f^2(\sigma))U(j(\sigma))} \\ &= \frac{\exp(j^2(\sigma) - f^2(\sigma))}{F'(f(\sigma)) + 2f(\sigma)\exp(-f^2(\sigma))U(j(\sigma))} \end{aligned}$$

Taking into account that Dawson's function satisfies [3]

$$\lim_{x\to+\infty} xF(x) = \frac{1}{2}\,,$$

$$\lim_{\sigma\to 0} f(\sigma)\exp(-f^2(\sigma))U(j(\sigma)) = \frac{\exp(-R)}{2}\,.$$

Then we have $F_1(0) = 1$. By applying L'Hopital's rule we show that $F_1(+\infty) = -\infty$.

We know that

$$F'(x) = 1 - 2xF(x)\begin{cases} > 0 & \text{if } 0 < x < x_1 \\ = 0 & \text{if } x = x_1 \\ < 0 & \text{if } x > x_1\,, \end{cases}$$

where $x_1 \simeq 0.924$, $F(x_1) \simeq 0.541$. Then if $R < 2x_1^2$, this is $\sqrt{R/2} < x_1$, we have $F'(\sqrt{R/2}) > 0$ and

$$F_1(R) = \frac{\exp(-R/2)}{F'(\sqrt{R/2})} > 0.$$

Since F_1 does not cancel and $F_1(+\infty) = -\infty$, there exists $\sigma_0 > R$ such that

$$F_1(\sigma) \begin{cases} > 0, & \sigma < \sigma_0 \\ < 0, & \sigma > \sigma_0, \end{cases}$$
$$F_1(\sigma_0^-) = +\infty, \quad F_1(\sigma_0^+) = -\infty.$$

If $R > 2x_1^2$ this is $\sqrt{R/2} > x_1$, we have $F_1(R) < 0$. Then, there exists $\sigma_0 < R$ such that

$$F_1(\sigma) \begin{cases} > 0, & 0 < \sigma < \sigma_0 \\ < 0, & \sigma_0 < \sigma, \end{cases}$$
$$F_1(\sigma_0^-) = +\infty, \quad F_1(\sigma_0^+) = -\infty$$

If $R = 2x_1^2$, then $F'(\sqrt{R/2}) = 0$ and

$$F_1(R^-) = +\infty, \quad F_1(R^+) = -\infty \tag{2.44}$$

In this case,

$$F_1(\sigma) \begin{cases} > 0, & \sigma < R \\ < 0, & \sigma > R\,, \end{cases}$$

and the claim holds. □

Lemma 2.8. *The function*

$$F_2(\sigma) = p(R - \sigma), \quad \sigma > 0 \tag{2.45}$$

satisfies

(a) *If $b + w_0 > l/c$ then $F_2(0) = pR < 0$ and $F_2'(\sigma) = -p > 0$*
(b) *If $b + w_0 < l/c$ then $F_2(0) = pR > 1$ and $F_2'(\sigma) = -p < 0$,*

and $F_2(R) = 0$.

Lemma 2.9. *(a) If $b + w_0 > l/c$ there is no solution $\mu < 0$ to* (2.29).
(b) If $b + w_0 < l/c$ there exists unique $-R < \overline{\mu} < 0$ solution to (2.29).

Proof. Taking into account properties of functions F_1 and F_2 it is easy to see that there exist at least one $0 < \overline{\sigma} < R$ solution of (2.37) only in the case $p < 0$ this is $b + w_0 < l/c$.

To prove uniqueness of the solution to (2.37) we rewrite it as

$$M_1(\sigma) = M_2(\sigma) \tag{2.46}$$

for $0 < \sigma < R$, where

$$M_1(\sigma) = \exp(j^2(\sigma)) + p(R - \sigma)2f(\sigma)U(f(\sigma)),$$
$$M_2(\sigma) = p(R - \sigma)\big[\exp(f^2(\sigma)) + 2f(\sigma)U(j(\sigma))\big].$$

This functions satisfy

$$M_1(0) = +\infty, \quad M_1(R) = 1, \quad M_1'(\sigma) < 0,$$
$$M_2(0) = +\infty, \quad M_2(R) = 0, \quad M_2'(\sigma) < 0, \quad M_2(\sigma) > M_1(\sigma)$$

if and only if $F_2(\sigma) > F_1(\sigma)$ when σ approaches 0, which is satisfied by the previous lemma.

Then there exists unique $0 < \overline{\sigma} < R$ solution of (2.37) and therefore there exists unique $-R < \overline{\mu} < 0$ solution to (2.29). □

Theorem 2.10. *Under assumption* (1.8)*, if $b + w_0 < l/c$, for each $R > 0$ then problem* (2.23)-(2.27) *has a unique solution given by*

$$T(z) = C_1\Big[U\Big(\frac{\overline{\mu}z + R}{\sqrt{-2\overline{\mu}}}\Big) - U\Big(\frac{R}{\sqrt{-2\overline{\mu}}}\Big)\Big] + D_1, \quad 0 < z < 1 \tag{2.47}$$

with

$$C_1 = \frac{R\gamma\sqrt{-2/\overline{\mu}}}{\exp(-\frac{R^2}{2\overline{\mu}}) + R\sqrt{\frac{2}{-\overline{\mu}}}\big[U(\frac{\overline{\mu}+R}{\sqrt{-2\overline{\mu}}}) - U(\frac{R}{\sqrt{-2\overline{\mu}}})\big]}, \tag{2.48}$$

$$D_1 = \frac{\exp(-\frac{R^2}{2\overline{\mu}})\gamma}{\exp(-\frac{R^2}{2\overline{\mu}}) + R\sqrt{\frac{2}{-\overline{\mu}}}\big[U(\frac{\overline{\mu}+R}{\sqrt{-2\overline{\mu}}}) - U(\frac{R}{\sqrt{-2\overline{\mu}}})\big]} \tag{2.49}$$

where $-R < \overline{\mu} < 0$ is the unique solution of (2.29)*.*

Theorem 2.11. *Assuming* (1.8)*, $b + w_0 < l/c$ and $r(t) = R/Y(t)$, with $R > 0$, there exists a unique similarity type solution to* (2.10)-(2.15) *given by*

$$\begin{aligned} v(y,t) &= T\big(\frac{y}{Y(t)}\big) \\ &= C_1\Big[U\Big(\frac{\overline{\mu}\frac{y}{Y(t)} + R}{\sqrt{-2\overline{\mu}}}\Big) - U\Big(\frac{R}{\sqrt{-2\overline{\mu}}}\Big)\Big] + D_1, \\ &\quad 0 < y < Y(t), \quad t > 0, \end{aligned} \tag{2.50}$$

where

$$Y(t) = \sqrt{2\overline{\mu}t + y_0^2}, \quad 0 \le t \le \frac{-y_0^2}{2\overline{\mu}} \tag{2.51}$$

is the free boundary and $-R < \overline{\mu} < 0$ is the solution of equation (2.29)*.*

The proof of the above theorem follows from Theorems 2.3 and 2.10, and (2.20). With these results we have proved that the free boundary problem (2.10)-(2.15), with a particular heat flow in the fixed face $x = 0$, has a unique similarity type solution. Then, taking into account the transformation (2.9) we can enunciate the following result.

Theorem 2.12. *If we assume* (1.8)*, $b + w_0 < l/c$ and $q^*(t) = \frac{R}{y_0 s(t))}$ with $R > 0$, then there exists a unique solution to problem* (2.2)-(2.7) *which is given by*

$$u(x,t) = C_1\Big[U\Big(\frac{\overline{\mu}\int_0^x \frac{d\eta}{u(\eta,t)}}{\sqrt{-2\overline{\mu}}\sqrt{2\overline{\mu}t + y_0^2}} + \frac{R}{\sqrt{-2\overline{\mu}}}\Big) - U\big(\frac{R}{\sqrt{-2\overline{\mu}}}\big)\Big] + D_1, \tag{2.52}$$

for $0 \le x \le s(t)$ where the coefficient $-R < \overline{\mu} < 0$ is the unique solution of (2.29)*, and the free boundary is*

$$s(t) = \frac{1}{y_0}\sqrt{y_0^2 + 2\overline{\mu}t}, \quad 0 \le t < \frac{-y_0^2}{2\overline{\mu}}. \tag{2.53}$$

Moreover the constant y_0 defined by (2.15) *satisfies*

$$y_0 = \frac{\overline{\mu}((b+w_0)c-l)}{c\sqrt{a}(\overline{\mu}+R)}. \tag{2.54}$$

Proof. Taking into account (2.9) and Theorem 2.11 we obtain (2.52)-(2.53). From the above theorem and (2.19) we have

$$s(t)-1 = \frac{(\overline{\mu}+R)c\sqrt{a}}{\overline{\mu}\big((b+w_0)c-l\big)}\Big(\sqrt{2\overline{\mu}t+y_0^2}-y_0\Big)$$

From (2.16), it follows that $s(\frac{y_0^2}{-2\overline{\mu}})=0$, then (2.54) holds. □

Corollary 2.13. *Under the claim of Theorem 2.11, the free boundary given by* (2.53) *satisfies*

$$\lim_{t\to(\frac{-y_0^2}{2\overline{\mu}})^-} s(t)=0, \qquad \lim_{t\to(\frac{-y_0^2}{2\overline{\mu}})^-} \dot{s}(t)=-\infty \tag{2.55}$$

so finite time blow-up occurs.

Remark 2.14. The higher value of R decreases the value of $\overline{\mu}$, this implies that the free boundary $s = s_R(t)$ given by (2.53) decreases, for each $t > 0$. This is $s_{R_1}(t) > s_{R_2}(t)$ when $R_1 < R_2$.

3. Solving the equivalent integral equation

In the previous section we have shown that if $u = u(x,t)$ is a solution to problem (2.2)-(2.7), then it must be solution in variable x, of the integral equation (2.52) and the free boundary $s = s(t)$ is given by (2.53). Reciprocally we will demonstrate that if u is solution of integral equation (2.52) then (u,s) is the solution of the free boundary problem (2.2)-(2.7).

Theorem 3.1. *Let $b+w_0 < l/c$ and $q^*(t) = \frac{R}{y_0 s(t)}$, for $R>0$ and y_0 defined by* (2.15)*. If $u = u(x,t)$ is a solution of the integral equation* (2.52) *with s given by* (2.53) *and function*

$$V(x,t) = \frac{\frac{\overline{\mu}\int_0^x \frac{d\eta}{u(\eta,t)}}{\sqrt{2\overline{\mu}\tau+y_0^2}}+R}{\sqrt{-2\overline{\mu}}}, \tag{3.1}$$

where $\overline{\mu}$ is the unique solution of (2.29)*, satisfies the following conditions*

$$\frac{\partial V}{\partial x}(x,t) = \frac{-\sqrt{-\overline{\mu}/2}}{\sqrt{2\overline{\mu}t+y_0^2}\Big(C_1\big[U(V(x,t))-U(\frac{R}{\sqrt{-2\overline{\mu}}})\big]+D_1\Big)}, \tag{3.2}$$

$$V(0,t) = \frac{R}{\sqrt{-2\overline{\mu}}}\,, \tag{3.3}$$

$$\frac{\partial V}{\partial t}(x,t) = \frac{\sqrt{-\overline{\mu}/2}}{(2\overline{\mu}t+y_0^2)}\big(-u_x(x,t)\sqrt{2\overline{\mu}t+y_0^2}-\sqrt{-2\overline{\mu}}V(x,t)\big)\,, \tag{3.4}$$

$$V(s(t),t) = \frac{\overline{\mu}+R}{\sqrt{-2\overline{\mu}}}\,, \tag{3.5}$$

$$V(x,0) = \frac{\sqrt{-\overline{\mu}}\int_0^x(b+g(z))dz}{\sqrt{2a}y_0}+\frac{R}{\sqrt{-2\overline{\mu}}}\,. \tag{3.6}$$

Then (u,s) is a solution of the free boundary problem (2.2)-(2.7).

The proof of the above theorem follows by taking into account the above developments and elementary computations. Following the ideas developed in [5], we will prove that the integral equation (2.52) has a unique solution, showing that it is equivalent to a Cauchy differential problem

Theorem 3.2. *The integral equation* (2.52) *has a unique solution for* $0 \leq t \leq t_1 < y_0^2/(-2\overline{\mu})$ *where* t_1 *is an arbitrary positive time.*

Proof. We define $V(x,t)$ by (3.1). Then (2.52) is equivalent to the Cauchy differential problem

$$\frac{\partial V}{\partial x}(x,t) = \frac{-\sqrt{-\overline{\mu}/2}}{\sqrt{2\overline{\mu}t + y_0^2\big(C_1\big[U(V(x,t)) - U(\frac{R}{\sqrt{-2\overline{\mu}}})\big] + D_1\big)}}\,, \tag{3.7}$$

$$V(0,t) = \frac{R}{\sqrt{-2\overline{\mu}}}\,, \tag{3.8}$$

with a parameter $0 \leq t \leq t_1 < -y_0^2/(2\overline{\mu})$, the coefficients C_1, D_1 are given by (2.48), (2.49). The claim follows by the same method as in [5]. □

Theorem 3.3. *The free boundary problem* (1.1)-(1.6) *with* $b+w_0 < l/c$ *and* $q^*(t) = R/(y_0 s(t))$*, for* $R > 0$ *and* y_0 *defined by* (2.17)*, has a unique similarity type solution* (w,s) *which is given by*

$$w(x,t) = \frac{\sqrt{a}}{u(x,t)} - b, \quad 0 < x < s(t),\ 0 \leq t \leq t_1 < -\frac{y_0^2}{2\overline{\mu}} \tag{3.9}$$

where $s(t)$ *is the free boundary given by* (2.53)*,* u *is solution of the integral equation* (2.52) *and* $-R < \overline{\mu} < 0$ *is the unique solution of equation* (2.29)*. Moreover we have that finite blow-up occurs at* $t = -y_0^2/(2\overline{\mu})$*.*

4. Determination of thermal coefficients

Taking into account the results obtained in previous sections we have that the solution given by Theorem 3.3 satisfies

$$\begin{aligned} w(0,t) &= \frac{\sqrt{a}}{\gamma}\big\{1 + \exp(R^2/2\overline{\mu})R\sqrt{\tfrac{2}{-\overline{\mu}}}\big[U\big(\tfrac{\overline{\mu}+R}{\sqrt{-2\overline{\mu}}}big) - U\big(\tfrac{R}{\sqrt{-2\overline{\mu}}}\big)\big]\big\} - b \\ &= (b+w_0)N - b\,, \end{aligned} \tag{4.1}$$

where

$$N = 1 + \exp(R^2/2\overline{\mu})R\sqrt{\tfrac{2}{-\overline{\mu}}}\big[U\big(\tfrac{\overline{\mu}+R}{\sqrt{-2\overline{\mu}}}\big) - U\big(\tfrac{R}{\sqrt{-2\overline{\mu}}}\big)\big] < 1$$

it is to see that the temperature at fixed face is not time dependent and it satisfies $w(0,t) < w_0$.

Therefore, if we add to problem (1.1)-(1.7) a Dirichlet condition of the type

$$w(0,t) = w_i < w_0 \tag{4.2}$$

we have an overspecified condition at fixed face $x = 0$ that allows us to determine one or two unknown thermal coefficients chosen among l, c, b, w_0, w_i as a function of data depending if $s = s(t)$ is a free boundary (unknown function) or a moving boundary (known function) , which satisfy the problem (1.1)-(1.7).

From (2.37) and (4.1) we have that the unknown thermal coefficients and $\sigma = -\mu$ (the coefficient which characterize the free boundary $s = s(t)$) must satisfy the system of the equations

$$\frac{w_i + b}{w_0 + b} = 1 + \tfrac{R}{\sqrt{2\sigma}} \exp(-\tfrac{R^2}{2\sigma})\big[U(\tfrac{R-\sigma}{\sqrt{2\sigma}}) - U(\tfrac{R}{\sqrt{2\sigma}})\big]\,, \tag{4.3}$$

$$\frac{\exp(\frac{\sigma}{2} - R)}{1 + \frac{R}{\sqrt{2\sigma}} \exp(-\frac{R^2}{2\sigma})\big[U(\frac{R-\sigma}{\sqrt{2\sigma}}) - U(\frac{R}{\sqrt{2\sigma}})\big]} = p(R - \sigma) \tag{4.4}$$

or the equivalent system of equations

$$\frac{w_i + b}{w_0 + b} = N(R, \sigma)\,, \tag{4.5}$$

$$F_1(\sigma) = \frac{l}{R[l - c(b + w_0)]}(R - \sigma) \tag{4.6}$$

where $N(R, \sigma)$ is the right hand of (4.3) which satisfies

$$N(R, \sigma) = \frac{\exp(\frac{\sigma}{2} - R)}{F_1(\sigma)} \tag{4.7}$$

Firstly, we assume that $s = s(t)$ is the free boundary, then from previous sections we have that

$$s(t) = \frac{1}{y_0}\sqrt{y_0^2 - 2\sigma t}$$

with unknown σ. Therefore, through (4.5)-(4.6) we can determine σ and one unknown thermal coefficient chosen among l, c, b, w_i.

From the results given in Section 3 is easy to see that there exists a unique $\overline{\sigma}$ solution to (4.5). Therefore, from (4.6) we determine an unknown thermal coefficient chosen among l, c, b. We get the following formula for them:

Case 1: Unknown l.

$$l = \frac{c(b + w_0)M(R, \overline{\sigma})}{M(R, \overline{\sigma}) - 1} \tag{4.8}$$

where

$$M(R, \sigma) = \frac{RF_1(\sigma)}{R - \sigma}. \tag{4.9}$$

Case 2: Unknown c.

$$c = \frac{l\,[M(R, \overline{\sigma}) - 1]}{(b + w_0)M(R, \overline{\sigma})} \tag{4.10}$$

Case 3: Unknown b.

$$b = \frac{l\,[M(R, \overline{\sigma}) - 1]}{c\,M(R, \overline{\sigma})N(R, \overline{\sigma})} - w_i \tag{4.11}$$

Case 4: Unknown w_i. For the case where the unknowns coefficients are σ and w_i we obtain a unique solution $\overline{\sigma}$ from (4.6) and w_i is determined from (4.5) and is

$$w_i = (b + w_0)N(R, \overline{\sigma}) - b\,. \tag{4.12}$$

Now, if we assume that $s(t)$ is known and it is defined by

$$s(t) = \frac{1}{y_0}\sqrt{y_0^2 + 2\mu t} = \frac{1}{y_0}\sqrt{y_0^2 - 2\sigma t}$$

for a given $-R < \mu < 0 \quad (0 < \sigma < R)$, then two unknown coefficients can be chosen among l, c, b, w_0, w_i which must satisfy equations (4.3)-(4.4). We get the following cases and formulas for them:

Case 5: Unknowns w_i and l.

$$w_i = N(R,\sigma)(b+w_0) - b, \quad l = \frac{c(b+w_0)M(R,\sigma)}{M(R,\sigma)-1} \tag{4.13}$$

Case 6: Unknowns w_0 and l.

$$w_0 = \frac{w_i + b}{N(R,\sigma)} - b, \quad l = \frac{c(b+w_0)M(R,\sigma)}{M(R,\sigma)-1} \tag{4.14}$$

Case 7: Unknowns b and l.

$$b = \frac{w_i - N(R,\sigma)w_0}{N(R,\sigma)-1}, \quad l = \frac{c(b+w_0)M(R,\sigma)}{M(R,\sigma)-1} \tag{4.15}$$

Case 8: Unknowns w_i and c.

$$w_i = N(R,\sigma)(b+w_0) - b, \quad c = \frac{l[M(R,\sigma)-1]}{(b+w_0)M(R,\sigma)} \tag{4.16}$$

Case 9: Unknowns w_0 and c.

$$w_0 = \frac{w_i + b}{N(R,\sigma)} - b, \quad c = \frac{l[M(R,\sigma)-1]}{(b+w_0)M(R,\sigma)} \tag{4.17}$$

Case 10: Unknowns b and c.

$$b = \frac{w_i - N(R,\sigma)w_0}{N(R,\sigma)-1}, \quad c = \frac{l[M(R,\sigma)-1]}{(b+w_0)M(R,\sigma)} \tag{4.18}$$

Conclusions. We studied a supercooled one-phase Stefan problem for a semi-infinite material with temperature-dependent diffusivity at the fixed face $x = 0$ and a Neumann type condition at fixed face. We established a necessary and sufficient condition for the heat flux $q^*(t) = Q_0/s(t)$ where $s(t)$ is the free boundary, in order to obtain existence and uniqueness of the solution of similarity type, local in time. This explicit solution was obtained through the unique solution of an integral equation with the time as a parameter. Moreover it is showed that finite time blow-up occurs. At last, an over-specified boundary condition of Dirichlet type is considered at the fixed face and one or two thermal coefficients are obtained, depending if $s = s(t)$ is a free boundary (unknown function) or a moving boundary (known function. Formulae for these thermal coefficients are given.

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