On finite MTL-algebras that are representable as poset products of archimedean chains

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Abstract

We obtain a duality between the category of locally unital finite MTL-algebras and the category of finite labeled trees. In addition we prove that certain poset products of MTL-algebras are in fact, sheaves of MTL-chains over Alexandrov spaces. Finally we give a concrete description for the studied poset products in terms of direct products and ordinal sums of finite MTL-algebras.

Introduction

In the literature there are several dualities for categories of residuated structures involving the use of categories of labeled trees. In [6], it is proved that the category of MV-algebras with finite spectrum is dual to the category of labeled root systems. Later, in [2], and [7], it is proved independently, that the category of finite BL-algebras, is dual to the category of finite labeled trees. The techniques employed in those papers have a combinatorial character. These facts allow to establish a precise description of the category of finite labeled trees.

In [14], Jipsen introduces the construction of poset products for algebras as a generalization for the dual of poset sum, introduced in [12] by Jipsen and Montagna for residuated lattices. The poset product was used originally to prove decomposition theorems for several kinds of ordered structures. In particular, in [3], Montagna and Busaniche studied the poset product of BL-algebras to obtain representation theorems in terms of poset products of MV-algebras and product algebras.

The theory of representation by sheaves has been used as a tool for developing decomposition results in terms of subdirect products. For the case of MV-algebras, see [8], for Heyting algebras see [5]. More recently, in [4] a representation theorem for integral rigs is given by using topos theoretic tools.

In [9] Esteva and Godo introduced MTL-logic as the basic fuzzy logic of left-continuous t-norms. Furthermore, a new class of algebras was defined, the variety of MTL-algebras. This variety constitutes an equivalent algebraic semantics for MTL-logic. MTL-algebras are essentially integral commutative residuated lattices with bottom satisfying the prelinearity equation:

$$(x \to y) \vee (y \to x) \approx 1$$

The main result of this work is a duality between the categories of locally unital finite MTL-algebras (see Definition 3) and the category of finite labeled trees (see Section 3). Such a duality is obtained by the extensive study of certain poset products of MTL-chains. Additionally, it is shown that the studied poset products of MTL-chains are indeed sheaves over Alexandrov spaces. Of independent interest, it turns out to be the description of the aforementioned poset products in terms of ordinal sums and direct products. The reader is assumed to be familiar with results of sheaf theory as presented in [15].

This paper is divided as follows. Section 1 is devoted to present the basic contents that are necessary to understand this work. In Section 2, we characterize the finite archimedean MTL-chains in terms of their nontrivial idempotent elements. In Section 3, we show that there exist a functor from the category of finite MTL-algebras to the category of finite labeled forests. We take advantage of the intimate relation between idempotent elements and filters, that is given for the case of finite MTL-algebras. In Section 4, we study the forest products of MTL-chains. We prove that such construction is, in fact, a sheaf over an Alexandrov space whose fibers are MTL-chains. In Section 5 we use the results obtained in Section 4 in order to establish a functor from the category of finite labeled forest to the category of finite MTL-algebras. We also bring a duality theorem between the category of locally unital finite MTL-algebras and finite labeled forest. Finally, we present a description of the forest product of finite MTL-algebras in terms of ordinal sums and direct products of finite MTL-algebras.

1 Preliminaries

The aim of the following section is to give a brief survey about the background in MTL-algebras required to read this work. We present some known definitions and some particular constructions for prelinear semihoops that naturally can be extended to MTL-algebras.

We write **Set** to denote the category whose objects are sets and their morphisms are set functions.

A *prelinear semihoop* is an algebra $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ of type $(2, 2, 2, 2, 0)$ such that $(A, \wedge, \vee)$ is lattice with 1 as greatest element, $(A, \cdot, 1)$ is a commutative monoid and for every $x, y, z \in A$ the following conditions hold:

<blockquote>

(residuation) $\qquad xy \leq z$ if and only if $x \leq y \to z$

(prelinearity) $\qquad (x \to y) \vee (y \to x) = 1$
</blockquote>

Equivalently, a prelinear semihoop is an *integral prelinear commutative residuated lattice.*

Remark 1. *Notice that in every prelinear semihoop $\mathbf{A}$, the following equations hold:*

1. $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

2. $x \wedge y = (x \cdot (x \to y)) \vee (y \cdot (y \to x))$.

In order to prove 1., we write $z = ((x \to y) \to y) \wedge ((y \to x) \to x)$. Then, from $x(x \to y) \leq y$ and $y \to x \leq 1$ we get $x \leq (x \to y) \to y$ and $x \leq (y \to x) \to x$, respectively. Thus, we can conclude $x \leq z$. In a similar way, we obtain $y \leq z$. Let us assume $x, y \leq c$. By monotonicity of the residual, we obtain $(x \to y) \to y \leq (x \to y) \to c$

and $(y \to x) \to x \leq (y \to x) \to c$. *Since in every commutative residuated lattice,* $(\bigvee_{i \in I} y_i) \to x = \bigwedge_{i \in I}(y_i \to x)$ *(c.f. [19]), using prelinearity we get*

$$z \leq ((x \to y) \to c) \wedge ((y \to x) \to c) = ((x \to y) \vee (y \to x)) \to c = 1 \to c = c.$$

On the other hand, for proving 2.*, we write* $w = (x \cdot (x \to y)) \vee (y \cdot (y \to x))$. *Recall that, since* $x \cdot (x \to y) \leq x \cdot 1 = x$ *and* $y \cdot (y \to x) \leq x$, *we get* $w \leq x$. *Similarly, we deduce that* $w \leq y$. *If we assume that* $c \leq x, y$; *by monotonicity of the product, we get that* $c \cdot (x \to y) \leq x \cdot (x \to y)$ *and* $c \cdot (y \to x) \leq y \cdot (y \to x)$. *Hence,* $(c \cdot (x \to y)) \vee (c \cdot (y \to x)) = c \cdot ((x \to y) \vee (y \to x)) \leq w$. *Since, by prelinearity, the leftmost term of this inequality is* c, *we get that* $c \leq w$.

A prelinear semihoop $\mathbf{A}$ is a *bounded* if $(A, \wedge, \vee, 1)$ has a least element 0. An *MTL-algebra* is a bounded prelinear semihoop, hence, MTL-algebras are bounded prelinear integral commutative residuated lattices, as usually defined [9, 11, 17]. An MTL-algebra $\mathbf{A}$ is an *MTL-chain* if it is totally ordered. Recall that any MTL-algebra is a subdirect product of MTL-chains (Corollary 4.4 of [13]), so MTL-algebras have distributive lattice reducts. Let $\mathbf{1}$ and $\mathbf{2}$ be the MTL-chains of one and two elements, respectively. For the rest of this paper we will refer to $\mathbf{1}$ as the *trivial MTL-chain*.

It is known that the class of MTL-algebras is a variety. We write $\mathcal{MTL}$ for the category of MTL-algebras and MTL-homomorphisms. In particular MTL-homomorphisms preserve 0.

Let $\mathbf{I} = (I, \leq)$ be a totally ordered set and $\mathcal{F} = \{\mathbf{A}_i\}_{i \in I}$ a family of semihoops. Let us assume that the members of $\mathcal{F}$ share (up to isomorphism) the same neutral element; i.e., for every $i \neq j$, $A_i \cap A_j = \{1\}$. The *ordinal sum* of the family $\mathcal{F}$, is the structure $\bigoplus_{i \in I} A_i$ whose universe is $\bigcup_{i \in I} A_i$ and whose operations are defined as:

$$x \cdot y = \begin{cases} x \cdot_i y, & \text{if } x, y \in A_i \\ y, & \text{if } x \in A_i, \text{ and } y \in A_j - \{1\}, \text{ with } i > j, \\ x, & \text{if } x \in A_i - \{1\}, \text{ and } y \in A_j, \text{ with } i < j, \end{cases}$$

$$x \to y = \begin{cases} x \to_i y, & \text{if } x, y \in A_i \\ y, & \text{if } x \in A_i, \text{ and } y \in A_j, \text{ with } i > j, \\ 1, & \text{if } x \in A_i - \{1\}, \text{ and } y \in A_j, \text{ with } i < j, \end{cases}$$

where the subindex i denotes the application of operations in A_i.

Moreover, if $\mathbf{I}$ has a minimum $\perp$, A_i is a totally ordered semihoop for every $i \in \mathbf{I}$ and $A_\perp$ is bounded then $\bigoplus_{i \in I} A_i$ becomes an MTL-chain.

Let M be an MTL-algebra. A submultiplicative monoid F of M is called a filter if it is an up-set with respect to the order of M. In particular, for every $x \in F$, we write $\langle x \rangle$ for the filter generated by x; i.e.,

$$\langle x \rangle = \{a \in M \mid x^n \leq a, \text{ for some } n \in \mathbb{N}\}.$$

For any filter F of M, we can define a binary relation $\sim_F$, on M by $a \sim_F b$ if and only if $a \to b \in F$ and $b \to a \in F$. A straightforward verification shows that $\sim_F$ is a congruence on M. It is well known that there is a bijective correspondence between filters and congruences on MTL-algebras (c.f. [17]), so we write M/F to denote the quotient

$M/\sim_F$. For every $a \in M$, we write $[a]_F$ for the equivalence class of a in M/F. If the context is clear, we simply write $[a]$. Let $h : A \to B$ be an MTL-morphism and $S \subseteq A$. As usual, we write $h|_S$ to denote the restriction of h to S. Recall that (Section 3 of [4]) the canonical homomorphism $h : M \to M/F$ has the *universal property* of forcing all the elements of M to be 1; i.e., for every MTL-algebra B and every MTL-morphism $f : M \to B$ such that $f(a) = 1$ for every $a \in F$, there exists a unique MTL-morphism $g : M/F \to B$ making the diagram below

$$M \xrightarrow{h} M/F$$
$$\searrow_{f} \quad \downarrow_{g}$$
$$B$$

commute. A filter F of M is *prime* if $0 \notin F$ and $x \vee y \in F$ entails $x \in F$ or $y \in F$, for every $x, y \in M$. The set of prime filters of an MTL-algebra M ordered by inclusion is called the *spectrum* and is denoted by $\mathrm{Spec}(M)$.

2 Finite archimedean MTL-chains

In this section we give a characterization of the finite archimedean MTL-chains in terms of their nontrivial idempotent elements. In addition we prove that every morphism of finite archimedean MTL-chains is injective.

A totally ordered MTL-algebra is said to be archimedean if for every $x \leq y < 1$, there exists $n \in \mathbb{N}$ such that $y^n \leq x$.

Lemma 1. *Let M be an MTL-chain. If there is an $a \in M$ such that for every $n \in \mathbb{N}$, $a^{n+1} < a^n$, then M is infinite.*

Proposition 2. *A finite MTL-chain M is archimedean if and only if $M = \mathbf{2}$ or M does not have nontrivial idempotent elements.*

Proof. If $M = \mathbf{2}$ the proof is trivial. If $M \neq \mathbf{2}$ and does not have nontrivial idempotents, we may assume that there exists some $a \in M$ such that $0 < a < 1$. Since M is finite, by Lemma 1, there exists $n \in \mathbb{N}$ such that $a^{n+1} = a^n$. If $a^n > 0$, $(a^n)^2 = a^n$, and hence, M has a nontrivial idempotent, in contradiction with the fact that M does not have nontrivial idempotents. Hence, there exists $n \in \mathbb{N}$ such that $a^n = 0$. Now, for $b < a$ in M, we have that $a^n \leq b$, from where we can conclude that M is archimedean. Conversely, let us assume that M is archimedean but there exists an idempotent element $a \neq 0, 1$. Hence, $a^n = a$ for every $n \in \mathbb{N}$. If $b < a$ (for example, if $b = 0$), we have that for every $n \in \mathbb{N}$, $b < a \leq a^n$, contradicting the archimedeanity of M. In consequence, no such idempotent can exist. $\square$

Corollary 3. *For any finite nontrivial MTL-chain M, the following are equivalent:*

i. M is archimedean,

ii. M is simple, and

iii. M does not have nontrivial idempotent elements.

In [11] Horčik and Montagna gave an equational characterization for the archimedean MTL-chains.

Lemma 4 (Lemma 6.6). *Let M be an MTL-chain. Then, M is archimedean if and only if for every $a, b \in M$,*

$$((a \to b) \to b)^2 \leq a \vee b.$$

The last part of this section is devoted to obtain a description of the morphisms between finite archimedean MTL-chains. Let $f : A \to B$ be a morphism of finite MTL-chains. As usual, we write K_f for the kernel of f; i.e.,

$$K_f = \{x \in A \mid f(x) = 1\}$$

Lemma 5. *Let $f : A \to B$ be a morphism of MTL-algebras. Then f is injective if and only if $f(x) = 1$ implies $x = 1$.*

Proof. Let f be an injective morphism of MTL-algebras, then $f(x) = 1 = f(1)$ implies $x = 1$. On the one hand, let us assume that $f(x) = 1$ implies $x = 1$. If $f(a) = f(b)$ then $f(a) \leq f(b)$ and $f(b) \leq f(a)$, thus by general properties of the residual it follows that $f(a) \to f(b) = 1$ and $f(b) \to f(a) = 1$ so $f(a \to b) = 1$ and $f(b \to a) = 1$. From the assumption we get that $a \to b = 1$ and $b \to a = 1$, thus $a \leq b$ and $b \leq a$. Hence, $a = b$ so f is injective. $\qquad \square$

Lemma 6. *Let $f : A \to B$ be a morphism of finite MTL-chains. If A is archimedean then $B = \mathbf{1}$ or f is injective.*

Proof. Since A is archimedean, by (ii) of Corollary 3 we get that if is also simple so $K_f = A$ or $K_f = \{1\}$. In the first case, we get that $f(a) = 1$ for every $a \in A$, so in particular $f(0) = 0 = 1$, hence $B = \mathbf{1}$. In the second case, it follows that $f(a) = 1$ implies $a = 1$, so by Lemma 5 f is injective. $\qquad \square$

Corollary 7. *Let $f : A \to B$ be a morphism of finite MTL-chains. If A is archimedean and $B \neq \mathbf{1}$ then $f(x) = 0$ implies $x = 0$.*

Proof. Suppose that there exists $a \in A$ such that $f(a) = 0$ but $a \neq 0$. Since A is archimedean, by Lemma 6, we get that f is injective so, from

$$f(a^2) = f(a)^2 = 0 = f(a),$$

we obtain that $a = a^2$. On the other hand, since $f(a) \neq 1$, again from Lemma 6, we get that $a \neq 1$ and consequently $0 < a < 1$. So A possesses a nontrivial idempotent which by (iii) of Corollary 3 is absurd. $\qquad \square$

Remark 2. *Observe that every morphism of MTL-algebras between finite archimedean MTL-chains is injective. Let $f : A \to B$ be a morphism of finite archimedean MTL-chains. Since A is archimedean, from Lemma 6 it follows that f is injective or $B = \mathbf{1}$, but B is archimedean by assumption so, from Proposition 2, it follows that B cannot be trivial, hence f must be injective.*

3 Finite labeled forests

It is a well known fact that if M is a BL-algebra, then, its dual spectrum is a forest. (c.f. Proposition 6 of [19]). Such relation has been used to establish functorial correspondences before (c.f. [2]) between BL-algebras and certain kind of labeled forests[1]. Motivated by these ideas, in this part we show that there exists a functor from the category of finite MTL-algebras to the category of finite labeled forests. To do so, we will take advantage of the intimate relation between idempotent elements and filters that is given for the case of finite MTL-algebras. This particular condition allows to describe the spectrum of a finite MTL-algebra in terms of its join-irreducible idempotent elements, as well as characterize the quotients that produce archimedean MTL-chains.

A *forest* is a poset X such that for every $a \in X$ the set

$$\downarrow a = \{x \in X \mid x \leq a\}$$

is a totally ordered subset of X.

This definition is motivated by the following result whose proof is similar to the dual of Proposition 6 of [19].

Lemma 8. *Let M be a (finite) MTL-algebra. Then $\mathrm{Spec}(M)^{op}$ is a (finite) forest.*

A *tree* is a forest with a least element. A *p-morphism* is a morphism of posets $f : X \to Y$ satisfying the following property: given $x \in X$ and $y \in Y$ such that $y \leq f(x)$ there exists $z \in X$ such that $z \leq x$ and $f(z) = y$. Let $f\mathcal{MTL}$ be the algebraic category of finite MTL-algebras. We write $fa\mathcal{MTL}$ for the algebraic category of finite archimedean MTL-algebras and $fa\mathcal{MTL}c$ for the full subcategory of finite archimedean MTL-chains. Let $\mathfrak{S}$ be the skeleton of $fa\mathcal{MTL}c$. A *labeled forest* is a function $l : \mathbf{F} \to \mathfrak{S}$, such that $\mathbf{F}$ is a forest and the collection of archimedean MTL-chains $\{l(i)\}_{i \in \mathbf{F}}$ (up to isomorphism) shares the same neutral element 1. Consider two labeled forests $l : \mathbf{F} \to \mathfrak{S}$ and $m : \mathbf{G} \to \mathfrak{S}$. A morphism $l \to m$ is a pair $(\varphi, \mathcal{F})$ such that $\varphi : \mathbf{F} \to \mathbf{G}$ is a p-morphism and $\mathcal{F} = \{f_x\}_{x \in \mathbf{F}}$ is a family of injective morphisms $f_x : (m \circ \varphi)(x) \to l(x)$ of MTL-algebras.

Let $(\varphi, \mathcal{F}) : l \to m$ and $(\psi, \mathcal{G}) : m \to n$ be two morphism between labeled forests. We define the composition $(\varphi, \mathcal{F})(\psi, \mathcal{G}) : l \to n$ as the pair $(\psi\varphi, \mathcal{M})$, where $\mathcal{M}$ is the family whose elements are the MTL-morphisms $f_x g_{\varphi(x)} : n(\psi\varphi)(x) \to l(x)$ for every $x \in \mathbf{F}$. We will call $f\mathcal{LF}$ the category of finite labeled forests and its morphisms. The details of checking that $f\mathcal{LF}$ is a category are left to the reader.

Let M be an MTL-algebra. We write $\mathcal{I}(M)$ for the poset of idempotent elements of M; i.e.,

$$\mathcal{I}(M) := \{x \in M \mid x^2 = x\}.$$

Lemma 9. *In any MTL-algebra M, the following are equivalent,*

 i. *$a \in \mathcal{I}(M)$, and*

 ii. *$\langle a \rangle = \ \uparrow a$*

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[1]Actually in [2] the authors use the name *weighted* instead of labeled.

Proof. Let us assume that $\langle a \rangle = \uparrow a$. Since $a^2 \in \langle a \rangle$ then $a^2 \in \uparrow a$, so $a \leq a^2$. Finally, from the integrality of M we conclude that $a^2 \leq a$. Therefore $a^2 = a$. The reverse direction follows directly from the definition. $\square$

Corollary 10. *Let M be a finite MTL-algebra and $F \subseteq M$ a filter in M. There exists a unique $a \in \mathcal{I}(M)$ such that $F = \uparrow a$.*

Proof. Since M is finite, every filter $F \subseteq M$ is principal, so by Lemma 9, $F = \uparrow a$ for some $a \in \mathcal{I}(M)$. If there exists $a' \in \mathcal{I}(M)$ such that $\uparrow a = \uparrow a'$, then $a \leq a'$ and $a' \leq a$. $\square$

Let M be a finite MTL-algebra. From Corollary 10, it follows that there is a bijection between $\mathcal{I}(M)$ and the filters of M. Let $\mathcal{J}(\mathcal{I}(M))$ be the subposet of join-irreducible elements of $\mathcal{I}(M)$. A direct application of Birkhoff's duality produces the following result.

Corollary 11. *Let M be a finite MTL-algebra and $P \in \mathrm{Spec}(M)$. Then, there exists a unique $e \in \mathcal{J}(\mathcal{I}(M))$ such that $P = \uparrow e$.*

Corollary 12. *Let M be a finite MTL-algebra. The posets $\mathrm{Spec}(M)^{op}$ and $\mathcal{J}(\mathcal{I}(M))$ are isomorphic.*

Proof. Let $\varphi : \mathcal{J}(\mathcal{I}(M)) \to Spec(M)$ be the mapping defined as $\varphi(e) = \uparrow e$. From Corollary 11, it follows that φ is bijective. The proofs of the antimonotonicity of φ and φ^{-1} are straightforward. $\square$

Lemma 13. *Let M be a MTL-algebra (not necessarily finite) and P be a prime filter of M. If F is a proper filter of M such that $P \subseteq F$, then F is prime.*

Proof. Let us assume that $x \vee y \in F$. Since $1 \in P$, from prelinearity we obtain that $(x \to y) \vee (y \to x) \in P$, so, since P is prime, we get that $x \to y \in P$ or $y \to x \in P$. Suppose $x \to y \in P$, then $x \to y \in F$. By Remark 1, it follows that $x \vee y \leq (x \to y) \to y$, thus, since F is an up-set, we get $(x \to y) \to y \in F$. Finally, since F is a multiplicative submonoid, from $(x \to y)((x \to y) \to y) \leq y$ we conclude $y \in F$. In a similar way we can deduce that $x \in F$. This concludes the proof. $\square$

Corollary 14. *Let M be a finite MTL-algebra and $x \in \mathcal{I}(M)$ such that $x \neq 0$. If there exists some $k \in \mathcal{J}(\mathcal{I}(M))$ such that $x \leq k$ then x is join-irreducible.*

Proof. Let $x \in \mathcal{I}(M)$ with $x \neq 0$. If there exists some $k \in \mathcal{J}(\mathcal{I}(M))$ such that $x \leq k$, then $\uparrow k \subseteq \uparrow x$. From Corollaries 10 and 11 we get that $\uparrow x$ is a proper filter and $\uparrow k$ is a prime filter of M, respectively. The result follows from applying Lemma 13 together with Corollary 11. $\square$

Remark 3. *It is well known that the congruence lattice of a finite residuated lattice is dually isomorphic to the lattice of central negative idempotents (Corollary 3.8 of [13]), so the join-irreducible idempotents in a finite MTL-algebra correspond to meet-irreducible congruences. It follows that $M/\uparrow e$ is subdirectly irreducible for any $e \in \mathcal{J}(\mathcal{I}(M))$, hence a chain.*

We write $m(M)$ for the minimal elements of $\mathcal{J}(\mathcal{I}(M))$.

Lemma 15. *Let M be a finite MTL-algebra and $e \in \mathcal{J}(\mathcal{I}(M))$. Then, there exists a unique $k \in \mathcal{J}(\mathcal{I}(M)) \cup \{0\}$ such that $k \prec e$, where $\prec$ denotes the covering relation in posets.*

Proof. Let $e \in \mathcal{J}(\mathcal{I}(M))$, then either $e \in m(M)$ or $e \notin m(M)$. In the first case, the result follows, since $0 \prec e$. In the second case, by Lemma 8 and Corollary 11 we get that $\downarrow\!e \cap \mathcal{J}(\mathcal{I}(M))$ is a finite chain. If we consider k as the coatom of the latter chain, the result holds. $\qquad\square$

Let $e \in \mathcal{J}(\mathcal{I}(M))$. In the following, we will write a_e to denote the join-irreducible element associated to e in Lemma 15[2]. Note that $a_e = 0$ if and only if $e \in m(M)$.

Lemma 16. *Let M be a finite MTL-algebra and $e \in \mathcal{J}(\mathcal{I}(M))$. Then $M/\!\!\uparrow\!e$ is archimedean if and only if $e \in m(M)$.*

Proof. Notice that $M/\!\!\uparrow\!e$ is simple if and only if $\uparrow\!e$ corresponds to a maximal proper meet-irreducible congruence, which is the case if and only if $e \in m(M)$. Hence, by the Corollary 3, the result follows. $\qquad\square$

Remark 4. *Let M be a finite MTL-algebra and $F \subseteq M$ a filter. Let us check that $(F, \vee, \wedge, 1, x)$ is a finite MTL-algebra such that $0_F = x$. By definition of F, $(F, \cdot, 1)$ is a commutative monoid so, if $a, b \in F$ then $ab \in F$. The integrality of M implies that $a \leq b \to a$ and $b \leq a \to b$, so since F is an up-set of M then for every $a, b \in F$, we get that $a \to b, b \to a \in F$. Similarly, since $ab \leq a \wedge b$, by applying the last argument we get that $a \wedge b \in F$. The proof for $a \vee b \in F$ is the same. Finally, due to Corollary 10 there exists a unique $x \in \mathcal{I}(M)$ such that $F = \uparrow\!x$, which is equivalent to saying that $x = 0_F$.*

Lemma 17. *Let M be a finite MTL-algebra, then $(\uparrow\!a_e)/(\uparrow\!e)$ is an archimedean MTL-chain for every $e \in \mathcal{J}(\mathcal{I}(M))$.*

Proof. Recall that by Remark 4 and Lemma 15, we get that $\uparrow\!a_e$ is a finite MTL-algebra whose least element is a_e. Since $a_e \prec e$, it follows that $\uparrow\!e$ is a proper filter of $\uparrow\!a_e$ with $e \in m(\uparrow\!a_e)$. Therefore, from Lemma 16 we get that $\uparrow\!a_e/\!\!\uparrow\!e$ is an archimedean MTL-chain. $\qquad\square$

Let M and N be finite MTL-algebras and $f : M \to N$ a morphism of MTL-algebras. It is a known fact (c.f. [16]) that the assignments $M \mapsto Spec(M)$ and $f \mapsto Spec(f) = f^{-1}$, determines a contravariant functor $\mathbf{Spec} : f\mathcal{MTL} \to f\mathbf{Coh}$ from the category of finite MTL-algebras into the category of finite coherent (or spectral) spaces.

Let φ_M be the isomorphism between $\mathcal{J}(\mathcal{I}(M))$ and $Spec(M)^{op}$ of Proposition 12.

Lemma 18. *Let M and N be finite MTL-algebras and $f : M \to N$ an MTL-algebra morphism. There exists a unique p-morphism $f^* : \mathcal{J}(\mathcal{I}(N)) \to \mathcal{J}(\mathcal{I}(M))$ making the following diagram*

$$
\begin{array}{ccc}
\mathcal{J}(\mathcal{I}(N)) & \xrightarrow{\;f^*\;} & \mathcal{J}(\mathcal{I}(M)) \\
{\scriptstyle\varphi_N}\downarrow & & \downarrow{\scriptstyle\varphi_M} \\
Spec(N) & \xrightarrow[Spec(f)]{} & Spec(M)
\end{array}
$$

commute.

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[2]The element a_e is also noted as e_* in [10].

Proof. Since φ_M is an isomorphism, we get that $f^* = \varphi_M^{-1} spec(f) \varphi_N$. Observe that this map is defined as $f^*(e) = min\ S_e$ where $S_e = f^{-1}(\uparrow e) \cap \mathcal{J}(\mathcal{I}(M))$. In order to check the monotonicity, let $e \leq g$ in $\mathcal{J}(\mathcal{I}(N))$, then $\uparrow g \subseteq \uparrow e$, thus $f^{-1}(\uparrow g) \subseteq f^{-1}(\uparrow e)$ so $\uparrow f^*(g) \subseteq \uparrow f^*(e)$. Thereby, $f^*(e) \subseteq f^*(g)$. It only remains to check that f^* is a p-morphism. To do so, let $g \in \mathcal{J}(\mathcal{I}(N))$ and $e \in \mathcal{J}(\mathcal{I}(M))$ such that $g \leq f^*(e)$. Since $\mathcal{J}(\mathcal{I}(N))$ is finite, we can consider $m = min\ S$, with

$$S = \{ k \in \mathcal{J}(\mathcal{I}(N)) \mid k \leq e,\ g \leq f^*(k) \}.$$

We will prove that $f^*(m) = g$. Let $x \in \mathcal{I}(N)$ be such that $g \leq x$. Since $e \leq f(g)$, it follows that $f(x) \neq 0$. Consider $\uparrow mf(x)$. Since $mf(x) \leq m$, by Corollary 14 we get that $mf(x) \in \mathcal{J}(\mathcal{I}(M))$. Let us very that $g \leq f^*(mf(x))$ by checking $f^{-1}(\uparrow mf(x)) \subseteq \uparrow g$. If $b \in f^{-1}(\uparrow mf(x))$, then $mf(x) \leq f(b)$ and hence $m \leq f(x) \to f(b) = f(x \to b)$. Therefore $x \to b \in f^{-1}(\uparrow m)$. By construction of m, we have that $g \leq f^*(m)$, so $f^{-1}(\uparrow m) \subseteq \uparrow g$. Consequently, $g \leq x \to b$. Since $g \leq x$, we obtain that $g \leq x(x \to b) \leq b$. Hence, $mf(x) = m$, because $mf(x) \in S$. Finally, since $mf(x) = m \leq f(x)$, it follows that $x \in f^{-1}(\uparrow m)$. Thus $\uparrow x \subseteq f^{-1}(\uparrow m)$ and since $x \leq g$, we obtain that $\uparrow g \subseteq f^{-1}(\uparrow m)$. Then we conclude that $f^*(m) \leq g$. This concludes the proof. $\square$

Let M be a finite MTL-algebra and consider the function

$$l_M : \mathcal{J}(\mathcal{I}(M)) \to \mathfrak{S}$$

defined as $l_M(e) = \uparrow a_e / \uparrow e$. Since from Lemma 8 we know that $\mathcal{J}(\mathcal{I}(M))$ is a finite forest, l_M is a finite labeled forest.

Let $\mathbf{F}$ be a finite forest and $X \subseteq F$. We write $Min(X)$ for the minimal elements of X.

Lemma 19. *Let $f : X \to Y$ be a p-morphism. If $x \in Min(X)$ then $f(x) \in Min(Y)$.*

Proof. Let us assume $x \in Min(X)$, and suppose that there exists $y \in Y$ such that $y < f(x)$. Since f is a p-morphism, there exists $z \in X$, with $z \leq x$ such that $y = f(z)$. Since $y \neq f(x)$, then $z \neq x$, so $x \notin Min(X)$. This fact is absurd by assumption. $\square$

Lemma 20. *Let M and N be finite MTL-algebras and $f : M \to N$ an MTL-algebra morphism. Then, for every $e \in \mathcal{J}(\mathcal{I}(N))$, f determines a morphism $\overline{f}_e : \uparrow a_{f^*(e)} \to \uparrow a_e$ such that there exists a unique MTL-algebra morphism $f_e : \uparrow a_{f^*(e)} / \uparrow f^*(e) \to \uparrow a_e / \uparrow e$ making the diagram*

$$
\begin{array}{ccc}
\uparrow a_{f^*(e)} & \xrightarrow{\ \overline{f}_e\ } & \uparrow a_e \\
\downarrow & & \downarrow \\
\uparrow a_{f^*(e)} / \uparrow f^*(e) & \xrightarrow[f_e]{} & \uparrow a_e / \uparrow e
\end{array}
$$

commute.

Proof. Let $e \in \mathcal{J}(\mathcal{I}(N))$. Then $e \notin m(N)$ or $e \in m(N)$. In the first case, it follows that $a_e > 0_N$ and thus, $\uparrow a_e \subset N$. Since $a_e \leq e$ and f^* is monotone then $f^*(a_e) \leq f^*(e)$. Since

$a_{f^*(e)} \prec f^*(e)$, $f^*(a_e) \le a_{f^*(e)}$. Thereby, $\uparrow a_{f^*(e)} \subseteq \uparrow f^*(a_e)$. Let us define $\overline{f}_e : \uparrow a_{f^*(e)} \to \uparrow a_e$ as

$$\overline{f}_e(x) = \begin{cases} f(x), & x > a_{f^*(e)} \\ a_e, & x = a_{f^*(e)} \end{cases}$$

From Lemma 18, we get that $\uparrow f^*(a_e) = f^{-1}(\uparrow a_e)$ and $\overline{f}_e$ is a well defined MTL-morphism. Let us consider $a_{f^*(e)} < f^*(e) \le x$, then, $\overline{f}_e(a_{f^*(e)}) < \overline{f}_e(f^*(e)) \le \overline{f}_e(x)$ since $\overline{f}_e$ is monotone. By definition of $\overline{f}_e$ we obtain that $a_e < f(f^*(e)) \le f(x)$. Then, applying Lemma 18, we get that $e \le f(f^*(e))$, so we can conclude that $e \le f(x)$. This means that $[\overline{f}_e(x)] = [1]$ in $\uparrow a_e/\uparrow e$. Hence, by the universal property of quotients in $\mathcal{MTL}$, there exists a unique MTL-morphism $f_e : \uparrow a_{f^*(e)}/\uparrow f^*(e) \to \uparrow a_e/\uparrow e$ making the diagram above commutes.

Finally, if $e \in m(N)$, we get that $a_e = 0_N$ and $\uparrow a_e = N$. Since f^* is a p-morphism, due to Lemma 19, $f^*(e) \in m(M)$, $a_{f^*(e)} = 0_M$ and consequently, $\uparrow a_{f^*(e)} = M$. Let $\overline{f}_e = f$. The proof of $[\overline{f}_e(x)] = [1]$ in $\uparrow a_e/\uparrow e$ is similar to the given for first case. The rest of the proof follows from the universal property of quotients in $\mathcal{MTL}$.

$\square$

Let $f : M \to N$ be an MTL-morphism between finite MTL-algebras and $\mathcal{F}_f := \{f_e\}_{e \in \mathcal{J}(\mathcal{I}(N))}$ be the family of MTL-morphisms obtained in Lemma 20.

Corollary 21. *Let M and N be finite MTL-algebras and $f : M \to N$ an MTL-algebra morphism. Then the pair $(f^*, \mathcal{F}_f)$ is a morphism between the labeled forests l_N and l_M.*

Theorem 22. *The assignments $M \mapsto l_M$ and $f \mapsto (f^*, \mathcal{F}_f)$ define a contravariant functor*

$$\mathcal{G} : f\mathcal{MTL} \to f\mathcal{LF}.$$

Proof. Let $f : M \to N$ and $g : N \to O$ be morphisms in $f\mathcal{MTL}$ and consider the diagram

$$\begin{array}{ccccc} Spec(O) & \xrightarrow{Spec(g)} & Spec(N) & \xrightarrow{Spec(f)} & Spec(M) \\ \varphi_O \uparrow & & \varphi_N \uparrow \downarrow \varphi_N^{-1} & & \varphi_M \uparrow \\ \mathcal{J}(\mathcal{I}(O)) & \xrightarrow{g^*} & \mathcal{J}(\mathcal{I}(N)) & \xrightarrow{f^*} & \mathcal{J}(\mathcal{I}(M)) \end{array}$$

associated to the composition $gf : M \to O$. Since **Spec** is contravariant,

$$(gf)^* = \varphi_M^{-1} Spec(gf)\varphi_O = (\varphi_M^{-1} Spec(f)\varphi_N)(\varphi_N^{-1} Spec(g)\varphi_O) = f^* g^*. \tag{1}$$

If id_M denotes the identity map of the MTL-algebra M, a straightforward calculation proves that $(id_M)^* = id_{\mathcal{J}(\mathcal{I}(M))}$. Conversely, let $e \in \mathcal{J}(\mathcal{I}(O))$. We will verify that $\overline{(gf)}_e = \overline{g}_e \overline{f}_{g^*(e)}$. From (1), we conclude that $a_{(gf)^*(e)} = a_{f^*(g^*(e))}$. If $x > a_{(gf)^*(e)}$, then $x > a_{f^*(g^*(e))}$. By the monotonicity of f and Lemma 18, we obtain that $a_{g^*(e)} \le f(a_{f^*(g^*(e))}) \le f(x)$. Hence $a_{g^*(e)} \le f(x)$. In a similar way, by the monotonicity of g and using again Lemma 18, we get that $a_e \le (gf)(x)$. Therefore, for every $x \in \uparrow a_{(gf)^*(e)}$ it

follows that $\overline{(gf)}_e(x) = \overline{g}_e \overline{f}_{g^*(e)}(x)$. Finally, from Lemma 20, we obtain that $g_e f_{g^*(e)} = (gf)_e$. So, for every $e \in \mathcal{J}(\mathcal{I}(N))$ the diagram below

$$
\begin{array}{ccccc}
& & \overline{(gf)}_e & & \\
& \xrightarrow{\hspace{6cm}} & & & \\
{\uparrow}a_{(gf)^*(e)} & \xrightarrow{(\overline{f})_{g^*(e)}} & {\uparrow}a_{g^*(e)} & \xrightarrow{\overline{g}_e} & {\uparrow}a_e \\
\downarrow & & \downarrow & & \downarrow \\
{\uparrow}a_{(gf)^*(e)}/{\uparrow}(gf)^*(e) & \xrightarrow{f_{g^*(e)}} & {\uparrow}a_{g^*(e)}/{\uparrow}g^*(e) & \xrightarrow{g_e} & {\uparrow}a_e/{\uparrow}e \\
& \xrightarrow[\hspace{6cm}]{} & & & \\
& & (gf)_e & &
\end{array}
$$

commutes. Therefore $(g^* f^*, \mathcal{F}) = ((gf)^*, \mathcal{F}_{gf})$, where $\mathcal{F}_{gf} = \{(gf)_e\}_{e \in \mathcal{J}(\mathcal{I}(O))}$ and $\mathcal{F} = \{g_e f_{g^*(e)}\}_{e \in \mathcal{J}(\mathcal{I}(O))}$. Hence $\mathcal{G}(gf) = \mathcal{G}(f)\mathcal{G}(g)$. From the definition of $\mathcal{G}$ it easily follows that $\mathcal{G}(id_M) = id_{\mathcal{G}(M)}$. $\qquad\square$

4 Forest Product of MTL-algebras

In this section we introduce the notion of forest product. It is simply a shorthand for a poset product as defined in [3] when restricted to posets which are forests. For the sake of completeness, we recall the necessary definition.

Definition 1. *Let $\boldsymbol{F} = (F, \leq)$ be a forest and let $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains such that, up to isomorphism, they all share the same neutral element 1. If $\left(\bigcup_{i \in \boldsymbol{F}} \boldsymbol{M}_i\right)^F$ denotes the set of functions $h : F \to \bigcup_{i \in \boldsymbol{F}} \boldsymbol{M}_i$ such that $h(i) \in \boldsymbol{M}_i$ for all $i \in \boldsymbol{F}$, the forest product $\bigotimes_{i \in \boldsymbol{F}} \boldsymbol{M}_i$ is the algebra $\boldsymbol{M}$ defined as follows:*

(1) *The elements of $\boldsymbol{M}$ are the $h \in \left(\bigcup_{i \in \boldsymbol{F}} \boldsymbol{M}_i\right)^F$ such that, for all $i \in \boldsymbol{F}$ if $h(i) \neq 0_i$ then for all $j < i$, $h(j) = 1$.*

(2) *The monoid operation and the lattice operations are defined pointwise.*

(3) *The residual is defined as follows:*

$$
(h \to g)(i) = \begin{cases} h(i) \to_i g(i), & \text{if for all } j < i, \ h(j) \leq_j g(j) \\ 0_i & \text{otherwise} \end{cases}
$$

where the subindex i denotes the application of operations and of order in $\boldsymbol{M}_i$.

The following result is a slight modification of Theorem 3.5.4 in [3].

Lemma 23. *The forest product of MTL-chains is an MTL-algebra.*

In the following if we refer to a collection $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ of MTL-chains indexed by a forest $\mathbf{F}$ we always will assume that it satisfies the conditions of Definition 1.

Lemma 24. *Let $\boldsymbol{F}$ be a forest and $\{\boldsymbol{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains. The following are equivalent:*

1. $h \in \bigotimes_{i \in \boldsymbol{F}} \boldsymbol{M}_i$,

2. For every $i < j$ in $\boldsymbol{F}$, $h(j) = 0_j$ or $h(i) = 1$,

3. For all $i \in \boldsymbol{F}$ if $h(i) \neq 1$ then for all $i < j$, $h(j) = 0_j$,

4. $\bigcup_{i \in \boldsymbol{F}} h^{-1}(0_i)$ is an up-set of $\boldsymbol{F}$, $h^{-1}(1)$ is a down-set of $\boldsymbol{F}$ and

$$C_h = \{i \in \boldsymbol{F} \mid h(i) \notin \{0_i, 1\}\},$$

is a (possibly empty) antichain of $\boldsymbol{F}$.

Proof. Since the implications $(1) \Rightarrow (2) \Rightarrow (3)$ follow straight from definition, we only prove the remaining implications. Let us start proving that (3) implies (4): To prove that $h^{-1}(1)$ is a down-set of $\mathbf{F}$ we proceed by contradiction. Suppose $i < j$ with $h(j) = 1$ but $h(i) \neq 1$. Thus, $h(j) = 0_j$, by assumption, which is absurd. To prove that $\bigcup_{i \in I} h^{-1}(0_j)$ is an up-set of $\mathbf{F}$ let us suppose that $i < j$ with $h(i) = 0_i$, then, since $h(i) \neq 1$, and (3), we get that $h(j) = 0_j$. If C_h is not an antichain, there exist $i, j \in C_h$ comparable. Without loss of generality, we can assume $i < j$, $h(i) \neq 1$ and $h(j) \neq 0_j$, then because of (3), we obtain that $h(j) = 0_j$, which is absurd. Finally, to prove that (4) implies (1), let $h \in \left(\bigcup_{i \in \mathbf{F}} \mathbf{M}_i\right)^F$ and suppose that $i < j$ with $h(i) \neq 1$. If $h(j) \neq 0_j$, thus $i, j \in C_h$, which is absurd, since C_h is by assumption an antichain. $\square$

Remark 5. *Let $\boldsymbol{F}$ be a chain and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Let us consider $h \in \bigotimes_{i \in \boldsymbol{F}} \boldsymbol{M}_i$ and $j \in \boldsymbol{F}$. Note that there are only two possible cases for $h(j)$, namely $h(j) \neq 0_j$ or $h(j) \neq 1$. If $h(j) \neq 1$, from (4) of Lemma 24, it follows that for every $j < i$, $h(i) = 0_i$ and due to (3) of the same lemma, $h(k) = 1$ for every $k < j$. If $h(j) \neq 0_j$, from Definition 1, we get that $h(k) = 0_k$ for every $j < k$ and by (4) of Lemma 24, we have that $h(i) = 1$ for every $i < j$.*

Lemma 25. *Let A be a nontrivial commutative integral residuated lattice. If A is totally ordered, then the top element of A is join-irreducible. If A is also prelinear then the converse holds.*

Proof. If A is totally ordered, then every element is join-irreducible; in particular its top element. For the converse, if 1 if join-irreducible, then prelinearity implies $1 = x \to y$ or $1 = y \to x$, hence $x \leq y$ or $y \leq x$. $\square$

Lemma 26. *Let $\boldsymbol{F}$ be a forest and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Then $\boldsymbol{F}$ is a totally ordered set if and only if $\bigotimes_{i \in \boldsymbol{F}} \boldsymbol{M}_i$ is an MTL-chain.*

Proof. Suppose $\mathbf{F}$ is a totally ordered set and let $g, h \in \bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$ be such that $(g \vee h) = 1$. Since the lattice operations in $\bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$ are calculated pointwise, for every $i \in \mathbf{F}$, $g(i) \vee h(i) = 1$. If $g, h \neq 1$, there exists some $j \in \mathbf{F}$ such that $g(j), h(j) \neq 1$. From Remark 5 it follows that $g(k) = h(k) = 0_k$, for every $j < k$ so we get that $(g \vee h)(k) = g(k) \vee h(k) = 0_k$, which contradicts our assumption. Hence, since every MTL-algebra is prelinear, from Lemma 25 we get that $\bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$ is an MTL-chain. On the one hand, let us assume that $\bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$ is an MTL-chain. If $\mathbf{F}$ is not a totally ordered set, then there exist two different elements n and m in F which are not comparable. Let us consider $g, h \in \bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$, defined as

$$g(i) = \begin{cases} 0_n, & \text{if } i \geq n \\ \\ 1, & \text{otherwise} \end{cases} \qquad h(i) = \begin{cases} 0_m, & \text{if } i \geq m \\ \\ 1, & \text{otherwise} \end{cases}$$

Observe that $g \vee h = 1$ but $g, h \neq 1$, which is a contradiction since, by Lemma 25, the top element of $\bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$ is join-irreducible. $\qquad\square$

Let $\mathbf{F}$ be a forest and $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains. We write $\mathcal{D}(\mathbf{F})$ for the collection of down-sets of $\mathbf{F}$. Let S be a proper down-set of $\mathbf{F}$ and consider

$$X_S := \{h \in \bigotimes_{i \in \mathbf{F}} \mathbf{M}_i \mid h|_S = 1\}$$

Observe that X_S is a proper filter of $\bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$. Since S^c is itself a forest, due to Lemma 23, $\bigotimes_{i \in \mathbf{S}^c} \mathbf{M}_i$ is an MTL-algebra. Using the fact that every filter of an MTL-algebra is a prelinear semihoop, we obtain the following result.

Lemma 27. *Let $\mathbf{F}$ be a forest and $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains and $S \in \mathcal{D}(\mathbf{F})$. Then X_S and $\bigotimes_{i \in \mathbf{S}^c} \mathbf{M}_i$ are isomorphic prelinear semihoops.*[3]

Proof. Let $g \in \bigotimes_{i \in \mathbf{S}^c} \mathbf{M}_i$. Define $\varphi : \bigotimes_{i \in \mathbf{S}^c} \mathbf{M}_i \to X_S$ as

$$\varphi(g)(i) = \begin{cases} g(i), & \text{if } i \notin S \\ \\ 1, & \text{if } i \in S \end{cases}$$

For this part of the proof we will write h to denote $\varphi(g)$. First, we prove that h is well defined. Let us take $i < j$ in F and suppose that $h(i) \neq 1$. By construction of h, we get that $i \notin S$, so $h(i) = g(i)$. If $h(j) \neq 0_j$, then $h(j) = 1$ or $0_j < h(j) < 1$. In the first case we obtain that $j \in S$ and since $i < j$ and $S \in \mathcal{D}(\mathbf{F})$, $i \in S$, which is absurd. In the second case, since S^c is an up-set of $\mathbf{F}$, from $i \notin S$ and $i < j$ it follows that $j \notin S$. Hence, $h(j) = g(j) \neq 1$. Therefore, there are $i, j \in C_h$ comparable, which by (4) of Lemma 24 is absurd. Consequently, $h(j) = 0_j$ and thus, by (3) of Lemma 24, we get that $h \in \bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$. By construction, it is clear that $h \in X_S$. In order to verify that φ is surjective, let $f \in X_S$ and consider $f|_{S^c}$. Since S^c is an up-set and $f \in \bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$, it is clear that $\varphi(f|_{S^c}) = f$. The injectivity of φ is immediate.

Since the monoid and lattice operations in X_S and $\bigotimes_{i \in \mathbf{S}^c} \mathbf{M}_i$ are defined pointwise it is clear that φ preserves such operations. We prove that φ preserves the residual. To do so, let $s, t \in \bigotimes_{i \in \mathbf{S}^c} \mathbf{M}_i$. Then,

$$(\varphi(s) \to \varphi(t))(i) = \begin{cases} \varphi(s)(i) \to_i \varphi(t)(i), & \text{if for all } j < i, \ \varphi(s)(j) \leq_j \varphi(t)(j) \\ \\ 0_i, & \text{otherwise} \end{cases} \tag{2}$$

and

$$\varphi(s \to t)(i) = \begin{cases} (s \to t)(i), & \text{if } i \notin S \\ \\ 1, & \text{if } i \in S \end{cases} \tag{3}$$

---

[3]From Remark 4, it is clear that every finite bounded prelinear semihoop is an MTL-algebra. Nevertheless, since in this section we are dealing with arbitrary MTL-chains, and in such a case not every filter is an MTL-algebra, we rather prefer to maintain the term prelinear semihoop in order to distinguish filters from MTL-algebras.

If $i \in S$ then $\varphi(s) = \varphi(t) = 1$ so $\varphi(s)(i) \to_i \varphi(t)(i) = 1$. Since $S \in \mathcal{D}(\mathbf{F})$, if $j < i$, $j \in S$. Therefore $\varphi(s)(j) = \varphi(t)(j) = 1$. Hence, for every $i \in S$, $\varphi(s \to t)(i) = (\varphi(s) \to \varphi(t))(i)$. On the other hand, if $i \notin S$ and $(\varphi(s) \to \varphi(t))(i) \neq 1$ then $(\varphi(s) \to \varphi(t))(i) = 0_i$ or $0_i < (\varphi(s) \to \varphi(t))(i) < 1$. In first case, from equation (2) it follows that there exists $j \in F$ with $j < i$ such that $\varphi(s)(j) \not\leq_j \varphi(t)(j)$. If $j \in S$, $\varphi(s)(j) = \varphi(t)(j) = 1$. Therefore $j \notin S$. Hence, since $\varphi(s)(j) = s(j)$ and $\varphi(t)(j) = t(j)$ we get that there exists $j \notin S$ with $j < i$ such that $s(j) \not\leq_j t(j)$. Then $(s \to t)(i) = 0_i = (\varphi(s) \to \varphi(t))(i)$. In second case, from equation (3) we get that $(\varphi(s) \to \varphi(t))(i) = \varphi(s)(i) \to_i \varphi(t)(i)$ so, by the definition of φ, we obtain that $\varphi(s)(i) \to \varphi(t)(i) = (\varphi(s) \to \varphi(t))(i)$. Finally, if $i \notin S$ and $(\varphi(s) \to \varphi(t))(i) = 1$ then $\varphi(s)(i) \leq_i \varphi(t)(i)$ and, in consequence $\varphi(s)(i) \to_i \varphi(t)(i) = 1$. Thus, for every $j < i$, $\varphi(s)(j) \leq_j \varphi(t)(j)$. In particular, if $j \notin S$ and $j < i$, $s(j) \leq_j t(j)$. Therefore $\varphi(s)(i) \to \varphi(t)(i) = (s \to t)(i)$. Hence, for every $i \notin S$, $(\varphi(s) \to \varphi(t))(i) = \varphi(s)(i) \to_i \varphi(t)(i)$. This concludes the proof. $\qquad\square$

Corollary 28. *Let $\boldsymbol{F}$ be a forest, $S, T \in \mathcal{D}(\boldsymbol{F})$ such that $S \subseteq T$ and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Take $X_S^T := \{h \in \bigotimes_{i \in \boldsymbol{T}} \boldsymbol{M}_i \mid h|_S = 1\}$. Then, X_S^T and $\bigotimes_{i \in \boldsymbol{S^c} \cap T} \boldsymbol{M}_i$ are isomorphic prelinear semihoops.*

Proof. Since $S \subseteq T$ and $S, T \in \mathcal{D}(\mathbf{F})$ we get that $S \in \mathcal{D}(\mathbf{T})$. The result follows from Lemma 27. $\qquad\square$

4.1 Forest products are sheaves

In every poset $\mathbf{F}$ the collection $\mathcal{D}(\mathbf{F})$ of down-sets of $\mathbf{F}$ defines a topology over F called the *Alexandrov topology* on $\mathbf{F}$. Let $S, T \in \mathcal{D}(\mathbf{F})$ be such that $S \subseteq T$ and $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ be a collection of MTL-chains. Observe that, if $h \in \bigotimes_{i \in \mathbf{T}} \mathbf{M}_i$ then the restriction $h|_S$ is an element of $\bigotimes_{i \in \mathbf{S}} \mathbf{M}_i$, so the assignment that sends $T \in \mathcal{D}(\mathbf{F})$ to $\bigotimes_{i \in \mathbf{T}} \mathbf{M}_i$ defines a presheaf $\mathcal{P} : \mathcal{D}(\mathbf{F})^{op} \to \mathcal{MTL}$.

Lemma 29. *Let $\boldsymbol{F}$ be a forest and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Then, for every $S \in \mathcal{D}(\boldsymbol{F})$*
$$\mathcal{P}(S) \cong \mathcal{P}(F)/X_S.$$

Proof. Let $r : \mathcal{P}(F) \to \mathcal{P}(S)$ be the restriction to S. It is clear that r is a surjective morphism of MTL-algebras such that $r(h) = 1$, for every $h \in X_S$. Then, by the universal property of the canonical homomorphism $\beta : \mathcal{P}(F) \to \mathcal{P}(F)/X_S$, there exists a unique morphism of MTL-algebras $\alpha : \mathcal{P}(F)/X_S \to \mathcal{P}(S)$ such that the diagram below

$$
\begin{array}{ccc}
\mathcal{P}(F) & \xrightarrow{\ \beta\ } & \mathcal{P}(F)/X_S \\
& \searrow{\scriptstyle r} & \big\downarrow{\scriptstyle \alpha} \\
& & \mathcal{P}(S)
\end{array}
$$

commutes. Observe that $\alpha\beta = r$, so since β is surjective, it follows that α is surjective too. The verification of the injectivity of α is straightforward. $\qquad\square$

Corollary 30. *Let $\boldsymbol{F}$ be a forest, $S, T \in \mathcal{D}(\boldsymbol{F})$ such that $S \subseteq T$ and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Then $\mathcal{P}(S) \cong \mathcal{P}(T)/X_S^T$.*

Proof. Due to Lemma 29, $\mathcal{P}(T) \cong \mathcal{P}(F)/X_T$. Observe that $X_S^T \cong X_S/X_T$, thus the result follows as a direct consequence of the second isomorphism theorem (Theorem 6.15 of [18]). $\qquad\square$

Lemma 31. *Let A be a nontrivial MTL-algebra. Then A/P is an MTL-chain if and only if P is a nontrivial prime filter.*

Lemma 32. *Let $\boldsymbol{F}$ be a forest, $S \in \mathcal{D}(\boldsymbol{F})$ and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Then X_S is prime if and only if S is totally ordered.*

Proof. Observe that asking S to be totally ordered, is equivalent, by Lemma 26, to asking $\bigotimes_{i \in \mathbf{S}} \mathbf{M}_i$ to be an MTL-chain. By Lemma 29, $\bigotimes_{i \in \mathbf{S}} \mathbf{M}_i \cong \bigotimes_{i \in \mathbf{F}} \mathbf{M}_i/X_S$. Hence, the result follows from the first observation and Lemma 31. $\qquad\square$

Let $\mathbf{Shv(P)}$ be the category of sheaves over the Alexandrov space $(P, \mathcal{D}(\mathbf{P}))$. Since the theory of MTL-algebras is algebraic, it is well-known that an MTL-algebra in $\mathbf{Shv(P)}$ is a functor $\mathcal{D}(\mathbf{P})^{op} \to \mathcal{MTL}$ such that the composite presheaf $\mathcal{D}(\mathbf{P})^{op} \to \mathcal{MTL} \to \mathbf{Set}$ is a sheaf.

Lemma 33. *Let $\boldsymbol{F}$ be a forest and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. The presheaf $\mathcal{P} : \mathcal{D}(\boldsymbol{P})^{op} \to \mathcal{MTL}$, with $\mathcal{P}(T) = \bigotimes_{i \in \boldsymbol{T}} \boldsymbol{M}_i$, is an MTL-algebra in $\mathbf{Shv(P)}$.*

Proof. Suppose that $T = \bigcup_{\alpha \in I} S_\alpha$, with $S_\alpha, T \in \mathcal{D}(\mathbf{F})$, for every $\alpha \in I$, and let $h_\alpha \in \mathcal{P}(S_\alpha)$ be a matching family. Thus, for every $\alpha \neq \beta$ in I:

$$h_\alpha|_{S_\alpha \cap S_\beta} = h_\beta|_{S_\alpha \cap S_\beta} \tag{4}$$

Let us consider the following function:

$$
\begin{array}{rccl}
h : & I & \to & \bigcup_{i \in \mathbf{T}} \mathbf{M}_i \\
& i & \mapsto & h_\alpha(i), \text{ if } i \in S_\alpha
\end{array}
$$

Observe that (4) guarantees that h is well defined. To check that $h \in \bigotimes_{i \in \mathbf{T}} \mathbf{M}_i$, let us suppose that $i \in T$ and $h(i) \neq 0_i$. If $j < i$ then, since $T = \bigcup_{\alpha \in I} S_\alpha$, there exists some $\beta \in I$ such that $i \in S_\beta$. In such case, $j \in S_\beta$, since $S_\beta \in \mathcal{D}(\mathbf{F})$. Then $h(i) = h_\beta(i)$ and $h(j) = h_\beta(j)$. Since $h_\beta \in \bigotimes_{i \in \mathbf{S}_\beta} \mathbf{M}_i$, we conclude that $h(j) = h_\beta(j) = 1$. Therefore h amalgamates $\{h_\alpha\}_{\alpha \in I}$. To verify the uniqueness of h, let us suppose that there exists $f \in \mathcal{P}(T)$ such that $f|_{S_\alpha} = h_\alpha$, for every $\alpha \in I$. Then,

$$(f|_{S_\alpha})|_{S_\alpha \cap S_\beta} = (h|_{S_\alpha})|_{S_\alpha \cap S_\beta} = (h|_{S_\beta})|_{S_\alpha \cap S_\beta} = (f|_{S_\beta})|_{S_\alpha \cap S_\beta}.$$

For $i \in S_\alpha$ we have that

$$f(i) = (f|_{S_\alpha})(i) = h_\alpha(i) = (h|_{S_\alpha})(i) = h(i).$$

Since this happens for every $\alpha \in I$, $f = h$. $\qquad\square$

Let $\mathbf{F}$ be a forest and $i \in \mathbf{F}$. Since $\mathcal{P}$ is a presheaf of MTL-algebras, its fiber over i is the set of *germs over i* and is written as $\mathcal{P}_i$ (c.f. II.5 [15]). Recall that $f, g \in \mathcal{P}(S)$ have the same germ at i if there exists some $R \in \mathcal{D}(\mathbf{F})$ with $i \in R$, such that $R \subseteq S \cap T$ and $f|_R = g|_R$. Hence, $\mathcal{P}_i$ results to be a "suitable quotient" of the MTL-algebra $\mathcal{P}(T)$. By

Lemma 33, $\mathcal{P}_i$ can be described as the filtering colimit over those $T \in \mathcal{D}(\mathbf{F})$ such that $i \in T$, i.e.,

$$\mathcal{P}_i = \underrightarrow{lim}_{i \in \mathbf{T}} \, \mathcal{P}(T).$$

Thereby, for every $T \in \mathcal{D}(\mathbf{F})$ the map $\varphi_T : \mathcal{P}(T) \to \mathcal{P}_i$ that sends $h \in \mathcal{P}(T)$ in its equivalence class "*modulo germ at i*" result to be a surjective morphism of MTL-algebras. We write $[h]_T$ for the equivalence class of h in $\mathcal{P}_i$.

Lemma 34. *Let $\boldsymbol{F}$ be a forest and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. For every $i \in F$, $\mathcal{P}(\downarrow i) \cong \mathcal{P}_i$ in $\mathcal{MTL}$.*

Proof. Let $i \in F$ and consider $\varphi_{\downarrow i} : \mathcal{P}(\downarrow i) \to \mathcal{P}_i$. From the above discussion it is clear that $\varphi_{\downarrow i}$ is surjective. To check that it is injective, let $f, g \in \mathcal{P}(\downarrow i)$ be such that $[f]_{\downarrow i} = [g]_{\downarrow i}$. There exists some $R \in \mathcal{D}(\mathbf{F})$ with $i \in R$, such that $R \subseteq \downarrow i$ and $f|_R = g|_R$. Since $\downarrow i$ is the smallest down-set to which i belongs, we get that $R = \downarrow i$. Then $f = g$. Hence $\varphi_{\downarrow i}$ is an isomorphism in $\mathcal{MTL}$. $\qquad \square$

Corollary 35. *Let $\boldsymbol{F}$ be a forest and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Then, the fibers of $\mathcal{P}$ are MTL-chains.*

Proof. Let $i \in \mathbf{F}$ and $\mathcal{P}_i$ be the fiber of $\mathcal{P}$ over i. From Lemma 34, $\mathcal{P}_i \cong \mathcal{P}(\downarrow i) = \bigotimes_{j \in \downarrow i} \mathbf{M}_j$. Since $\mathbf{F}$ is a forest, $\downarrow i$ is a chain, so by Lemma 26 we conclude that $\mathcal{P}_i$ is an MTL-chain. $\qquad \square$

Observe that the same argument used in Example 2 of [3] can be applied to prove that when the index set is a finite chain, the forest product and the ordinal sum of MTL-algebras coincide. The following result will be relevant for the last part of this paper.

Corollary 36. *Let $\boldsymbol{F}$ be a finite forest and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Then for every $j \in \boldsymbol{F}$, $\mathcal{P}_j \cong \bigoplus_{i \in \downarrow j} \boldsymbol{M}_i$.*

Proof. If $\mathbf{F}$ is a finite forest then $\downarrow j$ is a finite chain for every $j \in \mathbf{F}$. From the observed above respect to the forest product of MTL-algebras indexed by a finite chain, we conclude that $\mathcal{P}(\downarrow j) \cong \bigoplus_{i \in \downarrow j} \mathbf{M}_i$, which clearly is an MTL-chain. Therefore, from Lemma 34 the result follows. $\qquad \square$

We can now put together Lemma 33, and Corollary 35 in the following statement:

The forest product of MTL-chains is a sheaf of MTL-algebras over an Alexandrov space whose fibers are MTL-chains.

5   From finite forest products to MTL-algebras

In this section we show that a wide class of finite MTL-algebras can be represented as finite forest products of finite archimedean MTL-chains. To do so, we begin by showing that there exists a contravariant functor $\mathcal{H}$ from the category of finite labeled forests to the category of finite MTL-algebras. Moreover, we will prove that the functor $\mathcal{H}$ is right adjoint to the functor $\mathcal{G}$ of Theorem 22 and the counit of the adjunction is an isomorphism.

It is worth mentioning that this result is strongly based on the characterization of the join-irreducible elements of a finite forest product of finite archimedean MTL-chains.

In general, the unit of the adjoint pair $\mathcal{G} \dashv \mathcal{H}$ is not an isomorphism. In Section 5.1 we present a class of finite MLT-algebras which solves that problem. Finally in Section 5.2 we give a simple description of the forest product of finite MTL-chains in terms of ordinal sums and direct products.

Let $l : \mathbf{F} \to \mathfrak{S}$ and $m : \mathbf{G} \to \mathfrak{S}$ be finite labeled forests. If $(\varphi, \mathcal{F}) : l \to m$ is a morphism of finite labeled forests (see Section 3) then $\varphi : \mathbf{F} \to \mathbf{G}$ is a p-morphism and $\mathcal{F} = \{f_i\}_{i \in \mathbf{F}}$ is a family of morphisms $f_i : (m \circ \varphi)(i) \to l(i)$ of MTL-algebras. Recall that a morphism of posets is a p-morphism if and only if it is open respect to the Alexandrov topologies of the involved posets, so since $F \in \mathcal{D}(\mathbf{F})$ it follows that $\varphi(F) \in \mathcal{D}(\mathbf{G})$. From Lemma 23, we get that $\bigotimes_{k \in \varphi(F)} m(k)$ is an MTL-algebra. Notice that $m \circ \varphi : \mathbf{F} \to \mathfrak{S}$ is a finite labeled forest so we can consider the forest product $\bigotimes_{i \in \mathbf{F}} (m \circ \varphi)(i)$. Since $\bigcup_{k \in \varphi(F)} m(k) = \bigcup_{i \in \mathbf{F}} (m \circ \varphi)(i)$, we define, for every $h \in \bigotimes_{k \in \varphi(F)} m(k)$, the map $\gamma : \bigotimes_{k \in \varphi(F)} m(k) \to \bigotimes_{i \in \mathbf{F}} (m \circ \varphi)(i)$ as the composite

$$F \xrightarrow{\varphi} \varphi(F) \xrightarrow{h} \bigcup_{i \in \mathbf{F}} (m \circ \varphi)(i)$$
$$\underset{\gamma(h)}{}$$

Lemma 37. *The map γ, defined above, is a morphism of MTL-algebras.*

Proof. In order to check that γ is well defined, take $h \in \bigotimes_{k \in \varphi(F)} m(k)$ and consider $i \in F$ such that $\gamma(h)(i) \neq 0_{\varphi(i)}$. Assume $j < i$ in F. By the definition of γ, we get that $h(\varphi(i)) \neq 0_{\varphi(i)}$. From the monotonicity of φ, it follows that $\varphi(j) < \varphi(i)$. Then by assumption, when we have that $h(\varphi(j)) = 1$, and consequently $\gamma(h)(i) = 1$. By Definition 1, we have that $\gamma(h) \in \bigotimes_{i \in \mathbf{F}} (m \circ \varphi)(i)$. The proof of the fact that γ is an homomorphism is straightforward. $\square$

Notice, in addition that the family $\mathcal{F}$ induces a map $\alpha : \bigotimes_{i \in \mathbf{F}} (m \circ \varphi)(i) \to \bigotimes_{i \in \mathbf{F}} l(i)$ defined as $\alpha(g)(i) = f_i(g(i))$, for every $i \in \mathbf{F}$.

Lemma 38. *The map α, defined above, is a morphism of MTL-algebras.*

Proof. Let $g \in \bigotimes_{i \in \mathbf{F}} (m \circ \varphi)(i)$ be such that $\alpha(g)(i) \neq 1$. Let $j \in \mathbf{F}$ be such that $i < j$. Since, $f_i(g(i)) \neq 1$, we get that, by Lemma 5 $g(i) \neq 1$. Hence, by assumption $g(j) = 0_{\varphi(j)}$. Thereby $\alpha(g)(j) = f_j(g(j)) = 0_j$, and by (2) of Lemma 24, we have that $\alpha(g) \in \bigotimes_{i \in \mathbf{F}} l(i)$. The proof of the fact that α is an homomorphism is straightforward. $\square$

Lemmas 37 and 38 allow us to consider the following composite of morphisms of MTL-algebras:

$$\mathcal{P}_m(G) \xrightarrow{\beta} \bigotimes_{k \in \varphi(F)} m(k) \xrightarrow{\gamma} \bigotimes_{i \in \mathbf{F}} (m \circ \varphi)(i) \xrightarrow{\alpha} \mathcal{P}_l(F)$$

where $\mathcal{P}_m(G) = \bigotimes_{k \in \mathbf{G}} m(k)$, $\mathcal{P}_l(F) = \bigotimes_{i \in \mathbf{F}} l(i)$ and $\beta : \mathcal{P}_m(G) \to \bigotimes_{k \in \varphi(F)} m(k)$ is the restriction of $\mathcal{P}_m(G)$ to $\varphi(F)$.

Theorem 39. *The assignments $l \mapsto \mathcal{P}_l(F)$ and $(\varphi, \mathcal{F}) \mapsto \alpha\gamma\beta$ define a contravariant functor*

$$\mathcal{H} : f\mathcal{LF} \to f\mathcal{MTL}.$$

Proof. Let $l : \mathbf{F} \to \mathfrak{S}$, $m : \mathbf{G} \to \mathfrak{S}$ and $n : \mathbf{H} \to \mathfrak{S}$ be finite labeled forests, and $(\varphi, \mathcal{F}) : l \to m$ and $(\psi, \mathcal{G}) : l \to m$ be morphism of finite labeled forests. Let

$$\mathcal{M} = \{ f_i g_{\varphi(i)} : n(\psi(\varphi(i))) \to l(i) \mid i \in \mathbf{F} \}.$$

Consider $s \in \mathcal{H}(n)$ and $i \in \mathbf{F}$. Then from

$$
\begin{aligned}
\mathcal{H}[(\psi, \mathcal{G})(\varphi, \mathcal{F})](s)(i) &= \mathcal{H}(\psi\varphi, \mathcal{M})(s)(i) \\
&= (f_i g_{\varphi(i)})[s(\psi(\varphi(i)))] \\
&= f_i[g_{\varphi(i)}(s(\psi(\varphi(i))))] \\
&= f_i[\mathcal{H}(\psi, \mathcal{G})(s)(\varphi(i))] \\
&= \mathcal{H}(\varphi, \mathcal{F})[\mathcal{H}(\psi, \mathcal{G})(s)](i) \\
&= [\mathcal{H}(\varphi, \mathcal{F})\mathcal{H}(\psi, \mathcal{G})](s)(i)
\end{aligned}
$$

we conclude that $\mathcal{H}[(\psi, \mathcal{G})(\varphi, \mathcal{F})] = \mathcal{H}(\varphi, \mathcal{F})\mathcal{H}(\psi, \mathcal{G})$. Since $id_l = (id_F, \mathcal{I})$, where $\mathcal{I}$ is the family formed by the identities of $\{l(i)\}_{i \in \mathbf{F}}$, it is clear from the definition of $\mathcal{H}$ that $\mathcal{H}(id_l) = id_{\mathcal{H}(l)}$. $\square$

Let $\mathbf{F}$ be a finite forest and $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains. We recall that, in Lemma 24, for every $h \in \bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$ we defined $C_h = \{ i \in \mathbf{F} \mid h(i) \notin \{0_i, 1\} \}$.

Lemma 40. *Let $l : \boldsymbol{F} \to \mathfrak{S}$ be a finite labeled forest. Then $h \in \mathcal{I}(\mathcal{P}_l(F))$ if and only if $C_h = \emptyset$.*

Proof. Observe that $h \in \mathcal{I}(\mathcal{P}_l(F))$ if and only if $h(i)^2 = h(i)$, for every $i \in F$, which is equivalent to say that $h(i) \in \mathcal{I}(l(i))$. Since $l(i)$ is an archimedean MTL-chain, by Proposition 2, the only possible case is $h(i) = 0_i$ or $h(i) = 1$. This concludes the proof. $\square$

Let $\mathbf{F}$ be a finite forest and $S \subseteq F$. We write $Max(S)$ for the maximal elements of S.

Lemma 41. *Let $l : \boldsymbol{F} \to \mathfrak{S}$ be a finite labeled forest and $S \in \mathcal{D}(\boldsymbol{F})$, then*

$$X_S = \{ h \in \mathcal{P}_l(F) \mid h(i) = 1, \text{ for every } i \in Max(S) \}$$

Proof. Let $h \in \mathcal{P}_l(F)$ and suppose $h(i) = 1$ for every $i \in Max(S)$. If $j \in S$, there exists some $i \in Max(S)$ such that $j \leq i$. Since $h(i) \neq 0_i$, $h(j) = 1$ and $h \in X_S$. The other inclusion is straightforward. $\square$

Recall that, from Corollary 10 there exist a unique $h_S \in \mathcal{I}(\mathcal{P}_l(F))$ such that $X_S = \uparrow h_S$. As a direct consequence of Lemmas 40 and 41 we obtain the following result.

Lemma 42. *Let $l : \boldsymbol{F} \to \mathfrak{S}$ be a finite labeled forest and $S \in \mathcal{D}(\boldsymbol{F})$, then*

$$h_S(j) = \begin{cases} 1, & j \leq i, \text{ for some } i \in Max(S) \\ 0_j, & otherwise \end{cases}$$

Corollary 43. *Let $l : \boldsymbol{F} \to \mathfrak{S}$ be a finite labeled forest. The following holds for every $i \in F$:*

1. $X_{\downarrow i}$ is a prime filter of $\mathcal{P}_l(F)$,

2. $X_{\downarrow i} = \{h \in \mathcal{P}_l(F) \mid h(i) = 1\}$,

3. The map

$$h_{\downarrow i}(j) = \begin{cases} 1, & j \leq i \\ 0_j, & otherwise \end{cases}$$

   is a nonzero join-irreducible element of $\mathcal{P}_l(F)$.

Proof. A direct consequence of Lemmas 32, 41, 42 and Corollary 11. $\qquad \square$

Lemma 44. *Let $l : \boldsymbol{F} \to \mathfrak{S}$ be a finite labeled forest and $h \in \mathcal{I}(\mathcal{P}_l(F))$, then $h \in \mathcal{J}(\mathcal{I}(\mathcal{P}_l(F)))$ if and only if $h^{-1}(1)$ is a chain.*

Proof. Let us assume that $h^{-1}(1)$ is a chain, then, since F is finite, $h^{-1}(1) = \downarrow i$ for some $i \in F$. Suppose that there are $g, f \in \mathcal{I}(\mathcal{P}_l(F))$ such that $h = g \vee f$, then $h(k) = g(k) \vee f(k)$, for every $k \in F$. If $k \leq i$, we get that $g(k) \vee f(k) = 1$. Since the top element of $l(k)$ is join-irreducible, by Lemma 25, $g(k) = 1$ or $f(k) = 1$. Consequently, $g(k) = h(k)$ or $f(k) = h(k)$. If $h(k) = 0_k$, the result follows, since 0_k is join-irreducible in $l(k)$. Hence h is join-irreducible. Conversely, suppose that $h \in \mathcal{J}(\mathcal{I}(\mathcal{P}_l(F)))$. If $h^{-1}(1)$ is not a chain, there exist $i, j \in F$ not comparable such that $h(i) = h(j) = 1$. Let us consider the following functions:

$$g(k) = \begin{cases} h(k), & \text{if } k \neq i \\ 0_i, & \text{otherwise} \end{cases} \qquad f(k) = \begin{cases} h(k), & \text{if } k \neq j \\ 0_j, & \text{otherwise} \end{cases}$$

From Lemma 40, it follows that $g, f \in \mathcal{I}(\mathcal{P}_l(F))$. Thereby, $h = g \vee f$, which is in contradiction with the assumption. $\qquad \square$

Lemma 45. *Let $l : \boldsymbol{F} \to \mathfrak{S}$ be a finite labeled forest. There is a poset isomorphism between $\boldsymbol{F}$ and $\mathcal{J}(\mathcal{I}(\mathcal{P}_l(F)))$.*

Proof. Let us to consider $\mu : \mathcal{J}(\mathcal{I}(\mathcal{P}_l(F))) \to F$, defined as $\mu(h) = max\ h^{-1}(1) = i_h$. From Lemma 44, it follows that μ is well defined and is injective. To verify that μ is surjective, take $i \in F$ and define

$$h_i(j) = \begin{cases} 1, & j \leq i \\ 0_j, & \text{otherwise} \end{cases}$$

From Lemma 44, it follows that $h_i \in \mathcal{J}(\mathcal{I}(\mathcal{P}_l(F)))$. It is clear that $\mu(h_i) = i$. In order to check the monotonicity of μ, let us suppose that $h \leq g$, for $h, g \in \mathcal{J}(\mathcal{I}(\mathcal{P}_l(F)))$. From 3. of Corollary 43, we have that $h^{-1}(1) \subseteq g^{-1}(1)$, so $i_h \leq i_g$ and consequently $\mu(h) \leq \mu(g)$. The monotonicity of μ^{-1} is straightforward. $\qquad \square$

Let $l : \mathbf{F} \to \mathfrak{S}$ be a finite labeled forest. For every $i \in F$ we write h_i for the map of 3. in Corollary 43. Let us to consider the assignment $\varphi_l : \mathbf{F} \to \mathcal{J}(\mathcal{I}(\mathcal{P}_l(F)))$, defined as $\varphi_l(i) = h_i$.

Lemma 46. *The assignment φ_l is a p-morphism.*

Proof. The monotonicity of φ_l follows from 3. of Corollary 43. On the one hand, let $i \in F$ and suppose that $g \leq h_i$. Thus, $h(i) = 1$ implies that $g(i) = 1$ and, due to 2. of Corollary 43 we get that $g \in X_{\downarrow i}$. Therefore, $h_i \leq g$. In consequence, $g = \varphi_l(i)$, which was our aim. $\qquad \square$

Lemma 47. *The p-morphism φ_l is an isomorphism.*

Proof. We first prove the injectivity of φ_l. Let $i, j \in F$ be such that $\varphi_l(i) = \varphi_l(j)$. Then $h_i = h_j$. In particular $h_i(i) = h_j(i)$ and $h_j(j) = h_i(j)$, which, by definition of h_i and h_j means that $i \leq j$ and $j \leq i$. The surjectivity of φ_l follows from Lemma 44. Finally, we verify that φ_l^{-1} is also a p-morphism. To do so, notice that for every $h \in \mathcal{P}_l(F)$, $\varphi_l^{-1}(h)$ is just the $i \in F$ described in Lemma 44. We will denote such element as i_h. Let us suppose that $j \leq \varphi_l^{-1}(h)$, then $h(j) = h(i_h) = 1$ so, for every $k \leq j \leq i_h$ we get that $h(k) = h(j) = 1$. Take

$$g(k) = \begin{cases} 1, & k \leq j \\ 0_k, & \text{otherwise} \end{cases}$$

It is clear that $g \in \mathcal{P}_l(F)$, $g \leq h$ and $\varphi_l^{-1}(j) = g$. $\qquad \square$

Observe that, from Lemmas 15 and 46 we have that for every $i \in F$, there exists a unique $a_{\varphi_l(i)} \in \mathcal{J}(\mathcal{I}(\mathcal{P}_l(F)))$ such that $a_{\varphi_l(i)} \prec \varphi_l(i)$. Due to Lemma 17, $\uparrow a_{\varphi_l(i)}/\uparrow\varphi_l(i)$ is an archimedean MTL-chain. Let us consider the assignment $\tau_i : \uparrow a_{\varphi_l(i)} \to l(i)$, defined as $\tau_i(h) = h(i)$. It is clear, from the definition, that τ_i preserves all the binary monoid operations. Moreover, it preserves the residual. If $a_{\varphi_l(i)} \leq f, g$, then for every $j < i$, $f(j) = g(j) = 1$ so $f(j) \leq g(j)$, which means that $(f \to g)(i) = f(i) \to_i g(i)$, and consequently, $\tau_i(f \to g) = \tau_i(f) \to_i \tau_i(g)$. We have proved the following result:

Lemma 48. *The function τ_i, defined above, is a morphism of MTL-algebras.*

Notice that, from the universal property of quotients in $\mathcal{MTL}$, Lemma 48 implies that for every $i \in F$ there exists a unique morphism of MTL-algebras $f_i : (\uparrow a_{\varphi_l(i)})/(\uparrow\varphi_l(i)) \to l(i)$ such that the diagram

$$\uparrow(a_{\varphi_l(i)}) \longrightarrow \uparrow(a_{\varphi_l(i)})/\uparrow(\varphi_l(i))$$
$$\tau_i \searrow \qquad \downarrow f_i$$
$$l(i)$$

commutes.

Lemma 49. *For every $i \in F$, $f_i : (\uparrow a_{\varphi_l(i)})/(\uparrow\varphi_l(i)) \to l(i)$ is an isomorphism.*

Proof. To prove the injectivity of f_i, suppose that $f_i([f]) = f_i([g])$. Then $f(i) = g(i)$. Since $a_{\varphi_l(i)} \leq f, g$, for every $j < i$, $f(j) = g(j) = 1$. Hence, $f(i) \leq g(i)$, for every $i \leq j$, which is equivalent to say that $[f] = [g]$ in $(\uparrow a_{\varphi_l(i)})/(\uparrow\varphi_l(i))$. To check the surjectivity, take $x \in l(i)$. Define

$$h(k) = \begin{cases} 1, & k \leq i \\ x, & k = i \\ 0_k, & \text{otherwise} \end{cases}$$

It is clear that $f_i([h]) = x$. $\qquad \square$

Let $\mathcal{G}$ and $\mathcal{H}$ be the functors from Theorems 22 and 39, respectively.

Lemma 50. *Let $l : \mathbf{F} \to \mathfrak{S}$ be a finite labeled forest. Then $\mathcal{G} \circ \mathcal{H}$ is isomorphic to the identity functor in $f\mathcal{LF}$.*

Proof. A direct application of Lemmas 47 and 49. $\qquad\qquad\square$

Proposition 51. *The functor $\mathcal{G}$ is left adjoint to the functor $\mathcal{H}$. Moreover, $\mathcal{G}$ is full and faithful.*

Proof. Let M be a finite MTL-algebra, $l : \mathbf{F} \to \mathfrak{S}$ be a finite labeled forest and $f : M \to \mathcal{P}_l(F)$ be a morphism of MTL-algebras. By Lemma 18, there exists a unique p-morphism $f^* : \mathcal{J}(\mathcal{I}(\mathcal{P}_l(F))) \to \mathcal{J}(\mathcal{I}(M))$. Let $\varphi : F \to \mathcal{J}(\mathcal{I}(M))$ be defined as $\varphi = f^*\mu^{-1}$, where μ is the isomorphism between $\mathbf{F}$ and $\mathcal{J}(\mathcal{I}(\mathcal{P}_l(F)))$ given in Lemma 45. It is clear that φ is a p-morphism. On the other hand, if we write h_i for $\mu^{-1}(i)$, then from Lemma 20, it follows that for every $i \in F$ there exists a morphism of MTL-algebras $\overline{f}_{h_i} : \uparrow a_{f^*(h_i)} \to \uparrow a_{h_i}$ which determines a unique morphism of MTL-algebras $f_{h_i} : (\uparrow a_{f^*(h_i)})/(\uparrow f^*(h_i)) \to (\uparrow a_{h_i})/(\uparrow h_i)$. From Lemma 49 we get that $(\uparrow a_{h_i})/(\uparrow h_i) \cong l(i)$. Let us to consider $f_i : (\uparrow a_{f^*(h_i)})/(\uparrow h_i) \to l(i)$ as the composition of f_{h_i} with the isomorphism of Lemma 49. This concludes the proof.

$\qquad\qquad\square$

We conclude this section by noticing that the counit of the adjunction of Proposition 51, is not an isomorphism. In order to check this fact, consider the MTL-algebra A whose underlying poset is $0 < x < e < 1$ and whose product is given by the following table:

$\cdot$	1	e	x	0
1	1	e	x	0
e	e	e	0	0
x	x	0	0	0
0	0	0	0	0

Observe that $\mathcal{J}(\mathcal{I}(A)) = \{1, e\}$, and $\mathcal{G}(A) = l_A : \mathcal{J}(\mathcal{I}(A)) \to \mathfrak{S}$, is a finite labeled forest which satisfies $l_A(1) \cong l_A(e) \cong \mathbf{2}$. Since $\mathcal{J}(\mathcal{I}(A))$ is a chain, then, from Lemma 34 and Corollary 36 we obtain that

$$\mathcal{H}(\mathcal{G}(A)) = \mathcal{H}(l_A) = \mathcal{P}_{l_A}(\downarrow 1) = l_A(1) \otimes l_A(e) = l_A(1) \oplus l_A(e) \cong \mathbf{3},$$

where $\mathbf{3}$ is the MTL-chain of three elements.

5.1 The duality theorem

In this section we present a duality theorem between the class of locally unital finite MTL-algebras and finite labeled forests. To do so, we restrict the results obtained in Section 5 to the class of locally unital finite MTL-algebras.

Definition 2. *Let M be a finite MTL-algebra. An element $e \in \mathcal{I}(M)^*$ is said to be a local unit if for every $x \leq e$, $ex = x$.*

Lemma 52. *Let M be a finite MTL-algebra and $e \in \mathcal{I}(M)^*$. The following are equivalent:*

1. *e is a local unit.*

2. *$ey = e \wedge y$, for every $y \in M$.*

Proof. Let us assume that e is a *local unit* and $y \in M$. Since $e \wedge y \leq y$, $e(e \wedge y) \leq ey$. Hence $e \wedge y \leq ey$, because $e \wedge y \leq e$ and $e(e \wedge y) = e \wedge y$. The other case is straightforward. $\square$

Definition 3. *A finite MTL-algebra M is said to be locally unital if every nonzero idempotent satisfies any of the equivalent conditions of Lemma 52.*

Note that not every locally unital MTL-algebra is a BL-algebra, since, for example, every simple MTL-algebra is locally unital and not every simple MTL-algebra is a BL-algebra. Furthermore, it is the case that every BL-algebra is a locally unital MTL-algebra, since the divisibility condition implies that every idempotent different from 0 is a local unit. Moreover, there are even finite MTL-algebras which are not locally unital. As an example, one may consider the MTL-algebra A of four elements, which was presented at the end of the last section.

Let M be a locally unital MTL-algebra. For the rest of this section we will write F_M to denote $\mathcal{J}(\mathcal{I}(M))$.

Remark 6. *Observe that as a direct consequence of Definition 2, it follows that for every $e \in F_M$, $\uparrow a_e / \uparrow e \cong [a_e, e]$.*

Let M be a locally unital MTL-algebra and $m \in Max(F_M)$. In what follows we will denote the set $(\downarrow m) \cap F_M$ simply by $(\downarrow m)$.

Lemma 53. *Let M be a locally unital MTL-algebra and $m \in Max(F_M)$. Then, for every $x \in M$ there exists some $e \in \downarrow m$ such that $a_e \leq x$.*

Proof. Suppose that there exist $y > 0$ such that for every $e \in \downarrow m$ it holds that $a_e \not\leq y$. In particular, if $n = min(\downarrow m)$, it follows that $a_n = 0$; so $0 \not\leq x$, which is clearly absurd. $\square$

Lemma 54. *For every locally unital MTL-algebra M and $m \in Max(F_M)$, $M/(\uparrow m) \cong \bigoplus_{e \leq m}[a_e, e]$.*

Proof. In order to simplify the notation for this proof, we write B for $\bigoplus_{e \leq m}[a_e, e]$. Let us consider the map $f : M \to B$, defined as $f(x) = x \wedge m$. It is clear that f is a surjective morphism of MTL-algebras such that $f(m) = 1 \in B$. We stress that B has the universal property of $M/\uparrow m$. To prove it, let $g : M \to E$ be a morphism of MTL-algebras such that $g(m) = 1_E$. For every $z \in B$ define $\psi(z) = g(z)$. It easily follows that $\psi \circ f = g$, and since f is surjective, then ψ is unique. Hence $M/(\uparrow m) \cong B$. $\square$

Let M be a locally unital MTL-algebra. Recall that $\mathcal{G}(M) = l_M : F_M \to \mathfrak{S}$ is a finite labeled forest, so, from Lemma 45, it follows that $F_M \cong \mathcal{J}(\mathcal{I}(\mathcal{P}_{l_M}(F_M)))$. As a consequence, we get that $\bigoplus_{e \leq m}[a_e, e] \cong \mathcal{P}_{l_M}(\downarrow m)$. Explicitly, such assignment is defined for every $z \in \bigoplus_{e \leq m}[a_e, e]$ by:

$$h_z(c) = \begin{cases} m, & c \le e_z \\ z, & c = e_z \\ a_c, & c > e_z \end{cases} \tag{5}$$

Here e_z is the unique idempotent join-irreducible below m such that $a_{e_z} \le z \le e_z$.

Observe that $F_M = \bigcup_{m \in F_M}(\downarrow m)$. Hence the family $\mathcal{R} = \{\downarrow m\}_{m \in M}$ is a covering for F_M. Let $f_m : M \to \mathcal{P}_{l_M}(\downarrow m)$ be defined as $f_m(x) = h_{x \wedge m}$.

Lemma 55. *Let M be a locally unital MTL-algebra. For every $x \in M$, the family $\{f_m(x)\}_{m \in Max(F_M)}$ is a matching family for the covering $\mathcal{R}$.*

Proof. Let $m, n \in Max(F_M)$. Since F_M is a forest, $(\downarrow m) \cap (\downarrow n) = \emptyset$ or $(\downarrow m) \cap (\downarrow n) \ne \emptyset$. In the first case, the result holds, because by Lemma 33, $\mathcal{P}_{l_M}$ is a sheaf. In the second case, there exists an $e \in F_M$ with $e \le m, n$ such that $(\downarrow m) \cap (\downarrow n) = (\downarrow e)$. Observe that

$$(x \wedge m) \wedge e = x \wedge (m \wedge e) = x \wedge e = x \wedge (n \wedge e) = (x \wedge n) \wedge e.$$

Then, from the description of $h_{x \wedge m}$ and $h_{x \wedge n}$ of equation (5), we obtain that $h_{x \wedge m}|_{\downarrow e} = h_{x \wedge n}|_{\downarrow e}$. $\square$

Recall that Lemma 33 states that $\mathcal{P}_{l_M}$ is a sheaf so, from Lemma 55, we obtain that every $x \in M$ determines an amalgamation h_x for the family $\{f_m(x)\}_{m \in Max(F_M)}$. This fact allows us to consider the assignment $f_M : M \to \mathcal{P}_{l_M}(F_M)$, defined as $f(x) = h_x$. Observe that by construction f is a morphism of MTL-algebras.

Lemma 56. *For every locally unital MTL-algebra M, the assignment f_M is an isomorphism.*

Proof. Only remains to check that f_M is bijective. To prove the injectivity of f_M, suppose $h_x = h_y$, then, since h_x and h_y are the amalgamations of the families $\{f_m(x)\}_{m \in Max(F_M)}$ and $\{f_m(y)\}_{m \in Max(F_M)}$, respectively, it follows that $h_{x \wedge m} = h_{y \wedge m}$ for every $m \in Max(F_M)$. Then, from equation (5), it follows that $x \wedge m = y \wedge m$, for every $m \in Max(F_M)$. Then, $\bigvee_{m \in Max(F_M)}(x \wedge m) = \bigvee_{m \in Max(F_M)}(y \wedge m)$. Thereby, $x \wedge \bigvee_{m \in Max(F_M)} m = y \wedge \bigvee_{m \in Max(F_M)} m$. Since $\bigvee_{m \in Max(F_M)} m = 1$, thus $x = y$.

Finally, to prove the surjectivity of f_M, let $h \in \mathcal{P}(F_M)$. Then, $h|_{\downarrow m} \in \mathcal{P}(\downarrow m)$ for every $m \in Max(F_M)$. Since $\mathcal{P}(\downarrow m) \cong \bigoplus_{e \le m}[a_e, e]$ we will write z_m for the unique element of $\bigoplus_{e \le m}[a_e, e]$ which corresponds to $h|_{\downarrow m}$. Observe that for every $m, n \in Max(F_M)$ we have that

$$(h|_{\downarrow m})|_{\downarrow m \cap \downarrow n} = (h|_{\downarrow n})|_{\downarrow m \cap \downarrow n}$$

Thereby $z_m \wedge e_{mn} = z_n \wedge e_{mn}$, where e_{mn} is the greatest $e \in F_M$ below m and n. Hence

$$z_m \wedge n = (z_m \wedge e_{mn}) \wedge (m \wedge n) = (z_n \wedge e_{mn}) \wedge (m \wedge n) = z_n \wedge m, \tag{6}$$

since $z_m \le m$ and $z_n \le n$. If we consider $x = \bigvee_{m \in Max(F_M)} z_m$, then applying equation (6), in the following calculation

$$x \wedge n = \bigvee_{m \in Max(F_M)} (z_m \wedge n) = \bigvee_{m \in Max(F_M)} (z_n \wedge m) = z_n \wedge (\bigvee_{m \in Max(F_M)} m) = z_n,$$

we obtain that $h_x|_{\downarrow n} = h|_{\downarrow n}$ for every $n \in Max(F_M)$. Thereby, since h is the amalgamation of the family $\{h|_{\downarrow m}\}_{m \in Max(F_M)}$ and $\mathcal{P}_{l_M}$ is a sheaf, it follows that $h_x = h$. $\square$

Let us to consider the following quasi-equation in the language of MTL-algebras:

(LU) $(x^2 = x) \land (x \lor y = x) \Rightarrow (xy = x)$

By Definition 3, it is clear that (LU) holds for every finite locally unital MTL-algebra. Recall that if $l : \mathbf{F} \to \mathfrak{S}$ is a finite labeled forest and $i \in F$, then, by definition, $l(i)$ is an archimedean MTL-chain. From Corollary 3, this means that $\mathcal{I}(l(i)) = \{0_{l(i)}, 1\}$. By Definition 3, it follows that $l(i)$ is a finite locally unital MTL-algebra. Since $\mathcal{P}_l(F)$ is, by definition, a subalgebra of $\prod_{i \in F} l(i)$ and quasi-equations are preserved by direct products and subalgebras then, $\mathcal{P}_l(F)$ is a locally unital MTL-algebra. Hence, we have proved the following result:

Lemma 57. *For every finite labeled forest* $l : \mathbf{F} \to \mathfrak{S}$, *the finite MTL-algebra* $\mathcal{H}(l)$ *is locally unital.*

If we write $lu\mathcal{MTL}$ for the category of locally unital finite MTL-algebras, Lemma 57 allows us to restate Theorem 39 as follows:

Theorem 58. *The assignments* $l \mapsto \mathcal{P}_l(F)$ *and* $(\varphi, \mathcal{F}) \mapsto \alpha\gamma\beta$ *define a contravariant functor*

$$\mathcal{H} : f\mathcal{LF} \to lu\mathcal{MTL}.$$

Now, if we write $\mathcal{G}^*$ for the restriction of the functor $\mathcal{G}$ of Theorem 22 to the category $lu\mathcal{MTL}$, then, as a consequence of Proposition 51 and Theorem 58, we obtain that $\mathcal{G}^* \dashv \mathcal{H}$ and that the unit is an isomorphism. Notice that the assignment $f_M : M \to \mathcal{P}_{l_M}(F_M)$ is the counit of the latter adjunction and, by Lemma 56, it is an isomorphism. We have proved the main result of this paper:

Theorem 59. *The categories* $lu\mathcal{MTL}$ *and* $f\mathcal{LF}$ *are dually equivalent.*

5.2   An explicit description of finite forest products

The aim of this section is to bring a characterization of the forest product of finite MTL-algebras in terms of ordinal sums and direct products. Unlike the rest of this work, the methods used in this part are completely recursive and do not require any further knowledge. Finally, we recall that, throughout this section, the symbol $\oplus$ will be used indistinctly, to denote the ordinal sum of posets and the ordinal sum of MTL-algebras.

In [1], Aguzzoli suggest that every finite forest can be built recursively. We adapt those ideas in the following definition:

Definition 4. *The class of finite forests* **fFor**, *is the smallest collection of finite posets satisfying the following conditions:*

(F1) **1** $\in$ **fFor**,

(F2) *If* $\mathbf{F} \in$ **fFor**, *then* **1** $\oplus \mathbf{F} \in$ **fFor**,

(F3) *If* $\mathbf{F}_1, ..., \mathbf{F}_m \in$ **fFor** *then* $\biguplus_{k=1}^m \mathbf{F}_k \in$ **fFor**.

Recall that every finite forest $\mathbf{F}$ can be expressed as a finite disjoint union of finite trees. Hence each finite forest can be written as $\mathbf{F} = \biguplus_{k=1}^n \mathbf{T}_k$, where $\mathbf{T}_k$ is a finite tree. We call the family $\{\mathbf{T}_k\}$ the *family of component trees* of $\mathbf{F}$.

Lemma 60. *Let $\boldsymbol{F}$ be a finite forest and $l : \boldsymbol{F} \to \mathfrak{S}$ be a finite labeled forest. Then*

$$\mathcal{P}_l\left(\biguplus_{k=1}^{n} \boldsymbol{T}_k\right) \cong \mathcal{P}_{l_1}(\boldsymbol{T}_1) \times ... \times \mathcal{P}_{l_n}(\boldsymbol{T}_n)$$

where $l_i = l|_{\boldsymbol{T}_i}$, for every $i = 1, ..., n$.

Proof. Consider $\varphi : \mathcal{P}_l(\biguplus_{k=1}^n \mathbf{T}_k) \to \mathcal{P}_{l_1}(\mathbf{T}_1) \times ... \times \mathcal{P}_{l_n}(\mathbf{T}_n)$ defined as $\varphi(h) = (h|_{\mathbf{T}_1}, ..., h|_{\mathbf{T}_n})$. Also, consider $\tau : \mathcal{P}_{l_1}(\mathbf{T}_1) \times ... \times \mathcal{P}_{l_n}(\mathbf{T}_n) \to \mathcal{P}_l(\biguplus_{k=1}^n \mathbf{T}_k)$ defined as $\tau(h_1, ..., h_n) = h$, with $h(i) = h_j(i)$ if $i \in \mathbf{T}_j$. It is clear that φ and τ are well defined morphisms of MTL-algebras and that one is the inverse of the other. $\qquad\square$

Lemma 61. *Let $\boldsymbol{F}$ be a finite forest and $l : \boldsymbol{F} \to \mathfrak{S}$ be a finite labeled forest. If $\boldsymbol{F} = 1 \oplus \boldsymbol{F}_0$, where $\boldsymbol{F}_0$ is a finite forest, then*

$$\mathcal{P}_l(F) \cong l(\perp) \oplus \mathcal{P}_{l_0}(F_0)$$

where $\perp = Min(F)$ and $l_0 = l|_{\boldsymbol{F}_0}$.

Proof. Let us assume $\mathbf{F} = 1 \oplus \mathbf{F}_0$ and suppose that $h \in \mathcal{P}_l(F)$. If $A_h = \{i \in F \mid h(i) \neq 1\}$, then either $\perp \in A_h$, $\perp \notin A_h$ or $h(\perp) = 0_\perp$. In the first case, from the assumption it follows that $h(j) = 0_j$ for every $j \in F_0$. In the second case, again, by assumption, we obtain that $h(\perp) = 1$. The final case implies that $h = 0$. Thereby, h represents either an element of $l(\perp)$ (those with $h(\perp) \neq 1$) or an element of $\mathcal{P}_{l_0}(F_0)$ considered as $h = h|_{F_0}$. Based on this fact we consider $p : \mathcal{P}_l(F) \to l(\perp) \oplus \mathcal{P}_{l_0}(F_0)$, defined for every $h \in \mathcal{P}_l(F)$ as

$$p(h) = \begin{cases} h(\perp), & \text{if } h(\perp) \neq 1 \\ h|_{F_0}, & \text{if } h(\perp) = 1 \end{cases}$$

Let $a \in l(\perp) \oplus \mathcal{P}_{l_0}(F_0)$. Then, a is either an element of $l(\perp)$ or $a \in \mathcal{P}_{l_0}(F_0)$. Let us take $q : l(\perp) \oplus \mathcal{P}_{l_0}(F_0) \to \mathcal{P}_l(F)$ as $q(a) = h_a$, where $h_a(i) = a(i)$ if $i \in F_0$ or

$$h_a(\perp) = \begin{cases} a, & \text{if } a \in l(\perp) \\ 1, & \text{otherwise} \end{cases}$$

It is clear that p and q are well defined morphism of MTL-algebras such that one is the inverse of the other. This concludes the proof. $\qquad\square$

Let $\mathbf{T}$ be a finite tree, $i \in T$ and consider the *set of covering elements* of i:

$$\mathcal{C}_{\mathbf{T}}(i) = \{j \in T \mid i \prec j\}$$

where $\prec$ denotes the covering relation in posets.

Definition 5. *Let $\boldsymbol{T}$ be a finite tree and $l : \boldsymbol{T} \to \mathfrak{S}$ be a finite labeled forest. For every $i \in T$ let us define recursively the following MTL-algebra:*

$$M_{\boldsymbol{T}}(i) = \begin{cases} l(i), & i \in Max(T) \\ l(i) \oplus \Pi_{j \in \mathcal{C}_{\boldsymbol{T}}(i)} M_{\boldsymbol{T}}(j), & i \notin Max(T). \end{cases}$$

In the following, we will write $\mathcal{K}_l(\mathbf{T})$ for $M_{\mathbf{T}}(m)$, where m is the bottom element of $\mathbf{T}$.

Let $\mathbf{F}$ be a finite forest and $\{\mathbf{T}_k\}$ be its collection of component trees. If $l : \mathbf{F} \to \mathfrak{S}$ is a finite labeled forest let us consider the MTL-algebra:

$$\mathcal{K}_l(\mathbf{F}) = \prod_{k=1}^{n} \mathcal{K}_l(\mathbf{T}_k) \tag{7}$$

Proposition 62. *Let* $l : \boldsymbol{F} \to \mathfrak{S}$ *be a finite labeled forest. Then* $\mathcal{P}_l(F) \cong \mathcal{K}_l(\boldsymbol{F})$.

Proof. We prove the Proposition by induction over **fFor**. If $F = \mathbf{1}$, the conclusion is trivial. On the one hand, let us suppose that $\mathbf{F} = \mathbf{1} \oplus \mathbf{F}_0$, with $\mathbf{F}_0 \in$ **fFor** be such that $\mathcal{P}_l(F_0) \cong \mathcal{K}_l(\mathbf{F}_0)$. From Lemma 61 and the inductive hypothesis, it follows that

$$\mathcal{P}_l(F) \cong l(\bot) \oplus \mathcal{P}_{l_0}(F_0) \cong l(\bot) \oplus \mathcal{K}_{l_0}(\mathbf{F}_0).$$

Since $\mathbf{F}_0$ is a finite forest, $\mathbf{F}_0 = \biguplus_{k=1}^{r} \mathbf{T}_k^0$. Thus by equation (7):

$$\mathcal{K}_{l_0}(\mathbf{F}_0) = \prod_{k=1}^{r} \mathcal{K}_{l_0}(\mathbf{T}_k^0) = \prod_{k=1}^{r} M_{\mathbf{T}_k^0}(m_k),$$

with $m_k = Min(\mathbf{T}_k^0)$. Since $\mathcal{C}_{\mathbf{F}}(\bot) = \{m_1, ..., m_r\}$, by Definition 5, we have that $\mathcal{P}_l(F) \cong l(\bot) \oplus \mathcal{K}_{l_0}(\mathbf{F}_0) = \mathcal{K}_l(\mathbf{F})$. Finally, assume that $\mathbf{F} = \biguplus_{k=1}^{n} \mathbf{F}_k$, with $\mathbf{F}_k \in$ **fFor** be such that $\mathcal{P}_l(F_k) \cong \mathcal{K}_l(\mathbf{F}_k)$, for every $i = 1, ..., n$. Since $\mathbf{F}_k = \biguplus_{i=1}^{m_k} \mathbf{T}_{ki}$, with $\{\mathbf{T}_{ki}\}$ be the family of component trees of $\mathbf{F}_k$, then $\mathbf{F} = \biguplus_{k=1}^{n} \biguplus_{i=1}^{m_k} \mathbf{T}_{ki}$. By Lemma 60, we have that

$$\mathcal{P}_l(F) \cong \mathcal{P}_l(T_{11}) \times ... \times \mathcal{P}_l(T_{1m_1}) \times ... \times \mathcal{P}_l(T_{n1}) \times ... \times \mathcal{P}_l(T_{nm_n}). \tag{8}$$

Since the direct product of algebras is associative, by Lemma 60, $\mathcal{P}_l(F_k) \cong \prod_{i=1}^{m_k} \mathcal{P}_l(T_{ki})$. Then, from equation (8), we get that $\mathcal{P}_l(F) \cong \prod_{k=1}^{n} \mathcal{P}_l(F_k)$. Hence, by inductive hypothesis we have that $\mathcal{P}_l(F) \cong \prod_{k=1}^{n} \mathcal{K}_l(\mathbf{F}_k)$. From equation (7), we have that $\mathcal{K}_l(\mathbf{F}_k) = \prod_{i=1}^{m_k} \mathcal{K}_l(\mathbf{T}_{ki})$, so, since the family of component trees of $\mathbf{F}$ is $\bigcup_{k,i=1}^{n,m_k}\{\mathbf{T}_{ki}\}$, again, from equation (7), we conclude that $\mathcal{P}_l(F) \cong \mathcal{K}_l(\mathbf{F})$. $\square$

In the following example we illustrate how to build an MTL-algebra by applying equation (7) and Definition 5.

Example 1. *Let* $l : \boldsymbol{F} \to \mathfrak{S}$ *be a finite labeled forest and regard the following finite tree* $\boldsymbol{F}$:
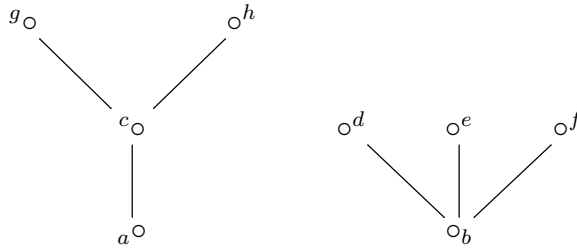
As we can see, $\boldsymbol{F} = \boldsymbol{T}_1 \uplus \boldsymbol{T}_2$, where $\boldsymbol{T}_1 = \{a, c, g, h\}$ and $\boldsymbol{T}_2 = \{b, d, e, f\}$. Since $Min(\boldsymbol{T}_1) = \{a\}$, $Min(\boldsymbol{T}_2) = \{b\}$, from equation (7), it follows that:

$$\mathcal{K}_l(\boldsymbol{F}) = \mathcal{K}_l(\boldsymbol{T}_1) \times \mathcal{K}_l(\boldsymbol{T}_2) = M_{\boldsymbol{T}_1}(a) \times M_{\boldsymbol{T}_2}(b).$$

Observe that $\mathcal{C}_{\boldsymbol{T}_1}(a) = \{c\}$ and $\mathcal{C}_{\boldsymbol{T}_2}(b) = \{d, e, f\}$. Since $a \notin Max(\boldsymbol{T}_1)$ and $b \notin Max(\boldsymbol{T}_2)$, applying Definition 5 to $M_{\boldsymbol{T}_1}(a)$ and $M_{\boldsymbol{T}_2}(b)$ respectively, we obtain:

$$\mathcal{K}_l(\boldsymbol{F}) = [l(a) \oplus M_{\boldsymbol{T}_1}(c)] \times [l(b) \oplus (M_{\boldsymbol{T}_2}(d) \times M_{\boldsymbol{T}_2}(e) \times M_{\boldsymbol{T}_2}(f))].$$

Since $c \notin Max(\boldsymbol{T}_1)$ and $\mathcal{C}_{\boldsymbol{T}_1}(c) = \{g, h\}$ but $d, e, f \in Max(\boldsymbol{T}_2)$; again, applying Definition 5 to $M_{\boldsymbol{T}_1}(c), M_{\boldsymbol{T}_2}(d)$, $M_{\boldsymbol{T}_2}(e)$ and $M_{\boldsymbol{T}_2}(f)$ respectively, we get:

$$\mathcal{K}_l(\boldsymbol{F}) = [l(a) \oplus (l(c) \oplus (M_{\boldsymbol{T}_1}(g) \times M_{\boldsymbol{T}_1}(h)))] \times [l(b) \oplus (l(d) \times l(e) \times l(f))].$$

Finally, since $g, h \in Max(\boldsymbol{T}_1)$, applying Definition 5 to $M_{\boldsymbol{T}_1}(g)$ and $M_{\boldsymbol{T}_1}(h)$, we conclude that:

$$\mathcal{K}_l(\boldsymbol{F}) = [l(a) \oplus (l(c) \oplus (l(g) \times l(h)))] \times [l(b) \oplus (l(d) \times l(e) \times l(f))].$$

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