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EXISTENCE AND UNIQUENESS OF THE P-GENERALIZED MODIFIED ERROR FUNCTION

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Abstract. In this article, we define a p-generalized modified error function as the solution to a non-linear ordinary differential equation of second order, with a Robin type boundary condition at $x = 0$. We prove existence and uniqueness of a non-negative C^∞ solution by using a fixed point argument. We show that the p-generalized modified error function converges to the p-modified error function defined as the solution to a similar problem with a Dirichlet boundary condition. In both problems, for $p = 1$, the generalized modified error function and the modified error function are recovered. In addition, we analyze the existence and uniqueness of solution to a problem with a Neumann boundary condition.

1. Introduction

Ceratani et al. [5] studied a fusion Stefan problem with variable thermal conductivity and a Robin boundary condition at the fixed face $x = 0$. They studied

$$\rho c\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\Big(k(T)\frac{\partial T}{\partial x}\Big), \quad 0 < x < s(t),\ t > 0, \tag{1.1}$$

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = \frac{h}{\sqrt{t}}[T(0,t) - T_0], \quad t > 0, \tag{1.2}$$

$$T(s(t),t) = T_f, \quad t > 0, \tag{1.3}$$

$$k\left(T(s(t),t)\right)\frac{\partial T}{\partial x}(s(t),t) = -\rho l\dot{s}(t), \quad t > 0, \tag{1.4}$$

$$s(0) = 0, \tag{1.5}$$

where the unknown functions are the temperature T and the free boundary s separating both phases. The parameters $\rho > 0$ (density), $l > 0$ (latent heat per unit mass), T_f (phase-change temperature), $T_0 > T_f$ (bulk temperature), $h > 0$ (coefficient that characterizes the heat transfer at $x = 0$), and c (specific heat) are all known constants.

Problem (1.1)–(1.5) is a phase-change problem known in the literature as a Stefan problem. It corresponds to the melting of a semi-infinite material which is initially solid at the phase-change temperature T_f. As $T_0 > T_f$, a phase-change interface

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$x = s(t)$, $t > 0$ is beginning at $t = 0$ with the initial position $s(0) = 0$. Then, the temperature of the liquid phase is $T = T(x,t)$ defined in the domain $0 < x < s(t)$, $t > 0$, and the temperature of the solid phase is $T = 0$ defined in the domain $x > s(t), t > 0$.

In [6], the thermal conductivity k is defined as

$$k(T) = k_0\Big(1 + \delta\Big(\frac{T - T_f}{T_0 - T_f}\Big)\Big), \tag{1.6}$$

where δ is a given positive constant and k_0 is the reference thermal conductivity. The existence of a solution to (1.1)–(1.5) when the thermal conductivity $k(T)$ is defined by (1.6) has been proved through the existence of what the authors in [5] called a *generalized modified error function* (GME), which is defined as the solution to the ordinary differential

$$[(1 + \delta y(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty, \tag{1.7a}$$

$$\big(1 + \delta y(0)\big)y'(0) - \gamma y(0) = 0, \tag{1.7b}$$

$$y(+\infty) = 1, \tag{1.7c}$$

where

$$\gamma = 2\,\mathrm{Bi}, \quad \mathrm{Bi} = \frac{h\sqrt{\alpha_0}}{k_0} \quad \text{(generalized Biot number)}, \tag{1.8}$$

$$\alpha_0 = \frac{k_0}{\rho c} \quad \text{(thermal diffusivity)}. \tag{1.9}$$

The solution to (1.1)–(1.5) is given as a function of the solution of (1.7) through the similarity variable $x/(2\sqrt{\alpha_0 t})$, see [5, 6, 12]. More explanations are given in [1, 9, 14].

Motivated by [10] we define a generalized thermal conductivity as

$$k(T) = k_0\Big(1 + \delta\Big(\frac{T - T_f}{T_0 - T_f}\Big)^p\Big), \quad p \geq 1. \tag{1.10}$$

Then the existence of a solution to (1.1)–(1.5) with k given by (1.10) will be studied through the *p-generalized modified error function* (p-GME) which we define as the solution to the nonlinear differential problem

$$[(1 + \delta y^p(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty, \tag{1.11a}$$

$$(1 + \delta y^p(0))y'(0) - \gamma y(0) = 0, \tag{1.11b}$$

$$y(+\infty) = 1\,. \tag{1.11c}$$

Note that when $p = 1$, we recover the problem studied in [4, 5] and originally defined in [6, 12]. Others studies for $p = 1$ can be found in [2, 13]. In that sense, the p-GME function constitutes a mathematical generalization of the GME function.

With the purpose of proving existence and uniqueness of the p-GME function, i.e. a solution to (1.11), we define a convenient contracting mapping, in Section 2. In Section 3, we study the asymptotic behavior of the p-GME function when $\gamma \to \infty$. We will show that this function converges to the solution of an ordinary differential equation that arises by changing the Robin condition at $x = 0$ [3] by a Dirichlet condition. Finally, in Section 4 we change the Robin condition by a Neumann condition in a solidification process and analyze the existence and uniqueness of a new ordinary differential problem. In conclusion, the aim of this paper is to prove existence and uniqueness of a solution to three ordinary differential problems that have been motivated by Stefan problems. This is done imposing different boundary conditions at the fixed face $x = 0$: Robin, Dirichlet and Neumann conditions.

2. Existence and uniqueness of the p-GME function

Let us define

$$X = \{h : \mathbb{R}_0^+ \to \mathbb{R} : \ h \text{ is a bounded and continuous real-valued function}\}, \quad (2.1)$$

$$K = \{h \in X : \|h\|_\infty \le 1, \ 0 \le h, \ h(+\infty) = 1\}. \quad (2.2)$$

We remark that K is a non-empty closed convex and bounded subset of the Banach space X with the norm

$$\|h\|_\infty = \sup_{x \in \mathbb{R}_0^+} |h(x)| < \infty;$$

see [7, page 2487], [8, page 152], [11, page 132].

In this section we prove existence and uniqueness of the p-GME function (problem (1.11)) by using the Banach fixed point theorem. First, we show that the ordinary differential problem (1.11) becomes equivalent to an integral equation. We consider that γ is a parameter for problem (1.11), and in Section 3 we will study the asymptotic behavior when $\gamma \to \infty$.

Theorem 2.1. *Let $\delta \ge 0$, $\gamma > 0$, $p \ge 1$. For each $\gamma > 0$, the function $y_\gamma \in K$ is a solution to problem* (1.11) *if and only if y_γ is a fixed point to the operator $T_\gamma : K \to K$ given by*

$$T_\gamma(h)(x) = \frac{1 + \gamma \int_0^x f_h(\eta) \mathrm{d}\eta}{1 + \gamma \int_0^\infty f_h(\eta) \mathrm{d}\eta}, \quad x \ge 0, \quad (2.3)$$

with

$$f_h(x) = \frac{1}{\Psi_h(x)} \exp\Big(-2 \int_0^x \frac{\xi}{\Psi_h(\xi)} \mathrm{d}\xi \Big), \quad \Psi_h(x) = 1 + \delta h^p(x). \quad (2.4)$$

Proof. Notice first that for each $y = y_\gamma \in K$ we can easily obtain

$$\frac{\exp(-\eta^2)}{1+\delta} \le f_y(\eta) \le \exp\Big(-\frac{\eta^2}{1+\delta} \Big), \quad (2.5)$$

from where it follows that

$$0 < \frac{\gamma\sqrt{\pi}}{2(1+\delta)} < 1 + \gamma \int_0^\infty f_y(\eta) \mathrm{d}\eta \le 1 + \frac{\gamma\sqrt{1+\delta}\sqrt{\pi}}{2}. \quad (2.6)$$

Taking into account (2.6), $T_\gamma(y)$ is a continuous function, since $y \in X$. Also, according to (2.1)–(2.3) and (2.6), $T_\gamma(y) \in K$.

Through the substitution $v = y'$, the ordinary differential equation (1.7a) is equivalent to

$$-\frac{\Psi_y'(x) + 2x}{\Psi_y(x)} = \frac{v'(x)}{v(x)},$$

from where we obtain

$$y(x) = y(0) + c_0 \int_0^x f_y(\eta) \mathrm{d}\eta.$$

Then, condition (1.7b) is satisfied if and only if $c_0 = \gamma y(0)$. In addition, from (1.7c) we obtain

$$y(0) = \Big(1 + \gamma \int_0^\infty f_y(\eta) \mathrm{d}\eta \Big)^{-1}. \quad (2.7)$$

Therefore, y is a solution to problem (1.11) if and only if y is a fixed point of the operator T_γ, i.e. $y(x) = T_\gamma(y)(x)$ for all $x \geq 0$. Conversely, if y is a fixed point of the operator T_γ we obtain immediately that (1.7c) is verified, and $y(0)$ is given by (2.7). Then, by differentiation (1.7a) and (1.7b) hold, and then y is a solution of (1.11). □

Remark 2.2. The notation y_γ, T_γ is adopted to emphasize the dependence of the solution to (1.11) on γ, although it also depends on p and δ. This fact is going to facilitate the subsequent analysis of the asymptotic behavior of y_γ when $\gamma \to \infty$, to be presented in Section 3.

By Theorem 2.1, we will focus on proving that T_γ is a contracting mapping on K. For that purpose, we need the following lemmas.

Lemma 2.3. *Let $y_1, y_2 \in K$, $\delta \geq 0, \gamma > 0$, $p \geq 1$ and $x \geq 0$. Then, the following estimates hold:*

$$\frac{\sqrt{\pi}}{2(1+\delta)} \leq \Big| \int_0^\infty f_{y_1}(\eta) \mathrm{d}\eta \Big| \leq \sqrt{1+\delta}\frac{\sqrt{\pi}}{2}, \tag{2.8}$$

$$\Big|\frac{1}{\Psi_{y_1}(\eta)} - \frac{1}{\Psi_{y_2}(\eta)}\Big| \leq \delta p \|y_1 - y_2\|_\infty, \tag{2.9}$$

$$\Big| \exp\Big(\int_0^\eta \frac{-2\xi}{\Psi_{y_1}(\xi)}\mathrm{d}\xi\Big) - \exp\Big(\int_0^\eta \frac{-2\xi}{\Psi_{y_2}(\xi)}\mathrm{d}\xi\Big)\Big| \leq \frac{2\delta p \eta^2}{\exp(\frac{\eta^2}{1+\delta})}\|y_1 - y_2\|_\infty, \tag{2.10}$$

$$\int_0^x |f_{y_1}(\eta) - f_{y_2}(\eta)| \mathrm{d}\eta \leq \frac{\sqrt{\pi}}{2}\delta p \sqrt{1+\delta}(2+\delta)\|y_1 - y_2\|_\infty, \tag{2.11}$$

$$\begin{aligned} &\Big|\frac{1}{1+\gamma\int_0^\infty f_{y_1}(\eta)\mathrm{d}\eta} - \frac{1}{1+\gamma\int_0^\infty f_{y_2}(\eta)\mathrm{d}\eta}\Big| \\ &\leq \frac{2(1+\delta)^{5/2}}{\gamma\sqrt{\pi}}\delta p(2+\delta)\|y_1 - y_2\|_\infty \end{aligned} \tag{2.12}$$

Proof. We follow the method was developed in [4].

Inequality (2.8) follows from integrating (2.5) in $(0, +\infty)$. For inequality (2.9) we note that from the Mean Value Theorem applied to the function $r(x) = x^p$ and the fact that $1 \leq \Psi_y(x) \leq 1+\delta$ for all $y \in K$, we obtain

$$\Big|\frac{1}{\Psi_{y_1}(\eta)} - \frac{1}{\Psi_{y_2}(\eta)}\Big| \leq \delta \big|y_2^p(\eta) - y_1^p(\eta)\big| \leq \delta p \|y_2 - y_1\|_\infty .$$

For inequality (2.10), applying the Mean Value Theorem to $r(x) = \exp(-2x)$ we have

$$\begin{aligned} &\Big| \exp\Big(\int_0^\eta \frac{-2\xi}{\Psi_{y_1}(\xi)}\mathrm{d}\xi\Big) - \exp\Big(\int_0^\eta \frac{-2\xi}{\Psi_{y_2}(\xi)}\mathrm{d}\xi\Big)\Big| \\ &\leq 2\exp\Big(-\frac{\eta^2}{1+\delta}\Big)\int_0^\eta \Big|\frac{\xi}{\Psi_{y_1}(\xi)} - \frac{\xi}{\Psi_{y_2}(\xi)}\Big|\mathrm{d}\xi \\ &\leq 2\exp\Big(-\frac{\eta^2}{1+\delta}\Big)\eta\int_0^\eta \Big|\frac{1}{\Psi_{y_1}(\xi)} - \frac{1}{\Psi_{y_2}(\xi)}\Big|\mathrm{d}\xi . \end{aligned}$$

Taking into account (2.9) we obtain the corresponding estimate. For inequality (2.11), from items (2.9) and (2.10) we obtain

$$\int_0^x \big|f_{y_1}(\eta) - f_{y_2}(\eta)\big|\mathrm{d}\eta$$

$$
\begin{aligned}
&\le \int_0^x \Big\{\Big|f_{y_1}(\eta) - \frac{\exp(-2\int_0^x \frac{\xi}{\Psi_{y_2}(\xi)}\mathrm{d}\xi)}{\Psi_{y_1}(\eta)}\Big| + \Big|\frac{\exp(-2\int_0^x \frac{\xi}{\Psi_{y_2}(\xi)}\mathrm{d}\xi)}{\Psi_{y_1}(\eta)} - f_{y_2}(\eta)\Big|\Big\}\mathrm{d}\eta \\
&\le \int_0^x \Big\{\frac{1}{\Psi_{y_1}(\eta)}\Big|\exp\Big(\int_0^\eta \frac{-2\xi}{\Psi_{y_1}(\xi)}\mathrm{d}\xi\Big) - \exp\Big(\int_0^\eta \frac{-2\xi}{\Psi_{y_2}(\xi)}\mathrm{d}\xi\Big)\Big| \\
&\quad + \exp\Big(\int_0^\eta \frac{-2\xi}{\Psi_{y_2}(\xi)}\mathrm{d}\xi\Big)\Big|\frac{1}{\Psi_{y_1}(\eta)} - \frac{1}{\Psi_{y_2}(\eta)}\Big|\Big\}\mathrm{d}\eta \\
&\le \|y_1 - y_2\|_\infty \delta p \int_0^x \exp\big(\frac{-\eta^2}{1+\delta}\big)(2\eta^2+1)\mathrm{d}\eta \\
&= \|y_1 - y_2\|_\infty \delta p\sqrt{1+\delta}\big[\frac{\sqrt{\pi}}{2}(2+\delta)\,\mathrm{erf}\big(\frac{x}{\sqrt{1+\delta}}\big) - x\sqrt{1+\delta}\exp\big(\frac{-x^2}{1+\delta}\big)\big] \\
&\le \frac{\sqrt{\pi}}{2}\delta p\sqrt{1+\delta}(2+\delta)\|y_1 - y_2\|_\infty.
\end{aligned}
$$

Inequality (2.12) follows immediately by using (2.6) and (2.11). □

Lemma 2.4. *Let $\gamma > 0$, $p \ge 1$ and*

$$
g_\gamma(x) = xp(1+x)^{3/2}\Big[(2+x)\big(1+(1+x)^{3/2}\big) + \frac{2}{\gamma\sqrt{\pi}}(1+x)\Big], \quad x \ge 0\,.
$$

Then there exist a unique $\delta_\gamma > 0$ such that $g_\gamma(\delta_\gamma) = 1$.

The above lemma follows immediately from the fact that g_γ is an increasing function, $g_\gamma(0) = 0$ and $\lim_{x\to\infty} g_\gamma(x) = +\infty$. Now, we are able to formulate the following result.

Theorem 2.5. *Let $\gamma > 0$ and $p \ge 1$. The problem (1.11) has a unique solution $y_\gamma \in K$ if and only if $0 \le \delta < \delta_\gamma$, where δ_γ is given by Lemma 2.4. Moreover, y_γ is a C^∞ function in $\mathbb{R}^+$ with the following properties:*

$$
y_\gamma'(x) > 0, \quad y_\gamma''(x) < 0, \quad \forall x \ge 0. \tag{2.13}
$$

Proof. Let y_1, $y_2 \in K$ and $x \ge 0$. Taking into account Lemma 2.3, we have

$$
\begin{aligned}
&|T_\gamma(y_1)(x) - T_\gamma(y_2)(x)| \\
&\le \Big|\frac{1+\gamma\int_0^x f_{y_1}(\eta)\mathrm{d}\eta}{1+\gamma\int_0^\infty f_{y_1}(\eta)\mathrm{d}\eta} - \frac{1+\gamma\int_0^x f_{y_2}(\eta)\mathrm{d}\eta}{1+\gamma\int_0^\infty f_{y_1}(\eta)\mathrm{d}\eta}\Big| \\
&\quad + \Big|\frac{1+\gamma\int_0^x f_{y_2}(\eta)\mathrm{d}\eta}{1+\gamma\int_0^\infty f_{y_1}(\eta)\mathrm{d}\eta} - \frac{1+\gamma\int_0^x f_{y_2}(\eta)\mathrm{d}\eta}{1+\gamma\int_0^\infty f_{y_2}(\eta)\mathrm{d}\eta}\Big| \\
&\le \frac{\gamma\int_0^x |f_{y_1}(\eta) - f_{y_2}(\eta)|\mathrm{d}\eta}{|1+\gamma\int_0^\infty f_{y_1}(\eta)\mathrm{d}\eta|} \\
&\quad + \Big|1+\gamma\int_0^x f_{y_2}(\eta)\mathrm{d}\eta\Big|\Big|\frac{1}{1+\gamma\int_0^\infty f_{y_1}(\eta)\mathrm{d}\eta} - \frac{1}{1+\gamma\int_0^\infty f_{y_2}(\eta)\mathrm{d}\eta}\Big| \\
&\le g_\gamma(\delta)\|y_1 - y_2\|_\infty.
\end{aligned}
$$

Then from Lemma 2.4, if $0 \le \delta < \delta_\gamma$ it follows that T_γ is a contracting mapping what allows to apply the Banach fixed point theorem. Therefore, the problem (1.11) has a unique non-negative continuous solution. Moreover, by differentiation and easy computation the solution is a C^∞ function in $\mathbb{R}^+$ with the useful properties (2.13). □

3. Asymptotic behavior of p-GME function when $\gamma \to \infty$

In this section if we consider the Stefan problem (1.1)–(1.5) and we change the Robin condition (1.2) by a Dirichlet condition.

$$T(0,t) = T_0 > 0, \tag{3.1}$$

we obtain the ordinary differential problem

$$[(1+\delta y^p(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty, \tag{3.2a}$$

$$y(0) = 0, \tag{3.2b}$$

$$y(+\infty) = 1\,. \tag{3.2c}$$

For the special case $p = 1$, the solution to this problem is called *modified error function* (ME) and was considered in [2, 4, 5, 6, 12]. In [4] the existence and uniqueness of the ME function was proved. Moreover, if it is considered $\delta = 0$, the classical error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2) dz, \quad x > 0, \tag{3.3}$$

arises as a solution.

In a similar way to the above section we can analyze the existence and uniqueness of the *p-modified error function* (p-ME), which is defined as the solution to problem (3.2) and constitutes a generalization of the ME function.

Now, let us define

$$K^* = \{h \in X : \|h\|_\infty \le 1,\ 0 \le h,\ h(0) = 0,\ h(+\infty) = 1\},$$

where X is given by (2.1). We remark that K^* is a non-empty closed convex and bounded subset of the Banach space X. We will show that the ordinary differential problem (3.2) becomes equivalent to an integral equation.

Theorem 3.1. *Let $\delta \ge 0$, $p \ge 1$. Then the function $y^* \in K^*$ is a solution to* (3.2) *if and only if y^* is a fixed point of the operator $T^* : K^* \to K^*$ given by:*

$$T^*(h)(x) = \frac{\int_0^x f_h(\eta) \mathrm{d}\eta}{\int_0^\infty f_h(\eta) \mathrm{d}\eta}, \quad x \ge 0, \tag{3.4}$$

with f_h defined by (2.4).

Proof. In a similar way as in the proof of Theorem 2.1, the operator T^* is well defined and it is easy to see that

$$y^*(x) = y^*(0) + c_0^* \int_0^x f_y^*(\eta) \mathrm{d}\eta,$$

with $y^*(0) = 0$ and $c_0^* = \big(\int_0^\infty f_h(\eta) \mathrm{d}\eta\big)^{-1}$. Then, using (3.2b) and (3.2c), we obtain (3.4). Therefore, y^* is a solution to (3.2) if and only if y^* is a fixed point of the operator T^*. □

To prove that the operator T^* is a contracting mapping on K^*, we enunciate the following lemmas which proofs are analogous to Lemma 2.3 and Lemma 2.4.

Lemma 3.2. *Let $y_1^*, y_2^* \in K^*$, $\delta \ge 0$, $p \ge 1$ and $x \ge 0$. Then*

$$\Big| \frac{1}{\int_0^\infty f_{y_1^*}(\eta) \mathrm{d}\eta} - \frac{1}{\int_0^\infty f_{y_2^*}(\eta) \mathrm{d}\eta} \Big| \le \frac{2(1+\delta)^{5/2}}{\sqrt{\pi}} \delta p (2+\delta) \|y_1^* - y_2^*\|_\infty.$$

Lemma 3.3. *Let $p \geq 1$ and*
$$g^*(x) = xp(1+x)^{3/2}(2+x)\big(1+(1+x)^{3/2}\big), \quad x \geq 0\,.$$
Then there exists a unique $\delta^* > 0$ such that $g^*(\delta^*) = 1$.

Theorem 3.4. *Problem (3.2) has a unique solution $y^* \in K$ if and only if $0 \leq \delta < \delta^*$, where δ^* is given by Lemma 3.3. Moreover, y^* is a C^∞ function in $\mathbb{R}^+$.*

Proof. Let $y_1^*,\ y_2^* \in K^*$ and $x \geq 0$. Taking into account Lemmas 2.3 and 3.2 we obtain
$$\begin{aligned}
&|T^*(y_1^*)(x) - T^*(y_2^*)(x)| \\
&\leq \Big|\frac{\int_0^x f_{y_1^*}(\eta)\mathrm{d}\eta}{\int_0^\infty f_{y_1^*}(\eta)\mathrm{d}\eta} - \frac{\int_0^x f_{y_2^*}(\eta)\mathrm{d}\eta}{\int_0^\infty f_{y_1^*}(\eta)\mathrm{d}\eta}\Big| + \Big|\frac{\int_0^x f_{y_2^*}(\eta)\mathrm{d}\eta}{\int_0^\infty f_{y_1^*}(\eta)\mathrm{d}\eta} - \frac{\int_0^x f_{y_2^*}(\eta)\mathrm{d}\eta}{\int_0^\infty f_{y_2^*}(\eta)\mathrm{d}\eta}\Big| \\
&\leq \frac{\int_0^x |f_{y_1^*}(\eta) - f_{y_2^*}(\eta)|\mathrm{d}\eta}{|\int_0^\infty f_{y_1^*}(\eta)\mathrm{d}\eta|} + \Big|\int_0^x f_{y_2^*}(\eta)\mathrm{d}\eta\Big|\Big|\frac{1}{\int_0^\infty f_{y_1^*}(\eta)\mathrm{d}\eta} - \frac{1}{\int_0^\infty f_{y_2^*}(\eta)\mathrm{d}\eta}\Big| \\
&\leq g^*(\delta^*)\|y_1^* - y_2^*\|_\infty.
\end{aligned}$$

Then from Lemma 3.3, if $0 \leq \delta < \delta^*$ it follows that T^* is a contracting mapping what allows to apply the Banach fixed point theorem. Therefore, the problem (3.2) has a unique non-negative continuous solution which is also a C^∞ function by simple differentiation in $\mathbb{R}^+$. □

To problem (1.11), we impose a Robin boundary condition characterized by the coefficient $\gamma > 0$ at $x = 0$. This condition constitutes a generalization of the Dirichlet condition, in the sense that taking the limit when $\gamma \to \infty$ in condition (1.7b), we obtain condition (3.2b). Now, we show that the solution to problem (1.11) converges to the solution to problem (3.2) when $\gamma \to \infty$. For this purpose, first, we need the following lemmas which proofs are immediate.

Lemma 3.5. *For every $p \geq 1$, when $\gamma \to \infty$, the following convergence results hold*
- (a) *$T_\gamma(h)(x) \to T^*(h)(x)$ for every $h \in K$ and $x \geq 0$.*
- (b) *$g_\gamma(x) \to g^*(x)$ for every $x \geq 0$.*
- (c) *$\delta_\gamma \to \delta^*$.*

In addition $g_\gamma(x) \geq g^*(x)$ and $\delta_\gamma < \delta^*$ for all $x \geq 0, \gamma > 0$.

Lemma 3.6. *Let $p \geq 1$ and*
$$C(x) = 2xp(1+x)^3(2+x), \quad x \geq 0. \tag{3.5}$$
Then there exists a unique $\hat{\delta} > 0$ such that $C(\hat{\delta}) = 1$.

Theorem 3.7. *Let $p \geq 1$ and $0 \leq \delta < \min\{\hat{\delta}, \delta_\gamma\}$. Then $\|y_\gamma - y^*\|_\infty \to 0$ when $\gamma \to \infty$. Furthermore, the order of convergence is $1/\gamma$ when $\gamma \to \infty$.*

Proof. First let us note that if $0 \leq \delta < \min\{\hat{\delta}, \delta_\gamma\}$, then as $\delta_\gamma < \delta^*$, we obtain that y_γ and y^* are well defined because of Theorems 2.5 and 3.4. Then for $x \geq 0$ we have
$$\begin{aligned}
&|y_\gamma(x) - y^*(x)| \\
&= \Big|\frac{\big(1+\gamma\int_0^x f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta\big) - \big(\int_0^x f_{y^*}(\eta)\mathrm{d}\eta\big)\big(1+\gamma\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)}{\big(1+\gamma\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta\big)}\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big|\Big[\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta+\gamma\big(\int_0^x f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta\big)-\int_0^x f_{y^*}(\eta)\mathrm{d}\eta\\
&\quad -\gamma\big(\int_0^x f_{y^*}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\Big]\Big/\Big[\big(1+\gamma\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta\big)\Big]\Big|\\
&=\Big|\Big[\int_x^\infty f_{y^*}(\eta)\mathrm{d}\eta+\gamma\big(\int_0^x f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta\big)\\
&\quad -\gamma\big(\int_0^x f_{y^*}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\Big]\Big/\Big[\big(1+\gamma\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta\big)\Big]\Big|\\
&\leq \Big|\Big[\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta+\gamma\big(\int_0^x f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta-\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\\
&\quad +\gamma\big(\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta-\int_0^x f_{y^*}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\Big]\\
&\quad \div\Big[\big(1+\gamma\int_0^\infty f_{y_\gamma}(\eta)\mathrm{d}\eta\big)\big(\int_0^\infty f_{y^*}(\eta)\mathrm{d}\eta\big)\Big]\Big|\\
&\leq \Big[\sqrt{1+\delta}\frac{\sqrt{\pi}}{2}+\gamma\sqrt{1+\delta}\frac{\sqrt{\pi}}{2}\big(\int_0^\infty |f_{y^*}(\eta)-f_{y_\gamma}(\eta)|\mathrm{d}\eta\big)\\
&\quad +\gamma\sqrt{1+\delta}\frac{\sqrt{\pi}}{2}\big(\int_0^x |f_{y^*}(\eta)-f_{y_\gamma}(\eta)|\mathrm{d}\eta\big)\Big]\Big/\Big[\frac{\gamma\sqrt{\pi}}{2(1+\delta)}\frac{\sqrt{\pi}}{2(1+\delta)}\Big]\\
&\leq \frac{\sqrt{1+\delta}\frac{\sqrt{\pi}}{2}+2\gamma\sqrt{1+\delta}\frac{\sqrt{\pi}}{2}\int_0^\infty|f_{y^*}(\eta)-f_{y_\gamma}(\eta)|\mathrm{d}\eta}{\frac{\gamma\pi}{4(1+\delta)^2}}\\
&\leq \frac{4(1+\delta)^2}{\gamma\pi}\Big(\sqrt{1+\delta}\frac{\sqrt{\pi}}{2}+\gamma\frac{\pi}{4}\delta p(1+\delta)(2+\delta)||y_\gamma-y^*||_\infty\Big)\\
&\leq \frac{2(1+\delta)^{5/2}}{\gamma\pi}+2(1+\delta)^3\delta p(2+\delta)\|y_\gamma-y^*\|_\infty\,.
\end{aligned}
$$

The above inequalities are obtained by applying Lemma 2.3, and they lead to

$$
(1-C(\delta))\|y_\gamma-y^*\|_\infty\leq\frac{1}{\gamma}\Big(\frac{2(1+\delta)^{5/2}}{\sqrt{\pi}}\Big),
$$

with C defined by (3.5). Finally, the desired convergence and order of convergence in Theorem 3.7 are obtained by noting that if $0\leq\delta<\hat{\delta}$, then $0\leq C(\delta)<1$ because of Lemma 3.6. □

4. Existence and uniqueness considering a Neumann condition

In this section we consider a solidification Stefan problem with a Neumann condition at the fixed face $x=0$, given by

$$
\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\Big(k(T)\frac{\partial T}{\partial x}\Big),\quad 0<x<s(t),\ t>0, \tag{4.1}
$$

$$
k(T(0,t))\frac{\partial T}{\partial x}(0,t)=\frac{q_0}{\sqrt{t}},\quad t>0, \tag{4.2}
$$

$$
T(s(t),t)=T_{_f},\quad t>0, \tag{4.3}
$$

$$
k(T(s(t),t))\frac{\partial T}{\partial x}(s(t),t)=\rho l\dot{s}(t),\quad t>0, \tag{4.4}
$$

$$
s(0)=0, \tag{4.5}
$$

where the unknown functions are the temperature T and the free boundary s separating both phases. The parameters $\rho > 0$ (density), $l > 0$ (latent heat per unit mass), T_f (phase-change temperature), $q_0 > 0$ (characterizes the heat flux on the fixed face $x = 0$ of the face-change material which can be determined experimentally) and $c > 0$ (specific heat) are all known constants. In this case, the thermal conductivity k is defined as

$$k(T) = k_0\Big(1 + \delta\big(\frac{T}{T_f}\big)^p\Big), \quad p \geq 1, \tag{4.6}$$

where δ is a given positive constant and k_0 is the reference thermal conductivity.

In a similar way as in previous sections, this Stefan problem leads us to the study the ordinary differential problem

$$[(1 + \delta y^p(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty, \tag{4.7a}$$

$$(1 + \delta y^p(0))y'(0) = \gamma^*, \tag{4.7b}$$

$$y(+\infty) = 1\,, \tag{4.7c}$$

where

$$\gamma^* = 2\,\mathrm{Bi}^* \quad \text{with} \quad Bi^* = \frac{q_0\sqrt{\alpha_0}}{k_0 T_f}\,. \tag{4.8}$$

In a similar way to the above sections we can state the following results:

Theorem 4.1. *Let $\delta \geq 0$, $p \geq 1$ and $0 < \gamma^* \leq \frac{2}{\sqrt{\pi(1+\delta)}}$. Then $y_{\gamma^*} \in K$ is a solution to (4.7) if and only if y_{γ^*} is a fixed point of the operator $T_{\gamma^*} : K \to K$ given by*

$$T_{\gamma^*}(h)(x) = 1 - \gamma^* \int_x^{+\infty} f_h(\eta)\mathrm{d}\eta, \quad x \geq 0, \tag{4.9}$$

with f_h defined by (2.4) and K given by (2.2).

Proof. Given $y_{\gamma^*} \in K$ and taking into account (2.5), we obtain

$$0 < \frac{\gamma^*\,\mathrm{erfc}(x)}{1+\delta} \leq \gamma^* \int_x^{\infty} f_y(\eta)\mathrm{d}\eta < \frac{\gamma^*\sqrt{1+\delta}\sqrt{\pi}}{2} \leq 1. \tag{4.10}$$

Note that from (4.9) we have that $T_{\gamma^*}(y_{\gamma^*})$ is an analytic function, since $y_{\gamma^*} \in X$. Also, according to (4.9) and (4.10), $T_{\gamma^*}(y_{\gamma^*}) \in K$.

In a similar way as in the proof of Theorem 2.1, y_{γ^*} is a solution to (4.7) if and only if y_{γ^*} is a fixed point of the operator T_{γ^*}. □

Theorem 4.2. *Let $p \geq 1$, $\delta > 0$ and $0 < \gamma^* \leq \frac{2}{\sqrt{\pi(1+\delta)}}$. Then (4.7) has a unique C^∞ solution $y_{\gamma^*} \in K$ if and only if $\delta < \delta_{\gamma^*}$ where δ_{γ^*} is the unique solution to the equation $g(x) = 1$, with*

$$g(x) = x\frac{p}{\sqrt{\pi}}\big[(1+x)\big(\sqrt{1+x}\exp(-\frac{1}{4}) + \sqrt{\pi}\big) + \sqrt{\pi}\big].$$

Proof. Let $y_{1_{\gamma^*}}$, $y_{2_{\gamma^*}} \in K$ and $x \geq 0$. Taking into account (2.9) and (2.10) we obtain

$$\begin{aligned}
&|T_{\gamma^*}(y_{1_{\gamma^*}})(x) - T_{\gamma^*}(y_{2_{\gamma^*}})(x)| \\
&\leq \|y_1 - y_2\|_\infty \delta p \gamma^* \Big[(1+\delta)^{3/2}\big(\frac{x}{\sqrt{1+\delta}} \exp\big(-\frac{x^2}{1+\delta}\big) + \frac{\sqrt{\pi}}{2}\big) + \sqrt{1+\delta}\frac{\sqrt{\pi}}{2}\Big]
\end{aligned}$$

$$\le \delta \frac{p}{\sqrt{\pi}} \big[(1+\delta)\big(\sqrt{1+\delta} \exp(-\frac{1}{4}) + \sqrt{\pi} \big) + \sqrt{\pi} \big] \| y_{1_{\gamma^*}} - y_{2_{\gamma^*}} \|_\infty$$
$$\le g(\delta) \| y_{1_{\gamma^*}} - y_{2_{\gamma^*}} \|_\infty .$$

Since g is an increasing function such that $g(0) = 0$ and $g(+\infty) = +\infty$, there exists a unique $\delta_{\gamma^*} > 0$ with $g(\delta_{\gamma^*}) = 1$.

Then, if $0 \le \delta < \delta_{\gamma^*}$ it follows that T_{γ^*} is a contracting mapping what allows to apply the Banach fixed point theorem. Therefore, the problem (4.7) has a unique non-negative continuous solution which is also a C^∞ function. □

Conclusion. In this article, the ordinary differential problems studied in [4, 5] have been generalized by defining what we call the p-GME function and the p-ME function corresponding to the case when a Robin or Dirichlet boundary condition are imposed at $x = 0$, respectively. In both problems, existence and uniqueness of C^∞ solution has been proved by defining convenient contracting mappings. In addition it has been studied the behavior of the p-GME function when the coefficient γ that characterizes the Robin condition goes to infinity, obtaining its convergence to the p-ME function with an order of convergence of the type $1/\gamma$ when $\gamma \to \infty$. Finally, existence and uniqueness of a solution to a solidification problem with a Neumann condition has been studied.

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