Soft Computing
https://doi.org/10.1007/s00500-018-3360-1

FOUNDATIONS

A topological duality for tense θ-valued Łukasiewicz–Moisil algebras

Aldo V. Figallo[1] · Inés Pascual[1,2] · Gustavo Pelaitay[1]

Abstract

In 2011, tense θ-valued Łukasiewicz–Moisil algebras (or tense LM_θ-algebras) were introduced by Chiriţă as an algebraic counterpart of the tense θ-valued Moisil propositional logic. In this paper we develop a topological duality for these algebras. In order to achieve this we extend the topological duality given in Figallo et al. (J Mult Valued Logic Soft Comput 16(3–5):303–322, 2010), for θ-valued Łukasiewicz–Moisil algebras. This new topological duality enables us to describe the tense LM_θ-congruences and the tense θLM_θ-congruences on a tense LM_θ-algebra and also to determine some properties of these algebras.

Keywords θ-valued Łukasiewicz–Moisil algebras · Tense θ-valued Łukasiewicz–Moisil algebras · Topological duality

1 Introduction

In 1940, the first system of many-valued logic was introduced by J.Łukasiewicz, and his motivation was of philosophical nature as he was looking for an interpretation of the concepts of possibility and necessity. Since then, plenty of research has been developed in this area. In 1968, when Gr.C. Moisil came across Zadeh's fuzzy set theory, he found the motivation he had been looking for in order to legitimate the introduction and study of infinitely valued Łukasiewicz algebras, so he defined θ-valued Łukasiewicz algebras (without negation) or LM_θ-algebras, for short, where θ is the order type of a chain. These structures were thought by Moisil as models of a logic with infinity nuances.

Propositional logics usually do not incorporate the dimension of time; consequently, in order to obtain a tense logic, a propositional logic is enriched by the addition of new unary operators (or connectives) which are usually denoted by G, H, F and P. Tense algebras (or tense Boolean algebras) are algebraic structures corresponding to the propositional tense logic (see Burges 1984; Kowalski 1998).

An algebra $\langle A, \vee, \wedge, \neg, G, H, 0, 1\rangle$ is a tense algebra if $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and G, H are unary operators on A which satisfy the following axioms for all $x, y \in A$:

$$G(1) = 1, H(1) = 1,$$
$$G(x \wedge y) = G(x) \wedge G(y), H(x \wedge y) = H(x) \wedge H(y),$$
$$x \leq GP(x), x \leq HF(x),$$

where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$.

Taking into account that tense algebras constitute the algebraic basis for the bivalent tense logic, Diaconescu and Georgescu (2007) introduced in the tense MV-algebras and the tense Łukasiewicz–Moisil algebras (or tense n-valued Łukasiewicz–Moisil algebras) as algebraic structures for some many-valued tense logics. In recent years, these two classes of algebras have become very interesting for several authors (see Botur et al. 2011; Chajda and Paseka 2015; Chiriţă 2010, 2011, 2012a; Figallo and Pelaitay 2011, 2015a, b). In particular, Chiriţă (2010, 2011) introduced tense θ-valued Łukasiewicz–Moisil algebras and proved an important representation theorem which made it possible to show the completeness of the tense θ-valued Moisil logic (see Chiriţă 2010). In Diaconescu and Georgescu (2007), the authors formulated an open problem about representation of

Communicated by A. Di Nola.

✉ Gustavo Pelaitay
gpelaitay@gmail.com

Aldo V. Figallo
aldofigallonavarro@gmail.com

Inés Pascual
ipascualdiz@gmail.com

1 Instituto de Ciencias Básicas, Universidad Nacional de San Juan, 5400 San Juan, Argentina

2 Departamento de Matemática, Universidad Nacional de San Juan, 5400 San Juan, Argentina

tense MV-algebras, and this problem was solved in Botur and Paseka (2015), Paseka (2013) for semisimple tense MV-algebras. Also, in Botur et al. (2011), tense basic algebras which are an interesting generalization of tense MV-algebras were studied.

The main purpose of this paper is to give a topological duality for tense θ-valued Łukasiewicz–Moisil algebras. In order to achieve this we will extend the topological duality given in Figallo et al. (2010), for θ-valued Łukasiewicz–Moisil algebras.

The paper is organized as follows: In Sect. 2, we briefly summarize the main definitions and results needed throughout this article. In Sect. 3, we develop a topological duality for tense θ-valued Łukasiewicz–Moisil algebras, extending the one obtained in Figallo et al. (2010) for θ-valued Łukasiewicz–Moisil algebras. In Sect. 4, the results of Sect. 3 are applied. We characterize congruences and θ-congruences on tense θ-valued Łukasiewicz–Moisil algebras by certain closed subsets of the space associated with them. Also, we determine some properties of these algebras. Finally, in Sect. 5, we will obtain another characterization of tense θ-congruences on a tense θ-valued Łukasiewicz–Moisil algebra.

2 Preliminares

The notions and results announced here will be used throughout the paper.

2.1 θ-valued ukasiewicz–Moisil algebras

The following assumption will be needed throughout the paper.

Let $\theta \geq 2$ be the order type of a totally ordered set J with least element 0, being $J = \{0\} + I$ (ordinal sum) and $(I, \leq)$ with first and last element.

Definition 1 (Boicescu et al. 1991) An algebra $\langle A, \vee, \wedge, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0, 1\rangle$ of type $(2, 2, \{1\}_{i\in I}, \{1\}_{i\in I}, 0, 0)$ is an θ-valued Łukasiewicz–Moisil algebra without negation (or LM_θ-algebra), if

(i) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,

(ii) $\varphi_i, \overline{\varphi}_i$, $i \in I$, are unary operations on A which satisfy the following conditions for any $i,\ j \in I$ and $x, y \in A$:

(L1) φ_i is an endomorphism of bounded distributive lattices,

(L2) $\varphi_i x \vee \overline{\varphi}_i x = 1,\ \varphi_i x \wedge \overline{\varphi}_i x = 0$,

(L3) $\varphi_i \varphi_j x = \varphi_j x$,

(L4) $i \leq j$ implies $\varphi_i x \leq \varphi_j x$,

(L5) $\varphi_i x = \varphi_i y$ for all $i \in I$ imply $x = y$.

An LM_θ-algebra $\langle A, \vee, \wedge, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0, 1\rangle$ will be denoted in the rest of this paper by A or by $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}\right)$.

Axiom (L5) in Definition 1 can be replaced by :

(L5*) $\varphi_i x \leq \varphi_i y$ for all $i \in I$ imply $x \leq y$.

Note, however, that we will use the same symbol 0 (symbol 1) for the least (greatest) elements of J and of the algebra A under investigation.

If $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}\right)$ is an LM_θ-algebra, then

(L6) for any $i,\ j \in I, \varphi_i \overline{\varphi}_j = \overline{\varphi}_i \varphi_j = \overline{\varphi}_j$ and $\overline{\varphi}_i \overline{\varphi}_j = \varphi_j$. If the set I has least element 0 and greatest element 1, then

(L7) $\varphi_0 a \leq a \leq \varphi_1 a$ for any $a \in A$.

It is well known that there are LM_θ-congruences (or congruences) on LM_θ-algebras such that the quotient algebra does not satisfy the determination principle (L5). That is the reason why a new notion was defined as follows:
"A θLM_θ-congruence (or θ-congruence) on an LM_θ-algebra is a bounded distributive lattice congruence ϑ such that $(x, y) \in \vartheta$ if and only if $(\varphi_i x, \varphi_i y) \in \vartheta$ for all $i \in I$". ([3, 12]).

The following characterizations of the Boolean elements of an LM_θ-algebra will be useful for the study of these algebras:

(L8) Let A be an LM_θ-algebra and let $\mathcal{B}(A)$ be the set of all Boolean elements of A. Then, for each $x \in A$, the following conditions are equivalent:

(i) $x \in \mathcal{B}(A)$,

(ii) there are $y \in A$ and $i \in I$ such that $x = \varphi_i y$ $(x = \overline{\varphi}_i y)$,

(iii) there is $i_0 \in I$ such that $x = \varphi_{i_0} x$ $(x = \overline{\varphi}_{i_0} x)$,

(iv) for all $i \in I$, $x = \varphi_i x$ $(x = \overline{\varphi}_i x)$.

2.2 A topological duality for LM_θ-algebras

In Figallo et al. (2010), extended Priestley duality to LM_θ-algebras considering θ-valued Łukasiewicz–Moisil spaces (or LM_θ-spaces) and LM_θ-functions. More precisely, these authors introduced the following notions:

Definition 2 A θ-valued Łukasiewicz–Moisil space (or LM_θ-space) is a pair $\left(X, \{f_i\}_{i\in I}\right)$ provided the following conditions are satisfied for all $i,\ j \in I$ and for all $x,\ y \in X$:

(lP1) X is a Priestley space (Priestley 1970),

(lP2) $f_i : X \longrightarrow X$ is a continuous function,

(lP3) $x \leq y$ implies $f_i(x) = f_i(y)$,

(lP4) $i \leq j$ implies $f_i(x) \leq f_j(x)$,
(lP5) $f_i \circ f_j = f_i$,
(lP6) $\bigcup_{i \in I} f_i(X)$ is dense in X (i.e. $\overline{\bigcup_{i \in I} f_i(X)} = X$, where $\overline{Z}$ denotes the closure of Z for all $Z \subseteq X$).

Definition 3 An LM_θ-function from an LM_θ-space $(X, \{f_i\}_{i \in I})$ into another $\left(X', \{f'_i\}_{i \in I}\right)$ is an increasing and continuous function f from X into X' satisfying $f'_i \circ f = f \circ f_i$ for all $i \in I$.

It is worth mentioning that condition (lP6) is equivalent to the following one:

(lP7) If U and V are increasing[1], closed and open subsets of X and $f_i^{-1}(U) = f_i^{-1}(V)$ for all $i \in I$, then $U = V$ (Figallo et al. 2010).

Besides, if $(X, \{f_i\}_{i \in I})$ is an LM_θ-space, then for all $x \in X$, the following properties are satisfied (Figallo et al. 2010):

(lP8) $x \leq f_i(x)$ or $f_i(x) \leq x$ for all $i \in I$.
If I has least element 0 and greatest element 1, then
(lP9) $f_0(x) \leq x$ and $f_0(x)$ is the unique minimal element in X that precedes

(lP10) $x \leq f_1(x)$ and $f_1(x)$ is the unique maximal element in X that follows x.

Furthermore, the above properties allowed them to assert that

(lP11) X is the cardinal sum of the sets $\uparrow \{f_i(x)\}_{i \in I} \cup \downarrow \{f_i(x)\}_{i \in I}$ for $x \in X$, where $\downarrow z = \{x \in X : x \leq z\}$ and $\uparrow z = \{x \in X : z \leq x\}$ for all $z \in X$.

If I has least element 0 and greatest element 1, then

(lP12) X is the cardinal sum of the sets $[f_0(x), f_1(x)]$ for $x \in X$, where $[y, z] = \{x \in X : y \leq x \leq z\}$ for all $y, z \in X$.

Although in Figallo et al. (2010) the authors developed a topological duality for LM_θ-algebras, next we will describe some results of it with the aim of fixing the notation we are about to use in this paper.
(A1) If $\left(X, \{f_i\}_{i \in I}\right)$ is an LM_θ-space and $D(X)$ is the lattice of all increasing, closed and open subsets of X, then $\left(D(X), \{\varphi_i^X\}_{i \in I}, \{\overline{\varphi}_i^X\}_{i \in I}\right)$ is an LM_θ-algebra, where for all $i \in I$, the operations φ_i^X and $\overline{\varphi}_i^X$ are defined for all $U \in D(X)$ by means of the formulas:

[1] Recall that W is an increasing subset of X iff $x \in W$ and $x \leq y$ imply $y \in W$.

$$\varphi_i^X(U) = f_i^{-1}(U) \text{ and } \overline{\varphi}_i^X(U) = X \backslash f_i^{-1}(U). \tag{1}$$

(A2) If $\left(A, \{\varphi_i\}_{i \in I}, \{\overline{\varphi}_i\}_{i \in I}\right)$ is an LM_θ-algebra and $X(A)$ is the set of all prime filters of A, ordered by inclusion relation and with the topology having as a sub-basis the sets

$$\sigma_A(a) = \{S \in X(A) : a \in S\} \text{ for each } a \in A, \tag{2}$$

and

$$X(A) \backslash \sigma_A(a) \text{ for each } a \in A.$$

Then, $\left(X(A), \{f_i^A\}_{i \in I}\right)$ is the LM_θ-space associated with A, where for all $i \in I$, the function $f_i^A : X(A) \longrightarrow X(A)$ is defined for all $S \in X(A)$ by the prescription:

$$f_i^A(S) = \varphi_i^{-1}(S). \tag{3}$$

(A3) Let $\left(A, \{\varphi_i\}_{i \in I}, \{\overline{\varphi}_i\}_{i \in I}\right)$ be an LM_θ-algebra and $\sigma_A : A \longrightarrow D(X(A))$ be the function defined as in (2), then σ_A is an LM_θ-isomorphism.
(A4) Let $\left(X, \{f_i\}_{i \in I}\right)$ be an LM_θ-space and $\varepsilon_X : X \longrightarrow X(D(X))$ be the function defined by

$$\varepsilon_X(x) = \{U \in D(X) : x \in U\}, \quad \text{for all } x \in X, \tag{4}$$

then ε_X is an isomorphism of LM_θ-spaces.
(A5) Let $h : \left(A, \{\varphi_i\}_{i \in I}, \{\overline{\varphi}_i\}_{i \in I}\right) \longrightarrow \left(A', \{\varphi'_i\}_{i \in I}, \{\overline{\varphi}'_i\}_{i \in I}\right)$ be an LM_θ-homomorphism. Then, the map $\Phi(h) : X(A') \longrightarrow X(A)$ is an LM_θ-function, where

$$\Phi(h)(S) = h^{-1}(S), \quad \text{for all } S \in X(A'). \tag{5}$$

(A6) Let $f : \left(X, \{f_i\}_{i \in I}\right) \longrightarrow \left(X', \{f'_i\}_{i \in I}\right)$ be an LM_θ-function. Then, the map $\Psi(f) : D(X') \longrightarrow D(X)$ is an LM_θ-homomorphism, where

$$\Psi(f)(U) = f^{-1}(U), \quad \text{for all } U \in D(X'). \tag{6}$$

Then, using the usual procedures they proved that the category $l_\theta\mathcal{P}$ of LM_θ-spaces and LM_θ-functions is naturally equivalent to the dual category of the category $l_\theta\mathcal{A}$ of LM_θ-algebras and their corresponding homomorphisms, where the isomorphisms σ_A and ε_X are the corresponding natural equivalences.

In addition, this duality allowed them to characterize the congruences and θ-congruences on these algebras for which they introduced these notions:

Definition 4 Let $\left(X, \{f_i\}_{i \in I}\right)$ be an LM_θ-space and let Y be a subset of X.

(i) Y is semimodal if $\bigcup_{i \in I} f_i(Y) \subseteq Y$, or equivalently $Y \subseteq \bigcup_{i \in I} f_i^{-1}(Y)$.

(ii) Y is modal if $Y = \bigcup_{i\in I} f_i^{-1}(Y)$.
(iii) Y is a θ-subset if $\bigcup_{i\in I} f_i(Y) \subseteq Y \subseteq \overline{\bigcup_{i\in I} f_i(Y)}$.

Then, the authors proved the lattice of all semimodal and closed subsets and the lattice of all closed θ-subsets of the LM_θ-space associated with an LM_θ-algebra play a fundamental role in the characterization of LM_θ-congruences and θLM_θ-congruences on these algebras, respectively, as we shall indicate next.

Theorem 1 (Figallo et al. 2010, Theorem 2.1.1) *Let $(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I})$ be an LM_θ-algebra and $Ł_\theta(A) = (X(A), \{f_i^A\}_{i\in I})$ be the LM_θ-space associated with A. Then, the lattice $\mathcal{C}_S(Ł_\theta(A))$ of all semimodal and closed subsets of $Ł_\theta(A)$ is anti-isomorphic to the lattice $Con_{LM_\theta}(A)$ of all LM_θ-congruences on A, and the anti-isomorphism is the function Θ_S defined by the prescription:*

$$\Theta_S(Y) = \{(a, b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\} \quad \textit{for all } Y \in \mathcal{C}_S(X(A)). \tag{7}$$

Theorem 2 (Figallo et al. 2010, Theorem 2.1.2) *Let $(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I})$ be an LM_θ-algebra and let $Ł_\theta(A) = (X(A), \{f_i^A\}_{i\in I})$ be the LM_θ-space associated with A. Then, the lattice $\mathcal{C}_\theta(Ł_\theta(A))$ of all closed θ-subsets of $Ł_\theta(A)$ is anti-isomorphic to the lattice $Con_{\theta LM_\theta}(A)$ of all θLM_θ-congruences on A, and the anti-isomorphism is the function Θ_θ defined as in Theorem 1.*

2.3 Tense θ-valued ukasiewicz–Moisil algebras

In Chiriţă (2011), made the first step in solving the problem to develop a tense logic based on the θ-valued Moisil logic. In order to do this, the author introduced the tense θ-valued Łukasiewicz–Moisil algebras by extending the tense Boolean algebras and the tense n-valued Łukasiewicz–Moisil algebras. The notion of tense θ-valued Łukasiewicz–Moisil is obtained by endowing θ-valued Łukasiewicz–Moisil algebra with two unary operations G and H similar to the tense operators on a Boolean algebra and the tense operators on an n-valued Łukasiewicz–Moisil algebra. Next we will indicate the basic definition, properties and examples of these algebras.

Definition 5 An algebra $\langle A, \vee, \wedge, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H, 0, 1\rangle$ is a tense θ-valued Łukasiewicz–Moisil algebra (or tense LM_θ-algebra) if $\langle A, \vee, \wedge, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0, 1\rangle$, is an LM_θ-algebra and G, H are two unary operators on A which satisfy the following properties for all $x, y \in A$ and for all $i \in I$:

(T1) $G(1) = 1$ and $H(1) = 1$,
(T2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,
(T3) $G(\varphi_i x) = \varphi_i G(x)$ and $H(\varphi_i x) = \varphi_i H(x)$,
(T4) $G(x) \vee y = 1$ iff $x \vee H(y) = 1$.

A tense LM_θ-algebra $\langle A, \vee, \wedge, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H, 0, 1\rangle$ will be denoted in the rest of this paper by (A, G, H) or by $(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H)$.

Proposition 1 (Chiriţă 2011, Proposition 3.1) *Let $(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H)$ be a tense LM_θ-algebra and $\mathcal{B}(A) = \{\varphi_i(x) : x \in A\}_{i\in I}$. Then, $(\mathcal{B}(A), G\mid_{\mathcal{B}(A)}, H\mid_{\mathcal{B}(A)})$ is a tense Boolean algebra.*

Definition 6 Let $(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H)$ be a tense LM_θ-algebra and let P, $F : \mathcal{B}(A) \longrightarrow \mathcal{B}(A)$ be the operators defined for all $x \in A$ and $i \in I$ by the prescriptions:

(i) $P(\varphi_i x) := \overline{\varphi}_i\, H(\overline{\varphi}_i x) = \neg H(\neg\varphi_i x)$,
(ii) $F(\varphi_i x) := \overline{\varphi}_i\, G(\overline{\varphi}_i x) = \neg G(\neg\varphi_i x)$,
where $\neg y$ is the complement of y for all $y \in \mathcal{B}(A)$.

From Definition 6 it follows that F and P are the operations on the tense Boolean algebra $(\mathcal{B}(A), G\mid_{\mathcal{B}(A)}, H\mid_{\mathcal{B}(A)})$.

Definition 7 Let $(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H)$ be a tense LM_θ-algebra and for each $i \in I$, let P_i, $F_i : A \longrightarrow A$ be the operators defined for all $x \in A$ by the prescriptions:

(i) $P_i(x) := P(\varphi_i x) = \overline{\varphi}_i\, H(\overline{\varphi}_i x) = \neg H(\overline{\varphi}_i x)$,
(ii) $F_i(x) := F(\varphi_i x) = \overline{\varphi}_i\, G(\overline{\varphi}_i x) = \neg G(\overline{\varphi}_i x)$.

Remark 1 From Definition 7 it follows that for all $i \in I$, $F_i\mid_{\mathcal{B}(A)} = F$ and $P_i\mid_{\mathcal{B}(A)} = P$, where F and P are the operations defined on $\mathcal{B}(A)$ in Definition 6.

Lemma 1 *Let (A, G, H) be a tense LM_θ-algebra. Then, the following properties hold for all $i, j \in I$, $x \in A$ and $S \in X(A)$:*

(i) *$F_i(x), P_i(x) \in \mathcal{B}(A)$,*
(ii) *$i \leq j$ implies that $F_i(x) \leq F_j(x)$ and $P_i(x) \leq P_j(x)$,*
(iii) *$i \leq j$ implies that $F_i^{-1}(S) \subseteq F_j^{-1}(S)$ and $P_i^{-1}(S) \subseteq P_j^{-1}(S)$,*
(iv) *$\varphi_j^{-1}\left(F_i^{-1}(S)\right) = F_j^{-1}(S)$ and $\varphi_j^{-1}\left(P_i^{-1}(S)\right) = P_j^{-1}(S)$.*

Proof (i): If follows immediately from Definition 6 and Remark 1.
(ii): Let $i, j \in I$, $i \leq j$, then by property (L4) of LM_θ-algebras we have that $\varphi_i(x) \leq \varphi_j(x)$ for any $x \in A$. From this last assertion and the fact that the operators F and P defined on the tense Boolean algebra $\mathcal{B}(A)$ are

monotonous, we infer that $F(\varphi_i(x)) \leq F(\varphi_j(x))$ and $P(\varphi_i(x)) \leq P(\varphi_j(x))$ for any $x \in A$, and therefore, $F_i(x) \leq F_j(x)$ and $P_i(x) \leq P_j(x)$ for any $x \in A$.

(iii): Let $i, j \in I$, $i \leq j$, and suppose that $F_i(x) \in S$. Since by (ii), $F_i(x) \leq F_j(x)$ and S is a filter of A it follows that $F_j(x) \in S$ and so $F_i^{-1}(S) \subseteq F_j^{-1}(S)$. The proof of the other inclusion is similar.

(iv): It is a consequence of the fact that each of the following statements is equivalent to the next one in the respective sequences, in which are taken into account Definition 7 and property (L3) of LM_θ-algebras:

$F_i(\varphi_j(a)) \in S$; $F\left(\varphi_i\left(\varphi_j a\right)\right) \in S$

$F\left(\varphi_j a\right) \in S$; $F_j\left(a\right) \in S$.

$P_i(\varphi_j(a)) \in S$; $P\left(\varphi_i\left(\varphi_j a\right)\right) \in S$; $P\left(\varphi_j a\right) \in S$;

$P_j\left(a\right) \in S$.

□

Proposition 2 (Chiriţă 2011, Proposition 3.4) *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be an LM_θ-algebra and let G, H be two unary operations on A that satisfy conditions* (T1), (T2) *and* (T3). *Then the condition* (T4) *is equivalent to the following one:*

(T4′) *$\varphi_i x \leq G(P(\varphi_i x))$ and $\varphi_i x \leq H(F(\varphi_i x))$ for all $x \in A$ and for all $i \in I$,*
or equivalently to the next one:

(T4″) *$\varphi_i x \leq G(P_i(x))$ and $\varphi_i x \leq H(F_i(x))$ for all $x \in A$ and for all $i \in I$.*

Corollary 1 *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra. Then for all $x \in A$ and for all $i \in I$ the following property holds:*

(T4*) *$x \leq G(P_i(x))$ and $x \leq H(F_i(x))$.*

Proof From property (T4′), we have that (1) $\varphi_i x \leq G(P(\varphi_i x))$ for all $i \in I$ and for all $x \in A$. Since $P(\varphi_i x) = P_i(x) \in \mathcal{B}(A)$, then from the characterizations (L8) of the Boolean elements of an LM_θ-algebra we obtain that $P_i(x) = \varphi_i(P_i(x))$. Therefore, (2) $G(P(\varphi_i x)) = G(\varphi_i(P(x)))$. Besides, from property (T3) we get that (3) $G(P(x)) = \varphi_i G(P_i(x))$. Then, from (1), (2) and (3) it follows that $\varphi_i x \leq \varphi_i G(P_i(x))$ for all $i \in I$ and for all $x \in A$. From this last statement and property (L5*) we conclude that $x \leq G(P_i(x))$ for all $x \in A$. In a similar way it can be proved that $x \leq H(F_i(x))$. □

In the following proposition, there are properties of tense LM_θ-algebras that are useful in what follows.

Proposition 3 (Chiriţă 2012b, Proposition 2.2.4) *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra. Then, the following properties hold for all $x, y \in A$ and for all $i \in I$:*

(T5) *$x \leq y$ implies that $G(x) \leq G(y)$ and $H(x) \leq H(y)$,*

(T6) *$x \leq y$ implies that $F_i(x) \leq F_i(y)$ and $P_i(x) \leq P_i(y)$,*

(T7) *$F_i(0) = 0$ and $P_i(0) = 0$,*

(T8) *$F_i(x \vee y) = F_i(x) \vee F_i(y)$ and $P_i(x \vee y) = P_i(x) \vee P(y)$,*

(T9) *$P_i(G(x)) \leq \varphi_i x$ and $F_i(H(x)) \leq \varphi_i x$,*

(T10) *$G(P_i(x)) \wedge F_i(y) \leq F_i(P_i(x) \wedge y)$ and $H(F_i(x)) \wedge P_i(y) \leq P_i(F_i(x) \wedge y)$,*

(T11) *$G(\varphi_i x) \wedge F_i(y) \leq F_i(x \wedge y)$ and $H(\varphi_i x) \wedge P_i(y) \leq P_i(x \wedge y)$,*

(T12) *$G(\varphi_i(x \vee y)) \leq G(\varphi_i x) \vee F_i(y)$ and $H(\varphi_i(x \vee y)) \leq H(\varphi_i x) \vee P_i(y)$,*

(T13) *$F_i(x) = F_i(\varphi_i(x))$ and $P_i(x) = P_i(\varphi_i(x))$.*

Corollary 2 *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra. Then, the following properties hold for all $x, y \in A$ and for all $i, j \in I$:*

(T9*) *$P_i(G(x)) \leq x$ and $F_i(H(x)) \leq x$,*

(T11*) *$G(x) \wedge F_i(y) \leq F_i(x \wedge y)$ and $H(x) \wedge P_i(y) \leq P_i(x \wedge y)$,*

(T12*) *$G(x \vee y) \leq G(x) \vee F_i(y)$ and $H(x \vee y) \leq H(x) \vee P_i(y)$,*

(T14) *$\varphi_j(F_i(x) = F_i(x) = F_i(\varphi_i(x))$ and $\varphi_j(P_i(x)) = P_i(x) = P_i(\varphi_i(x))$.*

Proof (T9*): From property (T9), we have that $P_i G(x) \leq \varphi_i x$ and $F_i(H(x)) \leq \varphi_i x$ for all $i \in I$ and for all $x \in A$. From Lemma 1 and property (L8) of Boolean elements of an LM_θ-algebra, we obtain that $P_i(G(x)) = \varphi_i P_i(G(x))$ and $F_i(H(x)) = \varphi_i F_i(H(x))$. Then, from the above assertion we infer that $\varphi_i P_i(G(x)) \leq \varphi_i x$ and $\varphi_i F_i(H(x)) \leq \varphi_i x$ for all $i \in I$ and for all $x \in A$, from which we conclude by property (L5*) that $P_i(G(x)) \leq x$ and $F_i(H(x)) \leq x$ for all $i \in I$ and for all $x \in A$.

(T11*): From property (T11), we have that $G(\varphi_i x) \wedge F_i(y) \leq F_i(x \wedge y)$ and $H(\varphi_i x) \wedge P_i(y) \leq P_i(x \wedge y)$ for all $i \in I$ and for all $x \in A$. Taking into account this last assertion, property (T3), Lemma 1, property (L2) and the characterizations (L8) of the Boolean elements of an LM_θ-algebra, we infer that $\varphi_i(G(x) \wedge F_i(y)) \leq \varphi_i(F_i(x \wedge y))$ and $\varphi_i(H(x) \wedge P_i(y)) \leq \varphi_i(P_i(x \wedge y))$ for all $i \in I$ and for all $x \in A$, and so from property (L5*) we can assert that $G(x) \wedge F_i(y) \leq F_i(x \wedge y)$ and $H(x) \wedge P_i(y) \leq P_i(x \wedge y)$ for all $i \in I$ and for all $x \in A$.

(T12*): From property (T12) we have that $G(\varphi_i(x \vee y)) \leq G(\varphi_i x) \vee F_i(y)$ and $H(\varphi_i(x \vee y)) \leq H(\varphi_i x) \vee P_i(y)$

for all $i \in I$ and for all $x, y \in A$. Then, from this last statement, Lemma 1 and properties (T3) and (L1) and the characterizations (L8) of the Boolean elements of an LM_θ-algebra, we infer that $\varphi_i G(x \vee y) \leq \varphi_i G(x) \vee \varphi_i F_i(y) = \varphi_i(G(x) \vee F_i(y))$ and $\varphi_i H(x \vee y) \leq \varphi_i H(x) \vee \varphi_i P_i(y) = \varphi_i(H(x) \vee P_i(y))$ for all $i \in I$ and for all $x, y \in A$, and therefore, from property (L5*), we conclude that $G(x \vee y) \leq G(x) \vee F_i(y)$ and $H(x \vee y) \leq H(x) \vee P_i(y)$ for all $i \in I$ and for all $x, y \in A$.

(T14): It is a direct consequence of (i) in Lemma 1, the characterizations (L8) of Boolean elements of an LM_θ-algebra and property (T13). □

3 A topological duality for tense LM_θ-algebras

In this section, we develop a topological duality for tense LM_θ-algebras, taking into account the results established by Figallo et al. (2010). In order to determine this duality, we introduce a topological category whose objects and their corresponding morphisms are described below.

Definition 8 A system $(X, \{f_i\}_{i\in I}, R)$ is a tense LM_θ-space if the following conditions are satisfied:

(i) $(X, \{f_i\}_{i\in I})$ is a LM_θ-space (Definition 2),

(ii) R is a binary relation on X and R^{-1} is the converse of R such that:

(tS1) for each $x \in X$, $R(x)$ and $R^{-1}(x)$ are closed subsets of X,

(tS2) for each $x \in X$, $R(x) =\downarrow R(x) \cap \uparrow R(x)$, $R^{-1}(x) =\downarrow R^{-1}(x) \cap \uparrow R^{-1}(x)$,

(tS3) $(x, y) \in R$ implies $(f_i(x), f_i(y)) \in R$ for any $i \in I$,

(tS4) $(f_i(x), y) \in R$, $i \in I$, implies that there exists $z \in X$ such that $(x, z) \in R$ and $f_i(z) \leq y$,

(tS5) $(y, f_i(x)) \in R$, $i \in I$, implies that there exists $z \in X$ such that $(z, x) \in R$ and $f_i(z) \leq y$,

(tS6) for each $U \in D(X)$, $G_R(U), H_{R^{-1}}(U) \in D(X)$, where G_R and $H_{R^{-1}}$ are operators on $\mathcal{P}(X)$, which are defined for all $Y \in \mathcal{P}(X)$ by the prescriptions:

$$G_R(Y) := \{x \in X : \downarrow z \cap R(x) \cap Y \neq \emptyset \text{for all} z \in R(x)\}, \tag{8}$$

$$H_{R^{-1}}(Y) := \{x \in X : \downarrow z \cap R^{-1}(x) \cap Y \neq \emptyset \text{for all} z \in R^{-1}(x)\}, \tag{9}$$

where $\downarrow z = \{x \in X : x \leq z\}$ for all $z \in X$.

Definition 9 A tense LM_θ-function f from a tense LM_θ-space $(X_1, \{f_i^1\}_{i\in I}, R_1)$ into another LM_θ-space $(X_2, \{f_i^2\}_{i\in I}, R_2)$ is a function $f : X_1 \longrightarrow X_2$ such that:

(i) $f : X_1 \longrightarrow X_2$ is an LM_θ-function (Definition 3),

(ii) $f : X_1 \longrightarrow X_2$ satisfies the following conditions, for all $x \in X_1$:

(tf1) $f(R_1(x)) \subseteq R_2(f(x))$ and $f(R_1^{-1}(x)) \subseteq R_2^{-1}(f(x))$,

(tf2) $R_2(f(x)) \subseteq \uparrow f(R_1(x))$,

(tf3) $R_2{}^{-1}(f(x)) \subseteq \uparrow f(R_1{}^{-1}(x))$.

The category that has tense LM_θ-spaces as objects and tense LM_θ-functions as morphisms will be denoted by $tLM_\theta S$, and the category of tense LM_θ-algebras and tense LM_θ-homomorphisms will be denoted by $tLM_\theta A$.

Our next task will be to determine that the category $tLM_\theta S$ is naturally equivalent to the dual category of $tLM_\theta A$. Firstly, we will determine some properties of tense LM_θ-spaces and tense LM_θ-functions, which will be quite useful in order to developed this duality and characterize the lattice of all congruences and the lattice of all θ-congruences of these algebras.

Proposition 4 *Let $(X, \{f_i\}_{i\in I}, R)$ be a tense LM_θ-space. For all $Y \in \mathcal{P}(X)$ such that Y is increasing,*

$$G_R(Y) = \{x \in X \mid R(x) \subseteq Y\}, \tag{10}$$

$$H_{R^{-1}}(Y) = \{x \in X \mid R^{-1}(x) \subseteq Y\}. \tag{11}$$

Proof Let $Y \subseteq X$ be increasing. It immediately follows that $\{x \in X \mid R(x) \subseteq Y\} \subseteq G_R(Y)$. Conversely, let $y \in G_R(Y)$ and $z \in R(y)$. Then, from prescription (8) we infer that there exists $w \in Y \cap R(y)$ such that $w \leq z$. Since Y is increasing, we obtain that $z \in Y$ and so, $R(y) \subseteq Y$. Therefore, $G_R(Y) = \{x \in X \mid R(x) \subseteq Y\}$. The proof that $H_{R^{-1}}(Y) = \{x \in X \mid R^{-1}(x) \subseteq Y\}$ is similar. □

Corollary 3 *Let $(X, \{f_i\}_{i\in I}, R)$ be a tense LM_θ-space. For any $U \in D(X)$,*

$$G_R(U) = \{x \in X \mid R(x) \subseteq U\}, \tag{12}$$

$$H_{R^{-1}}(U) = \{x \in X \mid R^{-1}(x) \subseteq U\}. \tag{13}$$

Proof It is a direct consequence of Proposition 4 and the fact that U is an increasing subset of X for all $U \in D(X)$. □

Definition 10 Let $(X, \{f_i\}_{i\in I}, R)$ be a tense LM_θ-space, $\mathcal{B}(D(X))$ be the Boolean algebra of all complemented elements of $D(X)$ and let F_R, $P_{R^{-1}}$ be the functions from $\mathcal{B}(D(X))$ into $\mathcal{B}(D(X))$, defined for all $U \in D(X)$ and for all $i \in I$ by the prescriptions:

$$F_R(f_i^{-1}(U)) := \left\{x \in X : R(f_i(x)) \cap f_i^{-1}(U) \neq \emptyset\right\}, \tag{14}$$

$$P_{R^{-1}}(f_i^{-1}(U)) := \left\{x \in X : R^{-1}(f_i(x)) \cap f_i^{-1}(U) \neq \emptyset\right\}. \tag{15}$$

Proposition 5 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. Then, for all $i \in I$ and for all $U \in D(X)$,*

$$F_R\left(f_i^{-1}(U)\right) = \left\{x \in X : R(x) \cap f_i^{-1}(U) \neq \emptyset\right\}, \tag{16}$$

$$P_{R^{-1}}\left(f_i^{-1}(U)\right) = \left\{x \in X : R^{-1}(x) \cap f_i^{-1}(U) \neq \emptyset\right\}. \tag{17}$$

Proof Let $x \notin F_R\left(f_i^{-1}(U)\right)$. Then, by Definition 10, we have that (1) $R(f_i(x)) \cap f_i^{-1}(U) = \emptyset$, from which it follows that (2) $R(x) \cap f_i^{-1}(U) = \emptyset$. Indeed, suppose that there is $z \in R(x)$ such that $f_i(z) \in U$. Then, from properties (tS3) and (lP5) we obtain the $f_i(z) \in R(f_i(x))$ and $f_i(f_i(z)) \in U$, and consequently $R(f_i(x)) \cap f_i^{-1}(U) \neq \emptyset$, which contradicts (1). Therefore, assertion (2) holds, which allows us to assert that $\left\{x \in X : R(x) \cap f_i^{-1}(U) \neq \emptyset\right\} \subseteq F_R\left(f_i^{-1}(U)\right)$. On the other hand, suppose that there exists $x \in X$ such that (3) $R(x) \cap f_i^{-1}(U) = \emptyset$ and (4) $R(f_i(x)) \cap f_i^{-1}(U) \neq \emptyset$. Then, there exists $y \in R(f_i(x))$ such that $f_i(y) \in U$, from which it follows by property (tS4) that there exists (5) $z \in R(x)$ such that $f_i(z) \leq y$. From this last assertion and property (lP5), we get that $f_i(y) = f_i(z)$ and so (6) $f_i(z) \in U$. By virtue of statements (5) and (6) we can assert that $R(x) \cap f_i^{-1}(U) \neq \emptyset$, which contradicts (3). Therefore, assertion (4) is not true, from which we conclude that $F_R\left(f_i^{-1}(U)\right) = \left\{x \in X : R(x) \cap f_i^{-1}(U) \neq \emptyset\right\}$. In a similar way we can proved (17). $\square$

Definition 11 Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space and let for each $i \in I$, $F_R^i : X \longrightarrow X$ and $P_{R^{-1}}^i : X \longrightarrow X$ be the functions defined for all $U \in \mathcal{P}(X)$ by the prescriptions:

$$F_R^i(U) := F_R\left(f_i^{-1}(U)\right), \tag{18}$$

$$P_{R^{-1}}^i(U) = P_{R^{-1}}\left(f_i^{-1}(U)\right). \tag{19}$$

Lemma 2 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. Then, for all $i \in I$ and for all $U \in D(X)$,*

(tS7) $F_R\left(\varphi_i^X(U)\right) = F_R\left(f_i^{-1}(U)\right) = \overline{\varphi}_i^X\left(G_R\left(\overline{\varphi}_i^X\left(f_i^{-1}(U)\right)\right)\right) = \overline{\varphi}_i^X\left(G_R\left(\overline{\varphi}_i^X(U)\right)\right)$,

(tS8) $P_{R^{-1}}\left(\varphi_i^X(U)\right) = P_{R^{-1}}\left(f_i^{-1}(U)\right) = \overline{\varphi}_i^X\left(H_{R^{-1}}\left(\overline{\varphi}_i^X\left(f_i^{-1}(U)\right)\right)\right) = \overline{\varphi}_i^X\left(H_{R^{-1}}\left(\overline{\varphi}_i^X(U)\right)\right)$,

(tS9) $F_R^i(U) = \overline{\varphi}_i^X\left(G_R\left(\overline{\varphi}_i^X\left(f_i^{-1}(U)\right)\right)\right) = \overline{\varphi}_i^X\left(G_R\left(\overline{\varphi}_i^X(U)\right)\right)$,

(tS10) $P_{R^{-1}}^i(U) = \overline{\varphi}_i^X\left(H_{R^{-1}}\left(\overline{\varphi}_i^X\left(f_i^{-1}(U)\right)\right)\right) = \overline{\varphi}_i^X\left(H_{R^{-1}}\left(\overline{\varphi}_i^X(U)\right)\right)$.

Proof It immediately follows taking into account Proposition 4, Definitions 10 and 11, the definitions of φ_i^X and $\overline{\varphi}_i^X$, $i \in I$, given in prescription (1), and property (L6) of LM_θ-algebras. $\square$

Lemma 3 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. Then, for all $i, j \in I$ and for all $U \in D(X)$,*

(i) $F_R^i(U), P_{R^{-1}}^i(U) \in \mathcal{B}(D(X))$,
(ii) $f_j^{-1}\left(F_R^i(U)\right) = F_R^i(U)$, $f_j^{-1}\left(P_{R^{-1}}^i(U)\right) = P_{R^{-1}}^i(U)$.

Proof (i): From statement (A1) and the characterizations (L8) of the Boolean elements of an LM_θ-algebra it follows that $\mathcal{B}(D(X)) = \left\{\varphi_i^X(U) : U \in D(X)\right\}_{i\in I} = \left\{\overline{\varphi}_i^X(U) : U \in D(X)\right\}_{i\in I}$, and consequently this last assertion and properties (tS6) (Definition 8), (tS9) and (tS10) (Lemma 2) of LM_θ-spaces allow us to complete the proof.
(ii): It is a direct consequence of (i), the characterizations (L8) of the Boolean elements of an LM_θ-algebra and the definition of φ_j^X, $j \in I$, given in (1). $\square$

Lemma 4 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. Then, for all $x, y \in X$, the following conditions are equivalent:*

(i) $(f_i(x), f_i(y)) \in R$ *for some* $i \in I$,
(ii) $(f_i(x), f_i(y)) \in R$ *for all* $i \in I$.

Proof It is a direct consequence of properties (lP5) and (tS3) of tense LM_θ-spaces. $\square$

Now, we will show a characterization of tense LM_θ-functions which will be useful later.

Lemma 5 *Let $\left(X_1, \{f_i^1\}_{i\in I}, R_1\right)$ and $\left(X_2, \{f_i^2\}_{i\in I}, R_2\right)$ be two tense LM_θ-spaces and let $f : X_1 \longrightarrow X_2$ be a tense LM_θ-function. Then, f satisfies the following conditions for any $x \in X$:*

(tf4) $\uparrow f(R_1(x)) = \uparrow R_2(f(x))$,
(tf5) $\uparrow f(R_1{}^{-1}(x)) = \uparrow R_2{}^{-1}(f(x))$.

Proof It can be proved using a similar technique to that used in the proof of Lemma 3.4 in Figallo et al. (2018). $\square$

Lemma 6 *Let* $\left(X_1, \{f_i^1\}_{i\in I}, R_1\right)$ *and* $\left(X_2, \{f_i^2\}_{i\in I}, R_2\right)$ *be two tense* LM_θ*-spaces, and let* $f : X_1 \longrightarrow X_2$ *be an* LM_θ*-function such that for all* $U \in D(X_2)$*:*

(tf6) $f^{-1}\left(G_{R_2}(U)\right) = G_{R_1}\left(f^{-1}(U)\right)$,
(tf7) $f^{-1}\left(H_{R_2^{-1}}(U)\right) = H_{R_1^{-1}}\left(f^{-1}(U)\right)$.

Then, f *satisfies the following conditions for all* $U \in D(X_2)$ *and for all* $i \in I$*:*

(tf8) $f^{-1}\left(F^i_{R_2}(U)\right) = F^i_{R_1}\left(f^{-1}(U)\right)$,
(tf9) $f^{-1}\left(P^i_{R_2^{-1}}(U)\right) = P^i_{R_1^{-1}}\left(f^{-1}(U)\right)$.

Proof (tf8): Let $U \in D(X_2)$. Then taking into account that $f : X_1 \longrightarrow X_2$ is an LM_θ-function, prescription (1) and properties (tS9) and (tf6) it follows that for all $i \in I$,

$$\begin{aligned} f^{-1}\left(F^i_{R_2}(U)\right) &= f^{-1}\left(\overline{\varphi}_i^{X_2}\left(G_{R_2}\left(\overline{\varphi}_i^{X_2}\left(f_i^{2^{-1}}(U)\right)\right)\right)\right) \\ &= \overline{\varphi}_i^{X_1}\left(G_{R_1}\left(\overline{\varphi}_i^{X_1}\left(f_i^{1^{-1}}\left(f^{-1}(U)\right)\right)\right)\right) \\ &= F^i_{R_1}\left(f^{-1}(U)\right). \end{aligned}$$

So, (tf8) holds.
(tf9): It can be proved in a similar way, taking into account properties (tS10) and (tf7). □

Lemma 7 *Let* $\left(X, \{f_i\}_{i\in I}, R\right)$ *be a tense* LM_θ*-space. Then for all* $x, y \in X$ *such that* $(x, y) \notin R$*, the following conditions are satisfied:*

(i) *There is* $U \in D(X)$ *such that* $y \notin U$ *and* $x \in G_R(U)$*, or there is* $V \in D(X)$ *such that* $y \in V$ *and* $x \notin F^0_R(V)$*,*
(ii) *there is* $W \in D(X)$ *such that* $y \notin W$ *and* $x \in H_{R^{-1}}(W)$*, or there is* $V \in D(X)$ *such that* $y \in V$ *and* $x \notin P^0_{R^{-1}}(V)$*.*

Proof (i): Let $x,\ y \in X$ such that $y \notin R(x)$. Then, from property (tS2) we have that $y \notin\uparrow R(x)$ or $y \notin\downarrow R(x)$. Suppose that $y \notin\uparrow R(x)$. Then, $z \not\leq y$ for all $z \in R(x)$. Since, by property (tS1), $R(x)$ is compact, then from the last assertion, we infer that there is $U \in D(X)$ such that $y \notin U$ and $R(x) \subseteq U$. Therefore, $x \in G_R(U)$. Suppose now that $y \notin\downarrow R(x)$. Then $y \not\leq z$ for all $z \in R(x)$. From the last statement and the fact that $R(x)$ is compact, we infer that there is $V \in D(X)$ such that $y \in V$ and (1) $R(x) \cap V = \emptyset$, which implies by property (lP9) that (2) $R(x) \cap f_0^{-1}(V) = \emptyset$. Indeed, if there is $z \in R(x)$ such that $f_0(z) \in V$, then from (lP9) we deduce that $z \in V$ and so $z \in R(x) \cap V$, which contradicts (1). Therefore, from (2) we conclude that $x \notin F^0_R(V)$.
(ii): It can be proved in a similar way to (i). □

Proposition 6 *Let* $\left(X_1, \{f_i^1\}_{i\in I}, R_1\right)$ *and* $\left(X_2, \{f_i^2\}_{i\in I}, R_2\right)$ *be two tense* LM_θ*-spaces. Then, the following conditions are equivalent:*

(i) $f : X_1 \longrightarrow X_2$ *is a tense* LM_θ*-function,*
(ii) $f : X_1 \longrightarrow X_2$ *is an* LM_θ*-function such that for all* $U \in D(X_2)$*:*

(tf6) $f^{-1}\left(G_{R_2}(U)\right) = G_{R_1}\left(f^{-1}(U)\right)$,
(tf7) $f^{-1}(H_{R_2^{-1}}(U)) = H_{R_1^{-1}}\left(f^{-1}(U)\right)$.

Proof The proof is similar in spirit to Lemma 3.6 of Figallo et al. (2018). □

Proposition 7 and Corollary 4 can be proved in a similar way to Lemma 3.8 and Corollary 3.9 of Figallo et al. (2018).

Proposition 7 *Let* $\left(X, \{f_i\}_{i\in I}, R\right)$ *be a tense* LM_θ*-space. Then, the following conditions are satisfied for any* $x, y, \in X$ *and* $i \in I$*:*

(tS11) $R(f_i(x)) \subseteq \bigcup_{y\in R(f_i(x))}\uparrow f_i(y)$,
(tS12) $R^{-1}(f_i(x)) \subseteq \bigcup_{y\in R^{-1}(f_i(x))}\uparrow f_i(y)$,
(tS13) $\uparrow f_i(R(x)) = \uparrow R(f_i(x))$,
(tS14) $\uparrow f_i(R^{-1}(x)) = \uparrow R^{-1}(f_i(x))$,
(tS15) $f_i^{-1}\left(G_R(U)\right) = G_R\left(f_i^{-1}(U)\right)$,
(tS16) $f_i^{-1}\left(H_{R^{-1}}(U)\right) = H_{R^{-1}}\left(f_i^{-1}(U)\right)$,
(tS17) $f_i^{-1}\left(F^i_R(U)\right) = F^i_R\left(f_i^{-1}(U)\right)$,
(tS18) $f_i^{-1}\left(P^i_{R^{-1}}(U)\right) = P^i_{R^{-1}}\left(f_i^{-1}(U)\right)$.

Corollary 4 *Let* $\left(X, \{f_i\}_{i\in I}, R\right)$ *be a tense* LM_θ*-space. Then, the conditions* (tS3), (tS4) *and* (tS5) *can be replaced by the following conditions:*

(tS15) $f_i^{-1}\left(G_R(U)\right) = G_R\left(f_i^{-1}(U)\right)$ *for any* $U \in D(X)$*,*
(tS16) $f_i^{-1}\left(H_{R^{-1}}(U)\right) = H_{R^{-1}}\left(f_i^{-1}(U)\right)$ *for any* $U \in D(X)$*.*

Proposition 8 *Let* $\left(X, \{f_i\}_{i\in I}, R\right)$ *be a tense* LM_θ*-space and* $D(X)$ *be the lattice of all increasing, closed and open subsets of* X*. Then,*

$$\Psi(X) = \left(D(X), \left\{\varphi_i^X\right\}_{i\in I}, \left\{\overline{\varphi}_i^X\right\}_{i\in I}, G_R, H_{R^{-1}}\right)$$

is a tense LM_θ*-algebra, where for all* $U \in D(X)$ *and all* $i \in I$*,* $\varphi_i^X(U)$*,* $\overline{\varphi}_i^X(U)$*,* $G_R(U)$ *and* $H_{R^{-1}}(U)$ *are defined as in* (1)*,* (10) *and* (11)*, respectively.*

Proof From statement (A1) we have that $\langle D(X), \cup, \cap, \{\varphi_i^X\}_{i\in I}, \{\overline{\varphi}_i^X\}_{i\in I}, \emptyset, X\rangle$ is an LM_θ-algebra. From the definition (10) and (11), we obtain that properties (T1) and (T2) hold. Let us prove that the remaining axioms are satisfied.

(T3): Since any $U \in D(X)$ satisfies properties (tS18) and (tS19) in Proposition 7, then we can assert that property (T3) holds.
(T4′): Let $U \in D(X)$, $i \in I$. Suppose that $x \in \varphi_i^X(U)$ and $y \in R(x)$. Since $x \in f_i^{-1}(U)$, it follows that $x \in R^{-1}(y) \cap f_i^{-1}(U)$. This last assertion and Proposition 5 imply that $y \in P_R(f_i^{-1}(U))$ and consequently $y \in P_R^i(U)$. Therefore, $R(x) \subseteq P_R^i(U)$, from which we obtain that $\varphi_i^X(U) \subseteq G_R\left(P_R^i(U)\right)$. In a similar way, we can prove that $\varphi_i^X(U) \subseteq H_{R^{-1}}\left(F_R^i(U)\right)$. And so property (T4′) holds too. From the above statements, we conclude that $\Psi(X)$ is a tense LM_θ-algebra. □

Lemma 8 *Let $f : \left(X_1, \{f_i^1\}_{i\in I}\right) \longrightarrow \left(X_2, \{f_i^2\}_{i\in I}\right)$ be a morphism of tense LM_θ-spaces. Then, the map $\Psi(f) : D(X_2) \longrightarrow D(X_1)$ defined by $\Psi(f)(U) = f^{-1}(U)$ for all $U \in D(X_2)$, is a tense LM_θ-homomorphism.*

Proof It follows from statement (A6) and Proposition 6. □

Proposition 8 and Lemma 8 show that Ψ is a contravariant functor from $tLM_\theta S$ to $tLM_\theta A$.

To achieve our goal we need to define a contravariant functor from $tLM_\theta A$ to $tLM_\theta S$.

Lemma 9 *Let (A, G, H) be a tense LM_θ-algebra. Then, the following conditions are equivalent for all $S, T \in X(A)$:*

(i) $G^{-1}(S) \subseteq T \subseteq F_0^{-1}(S)$,
(ii) $H^{-1}(T) \subseteq S \subseteq P_0^{-1}(T)$.

Proof (i)⇒(ii): Let S, T be two prime filters of A such that (1) $G^{-1}(S) \subseteq T \subseteq F_0^{-1}(S)$, and let us suppose that $H(x) \in T$. Then, from (1) it follows that $H(x) \in F_0^{-1}(S)$ and hence $F_0(H(x)) \in S$. From this last assertion and the definition of the operator F_0 (Definition 7) we obtain that (2) $F(\varphi_0(H(x))) \in S$. Taking into account that $F(\varphi_0(H(x))) \in \mathcal{B}(A)$ (Lemma 3) and the characterizations (L8) of the Boolean elements of an LM_θ-algebra, we infer that (3) $\varphi_0(F(\varphi_0(H(x))) = F(\varphi_0(H(x)))$. Besides, from property (T9), we have that (4) $F(\varphi_0(H(x)) \leq \varphi_0(x)$. Then, from (2), (3), (4) and property (L7) of LM_θ-algebras we get that $x \in S$. Therefore, $H^{-1}(T) \subseteq S$. On the other hand, suppose that $z \in S$. From property (T4*), we get that (5) $G(P_0(z)) \in S$ and so $P_0(z) \in G^{-1}(S)$. From this last statement and (1) we deduce that $P_0(z) \in T$, and hence $z \in P_0^{-1}(T)$. Therefore, $S \subseteq P_0^{-1}(T)$. The converse implication is similar. □

Lemma 10 *Let (A, G, H) be a tense LM_θ-algebra and let R^A be the relation defined on $X(A)$ by the prescription:*

$$(S, T) \in R^A \iff G^{-1}(S) \subseteq T \subseteq F_0^{-1}(S). \tag{20}$$

Then, for all $S, T \in X(A)$,

$$(S, T) \in R^A \iff H^{-1}(T) \subseteq S \subseteq P_0^{-1}(T). \tag{21}$$

Proof It is a direct consequence of Lemma 9. □

Remark 2 Lemma 10 means that we have two ways to define the relation R^A, either by using G and F_0, or by using H and P_0.

Corollary 5 *Let (A, G, H) be a tense LM_θ-algebra and let R^A be the relation defined on $X(A)$ by prescription (20) or equivalently by prescription (21). Then,*

$$(S, T) \in R^A \iff G^{-1}(S) \subseteq T \subseteq F_i^{-1}(S) \text{ for all } i \in I, \tag{22}$$

or equivalently,

$$(S, T) \in R^A \iff H^{-1}(T) \subseteq S \subseteq P_i^{-1}(T) \text{ for all } i \in I. \tag{23}$$

Proof From Lemma 1 it follows that for all $S, T \in X(A)$ and for all $i \in I$, $F_0^{-1}(S) \subseteq F_i^{-1}(S)$ and $P_0^{-1}(T) \subseteq P_i^{-1}(T)$ and so from Lemma 10 the proof is complete. □

Lemma 11 *Let (A, G, H) be a tense LM_θ-algebra and let R^A be the relation defined on $X(A)$ as in (20), or equivalently as in (21), then for all $S \in X(A)$,*

(i) $R^A(S) = \downarrow R^A(S) \cap \uparrow R^A(S)$,
(ii) $\left(R^A\right)^{-1}(S) = \downarrow \left(R^A\right)^{-1}(S) \cap \uparrow \left(R^A\right)^{-1}(S)$,
(iii) *$R^A(S)$ is closed in $X(A)$,*
(iv) *$\left(R^A\right)^{-1}(S)$ is closed in $X(A)$.*

Proof We will only prove (i) and (iii). Similarly we can prove (ii) and (iv).
(i): Since for all $S \in X(A)$, $R^A(S) \subseteq \downarrow R^A(S)$ and $R^A(S) \subseteq \uparrow R^A(S)$, we have that $R^A(S) \subseteq \downarrow R^A(S) \cap \uparrow R^A(S)$. On the other hand, let $T \in \uparrow R^A(S) \cap \downarrow R^A(S)$. Then, there are $S_1, S_2 \in X(A)$ such that $S \subseteq S_1$, $S_1 R^A T$, $S R^A S_2$ and $S_2 \supseteq T$. Hence, $G^{-1}(S) \subseteq G^{-1}(S_1) \subseteq T$ and $T \subseteq S_2 \subseteq F_i^{-1}(S)$ for any $i \in I$. Hence, from (T13) we deduce that $\varphi_i^{-1}(T) \subseteq F_i^{-1}(S)$ for all $i \in I$. Therefore, $T \in R^A(S)$.
(iii): Let us suppose that (1) $T \notin R^A(S)$. Then, by prescription (20) there is $x \in G^{-1}(S)$ such that $x \notin T$, or there is $y \in T$ such that $y \notin F_0^{-1}(S)$. In the first case, we have that (2) $T \notin \sigma_A(x)$ and (3) $G^{-1}(S) \in \sigma_A(x)$. Then, taking into account that $\sigma_A(x)$ is an increasing subset of $X(A)$, assertion (3) and prescription (20) we infer that $R^A(S) \subseteq \sigma_A(x)$. From this assertion and (2) we deduce that (4) $T \in \sigma_A(x)^c$ and $\sigma_A(x)^c \subseteq R^A(S)^c$. In the second case, $T \in \sigma_A(y)$ and

$F_0^{-1}(S) \in \sigma_A(y)^c$. Since $\sigma_A(y)^c$ is a decreasing set, then from the last statement and prescription (20) we infer that $R^A(S) \subseteq \sigma_A(y)^c$. So, we obtain that (5) $T \in \sigma_A(y)$ and $\sigma_A(y) \subseteq R^A(S)^c$. Therefore, assertions (1), (4) and (5) allow us to assert that $R^A(S)^c$ is an open subset of $X(A)$, which implies that $R^A(S)$ is a closed subset of $X(A)$. □

Lemma 12 *Let (A, G, H) be a tense LM_θ-algebra and let $S \in X(A)$ and $a \in A$. Then,*

(i) *$G(a) \notin S$ if and only if there exists $T \in X(A)$ such that $(S, T) \in R^A$ and $a \notin T$,*
(ii) *$H(a) \notin S$ if and only if there exists $T \in X(A)$ such that $(S, T) \in \left(R^A\right)^{-1}$ and $a \notin T$.*

Proof (i): Suppose that (1) $G(a) \notin S$. Let us consider the ideal $(\{a\} \cup (A \backslash F_0^{-1}(S))]$, and we will prove that

$$G^{-1}(S) \cap (\{a\} \cup (A \backslash F_0^{-1}(S))] = \emptyset. \tag{24}$$

Suppose the opposite. Then there exists (2) $b \in G^{-1}(S)$ and there exists (3) $c \in (A \backslash F_0^{-1}(S))$ such that $b \leq a \vee c$, Then, from properties (T5) and (T12) we have that $G(b) \leq G(a \vee c) \leq G(a) \vee F_0(c)$. From this last assertion, (1) and (2), we deduce that $F_0(c) \in S$, which contradicts (3). Thus, (24) holds. Therefore, from Birkhoff–Stone Theorem there is a prime filter T such that $G^{-1}(S) \subseteq T$ and $(\{a\} \cup (A \backslash F_0^{-1}(S))] \cap T = \emptyset$. Consequently $a \notin T$ and $T \subseteq F_0^{-1}(S)$. Therefore, $G^{-1}(S) \subseteq T \subseteq F_0^{-1}(S)$ and so from Lemma 10 we conclude that $(S, T) \in R^A$. The other implication is easy.
(ii): It can be proved in a similar way as in (i). □

Proposition 9 *Let $\left(A, \{\varphi_i\}_{i \in I}, \left\{\overline{\varphi}_i\right\}, G, H\right)$ be an LM_θ-algebra and let $\left(X(A), \{f_i^A\}_{i \in I}\right)$ be the LM_θ-space associated with A. Then, $\Phi(A) = \left(X(A), \{f_i^A\}_{i \in I}, R^A\right)$ is a tense LM_θ-space, where R^A is the relation defined on $X(A)$ as in (20) or (21). Besides, $\sigma_A : A \longrightarrow D(X(A))$, defined by prescription (2), is a tense LM_θ-isomorphism.*

Proof From statement (A2) it follows that $\left(X(A), \{f_i^A\}_{i \in I}\right)$ is an LM_θ-space and σ_A is an LM_θ-isomorphism. Also for all $a \in A$, $G_{R^A}(\sigma_A(a)) = \sigma_A(G(a))$ and $H_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(H(a))$. Indeed, let us take a prime filter S such that $G(a) \notin S$. Then, by Lemma 12, we infer that there exists $T \in X(A)$ such that $(S, T) \in R^A$ and $a \notin T$, which implies that $R^A(S) \nsubseteq \sigma_A(a)$. Hence $S \notin G_{R^A}(\sigma_A(a))$, and therefore, $G_{R^A}(\sigma_A(a)) \subseteq \sigma_A(G(a))$. On the other hand, let $S, T \in X(A)$ such that $G(a) \in S$ and $T \in R^A(S)$. Then $a \in G^{-1}(S)$ and $G^{-1}(S) \subseteq T$, from which it follows that $T \in \sigma_A(a)$. Therefore, $R^A(S) \subseteq \sigma_A(a)$, which allows us to assert that $S \in G_R(\sigma_A(a))$. And so, we get that $\sigma_A(G(a)) = G_{R^A}(\sigma_A(a))$. Similarly we can prove that $H_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(H(a))$ and consequently by Proposition 6, σ_A is a tense LM_θ-isomorphism. Besides, from Lemma 11 we have that properties (tS1) and (tS2) hold. Also, from Corollary 4 we obtain that conditions (tS4), (tS5) and (tS6) are satisfied. Therefore, we conclude that $\left(X(A), \{f_i^A\}_{i \in I}, R^A\right)$ is a tense LM_θ-space. □

Lemma 13 *Let (A_1, G_1, H_1) and (A_2, G_2, H_2) be two tense LM_θ-algebras and let $h : A_1 \longrightarrow A_2$ be a tense LM_θ-homomorphism. Then, the function $\Phi(h) : X(A_2) \longrightarrow X(A_1)$, defined by $\Phi(h)(S) = h^{-1}(S)$ for all $S \in X(A_2)$, is a tense LM_θ-function.*

Proof It is a direct consequence of statement (A5) and Proposition 6. □

Proposition 9 and Lemma 13 show that Φ is a contravariant functor from $tLM_\theta A$ to $tLM_\theta S$.

The following characterizations of isomorphisms in the category $tLM_\theta S$ will be used to determine the duality that we set out to prove.

Lemma 14 *Let $\left(X_1, \{f_i^1\}_{i \in I}, R_1\right)$ and $\left(X_2, \{f_i^2\}_{i \in I}, R_2\right)$ be two tense LM_θ-spaces. Then, the following conditions are equivalent, for every function $f : X_1 \longrightarrow X_2$:*

(i) *f is an isomorphism in the category $tLM_\theta S$,*
(ii) *f is a bijective function such that f and f^{-1} are tense LM_θ-functions.*

Proof It is routine. □

Proposition 10 *Let $\left(X_1, \{f_i^1\}_{i \in I}, R_1\right)$ and $\left(X_2, \{f_i^2\}_{i \in I}, R_2\right)$ be two tense LM_θ-spaces. Then, the following conditions are equivalent, for every function $f : X_1 \longrightarrow X_2$:*

(i) *f is an isomorphism in the category $tLM_\theta S$,*
(ii) *f is a bijective function such that f and f^{-1} are LM_θ-functions and for all $x, y \in X_1$:*

(itf) $(x, y) \in R_1 \Longleftrightarrow (f(x), f(y)) \in R_2$.

Proof (i) $\Rightarrow$ (ii): It follows immediately from the hypothesis (i), Lemma 14 and property (tf1) of tense LM_θ-functions.
(ii) $\Rightarrow$ (i): From the hypothesis (ii) and Lemma 14 it follows that f is a bijective function and f and f^{-1} are LM_θ-functions. Besides, f satisfies conditions (tf1), (tf2) and (tf3) as we will show next.
(tf1): From property (itf) and the fact that f is a bijective function it follows that for all $x \in X_1$, (1) $f(R_1(x)) = R_2(f(x))$ and (2) $f\left(R_1^{-1}(x)\right) = R_2^{-1}(f(x))$.
(tf2): From (1), we obtain that $R_2(f(x)) = f(R_1(x)) \subseteq\uparrow f(R_1(x))$ for all $x \in X_1$.

(tf3): From (2), we obtain that $R_2^{-1}(f(x)) = f\left(R_1^{-1}(x)\right) \subseteq \uparrow f\left(R_1^{-1}(x)\right)$ for all $x \in X_1$.

The proof that the function f^{-1} satisfies conditions (tf1), (tf2) and (tf3) is similar. Therefore, f and f^{-1} are tense bijective LM_θ-functions and so from Lemma 14 we conclude the proof. □

The following proposition allows us to assert that $\varepsilon_X : X \longrightarrow X(D(X))$, defined as in (4), is an isomorphism in the category $tLM_\theta S$, which is fundamental in the duality we are looking for.

Proposition 11 *Let $(X, \{f_i\}_{i\in I}, R)$ be a tense LM_θ-space, ε_X be the map from X onto $X(D(X))$ defined by prescription (4) and let $R^{D(X)}$ be the relation defined on $X(D(X))$ by means of the operators G_R, F_R^0, $H_{R^{-1}}$ and $P_{R^{-1}}^0$ as follows:*

$$(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Leftrightarrow G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y) \subseteq F_R^{0^{-1}}(\varepsilon_X(x)), \tag{25}$$

or equivalently

$$(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Leftrightarrow H_{R^{-1}}^{-1}(\varepsilon_X(y)) \subseteq \varepsilon_X(x) \subseteq P_{R^{-1}}^{0^{-1}}(\varepsilon_X(y)). \tag{26}$$

Then, the following property holds:

(tS19) $(x, y) \in R \iff (\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)}$.

Proof From Lemma 10 and statements (A1) and (A4) it follows that prescription (25) is equivalent to prescription (26).

Let $x, y \in X$ such that (1) $(\varepsilon_X(x), \varepsilon_X(y)) \notin R^{D(X)}$, then from the prescription (25) we infer that $G_R^{-1}(\varepsilon_X(x)) \nsubseteq \varepsilon_X(y)$ or $\varepsilon_X(y) \nsubseteq F_R^{0^{-1}}(\varepsilon_X(x))$. From the last statements it follows that there exists $U \in D(X)$ such that $x \in G_R(U)$ and $y \notin U$ or there exists $V \in D(X)$ such that $y \in V$ and $x \notin F_R^0(V)$. Consequently from Lemma 7 we obtain that $(x, y) \notin R$.

Besides, assertion (1), prescription (26) and Lemma 7 allow us to assert that $(x, y) \notin R$.

Conversely, let $x, y \in X$ such that $(x, y) \notin R$, then from Lemma 7 we have that the following conditions hold:

(3) There is $U \in D(X)$ such that $x \in G_R(U)$ and $y \notin U$, or there is $V \in D(X)$ such that $y \in V$ and $x \notin F_R^0(V)$,
(4) there is $W \in D(X)$ such that $y \in H_{R^{-1}}(W)$ and $x \notin W$, or there is $V \in D(X)$ such that $x \in V$ and $y \notin P_{R^{-1}}^0(V)$.

Suppose that (3) holds. Then $G_R^{-1}(\varepsilon_X(x)) \nsubseteq \varepsilon_X(y)$ or $\varepsilon_X(y) \nsubseteq F_R^{0^{-1}}(\varepsilon_X(x))$, and therefore, from prescription (25) we infer that $(\varepsilon_X(x), \varepsilon_X(y)) \notin R^{D(X)}$.

If (4) holds, then $H_{R^{-1}}^{-1}(\varepsilon_X(y)) \nsubseteq \varepsilon_X(x)$ or $\varepsilon_X(x) \nsubseteq P_{R^{-1}}^{o^{-1}}(\varepsilon_X(y))$ and so from prescription (26) we obtain that $(\varepsilon_X(x), \varepsilon_X(y)) \notin R^{D(X)}$. □

Corollary 6 *Let $(X, \{f_i\}_{i\in I}, R)$ be a tense LM_θ-space, ε_X be the map from X onto $X(D(X))$ defined by prescription (4) and let $R^{D(X)}$ be the relation defined on $X(D(X))$ by means of the operators G_R and F_R as follows:*

$$(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Leftrightarrow G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y) \subseteq F_R^{i^{-1}}(\varepsilon_X(x)) \text{ for all } i \in I, \tag{27}$$

or equivalently

$$(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \Leftrightarrow H_{R^{-1}}^{-1}(\varepsilon_X(y)) \subseteq \varepsilon_X(x) \subseteq P_{R^{-1}}^{i^{-1}}(\varepsilon_X(y)) \text{ for all } i \in I. \tag{28}$$

Then, the following property holds:

(tS19) $(x, y) \in R \iff (\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)}$.

Proof From Lemma 1 it follows that for all $x, y \in X$ and $i \in I$, $F_R^{0^{-1}}(\varepsilon_X(x)) \subseteq F_R^{i^{-1}}(\varepsilon_X(x))$ and $P_{R^{-1}}^{0^{-1}}(\varepsilon_X(y)) \subseteq P_{R^{-1}}^{i^{-1}}(\varepsilon_X(y))$. From these last assertions and Proposition 10 the proof is complete. □

Corollary 7 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. Then, the function $\varepsilon_X : X \longrightarrow X(D(X))$, defined by prescription (4), is an isomorphism in the category $tLM_\theta S$.*

Proof It follows from the results established in Figallo et al. (2010) and Propositions 10 and 11. □

The map $\varepsilon_X : X \longrightarrow X(D(X))$, defined as in (4), leads us to formulate another characterization of tense LM_θ-spaces as we will describe below:

Proposition 12 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space and let ε_X be the function from X onto $X(D(X))$ defined by prescription (4). If $R^{D(X)}$ is the relation defined on $X(D(X))$ by prescription (25) or (26), condition* (tS2) *in Definition 8 can be replaced by condition* (tS19).

Proof (tS2) $\Rightarrow$ (tS19): It follows from Proposition 11, which is a consequence of Lemma 7, and consequently it is a consequence of the fact that $\left(X, \{f_i\}_{i\in I}\right)$ is an LM_θ-space and the relation R satisfies properties (tS1) and (tS2).
(tS19) $\Rightarrow$ (tS2): We have to prove $\downarrow R(x) \cap \uparrow R(x) \subseteq R(x)$; the other inclusion always holds. Suppose that $y \in \downarrow R(x) \cap \uparrow R(x)$. Then there exists $z_1, z_2 \in X$ such that $y \leq z_1$, $(x, z_1) \in R$, $z_2 \leq y$ and $(x, z_2) \in R$. Hence, from the fact that ε_X is an order isomorphism and property (tS19), we infer that $\varepsilon_X(y) \subseteq \varepsilon_X(z_1)$, $(\varepsilon_X(x), \varepsilon_X(z_1)) \in R^{D(X)}$,

$\varepsilon_X(z_2) \subseteq \varepsilon_X(y)$ and $(\varepsilon_X(x), \varepsilon_X(z_2)) \in R^{D(X)}$. Thus, from the last statements and prescription (25), we obtain that $G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(z_2) \subseteq \varepsilon_X(y) \subseteq \varepsilon_X(z_1) \subseteq F_R^{0^{-1}}(\varepsilon_X(x))$. Therefore, $(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)}$, and so from property (tS19) we conclude that $(x, y) \in R$. □

Then, from the above results and using the usual procedures we can prove that the functors $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are naturally equivalent to the identity functors on $tLM_\theta S$ and $tLM_\theta A$, respectively, where the isomorphisms σ_A and ε_X are the corresponding natural equivalences, from which we conclude:

Theorem 3 *The category $tLM_\theta S$ is naturally equivalent to the dual category of the category $tLM_\theta A$.*

4 Congruences on tense LM_θ-algebras

In this section, our objective is the characterization of the congruence lattice and the θ-congruence lattice on a tense LM_θ-algebra by means of certain closed subsets of its associated tense LM_θ-space. With this purpose, we will start by introducing the following notion.

Definition 12 Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. A subset Y of X is a tense subset if it satisfies the following conditions:

(ts1) $Y \subseteq G_R(Y)$, where $G_R(Y)$ is defined by prescription (8), i.e., for all $y \in Y$ and for all $z \in R(y)$, $\downarrow z \cap R(y) \cap Y \neq \emptyset$,
(ts2) $Y \subseteq H_{R^{-1}}(Y)$, where $H_{R^{-1}}(Y)$ is defined by prescription (9), i.e. for all $y \in Y$ and for all $z \in R(y)$, $\downarrow z \cap R^{-1}(y) \cap Y \neq \emptyset$.

The notion of tense subset of a tense LM_θ-space has the following equivalent formulation, which will be useful later:

Lemma 15 *Let $(X, \{f_i\}_{i\in I}, R)$ be a tense LM_θ-space. If Y is a subset of X, then the following conditions are equivalent:*

(i) *Y is a tense subset,*
(ii) *$Y = G_R(Y) \cap Y \cap H_{R^{-1}}(Y)$.*

Proof It is immediate. □

In Figallo et al. (2012) the following characterizations of a modal subset of an LM_θ-space were obtained.

Proposition 13 (Figallo et al. 2012, Proposition 4.4, Corollary 4.1) *Let $\left(X, \{f_i\}_{i\in I}\right)$ be an LM_θ-space and let Y be a nonempty subset of X. Then, the following conditions are equivalent:*

(i) *Y is modal,*
(ii) *Y is decreasing and increasing,*
(iii) *Y is the cardinal sum of sets $[f_0(y), f_1(y)]$, $y \in Y$, where $[x, z] = \{w \in X : x \leq w \leq z\}$ for all $x, z \in X$.*

Corollary 8 *Let $(X, \{f_i\}_{i\in I})$ be an LM_θ-space. If $\{Y_i\}_{i\in I}$ is a family of modal subsets of X, then $\bigcap_{i\in I} Y_i$ is a modal subset of X*

Proof It is a direct consequence of Proposition 13. □

Proposition 14 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. If Y is a tense semimodal subset of X, then for all $i \in I$ and for all $y \in f_i^{-1}(Y)$, the following conditions are satisfied:*

(ts5) *$R(y) \subseteq f_i^{-1}(Y)$,*
(ts6) *$R^{-1}(y) \subseteq f_i^{-1}(Y)$.*

Proof (ts5): Let $i \in I$, (1) $y \in f_i^{-1}(Y)$ and (2) $z \in R(y)$. Then, from (2) and property (tS3) we obtain that $f_i(z) \in R(f_i(y))$. From (1), (2), the last assertion and the fact that Y is a tense subset of X, we infer that there is (3) $w \in Y$ such that $w \in R(f_i(y))$ and (4) $w \leq f_i(z)$. Then, from (4) and property (lP3) it follows that (5) $f_i(w) = f_i(z)$. Besides, from (3) and the fact that Y is semimodal, we obtain that $f_i(w) \in Y$ and so from (5), we conclude that $f_i(z) \in Y$, which allows us to assert that $R(y) \subseteq f_i^{-1}(Y)$.
(ts6): It can be proved in a similar way. □

Corollary 9 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. If Y is a tense and semimodal subset of X, then for all $i \in I$, $f_i^{-1}(Y)$ is a tense and modal subset of X.*

Proof From property (lP5) it follows that $f_i^{-1}(Y)$ is a modal subset of X for all $i \in I$. Then, from Proposition 13, we have that $f_i^{-1}(Y)$ is increasing for all $i \in I$. From this last assertion and Proposition 4, we infer that for all $i \in I$, $G_R(f_i^{-1}(Y)) = \{x \in X : R(x) \subseteq f_i^{-1}(Y)\}$ and $H_{R^{-1}}(f_i^{-1}(Y)) = \{x \in X : R^{-1}(x) \subseteq f_i^{-1}(Y)\}$ and so from Proposition 14, we conclude that $f_i^{-1}(Y) \subseteq G_R(f_i^{-1}(Y))$ and $f_i^{-1}(Y) \subseteq H_{R^{-1}}(f_i^{-1}(Y))$ for all $i \in I$. Therefore, the proof is complete. □

Corollary 10 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. If Y is a modal subset of X, then the following conditions are equivalent:*

(i) *Y is a tense subset,*
(ii) *for all $i \in I$ and for all $y \in Y$, the following conditions are satisfied:*

  (ts7) *$R(y) \subseteq Y$,*
  (ts8) *$R^{-1}(y) \subseteq Y$,*

(iii) *$Y = G_R(Y) \cap Y \cap H_{R^{-1}}(Y)$.*

Proof It is a direct consequence of Lemma 15 and Propositions 13 and 14. □

Proposition 15 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. If Y is a modal subset of X, then $G_R(Y)$ and $H_{R^{-1}}(Y)$ are also modal.*

Proof Let Y be a modal subset of X. From property (lP12) it follows immediately that (1) $G_R(Y) \subseteq \bigcup_{z\in G_R(Y)}[f_0(z), f_1(z)]$. Let (2) $z \in G_R(Y)$ and let (3) $w \in [f_0(z), f_1(z)]$, then from (3) and properties (lP3) and (lP5), we obtain that (4) $f_i(w) = f_i(z)$ for all $i \in I$. Let (5) $t \in R(w)$, then by (4), (5) and property (tS3), we infer that $f_0(t) \in R(f_0(z))$, and therefore, from properties (tS4), (lP3) and (lP5), we can assert that there exists $y \in X$ such that (5) $y \in R(z)$ and (6) $f_i(y) = f_i(t)$ for all $i \in I$. From (2) and (5) we get that $y \in Y$. Since Y is modal, then from this last assertion and (6) it results that $f_i(t) \in Y$ for all $i \in I$. Then, since Y is modal, we have that $t \in Y$, from which we deduce by (5) that $R(w) \subseteq Y$, which allows to assert that $w \in G_R(Y)$. Therefore, from (3) we can set that $\bigcup_{z\in G_R(Y)}[f_0(z), f_1(z)] \subseteq G_R(Y)$. Then, from (1) it follows that $G_R(Y) = \bigcup_{z\in G_R(Y)}[f_0(z), f_1(z)]$, and so from Proposition 13, we conclude that $G_R(Y)$ is modal. The proof that $H_{R^{-1}}(Y)$ is modal is similar. □

The characterization of tense subsets of a tense LM_θ-space, given in Lemma 15, prompts us to introduce the following definition:

Definition 13 Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space and let $d_X : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ defined by:

$$d_X(Z) = G_R(Z) \cap Z \cap H_{R^{-1}}(Z), \quad \text{for all } Z \in \mathcal{P}(X). \tag{29}$$

For each $n \in \omega$, let $d_X^n : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, defined by:

$$d_X^0(Z) = Z, \;\; d_X^{n+1}(Z) = d_X(d_X^n(Z)), \;\; \text{for all } Z \in \mathcal{P}(X). \tag{30}$$

By using the above functions d_X, d_X^n, $n \in \omega$, we obtain another equivalent formulation of the notion of tense subset of a tense LM_θ-space.

Lemma 16 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space. If Y is a modal subset of X, then the following conditions are equivalent:*

(i) *Y is a tense subset,*
(ii) *$Y = d_X^n(Y)$, for all $n \in \omega$,*
(iii) *$Y = \bigcap_{n\in\omega} d_X^n(Y)$.*

Proof It is a consequence of Corollary 10 and Definition 13. □

Proposition 16 *Let $(X, \{f_i\}_{i\in I}, R)$ be a tense LM_θ-space and let the structure $(D(X), G_R, H_{R^{-1}})$ be the tense LM_θ-algebra associated with X. Then, for all $n \in \omega$, for all $U, V \in D(X)$ and for all $i \in I$, the following conditions are satisfied:*

(d0) *$d_X^n(U) \in D(X)$,*
(d1) *$d_X^n(X) = X$ and $d_X^n(\emptyset) = \emptyset$,*
(d2) *$d_X^{n+1}(U) \subseteq d_X^n(U)$,*
(d3) *$d_X^n(U \cap V) = d_X^n(U) \cap d_X^n(V)$,*
(d4) *$U \subseteq V$ implies $d_X^n(U) \subseteq d_X^n(V)$,*
(d5) *$d_X^n(U) \subseteq U$,*
(d6) *$d_X^{n+1}(U) \subseteq G_R(d_X^n(U))$ and $d_X^{n+1}(U) \subseteq H_{R^{-1}}(d_X^n(U))$,*
(d7) *$d_X^n(f_i^{-1}(U)) = f_i^{-1}(d_X^n(U))$ for any $n \in \omega$ and $i \in I$,*
(d8) *if U is modal, then $d_X^n(U)$ is modal,*
(d9) *$\bigcap_{n\in\omega} d_X^n(f_i^{-1}(U))$ is a closed, modal and tense subset of X.*

Proof In a similar way to (Figallo et al. 2018, Proposition 4.11). □

As consequences of Proposition 16 and the above duality for tense LM_θ-algebras (Proposition 9) we obtain the following corollaries.

Corollary 11 (Chiriţă 2011, Proposition 5.1) *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra, $d : A \longrightarrow A$ be the function defined by $d(a) = G(a)\wedge a\wedge H(a)$ for all $a \in A$, and for each $n \in \omega$, let $d^n : A \longrightarrow A$ be the function defined by $d^0(a) = a$ and $d^{n+1}(a) = d(d^n(a))$ for all $a \in A$. Then, for all $n \in \omega$ and for all $a, b \in A$, the following conditions are satisfied:*

(d1) *$d^n(1) = 1$ and $d^n(0) = 0$,*
(d2) *$d^{n+1}(a) \leq d^n(a)$,*
(d3) *$d^n(a \wedge b) = d^n(a) \wedge d^n(b)$,*
(d4) *$a \leq b$ implies $d^n(a) \leq d^n(b)$,*
(d5) *$d^n(a) \leq a$,*
(d6) *$d^{n+1}(a) \leq G(d^n(a))$ and $d^{n+1}(a) \leq H(d^n(a))$,*
(d7) *for all $i \in I$ and $n \in \omega$, $d^n(\varphi_i(a)) = \varphi_i(d^n(a))$.*

Corollary 12 *Let (A, G, H) be a tense LM_θ-algebra and $\left(X(A), \{f_i^A\}_{i\in I}, R^A\right)$ be the tense LM_θ-space associated with A. If $\sigma_A : A \longrightarrow D(X(A))$ is the map defined by prescription (2), then $\sigma_A(d^n(a)) = d_{X(A)}^n(\sigma_A(a))$ for all $a \in A$ and $n \in \omega$.*

Proof It is a direct consequence of Proposition 9. □

It is worth mentioning that the operator d defined in Corollary 11 was previously defined in Kowalski (1998) for tense algebras, in Diaconescu and Georgescu (2007) for tense MV-algebras, in Figallo and Pelaitay (2014), Figallo et al. (2017) for IKt-algebras and in Chiriţă (2010), Chiriţă (2011) for tense θ-valued Łukasiewicz–Moisil algebras, respectively.

Lemma 17 *Let (A, G, H) be a tense LM_θ-algebra. If $\bigwedge_{j\in J} a_j$ exists, then the following conditions hold:*

(i) *$\bigwedge_{j\in J} G(a_j)$ exists and $\bigwedge_{j\in J} G(a_j) = G(\bigwedge_{j\in J} a_j)$,*
(ii) *$\bigwedge_{j\in J} H(a_j)$ exists and $\bigwedge_{j\in J} H(a_j) = H(\bigwedge_{j\in J} a_j)$,*
(iii) *$\bigwedge_{j\in J} d(a_j)$ exists and $\bigwedge_{j\in J} d^n(a_j) = d^n(\bigwedge_{j\in J} a_j)$ for all $n \in \omega$.*

Proof (i): Assume that $a_j \in A$ for all $j \in J$ and $\bigwedge_{j\in J} a_j$ exists. Since $\bigwedge_{j\in J} a_j \leq a_j$, we have by (T5) that $G\left(\bigwedge_{j\in J} a_j\right) \leq G(a_j)$ for each $j \in J$. Thus, $G\left(\bigwedge_{j\in J} a_j\right)$ is a lower bound of the set $\{G(a_j) : j \in J\}$. Assume now that b is a lower bound of the set $\{G(a_j) : j \in J\}$. From properties (T6) and (T9*) we have that $P_0(b) \leq P_0G(a_j) \leq a_j$ for each $j \in J$. So, $P_0(b) \leq \bigwedge_{j\in J} a_j$. Besides, the pair (G, P_0) is a Galois connection, this means that $x \leq G(y) \Longleftrightarrow P_0(x) \leq y$, for all $x, y \in A$. So, we can infer that $b \leq G\left(\bigwedge_{j\in J} a_j\right)$. This proves that $\bigwedge_{j\in J} G(a_j)$ exists and $\bigwedge_{j\in J} G(a_j) = G\left(\bigwedge_{j\in J} a_j\right)$.

(ii): The proof for the operator H is analogous to the proof for G.

(iii): It is a direct consequence of (i) and (ii). □

For invariance properties we have:

Lemma 18 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space and $\left(D(X), G_R, H_{R^{-1}}\right)$ be the tense LM_θ-algebra associated with X. Then, for all $U, V, W, Z \in D(X)$ such that $U = d_X(U)$, $V = d_X(V)$, $d_X\left(f_{i_0}^{-1}(W)\right) = f_{i_0}^{-1}(W)$ for some $i_0 \in I$, and $d_X\left(X\backslash f_{i_1}^{-1}(Z)\right) = X\backslash f_{i_1}^{-1}(Z)$ for some $i_1 \in I$, the following properties are satisfied:*

(i) *$U \cap V = d_X(U \cap V)$,*
(ii) *$U \cup V = d_X(U \cup V)$,*
(iii) *$d_X(f_i^{-1}(W)) = f_i^{-1}(W)$ for all $i \in I$,*
(iv) *$d_X(X\backslash f_i^{-1}(Z)) = X\backslash f_i^{-1}(Z)$ for all $i \in I$.*

Proof (i): It immediately follows from the definition of the function d_X and property (T2) of tense LM_θ-algebras.

(ii): Taking into account that $U = d_X(U)$ and $V = d_X(V)$ and the fact that the operations G_R and $H_{R^{-1}}$ are increasing, we infer that $U \cup V \subseteq G_R(U \cup V)$ and $U \cup V \subseteq H_{R^{-1}}(U \cup V)$, which imply that $U \cup V = d_X(U \cup V)$.

(iii): If $W \in D(X)$ and $d_X(f_{i_0}^{-1}(W)) = f_{i_0}^{-1}(W)$ for some $i_0 \in I$, then from (d7) it follows that $f_{i_0}^{-1}(d_X(W)) = f_{i_0}^{-1}(W)$. From the last assertion and property (lP5) we infer that $f_i^{-1}(d_X(W)) = f_i^{-1}(W)$ for all $i \in I$, and so from (d7), we get that $d_X(f_i^{-1}(W)) = f_i^{-1}(W)$ for all $i \in I$. □

(iv): It is proved in a similar way to (iii).

Corollary 13 *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra. Then, for all $a, b, c, d \in A$, such that $a = d(a)$, $b = d(b)$, $\varphi_{i_0}(c) = d(\varphi_{i_0}(c))$ for some $i_0 \in I$ and $\overline{\varphi}_{i_1}(d) = d(\overline{\varphi}_{i_1}(d))$ for some $i_1 \in I$, the following properties are satisfied:*

(i) *$d(a \wedge b) = a \wedge b$,*
(ii) *$d(a \vee b) = a \vee b$,*
(iii) *$\varphi_i(c) = d(\varphi_i(c))$ for all $i \in I$,*
(iv) *$\overline{\varphi}_i(d) = d(\overline{\varphi}_i(d))$ for all $i \in I$.*

Proof It is a direct consequence of Proposition 9 and Lemma 18. □

Lemma 19 *Let (A, G, H) be a tense LM_θ-algebra. Then, for all $a \in A$, the following conditions are equivalent:*

(i) *$a = d(a)$,*
(ii) *$a = d^n(a)$ for all $n \in \omega$.*

Proof It immediately follows from Corollary 11. □

Lemma 20 *Let (A, G, H) be a complete tense LM_θ-algebra or a finite tense LM_θ-algebra. Then, the following conditions are equivalent for any $a \in A$:*

(i) *$a = d(a)$,*
(ii) *$a = d^n(a)$ for all $n \in \omega$,*
(iii) *$a = \bigwedge_{n\in\omega} d^n(a)$,*
(iv) *$a = \bigwedge_{n\in\omega} d^n(b)$ for some $b \in A$.*

Proof Taking into accoun that $\bigwedge_{n\in\omega} d^n(a) \in A$ for any $a \in A$ and Lemma 17, it follows that $d\left(\bigwedge_{n\in\omega} d^n(a)\right) = \bigwedge_{n\in\omega} d^n(a)$ for any $a \in A$. This last assertion and Lemma 17 allows us to complete the proof. □

Proposition 17 (Chiriţă 2011, Proposition 5.2) *Let $(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H)$ be a tense LM_θ-algebra and let $\mathcal{C}(A) := \{a \in A : d(a) = a\}$. Then,*

$$\langle \mathcal{C}(A), \vee, \wedge, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0, 1\rangle$$

is an LM_θ-algebra.

Proof From Corollary 13 and property (d1) in Corollary 11, we have that $\langle \mathcal{C}(A), \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. Taking into account that $a = d(a)$ for all $a \in \mathcal{C}(A)$, and the properties (iii) and (iv) in Corollary 13 it follows that $\varphi_i(a) = \varphi_i(d(a)) = d(\varphi_i(a))$ and $\overline{\varphi}_i(a) =$

$\overline{\varphi}_i(d(a)) = d(\overline{\varphi}_i(a))$ for all $a \in \mathcal{C}(A)$ and $i \in I$. Therefore, $\varphi_i(a), \overline{\varphi}_i(a) \in \mathcal{C}(A)$ for all $a \in \mathcal{C}(A)$ and $i \in I$, from which we conclude that $\langle \mathcal{C}(A), \vee, \wedge, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0, 1\rangle$ is an LM_θ-algebra. □

Corollary 14 *Let (A, G, H) be a tense LM_θ-algebra. Then, $(\mathcal{B}(\mathcal{C}(A)), G, H)$ is a tense Boolean algebra, where $\mathcal{B}(\mathcal{C}(A))$ is the Boolean algebra of all complemented elements of $\mathcal{C}(A)$.*

Proof It is a direct consequence of the fact that $a \in \mathcal{B}(\mathcal{C}(A))$ iff $a \in \mathcal{C}(A)$ and $a = \varphi_i(a)$ for all $i \in I$ (see (L8), Proposition 17 and property (iii) in Corollary 13). □

Let us recall that a tense LM_θ-congruence of an LM_θ-algebra A is a lattice congruence ρ on A, which satisfied the following properties for all $x, y \in A$:

(i) if $(x, y) \in \rho$, then $(\varphi_i x, \varphi_i y) \in \rho$ for all $i \in I$ (i.e. ρ is an LM_θ-congruence on A),
(ii) if $(x, y) \in \rho$, then $(G(x), G(y)) \in \rho$ and $(H(x), H(y)) \in \rho$.

The tense, semimodal and closed subsets of the tense LM_θ-space associated with a tense LM_θ-algebra perform a fundamental role in the characterization of the tense LM_θ-congruences.

Theorem 4 *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra, and $\left(X(A), \{f_i^A\}_{i\in I}, R^A\right)$ be the tense LM_θ-space associated with A. Then, the lattice $\mathcal{C}_{ST}(X(A))$ of all tense, semimodal and closed subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{tLM_\theta}(A)$ of all tense LM_θ-congruences on A, and the anti-isomorphism is the function Θ_{ST} defined by the same prescription as in (7).*

Proof Since $\mathcal{C}_{ST}(X(A)) \subseteq \mathcal{C}_S(X(A))$, then from Theorem 1 it follows that for any $Y \in \mathcal{C}_{ST}(X(A))$, $\Theta_{ST}(Y)$ is an LM_θ-congruence on A. Let us prove that $\Theta_{ST}(Y)$ preserves G and H. Let (1) $(a, b) \in \Theta_{ST}(Y)$ and (2) $S \in \sigma_A(G(a)) \cap Y$. Since σ_A is a tense LM_θ-isomorphism it follows that $S \in G_{R^A}(\sigma_A(a)) \cap Y$. Hence, from the fact that $\sigma_A(a) \in D(X(A))$ and Corollary 3 we obtain that (3) $R^A(S) \subseteq \sigma_A(a)$. Suppose that $T \in R^A(S)$. Since Y is a tense subset of $X(A)$ and $S \in Y$, then from property (ts1) of these subsets we can assert that there is (4) $W \in Y$, such that $W \subseteq T$ and $W \in R^A(S)$. This last assertion and (3) allow us to infer that $W \in \sigma_A(a)$, from which we get by (4) that $W \in \sigma_A(a) \cap Y$, and so by (1) we conclude that $W \in \sigma_A(b) \cap Y$. Since $W \subseteq T$, we have that $T \in \sigma_A(b)$. Therefore, $R^A(S) \subseteq \sigma_A(b)$ and so, by virtue that $\sigma_A(b) \in D(X(A))$, Corollary 3 and (2), we infer that $S \in G_{R^A}(\sigma_A(b)) \cap Y$. Then, $\sigma_A(G(a)) \cap Y \subseteq \sigma_A(G(b)) \cap Y$. The other inclusion is proved in a similar way. Analogously, $\Theta_{ST}(Y)$ preserves H. Therefore, $\Theta_{ST}(Y) \in Con_{tLM_\theta}(A)$ for all $Y \in \mathcal{C}_{ST}(X(A))$.

Conversely, let $\vartheta \in Con_{tLM_\theta}(A)$ and let $h : A \longrightarrow A/\vartheta$ be the natural epimorphism. Since $\vartheta \in Con_{LM_\theta}(A)$, then from Theorem 1 it follows that $\vartheta = \Theta_S(Y)$, where $Y = \{\Phi(h)(S) : S \in X(A/\vartheta)\} = \{h^{-1}(S) : S \in X(A/\vartheta)\}$ and $Y \in \mathcal{C}_S(X(A))$. Besides, Y is a tense subset of $X(A)$. Indeed, let $T \in Y$ and (1) $Q \in R^A(T)$. Since there exists $S \in X(A/\vartheta)$ such that (2) $\Phi(h)(S) = h^{-1}(S) = T$, we obtain that $Q \in R^A(\Phi(h)(S))$. From Lemma 13, $\Phi(h)$ is a tense LM_θ-function, then from the last assertion and property (tf2) of tense LM_θ-functions, we infer that $Q \in \uparrow \Phi(h)\left(R^{A/\vartheta}(S)\right)$. Therefore, there exists $M \in X(A/\vartheta)$ such that (3) $M \in R^{A/\vartheta}(S)$ and $\Phi(h)(M) \subseteq Q$, and consequently $\Phi(h)(M) \in \downarrow Q$. Also, it is verified that $\Phi(h)(M) = h^{-1}(M) \in Y$. Besides, from (3) and property (tf1) of tense LM_θ-functions, we obtain that $\Phi(h)(M) \in R^A(\Phi(h)(S))$ and thus from (2), we get that $\Phi(h)(M) \in R^A(T)$. Therefore, $\Phi(h)(M) \in \downarrow Q \cap R^A(T) \cap Y$, from which we conclude by (1) that for all $T \in Y$, $\downarrow Q \cap R^A(T) \cap Y \neq \emptyset$ for all $Q \in R^A(T)$, which means that $Y \subseteq G_{R^A}(Y)$ and so, (ts1) holds. In a similar way it can be proved that property (ts2) holds. Finally, we conclude that Y is a tense, semimodal and closed subset of $X(A)$ and so $\vartheta = \Theta_{ST}(Y)$. □

The tense and closed θ-subsets of the tense LM_θ-space associated with a tense LM_θ-algebra enable us to characterize the tense θLM_θ-congruences on these algebras as Theorem 5 shows.

Theorem 5 *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra, and let $\left(X(A), \{f_i^A\}_{i\in I}, R^A\right)$ be the tense LM_θ-space associated with A. Then, the lattice $\mathcal{C}_{\theta T}(X(A))$ of closed and tense θ-subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{t\theta LM_\theta}(A)$ of tense θLM_θ-congruences on A, and the anti-isomorphism is the function $\Theta_{\theta T}$ defined by the same prescription as in (7).*

Proof It is a consequence of Theorems 2 and 4 and the fact that $Con_{t\theta LM_\theta}(A) \subseteq Con_{\theta LM_\theta}(A)$, $Con_{t\theta LM_\theta}(A) \subseteq Con_{tLM_\theta}(A)$, $\mathcal{C}_{\theta T}(X(A)) \subseteq \mathcal{C}_\theta(X(A))$ and $\mathcal{C}_{\theta T}(X(A)) \subseteq \mathcal{C}_{ST}(X(A))$. □

5 Other characterization of tense θLM_θ-congruences

In this section we will obtain another characterization of tense θLM_θ-congruences on a tense LM_θ-algebra. First, we will determine the filters such that the lattice of congruence associated with each of them is a tense LM_θ-congruence. For this, we will remember the notion of tense filter of a tense LM_θ-algebra and the notion of θ-filter of an LM_θ-algebra.

Definition 14 Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra. A filter S of A is a tense filter iff (tf) $d(a) \in S$ for all $a \in S$ or equivalently $d^n(a) \in S$ for all $a \in S$ and $n \in \omega$.

Definition 15 Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be an LM_θ-algebra. A filter S of A is a θ-filter iff

(sf) $\varphi_i(a) \in S$ for all $a \in S$ and $i \in I$, or equivalently $\varphi_0(a) \in S$ for all $a \in S$.

Lemma 21 *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra. If S is a filter of A, then the following conditions are equivalent:*

(i) *S is a tense filter of A,*
(ii) *$G(a) \in S$ and $H(a) \in S$ for all $a \in S$.*

Proof (i) $\Rightarrow$ (ii): Let S be a tense filter of A and $a \in S$. Since $d(a) \in S$, $d(a) \leq G(a)$ and $d(a) \leq H(a)$, we infer that $G(a) \in S$ and $H(a) \in S$.
(ii) $\Rightarrow$ (i): Let $a \in S$. Then, from the hypothesis (ii) and the fact that S is a filter we obtain that $d(a) \in S$, and therefore, S is a tense filter of A. □

Lemma 22 *Let $(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H)$ be a tense LM_θ-algebra. If S is a θ-filter of A, then the following conditions are equivalent:*

(i) *S is a tense filter of A,*
(ii) *$d^n(\varphi_i(a)) \in S$ for all $a \in S$, $n \in \omega$ and $i \in I$.*

Proof (i) $\Rightarrow$ (ii): Let $a \in S$, $n \in \omega$ and $i \in I$. Since S is a θ-filter of A, we have that $\varphi_i(a) \in S$. From this last assertion and the fact that S is a tense filter we conclude that $d^n(\varphi_i(a)) \in S$.
(ii) $\Rightarrow$ (i): From the hypothesis (ii) we obtain that for all $a \in S$ and $n \in \omega$, $d^n(\varphi_0(a)) \in S$. From the last assertion, properties (L7) and (d4) and the fact that S is a filter of A we infer that $d^n(a) \in S$ for all $a \in S$ and $n \in \omega$, and therefore, S is a tense filter of A. □

Definition 16 Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra. A filter S of A is a tense θ-filter iff S is a tense filter and a θ-filter of A.

We will denote by $\mathcal{F}_{T\theta}(A)$ the lattice of all tense θ-filters of a tense LM_θ-algebra (A, G, H).

Remark 3 Let us recall that under the Priestley duality, the lattice of all filters of a bounded distributive lattice is anti-isomorphic to the lattice of all increasing closed subsets of the dual space. Under that anti-isomorphism, any filter S of a bounded distributive lattice A corresponds to the increasing closed set

$$Y_S = \{T \in X(A) : T \subseteq S\} = \bigcap_{a\in S} \sigma_A(a) \tag{31}$$

and $\Theta_C(Y_S) = \Theta(S)$, where $\Theta_C(Y_S)$ is defined as in (7) and $\Theta(S)$ is the lattice congruence associated with S (i.e. $\Theta(S) = \{(a, b) \in A \times A : a \wedge s = b \wedge s$ for some $s \in S\}$).

Conversely any increasing closed subset Y of $X(A)$ corresponds to the filter

$$S_Y = \{a \in A : Y \subseteq \sigma_A(a)\} = \bigcap_{T\in Y} T, \tag{32}$$

and $\Theta(S_Y) = \Theta_C(Y)$, where $\Theta_C(Y)$ is defined as in (7), and $\Theta(S_Y)$ is the lattice congruence associated with S_Y.

Lemma 23 *Let $(X, \{f_i\}_{i\in I}, R)$ be a tense LM_θ-space. Then for every subset Y of X, the following conditions are equivalent:*

(i) *Y is increasing and semimodal,*
(ii) *Y is modal.*

Proof (i) $\Rightarrow$ (ii): Since Y is a semimodal subset of X, then (1) $Y \subseteq f_i^{-1}(Y)$ for all $i \in I$. On the other hand, let (2) $x \in f_0^{-1}(Y)$. Taking into account (2), property (lP9) and the fact that Y is increasing, we infer that $x \in Y$, and so by (1) we obtain that $Y = f_0^{-1}(Y)$. From this last assertion and property (lP5), we infer that $Y = f_i^{-1}(Y)$ for all $i \in I$, which means that Y is modal.
(i) $\Rightarrow$ (ii): It follows from Definition 4 and Proposition 13. □

Theorem 6 *Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra. If S is a filter of A, then the following conditions are equivalent:*

(i) *$\Theta(S) \in Con_{tLM_\theta}(A)$,*
(ii) *$S \in \mathcal{F}_{T_\theta}(A)$.*

Proof (i) $\Rightarrow$ (ii): Let S be a filter of A such that $\Theta(S) \in Con_{tLM_\theta}(A)$. Then, from Theorem 4 and Remark 3 it follows that $\Theta(S) = \Theta_{ST}(Y_S)$, where $\Theta(S)$ is the lattice congruence associated with S and (1) $Y_S = \{x \in X(A) : S \subseteq x\} = \bigcap_{a\in S} \sigma_A(a)$ is a tense, semimodal and closed subset of the tense LM_θ-space $X(A)$ associated with A. Since, Y_S is semimodal and also by Priestley duality, Y_S is increasing, we have by Lemma 23 that Y_S is modal. Besides, σ_A is an LM_θ-isomorphism, then taking into account the prescription (1) we obtain that $Y_S = f_i^{A^{-1}}(Y_S) = f_i^{A^{-1}}\left(\bigcap_{a\in S} \sigma_A(a)\right) = \bigcap_{a\in S} f_i^{A^{-1}}\left(\sigma_A(a)\right) = \bigcap_{a\in S} \sigma_A(\varphi_i(a))$, for any $i \in I$. From the last assertion, and taking into account that Y_S is a tense subset of $X(A)$, Corollaries 11 and 12 and Lemma 16, we infer that $Y_S = d_{X(A)}\left(\bigcap_{a\in S} \sigma_A(\varphi_i(a))\right) \subseteq \bigcap_{a\in S} d_{X(A)}(\sigma_A(\varphi_i(a))) = \bigcap_{a\in S} \sigma_A(d(\varphi_i(a))) \subseteq \bigcap_{a\in S} \sigma_A(\varphi_i(a)) = Y_S$, for any $i \in I$. Hence, (2) $Y_S = \bigcap_{a\in S} \sigma_A(d(\varphi_i(a)))$ for any $i \in I$, from which we get that $d(\varphi_i(a)) \in S$ for any $a \in S$ and $i \in I$. Indeed, assume that $a \in S$, then by the assertion (1), $a \in x$ for all $x \in Y_S$, from which it follows by the assertion (2) that $x \in \bigcap_{a\in S} \sigma_A(d(\varphi_i(a))$ for any $i \in I$, and thus $d(\varphi_i(a)) \in x$ for all $x \in Y_S$ and $i \in I$. Therefore, $d(\varphi_i(a)) \in \bigcap_{x\in Y_S} x$ for any $i \in I$, and taking into account that by Remark 3, $S = \bigcap_{x\in Y_S} x$, we deduce that $d(\varphi_i(a)) \in S$ for any $i \in I$, from which we conclude by Lemma 21 that $S \in \mathcal{F}_{T_\theta}(A)$.

(ii) $\Rightarrow$ (i): From Priestley duality and (31), we have that $\bigcap_{a\in S} \sigma_A(a) = Y_S = \{x \in X(A) : S \subseteq x\}$ is an increasing and closed subset of $X(A)$ and $\Theta(S) = \Theta_C(Y_S)$. By Theorem 4, it remains to show that Y_S is a semimodal and tense subset of $X(A)$. From the hypothesis (ii), we infer that for all $a \in S$, $i \in I$ and $x \in Y_S$, $d(\varphi_i(a)) \in x$, and consequently from Corollary 13, we obtain that $\varphi_i(d(a)) \in x$ for all $i \in I$ and all $x \in Y_S$, from which it follows that (1) $Y_S \subseteq \bigcap_{a\in S} \sigma_A(\varphi_i(d(a)))$ for all $i \in I$. And so, by property (d2) in Corollary 11, $Y_S \subseteq \bigcap_{a\in S} \sigma_A(\varphi_i(a))$ for all $i \in I$. From this assertion we have that $Y_S \subseteq \bigcap_{a\in S} \sigma_A(\varphi_1(a)) \subseteq \bigcap_{a\in S} \sigma_A(a) = Y_s$. Since σ_A is an LM_n-isomorphism, then by the prescription (1) we get that (2) $Y_s = \bigcap_{a\in S} \sigma_A(\varphi_1(a)) = \bigcap_{a\in S} f_1^{A^{-1}}(\sigma_A(a)) = f_1^{A^{-1}}\left(\bigcap_{a\in S} \sigma_A(a)\right) = f_1^{A^{-1}}(Y_S)$. Therefore, from the last statement and property (lP5) we conclude that $Y_S = f_i^A(Y_S)$ for all $i \in I$ and so, Y_S is modal. In addition, from (1), (2) and Corollary 11 we infer that $Y_S \subseteq \bigcap_{a\in S} \sigma_A(d(\varphi_1(a)) \subseteq \bigcap_{a\in S} \sigma_A(\varphi_1(a)) = Y_s$ and hence, $Y_S = \bigcap_{a\in S} \sigma_A(d(\varphi_1(a))$. Then, taking into account that $\bigcap_{a\in S} d_{X(A)}(\sigma_A(\varphi_1(a))) = d_{X(A)}\left(\bigcap_{a\in S} \sigma_A(\varphi_1(a))\right)$ and Corollary 12, we obtain that $Y_S = d_{X(A)}(Y_S)$, and thus, from Lemma 16 and the fact that Y_S is modal, we infer that Y_S is a tense subset of $X(A)$. Finally, since Y_S is a tense, modal and closed subset of $X(A)$ and $\Theta_{MT}(Y_S) = \Theta_C(Y_S) = \Theta(S)$, we conclude, from Theorem 4, that $\Theta(S) \in Con_{tLM_\theta}(A)$. ◻

The tense, modal and closed subsets of the tense LM_θ-space associated with a tense LM_θ-algebra play a key role in the characterization of the tense LM_θ-congruences associated with tense θ-filers of this algebras, as we will prove next.

Theorem 7 *Let (A, G, H) be a tense LM_θ-algebra, and $\left(X(A), \{f_i^A\}_{i\in I}, R^A\right)$ be the tense LM_θ-space associated with A. Then, the lattice $\mathcal{C}_{MT}(X(A))$ of tense, modal and closed subsets of $X(A)$ is anti-isomorphic to the lattice $Con_{tLM_{\theta\mathcal{F}_{T\theta}(A)}}(A)$ of tense LM_θ-congruences on A associated with some tense θ-filter of A, and the anti-isomorphism is the function Θ_{MT} defined by the same prescription as in (7).*

Proof Let $Y \in \mathcal{C}_{MT}(X(A))$. Thus, from Lemma 23, $Y \in \mathcal{C}_{ST}(X(A))$ and Y is increasing. Then, from Theorem 4 we obtain that $\Theta_{ST}(Y) \in Con_{tLM_\theta}(A)$, and from Remark 3, we get that $\Theta_{ST}(Y) = \Theta(S_Y)$, where S_Y is a filter of A defined as in (31). From this last assertion and Theorem 6, we infer that $S_Y \in \mathcal{F}_{T\theta}(A)$, from which we conclude that $\Theta_{ST} |_{\mathcal{C}_{MT}(X(A))} (Y) \in Con_{tLM_{\theta\mathcal{F}_{T\theta}(A)}}(A)$. Conversely, let $\vartheta \in Con_{tLM_{\theta\mathcal{F}_{T\theta}(A)}}(A)$. Then there is $S \in \mathcal{F}_{T\theta}(A)$ such that $\vartheta = \Theta(S)$. Since $Con_{tLM_{\theta\mathcal{F}_{T\theta}(A)}}(A) \subseteq Con_{tLM_\theta}(A)$, then from Theorem 4 and Remark 3, we infer that that $\Theta(S) = \Theta_{ST}(Y_S)$, where Y_S is defined as in (32). The fact that Y_S is increasing and Lemma 23 allow us to assert that $Y_S \in \mathcal{C}_{MT}(X(A))$ and $\vartheta = \Theta_{ST} |_{\mathcal{C}_{MT}(X(A))} (Y)$. Therefore, the restriction $\Theta_{ST} |_{\mathcal{C}_{MT}(X(A))}$ is a function from $\mathcal{C}_{MT}(X(A))$ onto $Con_{tLM_{\theta\mathcal{F}_{T\theta}(A)}}(A)$. If $\Theta_{MT} = \Theta_{ST} |_{\mathcal{C}_{MT}(X(A))}$, then from the last statement and Theorem 4, we conclude the proof. ◻

Our next objective is to prove that the θ-filters of a tense LM_θ-algebra allow us to characterize the tense θLM_θ-congruences on this algebra. For this purpose, we consider the following notion:

Definition 17 Let $\left(A, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, G, H\right)$ be a tense LM_θ-algebra and $\vartheta \in Con_{tLM_\theta}(A)$, the tense θLM_θ-congruence on A generated by ϑ is the smallest tense θLM_θ-congruence on A, in the sense of the inclusion relation, containing ϑ and it will be denoted by $\widehat{\vartheta}_\theta$.

In order to achieve the characterization that we have proposed we will take into account Theorems 5 and 7. First, we will obtain the greatest tense and closed θ-subset contained in an arbitrary tense, modal and closed subset of a tense LM_θ-space.

In Figallo et al. (2010), the following characterizations of closed θ-subsets of an LM_θ-space were obtained.

Proposition 18 (Figallo et al. 2010, Proposition 2.1.5) *Let $\left(X, \{f_i\}_{i\in I}\right)$ be an LM_θ-space and let Y be a subset of X. Then the following conditions are equivalent:*

(i) *Y is a closed θ-subset,*
(ii) *there is a subset Z of X such that $Y = \overline{\bigcup_{i\in I} f_i(Z)}$,*
(iii) *$Y = \overline{\bigcup_{i\in I} f_i(Y)}$.*

Proposition 19 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space and let Y be a tense, modal and closed subset of X. Then $\overline{\bigcup_{i\in I} f_i(Y)}$ is a tense and closed θ-subset.*

Proof By condition (ii) in Proposition 18 we have that $\overline{\bigcup_{i\in I} f_i(Y)}$ is a closed θ-subset of X. Now assume that (1) $y \in \overline{\bigcup_{i\in I} f_i(Y)}$. From (1), the fact that the subset Y is tense and modal and Corollary 10, it follows that $R(y) \subseteq Y$. Since, by Proposition 13, we have that Y is a decreasing subset of X, we obtain that (2) $\downarrow R(y) \subseteq Y$. From property (lP9), we deduce that for all $z \in R(y)$, $f_0(z) \in \downarrow R(y)$. Besides, from (2) we get that $f_0(z) \in Y$ and so from property (lP5) we obtain that $f_0(z) \in f_0(Y)$. Therefore,

$f_0(z) \in \overline{\bigcup_{i\in I} f_i(Y)} \cap \downarrow R(y)$, from which we conclude by (1) that $\downarrow R(y) \cap \overline{\bigcup_{i\in I} f_i(Y)} \neq \emptyset$ for all $y \in \overline{\bigcup_{i\in I} f_i(Y)}$, which allows us to assert that $\overline{\bigcup_{i\in I} f_i(Y)}$ is tense. □

Proposition 20 *Let $\left(X, \{f_i\}_{i\in I}, R\right)$ be a tense LM_θ-space associated and let Y be a tense, modal and closed subset of X. Then $\overline{\bigcup_{i\in I} f_i(Y)}$ is the largest tense and closed θ-subset, in the sense of the inclusion relation, contained in Y.*

Proof (i) From Proposition 19, we have that $\overline{\bigcup_{i\in I} f_i(Y)}$ is a tense and closed θ-subset.
(ii) Since Y is closed and modal, then $\overline{\bigcup_{i\in I} f_i(Y)} \subseteq Y$.
(iii) Let Z be a tense and closed θ-subset of X such that (1) $Z \subseteq Y$. Then, from Proposition 18 we have that (2) $Z = \overline{\bigcup_{i\in I} f_i(Z)}$. Besides, from (1) we obtain that $f_i(Z) \subseteq f_i(Y)$ for all $i \in I$, and consequently, $\overline{\bigcup_{i\in I} f_i(Z)} \subseteq \overline{\bigcup_{i\in I} f_i(Y)}$, from which we conclude by (2) that $Z \subseteq \overline{\bigcup_{i\in I} f_i(Y)}$.

Finally, from (i), (ii) and (iii) the proof is complete. □

Corollary 15 *Let (A, G, H) be a tense LM_θ-algebra and $\left(X(A), \{f_i^A\}_{i\in I}, R^A\right)$ be the tense LM_θ-space associated with A. Then, for every tense, modal and closed subset Y of $X(A)$, $\widehat{\Theta_{ST}(Y)}_\theta = \Theta_{\theta T}\left(\overline{\bigcup_{i\in Y} f_i^A(Y)}\right)$, where Θ_{ST} and $\Theta_{\theta T}$ are the anti-isomorphisms defined in Theorems 4 and 5, respectively.*

Proof It is a direct consequence of Theorems 4 and 5 and Proposition 20. □

Corollary 16 *Let (A, G, H) be a tense LM_θ-algebra and $\left(X(A), \{f_i^A\}_{i\in I}, R^A\right)$ be the tense LM_θ-space associated with A. Then for every $S \in \mathcal{F}_{T\theta}(A)$, $\widehat{\Theta(S)}_\theta = \widehat{\Theta_{MT}(Y_S)}_\theta = \Theta_{\theta T}\left(\overline{\bigcup_{i\in I} f_i^A(Y_S)}\right)$, where Y_S is defined as in (32), Θ_{MT} and $\Theta_{\theta T}$ are the anti-isomorphisms defined in Theorems 7 and 5, respectively.*

Proof It is a direct consequence of Remark 3, Theorems 6 and 7, Lemma 23 and Corollary 15. □

Theorem 8 *Let (A, G, H) be a tense LM_θ-algebra. Then, the lattice $\mathcal{F}_{T\theta}(A)$ of all tense θ-filters of A is isomorphic to the lattice $Con_{t\theta LM_\theta}(A)$ of tense θLM_θ-congruences of A, and the isomorphism is the function Ψ from $\mathcal{F}_{T\theta}(A)$ onto $Con_{t\theta LM_\theta}(A)$, defined for all $S \in \mathcal{F}_{T\theta}(A)$ by $\Psi(S) = \widehat{\Theta(S)}_\theta$, where $\Theta(S)$ is the LM_θ-congruence associated with S.*

Proof It is immediate that $\Psi(S) = \widehat{\Theta(S)}_\theta \in Con_{t\theta LM_\theta}(A)$ for all $S \in \mathcal{F}_{T\theta}(A)$. On the other hand, let $\vartheta \in Con_{t\theta LM_\theta}(A)$ and let $\left(X(A), \{f_i^A\}_{i\in I}, R^A\right)$ be the tense LM_θ-space associated with A, then from Theorem 5 it follows that there is $Y \in \mathcal{C}_{\theta T}(X(A))$ such that (1) $\vartheta = \Theta_{\theta T}(Y)$. Since, $Y = \overline{\bigcup_{i\in I} f_i^A(Y)}$, then $Z = f_1^{A^{-1}}(Y)$ is a modal and closed subset of $X(A)$ and (2) $Y = \overline{\bigcup_{i\in I} f_i^A(Z)}$. Also, taking into account that Y is tense and semimodal and Corollary 9, we infer that Z is tense. Therefore, from Theorem 7 it follows that (3) $\Theta_{ST}(Z) = \Theta(S_Z)$ and $S_Z \in \mathcal{F}_{T\theta}(A)$. Besides, from (1), (2) and Corollary 15, we obtain that $\widehat{\Theta_{ST}(Z)}_\theta = \Theta_{\theta T}\left(\overline{\bigcup_{i\in Y} f_i^A(Z)}\right) = \Theta_{\theta T}\left(\overline{\bigcup_{i\in Y} f_i^A(Y)}\right) = \vartheta$, from which we get by (3) that $\Psi(S) = \widehat{\Theta(S_Z)} = \vartheta$. In addition, let $S_1, S_2 \in \mathcal{F}_{T\theta}(A)$ such that $S_1 \subseteq S_2$, then $Y_{S_2} \subseteq Y_{S_1}$ and thus, $\overline{\bigcup_{i\in I} f_i^A(Y_{S_2})}, \overline{\bigcup_{i\in I} f_i^A(Y_{S_1})} \in \mathcal{C}_{\theta T}(X(A))$ and $\overline{\bigcup_{i\in I} f_i^A(Y_{S_2})} \subseteq \overline{\bigcup_{i\in I} f_i^A(Y_{S_1})}$. Consequently from Theorem 5 we have that $\Theta_{\theta T}\left(\overline{\bigcup_{i\in I} f_i^A(Y_{S_1})}\right) \subseteq \Theta_{\theta T}\left(\overline{\bigcup_{i\in I} f_i^A(Y_{S_2})}\right)$. Hence, from the last statement and Corollary 16, we deduce that $\widehat{\Theta(S_1)}_\theta \subseteq \widehat{\Theta(S_2)}_\theta$, and so we can assert that $\Psi(S_1) \subseteq \Psi(S_2)$. Conversely, let us assume that $S_1, S_2 \in \mathcal{F}_{T\theta}(A)$ and $\Psi(S_1) \subseteq \Psi(S_2)$, which means that $\widehat{\Theta(S_1)}_\theta \subseteq \widehat{\Theta(S_2)}_\theta$. Then, from Corollary 16 we infer that $\Theta_{\theta T}\left(\overline{\bigcup_{i\in Y} f_i^A(Y_{S_1})}\right) \subseteq \Theta_{\theta T}\left(\overline{\bigcup_{i\in Y} f_i^A(Y_{S_2})}\right)$, and so from Theorem 5, we obtain that $\overline{\bigcup_{i\in Y} f_i^A(Y_{S_2})} \subseteq \overline{\bigcup_{i\in Y} f_i^A(Y_{S_1})}$. Thus, $Y_{S_2} \subseteq Y_{S_1}$, which allows us to conclude that $S_1 \subseteq S_2$. Therefore, $\Psi : \mathcal{F}_{T\theta}(A) \longrightarrow Con_{t\theta LM_\theta}(A)$ is a lattice isomorphism. □

6 Conclusion and future works

Priestley spaces arise more naturally in relation with logics, as Priestley spaces incorporate the now widely used Kripke semantics in them. As a result, Priestley's duality became rather popular among logicians, and most dualities for distributive lattices with operators have been performed in terms of Priestley spaces. In particular, in this paper we have determined a topological duality for tense θ-valued Łukasiewicz–Moisil algebras, extending the one obtained for θ-valued Łukasiewicz–Moisil algebras in Figallo et al. (2010). By means of the above duality we have obtained properties of these algebras and also we have characterized the congruences and the θ-congruences on these algebras. In a future work we will use the previous characterizations to describe the simple and the subdirectly irreducible tense θ-valued Łukasiewicz–Moisil algebras and the simple and the subdirectly irreducible tense θ-valued Łukasiewicz–Moisil algebras by θ-congruences. Furthermore, by means of the aforementioned duality, we will prove a representation theorem for tense LM_θ-algebras, which was formulated and proved by a different method by Chiriţă (2011). The proof of this theorem could be of interest for people working in duality theory. We expect that our method can be easily applied to weak-tense operators or tense operators on θ-valued Łuaksiewicz–Moisil algebras with negation (see, Chiriţă (2012b), Chapter 6).

Acknowledgements The support of CONICET is gratefully acknowledged by Gustavo Pelaitay.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

Boicescu V, Filipoiu A, Georgescu G, Rudeanu S (1991) Łukasiewicz–Moisil Algebras, annals of discrete mathematics, vol 49. North-Holland, Amsterdam

Botur M, Chajda I, Halaš R, Kolařík M (2011) Tense operators on basic algebras. Int J Theor Phys 50(12):3737–3749

Botur M, Paseka J (2015) On tense MV-algebras. Fuzzy Sets Syst 259:111–125

Burges J (1984) Basic tense logic. In: Gabbay DM, Günter F (eds) Handbook of philosophical logic, vol II. Reidel, Dordrecht, pp 89–139

Chajda I (2011) Algebraic axiomatization of tense intuitionistic logic. Cent Eur J Math 9(5):1185–1191

Chajda I, Paseka J (2015) Algebraic approach to tense operators, research exposition 22 in mathematics, vol 35. Heldermann Verlag, Germany, p 204

Chiriţă C (2010) Tense θ-valued Moisil propositional logic. Int J Comput Commun Control 5:642–653

Chiriţă C (2011) Tense θ-valued Łukasiewicz–Moisil algebras. J Mult Valued Logic Soft Comput 17(1):1–24

Chiriţă C (2012a) Polyadic tense θ-valued Łukasiewicz–Moisil algebras. Soft Comput 16(6):979–987

Chiriţă C (2012b) Tense multiple-valued logical systems. Ph.D. thesis, University of Bucharest, Bucharest

Cignoli R (1970) Moisil algebras. Notas de Lógica Matemática 27. Inst. Mat. Univ. Nacional del Sur, Bahía Blanca

Diaconescu D, Georgescu G (2007) Tense operators on MV-algebras and Łukasiewicz–Moisil algebras. Fund Inf 81(4):379–408

Figallo AV, Pascual I, Ziliani A (2010) A duality for θ-valued Łukasiewicz–Moisil algebras and applications. J Mult Valued Logic Soft Comput 16(3–5):303–322

Figallo AV, Pascual I, Ziliani, A (2012) Principal and Boolean congruences on θ-valued Łukasiewicz–Moisil algebra. Logic without frontiers. Festschrift for Walter Alexandre Carnielli on the occasion of his 60th Birthday, 17, pp 215–237

Figallo AV, Pelaitay G (2011) Note on tense SHn-algebras. Ann Univ Craiova Ser Mat Inf 38(4):24–32

Figallo AV, Pelaitay G (2014) An algebraic axiomatization of the Ewald's intuitionistic tense logic. Soft Comput 18(10):1873–1883

Figallo AV, Pelaitay G (2015a) A representation theorem for tense $n \times m$-valued Łukasiewicz–Moisil algebras. Math Bohem 140(3):345–360

Figallo AV, Pelaitay G (2015b) Discrete duality for tense Łukasiewicz–Moisil algebras. Fund Inf 136(4):317–329

Figallo AV, Pascual I, Pelaitay G (2017) Subdirectly irreducible IKt-algebras. Stud Logica 105(4):673–701

Figallo AV, Pascual I, Gustavo P (2018) A topological duality for tense LM_n-algebras and applications. Logic J IGPL. https://doi.org/10.1093/jigpal/jzx056

Kowalski T (1998) Varieties of tense algebras. Rep Math Logic 32:53–95

Paseka J (2013) Operators on MV-algebras and their representations. Fuzzy Sets Syst 232:62–73

Priestley HA (1970) Representation of distributive lattices by means of ordered Stone spaces. Bull Lond Math Soc 2:186–190

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.