Asking infinite voters 'Who is a J?': Group Identification Problems in $\mathbb{N}$

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Abstract

We analyze the problem of classifing individuals in a group N taking into account their opinions about which of them should belong to a specific subgroup $N' \subseteq N$, in the case that $|N| > \infty$. We show that this problem is relevant in cases in which the group changes in time and/or is subject to uncertainty. The approach followed here to find the ensuing classification is by means of a *Collective Identity Function* (CIF) that maps the set of opinions into a subset of N. Kasher and Rubinstein (1997) characterized different CIFs axiomatically when $|N| < \infty$, in particular the Liberal and Oligarchic aggregators. We show that in the infinite setting the liberal result is still valid but the result no longer holds for the oligarchic case and give a characterization of all the aggregators satisfying the same axioms as the Oligarchic CIF. In our motivating examples, the solution obtained according to the alternative CIF is most cogent.
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1 Introduction

Many important problems in the real world involve classifying objects, properties and people in groups. Such classification can be simple and obvious, as for instance organizing countries by the continent to which they belong.

But if we want to classify the members of a class of individuals according to some subjective criterion, for example to determine which people in a rugby club are real fans of the team, the assessment of the individuals themselves matters for the final result. Many relevant social and economic problems can be stated in the light of this problem of finding a *consensus* on the classification of the members of a group.[1]

Unlike the usual situations in which objects have to be classified according to external criteria or their "objective" properties, such problems involve a certain circularity. That is, the final classfication of individuals is the result of aggregating the classifications conceived by the same individuals. Kasher (1993) and Kasher and Rubinstein (1997) analyzed this question, motivated by the problem of determining who is a Jew and is thus able to request the Israeli nationality under the *Law of Return*. After their contribution, the general problem is known as the question of "Who is a J?" and is seen as the search of appropriate opinion aggregation functions. Each individual in a society is assumed to have an opinion about which individuals, including himself, belong or not to a group and the identities of the individuals that belong to the group is to be determined by a function that takes their opinions as input. An entire branch of Social Choice theory grew out of the treatment of this question.[2]

We are interested in extensions of this problem. For instance consider the problem of classifying political leanings, assessed by the citizens themselves, in countries with boundaries subject to changes. If, say, the question were "what is the political inclination of the majority of citizens in Germany, left or right?", it matters whether the question was formulated previously to the reunification of 1990 or after an eventual merging in a European federation. While the categories of the answers are fixed (left, right) if the question is not framed in time, we face an *unbounded* number of individuals. For each single individual we have to create different avatars depending on the definition of the boundaries of the country. That is, given a citizen named Angela, we

[1]The problem of finding a consensus among classifications has been analyzed for many different structures. The second issue of volume 3 of the *Journal of Classification* was devoted to different aspects of this problem. The most relevant article, for our purposes, in that issue is Barthélemy et al. (1986) while other important contributions, in the case that the classifications constitute equivalence relations, are Mirkin (1975) and Fishburn and Rubinstein (1986). The specific case in which classifications have a hierarchical structure has been analyzed in McMorris and Powers (2008).

[2]See, for instance, Sung and Dimitrov (2005), Saporiti (2012), Cho and Saporiti (2015) or Cho and Ju (2017).

have to differentiate Angela in 1985 from Angela in 2018, since the former could have been a citizen of the Democratic Republic (East Germany) and the latter of the Federal Republic.[3]

The previous example can be, of course, handled as a case of continuous polling of opinions, taking averages or medians. But, as it is clear that opinions at time t matter for decisions that influence the opinions at any future $t' > t$, there are certain properties we seek in the aggregation related to the representativeness and fairness of outcome. More qualitative aggregation rules may be more appropriate to satisfy those requirements.

But beyond the social and economic motivations for this problem, technological advances pose alternative settings in which this problem may arise. For instance, in the *blockchain* technology, the basis for cryptocurrencies and smart contracts, the addition of new blocks to the collective ledger requires the validation by the nodes in a peer2peer network (Biais et al. 2018). This application shows that the original *Who is a J* problem can be extended to consider many situations in which the classification is obtained along time with an infinite horizon. Even if we assume that at each period the number of candidate blocks is finite, if time is unbounded and assignments are dated, it is equivalent from a mathematical viewpoint to consider countably infinite blocks to be added to the ledger. Alternatively, we can consider cases in which the classification of blocks is subject to uncertainty, with a countably infinite set of possible states of the network. Here, the addition of a block to the ledger depends on the state of the network. This again is equivalent to having a countably infinite number of blocks.

The treatment of the static and deterministic version of the problem involves (as shown by Kasher and Rubinstein) the axiomatization of aggregator functions able to yield the classification, called Collective Identity Functions (CIF). The main property that this functions should satisfy is "fairness". Three different candidate functions achieve this. One is the "Liberal" CIF, which assigns individuals to the categories to which they think they belong. On the other hand, the "Dictatorial" function is such that a single individual decides to which categories do all the individuals belong. Finally an "Oligarchic" function is such that this decision is made by a given group. Various results on the existence (or not) and uniqueness of CIFs can be found in the

[3]Similar problems arise even if the boundaries of the country do not change. Think about the problem of determining if Americans are pro or against immigration. Events like 9/11 and the election of Donald Trump as president of the USA indicate that the opinions of the citizens must be dated.

literature. Some of them have been obtained either by modifying Kasher and Rubinstein's axioms (Saporiti 2012), correcting previous results (Sung and Dimitrov 2003), working with the identification of more than two groups (Cho and Ju 2017) or even manipulating the incentives of voters (Cho and Saporiti 2015). But none of these contributions deals with the case of an infinite number of voters.

Our goal in this paper is to see if the CIFs presented in the literature do yield the appropriate classifications in (countable) infinite settings. We check whether an approach similar to Fishburn's (1970) is valid in this context.[4] While our findings are not so striking as those, they point towards a similarity between the finite and the infinite setting in the case of Strong Liberal CIFs, while a new class of CIF, which we call *anti-oligarchic* characterize axioms that, in the finite setting, are only characterized by oligarchic CIFs.

2 Liberalism

We consider a set $N = \mathbb{N}$ of individuals, i.e., each individual is identified with a natural number. Each individual i has an "opinion", namely a set $J_i \subseteq \mathbb{N}$ of individuals that i thinks belong to class J. By a slight abuse of language we denote by J a Collective Identity Function taking as arguments the profiles of opinions $(J_1, \ldots, J_n, \ldots)$ and yielding a set $J(J_1, \ldots, J_n, \ldots) \subseteq \mathbb{N}$, where $J(J_1, \ldots, J_n, \ldots)$ are the individuals deemed to belong to class J. For simplicity, we identify J with $J(J_1, \ldots, J_n, \ldots)$ when there is no chance of confusion. In this section we focus on the **Strong Liberal** CIF, introduced by K-R, defined as:

$$J = \{i \mid i \in J_i\}$$

i.e., J is the class of all the individuals that consider themselves to belong to J.

Let us now present axioms (originally presented in Kasher and Rubinstein, 1997) capturing properties that a fair CIF may satisfy:

- **Monotonicity** (MON): consider an individual $i \in J(J_1, \ldots, J_n, \ldots)$. Let $(J'_1, \ldots, J'_n \ldots)$ be a profile identical to $(J_1, \ldots, J_n, \ldots)$ except that there exists an individual k such that $i \notin J_k$ while $i \in J'_k$. Then $i \in J(J'_1, \ldots, J'_n, \ldots)$. Analogously, if $i \notin J(J_1, \ldots, J_n, \ldots)$ and $(J'_1, \ldots, J'_n, \ldots)$

[4] Fishburn shows that Arrow's Impossibility Theorem no longer holds when the number of agents is infinite.

is identical to $(J_1, \ldots, J_n, \ldots)$, except for the presence of a k such that if $i \in J_k$ and $i \notin J'_k$, then $i \notin J(J'_1, \ldots, J'_n, \ldots)$.

- **Independence** (I): consider an individual i and two profiles $(J_1, \ldots, J_n, \ldots)$ and $(J'_1, \ldots, J'_n, \ldots)$. If for every $k \neq i$, $k \in J$ if and only if $k \in J'$, and for all k (including i), $i \in J_k$ if and only if $i \in J'_k$, then $i \in J$ if and only if $i \in J'$.

- **Consensus** (C): if $j \in J_i$ for all i, then $j \in J$. Conversely, if $j \notin J_i$ for all i, then $j \notin J$.

- **Symmetry** (SYM): two individuals, j and k are symmetric in a profile $(J_1, \ldots, J_n, \ldots)$ if

  (i) $J_j - \{j, k\} = J_k - \{j, k\}$

  (ii) for all $i \in \mathbb{N} - \{j, k\}, j \in J_i$ iff $k \in J_i$

  (iii) $j \in J_j$ iff $k \in J_k$

  (iv) $j \in J_k$ iff $k \in J_j$

  Then, $j \in J$ if and only if $k \in J$.

- **Liberal Principle** (L): if there exists $i \in \mathbb{N}$ such that $i \in J_i$, then $J \neq \emptyset$, and if there exists i such that $i \notin J_i$, then $J \neq \mathbb{N}$.

According to Sung and Dimitrov (2003), these properties are not independent, in particular, (C) and (MON) can be derived from (SYM), (I) and (L). Furthermore, they prove the finite variant of the following claim:

Lemma 1 *If a CIF J satisfies (SYM), (I) and (L), then $J(S^{\mathbb{N}}) = S$ for each $S \subseteq \mathbb{N}$ where $S^{\mathbb{N}}$ is the profile where $J_i = S$ for all $i \in \mathbb{N}$.*

We will prove this result by a straightforward application of the following version of *transfinite induction* (Section 7.1 in Suppes, 1972):

Given a propositional function P defined over the class of subsets of $\mathbb{N}$, $2^{\mathbb{N}}$, such that

(i) $P(\emptyset)$ is true.

(ii) $P(S)$ implies $P(S^+)$, where $|S^+| = |S| + 1$.

(iii) If $P(S)$ is true for every finite $S \subseteq \mathbb{N}$, then $P(\bar{S})$ is true for every infinite $\bar{S}$ such that $S \subseteq \bar{S}$.

then $P(S)$ is true for every $S \in 2^{\mathbb{N}}$.

Then:
Proof of Lemma 1: *Sung and Dimitrov (2003) prove by finite induction on the cardinality $\mid S \mid$ of S that every CIF satisfying (SYM), (I) and (L) is such that*

$$J(S^N) = S \text{ and } J((N-S)^N) = N - S \text{ for every } S \subseteq N, \text{ where } |N| < \infty.$$

To show that this is valid in our setting just consider that this induction argument shares Steps (i) and (ii) with transfinite induction. On the other hand, step (iii) is satisfied by taking $\bar{S} = \mathbb{N}$. $\square$

A straightforward extension of Lemma 1 can be obtained considering any 4-partition of $\mathbb{N}$, (A_0, A_1, B_0, B_1):[5]

Lemma 2 *Given two profiles $(J_1^a, \ldots, J_n^a, \ldots)$ and $(J_1^b, \ldots, J_n^b, \ldots)$ such that for each $k \in \mathbb{N}$:*

$$J_k^a = \begin{cases} A_0 \cup A_1, \text{ if } k \in A_0 \cup B_0, \\ A_0 \cup A_1 \cup B_0, \text{ if } k \in A_1 \cup B_1 \end{cases}$$

and

$$J_k^b = \begin{cases} A_0, \text{ if } k \in A_0 \cup B_0, \\ A_0 \cup A_1, \text{ if } k \in A_1 \cup B_1 \end{cases}$$

if a CIF J satisfies (SYM), (I) and (L) then $J(J_1^a, \ldots, J_n^a, \ldots) = J(J_1^b, \ldots, J_n^b, \ldots) = A_0 \cup A_1$.

Then, we have:

Theorem 1 *The Strong Liberal CIF is the only CIF that satisfies (SYM), (I) and (L) for every set of voters $N \subseteq \mathbb{N}$.*

Proof of Theorem 1: *It is easy to check that the Strong Liberal CIF, which we denote J^L satisfies the axioms. For the converse, suppose by contradiction that there exist also another CIF satisfying the axioms, say J'. Then, there exists a profile $(J_1, \ldots, J_n, \ldots)$ such that $J'(J_1, \ldots, J_n, \ldots) \neq J^L(J_1, \ldots, J_n, \ldots)$. Thus there exists a $\imath \in \mathbb{N}$:*

- *$i \in J'(J_1, \ldots, J_n, \ldots)$ and $i \notin J^L(J_1, \ldots, J_n, \ldots)$, or*

[5]If the partition is $(S, \emptyset, \emptyset, \mathbb{N} \setminus S)$, for a $S \subseteq \mathbb{N}$, Lemma 2 yields Lemma 1.

- *$i \notin J'(J_1, \ldots, J_n, \ldots)$ and $i \in J^L(J_1, \ldots, J_n, \ldots)$.*

Four classes can be defined:

$$A_0 = \{k \in J'(J_1, \ldots, J_n, \ldots) \setminus \{i\} : i \notin J_k\}$$

$$A_1 = \{k \in J'(J_1, \ldots, J_n, \ldots) \setminus \{i\} : i \in J_k\}$$

$$B_0 = \{k \in (\mathbb{N} - J'(J_1, \ldots, J_n, \ldots)) \setminus \{i\} : i \notin J_k\}$$

$$B_1 = \{k \in (\mathbb{N} - J'(J_1, \ldots, J_n, \ldots)) \setminus \{i\} : i \in J_k\}$$

Then, if $i \in J'(J_1, \ldots, J_n, \ldots)$ but $i \notin J^L(J_1, \ldots, J_n, \ldots)$, we can build a new profile, $(\bar{J}_1, \ldots, \bar{J}_n, \ldots)$, where

$$\bar{J}_k = \begin{cases} A_0 \cup A_1, \textit{if } k \in A_0 \\ A_0 \cup A_1 \cup B_0 \cup \{i\}, \textit{if } k \in A_1 \\ A_0 \cup A_1, \textit{if } k \in B_0 \cup \{i\} \\ A_0 \cup A_1 \cup B_0 \cup \{i\}, \textit{if } k \in B_1 \end{cases}$$

According to Lemma 2, considering the partition $(A_0, A_1, B_0 \cup \{i\}, B_1)$, $J'(\bar{J}_1, \ldots, \bar{J}_n, \ldots)$=$J^L(\bar{J}_1, \ldots, \bar{J}_n, \ldots)$=$A_0 \cup A_1$=$J'(J_1, \ldots, J_n, \ldots) \setminus \{i\}$.

By definition, for every $k \in \mathbb{N} \setminus \{i\}$, $k \in J'(J_1, \ldots, J_n, \ldots)$ if and only if $k \in J'(\bar{J}_1, \ldots, \bar{J}_n, \ldots)$ while for every $k \in \mathbb{N}$, $i \in J_k$ if and only if $i \in \bar{J}_k$. According to (I), $i \in J'(J_1, \ldots, J_n, \ldots)$ if and only if $i \in J'(\bar{J}_1, \ldots, \bar{J}_n, \ldots)$. But by construction, $i \in J'(J_1, \ldots, J_n, \ldots)$ but $i \notin J'(\bar{J}_1, \ldots, \bar{J}_n, \ldots)$.

The remaining case, namely $i \notin J'(J_1, \ldots, J_n, \ldots)$ and $i \in J^L(J_1, \ldots, J_n, \ldots)$, allows a similar construction of an alternative profile $(\hat{J}_1, \ldots, \hat{J}_n, \ldots)$, where

$$\hat{J}_k = \begin{cases} A_0, \textit{if } k \in A_0 \\ A_0 \cup A_1 \cup \{i\}, \textit{if } k \in A_1 \cup \{i\} \\ A_0, \textit{if } k \in B_0 \\ A_0 \cup A_1 \cup \cup \{i\}, \textit{if } k \in B_1 \end{cases}$$

Again, an application of Lemma 2 leads to a contradiction with (I). Thus, there exists a single CIF satisfying the axioms, i.e. J^L. $\square$

3 Oligarchy

In this context we no longer work with individuals that have only opinions about who is in J, instead of this, each agent divides the society into different classes. Then, an aggregator function generates a partition of $\mathbb{N}$.

Formally, each $i \in \mathbb{N}$ specifies an equivalence relation on $\mathbb{N}$ denoted $\sim_i$, such that $j \sim_i k$ if i considers that j and k are in the same class. A CIF* is a function that assigns to each profile $(\sim_1, \ldots, \sim_i, \ldots)$ an equivalence relation $\sim$ over $\mathbb{N}$.
The axioms used here are the following:

- **Independence**(I^*): consider two profiles of equivalence relations, $(\sim_1, \ldots, \sim_i, \ldots)$ and $(\sim'_1, \ldots, \sim'_i, \ldots)$, such that for every i, j and k, $i \sim_k j$ if and only if $i \sim'_k j$, then $i \sim (\sim_1, \ldots, \sim_i, \ldots) j$ if and only if $i \sim (\sim'_1, \ldots, \sim'_i, \ldots) j$.

- **Consensus**(C^*): if $j \sim_i k$ for every $i \in \mathbb{N}$, then $i \sim j$.

An **oligarchic** CIF* is such that there exists a unique non-empty subset M verifying that $i \sim j$ if and only if $M \subseteq \{k \mid i \sim_k j\}$. In the case that $|N| < \infty$ the following result of Barthélemy et al. (1986) characterizes oligarchic aggregators:

Theorem 2 *If a CIF* satisfies (C*) and (I*) then it is either oligarchic or the constant CIF* $\sim = N$.*

When the number of individuals is infinite, this result does no longer hold. We have instead:

Theorem 3 *If $N = \mathbb{N}$, the oligarchic CIF* and the constant CIF* $\sim = \mathbb{N}$ are not the only CIF*s that satisfy (C*) and (I*).*

Proof of Theorem 3: *consider the CIF* defined as follows:*

$$i \sim j \textit{ if and only if } \{k|i \sim_k j\} = \mathbb{N} \textit{ or } \mathbb{N} - \{k|i \sim_k j\} \textit{ is infinite, with } \{k|i \sim_k j\} \neq \emptyset.$$

Clearly this CIF* satisfies both (C*) and (I*). But it is not an oligarchic CIF*. Suppose there is a proper subset of $\mathbb{N}$, M (finite or not) that constitutes an "oligarchy". It cannot determine the outcome by itself, because if everybody else (except a finite number of individuals) disagrees with the members of M, the choice of the oligarchy will not obtain. $\Box$
We can say that a CIF* is **anti-oligarchic** if there exists a unique subset $\overline{M}$ such that $i \sim j$ if and only if $\overline{M} \supseteq \mathbb{N} - \{k \mid i \sim_k j\}$. Again, this CIF* verifies (C*) and (I*).
A question posed by the existence of this anti-oligarchic CIF*, is whether there is a way to characterize the functions that satisfy (C*) and (I*). We can show the following result:

Theorem 4 *A CIF** *verifies (C*) and (I*) if and only if it is either oligarchic, anti-oligarchic or the constant CIF** $\sim = \mathbb{N}$.

Proof of Theorem 4: *the only if part is straightforward. The if part can be proven by contradiction. Assume that there exists a CIF* satisfying (C*) and (I*) and verifying the following two conditions:*

1. *for all* $M \subseteq \mathbb{N}$ *there exists* $i, j \in \mathbb{N}$ *such that* $i \sim J$ *and* $M \subsetneq \{k | i \sim_k j\}$; *and*

2. *for all* $\overline{M} \subseteq \mathbb{N}$ *there exists* $i, j \in \mathbb{N}$ *such that* $i \sim J$ *and* $\overline{M} \supsetneq \mathbb{N} - \{k | i \sim_k j\}$.

Consider a profile in which every agent thinks that all the agents are in the same class, i.e., $\sim_i = \mathbb{N}$ *for all* $i \in \mathbb{N}$. *Because the CIF* satisfies (C*), we have that* $\sim = \mathbb{N}$. *But then we have that* $M \subseteq \{k \mid i \sim_k j\} = \mathbb{N}$ *and* $\overline{M} \supseteq \mathbb{N} - \{k \mid i \sim_k j\} = \emptyset$ *for every* $i, j \in \mathbb{N}$. *Absurd.*
Now let us address the uniqueness part. Suppose there exist a set M *and* M' *that constitute two different 'oligarchies'. Consider an agent* h *such that* $h \in M$ *but* $h \notin M'$ *and take the profile where every agent except* h *thinks that* i *is in the same class as* j. *Because* M' *is an 'oligarchy', we have that* $i \sim j$, *but we have that* $M \subsetneq \{k \mid i \sim_k j\}$, *contradicting the fact that* M *is another 'oligarchy'. Analogously we can show the uniqueness of the 'anti-oligarchy'.* □

Proposition 1 $\overline{M} = \mathbb{N} - M$

Proof of Proposition 1: *it is clear from the definition that* $M \cap \overline{M} = \emptyset$. *We know that* $\overline{M} \supseteq \mathbb{N} - \{k \mid i \sim_k j\}$ *for all* i, j *such that* $i \sim j$. *Consider the profile where* $\{k \mid i \sim_k j\} = M$ *for every* i, j, *so we have* $\overline{M} \supseteq \mathbb{N} - M$. *Because of the uniqueness of* M, *we have that* $\overline{M}$ *must be* $\mathbb{N} - M$.□

4 Conclusion

In this work we studied the problem of a society of infinite numerable agents deciding which of them can be classified as J. Our goal was to determine which results in the original finite setting remain valid when the number of agents is $\mathbb{N}$. We showed the the Strong Liberal CIF keeps its main property in this extension, namely its uniqueness. On the contrary, in the case of the oligarchic CIF*, the result for finite settings no longer holds, because a new aggregation function arises in the infinite setting, the 'anti-oligarchic' CIF*. This function and the oligarchic one characterize the CIF*s that verify (C*) and (I*). We also found that, once determined a 'oligarchy', a corresponding 'anti-oligarchy' is immediately obtained. This result is highly relevant for the examples presented in the Introduction. In the case of the political leanings in Germany, the decision can be made either by all the individual or by a (rather small) class of lawmakers (say the members of the Bundestag since 1949). This last case seems to be the most sensible answer to the question.

Similarly, in the case of the blockchain, the anti-oligarchy is constituted only by the nodes (i.e. peers) that are really involved in running the *proof-of-work* to incorporate new blocks into the ledger. That is, yielding the so-called *permissioned* or private blockchain (Shorish 2018).

The only remaining original aggregator function, namely the dictatorial aggregator deserves to be analyzed in the infinite case, under the condition that the range of the function cannot be the entire society nor the empty set. This problem, that intuitively leads to results similar to Fishburn's (1970), is matter for further research.

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