DATA APPROXIMATION WITH TIME-FREQUENCY INVARIANT SYSTEMS

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ABSTRACT. In this paper we prove the existence of a time-frequency space that best approximates a given finite set of data. Here best approximation is in the least square sense, among all time-frequency spaces with no more than a prescribed number of generators. We provide a formula to construct the generators from the data and give the exact error of approximation. The setting is in the space of square integrable functions defined on a second countable LCA group and we use the Zak transform as the main tool.

1. INTRODUCTION AND MAIN RESULT

Time-frequency systems, also called Gabor or Weyl-Heisenberg systems in the literature, are used extensively in the theory of communication, to analyze continuous signals, and to process digital data such as sampled audio or images.

Time-frequency spaces try to represent features of both a function and its frequencies by decomposing the signal into time-frequency atoms given by modulations and translations of a finite number of functions [9]. If one looks at a musical score, on the horizontal axis the composer represents the time, and on the vertical axis the "frequency" given by the amplitude of the signal at that instant. Finding *sparse representations* (i.e. spaces generated by a small set of functions) will be useful for example in classification tasks.

In numerical applications to time-dependent phenomena, one often encounters uniformly sampled signals of finite length, i.e. vectors of d elements, such as audio signals with a constant sampling frequency. In this case the most direct approach is to consider Fourier analysis on the cyclic group $\mathbb{Z}_d$.

To include a large variety of situations, our setting will be that of a locally compact abelian (LCA) group. The general construction developed in this paper will be specialised to the cyclic group $\mathbb{Z}_d$ in Example 2.2.

In this paper $G = (G, +)$ will be a second countable LCA group, that is, an abelian group endowed with a locally compact and second countable Hausdorff topology for which $(x, y) \mapsto x - y$ is continuous from $G \times G$ into G. We denote by $\widehat{G}$ the dual group of G, formed by the characters of G: an element $\alpha \in \widehat{G}$ is a continuous homomorphism from G into $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The action of α on $x \in G$ will be denoted by $(x, \alpha) := \alpha(x)$, to reflect the fact that the dual of $\widehat{G}$ is isomorphic to G, and therefore x can also act on α. For $\alpha_1, \alpha_2 \in \widehat{G}$ the group law is denoted by $\alpha_1 \cdot \alpha_2$, so that $(x, \alpha_1 \cdot \alpha_2) = (x, \alpha_1)(x, \alpha_2)$.

A uniform lattice, $L \subset G$, is a subgroup of G whose relative topology is the discrete one and for which G/L is compact in the quotient topology. The annihilator

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of L is $L^\perp = \{\alpha \in \widehat{G} : (\ell, \alpha) = 1 \;\; \forall \ell \in L\}$. Since $L^\perp \approx \widehat{(G/L)}$ ([11], Theorem 2.1.2) and G/L is compact, $L^\perp$ is discrete ([11], Theorem 1.2.5). In particular, since G is second countable, $\widehat{G}$ is also second countable, so both discrete groups L and $L^\perp$ are countable.

Lel L be a uniform lattice in the LCA group G and $\mathcal{B} \subset L^\perp$ be a uniform lattice in the dual group $\widehat{G}$. For $f \in L^2(G)$, $\ell \in L$, and $\beta \in \mathcal{B}$ let $T_\ell f(x) = f(x-\ell), x \in G$, be the translation operator, and $M_\beta f(x) = (x, \beta) f(x), x \in G$, be the modulation operator. The collection

$$\{T_\ell M_\beta f : \ell \in L, \beta \in \mathcal{B}\},$$

is the time-frequency system generated by $f \in L^2(G)$.

Since $\mathcal{B} \subset L^\perp$, we have $T_\ell M_\beta f = M_\beta T_\ell f$ for all $f \in L^2(G)$, $\ell \in L$, and $\beta \in \mathcal{B}$. Thus $\Pi(\ell, \beta) := T_\ell M_\beta$ is a unitary representation of the abelian group $\Gamma := L \times \mathcal{B}$, with operation $(\ell_1, \beta_1) \cdot (\ell_2, \beta_2) = (\ell_1 + \ell_2, \beta_1 \cdot \beta_2)$, in $L^2(G)$.

A closed subspace V of $L^2(G)$ is said to be Γ-invariant (or time-frequency invariant) if for every $f \in V$, $\Pi(\ell, \beta) f \in V$ for every $(\ell, \beta) \in \Gamma$. All Γ-invariant subspaces V of $L^2(G)$ are of the form

$$V = S_\Gamma(\mathcal{A}) := \overline{\operatorname{span}\{T_\ell M_\beta \varphi : \varphi \in \mathcal{A}, (\ell, \beta) \in \Gamma\}}^{L^2(G)}$$

for some countable collection $\mathcal{A}$ of elements of $L^2(G)$. If $\mathcal{A}$ is a finite collection we say that $V = S_\Gamma(\mathcal{A})$ has finite length, and $\mathcal{A}$ is a set of generators of V. We call the length of V, denoted length(V), the minimum positive integer n such that V has a set of generators with n elements.

We now state our approximation problem. Let $\mathcal{F} = \{f_1, f_2, ..., f_m\} \subset L^2(G)$ be a set of functional data. Given a closed subspace V of $L^2(G)$ define

$$\mathcal{E}(\mathcal{F}; V) := \sum_{j=1}^{m} \|f_j - \mathbb{P}_V f_j\|^2_{L^2(G)} \tag{1.1}$$

as the error of approximation of $\mathcal{F}$ by V, where $\mathbb{P}_V$ denotes the orthogonal projection of $L^2(G)$ onto V.

Is it possible to find a Γ-invariant space of length at most $n < m$ that *best* approximates our functions, in the sense that

$$\mathcal{E}(\mathcal{F}; S_\Gamma\{\psi_1, ..., \psi_n\}) \leq \mathcal{E}(\mathcal{F}; V)$$

for all Γ-invariant subspaces V of $L^2(G)$ with length$(V) \leq n$?

This question is relevant in applications. For example, if $\{f_1, \ldots, f_m\}$ are audio signals, *the best* Γ-invariant space provides a time-frequency *optimal* model to represent these signals.

The answer to this question is affirmative, and is given by the main theorem of this work.

Theorem 1.1. *Let G be a second countable LCA group, L and $\mathcal{B}$ be uniform lattices in G and $\widehat{G}$ respectively, with $\mathcal{B} \subset L^\perp$. For each set of functional data $\mathcal{F} = \{f_1, f_2, ..., f_m\} \subset L^2(G)$ and each $n \in \mathbb{N}$, $n < m$, there exists $\{\psi_1, ..., \psi_n\} \subset L^2(G)$ such that*

$$\mathcal{E}(\mathcal{F}; S_\Gamma\{\psi_1, ..., \psi_n\}) \leq \mathcal{E}(\mathcal{F}; V)$$

for all Γ-invariant subspaces V of $L^2(G)$ with length$(V) \leq n$.

Remark 1.1. *Observe that, in the previous statement, some of the generators* $\{\psi_1, ..., \psi_n\}$ *may be zero. In this case, the length of* $S_\Gamma\{\psi_1, ..., \psi_n\}$ *would be strictly smaller than* n*.*

The proof of Theorem 1.1 will follow the ideas originally developed in [1] for approximating data in $L^2(\mathbb{R}^d)$ by shift-invariant subspaces of finite length, and which have also been used in [6, 3].

We reduce the problem of finding the collection $\{\psi_1, ..., \psi_n\}$, whose existence is asserted in Theorem 1.1, to solve infinitely many approximation problems for data in a particular Hilbert space of sequences. This is accomplished with the help of an isometric isomorphism H_Γ that intertwines the unitary representation Π with the characters of Γ. This isometry H_Γ generalizes the fiberization map of [4] used in [1], and has the properties of a Helson map as defined in [2](Definition 7). The definition and properties of H_Γ are given in Section 2.

The reduced problems are then solved by using Eckart-Young theorem as stated and proved in [1] (Theorem 4.1). The solutions of all of these reduced problems are patched together to finally obtain the proof of Theorem 1.1 in Section 3.

2. An isometric isomorphism

Let G be a second countable LCA group, L a uniform lattice in G, and $\mathcal{B} \subset L^\perp$ a uniform lattice in $\widehat{G}$ (see definitions in Section 1). With $\Gamma = L \times \mathcal{B}$, each Γ-invariant subspace V of $L^2(G)$ is of the form

$$V = S_\Gamma(\mathcal{A}) := \overline{\text{span}\{T_\ell M_\beta \varphi : \varphi \in \mathcal{A}, (\ell, \beta) \in \Gamma\}}^{L^2(G)}$$

for some countable set $\mathcal{A} \subset L^2(G)$. Therefore

$$V = S_L(\{M_\beta \varphi : \varphi \in \mathcal{A}, \beta \in \mathcal{B}\})$$

is also an L-invariant subspace, that is $T_\ell f \in V$ for all $\ell \in L$ whenever $f \in V$. The theory of shift-invariant spaces on LCA groups, as developed in [7], can be applied to this situation.

Let $T_{L^\perp} \subset \widehat{G}$ be a measurable cross-section of $\widehat{G}/L^\perp$. The set $T_{L^\perp}$ is in one to one correspondence with the elements of $\widehat{G}/L^\perp$, and $\{T_{L^\perp} + \lambda : \lambda \in L^\perp\}$ is a tiling of $\widehat{G}$.

Let $\widehat{f}(\omega) := \int_G f(x)\overline{(x, w)}dx$ denote the unitary Fourier transform of $f \in L^2(G) \cap L^1(G)$ and extended to $L^2(G)$ by density. By Proposition 3.3 in [7] the mapping $\mathscr{T} : L^2(G) \to L^2(T_{L^\perp}, \ell^2(L^\perp))$ given by

$$\mathscr{T} f(\omega) = \{\widehat{f}(\omega + \lambda)\}_{\lambda \in L^\perp}, \; f \in L^2(G), \tag{2.1}$$

is an isometric isomorphism. Moreover, since $V \subset L^2(G)$ is an L-invariant space, it has an associated measurable range function

$$J : T_{L^\perp} \longrightarrow \{\text{closed subspaces of } \ell^2(L^\perp)\}$$

such that (See Theorem 3.10 in [7])

$$J(\omega) = \overline{\text{span}\,\{\mathscr{T}(M_\beta \varphi)(\omega) : \beta \in \mathcal{B}, \, \varphi \in \mathcal{A}\}}^{\ell^2(L^\perp)}, \text{ a.e } \omega \in T_{L^\perp}. \tag{2.2}$$

Using the definition of $\mathscr{T}$ given in (2.1), for each $\beta \in \mathcal{B}$ and each $\varphi \in L^2(G)$ we have

$$\mathscr{T}(M_\beta \varphi)(\omega) = \{\widehat{M_\beta \varphi}\,(\omega + \lambda)\}_{\lambda \in L^\perp} = \{\widehat{\varphi}\,(\omega + \lambda - \beta)\}_{\lambda \in L^\perp} = t_\beta(\mathscr{T}\varphi(\omega)) \tag{2.3}$$

where $t_\beta : \ell^2(L^\perp) \longrightarrow \ell^2(L^\perp)$ is the translation of sequences in $\ell^2(L^\perp)$ by elements of $\beta \in \mathcal{B}$, that is $t_\beta(\{a(\lambda)\}_{\lambda \in L^\perp}) = \{a(\lambda - \beta)\}_{\lambda \in L^\perp}$. Therefore, $\mathscr{T}$ intertwines the modulations $\{M_\beta\}_{\beta\in\mathcal{B}}$ with the translations by $\mathcal{B}$ on $\ell^2(L^\perp)$.

By equations (2.2) and (2.3), for a. e. $\omega \in T_{L^\perp}$,

$$J(\omega) = \overline{\text{span}\,\{t_\beta(\mathscr{T}\varphi(\omega)) : \beta \in \mathcal{B},\, \varphi \in \mathcal{A}\}}^{\ell^2(L^\perp)}.$$

Therefore, $J(\omega)$ is a $\mathcal{B}$-invariant subspace of $L^2(L^\perp)$. We can apply the theory of shift-invariant spaces as developed in [7] to the discrete LCA group $L^\perp$ and its uniform lattice $\mathcal{B}$.

Let $\mathcal{B}^\perp$ be the annihilator of $\mathcal{B}$ in the compact group $\widehat{L^\perp} \subset G$, that is

$$\mathcal{B}^\perp = \{b \in \widehat{L^\perp} : (b, \beta) = 1 \;\forall \beta \in \mathcal{B}\}. \tag{2.4}$$

Observe that $\mathcal{B}^\perp$ is finite, because it is a discrete subgroup of a compact group.

Let $T_{\mathcal{B}^\perp} \subset \widehat{L^\perp}$ be a measurable cross-section of $\widehat{L^\perp}/\mathcal{B}^\perp$. The set $T_{\mathcal{B}^\perp}$ is in one to one correspondence with the elements of $\widehat{L^\perp}/\mathcal{B}^\perp$ and $\{T_{\mathcal{B}^\perp} + b : b \in \mathcal{B}^\perp\}$ is a tiling of $\widehat{L^\perp}$.

Example 2.1. . Let $G = \mathbb{R}, L = \mathbb{Z}$ and $\mathcal{B} = n\mathbb{Z} \subset L^\perp = \mathbb{Z} \subset \widehat{\mathbb{R}}$. Since $\widehat{L^\perp} = \widehat{\mathbb{Z}} \approx [0, 1)$, $\ell \in \mathcal{B}^\perp$ if and only if $\ell \in [0, 1)$ and $e^{2\pi i \ell \cdot nk} = 1$ for all $k \in \mathbb{Z}$. Hence

$$\mathcal{B}^\perp = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}.$$

We can take $T_{\mathcal{B}^\perp} = [0, \dfrac{1}{n})$. Notice that as a subgroup of $\widehat{\mathbb{R}}$ the annihilator of $\mathcal{B}$ is $\dfrac{1}{n}\mathbb{Z}$.

Example 2.2. Let $p, q \in \mathbb{N}$, $d = pq$, and $G = \mathbb{Z}_d = \{0, 1, \ldots, d-1\}$. Let $L = \{0, p, 2p, \ldots p(q-1)\} = \{np : n = 0, \ldots, q-1\} \approx \mathbb{Z}_q$. Its annihilator lattice is

$$\begin{aligned} L^\perp &= \Big\{\lambda \in \{0, 1, \ldots, d-1\} : e^{2\pi i \frac{\lambda n p}{d}} = 1 \;\forall\; n = 0, \ldots, q-1\Big\} \\ &= \{0, q, 2q, \ldots q(p-1)\} = \{kq : k = 0, \ldots, p-1\} \approx \mathbb{Z}_p. \end{aligned}$$

A fundamental set $T_{L^\perp}$ for $L^\perp$ in $\widehat{G} \approx \mathbb{Z}_d$ is $T_{L^\perp} = \{0, \ldots, q-1\} \approx \mathbb{Z}_q$. The characters $\omega \in \widehat{L^\perp} = \{\text{homomorphisms} : L^\perp \to \mathbb{T}\}$ of this group are of the form (see e.g. [8] Lemma 5.1.3) $\omega_\nu(\lambda) = e^{2\pi i \frac{\lambda\nu}{p}}$, $\lambda \in L^\perp$ for $\nu \in \{\frac{\ell}{q} : \ell = 0, \ldots, p-1\} \approx \mathbb{Z}_p$. S*uppose now that* $p = rs$ *for some* $r, s \in \mathbb{N}$, *and let* $\mathcal{B} \subset L^\perp$ *be*

$$\mathcal{B} = \{0, rq, 2rq, \ldots, (s-1)rq\} = \{jrq : j = 0, \ldots, s-1\} \approx \mathbb{Z}_s.$$

The annihilator of $\mathcal{B}$ *in* $\widehat{L^\perp}$ *thus reads*

$$\begin{aligned} \mathcal{B}^\perp &= \Big\{b \in \{\frac{\ell}{q} : \ell = 0, \ldots, p-1\} : e^{2\pi i \frac{bjrq}{p}} = 1 \;\forall j = 0, \ldots, s-1\Big\} \\ &= \{0, \frac{s}{q}, \frac{2s}{q}, \ldots, \frac{s(r-1)}{q}\} = \{h\frac{s}{q} : h = 0, \ldots, r-1\} \approx \mathbb{Z}_r. \end{aligned}$$

A fundamental set in $\widehat{L^\perp} = \{\frac{\ell}{q} : \ell = 0, \ldots, p-1\}$ *for* $\mathcal{B}^\perp$ *is*
$T_{\mathcal{B}^\perp} = \Big\{0, \frac{1}{q}, \ldots, \frac{s-1}{q}\Big\} \approx \mathbb{Z}_s.$

By Proposition 3.3 in [7], the mapping $\mathscr{K} : \ell^2(L^\perp) \to L^2(T_{\mathcal{B}^\perp}, \ell^2(\mathcal{B}^\perp))$ given by

$$\begin{aligned} \mathscr{K}(\{a(\lambda)\}_{\lambda\in L^\perp})(t) &= \{(\{a(\lambda)\}_{\lambda\in L^\perp})^\wedge\ (t+b)\}_{b\in\mathcal{B}^\perp} \\ &= \left\{ \sum_{\lambda\in L^\perp} a(\lambda)\overline{(t+b,\lambda)} \right\}_{b\in\mathcal{B}^\perp}, \end{aligned} \tag{2.5}$$

is an isometric isomorphism. Moreover, each $\mathcal{B}$-invariant subspace $J(\omega),\ \omega \in T_{L^\perp}$, has an associated measurable range function

$$J(\omega,\cdot) : T_{\mathcal{B}^\perp} \longrightarrow \{\text{closed subspaces of } \ell^2(\mathcal{B}^\perp)\},$$

such that for almost every $t \in T_{\mathcal{B}^\perp}$, $J(\omega,t) = \overline{\operatorname{span}\{\mathscr{K}(\mathscr{T}\varphi)(\omega))(t) : \varphi \in \mathcal{A}\}}^{\ell^2(\mathcal{B}^\perp)}$. From the definition of $\mathscr{T}$ given in (2.1) and the definition of $\mathscr{K}$ given in (2.5) we obtain

$$\mathscr{K}(\mathscr{T}\varphi)(\omega))(t) = \left\{ \sum_{\lambda\in L^\perp} \widehat{f}(\omega+\lambda)\overline{(t+b,\lambda)} \right\}_{b\in\mathcal{B}^\perp}, \tag{2.6}$$

when $f \in L^2(G)$, $\omega \in T_{L^\perp}$, and $t \in T_{\mathcal{B}^\perp}$.

For $f \in L^2(G)$, $\omega \in \widehat{G}$, and $t \in G$ define

$$\mathcal{Z}f(\omega,t) := \sum_{\lambda\in L^\perp} \widehat{f}(\omega+\lambda)\overline{(t,\lambda)}, \tag{2.7}$$

the Zak transform of $\widehat{f}$ with respect to the lattice $L^\perp$. Observe that in terms of this map, $\mathscr{K}(\mathscr{T}\varphi)(\omega))(t) = \{\mathcal{Z}f(\omega,t+b)\}_{b\in\mathcal{B}^\perp}$.

To simplify the statement of the next theorem we write X_β for the character on G associated to $\beta \in \mathcal{B}$, that is $X_\beta : G \longrightarrow \mathbb{T}$ with $X_\beta(x) = (x,\beta)$ for all $x \in G$. Similarly X_ℓ will denote the character on $\widehat{G}$ associated to $\ell \in L$, that is $X_\ell : \widehat{G} \longrightarrow \mathbb{T}$ with $X_\ell(\omega) = (\ell,\omega)$ for all $\omega \in \widehat{G}$.

Theorem 2.1. *Let G be a second countable LCA group, L and $\mathcal{B}$ be uniform lattices in G and $\widehat{G}$ repectively, with $\mathcal{B} \subset L^\perp$. Let $\Gamma = L \times \mathcal{B}$ and for $f \in L^2(G)$, $\omega \in T_{L^\perp}$, and $t \in T_{\mathcal{B}^\perp}$ define*

$$H_\Gamma f(\omega,t) = \{\mathcal{Z}f(\omega,t+b)\}_{b\in\mathcal{B}^\perp}. \tag{2.8}$$

Then

1) The map H_Γ intertwines Π with the characters of Γ, that is $H_\Gamma\Pi(\ell,\beta)f = X_{-\ell}X_{-\beta}H_\Gamma f$ for all $f \in L^2(G), \ell \in L, \beta \in \mathcal{B}$.

2) The map H_Γ defined in (2.8) is an isometric isomorphism from $L^2(G)$ onto $L^2(T_{L^\perp} \times T_{\mathcal{B}^\perp}, \ell^2(\mathcal{B}^\perp))$.

Proof. For each $b \in \mathcal{B}^\perp$, the definition of $\mathcal{Z}$ given in (2.7) and the properties of the Fourier transform give

$$\begin{aligned} \mathcal{Z}\Pi(\ell,\beta)f(\omega,t+b) &= \sum_{\lambda\in\Lambda^\perp} \widehat{T_\ell M_\beta f}(\omega+\lambda)\overline{(t+b,\lambda)} \\ &= \sum_{\lambda\in\Lambda^\perp} \overline{(\ell,\omega+\lambda)}\widehat{f}(\omega+\lambda-\beta)\overline{(t+b,\lambda)}. \end{aligned}$$

Using that $(\ell,\lambda)=1$ and the change of variables $\lambda-\beta=\lambda'\in L^\perp$ yields

$$\mathcal{Z}\Pi(\ell,\beta)f(\omega,t+b)=\overline{(\ell,\omega)}\sum_{\lambda'\in\Lambda^\perp}\widehat{f}(\omega+\lambda')\overline{(t+b,\lambda'+\beta)}\,.$$

Using that $(t+b,\beta)=(t,\beta)\cdot(b,\beta)=(t,\beta)$ we obtain

$$\begin{aligned}\mathcal{Z}\Pi(\ell,\beta)f(\omega,t+b)&=\overline{(\ell,\omega)}\ \overline{(t,\beta)}\sum_{\lambda'\in\Lambda^\perp}\widehat{f}(\omega+\lambda')\overline{(t+b,\lambda')}\\&=X_{-\ell}(\omega)X_{-\beta}(t)\mathcal{Z}f(\omega,t+b)\,.\end{aligned}$$

This proves 1). To prove 2) observe that by the definition of H_Γ given in (2.8) together with (2.6) and (2.7) we have

$$H_\Gamma f(\omega,t)=\mathscr{K}(\mathscr{T}f(\omega))(t)\,.$$

That H_Γ is an isometry now follows from the fact that $\mathscr{T}$ and $\mathscr{K}$ are isometries in their respective spaces.

We need to prove that H_Γ is onto. Since $\mathscr{K}:\ell^2(L^\perp)\to L^2(T_{\mathcal{B}^\perp},\ell^2(\mathcal{B}^\perp))$ is an isometric isomorphism between Hilbert spaces, by Lemma 4.1 in the Appendix, the map

$$Q_{\mathscr{K}}:L^2(T_{L^\perp},\ell^2(L^\perp))\longrightarrow L^2(T_{L^\perp},L^2(T_{\mathcal{B}^\perp},\ell^2(\mathcal{B}^\perp))$$

given by

$$(Q_{\mathscr{K}}f)(\omega)=\mathscr{K}(f(\omega)),\ \ f\in L^2(T_{L^\perp},\ell^2(L^\perp))$$

is an isometric isomorphism. Moreover, by Fubini's theorem, the Hilbert spaces $L^2(T_{L^\perp},L^2(T_{\mathcal{B}^\perp},\ell^2(\mathcal{B}^\perp))$ and $L^2(T_{L^\perp}\times T_{\mathcal{B}^\perp},L^2(\ell^2(\mathcal{B}^\perp))$ are also isomorphic and the isomorphism is given by $\Phi(f)(\omega,t)=f(\omega)(t)$, for $f\in L^2(T_{L^\perp},L^2(T_{\mathcal{B}^\perp},\ell^2(\mathcal{B}^\perp))$.

Let now $F\in L^2(T_{L^\perp}\times T_{\mathcal{B}^\perp},L^2(\ell^2(\mathcal{B}^\perp))$. Choose $g\in L^2(T_{L^\perp},\ell^2(L^\perp))$ such that $\Phi\circ Q_{\mathscr{K}}(g)=F$. Hence

$$F(\omega,t)=\Phi\circ Q_{\mathscr{K}}(g)(\omega,t)=Q_{\mathscr{K}}(g)(\omega)(t)=\mathscr{K}(g(\omega))(t).$$

Choose now $f\in L^2(G)$ such that $\mathscr{T}(f)=g$. Then

$$H_\Gamma f(\omega,t)=\mathscr{K}(\mathscr{T}f(\omega))(t)=F(\omega,t).$$

This finishes the proof of the theorem. □

Example 2.3. For the cyclic group of Example 2.2, recall that, for $f\in\mathbb{C}^d$

$$\widehat{f}(\omega)=\frac{1}{\sqrt{d}}\sum_{g=0}^{d-1}f(g)e^{-2\pi i\frac{g\omega}{d}},\ \ \omega\in\{0,\dots,d-1\}.$$

For $t\in T_{\mathcal{B}^\perp}=\left\{0,\frac{1}{q},\dots,\frac{s-1}{q}\right\}$, the Zak transform (2.7) thus reads

$$\begin{aligned}\mathcal{Z}f(\omega,t)&=\sum_{k=0}^{p-1}\widehat{f}(\omega+kq)e^{-2\pi i\frac{kqt}{p}}=\sum_{k=0}^{p-1}\frac{1}{\sqrt{d}}\sum_{g=0}^{d-1}f(g)e^{-2\pi i\frac{g(\omega+kq)}{d}}e^{-2\pi i\frac{kqt}{p}}\\&=\frac{1}{\sqrt{d}}\sum_{g=0}^{d-1}f(g)e^{-2\pi i\frac{g\omega}{d}}K(g+qt)=\frac{e^{2\pi i\frac{qt\omega}{d}}}{\sqrt{d}}\sum_{g=0}^{d-1}f(g-qt)e^{-2\pi i\frac{g\omega}{d}}K(g)\end{aligned}$$

where $K(g) = \sum_{k=0}^{p-1} \left(e^{-2\pi i \frac{g}{p}}\right)^k = \begin{cases} p & if \quad g \in L \\ 0 & if \quad g \notin L \end{cases}$. This gives

$$\mathcal{Z}f(\omega,t) = \sqrt{p}e^{2\pi i \frac{qt\omega}{d}} \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} f(pn - qt)e^{-2\pi i \frac{pn\omega}{q}}.$$

Before embarking in the proof of Theorem 1.1, which will be accomplished in Section 3, we need an additional result.

Let $V = S_\Gamma(\mathcal{A})$ be a Γ-invariant subspace of $L^2(G)$, where $\mathcal{A} \subset L^2(G)$. For each $(\omega,t) \in T_{L^\perp} \times T_{\mathcal{B}^\perp}$, consider the range function

$$J_V : \mathscr{T}_{L^\perp} \times T_{\mathcal{B}^\perp} \longrightarrow \{\text{closed subspaces of } \ell^2(\mathcal{B}^\perp)\}$$

given by

$$J_V(\omega,t) := \overline{\operatorname{span}\{H_\Gamma\varphi(\omega,t) : \varphi \in \mathcal{A}\}}^{\ell^2(\mathcal{B}^\perp)}. \tag{2.9}$$

Proposition 2.1. *With $V = S_\Gamma(\mathcal{A})$ as above, let $\mathcal{P}_{J_V(\omega,t)}$ be the orthogonal projection of $\ell^2(\mathcal{B}^\perp)$ onto $J_V(\omega,t)$. Then, for all $f \in L^2(G)$ and $(\omega,t) \in T_{L^\perp} \times T_{\mathcal{B}^\perp}$,*

$$\mathrm{H}_\Gamma \mathbb{P}_{S_\Gamma(\mathcal{A})} f(\omega,t) = \mathcal{P}_{J_V(\omega,t)}(H_\Gamma f(\omega,t)).$$

Proof. Observe first that, since H_Γ is an isometric isomorphism between Hilbert spaces, then

$$H_\Gamma \mathbb{P}_{S_\Gamma(\mathcal{A})} = \mathbb{P}_{H_\Gamma(S_\Gamma(\mathcal{A}))} H_\Gamma. \tag{2.10}$$

The set $\mathcal{D} := \{X_\ell X_\beta : (\ell,\beta) \in \Gamma\}$ of characters of Γ is a determining set for $L^1(T_{L^\perp} \times T_{\mathcal{B}^\perp})$ in the sense of Definition 2.2 in [5], because

$$\int_{T_{L^\perp} \times T_{\mathcal{B}^\perp}} f(\omega,t) X_\ell(\omega) X_\beta(t) d\omega dt = 0 \Rightarrow f = 0 \quad \forall\, f \in L^1(T_{L^\perp} \times T_{\mathcal{B}^\perp}).$$

Indeed, this is Fourier uniqueness theorem since $T_{L^\perp}$ and $T_{\mathcal{B}^\perp}$ are relatively compact.

By 1) of Theorem 2.1, for all $f \in L^2(G)$, $H_\Gamma(T_\ell M_\beta f) = X_{-\ell} X_{-\beta}(H_\Gamma f)$. Thus, $H_\Gamma(S_\Gamma(\mathcal{A}))$ is $\mathcal{D}$-multiplicative invariant in the sense of Definition 2.3 in [5]. Indeed, if $X_\ell X_\beta \in \mathcal{D}$, $F \in H_\Gamma(S_\Gamma(\mathcal{A}))$ writing $H_\Gamma f = F$ we have

$$X_\ell X_\beta F = X_\ell X_\beta (H_\Gamma f) = H_\Gamma(T_{-\ell} M_{-\beta} f) \in H_\Gamma(S_\Gamma(\mathcal{A})).$$

By Theorem 2.4 in [5], J_V is a measurable range function. By Proposition 2.2 in [5],

$$\mathbb{P}_{H_\Gamma(S_\Gamma(\mathcal{A}))}(H_\Gamma f)(w,t) = \mathcal{P}_{J_V(\omega,t)}(H_\Gamma f(\omega,t)).$$

The result now follows from (2.10). □

3. Solution to the approximation problem

This section is dedicated to the proof of Theorem 1.1. Let $\mathcal{F} = \{f_1, \ldots, f_m\} \subset L^2(G)$ be a collection of functional data. With the notation of Theorem 1.1, for each $n < m$ we need to find $\{\psi_1, \ldots, \psi_n\} \subset L^2(G)$ such that $\mathcal{E}(\mathcal{F}; S_\Gamma\{\psi_1, \ldots, \psi_n\}) \leq \mathcal{E}(\mathcal{F}; V)$ for any Γ-invariant subspace V of $L^2(G)$ of length less than or equal n. The definition of $\mathcal{E}(\mathcal{F}; V)$ is given in (1.1) and for convenience of the reader we recall it here.

$$\mathcal{E}(\mathcal{F}; V) := \sum_{j=1}^m \|f_j - \mathbb{P}_V f_j\|^2_{L^2(G)}.$$

For a.e. $(\omega, t) \in T_{L^\perp} \times T_{\mathcal{B}^\perp}$ consider

$$H_\Gamma(\mathcal{F})(w,t) := \{H_\Gamma f_1(\omega,t), \ldots, H_\Gamma f_m(\omega,t)\}.$$

Let $G_{\mathcal{F},\Gamma}(w,t)$ be the $m \times m$ $\mathbb{C}$-valued matrix whose (i,j) entry is given by

$$[G_{\mathcal{F},\Gamma}(w,t)]_{i,j} = \langle H_\Gamma f_i(\omega,t), H_\Gamma f_j(\omega,t)\rangle_{\ell^2(\mathcal{B}^\perp)}.$$

The matrix $G_{\mathcal{F},\Gamma}(w,t)$ is hermitian and its entries are measurable functions defined on $T_{L^\perp} \times T_{\mathcal{B}^\perp}$. Write

$$\lambda_1(\omega,t) \geq \lambda_2(\omega,t) \geq \ldots, \geq \lambda_m(\omega,t) \geq 0$$

for the eigenvalues of $G_{\mathcal{F},\Gamma}(w,t)$. By Lemma 2.3.5 in [10] the eigenvalues $\lambda_i(\omega,t)$, $i = 1,\ldots,m$, are measurable and there exist corresponding measurable vectors $y_i(\omega,t) = (y_{i,1}(\omega,t),\ldots,y_{i,m}(\omega,t))$ that are orthonormal left eigenvectors of the matrix $G_{\mathcal{F},\Gamma}(w,t)$. That is,

$$y_i(\omega,t)\, G_{\mathcal{F},\Gamma}(w,t) = \lambda_i(\omega,t)\, y_i(\omega,t), \quad i = 1,\ldots,m. \tag{3.1}$$

For $n \leq m$, define $q_1(\omega,t),\ldots,q_n(\omega,t) \in \ell^2(\mathcal{B}^\perp)$ by

$$q_i(\omega,t) = \widetilde{\sigma}_i(\omega,t) \sum_{j=1}^{m} y_{i,j}(\omega,t)\, H_\Gamma f_j(\omega,t) \quad i = 1,\ldots,n, \tag{3.2}$$

where

$$\widetilde{\sigma}_i(\omega,t) = \begin{cases} \frac{1}{\sqrt{\lambda_i(\omega,t)}} & \text{if} \quad \lambda_i(\omega,t) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

By the Eckart-Young Theorem (see the version stated and proved in Theorem 4.1 of [1]), $\{q_1(\omega,t),\ldots,q_n(\omega,t)\}$ is a Parseval frame for the space it generates $Q(\omega,t) := \operatorname{span}\{q_1(\omega,t),\ldots,q_n(\omega,t)\}$ and $Q(\omega,t)$ is optimal in the sense that

$$E(H_\Gamma(\mathcal{F})(w,t);Q(\omega,t)) := \sum_{i=1}^{m} \|H_\Gamma f_i(\omega,t) - \mathcal{P}_{Q(\omega,t)} H_\Gamma(f_i)(w,t)\|^2_{\ell^2(\mathcal{B}^\perp)}$$

$$\leq \sum_{i=1}^{m} \|H_\Gamma f_i(\omega,t) - \mathcal{P}_{Q'} H_\Gamma(\mathcal{F})(w,t)\|^2_{\ell^2(\mathcal{B}^\perp)} := E(H_\Gamma(f_i)(w,t);Q') \tag{3.3}$$

for any Q' subspace of $\ell^2(\mathcal{B}^\perp)$ of dimension less than or equal to n. Moreover,

$$E(H_\Gamma(\mathcal{F})(w,t);Q(\omega,t)) = \sum_{i=n+1}^{m} \lambda_i(\omega,t). \tag{3.4}$$

Before continuing with the proof, let us relate the pointwise errors that appear in (3.3) to the error defined in (1.1) for Γ-invariant subspaces.

Proposition 3.1. *For $V = S_\Gamma(\mathcal{A})$ as in Proposition 2.1,*

$$\mathcal{E}(\mathcal{F};V) = \int_{T_{L^\perp}} \int_{T_{\mathcal{B}^\perp}} E(H_\Gamma(\mathcal{F})(w,t);J_V(\omega,t))\, dt d\omega\,,$$

where $J_V(\omega,t)$ is defined in (2.9).

Proof. By 2) of Theorem 2.1, H_Γ is an isometry from $L^2(G)$ onto the space $L^2(T_{L^\perp}\times T_{\mathcal{B}^\perp}, \ell^2(\mathcal{B}^\perp))$. Therefore,

$$\begin{aligned}\mathcal{E}(\mathcal{F};V) &= \sum_{j=1}^m \|f_j - \mathbb{P}_V f_j\|^2_{L^2(G)} \\ &= \sum_{j=1}^m \|H_\Gamma f_j - H_\Gamma \mathbb{P}_V f_j\|^2_{L^2(T_{L^\perp}\times T_{\mathcal{B}^\perp},\ell^2(\mathcal{B}^\perp))} \\ &= \sum_{j=1}^m \int_{T_{L^\perp}}\int_{\mathcal{B}_{L^\perp}} \|H_\Gamma f_j(\omega,t) - H_\Gamma \mathbb{P}_V f_j(\omega,t)\|^2_{\ell^2(\mathcal{B}^\perp)}\, dt d\omega\,.\end{aligned}$$

By Proposition 2.1,

$$\begin{aligned}\mathcal{E}(\mathcal{F};V) &= \int_{T_{L^\perp}}\int_{\mathcal{B}_{L^\perp}} \sum_{j=1}^m \|H_\Gamma f_j(\omega,t) - \mathcal{P}_{J_V(\omega,t)}(H_\Gamma f_j(\omega,t))\|^2_{\ell^2(\mathcal{B}^\perp)}\, dt d\omega \\ &= \int_{T_{L^\perp}}\int_{\mathcal{B}_{L^\perp}} E(H_\Gamma(\mathcal{F})(w,t); J_V(\omega,t))\, dt d\omega\,. \qquad \square\end{aligned}$$

□

Let us now continue with the proof of Theorem 1.1. By definition (3.2), each $q_i(\omega,t)$ is measurable and defined on $T_{L^\perp}\times T_{\mathcal{B}^\perp}$ with values in $\ell^2(\mathcal{B}^\perp)$. Moreover,

$$\begin{aligned}&\|q_i(\omega,t)\|^2_{\ell^2(\mathcal{B}^\perp)} = \langle q_i(\omega,t), q_i(\omega,t)\rangle_{\ell^2(\mathcal{B}^\perp)} \\ &= \widetilde{\sigma}_i(\omega,t)^2 \sum_{b\in\mathcal{B}^\perp}\sum_{j=1}^m\sum_{s=1}^m y_{i,j}(\omega,t)\, \mathcal{Z}f_j(\omega,t+b)\, \mathcal{Z}f_s(\omega,t+b)\, \overline{y_{i,s}(\omega,t)} \\ &= \widetilde{\sigma}_i(\omega,t)^2 \sum_{j=1}^m y_{i,j}(\omega,t) \sum_{s=1}^m \langle Zf_j(\omega,t), Zf_s(\omega,t)\rangle_{\ell^2(\mathcal{B}^\perp)} \overline{y_{i,s}(\omega,t)}\,.\end{aligned}$$

In matrix form,

$$\|q_i(\omega,t)\|^2_{\ell^2(\mathcal{B}^\perp)} = \widetilde{\sigma}_i(\omega,t)^2 y_i(\omega,t)\, G_{\mathcal{F},\Gamma}(w,t)\, \overline{y_i(\omega,t)}^t\,.$$

By (3.1), the orthonormality of the vectors $y_i(\omega,t)$, and the definition of $\widetilde{\sigma}_i(\omega,t)$, we have

$$\|q_i(\omega,t)\|^2_{\ell^2(\mathcal{B}^\perp)} = \widetilde{\sigma}_i(\omega,t)^2 \lambda_i(\omega,t)\, \|y_i(\omega,t)\|^2 \le 1\,.$$

Since $T_{L^\perp}$ and $T_{\mathcal{B}^\perp}$ have finite measure, we conclude that for $i=1,\dots,n$, $q_i \in L^2(T_{L^\perp}\times T_{\mathcal{B}^\perp}, \ell^2(\mathcal{B}^\perp))$. The mapping H_Γ is onto by part 2) of Theorem 2.1. Therefore there exist $\psi_i \in L^2(G)$ such that

$$H_\Gamma(\psi_i) = q_i\,, \quad i=1,\dots,n.$$

It remains to show that the space $W := S_\Gamma(\psi_1,\dots,\psi_n)$ is the optimal one as required in the statement of Theorem 1.1.

By Proposition 3.1

$$\mathcal{E}(\mathcal{F};W) = \int_{T_{L^\perp}}\int_{T_{\mathcal{B}^\perp}} E(H_\Gamma(\mathcal{F})(w,t); J_W(\omega,t))\, dt d\omega\,.$$

By (3.3) and the definitions of ψ_i, $J_W(\omega,t) = Q(\omega,t)$. Therefore, we can write,

$$\mathcal{E}(\mathcal{F};W) = \int_{T_{L^\perp}}\int_{T_{\mathcal{B}^\perp}} E(H_\Gamma(\mathcal{F})(w,t); Q(\omega,t))\, dt d\omega\,. \tag{3.5}$$

Let now $V = S_\Gamma(\varphi_1, \dots, \varphi_r)$, $r \leq n$, be any Γ-invariant subspace of length less than or equal n. Since $J_V(\omega, t)$ has dimension less than or equal n, (3.3) gives

$$\mathcal{E}(\mathcal{F}; W) \leq \int_{T_{L^\perp}} \int_{T_{\mathcal{B}^\perp}} E(H_\Gamma(\mathcal{F})(w,t); J_V(\omega,t))\, dtd\omega = \mathcal{E}(\mathcal{F}; V)\,,$$

where the last equality is due to Proposition 3.1. Moreover, by (3.5) and (3.4)

$$\mathcal{E}(\mathcal{F}; W) = \sum_{i=n+1}^{m} \int_{T_{L^\perp}} \int_{T_{\mathcal{B}^\perp}} \lambda_i(\omega, t) d\omega dt.$$

This finishes the proof of Theorem 1.1. □

4. Appendix

We give the proof of the following Lemma that has been used in Section 2 to prove part 2) of Theorem 2.1.

Lemma 4.1. *Let $\sigma : \mathbb{H}_1 \longrightarrow \mathbb{H}_2$ be an isometric isomorphism between the Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$. For a measure spaces $(X, d\mu)$ the map $Q_\sigma : L^2(X, \mathbb{H}_1) \longrightarrow L^2(X, \mathbb{H}_2)$ given by $(Q_\sigma f)(x) = \sigma(f(x))$ is also an isometric isomorphism.*

Proof. Let f be a measurable vector function in $L^2(X, \mathbb{H}_1)$, that is, for every $y \in \mathbb{H}_1$ the scalar function $x \longrightarrow \langle f(x), y\rangle_{\mathbb{H}_1}$ is measurable. We must prove that Qf is also a measurable vector function in $L^2(X, \mathbb{H}_2)$. For $z \in \mathbb{H}_2$ we have

$$< Qf(x), z >_{\mathbb{H}_2} = < \sigma(f(x)), z >_{\mathbb{H}_2} = < f(x), \sigma^*(z) >_{\mathbb{H}_1}\, .$$

Since $\sigma^*(z) = \sigma^{-1}(z)$ is a general element of $\mathbb{H}_1$, this shows that Qf is measurable. Moreover, for $f, g \in L^2(X, \mathbb{H}_1)$,

$$\begin{aligned} < Qf, Qg >_{L^2(X,\mathbb{H}_2)} &= \int_X < \sigma(f(x)), \sigma(g(x)) >_{\mathbb{H}_2} d\mu(x) \\ &= \int_X < f(x), (g(x) >_{\mathbb{H}_1} d\mu(x) = < f, g >_{L^2(X,\mathbb{H}_1)}\, . \end{aligned}$$

This shows that if $f \in L^2(X, \mathbb{H}_1)$, $Q_\sigma f \in L^2(X, \mathbb{H}_2)$ and that Q_σ is and isometry.

Finally, it is easy to see that $R : L^2(X, \mathbb{H}_2) \to L^2(X, \mathbb{H}_1)$ defined by $Rg(x) = \sigma^{-1}(g(x)$ is the inverse and the adjoint of Q. Therefore, Q_σ is onto. □

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