A DUALITY RESULT ON FRAME AND RIESZ SEQUENCES

Pablo Calderon and Mariano Ruiz

Departamento de Matemática, FCE-UNLP & IAM-CONICET
pcalderon@mate.unlp.edu.ar, mruiz@mate.unlp.edu.ar

Abstract: In this article we present a short proof of a duality principle concerning frame and Riesz sequences due to Ulanovskii and Olevskii. Our proof is derived from a result on compression on orthogonal projections. As a consequence, we get a better lower frame (Riesz) bound for the sequences than the one deduced from the original proof.

Keywords: *frame sequence, Riesz sequence*
2000 AMS Subject Classification: 42C15 - 39B42

1 INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and I a countable index set. A family $\{f_i\}_{i\in I}$ of vectors in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exists $a, b > 0$ such that

$$a\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq b\|f\|^2, \text{ for all } f \in \mathcal{H} \tag{1}$$

The optimal constants a, b for which (1) is satisfied are called *frame bounds*. If only the upper inequality is satisfied, we say that $\{f_i\}_{i\in I}$ is a *Bessel sequence* for $\mathcal{H}$.

Frames constitutes a very useful tool in several areas of mathematics, such as signal processing, harmonic analysis, among others. Roughly speaking, their importance is based in that they conserve some properties of the orthonormal basis (basis expansion for example), but relaxing the uniqueness on the basis decomposition. For a detailed treatment of frame theory, we refer the reader to [1] and the references therein.

In the special case in which such redundancy does not occur, we say that the frame is a *Riesz basis*. Namely, a Riesz basis is a complete sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ such that there exists $a, b > 0$ such that

$$a\sum_{i\in J}|c_i|^2 \leq \|\sum_{i\in J} c_i f_i\|^2 \leq b\sum_{i\in J}|c_i|^2, \tag{2}$$

for each finite sequence $\{c_i\}_{i\in J}$ of scalars.

If the inequalities in (1) are satisfied only for f in the closure of the linear span of $\{f_i\}_{i\in I}$, we say that $\{f_i\}_{i\in I}$ is a *frame sequence*. Similarly, if a sequence $\{f_i\}_{i\in I}$ satisfies (2) but it is not necessary complete in $\mathcal{H}$, it is called *Riesz sequence*.

There are bounded linear operators between the Hilbert space $\ell^2(I) := \{\{c_i\}_{i\in I} : \sum_{i\in I}|c_i|^2 < \infty\}$ and $\mathcal{H}$, that can be associated to a Bessel sequence :

The *synthesis operator* $T : \ell^2(I) \to \mathcal{H}$, defined as

$$T(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i,$$

which is a well defined bounded linear operator whose operator norm is bounded by $\sqrt{b}$. The adjoint of T is the *analysis operator* which sends $f \in \mathcal{H}$ to the sequence $\{\langle f, f_i\rangle\}_{i\in I} \in \ell^2(I)$.

Finally, the frame operator is the composition $S = TT^*$, which is a bounded linear operator in $\mathcal{H}$. It is easy to see that $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ if and only if S is a positive definite operator whose spectrum lies in the interval $[a, b]$. Moreover, $\{f_i\}_{i\in I}$ is a Riesz basis if and only its synthesis operator is invertible.

Let us establish some notation used throughout the paper: given T a bounded operator in $\mathcal{H}$, by $N(T)$ and $R(T)$ we denote its nullspace and range respectively. By $Id_{\mathcal{H}}$ we denote the identity operator in $\mathcal{H}$. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then by $P_{\mathcal{M}}$ we mean the orthogonal projection onto $\mathcal{M}$, that is $P_{\mathcal{M}} = P_{\mathcal{M}}^2 = P_{\mathcal{M}}^*$ and $R(P_{\mathcal{M}}) = \mathcal{M}$. Finally, we consider the usual order in bounded operators: if A, B are bounded operators in $\mathcal{H}$, $B \geq A$ iff $B - A$ is a semidefinite operator.

There is a well-known characterization of being a Riesz (resp. frame) sequence in terms of the synthesis and analysis operators

Lemma 1 *Let $\{f_i\}_{i\in I}$ be a Bessel sequence in a Hilbert space $\mathcal{H}$, with synthesis operator T. Then,*

1. *$\{f_i\}_{i\in I}$ is a Riesz sequence iff there exists $a > 0$ such that*

$$T^*T \geq aId_{\ell^2(I)}$$

2. *$\{f_i\}_{i\in I}$ is a frame sequence iff there exists $a > 0$ such that*

$$TT^* \geq aP_{N(T^*)^\perp}$$

where $N(T^*)$ is the nullspace of T^*.

Proof. Suppose that $\{f_i\}_{i\in I}$ is a Riesz sequence in $\mathcal{H}$, with synthesis operator T. Then, there exist $a, b > 0$ such that (2) holds for each $\{c_i\}_{i\in I} \in \ell^2(I)$ finite. Since T is bounded, it is easy to see that (2) is satisfied for all $\{c_i\}_{i\in I}$ in $\ell^2(I)$. Therefore, if $\{c_i\}_{i\in I}$ then

$$\langle T^*T\{c_i\}_{i\in I}\,,\,\{c_i\}_{i\in I}\rangle = \|T\{c_i\}_{i\in I}\|^2 \geq a\|\{c_i\}_{i\in I}\|^2 = a\,\langle\{c_i\}_{i\in I}\,,\,\{c_i\}_{i\in I}\rangle$$

The converse is clear from the previous inequalities.

On the other hand, if $\{f_i\}_{i\in I}$ is a frame sequence then, T is a closed range operator simply from (1). Thus, as we did above, the frame inequalities in $R(T) = N(T^*)^\perp$ implies $TT^* \geq aP_{N(T^*)^\perp}$. The proof of the reverse implication is also similar.

□

In a finite dimensional Hilbert space, $\mathcal{H} = \mathbb{C}^n$, it is easy to see that a frame $\{f_i\}_{i=1}^m$ for $\mathcal{H}$ is just a set of generators. Also, by considering the canonical basis, we can consider the synthesis, analysis and frame operators as $n\times m$, $m\times n$ and $n\times n$ matrices over $\mathbb{C}$, respectively. Notice that in the finite case, the ratio b/a, of the (optimal) frame bounds is the condition number of the frame matrix. It is clear then that, for practical purposes, it is important to have good estimates for the optimal bounds for frame (Riesz) sequences.

2 Main result

Proposition 1 *Let $\mathcal{H}$ be a complex separable Hilbert space and let P and Q denote two orthogonal projections onto closed subspaces of $\mathcal{H}$. Let $0 < a \leq 1$. Then the following are equivalent*

i. $PQP \geq aP$.

ii. $Q^\perp P^\perp Q^\perp \geq aQ^\perp$.

Proof. Suppose that $PQP \geq aP$. Then, $PQ^\perp P = P(I-Q)P \leq (1-a)P$, which helds

$$(Q^\perp PQ^\perp)^2 = Q^\perp PQ^\perp PQ^\perp \leq (1-a)Q^\perp PQ^\perp. \tag{3}$$

Since $Q^\perp PQ^\perp$ is semidefinite positive, this implies $Q^\perp PQ^\perp \leq (1-a)I$. In particular,

$$Q^\perp PQ^\perp \leq (1-a)Q^\perp,$$

which yields

$$Q^\perp P^\perp Q^\perp = Q^\perp - Q^\perp PQ^\perp \geq aQ^\perp$$

as it was claimed. The other implication is identical. □

Now we are ready to present an alternative proof of the following result, due to A. Olevskii and A. Ulanovskii ([2, Prop. 1.23]):

Theorem 1 *Let $\{e_i\}_{i\in I}$ be a orthonormal basis of $\mathcal{H}$ and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Let us consider $\sigma \subset I$. Denote by $P_{\mathcal{M}}$ the orthonormal projection onto $\mathcal{M}$. Then, the following are equivalent*

i. $\{P_{\mathcal{M}}(e_i)\}_{i\in\sigma}$ *is a Riesz sequence, with lower bound $a > 0$.*

ii. $\{P^{\perp}_{\mathcal{M}}(e_i)\}_{i\in\sigma^c}$ *is a frame sequence, with lower bound $a > 0$.*

Proof. It is clear from definitions that $\{e_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$, whose synthesis operator is an isometric isomorphism $U : \ell^2(I) \to \mathcal{H}$. Denote by P_σ the orthonormal (diagonal) projection onto the closed space spanned by $\{b_i\}_{i\in\sigma}$, where $\{b_i\}_{i\in I}$ is the canonical orthonormal basis in $\ell^2(I)$.

Suppose that $\{P_{\mathcal{M}}(e_i)\}_{i\in\sigma}$ is a Riesz sequence, with lower bound $a > 0$. Let W be an isometric isomorphism between $\ell^2(\sigma)$ and $\overline{\operatorname{span}\{e_i\}_{i\in\sigma}} \subset \mathcal{H}$. Therefore, a synthesis operator T for the sequence $\{P_{\mathcal{M}}(e_i)\}_{i\in\sigma}$ is given by $T = P_{\mathcal{M}}UW$. It is clear that $P_{\mathcal{M}}UW = P_{\mathcal{M}}UP_\sigma W$. Then, by Lemma 1, we have that

$$T^*T = W^*P_\sigma U^*P_{\mathcal{M}}UP_\sigma W \geq aId_{\ell^2(\sigma)} = aW^*W$$

Thus

$$P_\sigma U^*P_{\mathcal{M}}UP_\sigma \geq aP_\sigma.$$

If we put $Q = U^*P_{\mathcal{M}}U$ and $P = P_\sigma$ then, by Prop. 1, we can conclude that $Q^\perp P^\perp Q^\perp \geq aQ^\perp$, which means that

$$U^*P_{\mathcal{M}^\perp}UP_{\sigma^c}U^*P_{\mathcal{M}^\perp}U \geq aU^*P_{\mathcal{M}^\perp}U,$$

so $P_{\mathcal{M}^\perp}UP_{\sigma^c}U^*P_{\mathcal{M}^\perp} \geq aP_{\mathcal{M}^\perp}$. Then, by using again Lemma 1, we have that $\{P^{\perp}_{\mathcal{M}}(e_i)\}_{i\in\sigma^c}$ is a frame sequence, with lower bound $a > 0$. The proof of the reverse implication is identical. □

Remark 2 *Our presentation of the previous result differs from [2] since we establish the equality of the lower frame bound a in both sequences.*

As we can derive from the proof of Olevskii and Ulanovskii, in their proof it is shown that if the lower constant of the frame sequence $\{P_{\mathcal{M}}e_i\}_{i\in\sigma}$ is a, then $\{P_{\mathcal{M}^\perp}e_i\}_{i\in\sigma^c}$ is a Riesz sequence with lower constant $\frac{a}{a+1}$ which is smaller than a.

It turns out that our result is sharp, as it is shown in the following example:

Example 1 *Let $\{e_i\}_{i\in I}$ be a orthonormal basis of $\mathcal{H}$. Consider $\mathcal{M} = \overline{\operatorname{span}\{e_i\}_{i\in\sigma}}$, for some $\sigma \subset I$. Then, it is clear that $\{P_{\mathcal{M}}e_i\}_{i\in\sigma} = \{e_i\}_{i\in\sigma}$ and $\{P_{\mathcal{M}^\perp}e_i\}_{i\in\sigma^c} = \{e_i\}_{i\in\sigma^c}$ are orthonormal basis on their respective generated subspaces so in both cases their lower frame bound is 1.*

REFERENCES

[1] O. CHRISTENSEN, *A*n introduction to frames and Riesz bases, Birkhäuser, Boston, (2003).

[2] A.M.OLEVSKII AND A. ULANOVSKII, *Functions with Disconnected Spectrum. Sampling, Interpolation, Translates*, University Lecture Series, vol. 65 (American Mathematical Society, Providence, RI 2016).