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The Gelfand problem for the Infinity Laplacian†

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Abstract: We study the asymptotic behavior as $p \to \infty$ of the Gelfand problem

$$\begin{cases} -\Delta_p u = \lambda\, e^u & \text{in } \Omega \subset \mathbb{R}^n \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Under an appropriate rescaling on u and λ, we prove uniform convergence of solutions of the Gelfand problem to solutions of

$$\begin{cases} \min\left\{|\nabla u| - \Lambda\, e^u, -\Delta_\infty u\right\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We discuss existence, non-existence, and multiplicity of solutions of the limit problem in terms of Λ.

Keywords: fully nonlinear; infinity Laplacian; Gelfand problem

Dedicated to the memory of Ireneo Peral, with love and admiration.

1. Introduction

We are interested in the asymptotic behavior as $p \to \infty$ of sequences of solutions of the problem

$$\begin{cases} -\Delta_p u = \lambda\, e^u & \text{in } \Omega \subset \mathbb{R}^n \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In the case $p = 2$, problem (1.1) is known as the Liouville-Bratu-Gelfand problem [5, 22, 37]; see also [15,26]. It appears in connection with prescribed Gaussian curvature problems [9,37], emission of electricity from hot bodies [40], and the equilibrium of gas spheres and the structure of stars [8,17,42]. Problem (1.1) with $p = 2$ was also studied by Barenblatt in relation to combustion theory in a volume edited by Gelfand [22]. For general p, problem (1.1) is often known in the literature as the "Gelfand problem" or a "Gelfand-type problem". It was studied by García-Azorero, Peral, and Puel in [20,21]; see also [7,24,41] and the references therein.

The asymptotic study of p-Laplacian problems as $p \to \infty$ offers a qualitative and quantitative understanding of their solution sets for large p, see [4, 10–13, 18, 30]. Additionally, they have been used in [23] to obtain optimal bounds for the diameter of manifolds in terms of their curvature.

In [4, 10–13, 18, 30], the authors study limits of p-Laplacian equations with power-type right-hand sides and combinations of these. In all these cases, the parameter λ is allowed to vary with p in order to get nontrivial limits of sequences $\{u_{\lambda,p}\}_p$ of solutions to the corresponding p-Laplacian problem; namely,

$$\lambda_p^{1/p} \to \Lambda \quad \text{and} \quad u_{\lambda_p,p} \to u \quad \text{as } p \to \infty.$$

With an exponential right-hand side, the solution sets change more drastically as $p \to \infty$ and more severe rescalings become necessary. To take limits in (1.1), we consider

$$\begin{cases} -\Delta_p u_{\lambda_p,p} = \lambda_p \, e^{u_{\lambda_p,p}} & \text{in } \Omega \\ \quad u_{\lambda_p,p} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with the rescaling

$$\frac{\lambda_p^{1/p}}{p} \to \Lambda \quad \text{as } p \to \infty. \tag{1.3}$$

Under this normalization, we prove that any uniform limit

$$\frac{u_{\lambda_p,p}}{p} \to u \quad \text{as } p \to \infty \tag{1.4}$$

is a viscosity solution of the limit problem

$$\begin{cases} \min\{|\nabla u| - \Lambda\, e^u, -\Delta_\infty u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

It is worth noting that in [38], the authors consider problem (1.1) without the rescalings (1.3) and (1.4). They obtain that, regardless of λ, the solutions u_p converge uniformly as $p \to \infty$ to the unique viscosity solution of

$$\begin{cases} \min\{|\nabla u| - 1, -\Delta_\infty u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is the distance function to the boundary of the domain. As the authors of [38] acknowledge in their paper, this result is not unexpected since for each nonnegative function $f \in L^\infty(\Omega)\backslash\{0\}$, the sequence of unique solutions of

$$\begin{cases} -\Delta_p v_p = f(x) & \text{in } \Omega \\ \quad v_p = 0 & \text{on } \partial\Omega, \end{cases}$$

converges uniformly in Ω to the distance function to the boundary of the domain; see [3, 27, 31]. This highlights a critical feature of these problems, a precise scaling between u and λ that balances reaction and diffusion and produces a nontrivial limit problem.

Therefore, in this paper, we prove passage to the limit of the sequence of minimal solutions of problem (1.2) under the rescaling (1.3), (1.4). Furthermore, we show that the resulting limit is a minimal solution of (1.5). Note that the fact that the limit solution is minimal is nontrivial; in principle, limit and minimal solutions could differ. To prove this, we use a comparison principle for "small solutions" of problem (1.5), which we prove in Section 4. As it turns out, minimal solutions to problem (1.5) are "small" in the sense of this comparison principle. To the best of our knowledge, no corresponding comparison and uniqueness results for small solutions were known in the literature for $p < \infty$.

In Section 8, we find a second solution to the limit problem (1.5) under certain geometric assumptions on the domain Ω. Furthermore, we show that both solutions lie on an explicit curve of solutions. Some examples of domains satisfying the geometric condition are the ball, the annulus, and the stadium (convex hull of two balls of the same radius); squares or ellipses do not verify the condition. We conjecture that this second solution is a limit of appropriately rescaled mountain-pass solutions of (1.2).

The paper is organized as follows. In Section 2, we provide some necessary preliminaries, and Section 3 formally introduces the limit problem. We have chosen to introduce the limit problem before proving any convergence results to streamline the presentation. In Section 4, we prove the comparison principle for small solutions of the limit equation (1.5). Section 5 concerns non-existence of solutions to (1.5) for large values of Λ. In Section 6, we find a branch of minimal solutions to (1.5) up to a maximal Λ. Section 7 discusses uniform convergence as $p \to \infty$ of p-minimal solutions to minimal solutions of (1.5). Finally, in Section 8, we show the multiplicity result and exhibit a curve of explicit solutions under a geometric condition on the domain.

2. Preliminaries

In this section, we state some necessary preliminaries and notation. First, let us recall that weak solutions of problem (1.1) are also viscosity solutions. The proof, which we omit here, follows [30, Lemma 1.8]; see also [3].

Lemma 2.1. *If u is a continuous weak solution of* (1.1)*, then it is a viscosity solution of the same problem, rewritten as*

$$\begin{cases} F_p(\nabla u, D^2 u) = \lambda\, e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$F_p(\xi, X) = -|\xi|^{p-2} \cdot \operatorname{trace}\left(\left(I + (p-2)\frac{\xi \otimes \xi}{|\xi|^2}\right)X\right). \tag{2.1}$$

The divergence form of the p-Laplacian, i.e., $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, is better suited for variational techniques, while the expanded form (2.1) is preferable in the viscosity framework. In the sequel, we will always consider the most suitable form without further mention.

In [31] the problem

$$\begin{cases} -\Delta_p v_p = 1 & \text{in } \Omega \\ v_p \in W_0^{1,p}(\Omega) \end{cases}$$

is studied in connection with torsional creep problems when Ω is a general bounded domain. Since we are interested in the case $p \to \infty$, we can assume $p > n$ without loss of generality. Then every function in $v_p \in W_0^{1,p}(\Omega)$ can be considered continuous in $\overline{\Omega}$ and 0 on the boundary in the classical sense. The existence result we will need below is the following. We refer the interested reader to [31] and [27, Theorem 3.11 and Remark 4.23] for the proof.

Proposition 2.2. *Let Ω be a bounded domain and $n < p < \infty$. Then, there exists a unique solution $v_p \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ of the p-torsion problem*

$$\begin{cases} -\Delta_p v_p = 1 & \textit{in } \Omega \\ \quad v_p = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.2}$$

and v_p converge uniformly as $p \to \infty$ to the unique viscosity solution to

$$\begin{cases} \min\{|\nabla v| - 1, -\Delta_\infty v\} = 0 & \textit{in } \Omega, \\ v = 0 & \textit{on } \partial\Omega. \end{cases}$$

Moreover, $v(x) = \operatorname{dist}(x, \partial\Omega)$.

The uniqueness of the solution in Proposition 2.2 follows from the following comparison principle.

Lemma 2.3. *Let $f : \Omega \to \mathbb{R}$ be a continuous, bounded, and positive function. Suppose that $u, v : \overline{\Omega} \to \mathbb{R}$ are bounded, u is upper semicontinuous and v is lower semicontinuous in $\overline{\Omega}$. If u and v are, respectively, a viscosity sub- and supersolution of*

$$\min\{|\nabla w| - f(x), -\Delta_\infty w\} = 0 \quad \textit{in } \Omega,$$

and $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.

We refer the interested reader for instance to [27, Theorem 4.18 and Remark 4.23] and also [25, Theorem 2.1] (for the proof of [27, Theorem 4.18], notice that every ∞-superharmonic function is Lipschitz continuous, see [35]).

We also need some facts about first eigenvalues and eigenfunctions of the p-Laplacian. Let us recall that the first eigenvalue $\lambda_1(p; \Omega)$ is characterized by the nonlinear Rayleigh quotient

$$\lambda_1(p; \Omega) = \inf_{\phi \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla \phi|^p \, dx}{\int_\Omega |\phi|^p \, dx}.$$

In [32] (see also [33]), it is proved that the first eigenvalue of the p-Laplacian is simple (that is, the first eigenfunction is unique up to multiplication by constants) when Ω is a bounded domain; see also [2, 19, 39] and the references in [32]. Moreover, it is also proved in [32] that in a bounded domain, only the first eigenfunction is positive and that the first eigenvalue is isolated (there exists $\epsilon > 0$ such that there are no eigenvalues in $(\lambda_1, \lambda_1 + \epsilon]$).

Proposition 2.4 ([32]). *Let Ω be a bounded domain and $n < p < \infty$. Then, there exists a solution $\psi_p \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ of*

$$\begin{cases} -\Delta_p \psi_p = \lambda_1(p;\Omega)\,|\psi_p|^{p-2}\psi_p & \text{in } \Omega \\ \psi_p = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\lambda_1(p;\Omega)$ is simple and isolated.

Lastly, we recall the behavior as $p \to \infty$ of the first eigenvalue of the p-Laplacian, see [30] for the proof.

Lemma 2.5. $\lim_{p\to\infty} \lambda_1(p,\Omega)^{1/p} = \Lambda_1(\Omega) = \|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty^{-1}$.

We denote the first ∞-eigenvalue by $\Lambda_1(\Omega)$, see [30].

3. The limit problem

In the present section, we characterize uniform limits of appropriate rescalings of solutions of (1.2) as solutions of a PDE. See [4, 10–13, 18, 30] for related results.

Proposition 3.1. *Consider a sequence $\{(\lambda_p, u_{\lambda_p,p})\}_p$ of solutions of* (1.2) *and assume*

$$\lim_{p\to\infty} \frac{\lambda_p^{1/p}}{p} = \Lambda.$$

Then, any uniform limit

$$u_\Lambda = \lim_{p\to\infty} \frac{u_{\lambda_p,p}}{p}$$

is a viscosity solution of the problem

$$\begin{cases} \min\left\{|\nabla u| - \Lambda\, e^u, -\Delta_\infty u\right\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Proof. Consider a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\Lambda - \phi$ has a strict local minimum at x_0. As u_Λ is the uniform limit of $u_{\lambda_p,p}/p$, there exists a sequence of points $x_p \to x_0$ such that $u_{\lambda_p,p} - p\,\phi$ attains a local minimum at x_p for each p. As $u_{\lambda_p,p}$ is a continuous weak solution of (1.2), it is also a viscosity solution and a supersolution. Then, we get

$$\begin{aligned} -(p-2)\,p^{p-1}\,|\nabla\phi(x_p)|^{p-4}\left\{\frac{|\nabla\phi(x_p)|^2}{p-2}\Delta\phi(x_p) + \langle D^2\phi(x_p)\nabla\phi(x_p), \nabla\phi(x_p)\rangle\right\} \\ = -p^{p-1}\Delta_p\phi(x_p) \geqslant \lambda_p\, e^{u_{\lambda,p}(x_p)}. \end{aligned}$$

Rearranging terms, we obtain

$$-(p-2)\left[\frac{|\nabla\phi(x_p)|}{\left(\frac{\lambda_p}{p^{p-1}}\, e^{u_{\lambda,p}(x_p)}\right)^{\frac{1}{p-4}}}\right]^{p-4}\left\{\frac{|\nabla\phi(x_p)|^2}{p-2}\Delta\phi(x_p) + \langle D^2\phi(x_p)\nabla\phi(x_p), \nabla\phi(x_p)\rangle\right\} \geqslant 1.$$

If we suppose that $|\nabla\phi(x_0)| < \Lambda e^{u_\Lambda(x_0)}$ we obtain a contradiction letting $p \to \infty$ in the previous inequality. Thus, it must be

$$|\nabla\phi(x_0)| - \Lambda e^{u_\Lambda(x_0)} \geqslant 0. \tag{3.2}$$

We also have that

$$-\Delta_\infty\phi(x_0) = -\langle D^2\phi(x_0)\nabla\phi(x_0), \nabla\phi(x_0)\rangle \geqslant 0, \tag{3.3}$$

because we would get a contradiction otherwise. Therefore, we can put together (3.2) and (3.3) writing

$$\min\left\{|\nabla\phi(x_0)| - \Lambda e^{u_\Lambda(x_0)}, -\Delta_\infty\phi(x_0)\right\} \geqslant 0,$$

and conclude that u_Λ is a viscosity supersolution of (3.1).

It remains to show that u_Λ is a viscosity subsolution of the limit equation (3.1). More precisely, we have to show that, for each $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u_\Lambda - \phi$ attains a strict local maximum at x_0 (note that x_0 and ϕ are not the same than before) we have

$$\min\left\{|\nabla\phi(x_0)| - \Lambda e^{u_\Lambda(x_0)}, -\Delta_\infty\phi(x_0)\right\} \leqslant 0.$$

We can suppose that

$$|\nabla\phi(x_0)| > \Lambda e^{u_\Lambda(x_0)},$$

since we are done otherwise. Again, the uniform convergence of $u_{\lambda,p}/p$ to u_Λ provides a sequence of points $x_p \to x_0$ which are local maxima of $u_{\lambda,p} - p\,\phi$. Recalling the definition of viscosity subsolution we have

$$-(p-2)\left[\frac{|\nabla\phi(x_p)|}{\left(\frac{\lambda_p}{p^{p-1}}e^{u_{\lambda,p}(x_p)}\right)^{\frac{1}{p-4}}}\right]^{p-4}\left\{\frac{|\nabla\phi(x_p)|^2}{p-2}\Delta\phi(x_p) + \langle D^2\phi(x_p)\nabla\phi(x_p), \nabla\phi(x_p)\rangle\right\} \leqslant 1,$$

for each p. Letting $p \to \infty$, we find $-\Delta_\infty\phi(x_0) \leqslant 0$, or else we get a contradiction. □

In the previous argument, the fact that $e^{u_\Lambda(x_0)}$ is strictly positive independently of the value of $u_\Lambda(x_0)$ makes a difference with the case with a power-type right-hand side (see [10–13, 18, 30]), where one needs to make sure that $u_\Lambda > 0$ in Ω. Furthermore, in the power-type right-hand side case, one can consider sign-changing solutions, see [10, 29] and get a more involved limit equation that takes into account sign changes. In the next result, we show that all solutions to the limit problem (3.1) are positive. Moreover, we show that solutions cannot be arbitrarily small for every given Λ and must grow (at least) linearly from the boundary.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\Lambda > 0$. Then, every solution u_Λ of* (3.1) *verifies*

$$u_\Lambda \geqslant \Lambda\,\mathrm{dist}(\cdot, \partial\Omega) \quad \text{in } \Omega.$$

In particular, every solution of (3.1) *is strictly positive and satisfies the estimate*

$$\|u_\Lambda\|_{L^\infty(\Omega)} \geqslant \Lambda\Lambda_1(\Omega)^{-1}.$$

Proof. Let u_Λ be a solution of (3.1). Then, $u_\Lambda \geqslant 0$ in Ω by Lemma 2.3. Let us show that

$$\min\{|\nabla u_\Lambda| - \Lambda, -\Delta_\infty u_\Lambda\} \geqslant 0 \quad \text{in } \Omega$$

in the viscosity sense. To see this, consider $x_0 \in \Omega$ and $\phi \in C^2$ such that $u_\Lambda - \phi$ has a minimum at x_0. Since $u_\Lambda(x)$ is a solution of (3.1), we have

$$\min\left\{|\nabla\phi(x_0)| - \Lambda\, e^{u_\Lambda(x_0)}, -\Delta_\infty\phi(x_0)\right\} \geqslant 0 \quad \text{in } \Omega.$$

We deduce $-\Delta_\infty\phi(x_0) \geqslant 0$ and $|\nabla\phi(x_0)| \geqslant \Lambda\, e^{u_\Lambda(x_0)} \geqslant \Lambda$ and we get

$$\min\{|\nabla\phi(x_0)| - \Lambda, -\Delta_\infty\phi(x_0)\} \geqslant 0 \quad \text{in } \Omega$$

as desired.

On the other hand, $v_\Lambda(x) = \Lambda\,\mathrm{dist}(x, \partial\Omega)$ is the unique viscosity solution of

$$\min\{|\nabla v_\Lambda| - \Lambda, -\Delta_\infty v_\Lambda\} = 0 \quad \text{in } \Omega.$$

Then, one gets $u_\Lambda \geqslant v_\Lambda = \Lambda\,\mathrm{dist}(\cdot, \partial\Omega)$ by comparison, see Lemma 2.3. □

4. Comparison for small solutions of the limit problem

In this section, we prove a comparison principle for small solutions of the limit equation (1.5). This result is interesting for two main reasons. Firstly, Eq (1.5) is not proper in the terminology of [14], a basic requirement for comparison. Secondly, based on the multiplicity results for the p-Laplacian equation (1.2), see [20,21], one cannot expect comparison to hold in general. The key idea is a change of variables that allows us to obtain a proper equation for solutions with $\|u\|_\infty < 1$. Remarkably, minimal solutions of (1.5) verify this condition (see Section 6 below), and we can conclude they are the only ones with $\|u\|_\infty < 1$. The change of variables we use here is the same that was used to prove comparison for the limit problem with concave right-hand side in [12].

We prove a more general result with a "right-hand" side $f(u)$ that satisfies a hypothesis reminiscent of the celebrated Brezis-Oswald condition, see [6] and Remark 4.2 below.

Theorem 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which there exist $c \in (0, \infty]$ and $q \in (0, 1)$ such that*

$$\frac{f(t)}{t^q} \quad \textit{is positive and non-increasing for all } t \in (0, c). \tag{4.1}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u, v \in C(\overline{\Omega})$ with $\max\{\|u\|_\infty, \|v\|_\infty\} < c$ be, respectively, a positive viscosity sub- and supersolution of

$$\min\{|\nabla w| - f(w), -\Delta_\infty w\} = 0 \quad \textit{in } \Omega. \tag{4.2}$$

Then, whenever $u \leqslant v$ on $\partial\Omega$, we have $u \leqslant v$ in $\overline{\Omega}$.

Remark 4.2. It is possible to prove a comparison principle for Eq (4.2) under the Brezis-Oswald [6] condition

$$\frac{f(t)}{t} \quad \text{is decreasing for all } t > 0.$$

Under this condition, the power-type change of variables used in [12] and in the proof of Theorem 4.1 no longer applies. Instead, we need a logarithmic change of variables, similarly to the comparison principle for the eigenvalue problem for the infinity Laplacian in [30]. However, a viscosity comparison principle obtained through a logarithmic change of variables requires that either the sub- or the supersolution are strictly positive in $\overline{\Omega}$ and does not allow us to conclude uniqueness of solutions for the Dirichlet problem with homogeneous boundary data, which our result does.

Before going into the proof of Theorem 4.1, let us discuss an important consequence of Theorem 4.1, the uniqueness of "small" solutions of problem (3.1).

Corollary 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For every $\Lambda > 0$, the problem*

$$\begin{cases} \min\left\{|\nabla u| - \Lambda\, e^u, -\Delta_\infty u\right\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

has at most one viscosity solution with $\|u\|_\infty < 1$.

Proof of Corollary 4.3. Suppose for the sake of contradiction that there are two viscosity solutions, u, v of (4.3) with $\max\{\|u\|_\infty, \|v\|_\infty\} < 1$. Notice that both u and v are strictly positive in Ω by Proposition 3.2. In this case we have $f(t) = \Lambda\, e^t$ and (4.1) is satisfied with $c = q$ for every $q \in (0, 1)$. Then, we can choose $q \in (0, 1)$ such that $\max\{\|u\|_\infty, \|v\|_\infty\} < q < 1$, and all the hypotheses of Theorem 4.1 are satisfied. Because $u = v$ on $\partial\Omega$, we conclude $u \equiv v$. $\square$

We devote the rest of the section to the proof of Theorem 4.1. In the next lemma we apply a change of variables to Eq (4.2) to obtain a proper equation for small solutions.

Lemma 4.4. *Let $q \in (0, 1)$ and let v be a positive viscosity supersolution (respectively, subsolution) of* (4.2) *in Ω. Then, $\tilde{v}(x) = v^{1-q}(x)$ is a viscosity supersolution (subsolution) of*

$$\min\left\{|\nabla\tilde{w}(x)| - (1-q)\,\frac{f\left(\tilde{w}(x)^{\frac{1}{1-q}}\right)}{\tilde{w}(x)^{\frac{q}{1-q}}}, -\Delta_\infty\tilde{w}(x) - \frac{q}{1-q}\frac{|\nabla\tilde{w}(x)|^4}{\tilde{w}(x)}\right\} = 0 \tag{4.4}$$

in every subdomain U compactly contained in Ω.

Proof. Let $\tilde{\phi} \in C^2(\Omega)$ touch $\tilde{v}$ from below at $x_0 \in \Omega$. If we define $\phi(x) = \tilde{\phi}(x)^{\frac{1}{1-q}}$, then ϕ touches v from below at x_0. Note that $\phi(x)$ is C^2 in a neighborhood of x_0, since $v > 0$ in Ω implies $\tilde{\phi}(x) > 0$ around x_0. Then

$$\begin{aligned} \nabla\phi(x_0) &= \frac{1}{1-q}\,\tilde{\phi}(x_0)^{\frac{q}{1-q}}\,\nabla\tilde{\phi}(x_0), \\ D^2\phi(x_0) &= \frac{1}{1-q}\,\tilde{\phi}(x_0)^{\frac{q}{1-q}}D^2\tilde{\phi}(x_0) + \frac{q}{(1-q)^2}\,\tilde{\phi}(x_0)^{\frac{2q-1}{1-q}}\,\nabla\tilde{\phi}(x_0)\otimes\nabla\tilde{\phi}(x_0). \end{aligned}$$

Because v is a viscosity supersolution of (4.2) and $\phi(x_0) = v(x_0) > 0$, we have

$$0 \;\leqslant\; \min\left\{|\nabla\phi(x_0)| - f\big(\phi(x_0)\big), -\langle D^2\phi(x_0)\nabla\phi(x_0), \nabla\phi(x_0)\rangle\right\}$$

$$
= \min\left\{ \frac{1}{1-q}\,\tilde\phi(x_0)^{\frac{q}{1-q}} \left(|\nabla\tilde\phi(x_0)| - (1-q)\,\frac{f\left(\tilde\phi(x_0)^{\frac{1}{1-q}}\right)}{\tilde\phi(x_0)^{\frac{q}{1-q}}} \right), \right.
$$

$$
\left. -\left(\frac{1}{1-q}\,\tilde\phi(x_0)^{\frac{q}{1-q}}\right)^3 \left(\Delta_\infty\tilde\phi(x_0) + \frac{q}{1-q}\,\frac{|\nabla\tilde\phi(x_0)|^4}{\tilde\phi(x_0)} \right) \right\}.
$$

Therefore,

$$
\min\left\{ |\nabla\tilde\phi(x_0)| - (1-q)\,\frac{f\left(\tilde\phi(x_0)^{\frac{1}{1-q}}\right)}{\tilde\phi(x_0)^{\frac{q}{1-q}}}, -\Delta_\infty\tilde\phi(x_0) - \frac{q}{1-q}\,\frac{|\nabla\tilde\phi(x_0)|^4}{\tilde\phi(x_0)} \right\} \geqslant 0,
$$

that is, $\tilde v$ is a viscosity supersolution of (4.4). The subsolution case is analogous. $\square$

Equation (4.4) is given by the functional

$$
\begin{aligned}
\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^n &\longrightarrow \mathbb{R} \\
(t, p, X) &\longrightarrow \min\left\{ |p| - (1-q) f\left(t^{\frac{1}{1-q}}\right) t^{-\frac{q}{1-q}}, -\langle Xp, p\rangle - \frac{q}{1-q}\,\frac{|p|^4}{t} \right\},
\end{aligned}
$$

which is degenerate elliptic and non-decreasing in t for $0 < t < c^{1-q}$ by hypothesis (4.1). Under these conditions, it is well-known (see [14, Section 5.C]) that it is possible to establish a comparison principle when the supersolution or the subsolution are strict. In the next lemma we show that we can find a perturbation of the supersolution that is a strict supersolution, see [12, 27, 30] for related constructions.

Lemma 4.5. *Consider a subdomain U compactly contained in Ω, and $q \in (0, 1)$, $c > 0$ as in* (4.1). *Let $\tilde v > 0$ with $\|\tilde v\|_\infty < c^{1-q}$ be a viscosity supersolution of* (4.4) *in U. Define*

$$
\tilde v_\epsilon(x) = (1+\epsilon)\big(\tilde v(x) + \epsilon\big). \tag{4.5}
$$

Then, $\tilde v_\epsilon \to \tilde v$ uniformly in $\overline{U}$ as $\epsilon \to 0$, and for every $\epsilon > 0$ small enough, there exists a positive constant $C = C(\epsilon, q, \|\tilde v\|_\infty)$ such that

$$
\min\left\{ |\nabla\tilde v_\epsilon(x)| - (1-q)\,\frac{f\left(\tilde v_\epsilon(x)^{\frac{1}{1-q}}\right)}{\tilde v_\epsilon(x)^{\frac{q}{1-q}}}, -\Delta_\infty\tilde v_\epsilon(x) - \frac{q}{1-q}\,\frac{|\nabla\tilde v_\epsilon(x)|^4}{\tilde v_\epsilon(x)} \right\} \geqslant C > 0 \quad \text{in } U, \tag{4.6}
$$

in the viscosity sense, that is, $\tilde v_\epsilon$ is a strict viscosity supersolution of (4.4) *in U with $\|\tilde v_\epsilon\|_\infty < c^{1-q}$.*

Proof. Let $\tilde\phi_\epsilon \in C^2$ touch $\tilde v_\epsilon(x)$ from below at $x_0 \in U$. Define

$$
\tilde\phi(x) = \frac{1}{1+\epsilon}\,\tilde\phi_\epsilon(x) - \epsilon,
$$

which clearly touches $\tilde v(x)$ from below at x_0. Then,

$$
\nabla\tilde\phi(x_0) = (1+\epsilon)^{-1}\,\nabla\tilde\phi_\epsilon(x_0) \quad \text{and} \quad D^2\tilde\phi(x_0) = (1+\epsilon)^{-1}\,D^2\tilde\phi_\epsilon(x_0). \tag{4.7}
$$

Since $\tilde{v}(x)$ is a viscosity supersolution of (4.4) in U, we deduce

$$|\nabla\tilde{\phi}(x_0)| \geqslant (1-q)\,\frac{f\left(\tilde{v}(x_0)^{\frac{1}{1-q}}\right)}{\tilde{v}(x_0)^{\frac{q}{1-q}}}, \tag{4.8}$$

and

$$-\left\langle D^2\tilde{\phi}(x_0)\nabla\tilde{\phi}(x_0), \nabla\tilde{\phi}(x_0)\right\rangle - \frac{q}{1-q}\,\frac{|\nabla\tilde{\phi}(x_0)|^4}{\tilde{v}(x_0)} \geqslant 0. \tag{4.9}$$

In the sequel we assume ϵ small enough so that $\|\tilde{v}\|_\infty < \|\tilde{v}_\epsilon\|_\infty = (1+\epsilon)\big(\|\tilde{v}\|_\infty + \epsilon\big) < c^{1-q}$. Then, from (4.1), (4.5), (4.7) and (4.8), we obtain

$$\begin{aligned}
&|\nabla\tilde{\phi}_\epsilon(x_0)| - (1-q)\,\frac{f\left(\tilde{v}_\epsilon(x_0)^{\frac{1}{1-q}}\right)}{\tilde{v}_\epsilon(x_0)^{\frac{q}{1-q}}}\\
&\qquad \geqslant \epsilon\,(1-q)\,\frac{f\left(\tilde{v}(x_0)^{\frac{1}{1-q}}\right)}{\tilde{v}(x_0)^{\frac{q}{1-q}}} + (1-q)\,\left(\frac{f\left(\tilde{v}(x_0)^{\frac{1}{1-q}}\right)}{\tilde{v}(x_0)^{\frac{q}{1-q}}} - \frac{f\left(\tilde{v}_\epsilon(x_0)^{\frac{1}{1-q}}\right)}{\tilde{v}_\epsilon(x_0)^{\frac{q}{1-q}}}\right)\\
&\qquad \geqslant \epsilon\,(1-q)\,\frac{f\left(\|\tilde{v}\|_\infty^{\frac{1}{1-q}}\right)}{\|\tilde{v}\|_\infty^{\frac{q}{1-q}}}.
\end{aligned} \tag{4.10}$$

Similarly, from (4.1), (4.5), (4.7), (4.8), and (4.9) we arrive at

$$\begin{aligned}
&-\left\langle D^2\tilde{\phi}_\epsilon(x_0)\nabla\tilde{\phi}_\epsilon(x_0), \nabla\tilde{\phi}_\epsilon(x_0)\right\rangle - \frac{q}{1-q}\,\frac{|\nabla\tilde{\phi}_\epsilon(x_0)|^4}{\tilde{v}_\epsilon(x_0)}\\
&\geqslant (1+\epsilon)^3\frac{q}{1-q}\left(\frac{1}{\tilde{v}(x_0)} - \frac{1}{\tilde{v}(x_0)+\epsilon}\right)|\nabla\tilde{\phi}(x_0)|^4\\
&\geqslant \frac{\epsilon(1+\epsilon)^3 q(1-q)^3}{\|\tilde{v}\|_\infty\left(\|\tilde{v}\|_\infty+\epsilon\right)}\left(\frac{f\left(\tilde{v}(x_0)^{\frac{1}{1-q}}\right)}{\tilde{v}(x_0)^{\frac{q}{1-q}}}\right)^4 \geqslant \frac{\epsilon(1+\epsilon)^3 q(1-q)^3}{\|\tilde{v}\|_\infty\left(\|\tilde{v}\|_\infty+\epsilon\right)}\left(\frac{f\left(\|\tilde{v}\|_\infty^{\frac{1}{1-q}}\right)}{\|\tilde{v}\|_\infty^{\frac{q}{1-q}}}\right)^4.
\end{aligned} \tag{4.11}$$

Finally, we get (4.6) from (4.10) and (4.11) as desired, which concludes the proof. □

Proof of Theorem 4.1. Since $u-v \in C(\overline{\Omega})$ and $\overline{\Omega}$ is compact, $u-v$ attains its maximum in $\overline{\Omega}$. Suppose, for the sake of contradiction, that $\max_{\overline{\Omega}}(u-v) > 0$. Let

$$\tilde{u}(x) = u(x)^{1-q}, \qquad \tilde{v}(x) = v(x)^{1-q},$$

and define $\tilde{v}_\epsilon(x)$ as in (4.5). Notice that $u - v \leqslant 0$ on $\partial\Omega$ gives

$$\tilde{u} - \tilde{v}_\epsilon = \tilde{u} - (1+\epsilon)\,\tilde{v} - (1+\epsilon)\,\epsilon < 0 \quad \text{on } \partial\Omega.$$

Moreover, by uniform convergence, we have $\max_{\overline{\Omega}}(\tilde{u} - \tilde{v}_\epsilon) > 0$ for ϵ small enough. Therefore, we can fix $\epsilon > 0$ small as in Lemma 4.5 for the rest of the proof and assume there exists U compactly contained in Ω that contains all maximum points of $\tilde{u}-\tilde{v}_\epsilon$. We have proved in Lemmas 4.4 and 4.5 that $\tilde{u}$ and $\tilde{v}_\epsilon$ are, respectively, a viscosity subsolution and strict supersolution of (4.4) in U.

For every $\tau > 0$, let (x_τ, y_τ) be a maximum point of $\tilde{u}(x)-\tilde{v}_\epsilon(y)-\frac{\tau}{2}|x-y|^2$ in $\overline{\Omega}\times\overline{\Omega}$. By the compactness of $\overline{\Omega}$, we can assume that $x_\tau \to \hat{x}$ as $\tau\to\infty$ for some $\hat{x}\in\overline{\Omega}$ (notice that also $y_\tau\to\hat{x}$). Then, [14, Proposition 3.7] implies that $\hat{x}$ is a maximum point of $\tilde{u}-\tilde{v}_\epsilon$ and, therefore, an interior point of U. We also have that

$$\lim_{\tau\to\infty}\left(\tilde{u}(x_\tau)-\tilde{v}_\epsilon(y_\tau)-\frac{\tau}{2}|x_\tau-y_\tau|^2\right)=\tilde{u}(\hat{x})-\tilde{v}_\epsilon(\hat{x})>0,$$

and, consequently, both x_τ and y_τ are interior points of U for τ large enough and

$$\tilde{u}(x_\tau)-\tilde{v}_\epsilon(y_\tau)-\frac{\tau}{2}|x_\tau-y_\tau|^2>0. \tag{4.12}$$

The definition of viscosity solution and the maximum principle for semicontinuous functions, see [14], imply that there exist symmetric matrices X_τ, Y_τ with $X_\tau\leqslant Y_\tau$ such that

$$\min\left\{\tau\,|x_\tau-y_\tau|-(1-q)\,\frac{f\left(\tilde{u}(x_\tau)^{\frac{1}{1-q}}\right)}{\tilde{u}(x_\tau)^{\frac{q}{1-q}}},-\tau^2\langle X_\tau(x_\tau-y_\tau),(x_\tau-y_\tau)\rangle-\frac{q}{1-q}\,\frac{\tau^4|x_\tau-y_\tau|^4}{\tilde{u}(x_\tau)}\right\}\leqslant 0,$$

and

$$\begin{aligned}\min\Bigg\{&\tau\,|x_\tau-y_\tau|-(1-q)\,\frac{f\left(\tilde{v}_\epsilon(y_\tau)^{\frac{1}{1-q}}\right)}{\tilde{v}_\epsilon(y_\tau)^{\frac{q}{1-q}}},\\&-\tau^2\langle Y_\tau(x_\tau-y_\tau),(x_\tau-y_\tau)\rangle-\frac{q}{1-q}\,\frac{\tau^4|x_\tau-y_\tau|^4}{\tilde{v}_\epsilon(y_\tau)}\Bigg\}\geqslant C(\epsilon,q,\|\tilde{v}\|_\infty)>0.\end{aligned}$$

Subtracting both equations, we get

$$\begin{aligned}0<C(\epsilon,q,\|\tilde{v}\|_\infty)\leqslant\min\Bigg\{&\tau\,|x_\tau-y_\tau|-(1-q)\,\frac{f\left(\tilde{v}_\epsilon(y_\tau)^{\frac{1}{1-q}}\right)}{\tilde{v}_\epsilon(y_\tau)^{\frac{q}{1-q}}},\\&-\tau^2\langle Y_\tau(x_\tau-y_\tau),(x_\tau-y_\tau)\rangle-\frac{q}{1-q}\,\frac{\tau^4|x_\tau-y_\tau|^4}{\tilde{v}_\epsilon(y_\tau)}\Bigg\} \qquad (4.13)\\-\min\Bigg\{&\tau\,|x_\tau-y_\tau|-(1-q)\,\frac{f\left(\tilde{u}(x_\tau)^{\frac{1}{1-q}}\right)}{\tilde{u}(x_\tau)^{\frac{q}{1-q}}},\\&-\tau^2\langle X_\tau(x_\tau-y_\tau),(x_\tau-y_\tau)\rangle-\frac{q}{1-q}\,\frac{\tau^4|x_\tau-y_\tau|^4}{\tilde{u}(x_\tau)}\Bigg\}. \qquad (4.14)\end{aligned}$$

We consider four cases, depending on the values where the minima in (4.13) and (4.14) are attained. In all cases we obtain a contradiction using that $X_\tau\leqslant Y_\tau$ and $\tilde{v}_\epsilon(y_\tau)\leqslant\tilde{u}(x_\tau)$, which follows from (4.12).

1) Both minima are attained by the first terms and (4.1) implies a contradiction, i.e.,

$$0 < C(\epsilon, q, \|\tilde{v}\|_\infty) \leqslant (1-q)\left(\frac{f\left(\tilde{u}(x_\tau)^{\frac{1}{1-q}}\right)}{\tilde{u}(x_\tau)^{\frac{q}{1-q}}} - \frac{f\left(\tilde{v}_\epsilon(y_\tau)^{\frac{1}{1-q}}\right)}{\tilde{v}_\epsilon(y_\tau)^{\frac{q}{1-q}}}\right) \leqslant 0.$$

2) Both minima are attained by the second terms. Then,

$$\begin{aligned} 0 &< C(\epsilon, q, \|\tilde{v}\|_\infty) \\ &\leqslant -\tau^2\Big\langle (Y_\tau - X_\tau)(x_\tau - y_\tau), (x_\tau - y_\tau)\Big\rangle + \frac{q}{1-q}\,\tau^4|x_\tau - y_\tau|^4\left(\frac{1}{\tilde{u}(x_\tau)} - \frac{1}{\tilde{v}_\epsilon(y_\tau)}\right) \leqslant 0, \end{aligned}$$

a contradiction.

3) The minima in (4.13) and (4.14) are attained by the second and first term, respectively. This case can be reduced to case (1) above and we again obtain a contradiction. Namely,

$$\begin{aligned} 0 < C(\epsilon, q, \|\tilde{v}\|_\infty) &\leqslant -\tau^2\big\langle Y_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\big\rangle - \frac{q}{1-q}\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{v}_\epsilon(y_\tau)} \\ &\quad - \tau\,|x_\tau - y_\tau| + (1-q)\frac{f\left(\tilde{u}(x_\tau)^{\frac{1}{1-q}}\right)}{\tilde{u}(x_\tau)^{\frac{q}{1-q}}} \\ &\leqslant (1-q)\left(\frac{f\left(\tilde{u}(x_\tau)^{\frac{1}{1-q}}\right)}{\tilde{u}(x_\tau)^{\frac{q}{1-q}}} - \frac{f\left(\tilde{v}_\epsilon(y_\tau)^{\frac{1}{1-q}}\right)}{\tilde{v}_\epsilon(y_\tau)^{\frac{q}{1-q}}}\right) \leqslant 0. \end{aligned}$$

4) Finally, if the minima in (4.13) and (4.14) are respectively attained by the first and second term, we obtain a contradiction as in case (2) above, i.e.,

$$\begin{aligned} 0 < C(\epsilon, q, \|\tilde{v}\|_\infty) &\leqslant \tau\,|x_\tau - y_\tau| - (1-q)\frac{f\left(\tilde{v}_\epsilon(y_\tau)^{\frac{1}{1-q}}\right)}{\tilde{v}_\epsilon(y_\tau)^{\frac{q}{1-q}}} \\ &\quad + \tau^2\big\langle X_\tau(x_\tau - y_\tau), (x_\tau - y_\tau)\big\rangle + \frac{q}{1-q}\frac{\tau^4|x_\tau - y_\tau|^4}{\tilde{u}(x_\tau)} \\ &\leqslant -\tau^2\Big\langle (Y_\tau - X_\tau)(x_\tau - y_\tau), (x_\tau - y_\tau)\Big\rangle + \frac{q}{1-q}\,\tau^4|x_\tau - y_\tau|^4\left(\frac{1}{\tilde{u}(x_\tau)} - \frac{1}{\tilde{v}_\epsilon(y_\tau)}\right) \leqslant 0. \end{aligned}$$

Since all the alternatives lead to a contradiction, the proof is complete. □

5. Non-existence of solutions with large Λ for the limit problem

We show here that due to the structure of the limit problem (3.1), there exists a threshold $\Lambda_{\max}$ beyond which the problem has no solutions.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Problem* (3.1) *has no solutions for $\Lambda > \Lambda_{\max}$, where*

$$\Lambda_{\max} = e^{-1}\Lambda_1(\Omega), \tag{5.1}$$

and $\Lambda_1(\Omega) = \|\mathrm{dist}(\cdot, \partial\Omega)\|_\infty^{-1}$ is the first ∞-eigenvalue, see [30].

Proof. Define $\mu = \Lambda_1(\Omega) + \epsilon$ with $\epsilon > 0$. Suppose for contradiction that problem (3.1) has a solution u_Λ for some $\Lambda > e^{-1}\mu$.

First we are going to use this u_Λ to construct a supersolution to the eigenvalue problem with parameter μ. More precisely, we are going to show that

$$\min\left\{|\nabla u_\Lambda| - \mu\, u_\Lambda, -\Delta_\infty u_\Lambda\right\} \geqslant 0 \quad \text{in } \Omega \tag{5.2}$$

in the viscosity sense. To this aim, let $x_0 \in \Omega$ and $\phi \in C^2$ such that $u_\Lambda - \phi$ has a minimum in x_0. Since $u_\Lambda(x)$ is a solution of problem (3.1) we have

$$\min\left\{|\nabla\phi(x_0)| - \Lambda\, e^{u_\Lambda(x_0)}, -\Delta_\infty\phi(x_0)\right\} \geqslant 0 \quad \text{in } \Omega.$$

We deduce that $-\Delta_\infty\phi(x_0) \geqslant 0$ and $|\nabla\phi(x_0)| \geqslant \Lambda\, e^{u_\Lambda(x_0)}$. Hence,

$$|\nabla\phi(x_0)| - \mu\, u_\Lambda(x_0) \geqslant \Lambda\, e^{u_\Lambda(x_0)} - \mu\, u_\Lambda(x_0).$$

To deduce (5.2) it is enough to show that

$$\min_{t\in\mathbb{R}} \Phi_\Lambda(t) \geqslant 0 \quad \text{where} \quad \Phi_\Lambda(t) = \Lambda\, e^t - \mu\, t.$$

It is elementary to check that the function Φ_Λ is convex and has a unique minimum point at $t_{\min} = \log(\mu\,\Lambda^{-1})$. Notice that $\lim_{t\to\pm\infty}\Phi_\Lambda(t) = +\infty$, and hence $t_{\min}$ is a global minimum. Then, it is easy to check that $\Lambda > e^{-1}\mu$ implies $\Phi_\Lambda(t_{\min}) \geqslant 0$.

Next, we notice that any first ∞-eigenfunction is a subsolution of the eigenvalue problem with parameter μ. So, let v be a first $\infty-$eigenfunction, that is, a solution of

$$\begin{cases} \min\left\{|\nabla v| - \Lambda_1(\Omega)\, v, -\Delta_\infty v\right\} = 0 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

normalized in such a way that $\|v\|_\infty < e^{-1}$. Clearly, by definition of μ,

$$\min\left\{|\nabla v| - \mu\, v, -\Delta_\infty v\right\} \leqslant 0 \quad \text{in } \Omega.$$

Now, we have to show that u_Λ and v are ordered, namely, that $0 < v \leqslant u_\Lambda$ in Ω. Indeed, using that $\|v\|_\infty < e^{-1}$ and $\Lambda_1(\Omega) < \mu < \Lambda e$, it is easy to see that

$$\min\left\{|\nabla v| - \Lambda, -\Delta_\infty v\right\} \leqslant 0 \quad \text{in } \Omega,$$

and using that $e^{u_\Lambda(x)} \geqslant 1$ in Ω one gets

$$\min\left\{|\nabla u_\Lambda| - \Lambda, -\Delta_\infty u_\Lambda\right\} \geqslant 0 \quad \text{in } \Omega.$$

As $v = u_\Lambda = 0$ on $\partial\Omega$, we get $0 < v \leqslant u_\Lambda$ by comparison, see Lemma 2.3.

So far, we have a subsolution v and a supersolution u_Λ of the eigenvalue problem

$$\min\left\{|\nabla w| - \mu\, w, -\Delta_\infty w\right\} = 0 \quad \text{in } \Omega \tag{5.3}$$

which verify $0 < v \leqslant u_\Lambda$. Next we claim that it is possible to construct a solution of (5.3) iterating between v and u_Λ. The argument finishes noticing that we have constructed a positive $\infty-$eigenfunction associated to $\mu = \Lambda_1 + \epsilon$, which is a contradiction with the fact that Λ_1 is isolated (see [29, Theorem 8.1] and [30, Theorem 3.1]). Since the argument above works for every $\epsilon > 0$, we conclude that there is no solution of (3.1) for $\Lambda > \Lambda_{\max}$.

We conclude by proving the claim. First, define $w_1(x)$, viscosity solution of

$$\begin{cases} \min\left\{|\nabla w_1| - \mu\, v, -\Delta_\infty w_1\right\} = 0 & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

To prove that such a w_1 exists, notice that v is a subsolution of the problem and that u_Λ is a supersolution, since, from (5.2) and $v \leqslant u_\Lambda$ we deduce

$$\min\left\{|\nabla u_\Lambda| - \mu\, v, -\Delta_\infty u_\Lambda\right\} \geqslant 0.$$

Then, we can apply the comparison principle in Lemma 2.3 as above and apply the Perron method ([14, Theorem 4.1]), to get a unique w_1 such that

$$v \leqslant w_1 \leqslant u_\Lambda \quad \text{in } \Omega.$$

Then, we define w_2, the solution of

$$\begin{cases} \min\left\{|\nabla w_2| - \mu\, w_1, -\Delta_\infty w_2\right\} = 0 & \text{in } \Omega \\ w_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, w_1 is a subsolution and u_Λ is a supersolution, since

$$\min\left\{|\nabla w_1| - \mu\, v, -\Delta_\infty w_1\right\} = 0 \quad \Rightarrow \quad \min\left\{|\nabla w_1| - \mu\, w_1, -\Delta_\infty w_1\right\} \leqslant 0,$$

while

$$\min\left\{|\nabla u_\Lambda| - \mu\, u_\Lambda, -\Delta_\infty u_\Lambda\right\} \geqslant 0 \quad \Rightarrow \quad \min\left\{|\nabla u_\Lambda| - \mu\, w_1, -\Delta_\infty u_\Lambda\right\} \geqslant 0.$$

As $w_1 = u_\Lambda = 0$ on $\partial\Omega$, by comparison and the Perron method, we obtain that there exists a unique w_2 satisfying

$$v \leqslant w_1 \leqslant w_2 \leqslant u_\Lambda \quad \text{in } \Omega.$$

Iterating this procedure, we construct a non-decreasing sequence

$$v \leqslant w_1 \leqslant w_2 \leqslant \ldots \leqslant w_{k-1} \leqslant w_k \leqslant u_\Lambda$$

of solutions of

$$\begin{cases} \min\left\{|\nabla w_k| - \mu\, w_{k-1}, -\Delta_\infty w_k\right\} = 0 & \text{in } \Omega \\ w_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Notice that $\|w_k\|_\infty$ is uniformly bounded by construction. On the other hand, as $-\Delta_\infty w_k \geqslant 0$ in Ω, we have (see [34, 35] and also [28] for a related construction) that

$$|\nabla w_k(x)| \leqslant \frac{w_k(x)}{\operatorname{dist}(x, \partial\Omega)} \leqslant \frac{u_\Lambda(x)}{\operatorname{dist}(x, \partial\Omega)} \qquad \text{a.e. } x \in \Omega,$$

for all $k > 1$. From there, both $\|w_k\|_\infty$ and $\|\nabla w_k\|_\infty$ are uniformly bounded in compact subsets of Ω. We observe that v, u_Λ are barriers in $\partial\Omega$ for each w_k. Hence by the Ascoli-Arzela theorem and the monotonicity of the sequence $\{w_k\}$, the whole sequence converges uniformly in Ω to some $w \in C(\Omega)$ which verifies $w = 0$ on $\partial\Omega$. Then, we can take limits in the viscosity sense in (5.4) and obtain that the limit w is a viscosity solution of (5.3), which proves the claim. □

6. Existence of a branch of minimal solutions for the limit problem

In this section we show that for every $\Lambda \in (0, \Lambda_{\max}]$ there is a minimal solution of the problem

$$\begin{cases} \min\left\{|\nabla u| - \Lambda\, e^u, -\Delta_\infty u\right\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

The proof is based on the ideas in [20], although our construction is different in order to take advantage of Corollary 4.3, our result of uniqueness for small solutions (the construction in [20] would only allow us to conclude that the minimal solution satisfies $\|u\|_\infty \leqslant \|u_{\Lambda_{\max}}\|_\infty = 1$, and Corollary 4.3 requires a strict inequality).

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, problem* (6.1) *has a minimal solution u_Λ for every $\Lambda \in (0, \Lambda_{\max}]$, where $\Lambda_{\max}$ is given by* (5.1)*. Moreover,*

1) We have the estimate
$$\Lambda\,\mathrm{dist}(x, \partial\Omega) \leqslant u_\Lambda(x) \leqslant e\Lambda\,\mathrm{dist}(x, \partial\Omega).$$
In particular, $\|u_\Lambda\|_\infty \leqslant e\Lambda\Lambda_1(\Omega)^{-1} < 1$ for $\Lambda \in (0, \Lambda_{\max})$.

2) For every $\Lambda \in (0, \Lambda_{\max})$, u_Λ is the only solution of (6.1) *with $\|u\|_\infty < 1$.*

3) The branch of minimal solutions is a non-decreasing continuum, in the sense that if $0 < \Lambda < \Upsilon < \Lambda_{\max}$, then $u_\Lambda \leqslant u_\Upsilon$ and whenever $\Upsilon \to \Lambda \in (0, \Lambda_{\max})$, then $u_\Upsilon \to u_\Lambda$ uniformly.

Proof. 1) Let $\underline{u}$ and $\overline{u}$ be the unique viscosity solutions of

$$\begin{cases} \min\left\{|\nabla \underline{u}| - \Lambda, -\Delta_\infty \underline{u}\right\} = 0 & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{6.2}$$

and

$$\begin{cases} \min\left\{|\nabla \overline{u}| - e\Lambda, -\Delta_\infty \overline{u}\right\} = 0 & \text{in } \Omega \\ \overline{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

respectively. By Proposition 2.2, we have the explicit expressions

$$\underline{u}(x) = \Lambda\,\mathrm{dist}(x, \partial\Omega) \qquad \text{and} \qquad \overline{u}(x) = e\Lambda\,\mathrm{dist}(x, \partial\Omega) \tag{6.4}$$

and $\underline{u} \leqslant \overline{u}$ follows trivially (alternatively, this can be proved by comparison, Lemma 2.3, using that $\overline{u}$ is a viscosity supersolution of (6.2)).

2) Define now u_1, viscosity solution of

$$\begin{cases} \min\left\{|\nabla u_1| - \Lambda\, e^{\underline{u}}, -\Delta_\infty u_1\right\} = 0 & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Let us show that

$$\underline{u} \leqslant u_1 \leqslant \overline{u} \quad \text{in } \Omega. \tag{6.6}$$

First, we prove $u_1 \leqslant \overline{u}$. We aim to show that $\min\left\{|\nabla u_1| - e\Lambda, -\Delta_\infty u_1\right\} \leqslant 0$ in the viscosity sense and then apply comparison for Eq (6.3), see Lemma 2.3. Therefore, let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u_1 - \phi$ attains a local maximum at x_0. We can assume that $-\Delta_\infty \phi(x_0) > 0$ because we are done otherwise. Then, from (6.5), (6.4), and (5.1), we have

$$|\nabla\phi(x_0)| \leqslant \Lambda\, e^{\underline{u}(x_0)} \leqslant \Lambda\, e^{\Lambda_{\max}\Lambda_1(\Omega)^{-1}} < e\Lambda.$$

In order to show that $u_1 \geqslant \underline{u}$, we prove that $\min\left\{|\nabla u_1| - \Lambda, -\Delta_\infty u_1\right\} \geqslant 0$ in the viscosity sense and then proceed by comparison for Eq (6.2). Indeed, since u_1 is a supersolution of (6.5), we have $-\Delta_\infty u_1 \geqslant 0$ and $|\nabla u_1| \geqslant \Lambda\, e^{\underline{u}} \geqslant \Lambda$ in the viscosity sense, as desired.

3) For each $k \geqslant 0$, we define u_{k+1} as the viscosity solution of

$$\begin{cases} \min\left\{|\nabla u_{k+1}| - \Lambda\, e^{u_k}, -\Delta_\infty u_{k+1}\right\} = 0 & \text{in } \Omega \\ u_{k+1} = 0 & \text{on } \partial\Omega \end{cases} \tag{6.7}$$

with $u_0 = \underline{u}$ and u_1 given by (6.5). Let us show that for all $k \geqslant 0$

$$\underline{u} \leqslant u_k \leqslant u_{k+1} \leqslant \overline{u} \qquad \text{in } \Omega, \tag{6.8}$$

that is, the sequence $\{u_k\}_{k\geqslant 0}$ is non-decreasing and uniformly bounded.

We prove (6.8) by induction. First, notice that (6.6) proves the case when $k = 0$. Assume (6.8) holds true for $k - 1$ and let us prove that $u_k \leqslant u_{k+1}$. Since u_{k+1} is, by definition, a viscosity supersolution of (6.7), we have $-\Delta_\infty u_{k+1} \geqslant 0$ and $|\nabla u_{k+1}| \geqslant \Lambda\, e^{u_k} \geqslant \Lambda\, e^{u_{k-1}}$ in the viscosity sense by the induction hypothesis. Therefore, u_{k+1} is a viscosity solution of

$$\min\left\{|\nabla u_{k+1}| - \Lambda\, e^{u_{k-1}}, -\Delta_\infty u_{k+1}\right\} \geqslant 0 \qquad \text{in } \Omega.$$

By definition, we have $\min\left\{|\nabla u_k| - \Lambda\, e^{u_{k-1}}, -\Delta_\infty u_k\right\} = 0$ and $u_k \leqslant u_{k+1}$ follows by comparison, see Lemma 2.3 (notice that $e^{u_{k-1}}$ is bounded, positive, and continuous, since the ∞-superharmonicity of u_{k-1} imply its Lipschitz continuity, see [35]).

To prove that $u_{k+1} \leqslant \overline{u}$, we show that

$$\min\left\{|\nabla u_{k+1}| - e\Lambda, -\Delta_\infty u_{k+1}\right\} \leqslant 0 \qquad \text{in } \Omega$$

and use comparison for Eq (6.3) (see Lemma 2.3). Therefore, let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $u_{k+1} - \phi$ attains a local maximum at x_0. Assume that $-\Delta_\infty \phi(x_0) > 0$ since we are done otherwise. Then, from (6.7), (6.4), (5.1), and the induction hypothesis we get

$$|\nabla\phi(x_0)| \leqslant \Lambda\, e^{u_k(x_0)} \leqslant \Lambda\, e^{\|\overline{u}\|_\infty} \leqslant e\Lambda.$$

4) We have obtained a non-decreasing sequence $\{u_k\}_{k\geqslant 0}$, uniformly bounded by $\underline{u}$ and $\overline{u}$ given by (6.4). Therefore, we can pass to the limit in the viscosity sense in the same way as in Proposition 5.1 and get a viscosity solution u_Λ of problem (6.1) as intended. It is also clear that the solution u_Λ we just found is minimal for every $\Lambda \in (0, \Lambda_{\max}]$, because any solution of (6.1) could be taken as $\overline{u}$ in the iteration (note that the function u_Λ does not depend on $\overline{u}$). Moreover, by (6.4) and (6.8), we have

$$\Lambda \operatorname{dist}(x, \partial\Omega) \leqslant u_\Lambda(x) \leqslant e\Lambda \operatorname{dist}(x, \partial\Omega) \qquad \text{in } \Omega.$$

Therefore, by Corollary 4.3, for every $\Lambda \in (0, \Lambda_{\max})$, u_Λ is the only solution of (6.1) with $\|u\|_\infty < 1$.
5) Let us prove that the branch of minimal solutions is non-decreasing, i.e., $u_\Lambda \leqslant u_\Upsilon$ whenever $0 < \Lambda < \Upsilon < \Lambda_{\max}$. To this aim, let us just observe that we can repeat the above construction taking $\overline{u} = u_\Upsilon$ and keeping $\underline{u}(x) = \Lambda \operatorname{dist}(x, \partial\Omega)$ as before. In this way, we recover the minimal solution u_Λ with the estimate $u_\Lambda \leqslant u_\Upsilon < 1$.

We conclude by showing that the branch of minimal solutions is a continuum. Arguing again as in the proof of Proposition 5.1, we see that, for every $\Lambda \in (0, \Lambda_{\max})$, the uniform limits

$$\widehat{u}_\Lambda = \lim_{\Upsilon\to\Lambda^+} u_\Upsilon, \qquad \text{and} \qquad \breve{u}_\Lambda = \lim_{\Upsilon\to\Lambda^-} u_\Upsilon$$

are both viscosity solutions of (6.1) with $\max\left\{\|\widehat{u}_\Lambda\|_\infty, \|\breve{u}_\Lambda\|_\infty\right\} < 1$. Therefore $\widehat{u}_\Lambda \equiv \breve{u}_\Lambda$ by Corollary 4.3, as desired. □

7. Minimal solutions achieved as limits of p-minimal solutions as $p \to \infty$

This section shows that uniform limits of appropriately scaled, minimal solutions of

$$\begin{cases} -\Delta_p u = \lambda\, e^u & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{7.1}$$

converge to the minimal solutions of the limit problem (6.1), found in Section 6. Observe that the fact that the limit solution is minimal is nontrivial; in principle, a limit solution could be different from the minimal one. Here is where we use the uniqueness results from Section 4. We prove the following.

Theorem 7.1. *Let $\Lambda \in (0, \Lambda_{\max})$, and $\{\lambda_p\}_p$ be a sequence such that*

$$\lim_{p\to\infty} \frac{\lambda_p^{1/p}}{p} = \Lambda.$$

For each λ_p, consider $u_{\lambda_p,p}$, the minimal solution of (7.1) *for $\lambda = \lambda_p$. Then,*

$$\frac{u_{\lambda_p,p}}{p} \to u_\Lambda, \quad \textit{uniformly as } p \to \infty,$$

where u_Λ is the minimal solution of the limit problem (6.1).

We devote the rest of the section to the proof of Theorem 7.1. In order to obtain estimates that allow us to pass to the limit, we provide an explicit construction of the branch of minimal solutions of (7.1). Although these are rather classic facts, see [20,21], some of our results appear to be new. Additionally, we provide a modified, more streamlined, and systematic construction that exhibits the dependences on p at each step, which is necessary in order to pass to the limit.

First, we show that problem (7.1) has a minimal solution up to a certain, explicit $\breve{\lambda}_p$.

Proposition 7.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p > n$. Then, problem* (7.1) *has a minimal solution $u_{\lambda,p}(x)$ for every $\lambda \in (0, \breve{\lambda}_p]$, where*

$$\breve{\lambda}_p = \left(\frac{p-1}{e\|v_p\|_\infty} \right)^{p-1} \tag{7.2}$$

and v_p is given by (2.2)*. Moreover,*

1) *For every $\lambda \leqslant \breve{\lambda}_p$, we have the estimate*

$$\lambda^{\frac{1}{p-1}} v_p(x) \leqslant u_{\lambda,p}(x) \leqslant e\,\lambda^{\frac{1}{p-1}} v_p(x) \quad \text{in } \overline{\Omega}. \tag{7.3}$$

2) *For every $\lambda \leqslant \breve{\lambda}_p$, the minimal solution $u_{\lambda,p}$ is the only solution of* (7.1) *with $\|u\|_\infty \leqslant p-1$.*
3) *The branch of minimal solutions is non-decreasing, in the sense that if $0 < \lambda < \mu \leqslant \breve{\lambda}_p$, then $u_{\lambda,p} \leqslant u_{\mu,p}$ in Ω.*

The uniqueness result in part 2 of Proposition 7.2 appears to be new. For the proof, we use the following comparison principle, an adaptation of [1, Lemma 4.1] to problems that are proper (in the sense of [14]) only for "small" sub- and supersolutions.

Lemma 7.3. *Let $p > 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function for which there exists $c \in (0, \infty]$ such that*

$$\frac{f(t)}{t^{p-1}} \quad \text{is non-increasing for all } t \in (0, c).$$

Assume that $u, v \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$ are positive in Ω, $\max\{\|u\|_\infty, \|v\|_\infty\} \leqslant c$ and

$$-\Delta_p u \leqslant f(u) \quad \text{and} \quad -\Delta_p v \geqslant f(v) \qquad \text{in } \Omega.$$

Then $u \leqslant v$ in Ω.

We omit the proof of the lemma since it is a straightforward modification of [1, Lemma 4.1] (note that $c = \infty$ in [1]). We proceed now with the proof of Proposition 7.2.

Proof of Proposition 7.2. 1) Consider $\underline{u}$ and $\overline{u}$, the respective solutions of

$$\begin{cases} -\Delta_p \underline{u} = \lambda & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta_p \overline{u} = \lambda\, e^{p-1} & \text{in } \Omega \\ \overline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

By the weak comparison principle for the p-Laplacian, we have that $0 \leqslant \underline{u} \leqslant \overline{u}$ in $\overline{\Omega}$. Define now u_1, solution of

$$\begin{cases} -\Delta_p u_1 = \lambda\, e^{\underline{u}} & \text{in } \Omega \\ \quad u_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.4}$$

We clearly have $-\Delta_p u_1 \geqslant \lambda = -\Delta_p \underline{u}$. On the other hand, we find $\underline{u} = \lambda^{\frac{1}{p-1}} v_p$ by rescaling, which together with (7.2) yields

$$-\Delta_p u_1 \leqslant \lambda\, e^{\|\underline{u}\|_\infty} \leqslant \lambda\, e^{(\check{\lambda}_p)^{1/(p-1)}\|v_p\|_\infty} = \lambda\, e^{(p-1)/e} \leqslant -\Delta_p \overline{u}.$$

Then, by the weak comparison principle we have $\underline{u} \leqslant u_1 \leqslant \overline{u}$ in $\overline{\Omega}$.
2) Now, for each $k \geqslant 1$ define u_{k+1}, solution of

$$\begin{cases} -\Delta_p u_{k+1} = \lambda\, e^{u_k} & \text{in } \Omega \\ \quad u_{k+1} = 0 & \text{on } \partial\Omega \end{cases}$$

with u_1 defined by (7.4). Let us show by induction that

$$\underline{u} \leqslant u_k \leqslant u_{k+1} \leqslant \overline{u} \quad \text{in } \overline{\Omega}$$

for all $k \geqslant 1$. It is easy to see that $\underline{u} \leqslant u_k \leqslant u_{k+1}$ by comparison. To prove $u_{k+1} \leqslant \overline{u}$, notice that the induction hypothesis, the rescaling $\overline{u} = \lambda^{\frac{1}{p-1}} e\, v_p$, and (7.2) yield

$$-\Delta_p u_{k+1} = \lambda\, e^{u_k} \leqslant \lambda\, e^{\|\overline{u}\|_\infty} \leqslant \lambda\, e^{(\check{\lambda}_p)^{1/(p-1)} e\, \|v_p\|_\infty} = \lambda\, e^{p-1} = -\Delta_p \overline{u}.$$

Then, $u_{k+1} \leqslant \overline{u}$ follows by comparison.
3) We have obtained an increasing sequence $\{u_k\}_{k \geqslant 0}$, uniformly bounded by $\underline{u}$ and $\overline{u}$. Therefore, we can pass to the limit and get a solution $u_{\lambda,p}$ that satisfies the bounds (7.3). It is also clear that $u_{\lambda,p}$ is minimal, because any solution of (7.1) could be taken as $\overline{u}$ in the iterative scheme (note that each u_k does not depend on $\overline{u}$). Similarly, we see that the branch of minimal solutions is non-decreasing, since whenever $\lambda < \mu$, we can take $\overline{u} = u_{\mu,p}$ in the construction of $u_{\lambda,p}$ and obtain $u_{\lambda,p} \leqslant u_{\mu,p}$.
4) Finally, let us denote $f(t) = \lambda\, e^t$. It is elementary to see that $f(t)/t^{p-1}$ is non-increasing for $0 < t < p-1$. Moreover, by (7.2) and (7.3) we have that $\|u_{\lambda,p}\|_\infty \leqslant p-1$. Therefore, we can apply Lemma 7.3 with $c = p-1$ and conclude that $u_{\lambda,p}$ is the only solution of (7.1) with $\|u\|_\infty \leqslant p-1$ for every $\lambda \in (0, \check{\lambda}_p]$. □

The next result states that problem (7.1) has no solution for large λ; that is, there is a value $\widehat{\lambda}_p > 0$ such that (7.1) has no weak solution with $\lambda > \widehat{\lambda}_p$.

Proposition 7.4 ([20, Theorem 2.1]). *Problem* (7.1) *does not have a solution for $\lambda > \widehat{\lambda}_p$, where*

$$\widehat{\lambda}_p = \lambda_1(p, \Omega) \cdot \max\left\{1, \left(\frac{p-1}{e}\right)^{p-1}\right\}. \tag{7.5}$$

At this point we can define

$$\lambda_{\max,p} = \sup\big\{\lambda > 0 \;:\; \text{problem (7.1) has a solution}\big\}. \tag{7.6}$$

In the next result we show that $\lambda_{\max,p}$ is well-defined, find its asymptotic behavior as $p \to \infty$, and complete the construction of the branch of minimal solutions.

Proposition 7.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p > n$. Then, $\lambda_{\max,p}$ given by* (7.6) *is well-defined (in the sense that it is positive and finite). Moreover,* (7.1) *has a minimal solution $u_{\lambda,p}(x)$ for every $\lambda \in (0, \lambda_{\max,p})$ and no solution for $\lambda > \lambda_{\max,p}$. In addition,*

$$\check{\lambda}_p \leqslant \lambda_{\max,p} \leqslant \widehat{\lambda}_p, \tag{7.7}$$

where $\check{\lambda}_p$ and $\widehat{\lambda}_p$ are respectively given by (7.2), (7.5)*, and*

$$\lim_{p\to\infty} \frac{\lambda_{\max,p}^{1/p}}{p} = \Lambda_{\max}$$

for $\Lambda_{\max}$ defined by (5.1).

Proof. By Propositions 7.2 and 7.4, we have that $0 < \check{\lambda}_p \leqslant \lambda_{\max,p} \leqslant \widehat{\lambda}_p < \infty$. Moreover, although we do not know $\lambda_{\max,p}$ explicitly, (7.2), (7.5), and (7.7), along with Proposition 2.2 and Lemma 2.5 provide its asymptotic behavior, namely,

$$\lim_{p\to\infty} \frac{\lambda_{\max,p}^{1/p}}{p} = \lim_{p\to\infty} \frac{\check{\lambda}_p^{1/p}}{p} = \lim_{p\to\infty} \frac{\widehat{\lambda}_p^{1/p}}{p} = e^{-1}\Lambda_1(\Omega) = \Lambda_{\max}.$$

Let us now complete the construction of the branch of minimal solutions. Since $\lambda_{\max,p} < \infty$ we can take μ arbitrarily close to $\lambda_{\max,p}$ and u_μ solution of

$$\begin{cases} -\Delta_p u_\mu = \mu\, e^{u_\mu} & \text{in } \Omega, \\ \quad u_\mu = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, for every $\lambda \in (\check{\lambda}_p, \mu]$ we can produce a minimal solution as in Proposition 7.2, taking $\overline{u} = u_\mu$ in the iteration. □

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. 1) We have that

$$\begin{cases} -\Delta_p u_{\lambda_p,p} = \lambda_p\, e^{u_{\lambda_p,p}} & \text{in } \Omega \\ \quad u_{\lambda_p,p} = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the equation by $u_{\lambda_p,p}$ and integrating by parts, we get

$$\int_\Omega |\nabla u_{\lambda_p,p}(x)|^p\, dx = \lambda_p \int_\Omega u_{\lambda_p,p}(x)\, e^{u_{\lambda_p,p}(x)}\, dx.$$

Let us fix $p > n+1$. Then, for every $x, y \in \Omega$, there exists a positive constant C independent of p (see [10, Lemma 3.3]) such that

$$\frac{|u_{\lambda_p,p}(x) - u_{\lambda_p,p}(y)|}{|x-y|^{1-\frac{n}{n+1}}} \leqslant C \left(\int_\Omega \left|\nabla u_{\lambda_p,p}\right|^{n+1} dx \right)^{1/(n+1)} \leqslant C\, |\Omega|^{\frac{1}{n+1}-\frac{1}{p}} \left(\int_\Omega \left|\nabla u_{\lambda_p,p}\right|^{p} dx \right)^{1/p}$$

$$= C\,|\Omega|^{\frac{1}{n+1}-\frac{1}{p}}\left(\lambda_p\int_\Omega u_{\lambda_p,p}\,e^{u_{\lambda_p,p}}\,dx\right)^{1/p} \leqslant C\,|\Omega|^{\frac{1}{n+1}}\left(\lambda_p\|u_{\lambda_p,p}\|_\infty\,e^{\|u_{\lambda_p,p}\|_\infty}\right)^{1/p}. \quad (7.8)$$

Let us now find estimates for $\|u_{\lambda_p,p}\|_\infty$.

2) Consider $\breve{\lambda}_p$, given by (7.2). Since

$$\lim_{p\to\infty}\frac{\breve{\lambda}_p^{1/p}}{p} = \Lambda_{\max} > \Lambda = \lim_{p\to\infty}\frac{\lambda_p^{1/p}}{p},$$

there exists p_0 such that $\lambda_p < \breve{\lambda}_p$ for all $p \geqslant p_0$. Then, by estimate (7.3), we have

$$\frac{\lambda_p^{\frac{1}{p-1}}}{p}v_p(x) \leqslant \frac{u_{\lambda_p,p}(x)}{p} \leqslant \frac{\lambda_p^{\frac{1}{p-1}}}{p}\,e\,v_p(x) \quad \text{in } \overline{\Omega},$$

where v_p is given by (2.2). Take $\epsilon > 0$ such that $(1+\epsilon)\Lambda < \Lambda_{\max}$. By Proposition 2.2, we know that $v_p \to \operatorname{dist}(\cdot,\partial\Omega)$ uniformly as $p\to\infty$ and we deduce that

$$\frac{\|u_{\lambda_p,p}\|_\infty}{p} \leqslant (1+\epsilon)\Lambda e\|\operatorname{dist}(\cdot,\partial\Omega)\|_\infty = (1+\epsilon)\Lambda\Lambda_{\max}^{-1} < 1 \quad (7.9)$$

for p large enough. Then, from (7.8) and the Arzelà-Ascoli theorem, we find that there exists a subsequence p' and a limit function u_Λ such that

$$\frac{u_{\lambda_{p'},p'}}{p'} \to u_\Lambda, \quad \text{uniformly as } p' \to \infty.$$

3) By Proposition 3.1, we have that u_Λ is a viscosity solution of the limit problem (6.1). Additionally, from estimate (7.9) we deduce $\|u_\Lambda\|_\infty \leqslant \Lambda\Lambda_{\max}^{-1} < 1$, and then Theorem 6.1 implies that u_Λ must be the minimal solution of the limit problem (6.1). Therefore, the whole sequence $u_{\lambda_p,p}$ converges, and not only a subsequence, which concludes the proof. □

8. Multiplicity results in special domains

This section proves that, under certain geometric assumptions on the domain Ω, it is possible to compute an explicit curve of solutions. Moreover, we establish a further non-existence result with the aid of this curve of solutions. To this aim, we consider the *ridge set* of Ω,

$$\begin{aligned}\mathcal{R} &= \{x\in\Omega : \operatorname{dist}(x,\partial\Omega) \text{ is not differentiable at } x\}\\ &= \{x\in\Omega : \exists\, x_1,x_2\in\partial\Omega,\ x_1\neq x_2, \text{ s.t. } |x-x_1| = |x-x_2| = \operatorname{dist}(x,\partial\Omega)\}\end{aligned}$$

and its subset $\mathcal{M}$, the *set of maximal distance* to the boundary,

$$\mathcal{M} = \left\{x\in\Omega : \operatorname{dist}(x,\partial\Omega) = \|\operatorname{dist}(\cdot,\partial\Omega)\|_\infty\right\}.$$

We have proved in Theorem 6.1 the existence of minimal solutions for the limit problem (1.5), as well as several non-existence results in Propositions 3.2 and 5.1. These results hold for general bounded domains Ω. In this section, we find a second solution to the limit problem (1.5) under the additional assumption $\mathcal{M} \equiv \mathcal{R}$. Furthermore, both solutions lie on an explicit curve of solutions (see Figure 1). Some examples of domains satisfying $\mathcal{M} \equiv \mathcal{R}$ are the ball, the annulus, and the stadium (convex hull of two balls of the same radius). A square or an ellipse does not verify the condition.

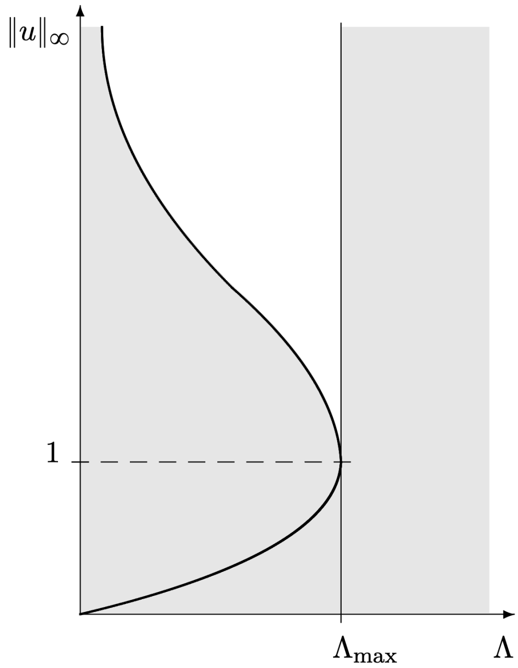

Figure 1. Curve of explicit solutions $\Lambda_1(\Omega)\,\|u_\Lambda\|_\infty - \Lambda\, e^{\|u_\Lambda\|_\infty} = 0$ in Theorem 8.1 and regions of non-existence derived from Proposition 5.1, Theorem 8.4, and the uniqueness result in Theorem 6.1.

8.1. A curve of explicit solutions

We have the following result.

Theorem 8.1. *Let* $\Lambda > 0$ *and* $\Lambda_{\max}$ *given by* (5.1)*. Assume that* $\Omega \subset \mathbb{R}^n$ *is a bounded domain that satisfies* $\mathcal{M} \equiv \mathcal{R}$*. Let us consider solutions of the form*

$$u(x) = \alpha \cdot \mathrm{dist}(x, \partial\Omega), \qquad \alpha > 0 \tag{8.1}$$

for the problem

$$\begin{cases} \min\left\{|\nabla u(x)| - \Lambda\, e^{u(x)}, -\Delta_\infty u(x)\right\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.2}$$

Then, problem (8.2)

i) *Has two solutions of the form* (8.1) *if* $0 < \Lambda < \Lambda_{\max}$*, corresponding to the two roots of*

$$\alpha - \Lambda\, e^{\alpha\, \|\mathrm{dist}(\cdot, \partial\Omega)\|_\infty} = 0. \tag{8.3}$$

ii) *Has one solution of the form* (8.1) *for* $\Lambda = \Lambda_{\max}$*, with* $\alpha = \|\mathrm{dist}(\cdot, \partial\Omega)\|_\infty^{-1}$.

iii) Has no solutions for $\Lambda > \Lambda_{\max}$, and only the trivial solution for $\Lambda = 0$.

Remark 8.2. By Theorem 6.1, for $0 < \Lambda < \Lambda_{\max}$ the solution of the form (8.1) with smallest α is the minimal solution of (8.2).

Proof. First of all, we are going to check that

$$-\Delta_\infty u(x) = 0 \qquad \text{in } \Omega \backslash \mathcal{R}$$

in the viscosity sense. Let $\phi \in C^2$ and $x_0 \in \Omega\backslash\mathcal{R}$ such that $u - \phi$ has a local maximum at x_0. We can assume $u(x_0) = \phi(x_0)$ and $\nabla\phi(x_0) \neq 0$. A Taylor expansion, and the fact that ϕ touches u from above at x_0 yield

$$-\frac{\Delta_\infty \phi(x_0)}{|\nabla\phi(x_0)|^2} + o(1) \leqslant \frac{1}{\epsilon^2}\left(2u(x_0) - \max_{y\in B_\epsilon(x_0)} u(y) - \min_{y\in B_\epsilon(x_0)} u(y)\right)$$

as $\epsilon \to 0$. From (8.1) we have that

$$\max_{y\in B_\epsilon(x_0)} u(y) = u(x_0) + \alpha\epsilon, \qquad \min_{y\in B_\epsilon(x_0)} u(y) = u(x_0) - \alpha\epsilon$$

and we deduce that u is ∞-subharmonic in $\Omega\backslash\mathcal{R}$. The proof that it is also ∞-superharmonic is analogous. Hence, we need make sure that

$$|\nabla u(x)| - \Lambda\, e^{u(x)} \geqslant 0 \qquad \forall x \in \Omega\backslash\mathcal{R}.$$

Indeed, we find that

$$|\nabla u(x)| - \Lambda\, e^{u(x)} = \alpha - \Lambda\, e^{\alpha\cdot \operatorname{dist}(x,\partial\Omega)}$$

(recall that $x \notin \mathcal{R}$ and the derivatives are classical). Since we can choose points $x \notin \mathcal{R} \equiv \mathcal{M}$ arbitrarily close to $\mathcal{M}$, we find the necessary condition

$$\alpha - \Lambda\, e^{\alpha\cdot\|\operatorname{dist}(\cdot,\partial\Omega)\|_\infty} \geqslant 0. \tag{8.4}$$

Next, we turn our attention to the ridge set $\mathcal{R}$. First, observe that cones as in (8.1) are always supersolutions of (8.2) in the ridge set, since they cannot be touched from below with C^2 functions at those points. Hence, we only have to consider the subsolution case. So, let $x_0 \in \mathcal{R}$ and $\phi \in C^2$ such that $u - \phi$ has a local maximum point at x_0. We aim to prove that

$$\min\left\{|\nabla\phi(x_0)| - \Lambda\, e^{u(x_0)}, -\Delta_\infty\phi(x_0)\right\} \leqslant 0. \tag{8.5}$$

It is well-known (see for instance [27, Lemma 6.10]) that

$$\min\Big\{|\nabla u(x)| - \alpha, -\Delta_\infty u(x)\Big\} = 0$$

in the viscosity sense. Thus, by definition of viscosity subsolution we have that either $|\nabla\phi(x_0)| \leqslant \alpha$ or $-\Delta_\infty\phi(x_0) \leqslant 0$. In the latter case, (8.5) holds and there is nothing to prove. Thus, we can assume in the sequel that $-\Delta_\infty\phi(x_0) > 0$ and $|\nabla\phi(x_0)| \leqslant \alpha$. Then, since $x_0 \in \mathcal{R} \equiv \mathcal{M}$, we have $u(x_0) = \alpha\,\|\operatorname{dist}(\cdot,\partial\Omega)\|_\infty$ and

$$|\nabla\phi(x_0)| - \Lambda\, e^{u(x_0)} \leqslant \alpha - \Lambda\, e^{\alpha\,\|\operatorname{dist}(\cdot,\partial\Omega)\|_\infty}.$$

Recalling (8.4), we discover that the only possibility is that (8.3) holds. The rest of the proof is devoted to study the number of positive solutions of equation (8.3).

Consider $\Phi_\Lambda(\alpha) = \Lambda\, e^{\alpha \|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty} - \alpha$. It is elementary to show that Φ_Λ is convex and has a global minimum at

$$\alpha_{\min} = -\|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty^{-1}\, \log\left(\Lambda\|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty\right).$$

This minimum value is

$$\min_{\alpha\in\mathbb{R}} \Phi_\Lambda(\alpha) = \Phi_\Lambda(\alpha_{\min}) = \|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty^{-1}\big(1 + \log\left(\Lambda\|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty\right)\big).$$

Whenever this minimum is strictly positive, Eq (8.3) has no solution. This happens when $\Lambda > \Lambda_{\max}$ (in fact, Proposition 5.1 gives a stronger result in this case). Furthermore, notice that if $\Lambda = 0$, then necessarily $\alpha = 0$. These facts amount to (iii). When the minimum equals 0, that is, when $\Lambda = \Lambda_{\max}$, then there exists a unique solution with $\alpha = \|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty^{-1}$. This is part (ii). And finally, for part (i), notice that when the minimum is strictly negative ($0 < \Lambda < \Lambda_{\max}$), equation (8.3) has two roots. □

Remark 8.3. Theorem 8.1 yields the following implicit curve of cone solutions

$$\Lambda_1(\Omega)\,\|u_\Lambda\|_\infty - \Lambda\, e^{\|u_\Lambda\|_\infty} = 0,$$

where $\Lambda_1(\Omega) = \|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty^{-1}$ is the first ∞-eigenvalue, see [30]. The same curve was deduced heuristically by Lions in the context of the Gelfand problem for the Laplacian in [36, p. 465, item (h) and Remark 2.4]. Unfortunately, Lions uses this example to caution against the heuristic reasoning since the bifurcation diagram is of corkscrew-type for dimensions $3 \leqslant n \leqslant 9$. One could wonder why we do not see a similar situation in Theorem 8.1. However, according to [16, Lemma 2.3], the corresponding corkscrew-type diagram for the p-Laplacian in the radial case occurs in the range

$$p < n < \frac{p\,(p+3)}{p-1},$$

which cannot happen as $p \to \infty$.

8.2. Further non-existence results

The following result shows that we can enlarge the region of nonexistence of solutions for certain domains by taking advantage of the curve of explicit solutions.

Theorem 8.4. *Let Ω be a bounded domain such that $\mathcal{M} \equiv \mathcal{R}$, and assume $\mathcal{M}$ is Lipschitz connected. Then, for every $\Lambda > 0$, the only solutions of the problem*

$$\begin{cases} \min\left\{|\nabla u_\Lambda(x)| - \Lambda\, e^{u_\Lambda(x)}, -\Delta_\infty u_\Lambda(x)\right\} = 0 & \text{in } \Omega,\\ u_\Lambda = 0 & \text{on } \partial\Omega \end{cases} \tag{8.6}$$

satisfying

$$\Lambda_1(\Omega)\,\|u_\Lambda\|_\infty - \Lambda\, e^{\|u_\Lambda\|_\infty} \geqslant 0, \tag{8.7}$$

are the explicit solutions found in Theorem 8.1, which satisfy (8.7) *with an equality.*

The idea of the proof of Theorem 8.4 is to show that any solution u_Λ satisfying (8.7) must necessarily be a cone and therefore belong to the curve of solutions given by Theorem 8.1. First, we show that solutions of (8.6) that satisfy (8.7) must lie below a cone with their same height.

Lemma 8.5. *Let Ω be a bounded domain and u_Λ be a viscosity solution of* (8.6) *satisfying* (8.7). *Then,*

$$u_\Lambda \leqslant \frac{\|u_\Lambda\|_\infty}{\|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty}\,\mathrm{dist}(\cdot,\partial\Omega) \quad \text{in } \Omega.$$

Proof. It is enough to prove that

$$\min\left\{|\nabla u_\Lambda(x)| - \Lambda_1(\Omega)\,\|u_\Lambda\|_\infty, -\Delta_\infty u_\Lambda(x)\right\} \leqslant 0 \quad \text{in } \Omega \tag{8.8}$$

in the viscosity sense. Then one gets $u_\Lambda(x) \leqslant \|u_\Lambda\|_\infty\,\|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty^{-1}\,\mathrm{dist}(x,\partial\Omega)$ in Ω by comparison (Lemma 2.3), and the result follows.

To prove (8.8), let $\phi \in C^2$ such that $u_\Lambda - \phi$ has a maximum at $x_0 \in \Omega$. As u_Λ is a viscosity solution of (8.6), it satisfies

$$\min\left\{|\nabla\phi(x_0)| - \Lambda\, e^{u_\Lambda(x_0)}, -\Delta_\infty\phi(x_0)\right\} \leqslant 0 \quad \text{in } \Omega.$$

If $-\Delta_\infty\phi(x_0) \leqslant 0$ we are done, so assume $-\Delta_\infty\phi(x_0) > 0$ and $|\nabla\phi(x_0)| - \Lambda\, e^{u_\Lambda(x_0)} \leqslant 0$. Using (8.7), we have

$$|\nabla\phi(x_0)| - \Lambda_1(\Omega)\,\|u_\Lambda\|_\infty \leqslant \Lambda\, e^{u_\Lambda(x_0)} - \Lambda_1(\Omega)\,\|u_\Lambda\|_\infty \leqslant 0,$$

and then

$$\min\left\{|\nabla\phi(x_0)| - \Lambda_1(\Omega)\,\|u_\Lambda\|_\infty, -\Delta_\infty\phi(x_0)\right\} \leqslant 0 \quad \text{in } \Omega$$

as desired. □

Remark 8.6. Lemma 8.5 holds for any bounded domain Ω without the assumption $\mathcal{M} \equiv \mathcal{R}$.

Next, we recall the following result from [43, Theorem 2.4, (i)], which is a crucial point in the proof of Theorem 8.4.

Lemma 8.7. *Let Ω be a bounded domain such that $\mathcal{M}$ is Lipschitz connected. If u is ∞-superharmonic (see [34, 35]) then,*

$$\left\{x \in \Omega :\ u(x) = \|u\|_{L^\infty(\Omega)}\right\} \equiv \mathcal{M}.$$

Now, we can complete the proof of Theorem 8.4.

Proof of Theorem 8.4. Consider u_Λ solution of (8.6) satisfying (8.7). Notice that

$$v(x) = \frac{\|u_\Lambda\|_\infty}{\|\mathrm{dist}(\cdot,\partial\Omega)\|_\infty}\,\mathrm{dist}(\cdot,\partial\Omega)$$

is the unique (see [25]) viscosity solution of the problem

$$\begin{cases} -\Delta_\infty v(x) = 0 & \text{in } \Omega\backslash\mathcal{M} \\ v(x) = \|u_\Lambda\|_\infty & \text{on } \mathcal{M} \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.9}$$

Since u_Λ is ∞-superharmonic, it is also a viscosity supersolution of (8.9) by Lemma 8.7. Then, we get $v \leqslant u_\Lambda$ by comparison (see [25]), and Lemma 8.5 yields $u_\Lambda \equiv v$. That is, u_Λ is of the form (8.1). Since all the solutions of (8.6) of the form (8.1) are given by Theorem 8.1, we find that there are no solutions with

$$\Lambda_1(\Omega)\,\|u_\Lambda\|_\infty - \Lambda\, e^{\|u_\Lambda\|_\infty} > 0.$$

Furthermore, if $\Lambda_1(\Omega)\,\|u_\Lambda\|_\infty - \Lambda\, e^{\|u_\Lambda\|_\infty} = 0$, then u_Λ must be one of the explicit solutions in Theorem 8.1. □

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Conflict of interest

The authors declare no conflict of interest.

References

1. B. Abdellaoui, I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian with a critical potential, *Ann. Mat. Pura Appl. IV. Ser.*, **182** (2003), 247–270. https://doi.org/10.1007/s10231-002-0064-y
2. A. Anane, J. L. Lions, Simplicitè et isolation de la premiere valeur propre du p-laplacien avec poids, *C. R. Acad. Sci. Paris Sér. I Math.*, **305** (1987), 725–728.
3. T. Bhattacharya, E. DiBenedetto, J. Manfredi, Limit as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems, *Rend. Sem. Mat. Univ. Politec. Torino*, 1989, 15–68.
4. M. Bocea, M. Mihăilescu, Existence of nonnegative viscosity solutions for a class of problems involving the ∞-Laplacian, *Nonlinear Differ. Equ. Appl.*, **23** (2016), 11. https://doi.org/10.1007/s00030-016-0373-2
5. G. Bratu, Sur les équations intégrales non linéaires, *Bulletin de la S. M. F.*, **42** (1914), 113–142.
6. H. Brezis, L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Anal. Theor.*, **10** (1986), 55–64. https://doi.org/10.1016/0362-546X(86)90011-8
7. X. Cabré, M. Sanchón, Semi-stable and extremal solutions of reaction equations involving the p-Laplacian, *Commun. Pure Appl. Anal.*, **6** (2007), 43–67. https://doi.org/10.3934/cpaa.2007.6.43
8. S. Chandrasekhar, *An introduction to the study of stellar structures*, New York: Dover, 1957.
9. S. Chanillo, M. Kiessling, Surfaces with prescribed Gauss curvature, *Duke Math. J.*, **105** (2000), 309–353. https://doi.org/10.1215/S0012-7094-00-10525-X
10. F. Charro, E. Parini, Limits as $p \to \infty$ of p-laplacian problems with a superdiffusive power-type nonlinearity: positive and sign-changing solutions, *J. Math. Anal. Appl.*, **372** (2010), 629–644. https://doi.org/10.1016/j.jmaa.2010.07.005

11. F. Charro, E. Parini, Limits as $p \to \infty$ of p-laplacian eigenvalue problems perturbed with a concave or convex term, *Calc. Var.*, **46** (2013), 403–425. https://doi.org/10.1007/s00526-011-0487-7

12. F. Charro, I. Peral, Limit branch of solutions as $p \to \infty$ for a family of sub-diffusive problems related to the p-laplacian, *Commun. Part. Diff. Eq.*, **32** (2007), 1965–1981. https://doi.org/10.1080/03605300701454792

13. F. Charro, I. Peral, Limits as $p \to \infty$ of p-Laplacian concave-convex problems, *Nonlinear Anal. Theor.*, **75** (2012), 2637–2659. https://doi.org/10.1016/j.na.2011.11.009

14. M. G. Crandall, H. Ishii, P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.*, **27** (1992), 1–67. https://doi.org/10.1090/S0273-0979-1992-00266-5

15. J. Dávila, Singular solutions of semi-linear elliptic problems, In: *Handbook of differential equations: stationary partial differential equations*, Amsterdam: Elsevier/North-Holland, 2008, 83–176. https://doi.org/10.1016/S1874-5733(08)80019-8

16. M. Del Pino, J. Dolbeault, M. Musso, Multiple bubbling for the exponential nonlinearity in the slightly supercritical case, *Commun. Pure Appl. Anal.*, **5** (2006), 463–482. https://doi.org/10.3934/cpaa.2006.5.463

17. R. Emden, *Gaskugeln: Anwendungen der mechanischen Wärmetheorie auf kosmologische und meteorologische Probleme (German)*, Germany: B. G. Teubner, 1907.

18. N. Fukagai, M. Ito, K. Narukawa, Limit as $p \to \infty$ of p-Laplace eigenvalue problems and L^∞-inequality of the Poincare type, *Differ. Integral Equ.*, **12** (1999), 183–206.

19. J. P. García Azorero, I. Peral, *Existence and nonuniqueness for the p-laplacian: nonlinear eigenvalues*, *Commun. Part. Diff. Eq.*, **12** (1987), 1389–1430.

20. J. García Azorero, I. Peral Alonso, On a Emden-Fowler type equation, *Nonlinear Anal. Theor.*, **18** (1992), 1085–1097. https://doi.org/10.1016/0362-546X(92)90197-M

21. J. García Azorero, I. Peral Alonso, J. P. Puel, Quasilinear problems with exponential growth in the reaction term, *Nonlinear Anal. Theor.*, **22** (1994), 481–498. https://doi.org/10.1016/0362-546X(94)90169-4

22. I. M. Gel'fand, Some problems in the theory of quasilinear equations, *Amer. Math. Soc. Trans.*, **29** (1963), 295–381.

23. J.-F. Grosjean, p-Laplace operator and diameter of manifolds, *Ann. Glob. Anal. Geom.*, **28** (2005), 257–270. https://doi.org/10.1007/s10455-005-6637-4

24. J. Jacobsen, K. Schmitt, The Liouville-Bratu-Gelfand problem for radial operators, *J. Differ. Equations*, **184** (2002), 283–298. https://doi.org/10.1006/jdeq.2001.4151

25. R. Jensen, Uniqueness of Lipschitz extensions: Minimizing the sup norm of the gradient, *Arch. Rational Mech. Anal.*, **123** (1993), 51–74. https://doi.org/10.1007/BF00386368

26. D. D. Joseph, T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Rational Mech. Anal.*, **49** (1973), 241–269. https://doi.org/10.1007/BF00250508

27. P. Juutinen, *Minimization problems for Lipschitz functions via viscosity solutions*, Helsinki: Suomalainen Tiedeakatemia, 1998.

28. P. Juutinen, Principal eigenvalue of a badly degenerate operator, *J. Differ. Equations*, **236** (2007), 532–550. https://doi.org/10.1016/j.jde.2007.01.020

29. P. Juutinen, P. Lindqvist, On the higher eigenvalues for the ∞-eigenvalue problem, *Calc. Var.*, **23** (2005), 169–192. https://doi.org/10.1007/s00526-004-0295-4

30. P. Juutinen, P. Lindqvist, J. Manfredi, The ∞-eigenvalue problem, *Arch. Rational Mech. Anal.*, **148** (1999), 89–105. https://doi.org/10.1007/s002050050157

31. B. Kawohl, On a family of torsional creep problems, *J. Reine Angew. Math.*, **410** (1990), 1–22.

32. P. Lindqvist, On the equation div $(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Proc. Amer. Math. Soc.*, **109** (1990), 157–164. https://doi.org/10.1090/S0002-9939-1990-1007505-7

33. P. Lindqvist, Addendum: "On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$" [Proc. Amer. Math. Soc. 109 (1990), no. 1, 157–164; MR1007505 (90h:35088)], *Proc. Amer. Math. Soc.*, **116** (1992), 583–584. https://doi.org/10.1090/S0002-9939-1992-1139483-6

34. P. Lindqvist, J. Manfredi, The Harnack inequality for ∞-harmonic functions, *Electron. J. Diff. Eqns.*, **5** (1995), 1–5.

35. P. Lindqvist, J. Manfredi, Note on ∞-superharmonic functions, *Revista Matemática de la Universidad Complutense de Madrid*, **10** (1997), 471–480.

36. P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.*, **24** (1982), 441–467. https://doi.org/10.1137/1024101

37. J. Liouville, Sur l'équation aux différences partielles $\frac{d^2 \log\lambda}{dudv} \pm \frac{\lambda}{2a^2} = 0$, *Journal de mathématiques pures et appliquées 1re série*, **18** (1853), 71–72.

38. M. Mihăilescu, D. Stancu-Dumitru, C. Varga, The convergence of nonnegative solutions for the family of problems $-\Delta_p u = \lambda e^u$ as $p \to \infty$, *ESAIM: COCV*, **24** (2018), 569–578. https://doi.org/10.1051/cocv/2017048

39. I. Peral, Some results on quasilinear elliptic equations: growth versus shape, In: *Proceedings of the second school of nonlinear functional analysis and applications to differential equations*, Trieste: World Scientific, 1998, 153–202.

40. O. W. Richardson, *The emission of electricity from hot Bodies*, India: Longmans, Green and Company, 1921.

41. M. Sanchón, Regularity of the extremal solution of some nonlinear elliptic problems involving the p-Laplacian, *Potential Anal.*, **27** (2007), 217–224. https://doi.org/10.1007/s11118-007-9053-5

42. G. W. Walker, Some problems illustrating the forms of nebulae, *Proc. R. Soc. Lond. A*, **91** (1915), 410–420. https://doi.org/10.1098/rspa.1915.0032

43. Y. Yu, Some properties of the ground states of the infinity Laplacian, *Indiana Univ. Math. J.*, **56** (2007), 947–964.