UNIVERSIDADE DE LISBOA

FACULDADE DE CIÊNCIAS

Geometry and Topology of Generalized Polygon Spaces

" Documento Definitivo"

Doutoramento em Matemática

Especialidade de Geometria e Topologia

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Tese orientada por:

Carlos Florentino, Leonor Godinho

Documento especialmente elaborado para a obtenção do grau de doutor

2021

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Com o apoio financeiro da Fundação para a Ciência e a Tecnologia (FCT)

2021

Resumo

O espaço clássico de polígonos em $\mathbb{R}^3$, designado por $Pol(\underline{\alpha})$ com $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^+$ foi introduzido por Klyachko em [Kl] e por Kapovich e Millson em [KM]. Posteriormente Hausmann e Knutson provaram em [HK1] e [HK2] muitos resultados importantes como por exemplo o cálculo dos possíveis polinómios de Poincaré e aneis de cohomologia.

Este espaço pode ser definido de diferentes formas:

1. Como o espaço dos polígonos de $\mathbb{R}^3$ com os lados de comprimentos $\alpha_1, \ldots, \alpha_n$, a menos de rotações e traslações e cujos pontos inicial e final coincidem.

2. Como o espaço das possíveis configurações ponderadas semiestáveis de n pontos de $\mathbb{CP}^1$ com pesos $\alpha_1, \ldots, \alpha_n$ a menos de uma transformação de Möbius, onde uma configuração é considerada semiestável se nenhum ponto tem um peso maior do que metade do peso total.

3. Como uma redução simplética do espaço de representações de um *quiver* em forma de estrela com *caracteristica* 2 e *torção n* num nível determinado por $\underline{\alpha}$.

Esta última construção pode ser generalizada a quociente Hyperkähler usando o quiver duplo associado, onde obtemos o *espaço de Hyperpolígonos.*

Os espaços de polígonos e Hyperpolígonos podem ser relacionados com muitas áreas da geometria e foram estudados por muitos autores. Por exemplo em [GM], foi provada a existência de um isomorfismo entre o espaço usual de Hyperpolígonos e o espaço de moduli de fibrados de Higgs parabólicos sobre $\mathbb{CP}^1$ com estrutura holomorfa trivial, característica 2, determinante fixo e campo de Higgs com traço nulo.

É então natural procurar generalizar estes espaços considerando polígonos com lados noutro espaço. Um caso especialmente interessante é o caso dos polígonos com lados no espaço tridimensional de Minkowski, que foi primeiro introduzido por Millson e estudado por Foth em

[F]. Estes espaços têm especial interesse dado que podem ser identificados com as componentes do conjunto de pontos fixos de uma involução natural no espaço de Hyperpolígonos (ver [BFG]).

Outra pergunta natural seria estudar o espaço de polígonos obtido ao mudar o grupo de Lie no quociente simpléctico.

O objectivo desta tese é o estudo da geometria e topologia do espaço obtido ao considerar o grupo $SU(m)$, construindo de forma análoga um espaço de polígonos. Dito de outro modo, vamos considerar o espaço reduzido $\prod \mathcal{O}^d_{\mathfrak{su}(m)}(\alpha_i) /\!\!/ SU(m)$, onde $\mathcal{O}^d_{\mathfrak{su}(m)}(\alpha_i)$ são órbitas degeneradas da ação coadjunta de $SU(m)$ em $\mathfrak{su}(m)^*$ e $/\!\!/$ denota a redução simplética. Neste caso, os espaços obtidos são espaços de polígonos com lados em $\mathbb{CP}^{m-1}$. Vamos ver também que estes espaços podem ser definidos como redução simplética de $(\mathbb{C}^m)^n$ pelo grupo $G := (U(m) \times (S^1)^n)/\Gamma$, onde Γ é o subcirculo diagonal de $U(m) \times (S^1)^n$. Esta construção é mais rica, dado que proporciona relações com outros espaços conhecidos tais como as Grassmannianas complexas.

No nosso estudo iremos usar *cohomologia equivariante.*

Em geral, se temos um grupo de Lie G a agir sobre uma variedade M, o quociente M/G pode não ser uma variedade, o que dificulta o estudo do seu anel de cohomologia. A cohomologia equivariante ou *cohomologia de Borel-Moore* é uma generalização do anel de cohomologia de M/G e pode ser definida mesmo quando os espaços quocientes referidos não são variedades. É por isso que é uma ferramenta especialmente valiosa quando consideramos ações de grupos de Lie sobre variedades diferenciáveis.

Foi introduzida primeiro por Borel e Moore em 1960 [BM] e depois estudada por muitos autores tais como Atiyah, Bott, Tymoczko, Kirwan ou Jeffrey (ver [AB], [T], [Kir], [JK]). Como os espaços de Polígonos e Hyperpolígonos podem ser obtidos como resultado de um quociente simplético, os resultados de Jeffrey e Kirwan vão ser especialmente relevantes no nosso estudo.

Outro metodo que será utilizado é o denominado por *wall crossing.*

Dado um grupo de Lie G e uma variedade simplética (M, ω) onde G age de forma hamiltoniana, muita informação sobre o espaço reduzido $M /\!\!/ G$ pode ser obtida através do espaço $M /\!\!/ T$, onde $T \subset G$ é o subtoro maximal de G tal como, por exemplo, o anel de cohomologia equivariante. Em particular, vamos ver que os conjuntos de pontos fixos dos subcírculos $S^1 \cong H \subset G = (U(m) \times (S^1)^n)/\Gamma$, correspondem a polígonos que se decompõem em dois subpolígonos contidos em subespaços complementares.

A imagem pela *aplicação momento* μ_G desses conjuntos, forma hyperplanos chamados *walls* ou *paredes*. Estas paredes dividem $\mathbb{R}^n$ em câmaras onde a classe de difeomorfismo do espaço $Pol_m(\underline{\alpha})$ se mantém constante.

Desta forma podemos obter muita informação ao estudar o modo como a passagem através de uma parede afecta o espaço de moduli. Tanto o polinómio de Poincaré como o volume simplético são aqui calculados por este método. No caso do volume usamos resultados de Shaun Martin em [Ma1].

A tese está estruturada como se segue:

Nos Capítulos 1 e 2 explicamos com mais detalhe a motivação do nosso trabalho e apresentamos resultados preliminares que serão utilizados mais tarde. Conceitos como cohomologia equivariante e *wall crossing* são formalmente definidos.

No Capítulo 3 analisamos as diferentes maneiras de construir $Pol_m(\underline{\alpha})$. Em particular, explícitamos a correspondência de Gelfand McPherson, que descreve $Pol_m(\underline{\alpha})$ como redução simplética a partir do espaço de Grassmannianas complexas $Gr(m,n)$. Obtemos assim o seguinte diagrama de reduções simpléticas:

$$\begin{array}{ccccc} & & ((\mathbb{C}^m)^n) & & \\ & \swarrow^{/\!\!/(S^1)^n} & \downarrow /\!\!/G & \searrow^{/\!\!/U(m)} & \\ ((\mathbb{C}^m)^n)/\!\!/_{-2\underline{\alpha}}(S^1)^n \cong \prod \mathbb{CP}^{m-1} & & & & ((\mathbb{C}^m)^n)/\!\!/_{\frac{\sum\alpha_i}{2m}Id}U(m)\cong Gr(m,n) \\ \downarrow /\!\!/SU(m) & & & & \downarrow /\!\!/(S^1)^{n-1} \\ \prod\mathbb{CP}^{m-1}/\!\!/_0(SU(m)/\mathbb{Z}_m) & \xrightarrow{\cong} & Pol_m(\underline{\alpha}) & \xleftarrow{\cong} & Gr(m,n)/\!\!/_{-\underline{\alpha}}(S^1)^{n-1}\end{array}$$

A dualidade entre espaços de Grassmannianas $Gr(m,n)$ e $Gr(n-m,n)$ e o diagrama acima sugerem uma dualidade entre espaços de polígonos com n lados em $\mathbb{CP}^{m-1}$ e polígonos com n lados em $\mathbb{CP}^{n-m-1}$. No Capítulo 4 estudamos esta dualidade.

No Capítulo 5 descrevemos as paredes no espaço de moduli. Mostramos que no interior das paredes externas o espaço $Pol_m(\underline{\alpha})$ é não vazio, e que, se $\underline{\alpha}$ não fica numa das paredes, é uma variedade diferenciável.

No Capítulo 6 calculamos o polinómio de Poincaré destes espaços de uma forma recursiva em m, obtendo uma fórmula fechada para os casos $m = 2$ e $m = 3$. No caso $m = 2$, comparamos a fórmula obtida com a dada por Hausmann e Knutson em [HK1].

Iremos aplicar o método de *wall crossing* para analisar a maneira na que o espaço de polígonos muda quando atravessamos uma parede.

Deste modo unicamente é necessário determinar um valor inicial $\underline{\beta}_{n,m}$ onde o polinómio de Poincaré do espaço $Pol_m(\underline{\beta}_{n,m})$ é conhecido e usar os resultados obtidos previamente. Em particular o espaço inicial $Pol_m(\underline{\beta}_{n,m})$ é uma *Bott tower* generalizada de $(m-1)$-passos, onde a fibra de cada passo é $\mathbb{CP}^{n-m-1}$ (ver [CMS]).

Existem resultados semelhantes em espaços relacionados com $Pol_m(\underline{\alpha})$. Por exemplo, Holla determinou em [Ho] uma fórmula fechada para o polinómio de Poincaré do espaço de moduli de fibrados parabólicos semiestáveis sobre uma superficie de Riemann. Goldin descreve em [G] como obter o anel de cohomologia de $Pol_m(\underline{\alpha})$. No entanto, esta descrição não é muito prática do ponto de vista computacional.

Na última secção do Capítulo 6, provamos que todos os espaços de polígonos estudados são simplesmente conexos.

No Capítulo 7, aplicamos os resultados de Shaun Martin e de Kirwan para determinar o volume simplético de $Pol_m(\underline{\alpha})$, obtendo uma formula explicita quando $m = 3$. Tal fórmula concorda com a fórmula obtida por Suzuki e Takakura em [ST].

Por último, nos Apêndices A e B, incluimos dois programas na linguagem Wolfram para calcular o polinómio de Poincaré no caso $m = 3$. O primeiro dos programas está baseado nas contas da Secção 6.3.2 e o segundo, na fórmula dada por Holla (ver [Ho]).

Palavras chave: Polígonos, redução simplética, polinómio de Poincaré.

Abstract

The classical space of polygons in $\mathbb{R}^3$, denoted by $Pol(\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbb{R}^+$, was first introduced by Klyachko in [Kl] and by Kapovich and Millson in [KM]. Hausmann and Knutson obtained in [HK1] and [HK2] many interesting results about these spaces such as the computation of the possible Poincaré polynomials and the cohomology rings.

It can be obtained in many different ways:

1. As the space of piecewise linear paths in $\mathbb{R}^3$ with n steps of length $\alpha_1, \ldots, \alpha_n$ modulo rotations and translations, whose start and endpoints agree.

2. As the space of semistable weighted configurations of n points in $\mathbb{CP}^1$, with weights $\alpha_1, \ldots, \alpha_n$, modulo Möbius transformations, where a configuration is considered unstable if more than half of the total weight is assigned to a single point.

3. As symplectic reduction on the space of representations of a *rank* 2 star-shaped quiver and *twist n* at a level determined by $(\alpha_1, \ldots, \alpha_n)$.

This last construction can be generalized to a Hyperkähler quotient, using the associated double quiver, thus obtaining what is known as the Hyperpolygon space.

Polygon and Hyperpolygon spaces can be related to many areas of geometry and have received the attention of many authors. For instance in [GM], Godinho and Mandini prove the existence of an isomorphism between Hyperpolygon spaces and moduli spaces of stable, rank-2, holomorphically trivial parabolic Higgs bundles over $\mathbb{CP}^1$ with fixed determinant and trace-free Higgs field.

It is natural to try to generalize these spaces, considering polygons ith edges in a different space. A particular interesting case is the space with edges in Minkowski 3-space, which was briefly introduced by Millson and studied by Foth in [F]. These spaces are specially interesting because they can be identified with the different components of the fixed point set of a natural involution on the space of Hyperpolygon (see [BFG]).

Since in the classical cases the Lie group $SO(3)$ plays a fundamental role, another natural question would be to wonder what happen if we change it by another matrix Lie group.

In this thesis we consider the group $SU(m)$ instead, and construct our space of polygons analogously. The corresponding space of polygons obtained is the one of polygons with edges in $\mathbb{CP}^{m-1}$. Our efforts are addressed to obtain information about these spaces.

Keywords: Polygon, symplectic reduction, Poincaré polynomial.

Contents

Chapter 1

Motivation

1.1 Actions and symplectic reduction

1.1.1 Actions and Orbits

Let G be a Lie group and, for a given $g \in G$, let us consider the action on G by conjugation

$$C_g(h) = ghg^{-1}, \quad \forall h \in G.$$

This diffeomorphism fixes the identity and its derivative at e is an invertible linear map

$$(dC_g)_e : T_eG \to T_eG.$$

Identifying T_eG with the Lie algebra $\mathfrak{g}$ of G, we denote by Ad_g the above isomorphism and we obtain what is known as the *Adjoint representation* of G

$$\begin{aligned} Ad : G &\to GL(\mathfrak{g}) \\ g &\mapsto Ad_g. \end{aligned}$$

Moreover, taking the derivative at e, we obtain the *adjoint representation* of $\mathfrak{g}$

$$\begin{aligned} ad : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ X &\mapsto ad_X. \end{aligned}$$

In particular, if G is a matrix group,

$$Ad_g(X) = gXg^{-1}$$

and

$$ad_x(Y) = [X,Y],$$

where $[\cdot,\cdot]$ is the Lie bracket.

From these representations we naturally obtain their dual representations. For the *adjoint representation* we get the *coadjoint representation,* defined as

$$\langle Ad_g^*\xi, X\rangle = \langle \xi, Ad_{g^{-1}}X\rangle, \quad \forall \xi \in \mathfrak{g}^*, g \in G, X \in \mathfrak{g},$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Similarly, we can define the *coadjoint representation*

$$ad_X^*(\xi) = \left(\frac{d}{dt}\right)_{|_{t=0}} \left(Ad^*_{\exp(tX)}(\xi)\right).$$

In general, the adjoint and coadjoint representations are not equivalent. However, in some cases, they are.

Proposition 1.1. Let ρ be a representation of G on a vector space V and let ρ^* be its dual representation. If there exists a G-invariant non-degenerate bilinear form on V, the two representations are equivalent.

In particular, for a semisimple Lie group, we can take the *Killing form* and then the Adjoint and the coadjoint representations are equivalent.

For $\xi \in \mathfrak{g}^*$ let us denote the coadjoint orbit through ξ by $\mathcal{O}(\xi)$. It is a homogeneous G-space with tangent space

$$T_\xi\mathcal{O}(\xi) = \{ad_X^*\xi \mid X \in \mathfrak{g}\}.$$

Moreover, there is a natural G-invariant symplectic structure ω on $\mathcal{O}(\xi)$ given by

$$\omega\left(ad_X^*\xi, ad_Y^*\xi\right) = \langle \xi, [X,Y]\rangle.$$

This form is known as the *Kirillov-Kostant-Souriau symplectic form* (KKS).

Example 1.2. Let m and n be two natural numbers with $m \le n$ and consider the space of

m-dimensional subspaces of $\mathbb{C}^n$ known as the complex Grassmannian $Gr(m,n)$. This space is isomorphic to the coadjoint orbit of the m-th fundamental weight [D] (i.e. the highest weight of the representation of $GL_n(\mathbb{C})$ on $\bigwedge^m(\mathbb{C}^n)$). The KKS symplectic form restricted to this orbit defines a symplectic form Ω on $Gr(m,n)$.

1.1.2 Moment map and symplectic reduction

Let (M,ω) be a symplectic manifold equipped with the action of a Lie group G.

Definition 1.3. The action is called *Hamiltonian* if there exists a map $\mu : M \to \mathfrak{g}^*$ such that

1. $$d\mu^X = \iota_{X^\sharp}\omega,$$

   where for each $X \in \mathfrak{g}$ the map $\mu^X : M \to \mathbb{R}$ is given by $\mu^X(p) = \langle \mu(p), X\rangle$ (the component of μ along X) and $X^\sharp$ is the vector field generated by the one parameter subgroup

   $$\{\exp(tX) \;\; t \in \mathbb{R}\}$$

   (note that μ^X is a Hamiltonian function for the vector field $X^\sharp$).

2. μ is G-equivariant with respect to the action on M and the coadjoint action on $\mathfrak{g}^*$, i.e.

   $$\mu(g \cdot p) = Ad_g^* \circ \mu(p), \text{ for every } p \in M \text{ and } g \in G.$$

The map μ is known as a moment map for this action of G.

We also recall some important results such as the *Marsden-Weinstein-Meyer Theorem.*

Theorem 1.4. Let (M,ω) be a symplectic manifold equipped with a Hamiltonian action of a Lie group G with moment map μ. Assume that G acts freely on the level set $\mu^{-1}(0) \hookrightarrow M$. Then the orbit space $M_{red} := \mu^{-1}(0)/G$ is a manifold and $\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle. Moreover, there is a symplectic form ω_{red} on M_{red} such that $i^*\omega = \pi^*\omega_{red}$.

The symplectic manifold (M_{red}, ω_{red}) is called the *reduced space* at 0 and is denoted by $M /\!\!/_0 G$.

Remark 1.5. Theorem 1.4 is stated for the reduction at 0. If we want to take the reduced space at another regular value $\xi \in \mathfrak{g}^*$, it is necessary that $\mu^{-1}(\xi)$ is preserved by G, or equivalently that $Ad_g^* \xi = \xi$ for all $g \in G$. Note that this condition is clearly satisfied by $\xi = 0$. If G is a torus, any level set is preserved by G and taking a reduced space at ξ is equivalent to reducing at 0 using a shifted moment map $\mu' := \mu - \xi$.

Example 1.6. Consider the S^1-action on $\mathbb{C}^m$ defined by

$$\lambda \cdot z = z\lambda^{-1}.$$

This action is effective and Hamiltonian, with moment map

$$\mu(z) = -\frac{1}{2}|z|^2.$$

Then

$$\mathbb{C}^m /\!\!/_{-\frac{1}{2}} S^1 = \mu^{-1}\left(-\frac{1}{2}\right)/S^1 \cong \mathbb{CP}^{m-1}.$$

Moreover, the reduced symplectic form agrees with the *Fubini-Study* form on $\mathbb{CP}^{m-1}$.

We also recall a result that will be useful (for a proof see [LS]).

Proposition 1.7. Under the conditions of Theorem 1.4, if $G = G_1 \times G_2$ with G_1, G_2 connected Lie groups and such that the actions of G_1 and G_2 commute, then it is possible to perform reduction in stages, that is,

$$M /\!\!/_{(0,0)} G \simeq (M /\!\!/_0 G_1) /\!\!/_0 G_2.$$

1.2 Space of polygons in $\mathbb{R}^3$

As it was pointed out in the abstract, the space of polygons in $\mathbb{R}^3$, denoted by $Pol(\alpha_1, \ldots, \alpha_n)$ can be obtained in many different ways. We will discuss later several possible constructions, but in this preliminary section we will see it as the set of piecewise linear paths in $\mathbb{R}^3$ starting and ending at the origin modulo rotations, and such that the i-th step has length α_i, for $i = 1, \ldots, n$.

Assume that $S^2_{\alpha_i}$ is the sphere in $\mathbb{R}^3$ of radius α_i and consider the manifold of n-sided polygonal paths in $\mathbb{R}^3$

$$\prod_{1 \le i \le n} S^2_{\alpha_i} \subset (\mathbb{R}^3)^n.$$

Then we define $Pol(\alpha_1,\ldots,\alpha_n)$ as

$$Pol(\alpha_1,\ldots,\alpha_n) := \left\{(v_1,\ldots,v_n) \in \prod_{1\le i\le n} S^2_{\alpha_i} : \sum_i v_i = 0\right\}/SO(3), \tag{1.2.1}$$

where $SO(3)$ acts diagonally.

If the equation $\sum_{i=1}^m \varepsilon_i\alpha_i = 0$ has no solution with $\varepsilon_i = \pm 1$ this space is a smooth Kähler manifold of (real) dimension $2(n-3)$. This condition means that there are no polygons in $Pol(\alpha_1,\ldots,\alpha_n)$ contained in a line.

Identifying $\mathbb{R}^3$ with $\mathfrak{so}(3)^*$, the spheres $S^2_{\alpha_i}$ can be seen as coadjoint orbits and the KKS form gives a symplectic form with symplectic volume $2\alpha_i$. Consequently, $\prod_{1\le i\le n} S^2_{\alpha_i}$ can be seen as the product of n-coadjoint orbits. The diagonal coadjoint action of $SO(3)$ on this space has moment map

$$\mu(v_1,\ldots,v_n) = \sum_{1\le i\le n} v_i, \tag{1.2.2}$$

and so

$$Pol(\alpha_1,\ldots,\alpha_n) = \prod_{1\le i\le n} S^2_{\alpha_i} /\!\!/_0\, SO(3).$$

Note that, using the isomorphism $SU(2)/\mathbb{Z}_2 \simeq SO(3)$, this space can also be obtained as a symplectic reduction by $SU(2)/\mathbb{Z}_2$.

This construction can be generalized to any other Lie group G taking its coadjoint orbits and the diagonal coadjoint action. Such action is Hamiltonian with moment map as in 1.2.2, and so we can define the corresponding polygon space as

$$Pol_G(\underline{\alpha}) := \prod_{i=1}^n \mathcal{O}(\alpha_i) /\!\!/_0\, G$$

for any regular value $\underline{\alpha} \in \oplus_{i=1}^n \mathfrak{g}^*$.

As an example we present in this preliminary chapter the case $G = SO(4)$. In the next chapters we will study the more interesting case of $G = PU(m) = SU(m)/\mathbb{Z}_m$.

1.3 Polygon spaces for $G = SO(4)$

Let

$$SO(4) = \{A \in \mathrm{GL}(4,\mathbb{R}) : A^T = A^{-1},\ \det(A) = 1\}$$

be the special orthogonal group and let $O_{\rho,\kappa}$ be the coadjoint orbit in $\mathfrak{so}(4)^*$ of

$$I_{\rho,\kappa} = \begin{pmatrix} 0 & \rho & 0 & 0 \\ -\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa \\ 0 & 0 & -\kappa & 0 \end{pmatrix}.$$

This orbit is the space of skew-symmetric matrices that can be obtaind from $I_{\rho,\kappa}$ by conjugation by elements of $SO(4)$. Note that we are using the identification between adjoint and coadjoint orbits.

Proposition 1.8. Let $Pol_{SO(4)}(\underline{\rho},\underline{\kappa})$ be the reduced space

$$Pol_{SO(4)}(\underline{\rho},\underline{\kappa}) = \left(\prod_{i=1}^{n} O_{\rho_i,\kappa_i}\right) /\!\!/_0\, SO(4)$$

for the diagonal coadjoint action of $SO(4)$. Then

$$Pol_{SO(4)}(\underline{\rho},\underline{\kappa}) \simeq Pol_{SO(3)}(\underline{\alpha}') \times Pol_{SO(3)}(\underline{\alpha}''),$$

where

$$\alpha_i' := |\rho_i + \kappa_i| \quad \text{and} \quad \alpha_i'' := |\rho_i - \kappa_i|.$$

Proof. The proof has two steps. First we show that the coadjoint orbits of $SO(4)$ are isomorphic to the product of two spheres in $\mathbb{R}^3$ and so the coadjoint action of $SO(4)$ can be extended to $S^2 \times S^2$. In the second step, we consider the covering map $SO(4) \to SO(3) \times SO(3)$. This map gives us a second action of $SO(4)$ on $S^2 \times S^2$. Since such map is a covering, it is surjective and therefore this action is equivalent to the action of $SO(3) \times SO(3)$ on $S^2 \times S^2$. Finally, we show that the two actions of $SO(4)$ are equivalent.

By [BCR, Section 3] there is an isomorphism

$$\begin{aligned} \psi : O_{\rho,\kappa}(A) \to \quad & S^2_{|\rho+\kappa|} \times S^2_{|\rho-\kappa|} \\ A \mapsto \quad & \left(\begin{pmatrix} -a_{14} - a_{23} \\ a_{13} - a_{24} \\ -a_{12} - a_{34} \end{pmatrix}, \begin{pmatrix} a_{14} - a_{23} \\ a_{13} + a_{24} \\ a_{12} - a_{34} \end{pmatrix} \right), \end{aligned}$$

where $A = (a_{ij}) \in \mathcal{O}_{\rho,\kappa}$. Note that the values $|\rho+\kappa|$ and $|\rho-\kappa|$ are given by $\sqrt{2c_1+2c_2}$ and $\sqrt{2c_1-2c_2}$ respectively, where c_1, c_2 are the values of the Casimir functions at A, which are defined as

$$c_1(A) = \frac{1}{2}\sum a_{ij}^2 \quad \text{and} \quad c_2(A) = -Pf(A).$$

Now we take the map $\varphi : SO(4) \to SO(3)\times SO(3)$ given by

$$X \mapsto \left(\begin{pmatrix} X_{11,44}+X_{12,43} & X_{12,44}-X_{11,43} & X_{11,42}+X_{13,44} \\ -X_{12,33}+X_{11,34} & X_{11,33}-X_{12,34} & -X_{11,32}-X_{13,34} \\ X_{11,24}+X_{12,23} & X_{12,24}-X_{11,23} & X_{11,22}+X_{13,24} \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} X_{11,44}-X_{12,43} & X_{12,44}+X_{11,43} & X_{11,42}-X_{13,44} \\ -X_{12,33}+X_{11,34} & X_{11,33}+X_{12,34} & -X_{11,32}+X_{13,34} \\ -X_{11,24}+X_{12,23} & -X_{12,24}-X_{11,23} & X_{11,22}-X_{13,24} \end{pmatrix} \right),$$

where, for every matrix $X \in SO(4)$, we let $X_{ij,kl}$ denote the 2×2 minor

$$X_{ij,kl} := \begin{vmatrix} x_{ij} & x_{il} \\ x_{kj} & x_{kl} \end{vmatrix}.$$

We just need to show that ψ is equivariant for the actions of $SO(4)$ on $\mathcal{O}_{\rho,\kappa}$ and on

$$S^2_{|\rho+\kappa|} \times S^2_{|\rho-\kappa|}$$

via φ.

A matrix $X = (x_{ij})_{ij} \in SO(4)$ acts on an element $A = (a_{kl})_{kl} \in \mathfrak{so}(4)^*$ as

$$\mathrm{Ad}_X(A) = XAX^T = \left(\sum_{1\le k<l\le 4} X_{ik,jl} a_{kl} \right)_{ij},$$

where we used the fact that $X^{-1} = X^T$ and $A^T = -A$. It is known that for any matrix $X \in SO(4)$, if

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then $\det(A) = \det(D)$ and therefore $X_{11,22} = X_{33,44}$. Moreover, multiplying X on both sides by matrices E_{ij} obtained from the identity matrix by exchanging the rows i and j, we get a new matrix in $SO(4)$ with the same entries as X, but with the rows/columns exchanged. Hence, we get that

$$X_{\sigma(1)\tau(1),\sigma(2)\tau(2)} = X_{\sigma(3)\tau(3),\sigma(4)\tau(4)}$$

for any pair of even permutations σ, τ. Therefore

$$\psi(\mathrm{Ad}_X(A)) = \psi(XAX^T) = (\varphi_1(X) \cdot \psi_1(A), \varphi_2(X) \cdot \psi_2(A))$$

as we wanted to show. $\square$

The next natural case is the matrix group $SU(3)$ and, more generally, $SU(m)$. This will be studied in the following chapters.

Chapter 2

Preliminaries

2.1 Generalities on $SU(m)$

In this section we recall some well-known facts about special unitary groups (see for example [A, Sections 3.2, 4.2] for additional details).

Let

$$U(m) = \{A \in \mathcal{M}_{m\times m}(\mathbb{C}) : \ A^{-1} = A^*\},$$

be the *unitary group of degree m* and let $SU(m)$ be the *special unitary group of degree m,* i.e. the subgroup of $U(m)$ of matrices with determinant 1.

Both groups are Lie groups with Lie algebras

$$\mathfrak{u}(m) = \{\xi \in \mathcal{M}_{m\times m}(\mathbb{C}) : \ \xi^* = -\xi\}$$

and

$$\mathfrak{su}(m) = \{\xi \in \mathfrak{u}(m) : \ \mathrm{tr}(\xi) = 0\}.$$

Let $\mathcal{H}_m$ be vector space of $m \times m$ Hermitian matrices. We can identify this space with $\sqrt{-1}\mathfrak{u}(m)$. Using the pairing $\langle \xi, X\rangle = \mathrm{Im}\ \mathrm{tr}(\xi X)$, where $\xi \in \mathfrak{u}(m)$ and $X \in \mathcal{H}_m$, we obtain the identification $\sqrt{-1}\mathfrak{u}(m) \cong \mathcal{H}_m \cong \mathfrak{u}(m)^*$ as well.

Analogously, we can consider $\mathcal{H}_m^0$ the subspace of $\mathcal{H}_m$ of traceless matrices, and identify it with $\sqrt{-1}\mathfrak{su}(m) \cong \mathcal{H}_m^0 \cong \mathfrak{su}(m)^*$.

Remark 2.1. In both cases such pairing agrees with the Killing form, which is degenerate for $U(m)$ and non-degenerate for $SU(m)$.

2.1.1 Principal Weyl Chamber in $SU(m)$

Recall that the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{su}(m)$ is defined as

$$\mathfrak{h} = \left\{ \sqrt{-1}H : \ H \text{ diagonal traceless matrix with entries in } \mathbb{R} \right\} \cong \mathbb{R}^{m-1}.$$

First we want to choose a basis of $\mathfrak{h}$ and use it to find nice expressions for the principal Weyl chamber and its walls, which is defined as

$$\overline{\Delta} = \{H \in \mathfrak{h} | \ \ 0 \leq \langle H, H_\gamma \rangle \text{ for all simple roots } \gamma\}.$$

Proposition 2.2. The set of roots is $\{\gamma_{ij}\}$, where

$$\gamma_{ij}(H) := H_{ii} - H_{jj}, \quad \forall i,j \in \{1,\ldots,m\},\ i \neq j.$$

Proof. Since $\mathfrak{g}$ is a simple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$, it admits a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\gamma \mathfrak{g}_\gamma,$$

where γ runs over all the non-zero roots and

$$\mathfrak{g}_\gamma := \{X \in \mathfrak{su}(m) : \ [H,X] = \mathrm{ad}_H X = \gamma(H)X,\ \forall H \in \mathfrak{h}\}.$$

Let $\gamma \in \mathfrak{h}^*, H \in \mathfrak{h}, X \in \mathfrak{su}(m)$, then $\gamma(H) = \sum_{i=1}^{m-1} a_i H_{ii}$ for some $a_1,\ldots,a_{m-1} \in \mathbb{R}$ and, since

$$([H,X])_{ij} = (H_{ii} - H_{jj})X_{ij},$$

the subspace $\mathfrak{g}_\gamma$ is nontrivial if and only if $\gamma(H) = H_{ii} - H_{jj}$ for some $i,j \in \{1,\ldots,m\}$ with $i \neq j$. Denote this root by γ_{ij}. □

Note that, in particular, $\gamma_{ji} = -\gamma_{ij}$ and $\gamma_{ij} = \gamma_{ik} + \gamma_{kj}$ for $i < k < j$. In this case the eigenvectors of γ_{ij} are τE^{ij}, where $\tau \in \mathbb{R}^+$ and E^{ij} is the matrix with zeros everywhere and a one in the $(i,j)-$position.

In order to find a proper basis for $\mathfrak{h}$, we need to compute the Lie brackets of pairs of eigenvectors of $\gamma_{ij}, -\gamma_{ij}$ such that their Killing form is $2m$. Hence

$$\langle \tau E^{ij}, \tau E^{ji} \rangle = 2m \operatorname{tr}(\tau^2 (E^{ij} E^{ji})) = 2m\tau^2 = 2m,$$

i.e. $\tau = 1$ and so $H^{ij} := [E^{ij}, E^{ji}] = E^{ii} - E^{jj}$. Then set

$$\{iH^{jj+1};\ 1 \leq j \leq m-1\}$$

as our basis for $\mathfrak{h}$.

Recall that the principal Weyl chamber is a convex space limited by its walls. Therefore any point in its interior can be obtained as a linear combination of points on the walls.

Let $\lambda_1, \ldots, \lambda_{m-1} \in \mathbb{R}$, then

$$H = \sqrt{-1} \sum_{j=1}^{m-1} \lambda_j H^{jj+1} = \sqrt{-1} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 - \lambda_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda_{m-1} \end{pmatrix}$$

and so we have $m-1$ inequalities

$$\begin{aligned} 0 \leq \langle H, iH^{12} \rangle &= -(2\lambda_1 - \lambda_2) \Leftrightarrow \lambda_2 \geq 2\lambda_1 \\ 0 \leq \langle H, iH^{jj+1} \rangle &= -(2\lambda_j - \lambda_{j+1} - \lambda_{j-1}) \Leftrightarrow \lambda_{j+1} + \lambda_{j-1} \geq 2\lambda_j \\ 0 \leq \langle H, iH^{m-1m} \rangle &= -(2\lambda_{m-1} - \lambda_{m-2}) \Leftrightarrow \lambda_{m-2} \geq 2\lambda_{m-1}. \end{aligned}$$

In particular, $\lambda_j \leq 0$ and the walls of the Weyl chamber are reached when $m-2$ of the inequalities are equalities. If the j-th inequality is the one that is not an equality, the corresponding Weyl wall is given by the matrices

$$H = -\sqrt{-1}\lambda \begin{pmatrix} (m-j)Id_j & 0 \\ 0 & -jId_{m-j} \end{pmatrix} \text{ for } \lambda > 0 \text{ and } 1 \leq j \leq m-1.$$

Proposition 2.3. ([A, Proposition 4.1.2]) Each adjoint orbit intersects the principal Weyl chamber in exactly one point Δ.

2.2 Quiver Varieties

We will see later that polygon spaces can be obtained from quiver varieties. In this section we recall the basic definitions related to quiver varieties based on [Kam].

Definition 2.4. A *quiver* is a directed graph $Q = (V, E)$, where $V = \{1, \ldots, n\}$ is the set of vertices and $E \subset V \times V$ is the set of oriented edges. For each edge (i, j), we will say that i is the *source* and j is the *target.*

A *representation* of a quiver Q, denoted by Rep (Q), is a choice of finite dimensional complex vector spaces V_i for each vertex $i \in V$ and linear maps $E_{ij} : V_i \longrightarrow V_j$ for each edge $(i, j) \in E$.

This definition generalizes, in some sense, the one of representation for a group. As in the case of finite group representations, we can build a functor between quivers and its representations (up to isomorphism).

Let $d_i := \dim V_i$ and

$$Hom(V) := \oplus_{(i,j)\in E} Hom(V_i, V_j).$$

Since V_i, V_j are Hermitian vector spaces, $Hom(V_i, V_j)$ has a Hermitian form given by $\langle A, B\rangle = \mathrm{tr}(AB^*)$, where B^* is the Hermitian adjoint of B. Therefore $Hom(V)$ admits a symplectic structure defined as

$$\omega(A, B) = 2\,\mathrm{Im}\,(\mathrm{tr}\ (AB^*))$$

and so it is a symplectic vector space.

Then $U(V) := U(d_1) \times \cdots \times U(d_n)$ acts on $Hom(V)$ in a Hamiltonian way via

$$(g_1, \ldots, g_n) \cdot (E_{ij})_{ij} = (g_j E_{ij} g_i^{-1})_{ij}.$$

A moment map for this action is given by

$$\mu_{U(V)}\left((B_{ij})_{ij}\right) = \left(\sum_{(j,1)\in E} B_{j1}B_{j1}^* - \sum_{(1,i)\in E} B_{1i}^* B_{1i}, \ldots, \sum_{(j,n)\in E} B_{jn}B_{jn}^* - \sum_{(n,i)\in E} B_{ni}^* B_{ni}\right).$$

Let $U(1) \subset U(V)$ be the subcircle formed by scalar matrices (a single eigenvalue). Since the action of $U(1)$ is trivial, we consider the action of $G := U(V)/U(1)$ and perform symplectic reduction.

Definition 2.5. Let $\underline{\alpha} \in (\mathbb{R}^+)^n$. Then the *quiver variety* for Q (of dimension d) at $\underline{\alpha}$ is

$$Hom(V) /\!\!/_{\underline{\alpha}} G := \mu_{U(V)}^{-1}(\underline{\alpha})/G.$$

Remark 2.6. Quiver varieties are actually complex algebraic varieties. This can be seen from another construction that identifies $Hom(V) /\!\!/_{\underline{\alpha}} G$ with a geometric invariant theory quotient of $Hom(V)$ by $G_{\mathbb{C}} = GL(d_1,\mathbb{C}) \times \cdots \times GL(d_n,\mathbb{C})$ (the complexification of $U(V)$), under a suitable stability condition (see [Ki]).

2.3 Equivariant Cohomology

In this section we compile some results that will be useful later. Although we will use them adjusted to our case, they are stated on a more general setting.

Let M be a manifold equipped with a smooth action of a Lie group G. The *G-equivariant cohomology* of M can be defined in different ways. We present here the most general definition, but it can also be defined as the usual de Rham cohomology with G-equivariant differential forms, see [GZ], [K] oder [L]. The two definitions agree when G is a connected compact Lie group and M is a differentiable manifold.

Let BG be the classifying space for G-principal bundles and let $EG \to BG$ be a fixed universal G-bundle, i.e. a contractible space such that every G-bundle over M can be obtained as a pull-back bundle f^*EG for a suitable map $f : M \to BG$.

Consider

$$M_G := EG \times_G M = EG \times M/\sim$$

where $(pg^{-1}, q) \sim (p, gq)$ for $p \in EG, q \in M, g \in G$ and we assume that G acts on EG on the right and on M on the left. Thus $\pi : M_G \to BG$ is the bundle with fiber M over the classifying space $BG = EG/G$ and π is the natural projection. Then the G-equivariant cohomology of M is defined as

$$H_G^*(M) := H^*(M_G)$$

and it is a contravariant functor from G-spaces to modules over the base ring

$$H_G^* = H_G^*(\{pt\}) = H^*(BG).$$

Remark 2.7.

1. The above definition arises as a generalization of the cohomology of the quotient space, since M_G is homotopy equivalent to M/G when G acts freely. This is because, if

$$\begin{aligned} \sigma : M_G &\to M/G \\ [(p,q)] &\mapsto [q], \end{aligned}$$

then

$$\sigma^{-1}(Gm) = EG/G_m \simeq BG_m,$$

where G_m is the stabilizer of m. In this case $G_m = \{e_g\}$ and hence σ is a fibration with fiber EG. Since EG is contractible we get $M_G \simeq M/G$.

2. Note that ordinary data on M should be thought of as being given on the fiber over the basepoint of BG, whereas equivariant data extend these to all of M_G.

3. Let $H \subset G$ be a subgroup. Then the restriction of the G-action gives a H-action. Therefore the H-equivariant cohomology ring is well defined and there is a natural restriction map

$$r_G^H : H_G^*(M) \to H_H^*(M). \tag{2.3.1}$$

Example 2.8. Let $G = S^1$ and $M = \{pt\}$. It can be showed that $EG \to BG$, with

$$EG = S^\infty$$

and $BG = EG/G = \mathbb{CP}^\infty$ is a universal S^1-bundle. Hence

$$H_{S^1}^*(\{pt\};\mathbb{R}) = H^*(\mathbb{CP}^\infty;\mathbb{R}) = \mathbb{R}[u].$$

Analogously, it can be seen that

$$H_{(S^1)^n}^*(\{pt\}) = \mathbb{R}[u_1,\dots,u_n].$$

2.3.1 Functoriality of the equivariant constructions

The functorial nature of the construction $M \to M_G$, enables one to extend all the concepts of ordinary cohomology to the equivariant one.

For example, let V be a vector bundle over M, on which G acts (lifting the action of G on M). Then it is straightforward to define a vector bundle V_G, over M_G, which extends the original V to all M_G. The characteristic classes of such bundles naturally take values in $H^*_G(M)$ and incorporate the lifting data.

In particular, we introduce the *equivariant Chern class,* which will be used later.

Equivariant Chern class

Let V be a G-equivariant complex vector bundle over M and extend it to

$$EG \times_G V = V_G \to M_G = M \times_G EG.$$

Then V_G is a complex vector bundle with the same rank as V and therefore we can consider the *equivariant Chern classes*

$$c_i^G(V) := c_i(V_G) \in H^*(M_G) = H^*_G(M),$$

defined by $c_i(V_G) = f^*(c_i)$, where $c_i \in H^*(BG)$ is the i^{th} universal Chern class and $f : M_G \to BG$ is the map defining $V_G \to M_G$.

Remark 2.9. Although the construction is analogous to the one for ordinary cohomology, it presents some differences. For example, the equivariant Chern class may not vanish even if $M = \{pt\}$ and the fiber bundle is trivial.

Other properties from usual cohomology that are satisfied in equivariant cohomology are the following:

1. excision,
2. the Mayer-Vietoris sequence,
3. the Künneth formula and
4. Poincaré duality (for M smooth orientable).

2.3.2 Kirwan Map

Until the end of this section, assume that (M,ω) is a symplectic manifold and G is a Lie group, which acts hamiltonianly on M with moment map μ.

Then, for a regular value α of μ, there is a natural map

$$H^*_G(M) \to H^*_G(\mu^{-1}(\alpha)).$$

If the action of G is free on the level set $\mu^{-1}(\alpha)$, we have

$$H^*_G(\mu^{-1}(\alpha)) = H^*(\mu^{-1}(\alpha)/G) = H^*(M /\!\!/_\alpha G).$$

The map

$$H^*_G(M) \to H^*(M /\!\!/_\alpha G)$$

is known as the *Kirwan map.*

Kirwan shows in [Kir, Sections 7, 8] that if M is compact then this map is surjective (for rational coefficients).

2.4 Wall Crossing

Let M be a smooth manifold with a Hamiltonian action of a torus T, and moment map μ and let $\hat{T} \subset T$ be a subtorus of T.

First we need to recall the following result (for a proof see [C]).

Proposition 2.10. Let G be a Lie group acting on the smooth manifold M. Then if the action is proper, the subset of fixed points M^G is a smooth submanifold of M. Moreover, if the action is symplectic, M^G is a symplectic submanifold.

The short exact sequence of groups $\hat{T} \hookrightarrow T \twoheadrightarrow T/\hat{T}$ induces the following exact sequences of Lie algebras and of their duals

$$\begin{array}{ccccc} Lie(\hat{T}) & \hookrightarrow & \mathfrak{t} & \twoheadrightarrow & Lie(T/\hat{T}) \\ Lie(\hat{T})^* & \twoheadleftarrow & \mathfrak{t}^* & \hookleftarrow & Lie(T/\hat{T})^*. \end{array}$$

Hence for any subtorus $\hat{T} \subset T$ we will consider $Lie(T/\hat{T})^*$ to be a subspace of $\mathfrak{t}^*$.

Proposition 2.11. ([Ma1, Fact 1.1]) Let $\hat{T} \subset T$ be a subtorus of T. Then $\hat{T}$ fixes a point $p \in M$ if and only if

$$d\mu(T_pM) \subset Lie(T/\hat{T})^*.$$

This result has a global consequence.

Proposition 2.12. ([Ma1, Fact 1.2]) The moment map μ maps each component of the fixed point set $M^{\hat{T}}$ to an affine translate of $Lie(T/\hat{T})^*$ in $\mathfrak{t}^*$.

Definition 2.13. Let M^H be the submanifold of fixed points for some subcircle $H \cong S^1$ of T. Then each connected component of its image by μ, will be called a *wall* in $\mathfrak{t}^*$.

Remark 2.14. From Proposition 2.10, a wall is contained in an affine hyperplane in $\mathfrak{t}^*$ parallel to $Lie(T/\hat{T})^*$. For any x in a wall, we will say x lies in the *interior* of the wall if the stabilizer of any point $p \in \mu^{-1}(x)$ is contained in a circle.

The image of the moment map μ_T (the moment polytope) is a convex subspace of $\mathfrak{t}^* \cong \mathbb{R}^n$ bounded by the *outer walls.* Note that, in the case of toric varieties, every wall is an outer wall.

Thus for every $\underline{\alpha}_1, \underline{\alpha}_2$ in the interior of the moment polytope, there exists a path Z connecting $\underline{\alpha}_1$ and $\underline{\alpha}_2$, which is contained in $Im(\mu_T)$.

Given a path Z that crosses a wall $\mu(M^H)$ at a point x, we can define an orientation on H as follows: we orient Z so that the positive direction goes from $\underline{\alpha}_1$ to $\underline{\alpha}_2$. Then a positive tangent vector in $T_xZ \subset \mathfrak{t}^*$, defines a linear functional on $\mathfrak{t}$, and this restricts to a nonzero functional on $\mathfrak{h}$; we then orient H to be positive with respect to this functional.

Moreover, such path can be chosen to be transverse to the walls that it crosses and so $\mu_T^{-1}(Z)$ is a submanifold of M.

The following results are due to Martin (see [Ma1, Section 1]).

Proposition 2.15. The path Z is transverse to the walls if and only if their intersection is transversal and in the interior of the walls.

Proof. Let x be a point in the intersection of Z with a wall and let $p \in \mu^{-1}(x)$. Since x lies on a wall, the point p is fixed by some subtorus $\hat{T}$ of T. The intersection is transversal if and only if

$$T_x\mathfrak{t}^* = d\mu(T_pM) \oplus T_xZ.$$

Since $T_x\mathfrak{t}^* = \mathfrak{t}^*$ and $\dim Z = 1$, we have

$$\dim d\mu(T_pM) = \dim T - 1.$$

Using Proposition 2.11, we have $d\mu(T_pM) \subset Lie(T/\hat{T})^*$, and so the intersection is transversal if and only if $\dim \hat{T} = 1$. We conclude that p is not fixed by any other subcircle and so x cannot lie in the intersection of two walls. □

Remark 2.16. In particular, for every $p \in \mu^{-1}(Z)$, the stabilizer subgroup of p is either finite or 1-dimensional. Moreover, the composition

$$T_pM^H \xrightarrow{(d\mu)_p} T_x\mathfrak{t}^* \to \nu_x Z \tag{2.4.1}$$

is surjective. Now we can identify the normal bundles

$$\mu^* : \nu_Z \longrightarrow \nu_{\mu^{-1}(Z)},$$

where μ^* is the pullback map. Then the map in (2.4.1) can be factored as

$$T_pM^H \hookrightarrow T_pM \to \nu_p\mu^{-1}(Z) \to \nu_x Z$$

and, since the map is surjective, we conclude that $\mu^{-1}(Z)$ is transverse to the submanifold of fixed points M^H.

Note that x is not a regular value of μ. However, since H acts trivially on the manifold M^H and $\mu(M^H)$ lies in an affine hyperplane parallel to $Lie(T/H)^*$, we can consider the reduction with T/H instead. In this case, x is a regular value in such hyperplane for μ_T thought of as a map to $\mu(M^H)$. The fact that x is a regular value is equivalent to the condition that Z crosses each wall in its interior. In particular, from [Ma1, Section 2] we have the following result.

Theorem 2.17. Under the above assumptions, $\mu^{-1}(x) \cap M^H$ is a compact closed submanifold of M^H and its quotient $M^H /\!\!/_x T$ is a compact symplectic orbifold.

Chapter 3

Construction of polygon spaces with edges in $\mathbb{CP}^{m-1}$

This chapter is devoted to the construction of our object of study, the space of polygons with edges in $\mathbb{CP}^{m-1}$. There are different ways to obtain it and we will focus on the most useful for our purposes.

3.1 Reduction in stages and Gelfand McPherson correspondence

The *Gelfand-MacPherson* correspondence plays a crucial role in our study of polygon spaces. In this section we study this correspondence applied to our case.

A symplectic version for polygons with edges in $\mathbb{CP}^1$ was introduced in [HK1] by Hausmann and Knutson. For higher dimensional projective spaces this correspondence is used by Flaschka and Millson in [FM1] and [FM2].

3.1.1 Degenerate orbits and projective spaces

For $\alpha \in \mathbb{R}^+$, let $O^d_{\mathfrak{su}(m)}(\alpha)$ denote the degenerate coadjoint orbit of

$$\frac{\alpha}{m}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -(m-1) \end{pmatrix} \tag{3.1.1}$$

in $\mathfrak{su}(m)^*$. This coadjoint orbit can be identified with a projective space.

Proposition 3.1. There is a symplectomorphism

$$\left(O^d_{\mathfrak{su}(m)}(\alpha), \omega_{KKS}\right) \cong \left(\mathbb{CP}^{m-1}, (2\alpha)\omega_{FS}\right),$$

where ω_{KKS} is the *Kostant-Kirilov-Soriau* form and ω_{FS} is the usual *Fubini-Study* form.

Proof. Let A be the matrix in (3.1.1). The stabilizer of A for the $SU(m)$ action on $\mathfrak{su}(m)^*$ is $SU(m-1) \times S^1$ and so

$$O^d_{\mathfrak{su}(m)}(\alpha) \cong SU(m)/(SU(m-1) \times S^1).$$

Now $SU(m)$ acts transitively on S^{2m-1} and the stabilizer of a point is $SU(m-1)$. Consequently $SU(m)/SU(m-1) \cong S^{2m-1}$ and so

$$O^d_{\mathfrak{su}(m)}(\alpha) \cong S^{2m-1}/S^1 \cong \mathbb{CP}^{m-1}.$$

It remains to show the relation between the two symplectic forms. First we need to describe explicitly the diffeomorphism between $O^d_{\mathfrak{su}(m)}(\alpha)$ and $\mathbb{CP}^{m-1}$. In order to find this map we compute the moment map for the action of $SU(m)$ on $(\mathbb{CP}^{m-1}, \omega_{FS})$.

Claim: The moment map for the action of $SU(m)$ on $(\mathbb{CP}^{m-1}, \omega_{FS})$ is

$$\begin{aligned} \phi_{SU(m)} : \mathbb{CP}^{m-1} &\to \mathfrak{su}(\mathfrak{m})^* \\ [x] &\mapsto -\frac{1}{2|x|^2}\left(xx^* - \frac{|x|^2}{m} Id_m\right). \end{aligned}$$

Proof of claim: Since the action of $SU(m)$ on $\mathbb{CP}^{m-1}$ is transitive, it is enough to check that

$$\langle (d\phi_{SU(m)})_{[x]}(v), A\rangle = \text{Im tr}\,((d\phi_{SU(m)})_{[x]}(v)A) = (\omega_{FS})_{[x]}(Ax, v),$$

for every $v \in T_{[x]}\mathbb{CP}^{m-1}$ and $A \in \mathfrak{su}(m)$ at $[x] = [0 : \cdots : 0 : 1] \in \mathbb{CP}^{m-1}$.

We can identify

$$T_{[x]}\mathbb{CP}^{m-1} \cong \{(z_1, \ldots, z_{m-1}, 0) \in \mathbb{C}^m\} \cong \mathbb{C}^{m-1}$$

and so in the coordinates

$$(z_1, \ldots, z_{m-1}) \mapsto [z_1 : \cdots : z_{m-1} : 1]$$

at $[x]$, the Fubini-Study symplectic form is given by

$$(\omega_{FS})_{[x]}(v, w) = \langle v, w\rangle,$$

where $\langle v, w\rangle = \text{Im tr}\,(w^* v)$ is the Hermitian product.

Therefore, by the choice of $[x]$, we have that

$$(d\phi_{SU(m)})_{[x]}(v) = vx^* + xv^*, \tag{3.1.2}$$

and so

$$\left\langle (d\phi_{SU(m)})_{[x]}(v), A\right\rangle = \frac{1}{2}\text{Im tr}\,((vx^* + xv^*)A).$$

Note that

$$\text{tr}\,(vx^*A) = \overline{\text{tr}\,(A^*xv^*)} = -\overline{\text{tr}(Axv^*)} = -\overline{\text{tr}(v^*Ax)} = -\overline{\langle Ax, v\rangle} = -\langle v, Ax\rangle$$

and analogously $\text{tr}(xv^*A) = -\overline{\langle Av, x\rangle} = -\langle x, Av\rangle$.

Combining this with the fact that $\langle Av, w\rangle = -\langle v, Aw\rangle$, we get

$$\left\langle (d\phi_{SU(m)})_{[x]}(v), A\right\rangle = \frac{1}{2}\text{Im}\left(\langle Ax, v\rangle - \overline{\langle Ax, v\rangle}\right) = (\omega_{FS})_{[x]}(Ax, v). \tag{3.1.3}$$

□

Now the map

$$\phi := (2\alpha)\phi_{SU(m)} : \left(\mathbb{CP}^{m-1}, (2\alpha)\omega_{FS}\right) \to \left(O^d_{\mathfrak{su}(m)}(\alpha), \omega_{KKS}\right)$$

is a symplectomorphism. Indeed, since $\phi_{SU(m)}$ is $SU(m)$-equivariant, it is again enough to check that

$$(2\alpha)\omega_{FS} = \phi^*\omega_{KKS}$$

at $[x] = [0 : \cdots : 0 : 1] \in \mathbb{CP}^{m-1}$. For this, let $v, w \in T_{[x]}\mathbb{CP}^{m-1}$. Then there exist $X, Y \in \mathfrak{su}(m)$ such that

$$(d\phi_{SU(m)})_{[x]}(v) = X \cdot \phi_{SU(m)}([x]) = [\phi_{SU(m)}([x]), X] = \alpha(Xxx^* - xx^*X)$$
$$(d\phi_{SU(m)})_{[x]}(w) = Y \cdot \phi_{SU(m)}([x]) = [\phi_{SU(m)}([x]), Y] = \alpha(Yxx^* - xx^*Y).$$

Moreover, from (3.1.2), we have that $Xx = v$ and $Yx = w$, and so

$$X = \begin{pmatrix} 0 & \dots & 0 & v_1 \\ \vdots & \ddots & \vdots & \vdots \\ -\overline{v_1} & \dots & -\overline{v_{m-1}} & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & \dots & 0 & w_1 \\ \vdots & \ddots & \vdots & \vdots \\ -\overline{w_1} & \dots & -\overline{w_{m-1}} & 0 \end{pmatrix}.$$

Therefore,

$$(\phi^*\omega_{KKS})_{[x]}(v, w) = \mathrm{Im}\,\mathrm{tr}\,(\phi([x])[X, Y]) = \alpha \mathrm{Im}\left(\sum (v_j\overline{w_j} - \overline{v_j}w_j)\right) = 2\alpha(\omega_{FS})_{[x]}(v, w).$$

□

Remark 3.2.

1. Another way to check the above isomorphism is the following. Abusing notation, let α also denote the matrix that defines the orbit and let $K = S(U(m-1) \times S^1) \subset SU(m)$ be its stabilizer. Then there is an isomorphism

$$\mathbb{CP}^{m-1} \cong GL_m(\mathbb{C})/P_{m-1,1} \cong SU(m)/K \cong O^d_{\mathfrak{su}(m)}(\alpha), \tag{3.1.4}$$

where $P_{m-1,m} \subset GL_m(\mathbb{C})$ is the parabolic subgroup which preserves the subspace of $\mathbb{C}^m$ defined by $x_1 = \cdots = x_{m-1} = 0$. The first and third isomorphisms in (3.1.4) come from the *orbit-stabilizer Theorem for Lie groups* (see for instance [Z]) and the second is the identification between compact and complex homogeneous spaces.

2. It is also possible to use the equations in (2.1.1) in order to check that the image of the map ϕ is precisely the orbit $O^d_{\mathfrak{su}(m)}(\alpha)$. Since ϕ is $SU(m)$-equivariant, it is enough to see that the image of $[0:\cdots:0:1]$ is the matrix

$$\frac{\alpha}{m}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -(m-1) \end{pmatrix},$$

which is the only point of intersection between $\mathrm{Im}(\phi)$ and the closure of the Weyl chamber $\overline{\Delta}$.

3. If we consider the action of $U(m)$ on $(\mathbb{CP}^{m-1}, \omega_{FS})$ instead of the action of $SU(m)$, we obtain a Hamiltonian action with moment map

$$\phi_{U(m)}([x]) \mapsto \frac{1}{2|x|^2}(xx^*).$$

This action is not effective since the center $Z(U(m)) = \{\lambda Id_m : \lambda \in S^1\} \cong S^1$ acts trivially. Note that, in this case, the image of $\widetilde{\phi} := (2\alpha)\phi_{U(m)}$ is the orbit $O^d_{\mathfrak{u}(m)}(\alpha)$ in $\mathfrak{u}(m)^*$ of the matrix

$$-\alpha\begin{pmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{pmatrix},$$

whose projection on $\mathfrak{su}(m)^*$ (which consists in taking the traceless parts) gives $O^d_{\mathfrak{su}(m)}(\alpha)$.

3.1.2 $Pol_m(\underline{\alpha})$ as reduced spaces of degenerate coadjoint orbits of $SU(m)$

Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}^+)^n$ and consider the diagonal action of $SU(m)$ on the product $\prod_{i=1}^n O^d_{\mathfrak{su}(m)}(\alpha_i)$ of coadjoint degenerate orbits. This action is Hamiltonian with moment map

$$\begin{array}{rcl} \mu_{SU(m)} : \prod_{i=1}^n O^d_{\mathfrak{su}(m)}(\alpha_i) & \to & \mathfrak{su}(m)^* \\ (A_1, \ldots, A_n) & \mapsto & \sum_{i=1}^n A_i. \end{array}$$

This action is not effective since the center

$$Z(SU(m)) = \{\xi_m Id_m, \text{ with } \xi_m \text{ a primitive } m \text{ root of unity}\} \cong \mathbb{Z}_m$$

of $SU(m)$ acts trivially.

Definition 3.3. The space $Pol_m(\underline{\alpha})$ of polygons with edges in $O^d_{\mathfrak{su}(m)}(\alpha_i) \simeq \mathbb{CP}^{m-1}$ is the symplectic reduced space

$$Pol_m(\underline{\alpha}) := \left(\prod_{i=1}^n O^d_{\mathfrak{su}(m)}(\alpha_i) \right) /\!\!/_0 \, (SU(m)/\mathbb{Z}_m).$$

Note that since $(\prod_{i=1}^n O^d_{\mathfrak{su}(m)}(\alpha_i), \omega_{KKS})$ is symplectomorphic to a product of projective spaces with the weighted symplectic form $(2\alpha_1)\omega_{FS} \oplus \cdots \oplus (2\alpha_n)\omega_{FS}$, the space $Pol_m(\underline{\alpha})$ can be seen as the space of polygons with edges in $\mathbb{CP}^{m-1}$.

We can obtain $Pol_m(\underline{\alpha})$ directly from $(\mathbb{C}^m)^n$ again by symplectic reduction. For that, consider the action of $(S^1)^n$ on $(\mathbb{C}^m)^n$ given by

$$\lambda \cdot x = (x_1 \lambda_1^{-1}, \ldots, x_n \lambda_n^{-1}),$$

where $x_j \in \mathbb{C}^m$ is a column vector, for $j = 1, \ldots, n$. This action is Hamiltonian with moment map

$$\mu_{(S^1)^n}(x) = -\frac{1}{2}(|x_1|^2, \ldots, |x_n|^2)$$

and, for a regular value α, we obtain

$$(\mathbb{C}^m)^n /\!\!/_{-\alpha} \, (S^1)^n = \{x \in (\mathbb{C}^m)^n : \, -|x_i|^2 = -2\alpha_i\}/(S^1)^n \cong (\mathbb{CP}^{m-1})^n.$$

From Example 1.6, we have that the corresponding symplectic form is

$$(2\alpha_1)\omega_{FS} \oplus \cdots \oplus (2\alpha_n)\omega_{FS}.$$

Hence we have that

$$Pol_m(\underline{\alpha}) = \left((\mathbb{C}^m)^n /\!\!/_{-2\alpha} (S^1)^n \right) /\!\!/_0 (SU(m)/\mathbb{Z}_m).$$

Note that the action of $SU(m) \times (S^1)^n$ on $(\mathbb{C}^m)^n$ defined by

$$(A; e_1, \ldots, e_n) \cdot (x_1, \ldots, x_n) = (A \cdot x_1 \cdot e_1^{-1}, \ldots, A \cdot x_n \cdot e_n^{-1}) \tag{3.1.5}$$

is not effective as

$$\{(\xi_m Id_m; \xi_m, \ldots, \xi_m), \text{ with } \xi_m \text{ a primitive } m\text{-root of unity}\} \cong \mathbb{Z}_m$$

fixes every point. Hence, considering the Hamiltonian actions of $\widetilde{G} := (SU(m) \times (S^1)^n)/\mathbb{Z}_m$ with moment map

$$(x_1, \ldots, x_n) \overset{\mu_{\widetilde{G}}}{\longmapsto} \left(\frac{1}{2}(xx^*)_0; -\frac{1}{2}|x_1|^2, \ldots, -\frac{1}{2}|x_n|^2 \right),$$

we can use reduction in stages to conclude that $Pol_m(\underline{\alpha})$ can be obtained directly from $(\mathbb{C}^m)^n$ by symplectic reduction as

$$Pol_m(\underline{\alpha}) = \mu_{\widetilde{G}}^{-1}(0; -\alpha_1, \ldots, -\alpha_n)/\widetilde{G} = (\mathbb{C}^m)^n /\!\!/_{(0;-\alpha_1,\ldots,-\alpha_n)} \widetilde{G}.$$

Remark 3.4.

1. We can instead consider the Hamiltonian action of $U(m) \times (S^1)^n$ with moment map

$$(x_1, \ldots, x_n) \overset{\mu_G}{\longmapsto} \left(\frac{1}{2}(xx^*); -\frac{1}{2}|x_1|^2, \ldots, -\frac{1}{2}|x_n|^2 \right).$$

   This action is not effective as the subgroup

$$\Gamma := \{(\lambda Id_m; \lambda, \ldots, \lambda) : \lambda \in S^1\}$$

fixes every point. Thus, taking $G := (U(m)\times(S^1)^n)/\Gamma$ and $p_{\underline{\alpha}} := \left(\frac{\sum\alpha_i}{2m}Id_m; -\alpha_1,\ldots,-\alpha_n\right)$, we obtain

$$Pol_m(\underline{\alpha}) = \mu_G^{-1}(p_{\underline{\alpha}})/G = (\mathbb{C}^m)^n /\!\!/_{p_{\underline{\alpha}}} G.$$

We fix this notation for the incoming chapters.

2. Let μ_G be the moment map for the action of G on $(\mathbb{C}^m)^n$ and φ the map

$$\begin{aligned} \varphi : (\mathbb{C}^m)^n &\rightarrow \oplus_{i=1}^n \mathfrak{su}(m)^* \\ x &\mapsto (xx^*)_0. \end{aligned}$$

According to [FR, Section 2.2], the image of $\varphi|_{\mu_G^{-1}(0)}$ is in $\mu_{SU(m)}^{-1}(0)$, where $\mu_{SU(m)}$ is the moment map for the action of $SU(m)$ on $\prod_{i=1}^n \mathfrak{su}(m)^*$.

Therefore we can embed the polygon spaces obtained from $(\mathbb{C}^m)^n$ and the action of G inherited from (3.1.5) inside the polygon space obtained from $\mathfrak{su}(m)^*$ and the coadjoint action of $SU(m)$. However, it is not possible to construct all the polygons in $\mathfrak{su}(m)^*$ this way since many elements of $\mathfrak{su}(m)^*$ are not in the image of φ. In fact only the degenerate orbits considered in the Proposition 3.1 are in the image of φ.

3.1.3 $Pol_m(\underline{\alpha})$ as reduced spaces from complex Grassmannians

Since $G = \left(U(m)\times(S^1)^n\right)/\Gamma$, we can perform reduction in stages in the opposite order obtaining what is known as the Gelfand-McPherson correspondence [GGMS].

Let us consider the space $\mathcal{M}_{m\times n}(\mathbb{C}) \cong (\mathbb{C}^m)^n$ of $m\times n$ complex matrices. The group $U(m)\times U(n)$ acts on $\mathcal{M}_{m\times n}(\mathbb{C})$ by

$$(A,B)\cdot N = ANB^{-1}, \text{ for } A\in U(m), B\in U(n),$$

where we take the natural symplectic structure given by the imaginary part of the standard Hermitian product on $(\mathbb{C}^m)^n$. The moment maps for the actions of $U(m), U(n)$ are respectively

$$\mu_{U(m)}(N) = \frac{1}{2}NN^* \quad \text{and} \quad \mu_{U(n)}(N) = -\frac{1}{2}N^*N.$$

The level set

$$\mu_{U(m)}^{-1}\left(\frac{\sum\alpha_i}{2m}Id\right)=\left\{\begin{pmatrix}v_1\\ \vdots\\ v_m\end{pmatrix}\in\mathcal{M}_{m\times n}(\mathbb{C}):\ |v_j|^2=\frac{\sum\alpha_i}{m}\text{ for }j=1,\ldots,m\text{ and }\langle v_i,v_j\rangle=0\text{ for }i\neq j\right\}$$

can be identified with the set of $m\times n$ complex matrices whose m rows define a unitary m frame in $\mathbb{C}^n$. The symplectic quotient

$$\mu_{U(m)}^{-1}\left(\frac{\sum\alpha_i}{2m}Id\right)/U(m)$$

is the Grassmannian of m planes in $\mathbb{C}^n$ and according to [Hos, Example 9.9], we obtain the following result.

Proposition 3.5. There exists a symplectomorphism

$$\left((\mathbb{C}^m)^n /\!\!/_{\frac{1}{2m}\sum\alpha_i Id_m} U(m),\omega_{red}\right)\cong\left(Gr(m,n),\frac{\sum\alpha_i}{m}\Omega\right).$$

Proof. Let Ω be the symplectic form introduced in Example 1.2. Then

$$\left((\mathbb{C}^m)^n /\!\!/_{\frac{1}{2}Id_m} U(m),\omega_{red}\right)=(Gr(m,n),\Omega).$$

Therefore

$$\left((\mathbb{C}^m)^n /\!\!/_{\frac{1}{2m}\sum\alpha_j Id_m} U(m),\omega_{red}\right)\cong\left(Gr(m,n),\frac{\sum\alpha_i}{m}\Omega\right).$$

□

The $U(n)$ action descends to the quotient and so does the action of the torus $(S^1)^n\subset U(n)$. This action is no longer effective as the diagonal circle of $(S^1)^n$ acts trivially on $\left(Gr(m,n),\frac{\sum\alpha_i}{m}\Omega\right)$.

Moreover, its moment map can be obtained by composing the moment map

$$\mu_{U(m)}^{Gr(m,n)}:Gr(m,n)\to\mathfrak{u}(m)^*,$$

(induced by $\mu_{U(n)}$) with the projection

$$\mathfrak{u}(m)^*\to\mathbb{R}^m$$

onto the diagonal entries.

Hence if a m-plane in $Gr(m,n)$ is generated by the vectors $v_1,\dots,v_m \in \mathbb{C}^m$ with $|v_j|^2 = \frac{\sum \alpha_i}{m}$ and $\langle v_i, v_j\rangle = 0$ for $i \neq j$ we have the map

$$\mu_{(S^1)^n}(\mathcal{L}\{v_1,\dots,v_m\}) = -\frac{1}{2}\left(\sum_{j=1}^{m}|v_{1j}|^2,\dots,\sum_{j=1}^{m}|v_{nj}|^2\right), \tag{3.1.6}$$

whose image is the hypersimplex

$$\left\{-(r_1,\dots,r_n)\in\mathbb{R}:\ 0\leq r_j\leq \frac{\sum\alpha_i}{2m} \text{ and } 2\sum r_i = \sum \alpha_i\right\}.$$

For appropriate values of $\underline{\alpha}$, which we will determine in Chapter 5, the group $(S^1)^n$ acts freely on the level set $\mu^{-1}_{(S^1)^n}(\underline{\alpha})$ and we recover our polygon space as the symplectic quotient

$$Pol_m(\underline{\alpha}) \cong \mu^{-1}_{(S^1)^n}(-\underline{\alpha})/(S^1)^n = \left(Gr(m,n), \frac{1}{m}\sum \alpha_i\Omega\right) /\!\!/_{-\underline{\alpha}} ((S^1)^n/S^1).$$

We conclude that we have two ways to obtain our space of polygons

$$\begin{array}{ccccc}
 & & ((\mathbb{C}^m)^n) & & \\
 & \swarrow^{/\!\!/(S^1)^n} & \downarrow /\!\!/G & \searrow^{/\!\!/U(m)} & \\
((\mathbb{C}^m)^n)/\!\!/_{-2\underline{\alpha}}(S^1)^n \cong \prod\mathbb{CP}^{m-1} & & & & ((\mathbb{C}^m)^n)/\!\!/_{\frac{\sum\alpha_i}{2m}Id}U(m)\cong Gr(m,n) \\
\downarrow /\!\!/SU(m) & & & & \downarrow /\!\!/(S^1)^{n-1} \\
\prod\mathbb{CP}^{m-1}/\!\!/_0(SU(m)/\mathbb{Z}_m) & \xrightarrow{\cong} & Pol_m(\underline{\alpha}) & \xleftarrow{\cong} & Gr(m,n)/\!\!/_{-\underline{\alpha}}(S^1)^{n-1}
\end{array}$$

In the following chapters we will use properties from both constructions.

3.2 Construction via Quiver representations

The space $Pol_m(\underline{\alpha})$ can also be obtained as a quiver variety. It arises as a Kähler version of a Nakajima variety coming from a star-shaped quiver. Such construction appears in [HP] and in [FR], but was first introduced by Konno.

Let us consider a star-shaped quiver Q of rank m, i.e. a directed graph with $n+1$ vertices labeled from 0 to n, such that for each $i \in \{1,\ldots,n\}$ there is an arrow from i to 0.

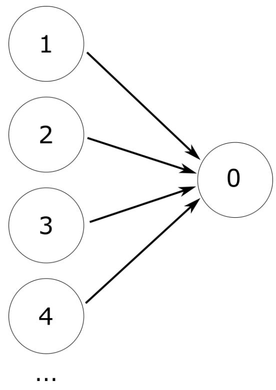

Figure 3.2.1: Quiver associated to Polygon spaces.

Given a collection $\{V_i, i \in \{0,\ldots,n\}\}$ of finite dimensional vector spaces such that $V_i = \mathbb{C}$ for $i = 1,\ldots,n$ and $V_0 = \mathbb{C}^m$, a representation of Q is a collection of maps from V_j to V_i for every pair of vertices connected by an arrow. The space of subrepresentations of Q is then, using the notation of Section 2.2,

$$Hom(V) \cong \bigoplus_{i=1}^{n} Hom(\mathbb{C},\mathbb{C}^m) \cong \mathbb{C}^{n\times m}.$$

The group $G = (U(m)\times(U(1))^n)/U(1)$ from Section 2.2 is isomorphic to $\left(SU(m)\times(S^1)^n\right)/\mathbb{Z}_m$ where $\mathbb{Z}_m$ acts diagonally. Taking an element $p_{\underline{\alpha}} \in \mathfrak{su}(m)^* \oplus (\mathfrak{u}(1)^n)^*$, the reduced space $Hom(V) /\!\!/_{p_{\underline{\alpha}}} G$ is diffeomorphic to $Pol_m(\underline{\alpha})$

$$Pol_m(\underline{\alpha}) = Hom(V) /\!\!/_{p_{\underline{\alpha}}} G.$$

Chapter 4

Duality between Polygon spaces

Since $Gr(m,n)$ is diffeomorphic to $Gr(n-m,n)$, from the description of the polygon space given in Section 3.1.3, we obtain a duality between the corresponding polygon spaces. This chapter explains this duality and is based on work of Howard and Millson (see [HM]).

Let $Gr(m,n)$ be the complex Grassmannians of m-planes in $\mathbb{C}^n$. The general linear group $GL_n(\mathbb{C})$ acts transitively on the space of m-planes in $\mathbb{C}^n$ and the stabilizer of any of these spaces can be identified with the parabolic subgroup

$$P_{m,n-m} = \left\{ \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} : X \in GL_m(\mathbb{C}), Z \in GL_{n-m}(\mathbb{C}) \right\}.$$

Consequently $Gr(m,n)$ is diffeomorphic to

$$M := GL_n(\mathbb{C})/P_{m,n-m}.$$

Moreover, as we saw in Section 3.1.3, if M is equipped with the symplectic form $\frac{1}{m}\sum \alpha_i \Omega$ and $T = (S^1)^n/S^1$, there is a symplectomorphism

$$Pol_m(\underline{\alpha}) \cong M /\!\!/_{-\underline{\alpha}} T.$$

4.1 Chevalley involution

Let us consider the Chevalley involution

$$\begin{aligned} \theta : GL_n(\mathbb{C}) &\rightarrow GL_n(\mathbb{C}) \\ A &\mapsto (A^T)^{-1}. \end{aligned}$$

Note that

$$\theta(P_{m,n-m}) = \left\{ \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} : X \in GL_m(\mathbb{C}),\ Z \in GL_{n-m}(\mathbb{C}) \right\}.$$

The Chevalley involution induces a map

$$\Theta : GL_n(\mathbb{C})/P_{m,n-m} \rightarrow GL_n(\mathbb{C})/\theta(P_{m,n-m})$$

defined by

$$\Theta(AP_{m,n-m}) = \theta(A)\theta(P_{m,n-m}).$$

Remark 4.1. We can consider a representative $n(\omega_0)$ of the longest class in the Weyl group and define a map

$$R : GL_n(\mathbb{C})/\theta(P_{m,n-m}) \rightarrow GL_n(\mathbb{C})/P_{n-m,m}$$

given by

$$R(A\theta(P_{m,n-m})) = An(\omega_0)P_{n-m,m}.$$

Composing this map with θ, we obtain a new map, which we also denote by Θ

$$\begin{aligned} M := GL_n(\mathbb{C})/\theta(P_{m,n-m}) &\rightarrow N := GL_n(\mathbb{C})/P_{n-m,n} \\ AP_{m,n-m} &\mapsto \theta(A)n(\omega_0)P_{n-m,n}. \end{aligned}$$

Remark 4.2. Since the principal Weyl walls of $U(n)$ are defined by the elements

$$\lambda\zeta_j = -\sqrt{-1}\lambda \begin{pmatrix} n-j & & & & & \\ & \ddots & & & & \\ & & n-j & & & \\ & & & -j & & \\ & & & & \ddots & \\ & & & & & -j \end{pmatrix},$$

with $\lambda > 0$, it is easy to check that $-\zeta_j$ belongs to the orbit of ζ_{n-j}. Therefore the longest class ω_0 in the Weyl group will intertwine these orbits for every $1 \leq j \leq n-1$, and so

$$\omega_0 = (1,n)(2,n-1)\dots\left(\left\lfloor\frac{n}{2}\right\rfloor, n-\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\right).$$

In the following example we describe explicitly how the map Θ works.

Example 4.3. For $n = 4$, let

$$g = \left(\begin{array}{c|ccc} g_{11} & g_{12} & g_{13} & g_{14} \\ \hline 0 & g_{22} & g_{23} & g_{24} \\ 0 & g_{32} & g_{33} & g_{34} \\ 0 & g_{42} & g_{43} & g_{44} \end{array}\right) \in P_{1,4}.$$

Then we identify,

$$h = \left(\begin{array}{c|ccc} h_{11} & 0 & 0 & 0 \\ \hline h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{array}\right) = (g^T)^{-1} = \theta(g)$$

with

$$\left(\begin{array}{ccc|c} h_{44} & h_{43} & h_{42} & h_{41} \\ h_{34} & h_{33} & h_{32} & h_{31} \\ h_{24} & h_{23} & h_{22} & h_{21} \\ \hline 0 & 0 & 0 & h_{11} \end{array}\right) \in P_{3,4}.$$

Remark 4.4. The map

$$\Theta : Gr(m,n) \to Gr(n-m,n)$$

can be seen as sending each m plane in $\mathbb{C}^n$ to its orthogonal complement with respect to the standard Hermitian product in $\mathbb{C}^n$. Indeed note that the j-th column of $\theta(g)$ is the j-th row of g^{-1}. Hence, the last $n-m$ columns of $\theta(g)$ are orthogonal to the first m columns of g.

Let us consider the symplectic forms ω_M, ω_N on $GL_n(\mathbb{C})/P_{m,n-m}, GL_n(\mathbb{C})/P_{n-m,m}$ given by the trace form. These forms agree with the form Ω defined in Example 1.2.

Theorem 4.5. ([HM, Lemma 2.7]) The map $\Theta : (M, \omega_M) \to (N, \omega_N)$ is a symplectomorphism.

Proof. The map $\theta : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is holomorphic and so

$$\Theta : M \cong Gr(m,n) \to N \cong Gr(n-m,n)$$

is holomorphic.

Then, since $(d\Theta)_{e_m} X = -X^T$ for any $X \in T_{e_m} M$, we have

$$(\Theta^* \omega_N)_{e_m}(X,Y) = (\omega_N)_{e_{n-m}}((d\Theta)_{e_m} X, (d\Theta)_{e_m} Y) = (\omega_M)_{e_m}(X,Y)$$

where $X, Y \in T_{e_m} M$. □

Remark 4.6. In fact Θ is a Kähler isomorphism between the two manifolds.

4.2 Duality between polygon spaces

Consider on the spaces M, N the natural action of the torus $T = (S^1)^n / S^1$ on $\mathbb{C}^n$ which takes subspaces to subspaces. Then the map Θ is T-equivariant.

Proposition 4.7. ([HM, Prop. 2.8]) Let μ_M, μ_N be the moment maps of the torus action on M and N respectively. Then there is a linear functional $\Lambda \in (\mathfrak{t}^*)^W$, invariant under the Weyl group, such that

$$\Theta^* \mu_N = \Lambda - \mu_M.$$

Proof. For $X \in \mathfrak{t}$ (so that $X = X^T$), let X_M and X_N be the associated fundamental vector fields on M and N respectively. Then, for $p \in M$, $Y \in T_pM$ we have

$$\begin{aligned}(d\langle\mu_M, X\rangle)_p(Y) &= -(\omega_M)_p(X_M(p), Y) = -(\omega_N)_{\Theta(p)}\left((d\Theta)_p(X_M(p)), (d\Theta)_p(Y)\right) \\ &= -(\omega_N)_{\Theta(p)}\left(-X_N(\Theta(p)), (d\Theta)_p(Y)\right) = i_{X_N(\Theta(p))}(\omega_N)_{\Theta(p)}((d\Theta)_pY) \\ &= -d(\langle\mu_N(\Theta(p)), X\rangle)((d\Theta)_pY) = -(\Theta^* d\langle\mu_N, X\rangle)_p(Y).\end{aligned}$$

Thus $\Theta^* d\langle\mu_N, X\rangle = -d\langle\mu_M, X\rangle$ and so

$$\Theta^*\langle\mu_N, X\rangle = -\langle\mu_M, X\rangle + \Lambda(X),$$

where $\Lambda : \mathfrak{t} \to \mathbb{R}$ is a linear functional on $\mathfrak{t}$ (i.e. $\Lambda \in \mathfrak{t}^*$). □

Remark 4.8. Note that since $X \in \mathfrak{t}$, we have

$$(d\Theta)_p\left(X_M^{(p)}\right) = -X_N(\Theta(p)).$$

Since $\Lambda \in \mathfrak{t}^*$ is invariant under the Weyl group W (as both μ_M and $\Theta^*\mu_N$ are W-equivariant), it has to be a multiple of the identity. We conclude the following result.

Theorem 4.9. The map $\Theta : M \to N$ induces a Kähler isomorphism

$$Pol_m(\underline{\alpha}) = M /\!\!/_{-\underline{\alpha}} T \to N /\!\!/_{-(\Lambda-\underline{\alpha})} T = Pol_{n-m}(\Lambda - \underline{\alpha})$$

when these reduced spaces are smooth.

In order to use Theorem 4.9 we need to determine Λ. Since it is constant, it is enough to compute

$$\Lambda = (\Theta^*\mu_N + \mu_M)([A])$$

for some $[A] \in M$.

Moreover, if we consider the multiples

$$\frac{\sum \alpha_i}{m}\omega_M \quad \text{and} \quad \frac{\sum \alpha_i}{m}\omega_N$$

of the symplectic forms on M and N given by the trace forms and the corresponding T-moment maps μ_M and μ_N, then μ_M coincides with the map

$$\mu_{(S^1)^n} : Gr(m,n) \to \mathfrak{t}^*$$

defined in (3.1.6). Hence, considering for instance $[A] \in M$ with

$$A = k\,\mathrm{Id}_n$$

and $k = \frac{\sum \alpha_i}{m}$, we obtain

$$\mu_M([A]) = \mu_{(S^1)^n}\left(\begin{pmatrix} k\,\mathrm{Id}_m \\ 0 \end{pmatrix}\right) = \Big(\underbrace{k,\ldots,k}_{m},0,\ldots,0\Big)$$

and

$$\mu_N(\Theta([A])) = \mu_{(S^1)^n}\left(\begin{pmatrix} 0 \\ k\,\mathrm{Id}_{n-m} \end{pmatrix}\right) = \Big(0,\ldots,0,\underbrace{k,\ldots,k}_{n-m}\Big)$$

and so

$$\Lambda = k(1,\ldots,1) = \frac{\sum \alpha_i}{m}(1,\ldots,1).$$

We conclude the following result

Theorem 4.10. Let $\underline{\alpha} \in (\mathbb{R}^+)^n$ and write $\underline{\alpha}^D = \left(\frac{\sum \alpha_i}{m} - \alpha_1, \ldots, \frac{\sum \alpha_i}{m} - \alpha_n\right)$. Then, when smooth, $Pol_m(\underline{\alpha})$ and $Pol_{n-m}(\underline{\alpha}^D)$ are isomorphic.

Example 4.11.

1. By [HK1] we know that the polygon space $Pol_2(\underline{\alpha})$ for $\underline{\alpha} = (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in (\mathbb{R}^+)^4$ with $\alpha_1 \neq \alpha_2$ and $\alpha_3 \neq \alpha_4$ is a 2-sphere. This sphere admits a Hamiltonian circle action and the image of the corresponding moment map is the interval $\Delta_{\underline{\alpha}} = I_1 \cap I_2$, where

   $$I_1 = [|\alpha_1 - \alpha_2|, \alpha_1 + \alpha_2] \quad \text{and} \quad I_2 = [|\alpha_3 - \alpha_4|, \alpha_3 + \alpha_4],$$

   i.e.

   $$\Delta_{\underline{\alpha}} = [\max\{|\alpha_1 - \alpha_2|, |\alpha_3 - \alpha_4|\}, \min\{\alpha_1 + \alpha_2, \alpha_3 + \alpha_4\}]\,.$$

Taking $n = 4, m = 2$, we have that $Pol_2(\underline{\alpha}^D)$ with

$$\underline{\alpha}^D = \left(\frac{\sum \alpha_i}{2} - \alpha_1, \ldots, \frac{\sum \alpha_i}{2} - \alpha_4 \right)$$

is symplectomorphic to $Pol_2(\underline{\alpha})$. This fact is easy to check since $Pol_2(\underline{\alpha}^D)$ is also a sphere and admits a Hamiltonian circle action whose moment map has image

$$\Delta_{\underline{\alpha}^D} = \left[\max\{|\alpha_1^D - \alpha_2^D|, |\alpha_3^D - \alpha_4^D|\}, \min\{\alpha_1^D + \alpha_2^D, \alpha_3^D + \alpha_4^D\} \right] = \Delta_{\underline{\alpha}},$$

since

$$|\alpha_i^D - \alpha_j^D| = |\alpha_i - \alpha_j|$$

and

$$|\alpha_1^D + \alpha_2^D| = |\alpha_3 + \alpha_4|, \ |\alpha_3^D + \alpha_4^D| = |\alpha_1 + \alpha_2|.$$

2. Another interesting example is when $m = 3, n = 5$. In this case, the resulting polygon space is, symplectomorphic to the space of pentagons with edges in $\mathbb{R}^3$.

   Such spaces are well known and were completely classified in [HK2]. In particular, in the equilateral case, the polygon space $Pol_2(1,1,1,1,1)$ is a *Del Pezzo surface of degree* 5 (see [DO, Chapter 2, Example 4]).

Chapter 5

On the Moduli Space

The goal of this chapter is to determine the values of $\underline{\alpha}$ for which the polygon space described in Chapter 3 is a smooth Kähler manifold of real dimension $2(n-m-1)(m-1)$.

5.1 Nonemptiness of the moduli space

Along this section we will determine conditions on $\underline{\alpha} \in (\mathbb{R}^+)^n$ to ensure that the moduli space is nonempty. For that, we first need to define a new space.

Definition 5.1. The $\underline{\alpha}$-*path space* is the product

$$\widetilde{P}_m(\underline{\alpha}) = \prod_{j=1}^{n} \mathcal{O}^d_{\mathfrak{su}(m)}(\alpha_j)$$

and a *path of length n* is an element $\underline{A} = (A_j) \in \widetilde{P}_m(\underline{\alpha})$.

Proposition 5.2. If $\mu^{-1}_{SU(m)}(0) \neq \emptyset$, then

$$(m-1)\alpha_j \leq \sum_{i \neq j} \alpha_i, \quad \text{for } 1 \leq j \leq n.$$

These inequalities will be called *strong triangle inequalities.*

Proof. Let $X = (x_{ij}) \in SU(m)$ and let

$$A = X\,\frac{\alpha}{m}\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -(m-1) \end{pmatrix} X^*$$

be an element of $O^d_{\mathfrak{su}(m)}(\alpha)$. Then

$$A_{ii} = \frac{\alpha}{m}\left(\sum_{j=1}^{n-1} |x_{ij}|^2 - (m-1)|x_{in}|^2\right)$$

for every $1 \le i \le m$. Since $X \in SU(m)$, its rows are unitary vectors and so

$$A_{ii} \in \left[-\frac{(m-1)\alpha}{m}, \frac{\alpha}{m}\right]$$

for every $1 \le i \le m$. Thus if $(A_1, \ldots, A_n) \in \mu^{-1}_{SU(m)}(0)$, we have that

$$\left[-\frac{(m-1)\alpha_j}{m}, \frac{\alpha_j}{m}\right] \subset \left[-\frac{\sum_{i \neq j} \alpha_i}{m}, \frac{(m-1)\sum_{i \neq j} \alpha_i}{m}\right]$$

for every $1 \le j \le n$. In particular, if $\mu^{-1}_{SU(m)}(0)$ is nonempty, we have $(m-1)\alpha_j \le \sum_{i \neq j} \alpha_i$ for every $1 \le j \le n$.

□

In order to prove sufficiency of the strong triangle inequalities we give the following definition.

Definition 5.3. Let $\{p_1, \ldots, p_n\} \subset \mathbb{CP}^{m-1}$ and $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}^+)^n$. Then

$$(p_1, \ldots, p_n) \in \widetilde{P}_m(\underline{\alpha})$$

is said to be *$\underline{\alpha}$-semi-stable* if, for every proper projective subspace L of $\mathbb{CP}^{m-1}$ we have

$$\sum_{i \in I_L} \alpha_i \le \frac{\dim L + 1}{m} \sum_{i=1}^{n} \alpha_i, \tag{5.1.1}$$

where I_L is the index set of the points $p_1, \ldots, p_n$ that are in L.

The point $(p_1, \dots, p_n)$ is said to be $\underline{\alpha}$*-stable* if the strict inequalities in (5.1.1) hold.

The motivation of Definition 5.3 will appear later, when we relate it with the results from Geometric Invariant Theory (see for example [DO] or [Mu]).

To state them, assume $\underline{\alpha} \in (\mathbb{Z}^+)^n$ and consider the line bundle $L_{\underline{\alpha}}$ on $\widetilde{P}_m(\mathbb{C})$

$$L_{\underline{\alpha}} := \bigotimes_{i=1}^{n} \mathrm{pr}_i^*(O_{\mathbb{CP}^{m-1}}(1)^{\otimes \alpha_i}),$$

where $\mathrm{pr}_i : \widetilde{P}_m(\underline{\alpha}) \to O^d_{\mathfrak{su}(m)}(\alpha_i)$ is the natural projection.

This bundle defines a projective embedding

$$\widetilde{P}_m(\underline{\alpha}) \to \prod_{i=1}^{n} \mathbb{CP}^{\binom{m-1+\alpha_i}{m-1}-1} \to \mathbb{CP}^{\prod_{i=1}^{n}\binom{m-1+\alpha_i}{m-1}-1}$$

which is the composition of the Segre and Veronese embeddings. The bundle $L_{\underline{\alpha}}$ admits a $GL_m(\mathbb{C})$-linearization, which restricts to the action of $SU(m)$ on $\widetilde{P}_m(\underline{\alpha})$.

Definition 5.4 (The Hilbert-Mumford Numerical Criterium)**.** Let $p \in \widetilde{P}_m(\underline{\alpha})$ and p^* a representative in the total space of $L_{\underline{\alpha}}$. Then p is said to be *semi-stable* if either $p \in \left(\widetilde{P}_m(\underline{\alpha})\right)^{\mathbb{C}^*}$ or for any 1-parameter subgroup $\lambda : \mathbb{C}^* \to GL_m(\mathbb{C})$ the limit

$$\lim_{t\to 0} \lambda(t) \cdot p^* \tag{5.1.2}$$

does not exist.

If for any such 1-parameter subgroup the limit in (5.1.2) does not exist and $p \notin \left(\widetilde{P}_m(\underline{\alpha})\right)^{\mathbb{C}^*}$, then it is called *stable.*

The notions of stable and semi-stable points play an important role due to the *Kempf-Ness Theorem* ([KN]).

Theorem 5.5 (Kempf-Ness)**.** Let G be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{CP}^N$ such that its maximal compact subgroup K acts unitarily with moment map $\mu : X \to K^*$.

Then p is semi-stable if and only if

$$\overline{G \cdot p} \cap \mu^{-1}(0) \neq \emptyset.$$

Hence, if we know that the set of semi-stable points is nonempty, we have that $\mu^{-1}(0)$ is also nonempty. Let us then see that the set of semi-stable points is nonempty.

Proposition 5.6. Let $\underline{\alpha} \in (\mathbb{Z}^+)^n$ and $p = (p_1, \ldots, p_n) \in \widetilde{P}_m(\underline{\alpha})$. If p is $\underline{\alpha}$-semi-stable ($\underline{\alpha}$-stable) then it is semi-stable (stable).

Proof. Let $\lambda : \mathbb{C}^* \to GL_m(\mathbb{C})$ be a 1-parameter subgroup of $GL_m(\mathbb{C})$. We can choose coordinates in $\mathbb{CP}^{m-1}$ in such a way that the action of $\lambda(\mathbb{C}^*)$ is diagonalized:

$$\lambda(t) \cdot [z_0 : \cdots : z_{m-1}] = [t^{k_0} z_0 : \cdots : t^{k_{m-1}} z_{m-1}]$$

for some integers k_i that we may assume satisfy

$$k_0 \geq \cdots \geq k_{m-1}, \ \sum_{i=1}^{n} k_i = 0, \ k_0 > 0.$$

Note that the condition $\sum_{i=1}^n k_i = 0$ is equivalent to $\lambda(t) \in SL_m(\mathbb{C})$ for every t, but this is no restriction since the action of $GL_m(\mathbb{C})$ restricts to $SL_m(\mathbb{C})$, i.e. we can consider the $SL_m(\mathbb{C})$-linearization instead.

For each $k = 0, \ldots, m-1$, let $L_k \subset \mathbb{CP}^{m-1}$ be the projective subspace defined by

$$z_0 = \cdots = z_{m-k-2} = 0,$$

let I_k be the index set of the points $p_1, \ldots, p_n$ in L_k and let $a_k = \sum_{i \in I_k} \alpha_i$ (with $a_{m-1} = \sum_{i=1}^n \alpha_i$). Note that $\dim L_k = k$ and that, if p_i is in L_k, than p_i is in L_{k+1} and so $a_{m-1} \geq \cdots \geq a_0$ (as $\alpha_i > 0$ for every i).

Consider the monomial

$$z_0^{a_{m-1}-a_{m-2}} z_1^{a_{m-2}-a_{m.3}} \cdots z_{m-1}^{a_0}$$

(this monomial corresponds to a coordinate of the Veronese embedding of $\mathbb{CP}^{m-1}$ of degree n).

The monomial

$$X = \prod_{i=1}^{n} \left(z_0^{(1)} \cdots z_0^{(n)} \right)^{a_{m-1}-a_{m-2}} \cdots \left(z_{m-2}^{(1)} \cdots z_{m-2}^{(n)} \right)^{a_1 - a_0} \left(z_{m-1}^{(1)} \cdots z_{m-1}^{(n)} \right)^{a_0}$$

is a coordinate of the Segre-Veronese embedding of $\widetilde{P}_m(\underline{\alpha})$ given by the bundle $L_{\underline{\alpha}}$.

A 1-parameter subgroup acts on these monomials X by

$$\lambda(t) \cdot X = \prod_{i=1}^{n} t^{w} X$$

with

$$\begin{aligned} w &= \textstyle\sum_{i=1}^{n} \alpha_i \left(k_0(a_{m-1} - a_{m-2}) + \cdots + k_{m-2}(a_1 - a_0) + k_{m-1}a_0\right) \\ &= \textstyle\sum_{i=1}^{n} \alpha_i \left(k_0 a_{m-1} - (k_0 - k_1)a_{m-2} - \cdots - (k_{m-2} - k_{m-1})a_0\right). \end{aligned}$$

Note that by (5.1.1) we have

$$a_k \leq \frac{k+1}{m} \sum_{i=1}^{n} \alpha_i$$

and so

$$\begin{aligned} w \leq & \left(\sum_{i=1}^{n} \alpha_i\right)^2 \left(k_0 - \frac{m-1}{m}(k_0 - k_1) - \frac{m-2}{m}(k_1 - k_2) - \cdots - \frac{1}{m}(k_{m-2} - k_{m-1})\right) \\ = & \frac{(\sum \alpha_i)^2}{m} (k_0 + \cdots + k_{m-1}) = 0. \end{aligned}$$

This shows that either

$$\lambda(t) \cdot p^*$$

does not have a limit or p is contained in the fixed point set of $\mathbb{C}^*$, implying that if (5.1.1) is satisfied, then p is semi-stable and, if the strict inequalities hold on 5.1.1, then it is stable. □

Remark 5.7. Since $Pol_m(\underline{\alpha})$ and $Pol_m(\lambda \cdot \underline{\alpha})$ are diffeomorphic for every $\lambda \in \mathbb{R}^+$, the above proof can be naturally extended to $\underline{\alpha} \in \lambda \cdot (\mathbb{Q}^+)^n$ with $\lambda \in \mathbb{R}^+$.

For a proof in the case $\underline{\alpha} \in (\mathbb{R}^+)^n$ see [FM2, Chapter 4]. Therefore, from now on we consider $\underline{\alpha} \in (\mathbb{R}^+)^n$.

Proposition 5.8. If the strong triangle inequalities are satisfied, then $\mu_{SU(m)}^{-1}(0) \neq \emptyset$.

Proof. If the strong triangular inequalities hold, then the set of semi-stable points is nonempty and so the Kempf Ness Theorem implies that $\mu^{-1}(0) \neq \emptyset$. □

Definition 5.9. $\underline{\alpha} \in (\mathbb{R}^+)^n$ is said to be *m-admissible* if, for every $1 \leq j \leq n$,

$$(m-1)\alpha_j \leq \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i.$$

Corollary 5.10. Each inequality of the strong triangle inequalities defines a half-space on the space of parameters $\underline{\alpha}$ bounded by the hyperplane

$$(m-1)\alpha_j = \sum_{i \neq j} \alpha_i.$$

The intersection of all these half-spaces is a convex set. In the above propositions we have shown that polygon spaces will be nonempty for values of $\underline{\alpha}$ in this set.

5.2 Smoothness of the moduli space

From now on we assume that $\underline{\alpha}$ is m-admissible and so the corresponding moduli space $Pol_m(\underline{\alpha})$ is nonempty.

As it was seen in (1.1.2), for

$$Pol_m(\underline{\alpha}) = (\mathbb{C}^m)^n /\!\!/_{(0,-\underline{\alpha})} \widetilde{G}$$

with

$$\widetilde{G} = (SU(m) \times (S^1)^n)/\mathbb{Z}_m$$

to be a smooth manifold, we need the action of $\widetilde{G}$ on $\mu_{\widetilde{G}}^{-1}(0, -\underline{\alpha})$ to be free.

We can consider the analogue of (5.1.1) for points in $(\mathbb{C}^m)^n$ and obtain the following definition.

Definition 5.11. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}^+)^n$. A point $q = (q_1, \ldots, q_n) \in (\mathbb{C}^m)^n$ is $\underline{\alpha}$*-stable* if for every proper subspace L of $\mathbb{C}^m$ we have

$$\sum_{i \in I_L} \alpha_i < \frac{\dim L}{m} \sum_{i=1}^{n} \alpha_i.$$

Then we have the following result.

Proposition 5.12. If $q = (q_1, \ldots, q_n) \in (\mathbb{C}^m)^n$ is $\underline{\alpha}$-stable, then there are no disjoint proper subspaces V, W of $\mathbb{C}^m$ such that every q_i is either in V or in W.

Proof. Let us assume that there exist two disjoint proper subspaces $V, W \subset \mathbb{C}^m$ such that q_i is either in V or in W. Then denoting by I the index set $I = \{i \in \{1, \ldots, n\} : q_i \in V\}$, we have

$$1 = \frac{\sum_I \alpha_i}{\sum_{i=1}^n \alpha_i} + \frac{\sum_{I^c} \alpha_i}{\sum_{i=1}^n \alpha_i} < \frac{\dim V}{m} + \frac{\dim W}{m} = \frac{\dim(V \oplus W)}{m} \leq \frac{m}{m} = 1,$$

which is impossible. □

The above proposition can be rewritten in terms of the stabilizer of q for the action of $\widetilde{G}$.

Proposition 5.13. Let $q = (q_1, \ldots, q_n) \in (\mathbb{C}^m)^n$ with $q_i \in \mathbb{C}^m$ for $i = 1, \ldots, n$. The stabilizer of q for the action of $\widetilde{G}$ is trivial if and only if there are no disjoint proper subspaces V, W of $\mathbb{C}^m$ such that every q_i is either in V or in W.

Proof. If $q \in (\mathbb{C}^m)^n$ is fixed by a nontrivial element $[A, (e_1, \ldots, e_n)]$ of $\widetilde{G}$, where $A \in SU(m)$ and $e_i \in S^1$, then

$$A q_i e_i^{-1} = q_i,$$

i.e. $Aq_i = e_i q_i$. Consequently q_i must be an eigenvector of A with eigenvalue e_i.

Since $A \in SU(m)$, it is diagonalizable. Let $1 \leq l \leq m$ be the number of distinct eigenvalues of A. Then $\mathbb{C}^m$ has an orthogonal decomposition as

$$\mathbb{C}^m = W_1 \oplus \cdots \oplus W_l$$

where $W_1, \ldots, W_l$ are the different eigenspaces of A.

Then, for each $i \in \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, l\}$ such that $q_i \in W_j$. Taking $V = W_1 \oplus \cdots \oplus W_{l-1}$ and $W = W_l$ we get that q_i is either in V or in W.

Conversely, let us take two disjoint proper subspaces V, W of $\mathbb{C}^m$. We can assume without loss of generality that

$$\begin{aligned} V &\subset \langle e_{k+1}, \ldots, e_m \rangle \\ W &\subset \langle e_1, \ldots, e_k \rangle \end{aligned}$$

for some $1 \leq k \leq m-1$. If, for every $i \in \{1, \ldots, n\}$ we have that q_i is either in V or in W, then the subgroup

$$\left\{ \left[\begin{pmatrix} aId_k & \\ & bId_{m-k} \end{pmatrix}, (\underbrace{a^{-1}, \ldots, a^{-1}}_{\sharp\{q_i \in W\}}, \underbrace{b^{-1}, \ldots, b^{-1}}_{\sharp\{q_i \in V\}}) \right] : a^k b^{m-k} = 1 \right\} \subset \widetilde{G}$$

fixes q and so, q has nontrivial stabilizer. □

As a direct consequence of Propositions 5.12 and 5.13, if q is $\underline{\alpha}$-stable then it has trivial stabilizer for the action of $\widetilde{G}$. Hence, to ensure that $q \in \mu_G^{-1}(p_{\underline{\alpha}})$ has trivial stabilizer, we have to consider only values of $\underline{\alpha}$ that satisfy

$$\sum_I \alpha_i < \frac{k}{m} \sum_{i=1}^n \alpha_i$$

for every $k \in \{1, \ldots, m-1\}$.

Remark 5.14. Note that, since $\underline{\alpha}$ is m-admissible we cannot have

$$\sum_I \alpha_i > \frac{k}{m} \sum_{i=1}^n \alpha_i$$

for any $k \in \{1, \ldots, m-1\}$.

If for some m-admissible $\underline{\alpha}$ we have

$$\sum_I \alpha_i = \frac{k}{m} \sum_{i=1}^n \alpha_i$$

or, equivalently,

$$(m-k) \sum_I \alpha_i = k \sum_{I^c} \alpha_i$$

for some $k \in \{1, \ldots, m-1\}$, then, taking $q_i \in \langle e_1, \ldots, e_k \rangle$ for $i \in I$ and $q_i \in \langle e_{k+1}, \ldots, e_m \rangle$ for $i \in I^c$ with $|q_i|^2 = \alpha_i$, we have

$$(q_i q_i^*)_0 = \left(\begin{array}{c|c} A_i & 0 \\ \hline 0 & -\frac{\alpha_i}{m} Id_{m-k} \end{array} \right), \quad \text{for } i \in I$$

and

$$(q_i q_i^*)_0 = \left(\begin{array}{c|c} -\frac{\alpha_i}{m} Id_k & 0 \\ \hline 0 & B_i \end{array}\right), \quad \text{for } i \in I^c$$

with

$$\sum_I A_i = \sum_{I^c} \frac{\alpha_i}{m} \mathrm{Id}_k = \frac{m-k}{mk} \sum_I \alpha_i \mathrm{Id}_k = \sum_I \frac{\widetilde{\alpha}_i}{k} \mathrm{Id}_k$$

for $\widetilde{\alpha}_i = \frac{m-k}{m} \alpha_i$ with $i \in I$ and

$$\sum_{I^c} B_i = \sum_I \frac{\alpha_i}{m} \mathrm{Id}_{m-k} = \frac{k}{m(m-k)} \sum_{I^c} \alpha_i \mathrm{Id}_{m-k} = \sum_{I^c} \frac{\widetilde{\alpha}_i}{m-k} \mathrm{Id}_{m-k}$$

for $\widetilde{\alpha}_i = \frac{k}{m} \alpha_i$ with $i \in I^c$.

We conclude that the polygon obtained from q decomposes into two closed polygons, one in $\overline{P}_k((\widetilde{\alpha}_i)_I)$ and the other in $\overline{P}_{m-k}((\widetilde{\alpha}_i)_{I^c})$, where $\overline{P}_l(\underline{\beta}) := \mu^{-1}(0, -\underline{\beta})$ for $l \in \mathbb{N}$ and $\underline{\beta} \in (\mathbb{R}^+)^l$. Such polygons will be called *decomposable.*

Definition 5.15. For $k \in \{1, \ldots, m\}$ and a nonempty set $I \subset \{1, \ldots, n\}$, let $\mathcal{H}_{(I,k)}$ be the hyperplane

$$\mathcal{H}_{(I,k)} := \left\{ \underline{\alpha} \in (\mathbb{R}^+)^n : \ (m-k) \sum_{i \in I} \alpha_i = k \sum_{i \in I^c} \alpha_i \right\}.$$

The corresponding wall $W_{(I,k)}$ is the intersection of $\mathcal{H}_{(I,k)}$ with the set of m-admissible values of $\underline{\alpha}$. Note that in particular $W_{(I,k)} = W_{(I^c, m-k)}$.

Remark 5.16. If $|I| = k = 1$ the equations that define $\mathcal{H}_{(I,1)}$ are the strong triangle inequalities. In this case, the wall is an outer wall of the set of m-admissible values of $\underline{\alpha}$.

Proposition 5.17. If $\underline{\alpha} \in W_{(I,k)}$ then $\widetilde{\alpha}_I := \frac{m-k}{m} (\alpha_i)_{i \in I}$ and $\widetilde{\alpha}_{I^c} := \frac{k}{m} (\alpha_i)_{i \in I^c}$ are respectively k-admissible and $(m-k)$-admissible.

Proof. If $\underline{\alpha} \in W_{(I,k)}$, then

$$(m-k) \sum_I \alpha_i = k \sum_{I^c} \alpha_i$$

and

$$m\alpha_j \leq \sum_{i=1}^n \alpha_i \text{ for all } j \in \{1, \ldots, n\},$$

hence,

$$k\alpha_j \leq \frac{k}{m} \sum_{i=1}^n \alpha_i = \frac{k}{m} \sum_I \alpha_i + \frac{k}{m} \sum_{I^c} \alpha_i = \frac{k}{m} \sum_I \alpha_i + \frac{m-k}{m} \sum_I \alpha_i = \sum_I \alpha_i$$

and so $\widetilde{\alpha}_I$ is k-admissible.

Similarly, we have that $\widetilde{\alpha}_{I^c}$ is $(m-k)$-admissible. □

From Proposition 5.13 it is clear that $\overline{P}_m(\underline{\alpha})$ contains a decomposable polygon if and only if $\underline{\alpha}$ lies on a wall. Moreover, if a polygon is not decomposable then its stabilizer is trivial. Hence a necessary and sufficient condition for the smoothness of the reduced space is that $\underline{\alpha}$ is not on any wall.

Definition 5.18. An m-admissible $\underline{\alpha} \in (\mathbb{R}^+)^n$ will be called *m-generic* if, for all $1 \leq k \leq \frac{m}{2}$ and $I \subset \{1, \ldots, n\}$,

$$k\left(\sum_{I^c} \alpha_i\right) - (m-k)\left(\sum_{I} \alpha_i\right) \neq 0.$$

Remark 5.19. Note that since $W_{(I,k)} = W_{(I^c, m-k)}$, it is enough to consider k with $k \leq \lfloor \frac{m}{2} \rfloor$ in order to make sure that an m-generic $\underline{\alpha}$ is not on any wall.

We conclude the following result.

Theorem 5.20. Let $\underline{\alpha}$ be m-generic. Then the *Polygon space*

$$Pol_m(\underline{\alpha}) = (\mathbb{C}^m)^n /\!\!/_{(0,-\underline{\alpha})} \widetilde{G}$$

is a smooth manifold of (real) dimension $2(m-1)(n-m-1)$.

Proof. By construction it is a smooth manifold, so we only need to compute its dimension.

The dimension of a reduced product of coadjoint orbits $\prod_{i=1}^n G \cdot x$ of a group G is

$$n\,(\dim G - \dim\ Stab(x)) - 2\dim G.$$

Since in our case $\dim G = \dim SU(m) = m^2 - 1$ and $\dim\ Stab(x) = (m-1)^2$, we obtain that

$$\begin{aligned} \dim\left(\prod_{i=1}^{n} \mathcal{O}^d_{\mathfrak{su}(m)}(\alpha_i) /\!\!/_0 (SU(m)/\mathbb{Z}_m)\right) &= n\left(m^2 - 1 - (m-1)^2\right) - 2(m^2-1) \\ &= 2(m-1)(n-m-1). \end{aligned}$$

□

Chapter 6

Topology of $Pol_m(\underline{\alpha})$

As we have seen in Chapter 5, the moduli space of m-admissible n-tuples $\underline{\alpha} \in (\mathbb{R}^+)^n$ is divided in chambers separated by walls. Within these chambers the polygon spaces $Pol_m(\underline{\alpha})$ for different values of $\underline{\alpha}$ are diffeomorphic. However, the diffeotype of these spaces changes where crossing a wall. The case $m = 2$, i.e. the usual polygon space with edges in $\mathbb{R}^3$, has been studied using wall crossing theory by several authors such as S. Martin [Ma1] or A. Mandini [M].

6.1 Variation of $Pol_m(\underline{\alpha})$ when crossing a wall

As we have seen in Chapter 5, a wall $W_{(I,k)}$ is the intersection of the set of m-admissible values of $\underline{\alpha}$ and the hyperplane

$$\mathcal{H}_{(I,k)} = \left\{ \underline{\alpha} \in (\mathbb{R}^+)^n : \ (m-k)\sum_{I} \alpha_i = k \sum_{I^c} \alpha_i \right\}$$

where $I \subset \{1, \dots, n\}$ and $1 \le k \le \lfloor \frac{m}{2} \rfloor$.

Definition 6.1. Given an m-admissible value $\underline{\alpha}$, a set $I \subset \{1, \dots, n\}$ is said to be k-*short* if

$$(m-k)\sum_{i \in I} \alpha_i < k \sum_{i \in I^c} \alpha_i$$

and k-*long* otherwise.

Given $\underline{\alpha}$ and a wall $W_{(I,k)}$, for a set I to be k-short/long, indicates the side of the wall where $\underline{\alpha}$ lies.

Note that for a set I to be k-short is equivalent to I^c being $(m-k)$-long. Hence, even though $W_{(I,k)} = W_{(I^c,m-k)}$, being k-short (for I) is exactly the opposite of being $(m-k)$-short (for I^c).

Let us recall the description of the polygon space $Pol_m(\underline{\alpha})$ as a symplectic quotient of $Gr(m,n)$ by the n torus $(S^1)^n$ (cf. Section 3.1.3) and let $\underline{\alpha}^c$ be in a single wall $W_{(I,k)}$. We have seen in Section 5.2 that there are polygons in $\overline{P}_m(\underline{\alpha}^c)$ that are decomposable. These correspond to m-planes $V \subset \mathbb{C}^n$ in $Gr(m,n)$ given by direct sums $V = V_1 \oplus V_2$ with V_1 and V_2 respectively a k-plane and a $(m-k)$-plane in $\mathbb{C}^n$.

Since the action of the n-torus on $Gr(m,n)$ is not effective (the diagonal circle Γ in $(S^1)^n$ fixes every point in $Gr(m,n)$), we consider the action of $(S^1)^n/\Gamma$ on $Gr(m,n)$ instead.

The stabilizer of the points in $Gr(m,n)$ corresponding to m-planes

$$V = V_1 \oplus V_2,$$

where $V_1 = \langle v_i \rangle_{i\in I}, V_2 = \langle v_i \rangle_{i\in I^c}$ (with $\dim V_1 = k, \dim V_2 = m-k$), for the action of $(S^1)^n/\Gamma$ is the circle

$$H_{(I,k)} := \{diag(a_1,\ldots,a_n), \text{ where } a_j = e^{\sqrt{-1}\theta} \text{ if } j \in I \text{ and } a_j = 1 \text{ otherwise}\}.$$

Let us assume that $l := |I| > k$ and $n-l = |I^c| > m-k$.

Let Y_C be the set of singular points in

$$Pol_m(\underline{\alpha}^c) := \mu^{-1}_{(S^1)^n/\Gamma}(\underline{\alpha}^c)/((S^1)^n/\Gamma)$$

i.e. those that correspond to orbits of points in $\mu^{-1}_{(S^1)^n/\Gamma}(\underline{\alpha}^c) \subset Gr(m,n)$ with $H_{(I,k)}$ as stabilizer. Then Y_C can be identified with

$$Y_C = \frac{\mu^{-1}_{(S^1)^l}(\underline{\widetilde{\alpha}}^c_I) \times \mu^{-1}_{(S^1)^{n-l}}(\underline{\widetilde{\alpha}}^c_{I^c})}{(S^1)^n} = \frac{\mu^{-1}_{(S^1)^l}(\underline{\widetilde{\alpha}}^c_I)}{(S^1)^l} \times \frac{\mu^{-1}_{(S^1)^{n-l}}(\underline{\widetilde{\alpha}}^c_{I^c})}{(S^1)^{n-l}}.$$

Now the action of $(S^1)^l$ on $\mu^{-1}_{(S^1)^l}(\underline{\widetilde{\alpha}}^c_I)$ is not effective since the diagonal circle $\Gamma_l \subset (S^1)^l$ fixes every point of the level set, so we consider the action of $(S^1)^l/\Gamma_l$ instead.

Similarly, the action of $(S^1)^{n-l}$ on $\mu^{-1}_{(S^1)^{n-l}}(\underline{\widetilde{\alpha}}^c_{I^c})$ is not effective so we have to consider the action of $(S^1)^{n-l}/\Gamma_{n-l}$ instead, where Γ_{n-l} is the diagonal circle in $(S^1)^{n-l}$.

Hence,

$$Y_C = \frac{\mu^{-1}_{(S^1)^l/\Gamma_l}(\widetilde{\underline{\alpha}}^c_I)}{(S^1)^l/\Gamma_l} \times \frac{\mu^{-1}_{(S^1)^{n-l}/\Gamma_{n-l}}(\widetilde{\underline{\alpha}}^c_{I^c})}{(S^1)^{n-l}/\Gamma_{n-l}} = Pol_k(\widetilde{\underline{\alpha}}^c_I) \times Pol_{m-k}(\widetilde{\underline{\alpha}}^c_{I^c}). \tag{6.1.1}$$

Remark 6.2. By Proposition 5.17 we know that $\widetilde{\underline{\alpha}}^c_I$ is k-admissible and $\widetilde{\underline{\alpha}}^c_{I^c}$ is $(m-k)$-admissible, so Y_C is nonempty.

Note also that Y_C is formed by the orbit classes of decomposable polygons in $\overline{P}_m(\underline{\alpha}^c)$.

Theorem 6.3. Let $\underline{\alpha}_0, \underline{\alpha}_1 \in (\mathbb{R}^+)^n$ be m-generic lying on two adjacent chambers separated by a unique wall $W_{(I,k)}$. Assume further that I is k-long for $\underline{\alpha}_0$ and k-short for $\underline{\alpha}_1$.

Then $Pol_m(\underline{\alpha}_0)$ can be obtained from $Pol_m(\underline{\alpha}^c)$ by a partial blow up along Y_C, where the exceptional divisor is a projective bundle over a copy of Y_C with fiber $\mathbb{CP}^{(m-k)(|I|-k)-1}$.

Similarly $Pol_m(\underline{\alpha}_1)$ can be obtained from $Pol_m(\underline{\alpha}^c)$ by a partial blow down along Y_C, where the exceptional divisor is a projective bundle over a copy of Y_C with fiber $\mathbb{CP}^{k(|I^c|-(m-k))-1}$.

Proof. We can assume without loss of generality that $I = \{1, \ldots, l\}$ and consider the group

$$H_{(I,k)} = \{\mathrm{diag}(e^{\sqrt{-1}\theta}, \ldots, e^{\sqrt{-1}\theta}, 1, \ldots, 1)\}$$

associated to the wall $W_{(I,k)}$.

Let us consider a line segment between $\underline{\alpha}_0$ and $\underline{\alpha}_1$ in the space of m-admissible values of $\underline{\alpha}$ and let $\underline{\alpha}^c$ be the point where it intersects the wall $W_{(I,k)}$.

In the level set $\mu^{-1}_{(S^1)^n/\Gamma}(\underline{\alpha}^c)$ there are points with $H_{(I,k)}$ as stabilizer, namely those corresponding to m-planes $V = V_1 \oplus V_2$, where $\dim V_1 = k$ and $\dim V_2 = m-k$, represented by matrices

$$\left(\begin{array}{c|c} (x_{ij})_{\substack{1\le i\le k \\ 1\le j\le l}} & 0 \\ \hline 0 & (x_{ij})_{\substack{k+1\le i\le m \\ l+1\le j\le n}} \end{array}\right).$$

By (6.1.1) the fixed point set Y_C of $H_{(I,k)}$ on $\mu^{-1}_{(S^1)^n/\Gamma}(\underline{\alpha}^c)$ can be identified with

$$Y_C \simeq Pol_k(\widetilde{\underline{\alpha}}^c_I) \times Pol_{m-k}(\widetilde{\underline{\alpha}}^c_{I^c}),$$

where $\widetilde{\underline{\alpha}}_I := \frac{m-k}{m}(\alpha_i)_{i\in I}$ and $\widetilde{\underline{\alpha}}_{I^c} = \frac{k}{m}(\alpha_i)_{i\in I^c}$ (it is formed by the decomposable polygons in $Pol_m(\underline{\alpha}^c)$).

Let $P_C \in \mu^{-1}_{(S^1)^n/\Gamma}(\underline{\alpha}^c) \subset Gr(m,n)$ be one of these fixed points and let

$$H = \{\text{diag}(1, e^{\sqrt{-1}\theta_2}, \dots, e^{\sqrt{-1}\theta_{n-1}}, 1),\ e^{\sqrt{-1}\theta_j} \in S^1\}$$

be a complement of $H_{(I,k)}$ in

$$(S^1)^n/\Gamma \simeq \{\text{diag}(e^{\sqrt{-1}\theta_1}, \dots, e^{\sqrt{-1}\theta_{n-1}}, 1),\ e^{\sqrt{-1}\theta_j} \in S^1\}.$$

We will now study the residual action of $H_{(I,k)}$ on a neighbourhood $U \subset \mu_H^{-1}(\underline{\alpha}^c)/H$ of $[P_C]_H$.

Given a point $[q] \in U$ with $q = (q_1, \dots, q_n)$ generating an m-plane in $\mathbb{C}^n$ we can, as usual, consider the $(m \times n)$-matrix $A = \left(q_1, \dots, q_n\right)$ which has characteristic m.

Sufficiently close to $[P_C]$, we can assume that the $(m \times m)$ minor M formed by the first k columns and the last $m-k$ columns of A,

$$M = \left(q_1, \dots, q_k, q_{n-(m-k-1)}, \dots, q_n\right)$$

is non singular. Then, using the action of $Gl_m(\mathbb{C})$ on $Gr(m,n)$, we can choose a $m \times n$ matrix representing A of the form

$$\left(\begin{array}{c|c|c|c} D_1 & (v_{ij})_{\substack{1\le i\le k \\ k+1\le j\le l}} & (w_{ij})_{\substack{1\le i\le k \\ l+1\le j\le n-m+k}} & 0_{k\times(m-k)} \\ \hline 0_{(m-k)\times k} & (w_{ij})_{\substack{k+1\le i\le m \\ k+1\le j\le l}} & (v_{ij})_{\substack{k+1\le i\le m \\ l+1\le j\le n-m+k}} & D_2 \end{array}\right), \tag{6.1.2}$$

where $D_1 = \text{diag}\left(\sqrt{2\alpha_1^c}, \dots, \sqrt{2\alpha_k^c}\right)$ and $D_2 = \text{diag}\left(\sqrt{2\alpha^c_{n-(m-k)+1}}, \dots, \sqrt{2\alpha_n^c}\right)$.

Indeed, we just have to take $B \in GL_m(\mathbb{C})$ such that

$$B = \left(\begin{array}{c|c} D_1 & 0_{(m-k)\times k} \\ \hline 0_{k\times(m-k)} & D_2 \end{array}\right) M^{-1}$$

and then BA is of the form (6.1.2).

Using the action of the torus H we can assume that the entries in the first line and column of the matrix

$$C_{k\times(m-k)} := \left(v_{ij}\right)_{\substack{1\le i\le k \\ k+1\le j\le l}}$$

are real numbers. Moreover, knowing that $q \in \mu^{-1}_{(S^1)^n/\Gamma}(\underline{\alpha}^c)$, the first column and the first line of

$C_{k\times(m-k)}$ are completely determined by the remaining entries of $C_{k\times(m-k)}$. We are then left with

$$(l-k)k-(l-1)=(k-1)(l-k-1)$$

free complex coordinates.

Similarly, we obtain $(m-k-1)(n-l-m+k-1)$ free coordinates in

$$D_{(m-k)\times(n-m+k-l)} := \left(v_{ij}\right)_{\substack{k+1\le i\le m\\ l+1\le j\le n-m+k}}.$$

Moreover, we have $(n-l-m+k)k$ free coordinates in

$$Z_{k\times(n-m+k-l)} = \left(w_{ij}\right)_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}}$$

and $(l-k)(m-k)$ coordinates in

$$W_{(m-k)\times(l-k)} := \left(w_{ij}\right)_{\substack{k+1\le i\le m\\ k+1\le j\le l}}.$$

Hence on U we have coordinates

$$\Bigg(\underbrace{\left(v_{ij}\right)_{\substack{1\le i\le k\\ k+1\le j\le l}}}_{(k-1)(l-k-1)},\underbrace{\left(w_{ij}\right)_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}}}_{(n-l-m+k)k},\underbrace{\left(w_{ij}\right)_{\substack{k+1\le i\le m\\ k+1\le j\le l}}}_{(l-k)(m-k)},\ \underbrace{\left(v_{ij}\right)_{\substack{k+1\le i\le m\\ l+1\le j\le n-m+k}}}_{(m-k-1)(n-l-m+k-1)}\Bigg).$$

Let us now see how the action of $H_{(I,k)}$ behaves on these coordinates

$$\left(\begin{array}{c|c|c|c} D_1 & \left(v_{ij}\right)_{\substack{1\le i\le k\\ k+1\le j\le l}} & \left(w_{ij}\right)_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}} & 0_{k\times(m-k)} \\ \hline 0_{(m-k)\times k} & \left(w_{ij}\right)_{\substack{k+1\le i\le m\\ k+1\le j\le l}} & \left(v_{ij}\right)_{\substack{k+1\le i\le m\\ l+1\le j\le n-m+k}} & D_2 \end{array}\right)\cdot\left(\begin{array}{c|c} e^{-\sqrt{-1}\theta}Id_l & 0 \\ \hline 0 & Id_{n-l}\end{array}\right)$$

$$=\left(\begin{array}{c|c|c|c} e^{-\sqrt{-1}\theta}D_1 & e^{-\sqrt{-1}\theta}\left(v_{ij}\right)_{\substack{1\le i\le k\\ k+1\le j\le l}} & \left(w_{ij}\right)_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}} & 0_{k\times(m-k)} \\ \hline 0_{(m-k)\times k} & e^{-\sqrt{-1}\theta}\left(w_{ij}\right)_{\substack{k+1\le i\le m\\ k+1\le j\le l}} & \left(v_{ij}\right)_{\substack{k+1\le i\le m\\ l+1\le j\le n-m+k}} & D_2 \end{array}\right)$$

$$\sim\left(\begin{array}{c|c|c|c} D_1 & \left(v_{ij}\right)_{\substack{1\le i\le k\\ k+1\le j\le l}} & e^{\sqrt{-1}\theta}\left(w_{ij}\right)_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}} & 0_{k\times(m-k)} \\ \hline 0_{(m-k)\times k} & e^{-\sqrt{-1}\theta}\left(w_{ij}\right)_{\substack{k+1\le i\le m\\ k+1\le j\le l}} & \left(v_{ij}\right)_{\substack{k+1\le i\le m\\ l+1\le j\le n-m+k}} & D_2 \end{array}\right).$$

Hence, the action of $H_{(I,k)}$ on U becomes

$$e^{\sqrt{-1}\theta}\cdot\left(\left(v_{ij}\right)_{\substack{1\le i\le k\\ k+1\le j\le l}},\left(w_{ij}\right)_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}},\left(w_{ij}\right)_{\substack{k+1\le i\le m\\ k+1\le j\le l}},\left(v_{ij}\right)_{\substack{k+1\le i\le m\\ l+1\le j\le n-m+k}}\right)$$
$$=\left(\left(v_{ij}\right)_{\substack{1\le i\le k\\ k+1\le j\le l}},e^{\sqrt{-1}\theta}\left(w_{ij}\right)_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}},e^{-\sqrt{-1}\theta}\left(w_{ij}\right)_{\substack{k+1\le i\le m\\ k+1\le j\le l}},\left(v_{ij}\right)_{\substack{k+1\le i\le m\\ l+1\le j\le n-m+k}}\right),$$

and the moment map is, in these coordinates,

$$\mu_{H_{(I,k)}}\left(\left(v_{ij}\right)_{\substack{1\le i\le k\\ k+1\le j\le l}},\left(w_{ij}\right)_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}},\left(w_{ij}\right)_{\substack{k+1\le i\le m\\ k+1\le j\le l}},\left(v_{ij}\right)_{\substack{k+1\le i\le m\\ l+1\le j\le n-m+k}}\right)$$
$$=\frac{1}{2}\left(\sum_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}}|w_{ij}|^2-\sum_{\substack{k+1\le i\le m\\ k+1\le j\le l}}|w_{ij}|^2\right).$$

The values of the variables w_{ij} at $[P_C]$ are zero and so, at the critical level set we have

$$\sum_{\substack{1\le i\le k\\ l+1\le j\le n-m+k}}|w_{ij}|^2=\sum_{\substack{k+1\le i\le m\\ k+1\le j\le l}}|w_{ij}|^2.$$

Note that the fixed point set of $H_{(I,k)}$ has dimension

$$2(k-1)(l-k-1)+2(m-k-1)(n-l-m+k-1)=\dim Pol_k(\underline{\widetilde{\alpha}}_I)+\dim Pol_{m-k}(\underline{\widetilde{\alpha}}_{I^c}),$$

where $\underline{\widetilde{\alpha}}_I:=\frac{m-k}{m}(\alpha_i)_{i\in I}$ and $\underline{\widetilde{\alpha}}_{I^c}=\frac{k}{m}(\alpha_i)_{i\in I^c}$ (see (6.1.1)).

Therefore, a transverse slice to Y_C at a point $[P_C]$ is a cone C_W over a link

$$W\simeq\left(S^{2(m-k)(l-k)-1}\times S^{2k((n-l)-(m-k))-1}\right)/S^1.$$

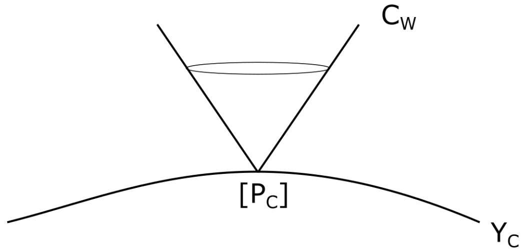

Figure 6.1.1: Cone C_W at $[P_C]$.

As $[P_C]$ varies, the slice changes continuously, giving rise to a fibration $E\to Y_C$ of a neighbourhood E of Y_C. The fibers of this fibration are C_W.

In particular, the link W of a fiber C_W over a point $[P_C] \in Y_C$ can be seen as a fiber bundle over $\mathbb{CP}^{(m-k)(l-k)-1}$ or a fiber bundle over $\mathbb{CP}^{k((n-l)-(m-k))-1}$. Corresponding to these two ways of describing W there are two possible ways of desingularizing the critical reduced space. Indeed, the action $w \mapsto e^{\sqrt{-1}\theta}w$ of S^1 on $S^{(m-k)(l-k)-1}$ extends to a linear action on $\mathbb{C}^{(m-k)(l-k)}$ and so we can consider the associated bundle

$$W_+ = \mathbb{C}^{(m-k)(l-k)} \times_{S^1} S^{2k((n-l)-(m-k))-1} \tag{6.1.3}$$

over $\mathbb{CP}^{k((n-l)-(m-k))-1}$. On this bundle there is a blow down map

$$\beta_+ : W_+ \to C_W$$

defined by $\beta_+(w_1, w_2) = (w_1, |w_1|w_2)$ and an embedding $\iota : \mathbb{CP}^{k((n-l)-(m-k))-1} \to W_+$ as the zero section of the bundle in (6.1.3). The image of ι gets blown down to 0 by β_+ and β_+ is a diffeomorphism from the complement of this image to the complement of $\{0\}$ in C_W.

Similarly, we obtain a different desingularization W_- of C_W and a blow down map

$$\beta_- : W_- \to C_W,$$

with

$$W_- = S^{2(m-k)(l-k)-1} \times_{S^1} \mathbb{C}^{k((n-l)-(m-k))}$$

where now there is a $\mathbb{CP}^{(m-k)(l-k)-1}$ that is blown down to $\{0\}$.

To simplify notation, let us write $p = (m-k)(l-k)$ and $q = k((n-l)-(m-k))$. We conclude that as $\underline{\alpha}$ goes from the critical level $\underline{\alpha}^c$ to $\underline{\alpha}_0$, the reduced space $Pol_m(\underline{\alpha})$ remains unchanged except in a neighbourhood of Y_C which suffers a partial blow up along Y_C. The exceptional divisor E^+ is a fiber bundle over Y_C with fiber $\mathbb{CP}^{q-1}$.

Similarly, as $\underline{\alpha}$ goes from the critical level $\underline{\alpha}^c$ to $\underline{\alpha}_1$, the reduced space $Pol_m(\underline{\alpha})$ is unchanged except in a neighbourhood of Y_C which suffers a partial blow up. The exceptional divisor E^- is a fiber bundle over Y_C with fiber $\mathbb{CP}^{p-1}$. □

Let us now see how these partial blow ups affect the cohomology of the reduced spaces.

Theorem 6.4. Let $\underline{\alpha}_0, \underline{\alpha}_1, \underline{\alpha}^c$ be as in Theorem 6.3. Let

$$p = (m-k)(|I|-k) \quad \text{and} \quad q = k(|I^c| - (m-k)).$$

Then the Poincaré polynomials of $Pol_m(\underline{\alpha}_0)$ and $Pol_m(\underline{\alpha}_1)$ satisfy

$$P_t(Pol_m(\underline{\alpha}_1)) - P_t(Pol_m(\underline{\alpha}_0)) = P_t(Y_C)\left(P_t\left(\mathbb{CP}^{p-1}\right) - P_t\left(\mathbb{CP}^{q-1}\right)\right).$$

Proof. As we have seen in Theorem 6.3 we know that

$$Pol_m(\underline{\alpha}_0) = \widetilde{Pol_m(\underline{\alpha}^c)}^+_{Y_C}$$

is a partial blow up of $Pol_m(\underline{\alpha}^c)$ along Y_C. Let $U \subset Pol_m(\underline{\alpha}^c)$ be a neighbourhood of Y_C which fibers over Y_C with fiber the cone C_W.

We have a globally defined blow down map

$$\beta_+ : Pol_m(\underline{\alpha}_0) \to Pol_m(\underline{\alpha}^c).$$

Let

$$\widetilde{U} = \beta_+^{-1}(U), \quad U^* = U \backslash Y_C, \quad \widetilde{U}^* = \widetilde{U} \backslash E^+,$$
$$Pol_m(\underline{\alpha}^c)^* = Pol_m(\underline{\alpha}^c) \backslash Y_C, \quad Pol_m(\underline{\alpha}_0)^* = Pol_m(\underline{\alpha}_0) \backslash E^+,$$

where E^+ is the exceptional divisor in $Pol_m(\underline{\alpha}_0)$, and compare the Mayer-Vietoris sequences for

$$Pol_m(\underline{\alpha}^c) = U \cup Pol_m(\underline{\alpha}^c)^*$$

and

$$Pol_m(\underline{\alpha}_0) = Pol_m(\underline{\alpha}_0)^* \cup \widetilde{U}$$

(see [GH, pg. 605] for details).

Since U and $\widetilde{U}$ respectively retract to Y_C and E^+, we have isomorphisms

$$H^*(U) \cong H^*(Y_C) \quad \text{and} \quad H^*(\widetilde{U}) \cong H^*(E^+).$$

Moreover, by the blow up construction we have isomorphisms

$$\begin{aligned} H^*(U^*) &\cong H^*(\widetilde{U}^*) \\ H^*(Pol_m(\underline{\alpha}^c)^*) &\cong H^*(Pol_m(\underline{\alpha}_0)^*). \end{aligned}$$

Hence we obtain

$$\begin{array}{ccccccc} H^{i-1}(\widetilde{U}^*) & \longrightarrow & H^i(Pol_m(\underline{\alpha}_0)) & \longrightarrow & H^i(Pol_m(\underline{\alpha}_0)^*) \oplus H^i(E^+) & \longrightarrow & H^i(\widetilde{U}^*) \\ \| & & \uparrow{\scriptstyle (\beta^+)^*} & & \| & & \| \\ H^{i-1}(U^*) & \longrightarrow & H^i(Pol_m(\underline{\alpha}^c)) & \longrightarrow & H^i(Pol_m(\underline{\alpha}^c)^*) \oplus H^i(Y_C) & \longrightarrow & H^i(U^*) \end{array}$$

and since $(\beta^+)^* : H^i(Pol_m(\underline{\alpha}^c)) \to H^i(Pol_m(\underline{\alpha}_0))$ is injective, we have that (additively),

$$H^*(Pol_m(\underline{\alpha}_0)) = (\beta^+)^* H^*(Pol_m(\underline{\alpha}^c)) \oplus H^*(E^+)/(\beta^+)^* H^*(Y_C).$$

We conclude that the Poincaré polynomial $P_t(Pol_m(\underline{\alpha}_0))$ of $Pol_m(\underline{\alpha}_0))$ satisfies

$$P_t(Pol_m(\underline{\alpha}_0)) = P_t(Pol_m(\underline{\alpha}^c)) + P_t(E^+) - P_t(Y_C).$$

Similarly, we conclude that

$$P_t(Pol_m(\underline{\alpha}_1)) = P_t(Pol_m(\underline{\alpha}^c)) + P_t(E^-) - P_t(Y_C)$$

and so

$$P_t(Pol_m(\underline{\alpha}_1)) - P_t(Pol_m(\underline{\alpha}_0)) = P_t(E^-) - P_t(E^+).$$

Now E^-, E^+ are projective bundles over Y_C respectively with fiber $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ so they are projectivizations $\mathbb{P}(\widetilde{E}^-), \mathbb{P}(\widetilde{E}^+)$ of vector bundles $\widetilde{E}^-$ and $\widetilde{E}^+$ of rank p and q respectively (see [Har, Chapter II]).

Consequently, since the Poincaré polynomial of any projectivization bundle splits, we have

$$P_t(E^-) = P_t\left(\mathbb{CP}^{p-1}\right) P_t(Y_C)$$

and

$$P_t(E^+) = P_t\left(\mathbb{CP}^{q-1}\right) P_t(Y_C).$$

We conclude that

$$P_t(Pol_m(\underline{\alpha}_1)) - P_t(Pol_m(\underline{\alpha}_0)) = P_t(E^-) - P_t(E^+) = \left(P_t\left(\mathbb{CP}^{p-1}\right) - P_t\left(\mathbb{CP}^{q-1}\right)\right) P_t(Y_C).$$

Note that

$$P_t\left(\mathbb{CP}^{p-1}\right) - P_t\left(\mathbb{CP}^{q-1}\right) = \begin{cases} t^{2q} + \cdots + t^{2p-2} & \text{if } p > q \\ t^{2p} + \cdots + t^{2q-2} & \text{if } p < q \\ 0 & \text{if } p = q. \end{cases}$$

□

6.2 Wall crossing and the Poincaré polynomial

Given a value of $\underline{\alpha}$ that is m-generic, our goal is to obtain the Poincaré polynomial of $Pol_m(\underline{\alpha})$. If we are able to compute it for some generic value $\underline{\beta}$, we can consider a path ζ in the moduli space from $\underline{\beta}$ to $\underline{\alpha}$ and use the results in the previous section to compute the Poincaré polynomial of $Pol_m(\underline{\alpha})$.

We can choose a transverse path ζ intersecting only one wall at a time from $\underline{\beta}$ to $\underline{\alpha}$. This path crosses a wall $W_{(I,k)}$ if and only if I is k-short/long for $\underline{\beta}$ and k-long/short for $\underline{\alpha}$.

We will choose $\underline{\beta}$ in such a way that we know its relative position to every wall and the Poincaré polynomial of $Pol_m(\underline{\beta})$. We will call such polynomial the *initial Poincaré polynomial.*

In this section we will always assume that $\alpha_1 \leq \cdots \leq \alpha_n$. Note that this assumption supposes no constraint, since we can always rearrange $\underline{\alpha}$.

Proposition 6.5. Let $\underline{\alpha}$ be m-admissible and not on an outer wall. Let $I \subset \{1, \ldots, n\}$ and consider an integer $k \in \{1, \ldots, m\}$. If $|I| \leq k$, then I is k-short for $\underline{\alpha}$.

Proof. Since $\alpha_1 \leq \cdots \leq \alpha_n$, then it is enough to prove the statement for $I = \{n-k+1, \ldots, n\}$ (the set where the sum of the α_is is maximal).

If I is not k-short for $\underline{\alpha}$, then

$$(m-k)(\alpha_{n-k+1} + \cdots + \alpha_n) \geq k(\alpha_1 + \cdots + \alpha_{n-k})$$

and so

$$m(\alpha_{n-|J|+1}+\cdots+\alpha_{n-1})+(m-k)\alpha_n \ \geq \ k(\alpha_1+\cdots+\alpha_{n-1}).$$

Then, we have

$$(m(k-1)+m-k)\alpha_n = (m-1)k\alpha_n \geq m(\underbrace{\alpha_{n-k+1}+\cdots+\alpha_{n-1}}_{k-1})+(m-k)\alpha_n \geq k(\alpha_1+\cdots+\alpha_{n-1})$$

and so

$$(m-1)\alpha_n \geq \alpha_1+\cdots+\alpha_{n-1}.$$

Contradicting the strong triangle inequalities and the fact that $\underline{\alpha}$ is not on an outer wall. □

Remark 6.6. From Proposition 6.5, it follows that if $m-k \geq |I^c|$, then I^c is $(m-k)$-short for $\underline{\alpha}$ or, equivalently, that I is k-long for $\underline{\alpha}$ (i.e. if $|I| \geq n-m+k$ then I is always k-long for $\underline{\alpha}$).

Since we are assuming that $\alpha_1 \leq \cdots \leq \alpha_n$, Proposition 6.5 implies that if we want to study the effect of crossing a wall on polygon spaces $Pol_m(\underline{\alpha})$, it is enough to consider the set of inner walls $W_{(I,k)}$ with (I,k) in the set $\mathcal{W}_m$, where

$$\mathcal{W}_m := \left\{(I,k) : I \subset \{1,\ldots,n\} \text{ and } k \in \left\{1,\ldots,\frac{m-1}{2}\right\},\ 1 \leq k < |I| < n-m+k \leq n-1\right\}$$

if m is odd and

$$\begin{aligned}\mathcal{W}_m := &\left\{(I,k) : I \subset \{1,\ldots,n\} \text{ and } k \in \left\{1,\ldots,\frac{m}{2}-1\right\},\ 1 \leq k < |I| < n-m+k \leq n-1\right\}\\ &\bigcup \left\{\left(I,\frac{m}{2}\right) : I \subset \{1,\ldots,n-1\} \text{ and } 1 \leq \frac{m}{2} < |I| < n-\frac{m}{2} \leq n-1\right\}\end{aligned}$$

if m is even. Here we used the fact that $W_{(I,k)} = W_{(I^c,m-k)}$.

Remark 6.7.

1. Note that the outer walls (i.e. those that bound the set of m-admissible values of $\underline{\alpha}$) are those defined by the equations

$$(m-1)\alpha_i = \sum_{j\neq i}\alpha_j \quad \text{for } i=1,\ldots,n$$

and so they are the walls $W_{(I,1)}$ with $|I|=1$ or $W_{(I,m-1)}$ with $|I|=m-1$.

2. When m is even and $k = \frac{m}{2}$, we have $W_{(I,\frac{m}{2})} = W_{(I^c,\frac{m}{2})}$, so we only consider sets $I \subset \{1,\dots,n-1\}$ to avoid repetitions.

We will start y taking an initial m-generic value $\underline{\beta}$ with $\beta_1 \le \cdots \le \beta_n$ and such that

1. a set $I \subset \{1,\dots,n\}$ with $1 \le k < |I| < n-m+k \le n-1$ is k-short for $\underline{\beta}$ for some $k \in \{1,\dots,\lfloor \frac{m}{2} \rfloor\}$ if and only if

$$|I \cap \{n-m+2,\dots,n\}| < k$$

and

2. a set $I \subset \{1,\dots,n\}$ with $1 \le k < |I| < n-m+k \le n-1$ is k-long for $\underline{\beta}$ for some $k \in \{1,\dots,\lfloor \frac{m}{2} \rfloor\}$ if and only if

$$|I \cap \{n-m+2,\dots,n\}| \ge k.$$

Then, if for any m-generic value $\underline{\alpha}$, we consider the sets

$$\mathcal{L}^{\underline{\alpha}}_{n,m,k} := \left\{(I,k) \in \mathcal{W}_m : \ |I \cap \{n-(m-2),\dots,n\}| \le k-1 \quad \text{and } I \text{ is } k\text{-long for } \underline{\alpha}\right\},$$

$$\mathcal{S}^{\underline{\alpha}}_{n,m,k} := \left\{(I,k) \in \mathcal{W}_m : \ |I \cap \{n-(m-2),\dots,n\}| \ge k \quad \text{and } I \text{ is } k\text{-short for } \underline{\alpha}\right\},$$

$$\mathcal{L}^{\underline{\alpha}}_{n,m} = \bigcup_{k=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{L}^{\underline{\alpha}}_{n,m,k} \quad \text{and} \quad \mathcal{S}^{\underline{\alpha}}_{n,m} = \bigcup_{k=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{S}^{\underline{\alpha}}_{n,m,k}, \tag{6.2.1}$$

we have that $\mathcal{L}^{\underline{\beta}}_{n,m} \cup \mathcal{S}^{\underline{\beta}}_{n,m} = \emptyset$ and so

$$\mathcal{L}^{\underline{\alpha}}_{n,m} \cup \mathcal{S}^{\underline{\alpha}}_{n,m}$$

will give us the collection of walls that the path ζ from $\underline{\beta}$ to $\underline{\alpha}$ has to cross.

6.2.1 Choice of the initial value $\underline{\beta}$

Proposition 6.8. Let $\beta_1,\dots,\beta_{n-m+1} \in \mathbb{R}^+$ be such that $\beta_1 \le \cdots \le \beta_{n-m+1}$ and let $S := \sum_{j=1}^{n-m+1} \beta_j$. Then for any $0 < \varepsilon_1 < \beta_1$, the initial value

$$\underline{\beta}_{n,m} := (\beta_1,\dots,\beta_{n-m+1}, S-\varepsilon_1,\dots,S-\varepsilon_{m-1})$$

with

$$\varepsilon_i := \frac{m^2+m-2i}{m^2+m-2}\varepsilon_1,$$

for $i = 1,\dots,m-1$, is m-admissible.

Remark 6.9. Note that

$$\varepsilon_{i-1} > \varepsilon_i > \frac{\varepsilon_1+\cdots+\varepsilon_{i-1}}{i} \tag{6.2.2}$$

for $i = 2,\dots,m-1$. Indeed,

$$\begin{aligned} i\varepsilon_i - \sum_{j=1}^{i-1}\varepsilon_j =& i\left(\frac{m^2+m-2i}{m^2+m-2}\right)\varepsilon_1 - \left(\frac{(m^2+m)(i-1)-i(i-1)}{m^2+m-2}\right)\varepsilon_1 \\ =& i\left(\frac{m^2+m-2i}{m^2+m-2}\right)\varepsilon_1 - \left(\frac{m^2+m-i}{m^2+m-2}\right)\varepsilon_1(i-1) \\ =& \frac{\varepsilon_1}{m^2+m-2}(m-i)(m+i+1) > 0. \end{aligned}$$

Moreover $\beta_1 \le \cdots \le \beta_n$, since $\beta_{n-m+1} < S-\varepsilon_1$.

Proof. Since $\beta_1 \le \cdots \le \beta_n$ (cf. Remark 6.9), it is enough to show that $\{n\}$ is 1-short for $\underline{\beta}_{n,m}$, i.e.

$$(m-1)\beta_n < \beta_1+\cdots+\beta_{n-1}.$$

This is true by (6.2.2) since

$$(m-1)(S-\varepsilon_{m-1}) < S+(m-2)S-\sum_{i=1}^{m-2}\varepsilon_i$$

holds if and only if

$$(m-1)\varepsilon_{m-1} > \sum_{i=1}^{m-2}\varepsilon_i.$$

We conclude that $\underline{\beta}_{n,m}$ is m-admissible for every $m \ge 1$. □

Remark 6.10. Knowing that $\underline{\beta}_{n,m}$ is m-admissible, one can choose $0 < \varepsilon_1 < \beta_1$ such that $\underline{\beta}_{n,m}$ is also m-generic.

Proposition 6.11. Let $\underline{\beta}_{n,m} := (\beta_1,\dots,\beta_{n-m+1}, S-\varepsilon_1, S-\varepsilon_2,\dots,S-\varepsilon_{m-1})$ be as in Proposition 6.8 for some $0 < \varepsilon_1 < \beta_1$ so that $\underline{\beta}_{n,m}$ is m-generic. Then,

$$\mathcal{S}^{\underline{\beta}}_{n,m} \cup \mathcal{L}^{\underline{\beta}}_{n,m} = \emptyset,$$

where $\mathcal{S}^{\underline{\beta}}_{n,m}$ and $\mathcal{L}^{\underline{\beta}}_{n,m}$ are the sets defined in 6.2.1.

Proof. Let us first show that $\mathcal{L}^{\underline{\beta}}_{n,m} = \emptyset$ for every $1 \le k \le \lfloor \frac{m}{2} \rfloor$ or, equivalently, that if a set $I \subset \{1,\dots,n\}$ with $1 \le k < |I| < n-m+k \le n-1$ satisfies $|I \cap \{n-m+2,\dots,n\}| \le k-1$, then it must be k-short for $\underline{\beta}$. (Note that, in this case, I has at most $k-1$ elements in $\{n-m+2,\dots,n\}$.)

For that, since $\beta_1 \le \cdots \le \beta_n$, it is enough to show that

$$I = \{2,\dots,n-m+1, n-k+2,\dots,n\}$$

is k-short for $\underline{\beta}$ (note that $|I| = n-m+k-1$). This follows from the fact that

$$(m-k)\sum_I \beta_i < k\sum_{I^c}\beta_i \Leftrightarrow (m-k)\left(S-\beta_1+(k-1)S-\sum_{i=m-k+1}^{m-1}\varepsilon_i\right) < k\left(\beta_1+(m-k)S-\sum_{i=1}^{m-k}\varepsilon_i\right)$$

$$\Leftrightarrow 0 < m\beta_1 + (m-k)\sum_{i=m-k+1}^{m-1}\varepsilon_i - k\sum_{i=1}^{m-k}\varepsilon_i$$

$$\Leftrightarrow 0 < m\beta_1 + m\sum_{i=m-k+1}^{m-1}\varepsilon_i - k\sum_{i=1}^{m-1}\varepsilon_i$$

$$\Leftrightarrow 0 < m\beta_1 + \left(\frac{m(k-1)(m^2-m+k)-km^2(m-1)}{m^2+m-2}\right)\varepsilon_1$$

$$\Leftrightarrow 0 < m\beta_1 + m\left(\frac{k^2-m^2+m-k}{m^2+m-2}\right)\varepsilon_1$$

$$\Leftrightarrow 0 < m\beta_1 - m\left(\frac{m^2-m+k-k^2}{m^2+m-2}\right)\varepsilon_1$$

$$\Leftrightarrow 0 < m\left(\beta_1 - \left(\frac{m^2-m+k-k^2}{m^2+m-2}\right)\varepsilon_1\right),$$

which is true since

$$0 < \beta_1 - \varepsilon_1 < \beta_1 - \left(\frac{m(m-1)-k(k-1)}{m^2+m-2}\right)\varepsilon_1 < \beta_1,$$

as
$$0 < \frac{m(m-1)-k(k-1)}{m^2+m-2} < 1.$$

Let us now show that $\mathcal{S}^{\underline{\beta}}_{n,m} = \emptyset$ for every $1 \le k \le \lfloor \frac{m}{2} \rfloor$ or, equivalently, that if a set $I \subset \{1,\dots,n\}$ with $1 \le k < |I| < n-m+k \le n-1$ satisfies $|I \cap \{n-m+2,\dots,n\}| \ge k$, then it must be k-long for $\underline{\beta}$. (Note that, in this case, I has at least k elements in $\{n-m+2,\dots,n\}$.)

Since $\beta_1 \le \cdots \le \beta_n$, it is enough to show that
$$I = \{1, n-m+2, \dots, n-m+k+1\}$$
is k-long for $\underline{\beta}$ (note that $|I| = k+1$). This follows from the fact that
$$(m-k)\left(\beta_1 - kS - \sum_{i=1}^{k} \varepsilon_i\right) > k(S - \beta_1 + (m-1-k)S - \sum_{i=k+1}^{m-1} \varepsilon_i)$$
$$\Leftrightarrow (m-k)\beta_1 - (m-k)\sum_{i=1}^{k} \varepsilon_i > -k\beta_1 - k\sum_{i=k+1}^{m-1} \varepsilon_i$$
$$\Leftrightarrow m\beta_1 - m\sum_{i=1}^{k} \varepsilon_i + k\sum_{i=1}^{m-1} \varepsilon_i > 0$$
$$\Leftrightarrow m\beta_1 - mk\left(\frac{m^2+m-k-1}{m^2+m-2}\right)\varepsilon_1 + k\frac{m^2(m-1)}{m^2+m-2}\varepsilon_1 > 0$$
$$\Leftrightarrow m\beta_1 - mk\left(\frac{2m-k-1}{m^2+m-2}\right)\varepsilon_1 > 0$$
$$\Leftrightarrow m\left(\beta_1 - k\frac{2m-k-1}{m^2+m-2}\varepsilon_1\right) > 0,$$
which is true since
$$0 < \frac{k(2m-k-1)}{m^2+m-2} < 1,$$

as

$$2mk - k^2 - k < m^2 + m - 2 \Leftrightarrow m^2 + m(1-2k) + k(k+1) - 2 > 0$$
$$\Leftrightarrow m(m-2k+1) + k(k+1) - 2 > 0.$$

□

Proposition 6.12. Let $\underline{\beta}_{n.m} := (\beta_1, \ldots, \beta_{n-m+1}, S - \varepsilon_1, \ldots, S - \varepsilon_{m-1})$ be as in Proposition 6.8. Then there exists $0 < \varepsilon_1 < \beta_1$ such that $\underline{\beta}_{n,m}$ is m-generic and

$$P_t\left(Pol_m(\underline{\beta}_{n,m})\right) = \left(P_t\left(\mathbb{CP}^{n-m-1}\right)\right)^{m-1}.$$

Proof. We know from Proposition 6.8 that $\underline{\beta}_{n,m}$ is m-admissible. Moreover, we can choose ε_1 appropriately so that it is also m-generic.

We will now show by induction on m that

$$P_t\left(Pol_m(\underline{\beta}_{n,m})\right) = \left(P_t\left(\mathbb{CP}^{n-m-1}\right)\right)^{m-1} \text{ for all } m \geq 1 \text{ and } n \geq m+1. \tag{6.2.3}$$

Note that (6.2.3) is trivially true for $m = 1$.

Let us assume then that (6.2.3) is true for $m-1$, i.e. for all $n \geq m$,

$$P_t\left(Pol_{m-1}(\underline{\beta}_{n,m-1})\right) = \left(P_t\left(\mathbb{CP}^{n-m}\right)\right)^{m-2}$$

and show that it is true for m.

For that let us consider the following m-admissible $\underline{\beta}^c_{n,m}$,

$$\underline{\beta}^c_{n,m} := (\beta_1, \ldots, \beta_{n-m+1}, S - \varepsilon_1 \ldots, S - \varepsilon_{m-2}, S - \lambda_m)$$

where

$$\lambda_m = \frac{1}{m-1} \sum_{i=1}^{m-2} \varepsilon_i.$$

Note that $\underline{\beta}^c_{n,m} \in W_{\{n\},1}$, since

$$(m-1)(S - \lambda_m) = S + (m-2)S - \sum_{i=1}^{m-2} \varepsilon_i \Leftrightarrow (m-1)\lambda_m = \sum_{i=1}^{m-2} \varepsilon_i.$$

We can choose ε_1 small enough such that $\underline{\beta}_{n,m}$ is close to $\underline{\beta}^c_{n,m}$ (so that there are no walls between them), and then, by Theorem 6.4, we have

$$P_t(Pol_m(\underline{\beta}_{n,m})) = P_t(Y_C)P_t\left(\mathbb{CP}^{n-m-1}\right),$$

where Y_C is the set of critical points in $Pol_m(\underline{\beta}^c_{n,m})$. By (6.1.1) we know Y_C is a copy of

$$Pol_{m-1}(\underline{\beta}^c_{n-1,m-1}),$$

where $\underline{\beta}^c_{n-1,m-1} = (\beta_1, \dots, \beta_{n-m+1}, S-\varepsilon_1, \dots, S-\varepsilon_{m-2})$. Note that we can still assume, without loss of generality, that $\underline{\beta}^c_{n-1,m-1}$ is $(m-1)$-generic. (In fact, given m, we can choose ε_1 such that $\underline{\beta}^c_{n-j,m-j}$ is $(m-j)$-generic for every $0 \le j \le m-1$.)

By the induction hypothesis we have

$$\begin{aligned} P_t\left(Pol_m(\underline{\beta}_{n,m})\right) &= P_t(Y_C)P_t\left(\mathbb{CP}^{n-m-1}\right) = P_t\left(Pol_{m-1}(\underline{\beta}_{n-1,m-1})\right)P_t\left(\mathbb{CP}^{n-m-1}\right) \\ &= P_t\left(\mathbb{CP}^{n-1-m}\right)^{m-2} P_t\left(\mathbb{CP}^{n-m-1}\right) = P_t\left(\mathbb{CP}^{n-m-1}\right)^{m-1}. \end{aligned}$$

□

Remark 6.13. We can easily adapt this proof to show that, in fact, $Pol_m(\underline{\beta}_{n,m})$ is a $(m-1)$-stage generalized Bott tower, where the fiber of each stage is $\mathbb{CP}^{n-m-1}$, i.e. it is the final space of a sequence of toric manifolds

$$B_{m-1} \xrightarrow{\pi_{m-1}} B_{m-2} \xrightarrow{\pi_{m-2}} \dots \xrightarrow{\pi_1} B_0 = \{pt\}$$

where the fiber of $\pi_i : B_i \longrightarrow B_{i-1}$ for $i = 1, \dots, m-1$ is $\mathbb{CP}^{n-m-1}$ (see [CMS]).

6.3 The Poincaré polynomial

Let us see how to compute the Poincaré polynomial of $Pol_m(\underline{\alpha})$ for some m-generic value $\underline{\alpha}$.

Theorem 6.14. Let $\underline{\alpha} \in (\mathbb{R}^+)^n$ with $\alpha_1 \le \dots \le \alpha_n$ be m-generic and let ζ be the line segment from $\underline{\beta}_{n,m}$ to $\underline{\alpha}$ (contained in the set of m-admissible values).

Let $\mathcal{S}^{\underline{\alpha}}_{n,m}$ and $\mathcal{L}^{\underline{\alpha}}_{n,m}$ be the sets defined in 6.2.1. Then the Poincaré polynomial of $Pol_m(\underline{\alpha})$ is

$$P_t(Pol_m(\underline{\alpha})) = \left(P_t\left(\mathbb{CP}^{n-m-1}\right)\right)^{m-1}$$

$$+ \sum_{(I,k)\in\mathcal{S}^{\underline{\alpha}}_{n,m}\cup\mathcal{L}^{\underline{\alpha}}_{n,m}} (-1)^{\chi_{\mathcal{L}^{\underline{\alpha}}_{n,m}}(I,k)} P_t\left(Pol_k(\zeta^c_I)\right) P_t\left(Pol_{m-k}(\zeta^c_{I^c})\right) Q_{(I,k)}(t),$$

where

$$Q_{(I,k)}(t) = P_t\left(\mathbb{CP}^{k(|I^c|-m+k)-1}\right) - P_t\left(\mathbb{CP}^{(m-k)(|I|-k)-1}\right),$$

ζ^c_I and $\zeta^c_{I^c}$ are formed by the coordinates of $\zeta(s)$ with indices respectively in I and I^c when it intersects the wall $W_{(I,k)}$ and $\chi_{\mathcal{L}^{\underline{\alpha}}_{n,m}}$ is the characteristic function of the set $\mathcal{L}^{\underline{\alpha}}_{n,m}$.

Proof. We already know that the set $\mathcal{S}^{\underline{\alpha}}_{n,m} \cup \mathcal{L}^{\underline{\alpha}}_{n,m}$ gives us the set of walls crossed by the path ζ from $\underline{\beta}_{n,m}$ to $\underline{\alpha}$.

From Theorem 6.4 we know that when ζ crosses a wall $W_{(I,k)}$, the Poincaré polynomial changes by

$$(-1)^{\chi_{\mathcal{L}^{\underline{\alpha}}_{n,m}}(I,k)} P_t\left(Pol_k(\zeta^c_I)\right) P_t\left(Pol_{m-k}(\zeta^c_{I^c})\right) Q_{(I,k)}(t),$$

where

$$\chi_{\mathcal{L}^{\underline{\alpha}}_{n,m}}(I,k) = \begin{cases} 1 \text{ if } (I,k) \in \mathcal{L}^{\underline{\alpha}}_{n,m} \\ 0 \text{ if } (I,k) \notin \mathcal{L}^{\underline{\alpha}}_{n,m} \end{cases}$$

i.e. the sign of the additional polynomial depends on whether the set $I \in \mathcal{S}^{\underline{\alpha}}_{n,m} \cup \mathcal{L}^{\underline{\alpha}}_{n,m}$ is short or long for $\underline{\beta}_{n,m}$. □

Corollary 6.15. Let $\underline{\alpha} \in (\mathbb{R}^+)^n$, then $Pol_m(\underline{\alpha})$ is an even cohomology space.

Corollary 6.16. Let $\underline{\alpha} \in (\mathbb{R}^+)^n$ with $\alpha_1 \leq \cdots \leq \alpha_n$ be m-generic. With the notation of Theorem 6.14, the Euler Characteristic of $Pol_m(\underline{\alpha})$ is

$$\chi\left(Pol_m(\underline{\alpha})\right) = (n-m)^{m-1} + \sum_{(I,k)\in\mathcal{I}^{\underline{\alpha}}\cup\mathcal{S}^{\underline{\alpha}}} (-1)^{\chi_{\mathcal{L}^{\underline{\alpha}}_{n,m}}(I,k)} \chi\left(Pol_k(\zeta^c_I)\right) \chi\left(Pol_{m-k}(\zeta^c_{I^c})\right) (kn - n|I|),$$

where $\chi\left(Pol_k(\zeta^c_I)\right)$ and $\chi\left(Pol_{m-k}(\zeta^c_{I^c})\right)$ are the Euler Characteristics of $Pol_k\left(\zeta^c_I\right)$ and $Pol_{m-k}\left(\zeta^c_{I^c}\right)$ respectively.

6.3.1 Closed formula for $m = 2$

In this section we always assume that $m = 2$ and so k can only be equal to 1. We will therefore omit it abusing notation.

Using Theorem 6.14 we will recover the formula proved by Hausmann and Knutson in [HK2] for the Poincaré polynomial of $Pol_2(\underline{\alpha})$.

Theorem 6.17. ([HK2, Corollary 4.3]) Let $\underline{\alpha}$ be 2−generic. Then

$$P_{Pol_2(\underline{\alpha})}(t) = \frac{1}{1-t^2} \sum_{I \in S_n} \left(t^{2|I|} - t^{2(n-|I|-2)} \right),$$

where

$$S_n := \{I \in \{1, \ldots, n-1\} : \ I \cup \{n\} \text{ is 1-short for } \underline{\alpha}\}.$$

Proof. Using the fact that $Pol_1(\underline{\alpha})$ is always a point for every $\underline{\alpha}$, we have from Theorem 6.14 that

$$\begin{aligned} P_t(Pol_2(\underline{\alpha})) &= P_t\left(\mathbb{CP}^{n-3}\right) + \sum_{I \in \mathcal{S}^{\underline{\alpha}}_{n,2} \cup \mathcal{L}^{\underline{\alpha}}_{n,2}} (-1)^{\chi_{\mathcal{L}^{\underline{\alpha}}_{n,2}}(I)} P_t(Pol_1(\zeta_I^c)) P_t(Pol_1(\zeta_{I^c}^c)) Q_I(t) \\ &= P_t\left(\mathbb{CP}^{n-3}\right) + \sum_{I \in \mathcal{S}^{\underline{\alpha}}_{n,2} \cup \mathcal{L}^{\underline{\alpha}}_{n,2}} (-1)^{\chi_{\mathcal{L}^{\underline{\alpha}}_{n,2}}(I)} Q_I(t), \end{aligned}$$

where $Q_I(t) = P_t\left(\mathbb{CP}^{|I^c|-2}\right) - P_t\left(\mathbb{CP}^{|I|-2}\right)$.

Let $\underline{\beta}_n = (\beta_1, \ldots, \beta_{n-1}, S - \varepsilon_1)$ be as in (6.8), where $S = \sum_{i=1}^{n-1} \beta_i$ and $0 < \varepsilon_1 < \beta_1$. Since $\beta_1, \ldots, \beta_{n-1}$ are arbitrary, we can take $\beta_i = \alpha_i$ for $i = 1, \ldots, n-1$.

Consider the path $\zeta(\lambda) = \underline{\beta} + \lambda(\underline{\alpha} - \underline{\beta})$. Since the set of m-admissible values is convex, all the values $\zeta(\lambda)$ are m-admissible. By choosing another $\underline{\alpha}'$ in the same chamber as $\underline{\alpha}$ if necessary, we can assume that the path ζ always crosses one wall at a time. Moreover, the sets $\mathcal{W}_2$, $\mathcal{S}^{\underline{\alpha}}_{n,2}$ and $\mathcal{L}^{\underline{\alpha}}_{n,2}$ defined in Section 6.2 are

$$\begin{aligned} \mathcal{W}_2 &= \{I \subset \{1, \ldots, n-1\}, 2 \leq |I| \leq n-2\} \\ \mathcal{S}^{\underline{\alpha}}_{n,2} &= \emptyset \\ \mathcal{L}^{\underline{\alpha}}_{n,2} &= \{I \in \mathcal{W}_2 : \ I \text{ is 1-long for } \underline{\alpha}\}. \end{aligned}$$

Hence,

$$\begin{aligned}
P_t(Pol_2(\underline{\alpha})) &= P_t\left(\mathbb{CP}^{n-3}\right) + \sum_{I\in\mathcal{S}^{\alpha}_{n,2}\cup\mathcal{L}^{\alpha}_{n,2}} (-1)^{\chi_{\mathcal{L}^{\alpha}_{n,2}}(I)} \left(P_t\left(\mathbb{CP}^{|I^c|-2}\right) - P_t\left(\mathbb{CP}^{|I|-2}\right)\right)\\
&= P_t\left(\mathbb{CP}^{n-3}\right) - \sum_{I\in\mathcal{L}^{\alpha}_{n,2}} \left(P_t\left(\mathbb{CP}^{|I^c|-2}\right) - P_t\left(\mathbb{CP}^{|I|-2}\right)\right)\\
&= 1+t^2+\cdots+t^{2(n-3)} - \sum_{I\in\mathcal{L}^{\alpha}_{n,2}} \left(1+t^2+\cdots+t^{2(|I^c|-2)} - \left(1+t^2+\cdots+t^{2(|I|-2)}\right)\right).
\end{aligned}$$

If a set J is in $\mathcal{L}^{\alpha}_{n,2}$, then $J\subset\{1,\ldots,n-1\}$, $2\le |J|\le n-2$ and J is 1-long for $\underline{\alpha}$, implying that $n\in J^c$, $2\le |J^c|\le n-2$ and J^c is 1-short for $\underline{\alpha}$.

Let S_n be the set of subsets I of $\{1,\ldots,n-1\}$ such that $I\cup\{n\}$ is 1-short for $\underline{\alpha}$. Then

$$S_n = \left\{I^c\backslash\{n\} : \ I\in\mathcal{L}^{\alpha}_{n,2}\right\}\cup\{\emptyset\}.$$

Note that if a set $A\cup\{1,\ldots,n-1\}$ is such that $|A| = n-2$, then $A\cup\{n\}$ is always 1-long.

Therefore

$$P_t(Pol_2(\underline{\alpha})) = 1+t^2+\cdots+t^{2(n-3)} - \sum_{I\in\mathcal{L}^{\alpha}_{n,2}} \left(1+t^2+\cdots+t^{2(|I^c|-2)} - (1+t^2+\cdots+t^{2(|I|-2)})\right)$$

$$= 1+t^2+\cdots+t^{2(n-3)} - \sum_{J\in S_n\backslash\{n\}} \left(1+t^2+\cdots+t^{2(|J|-1)} - (1+t^2+\cdots+t^{2(n-|J|-3)})\right).$$

Note that if $J\in S_n$, then $I := (J\cup\{n\})^c \in\mathcal{L}^{\alpha}_{n,2}$ and so $J = I^c\backslash\{n\}$ and $|J| = |I^c|-1$.

If $|J|-1 > n-|J|-3$, then

$$\begin{aligned}
\left(1+t^2+\cdots+t^{2(|J|-1)}\right) - \left(1+t^2+\cdots+t^{2(n-|J|-3)}\right) &= t^{2(n-|J|-2)}+\cdots+t^{2(|J|-1)}\\
&= \frac{1}{1-t^2}\left(t^{2(n-|J|-2)} - t^{2|J|}\right).
\end{aligned}$$

Otherwise we have

$$\begin{aligned}
\left(1+t^2+\cdots+t^{2(|J|-1)}\right) - \left(1+\cdots+t^{2(n-|J|-3)}\right) &= -\left(t^{2|J|}+\cdots+t^{2(n-|J|-3)}\right)\\
&= -\frac{1}{1-t^2}\left(t^{2|J|} - t^{2(n-|J|-2)}\right).
\end{aligned}$$

Hence

$$\begin{aligned} P_t(Pol_2(\underline{\alpha})) &= 1+t^2+\cdots+t^{2(n-3)}+\sum_{J\in S_n\setminus\{\emptyset\}}\frac{1}{1-t^2}\left(t^{2|J|}-t^{2(n-|J|-2)}\right) \\ &= \sum_{J\in S_n}\frac{1}{1-t^2}\left(t^{2|J|}-t^{2(n-|J|-2)}\right). \end{aligned}$$

□

Corollary 6.18. The Euler Characteristic of $Pol_2(\underline{\alpha})$ is

$$\chi(Pol_2(\underline{\alpha})) = n-2-\sum_{J\in S_n\setminus\{n\}}(2|J|+2-n).$$

6.3.2 Closed formula and examples for $m = 3$

In this section we always assume that $m = 3$ and so k can only be equal to 1. We will therefore omit it, abusing notation.

Using Theorem 6.14 we have the following result.

Theorem 6.19. Let $\underline{\alpha}$ be 3-generic. Then

$$P_t(Pol_3(\underline{\alpha})) = \left(1+t^2+\cdots+t^{2(n-4)}\right)^2+$$
$$+\frac{1}{(1-t^2)^2}\left(\sum_{\substack{J\in\mathcal{L}_{n-1,n}\cup\mathcal{S}_{n-1,n}\\ J\neq\{\{n-1,n\}\}}}(-1)^{\chi_{\mathcal{S}_{n-1,n}}(J)}\sum_{A\in S_{\max}(J)}\left(t^{2|A|}-t^{2(|J|-|A|-2)}\right)\left(t^{2(2n-2|J|-2)}-t^{2(|J|-2)}\right)\right),$$

where

$$\mathcal{S}_{n-1,n} = \{I\subset\{1,\ldots,n\} :\ \{n-1,n\}\subset I \text{ and } I \text{ is 2-short for } \underline{\alpha}\},$$

$$\mathcal{L}_{n-1,n} = \{J\subset\{1,\ldots,n\} :\ 3\le|J|\le n-2, \{n-1,n\}\not\subset J \text{ and } J \text{ is 2-long for } \underline{\alpha}\}$$

$$S_{\max}(J) = \{A\subset J\setminus\{\max J\} :\ A\cup\{\max J\} \text{ is 1-short for } \underline{\alpha}^c_J\},$$

with

$$(\underline{\alpha}^c_J)_i = \begin{cases} \alpha_i, & \text{if } j\in J\setminus\{n-1,n\} \\ \lambda_{J^c}\alpha_{n-1}+(1-\lambda_{J^c})(S-\varepsilon_1), & \text{if } i=n-1 \\ \lambda_{J^c}\alpha_n+(1-\lambda_{J^c})(S-\frac{4}{5}\varepsilon_1), & \text{if } i=n \end{cases}$$

where

$$\lambda_{J^c} = \frac{3}{2 - a_J - b_J + a_J b_J} \cdot$$

$$\cdot \frac{(2-a_J-b_J)S+(-1)^{a_J b_J - 1}\sum_{J\setminus\{n-1,n\}}\alpha_i-(a_J+2b_J+6(1-a_J)(1-b_J))\frac{\varepsilon_1}{5}}{\left(2-a_J-b_J+2\left\lfloor\frac{a_J+b_J}{2}\right\rfloor\right)\sum_{i=1}^{n}\alpha_i-3\left((1+a_J b_J)(\alpha_{n-1}+\alpha_n)-a_J\alpha_{n-1}-b_J\alpha_n+(a_J+2b_J+3(1-a_J)(1-b_J))\frac{\varepsilon_1}{5}\right)},$$

with

$$\begin{aligned} a_J &= |J \cap \{n-1\}|, \\ b_J &= |J \cap \{n\}| \text{ and} \\ S &= \sum_{i=1}^{n-2} \alpha_i. \end{aligned}$$

Proof. Since k is always 1, abusing notation we can simply write $\mathcal{L}^{\underline{\alpha}}, \mathcal{S}^{\underline{\alpha}}$ for their projections on the first factor.

Let

$$\underline{\beta} = (\beta_1, \ldots, \beta_{n-2}, S - \varepsilon_1, S - \varepsilon_2)$$

be as in Proposition 6.8, where $S = \sum_{i=1}^{n-2} \beta_i, 0 < \varepsilon_1 < \beta_1$ and $\varepsilon_2 = \frac{4}{5}\varepsilon_1$. Since $\beta_1, \ldots, \beta_{n-2}$ are arbitrary, we can take $\beta_i = \alpha_i$ for $i = 1, \ldots, n-2$.

Consider now the path $\zeta(\lambda) = \lambda\underline{\alpha} + (1-\lambda)\underline{\beta}$. Since the set of m-admissible values is convex, all the values $\zeta(\lambda)$ are m-admissible.

By choosing another $\underline{\alpha}'$ in the same chamber as $\underline{\alpha}$ if necessary, we can assume that the path ζ always crosses one wall at a time.

Let

$$\mathcal{W} = \{I \subset \{1, \ldots, n\} : \ 1 < |I| < n-2\},$$

$$\mathcal{S}^{\underline{\alpha}} = \{I \in \mathcal{W} : \ I \cap \{n-1, n\} \neq \emptyset \text{ and } I \text{ is 1-short for } \underline{\alpha}\}$$

and

$$\mathcal{L}^{\underline{\alpha}} = \{I \in \mathcal{W} : \ I \cap \{n-1, n\} = \emptyset \text{ and } I \text{ is 1-long for } \underline{\alpha}\}.$$

Now we have different cases:

- Let $I \in \mathcal{L}^{\underline{\alpha}}$ and let $J = I^c \cap \{1, \ldots, n-2\}$. Note that $\chi_{\mathcal{L}^{\underline{\alpha}}}(I) = 1$ and that, since $|I^c| \geq 3$, we have $|J| \geq 1$. Then

$$2\sum_I \alpha_i > \sum_J \alpha_i + \alpha_{n-1} + \alpha_n \Leftrightarrow 2S - \alpha_{n-1} - \alpha_n > 3\sum_J \alpha_i \geq 3\alpha_1 > 3\varepsilon_1 > 0. \qquad (6.3.1)$$

Moreover, the critical value ζ_I^c given by the intersection of the path ζ with the wall $W_{(I,1)}$, satisfies

$$\begin{aligned} 2\sum_I \zeta_i^c = \sum_{I^c} \zeta_i^c &\Leftrightarrow 2\sum_I (\beta_i + \lambda_I(\alpha_i - \beta_i)) = \sum_{I^c} (\beta_i + \lambda_I(\alpha_i - \beta_i)) \\ &\Leftrightarrow 2\sum_I \alpha_i = \sum_J \alpha_i + \beta_{n-1} + \lambda_I(\alpha_{n-1} - \beta_{n-1}) + \beta_n + \lambda_I(\alpha_n - \beta_n), \end{aligned}$$

with $\beta_{n-1} = S - \varepsilon_1$, $\beta_n = S - \varepsilon_2 = S - \frac{4}{5}\varepsilon_1$ and $S = \sum_{i=1}^{n-2} \alpha_i$. Then,

$$\begin{aligned} \lambda_I(\alpha_{n-1} + \alpha_n - \beta_{n-1} - \beta_n) &= 2\sum_I \alpha_i - \sum_J \alpha_i - \beta_{n-1} - \beta_n \\ \Leftrightarrow \lambda_I\left(\alpha_{n-1} + \alpha_n - 2S + \frac{9}{5}\varepsilon_1\right) &= 2\sum_I \alpha_i - \sum_J \alpha_i - 2S + \frac{9}{5}\varepsilon_1 = -3\sum_J \alpha_i + \frac{9}{5}\varepsilon_1, \end{aligned}$$

and so $\lambda_I = \frac{a}{b}$, where $a = 3\sum_J \alpha_i - \frac{9}{5}\varepsilon_1$ and $b = 2S - \alpha_{n-1} - \alpha_n - \frac{9}{5}\varepsilon_1$. Note that by (6.3.1), we have $b > a > 0$ and so $\lambda_I \in (0,1)$. We obtain

$$\lambda_I = 3\frac{\sum_{I^c \setminus \{n-1,n\}} \alpha_i - \frac{3}{5}\varepsilon_1}{2\sum_{i=1}^n \alpha_i - 3(\alpha_{n-1} + \alpha_n + \frac{3}{5}\varepsilon_1)}.$$

We conclude that the corresponding critical value ζ_I^c is given by

$$\zeta_I^c = \underline{\beta} + \lambda_I(\underline{\alpha} - \underline{\beta})$$

and

$$\underline{\alpha}_I^c = (\beta_i + \lambda_I(\alpha_i - \beta_i))_{i \in I} = (\alpha_i)_{i \in I}, \qquad (6.3.2)$$

while

$$(\underline{\alpha}_{I^c}^c)_i = \begin{cases} \alpha_i, & \text{if } i \in I^c \setminus \{n-1,n\} \\ \lambda_I \alpha_{n-1} + (1-\lambda_I)(S - \varepsilon_1), & \text{if } i = n-1 \\ \lambda_I \alpha_n + (1-\lambda_I)(S - \frac{4}{5}\varepsilon_1), & \text{if } i = n. \end{cases} \qquad (6.3.3)$$

Note that the actual values of $\underline{\alpha}_I^c$ and $\underline{\alpha}_{I^c}^c$ given by Proposition 5.17 are the ones in

(6.3.2) and (6.3.3) multiplied by 2/3 and 1/3 respectively. Nevertheless, since we are only interested in the Poincaré polynomials we can forget about these extra factors as the corresponding polygon spaces are diffeomorphic to the ones obtained using (6.3.2) and (6.3.3).

- Let $I \in \mathcal{S}_{\underline{\alpha}}$ and let $J = I^c \cap \{1,\ldots,n-2\}$. Now we have three possibilities

  1. If $n-1 \in I \subset \{1,\ldots,n-1\}$ then $n \in I^c$ and $|J| \geq 2$, and

$$\begin{aligned} 2\sum_I \alpha_i < \sum_{I^c} \alpha_i \quad &\Leftrightarrow \quad 2\sum_{I\setminus\{n-1\}} \alpha_i + 2\alpha_{n-1} < \sum_{I^c\setminus\{n\}} \alpha_i + \alpha_n \\ &\Leftrightarrow \quad 2S - 2\sum_J \alpha_i + 2\alpha_{n-1} < \sum_J \alpha_i + \alpha_n \\ &\Leftrightarrow \quad 2S + 2\alpha_{n-1} - \alpha_n < 3\sum_J \alpha_i. \end{aligned} \tag{6.3.4}$$

  Moreover, the critical value ζ_I^c given by the intersection of the path ζ with the wall $W_{(I,1)}$, satisfies

$$\begin{aligned} &2\sum_I \zeta_i^c = \sum_{I^c} \zeta_i^c \\ &\Leftrightarrow 2\sum_{I\setminus\{n-1\}} (\beta_i + \lambda_I(\alpha_i - \beta_i)) + 2(S - \varepsilon_1 + \lambda_I(\alpha_{n-1} - S + \varepsilon_1)) \\ &\qquad = \sum_{I^c\setminus\{n\}} (\beta_i + \lambda_I(\alpha_i - \beta_i)) + S - \frac{4}{5}\varepsilon_1 + \lambda_I(\alpha_n - S + \frac{4}{5}\varepsilon_1) \\ &\Leftrightarrow 2\sum_{I\setminus\{n-1\}} \alpha_i + 2S - 2\varepsilon_1 + 2\lambda_I(\alpha_{n-1} - S + \varepsilon_1) \\ &\qquad = \sum_{I^c\setminus\{n\}} \alpha_i + S - \frac{4}{5}\varepsilon_1 + \lambda_I(\alpha_n - S + \frac{4}{5}\varepsilon_1) \\ &\Leftrightarrow \lambda_I(2\alpha_{n-1} - \alpha_n - S + \frac{6}{5}\varepsilon_1) = \sum_{I^c\setminus\{n\}} \alpha_i - 2\sum_{I\setminus\{n-1\}} \alpha_i - S + \frac{6}{5}\varepsilon_1 \\ &\qquad = \sum_J \alpha_i - 2(S - \sum_J \alpha_i) - S + \frac{6}{5}\varepsilon_1 \\ &\qquad = 3\sum_J \alpha_i - 3S + \frac{6}{5}\varepsilon_1 = -3\sum_{I\setminus\{n-1\}} \alpha_i + \frac{6}{5}\varepsilon_1, \end{aligned}$$

  and so $\lambda_I = \frac{a}{b}$, where $a = 3\sum_{I\setminus\{n-1\}} \alpha_i - \frac{6}{5}\varepsilon_1$ and $b = S + \alpha_n - 2\alpha_{n-1} - \frac{6}{5}\varepsilon_1$. Note that since $|I| \geq 2$, we have

$$a \geq 3\alpha_1 - \frac{6}{5}\varepsilon_1 > 0$$

and by (6.3.4) we have

$$S+\alpha_n-2\alpha_{n-1}>3\sum_{I\backslash\{n-1\}}\alpha_i>3\alpha_1>\frac{6}{5}\varepsilon_1$$

and so $b>a>0$. We then obtain

$$\lambda_I=3\frac{\sum_{I\backslash\{n-1\}}\alpha_i-\frac{2}{5}\varepsilon_1}{\sum_{i=1}^n\alpha_i-3(\alpha_{n-1}+\frac{2}{5}\varepsilon_1)}\in(0,1).$$

We conclude that the corresponding critical value ζ_I^c is given by

$$\zeta_I^c=\underline{\beta}+\lambda_I(\underline{\alpha}-\underline{\beta})$$

and $\underline{\alpha}_I^c=(\beta_i+\lambda_I(\alpha_i-\beta_i))_{i\in I}$ is given by

$$(\underline{\alpha}_I^c)_i=\begin{cases}\alpha_i, & \text{if } i\in I\backslash\{n-1\}\\ \lambda_I\alpha_{n-1}+(1-\lambda_I)(S-\varepsilon_1), & \text{if } i=n-1\end{cases}$$

while

$$(\underline{\alpha}_{I^c}^c)_i=\begin{cases}\alpha_i, & \text{if } i\in I^c\backslash\{n\}\\ \lambda_I\alpha_n+(1-\lambda_I)(S-\frac{4}{5}\varepsilon_1), & \text{if } i=n.\end{cases}$$

Note also that, in this case, we have

$$\chi_{\mathcal{L}^{\underline{\alpha}}}(I)=0.$$

2. If $n\in I\subset\{1,\dots,n-2,n\}$, then $n-1\in I^c$ and $|J|\geq 2$ and

$$\begin{aligned}2\sum_I\alpha_i<\sum_{I^c}\alpha_i\Leftrightarrow&2\sum_{I\backslash\{n\}}\alpha_i+2\alpha_n<\sum_{I^c\backslash\{n-1\}}\alpha_i+\alpha_{n-1}\\ \Leftrightarrow&2S-2\sum_J\alpha_i+2\alpha_n<\sum_J\alpha_i+\alpha_{n-1}\\ \Leftrightarrow&2S-\alpha_{n-1}+2\alpha_n<3\sum_J\alpha_i.\end{aligned}\tag{6.3.5}$$

Moreover, the critical value ζ_I^c given by the intersection of the path ζ with the wall

$W_{(I,1)}$, satisfies

$$
\begin{aligned}
2\sum_{I}\zeta_i^c = \sum_{I^c}\zeta_i^c \\
&\Leftrightarrow 2\sum_{I\setminus\{n\}}(\beta_i+\lambda_I(\alpha_i-\beta_i))+2(S-\frac{4}{5}\varepsilon_1+\lambda_I(\alpha_n-S+\frac{4}{5}\varepsilon_1)) \\
&\qquad\qquad = \sum_{I^c\setminus\{n-1\}}(\beta_i+\lambda_I(\alpha_i-\beta_i))+S-\varepsilon_1+\lambda_I(\alpha_{n-1}-S+\varepsilon_1) \\
&\Leftrightarrow 2\sum_{I\setminus\{n\}}\alpha_i+2S-\frac{8}{5}\varepsilon_1+2\lambda_I(\alpha_n-S+\frac{4}{5}\varepsilon_1) \\
&\qquad\qquad = \sum_{I^c\setminus\{n-1\}}\alpha_i+S-\varepsilon_1+\lambda_I(\alpha_{n-1}-S+\varepsilon_1) \\
&\Leftrightarrow \lambda_I(2\alpha_n-\alpha_{n-1}-S+\frac{3}{5}\varepsilon_1) = \sum_{I^c\setminus\{n-1\}}\alpha_i-2\sum_{I\setminus\{n\}}\alpha_i-S+\frac{3}{5}\varepsilon_1 \\
&\qquad\qquad = \sum_{J}\alpha_i-2(S-\sum_{J}\alpha_i)-S+\frac{3}{5}\varepsilon_1 = 3\sum_{J}\alpha_i-3S+\frac{3}{5}\varepsilon_1 \\
&\qquad\qquad = -3\sum_{I\setminus\{n\}}\alpha_i+\frac{3}{5}\varepsilon_1,
\end{aligned}
$$

and so $\lambda_I = \frac{a}{b}$, where $a = 3\sum_{I\setminus\{n\}}\alpha_i - \frac{3}{5}\varepsilon_1$ and $b = S+\alpha_{n-1}-2\alpha_n-\frac{3}{5}\varepsilon_1$. Note that since $|I|\geq 2$, we have

$$a \geq 3\alpha_1 - \frac{3}{5}\varepsilon_1 > 0$$

and by (6.3.5) we have

$$S+\alpha_{n-1}-2\alpha_n > 3\sum_{I\setminus\{n\}}\alpha_i > 3\alpha_1 > \frac{3}{5}\varepsilon_1$$

and so $b > a > 0$. We then obtain

$$\lambda_I = 3\frac{\sum_{I\setminus\{n\}}\alpha_i-\frac{1}{5}\varepsilon_1}{\sum_{i=1}^n\alpha_i-3(\alpha_n+\frac{1}{5}\varepsilon_1)} \in (0,1).$$

We conclude that the corresponding critical value ζ_I^c is given by

$$\zeta_I^c = \underline{\beta}+\lambda_I(\underline{\alpha}-\underline{\beta})$$

and $\underline{\alpha}^c_I = (\beta_i + \lambda_I(\alpha_i - \beta_i))_{i\in I}$ is given by

$$(\underline{\alpha}^c_I)_i = \begin{cases} \alpha_i, & \text{if } i \in I\backslash\{n\} \\ \lambda_I \alpha_n + (1-\lambda_I)(S - \frac{4}{5}\varepsilon_1), & \text{if } i = n, \end{cases}$$

while

$$(\underline{\alpha}^c_{I^c})_i = \begin{cases} \alpha_i, & \text{if } i \in I^c\backslash\{n-1\} \\ \lambda_I \alpha_{n-1} + (1-\lambda_I)(S - \varepsilon_1), & \text{if } i = n-1. \end{cases}$$

Note also that, in this case, we have

$$\chi_{\mathcal{L}^{\underline{\alpha}}}(I) = 0.$$

3. If $\{n-1,n\} \subset I$, then $I^c = J \subset \{1,\ldots,n-2\}$ and

$$\begin{aligned} 2\sum_I \alpha_i < \sum_{I^c}\alpha_i &\Leftrightarrow 2\sum_{I\backslash\{n-1,n\}} \alpha_i + 2\alpha_{n-1} + 2\alpha_n < \sum_{I^c}\alpha_i \\ &\Leftrightarrow 2S - 2\sum_J \alpha_i + 2\alpha_{n-1} + 2\alpha_n < \sum_J \alpha_i \\ &\Leftrightarrow 2S + 2\alpha_{n-1} + 2\alpha_n < 3\sum_J \alpha_i = 3(S - \sum_{I\backslash\{n-1,n\}} \alpha_i). \end{aligned} \tag{6.3.6}$$

Moreover, the critical value ζ^c_I given by the intersection of the path ζ with the wall $W_{(I,1)}$, satisfies

$$\begin{aligned} &2\sum_I \zeta^c_i = \sum_{I^c}\zeta^c_i \\ &\Leftrightarrow 2\sum_{I\backslash\{n-1,n\}} (\beta_i + \lambda_I(\alpha_i - \beta_i)) + 2(2S - \varepsilon_1 - \frac{4}{5}\varepsilon_1 + \lambda_I(\alpha_{n-1} + \alpha_n - 2S + \varepsilon_1 + \frac{4}{5}\varepsilon_1)) \\ &\qquad\qquad = \sum_{I^c}(\beta_i + \lambda_I(\alpha_i - \beta_i)) \\ &\Leftrightarrow 2\sum_{I\backslash\{n-1,n\}} \alpha_i + 4S - \frac{18}{5}\varepsilon_1 + 2\lambda_I(\alpha_{n-1} + \alpha_n - 2S + \frac{9}{5}\varepsilon_1) = \sum_{I^c}\alpha_i \\ &\Leftrightarrow 2\lambda_I(\alpha_{n-1} + \alpha_n - 2S + \frac{9}{5}\varepsilon_1) = \sum_{I^c}\alpha_i - 2\sum_{I\backslash\{n-1,n\}}\alpha_i - 4S + \frac{18}{5}\varepsilon_1 \\ &\qquad\qquad = 3(-S - \sum_{I\backslash\{n-1,n\}} \alpha_i + \frac{6}{5}\varepsilon_1), \end{aligned}$$

and so $\lambda_I = \frac{a}{b}$, where $a = 3S + 3\sum_{I\backslash\{n-1,n\}}\alpha_i - \frac{18}{5}\varepsilon_1$ and $b = 2(2S - \alpha_{n-1} - \alpha_n - \frac{9}{5}\varepsilon_1)$.

Note that

$$a = 3(S-\varepsilon_1)+3\sum_{I\backslash\{n-1,n\}}\alpha_i - \frac{3}{5}\varepsilon_1 \geq 3\alpha_{n-2} - \frac{3}{5}\varepsilon_1 \geq 3(\alpha_1 - \frac{\varepsilon_1}{5}) > 0$$

and by (6.3.6) we have

$$4S - 2\alpha_{n-1} - 2\alpha_n > 3S + 3\sum_{I\backslash\{n-1,n\}}\alpha_i$$

and so $b > a > 0$. We then obtain

$$\lambda_I = \frac{3}{2}\cdot\frac{2S - \sum_{I^c\backslash\{n-1,n\}}\alpha_i - \frac{6}{5}\varepsilon_1}{2\sum_{i=1}^n \alpha_i - 3(\alpha_{n-1}+\alpha_n+\frac{3}{5}\varepsilon_1)} \in (0,1).$$

We conclude that the corresponding critical value ζ_I^c is given by

$$\zeta_I^c = \underline{\beta} + \lambda_I(\underline{\alpha} - \underline{\beta})$$

and $\underline{\alpha}_I^c = (\beta_i + \lambda_I(\alpha_i - \beta_i))_{i\in I}$ is given by

$$(\underline{\alpha}_I^c)_i = \begin{cases} \alpha_i, & \text{if } i \in I\backslash\{n-1,n\} \\ \lambda_I\alpha_{n-1} + (1-\lambda_I)(S-\varepsilon_1) & \text{if } i = n-1 \\ \lambda_I\alpha_n + (1-\lambda_I)(S-\frac{4}{5}\varepsilon_1), & \text{if } i = n \end{cases}$$

while

$$(\underline{\alpha}_{I^c}^c)_i = (\alpha_i)_{i\in I^c}.$$

Note also that, in this case, we have

$$\chi_{\mathcal{L}^{\underline{\alpha}}}(I) = 0.$$

From the above computations we obtain the following general formulas for λ_I and $\underline{\alpha}_{I^c}$,

$$\lambda_I = \frac{3}{2 - a_{I^c} - b_{I^c} + a_{I^c}b_{I^c}}\cdot$$
$$\cdot\frac{(2-a_{I^c}-b_{I^c})S+(-1)^{a_{I^c}b_{I^c}-1}\sum_{I^c\backslash\{n-1,n\}}\alpha_i-(a_{I^c}+2b_{I^c}+6(1-a_{I^c})(1-b_{I^c}))\frac{\varepsilon_1}{5}}{\left(2-a_{I^c}-b_{I^c}+2\left\lfloor\frac{a_{I^c}+b_{I^c}}{2}\right\rfloor\right)\sum_{i=1}^n\alpha_i-3\Big((1+a_{I^c}b_{I^c})(\alpha_{n-1}+\alpha_n)-a_{I^c}\alpha_{n-1}-b_{I^c}\alpha_n+(a_{I^c}+2b_{I^c}+3(1-a_{I^c})(1-b_{I^c}))\frac{\varepsilon_1}{5}\Big)}$$

where

$$a_{I^c} = |I^c \cap \{n-1\}|$$
$$b_{I^c} = |I^c \cap \{n\}|$$
$$S = \sum_{i=1}^{n-2} \alpha_i$$

and

$$(\underline{\alpha}^c_{I^c})_i = \begin{cases} \alpha_i, & \text{if } i \in I^c \setminus \{n-1,n\} \\ \lambda_I \alpha_{n-1} + (1-\lambda_I)(S-\varepsilon_1), & \text{if } i = n-1 \\ \lambda_I \alpha_n + (1-\lambda_I)(S-\frac{4}{5}\varepsilon_1), & \text{if } i = n. \end{cases} \tag{6.3.7}$$

Then, from Theorem 6.14 we have

$$\left(P_t(\mathbb{CP}^{n-4})\right)^2 - \sum_{\substack{I \in \mathcal{L}^{\underline{\alpha}} \\ I \subset \{1,\dots,n-2\}}} P_t(Pol_2(\underline{\alpha}^c_{I^c}))Q_I(t) + \sum_{\substack{I \in \mathcal{S}^{\underline{\alpha}} \\ \{n\} \subset I \subset \{1,\dots,n-2,n\}}} P_t(Pol_2(\underline{\alpha}^c_{I^c}))Q_I(t)$$

$$+ \sum_{\substack{I \in \mathcal{S}^{\underline{\alpha}} \\ \{n-1\} \subset I \subset \{1,\dots,n-1\}}} P_t(Pol_2(\underline{\alpha}^c_{I^c}))Q_I(t) + \sum_{\substack{I \in \mathcal{S}^{\underline{\alpha}} \\ \{n-1,n\} \subset I}} P_t(Pol_2(\underline{\alpha}^c_{I^c}))Q_I(t) = P_t(Pol_3(\underline{\alpha})),$$

where

$$Q_I(t) = P_t\left(\mathbb{CP}^{|I^c|-3}\right) - P_t\left(\mathbb{CP}^{2|I|-3}\right)$$

and $\underline{\alpha}^c_{I^c}$ is given by 6.3.7. Now

$$\begin{aligned} \mathcal{L}^{\underline{\alpha}} &= \{I \subset \{1,\dots,n-2\} : 2 \le |I| \le n-3 \text{ and } I \text{ is 1-long for } \underline{\alpha}\} \\ &= \{I \subset \{1,\dots,n\} :\ 3 \le |I^c| \le n-2,\ \{n-1,n\} \subset I^c \text{ and } I^c \text{ is 2-short for } \underline{\alpha}\} \end{aligned}$$

and so

$$\begin{aligned} \sum_{I \in \mathcal{L}^{\underline{\alpha}}} P_t(Pol_2(\underline{\alpha}^c_{I^c}))Q_I(t) &= \sum_{\substack{3 \le |J| \le n-2, \{n-1,n\} \subset J \\ J \text{ is 2-short for } \underline{\alpha}}} P_t(Pol_2(\underline{\alpha}^c_J))\left(P_t\left(\mathbb{CP}^{|J|-3}\right) - P_t\left(\mathbb{CP}^{2|J^c|-3}\right)\right) \\ &= \sum_{I \in \mathcal{S}_{n-1,n} \setminus \{n-1,n\}} P_t(Pol_2(\underline{\alpha}^c_I))\left(P_t\left(\mathbb{CP}^{|I|-3}\right) - P_t\left(\mathbb{CP}^{2n-2|I|-3}\right)\right), \end{aligned}$$

where

$$\mathcal{S}_{n-1,n} = \{I \subset \{1,\dots,n\} : \{n-1,n\} \subset I \text{ and } I \text{ is 2-short for } \underline{\alpha}\}.$$

Remark 6.20. Note that since $\underline{\alpha}$ is 3-admissible, there are no sets J with $\{n-1,n\} \subset J$ and $|J| \geq n-1$ such that J is 2-short for $\underline{\alpha}$.

Moreover

$$\begin{aligned} \mathcal{S}^{\underline{\alpha}} &= \{I \subset \{1,\dots,n\} : 2 \leq |I| \leq n-3,\ I \cap \{n-1,n\} \neq \emptyset \text{ and } I \text{ is 1-short for } \underline{\alpha}\} \\ &= \{I^c \subset \{1,\dots,n\} :\ 3 \leq |I^c| \leq n-2,\ \{n-1,n\} \not\subset I^c \text{ and } I^c \text{ is 2-long for } \underline{\alpha}\} \end{aligned}$$

and so

$$\sum_{\substack{I \in \mathcal{S}^{\underline{\alpha}} \\ n \in I \text{ and } n-1 \notin I}} P_t(Pol_2(\underline{\alpha}^c_{I^c}))Q_I(t) + \sum_{\substack{I \in \mathcal{S}^{\underline{\alpha}} \\ n-1 \in I \text{ and } n \notin I}} P_t(Pol_2(\underline{\alpha}^c_{I^c}))Q_I(t) + \sum_{\substack{I \in \mathcal{S}^{\underline{\alpha}} \\ \{n-1,n\} \subset I}} P_t(Pol_2(\underline{\alpha}^c_{I^c}))Q_I(t)$$

$$= \sum_{J \in \mathcal{L}_{n-1,n}} P_t(Pol_2(\underline{\alpha}^c_J)) \left(P_t\left(\mathbb{CP}^{|J|-3}\right) - P_t\left(\mathbb{CP}^{2n-2|J|-3}\right)\right),$$

where

$$\mathcal{L}_{n-1,n} = \{J \subset \{1,\dots,n\} : 3 \leq |J| \leq n-2, \{n-1,n\} \not\subset J \text{ and } J \text{ is 2-long for } \underline{\alpha}\}.$$

Hence,

$$\begin{aligned} P_t(Pol_3(\underline{\alpha})) &= \left(1+\cdots+t^{2(n-4)}\right)^2 \\ &\quad - \sum_{J \in \mathcal{S}_{n-1,n} \setminus \{\{n-1,n\}\}} P_t(Pol_2(\underline{\alpha}^c_J)) \left(P_t\left(\mathbb{CP}^{|J|-3}\right) - P_t\left(\mathbb{CP}^{2n-2|J|-3}\right)\right) \\ &\quad + \sum_{J \in \mathcal{L}_{n-1,n}} P_t(Pol_2(\underline{\alpha}^c_J)) \left(P_t\left(\mathbb{CP}^{|J|-3}\right) - P_t\left(\mathbb{CP}^{2n-2|J|-3}\right)\right) \end{aligned}$$

with

$$\begin{aligned} \mathcal{S}_{n-1,n} &= \{I \subset \{1,\dots,n\} :\ \{n-1,n\} \subset I \text{ and } I \text{ is 2-short for } \underline{\alpha}\} \\ \mathcal{L}_{n-1,n} &= \{J \subset \{1,\dots,n\} : 3 \leq |J| \leq n-2, \{n-1,n\} \not\subset J \text{ and } J \text{ is 2-long for } \underline{\alpha}\} \end{aligned}$$

and so

$$P_t(Pol_3(\underline{\alpha})) = \left(1+\cdots+t^{2(n-4)}\right)^2$$

$$-\frac{1}{1-t^2}\sum_{J\in\mathcal{S}_{n-1,n}\backslash\{\{n-1,n\}\}}\sum_{A\in S_{\max}(J)}\left(t^{2|A|}-t^{2(|J|-|A|-2)}\right)\left(P_t\left(\mathbb{CP}^{|J|-3}\right)-P_t\left(\mathbb{CP}^{2n-2|J|-3}\right)\right)$$

$$+\frac{1}{1-t^2}\sum_{J\in\mathcal{L}_{n-1,n}}\sum_{A\in S_{\max}(J)}\left(t^{2|A|}-t^{2(|J|-|A|-2)}\right)\left(P_t\left(\mathbb{CP}^{|J|-3}\right)-P_t\left(\mathbb{CP}^{2n-2|J|-3}\right)\right),$$

where

$$S_{\max}(J)=\{A\subset J\backslash\{\max J\}: A\cup\{\max J\} \text{ is 1-short for } \underline{\alpha}^c_J\}.$$

We conclude that

$$P_t(Pol_3(\underline{\alpha})) = \left(1+\cdots+t^{2(n-4)}\right)^2$$

$$-\frac{1}{(1-t^2)^2}\sum_{J\in\mathcal{S}_{n-1,n}\backslash\{\{n-1,n\}\}}\sum_{A\in S_{\max}(J)}\left(t^{2|A|}-t^{2(|J|-|A|-2)}\right)\left(t^{2(2n-2|J|-2)}-t^{2(|J|-2)}\right)$$

$$+\frac{1}{(1-t^2)^2}\sum_{J\in\mathcal{L}_{n-1,n}}\sum_{A\in S_{\max}(J)}\left(t^{2|A|}-t^{2(|J|-|A|-2)}\right)\left(t^{2(2n-2|J|-2)}-t^{2(|J|-2)}\right)$$

and the result follows □

Remark 6.21. Holla computed in [Ho] a closed formula for the Poincaré polynomial for the moduli spaces of semi-stable parabolic bundles on a curve by a completly different method.

Due to the tedious computations, such formula is not the best way to compute the Poincaré polynomial. However, it can be programmed and leave that task to a computer.

We include in Appendix A a program for Wolfram Mathematica that computes the Poincaré polynomial of $Pol_3(\underline{\alpha})$ using our results and in Appendix B, a program using Hollas results.

Here are some examples for $m=3$ and $n=5$, in this case $|I|=2$ for every $I\in\mathcal{L}^{\underline{\alpha}}\cup\mathcal{S}^{\underline{\alpha}}$, so either $P_t(Pol_2(\alpha^c_{I^c}))=1$ or $P_t(Pol_2(\alpha^c_{I^c}))=0$. Notice that $P_t(Pol_2(\alpha^c_{I^c}))=0$ if and only if $Pol_2(\alpha^c_{I^c})=\emptyset$, but from Proposition 5.17, this is not possible. On the other hand, $P_t(\mathbb{CP}^{2|I|-3})-P_t(\mathbb{CP}^{n-|I|-3})=t^2$, so the Poincaré polynomial is $(1+t^2)^2+(|\mathcal{L}^{\underline{\alpha}}|-|\mathcal{S}^{\underline{\alpha}}|)t^2$.

Example 6.22.

1. Let $0 < \varepsilon$ be small enough and $\underline{\alpha} = (3.25, 4.25, 5.25, 7.75, 8+\varepsilon)$. Then $\mathcal{S}^{\underline{\alpha}} = \mathcal{L}^{\underline{\alpha}} = \emptyset$ and so
$$P_{Pol_3(\underline{\alpha})}(t) = (1+t^2)^2.$$

2. Let $0 < \varepsilon$ be small enough and $\underline{\alpha} = (1-\varepsilon, 2, 2, 2, 2)$, then $\mathcal{L}^{\underline{\alpha}} = \{\{2,3\}\}$ and $\mathcal{S}^{\underline{\alpha}} = \{\{1,4\},\{1,5\}\}$. Therefore
$$P_{Pol_3(\underline{\alpha})}(t) = 1+t^2+t^4.$$

   Using the notation of Chapter 4, $\Lambda = (3-\frac{\varepsilon}{3})(1,1,1,1,1)$ and $\Lambda - \underline{\alpha} = (2+\frac{2\varepsilon}{3}, 1-\frac{\varepsilon}{3}, 1-\frac{\varepsilon}{3}, 1-\frac{\varepsilon}{3}, 1-\frac{\varepsilon}{3})$. As we saw, the Poincaré polynomial of $Pol_3(\underline{\alpha})$ is $P_{Pol_3(\underline{\alpha})}(t) = 1+t^2+t^4$. On the other hand by [HK2, Corollary 4.2],
$$P_{Pol_2(\Lambda-\underline{\alpha})}(t) = \frac{1-t^6+3(t^2-t^4)+3(t^4-t^2)}{1-t^2} = 1+t^2+t^4.$$

6.4 Fundamental group of $Pol_m(\underline{\alpha})$

We will now show that all polygon spaces are simply connected.

Theorem 6.23. Let $\underline{\alpha} \in (\mathbb{R}^+)^n$ be m-generic. Then the fundamental group of $Pol_m(\underline{\alpha})$ is trivial.

Proof. For any m, n, we can choose $\underline{\beta}_{n,m}$ m-generic such that $Pol_m(\underline{\beta}_{n,m})$ is a $(m-1)$-stage generalized Bott tower (see Remark 6.13). In particular, $Pol_m(\underline{\beta}_{n,m})$ is a toric manifold and so it is simply connected. Now for any m-generic $\underline{\alpha}$, we can find a path ζ from $\underline{\beta}_{n,m}$ to $\underline{\alpha}$ such that crosses one wall at a time. Without loss of generality, we can assume that ζ only crosses one wall $W_{(I,k)}$ for some $I \subset \{1,\ldots,n\}$ and $k < m$.

By Theorem 6.3 we have that $Pol_m(\underline{\alpha})$ can be obtained from $Pol_m(\underline{\beta}_{n,m})$ by a partial blow-down followed by a partial blow-up or vice-versa, where the exceptional divisors involved are projective bundles over Y_C.

Now we can apply [R, Proposition 2.3] and so we have that the fundamental group of $Pol_m(\underline{\alpha})$ remains invariant as $\underline{\alpha}$ crosses inner walls. Since $Pol_m(\underline{\beta}_{n,m})$ has trivial fundamental group, the result follows. □

Chapter 7

Symplectic Volume

For $m = 2$, the symplectic volume of $Pol_2(\underline{\alpha})$ has been determined by many authors. In particular in [M, Section 3.2], it is computed using the wall crossing method.

In this chapter we generalize this method to compute the symplectic volume for every m and give an explicit formula for $m = 3$. From now on we assume that $\underline{\alpha}$ is m-generic. We will use the description of $Pol_3(\underline{\alpha})$ given in Section 3.1.2.

7.1 Equivariant Integration Formula

First we will fix notation and recall some results. Let us denote by $\mathbb{C}_{(\gamma)}$ the complex space $\mathbb{C}^m$ endowed with the S^1-action given by

$$\lambda \cdot (z_1, \dots, z_m) = (\lambda^{\gamma_1} z_1, \dots, \lambda^{\gamma_m} z_m), \ \lambda \in S^1, \ \gamma \in \mathbb{Z}^m$$

and let $\underline{\mathbb{C}}_{(\gamma)} := M \times \mathbb{C}_{(\gamma)}$ be the total space of an equivariant line bundle over a symplectic manifold M.

Our computations of the volume are based on the following theorem (see [Ma2] for a detailed proof).

Theorem 7.1. ([Ma2]) Let M be a smooth symplectic manifold equipped with the action of a Lie group G with maximal torus T. Assume that the action is Hamiltonian and that the reduced space $M /\!\!/_p G$ is a smooth manifold. Then, for every equivariant class $a \in H^*_G(M)$,

$$\int_{M /\!\!/_p G} k_G(a) = \frac{1}{|W|} \int_{M /\!\!/_{p'} T} k_T \left(r^G_T(a) \smile \prod_{\gamma} c^T_1(\underline{\mathbb{C}}_{(\gamma)}) \right),$$

where k_G, k_T are the corresponding kirwan maps

$$k_G : H^*_G(M) \to H^*(M /\!\!/_p G) \tag{7.1.1}$$

$$k_T : H^*_T(M) \to H^*(M /\!\!/_p T) \tag{7.1.2}$$

defined in Section 2.3.2, W is the Weyl group of G, p' is the projection of p on $\mathfrak{t}^*$, r^G_T is the restriction map defined in (2.3.1) and the product runs over all the roots γ of G.

Remark 7.2. Let $\pi_i : \prod O^d_{\mathfrak{su}(m)}(\alpha_i) \to O^d_{\mathfrak{su}(m)}(\alpha_i)$ be the projection on the i-th coadjoint orbit. Then the symplectic form on $\prod O^d_{\mathfrak{su}(m)}(\alpha_i)$ is given by $\omega = \sum \pi_i^* \omega_i$, where ω_i is the KKS symplectic form on each orbit.

As we have seen in Section 3.1.2, every orbit $(O^d_{\mathfrak{su}(m)}(\alpha_i), \omega_i)$ is symplectomorphic to $(\mathbb{CP}^{m-1}, (2\alpha_i)\omega_{FS})$.

Therefore, on each coadjoint orbit we can consider the line bundle

$$\mathcal{L}_i : O_{\mathbb{CP}^{m-1}}(2\alpha_i) \to \mathbb{CP}^{m-1} \cong O^d_{\mathfrak{su}(m)}(\alpha_i)$$

and its pullback to $\prod O^d_{\mathfrak{su}(m)}(\alpha_i)$ by the projection

$$p_i : \prod_{i=1}^{n} O^d_{\mathfrak{su}(m)}(\alpha_i) \to O^d_{\mathfrak{su}(m)}(\alpha_i).$$

The tensor product of these pullbacks defines a line bundle

$$\mathcal{L} := p_1^* \mathcal{L}_1 \otimes \cdots \otimes p_n^* \mathcal{L}_n,$$

known as the *prequantum line bundle* of the product. Note that $c_1(\mathcal{L}) = \left[\frac{\omega}{2\pi}\right]$.

Since

$$\dim Pol_m(\underline{\alpha}) = 2(n-m-1)(m-1),$$

the volume of $Pol_m(\underline{\alpha})$ is given by the evaluation of the class

$$\frac{(2\pi c_1(\mathcal{L}))^{(n-m-1)(m-1)}}{((n-m-1)(m-1))!}$$

on the fundamental class $[Pol_m(\underline{\alpha})]$. Hence the volume of $Pol_m(\underline{\alpha})$ can be obtained from

Theorem 7.1, using the form

$$a := \frac{(2\pi)^{(n-m-1)(m-1)}}{((n-m-1)(m-1))!} \left(c_1^{PU(m)}(\mathcal{L})\right)^{(n-m-1)(m-1)}, \tag{7.1.3}$$

where $c_1^{PU(m)}$ is the equivariant first Chern class for the action of $PU(m) \cong SU(m)/\mathbb{Z}_m$ on $\mathbb{CP}^{m-1}$ (see Section 2.3).

Note that the above construction is only possible for $2\alpha_i \in \mathbb{Z}$, but this supposes no restriction since, by rescaling, we have

$$Vol(Pol_m(\lambda\underline{\alpha})) = \lambda^{2(m-1)(n-m-1)} Vol(Pol_m(\underline{\alpha})), \quad \forall \lambda \in \mathbb{R}^+.$$

7.2 Wall decomposition and localization theorem

In order to compute the right hand side of (7.1.1) for a as in (7.1.3), we need to introduce more results related to wall crossing. These are due to S. Martin and detailed proofs can be found in [Ma1, Chapters 5 to 9].

Let $\hat{T}$ be a subtorus of the maximal torus T and let $M^{\hat{T}}$ be the corresponding fixed point set. The T-action on each connected component $M_i^{\hat{T}}$ of $M^{\hat{T}}$ induces a Hamiltonian action of $T/\hat{T}$ on $M_i^{\hat{T}}$ whose image lies in an affine translate S_i of $Lie(T/\hat{T}) \subset \mathfrak{t}^*$.

Definition 7.3. Let $\xi \in \mathfrak{t}^*$ and let S be the affine translate

$$S := \xi + Lie(T/\hat{T})^* \subset \mathfrak{t}^*.$$

We say that ξ is *$\hat{T}$-regular* if μ_T maps some connected component of $M^{\hat{T}}$ to S and for this component ξ is a regular value for μ_T (thought as a map to S).

Remark 7.4. If $\hat{T}$ is a subcircle of T and ξ is in a wall corresponding to $\hat{T}$, then ξ is $\hat{T}$-regular if it lies in the interior of this wall.

Definition 7.5. Let S be an affine translate of $Lie(T/\hat{T})^*$, let $\xi_0, \xi_1 \in S$ be two $\hat{T}$-regular values and let Z be a path between them. Then Z is said to be *$\hat{T}$-transverse* if it is contained in S and for each connected component of $\mu_T(M^{\hat{T}})$ that is mapped to S, the path Z is transverse to the restriction of μ_T (thought as a map to S).

Definition 7.6. Let $Z \subset S$ be a $\hat{T}$-transverse path with endpoints ξ_0, ξ_1 (two $\hat{T}$-regular values). The *wall crossing data* for Z is the set

$$\mathrm{data}(Z) = \left\{(H,\xi) : \ H \text{ is a subtorus of } T \text{ containing } \hat{T} \text{ and } \xi \in Z \cap \mu_T(M^H)\right\}.$$

Definition 7.7. For a given subtorus $\hat{T} \subset T$, we say that a collection of subtori

$$\Theta = (1 = H_0 \subset H_1 \subset \cdots \subset H_k = \hat{T} \subset T)$$

is a $\hat{T}$-flag of subtori if H_i is an i-torus and $H_i/H_{i-1} \cong S^1$ for every $i = 1, \ldots, k$.

Definition 7.8. Let $\hat{T} \subset T$ be a subtorus, then

$$\mathcal{A}_{\hat{T}} := \bigoplus \mathbb{Z}(\Theta, \xi)$$

is the set of formal linear combinations of pairs (Θ, ξ), where Θ is a $\hat{T}$-flag of subtori and ξ a $\hat{T}$-regular value.

Let us consider the $\mathbb{Z}$-module

$$\mathcal{A} := \bigoplus_{\hat{T}} \mathcal{A}_{\hat{T}},$$

where the sum runs over all the subtori $\hat{T} \subset T$.

Remark 7.9. Only a finite number of sets $\mathcal{A}_{\hat{T}}$ are nontrivial and so the sum in $\mathcal{A}$ is finite.

Definition 7.10. Let $\mathcal{R} \subset \mathcal{A}$ be the submodule of relations defined by

1. For every $\hat{T} \subset T$, let ξ_0, ξ_1 be two $\hat{T}$-regular values, Z a $\hat{T}$-transversal path connecting them and Θ a $\hat{T}$-flag of subtori $(1 = H_0 \subset H_1 \subset \cdots \subset H_k = \hat{T})$. Then we have the relation

$$-(\Theta, \xi_0) + (\Theta, \xi_1) + \sum_{(H,\xi) \in \mathrm{data}(Z)} (\Theta \cup H, \xi),$$

   where $\Theta \cup H$ is the flag defined as

$$(1 \subset H \subset H_1 \cup H \subset \cdots \subset \hat{T} \cup H \subset T).$$

2. Let ξ be a $\hat{T}$-regular value such that $\xi \notin \mu_T(M^{\hat{T}})$, then for every $\hat{T}$-flag Θ of subtori, we have the relation (Θ, ξ).

Remark 7.11. Since M is compact, for any regular value $\xi_0 \in \mathfrak{t}^*$ there is always a path Z starting at ξ_0 and ending outside the image of the moment map μ_T. The same holds for every subtori of T.

7.2.1 Localization theorem

Let $\hat{T}$ be a subtorus of T and let Θ be a $\hat{T}$-flag of subtori

$$\Theta = (1 = H_0 \subset \cdots \subset H_k = \hat{T} \subset T).$$

We define the *localization map*

$$\lambda_\Theta : H_T^*(M) \to H_{T/\hat{T}}^*(M^{\hat{T}})$$

as follows.

If $\hat{T} = \{1\}$ (and consequently $\Theta = (1 \subset T)$ is the trivial flag), then λ_Θ is the identity map. Otherwise,

$$\lambda_\Theta = \lambda_{H_k/H_{k-1}} \circ \cdots \circ \lambda_{H_2/H_1} \circ \lambda_{H_1},$$

where

$$\lambda_{H_i/H_{i-1}} : H_{T/H_{i-1}}^*(M^{H_{i-1}}) \to H_{(T/H_{i-1})/(H_i/H_{i-1})}^*(M^{H_i/H_{i-1}}) \cong H_{T/H_i}^*(M^{H_i/H_{i-1}})$$

is the localization map defined in the following way.

Let $S(\nu_{M^{H_i/H_{i-1}}})$ be the sphere bundle of the normal bundle $\nu_{M^{H_i/H_{i-1}}}$ of $M^{H_i/H_{i-1}}$ inside $M^{H_{i-1}}$. Moreover, let p and π be the projections

$$\begin{array}{ccc} S(\nu_{M^{H_i/H_{i-1}}}) & \xrightarrow{\ /(H_i/H_{i-1})\ } & S(\nu_{M^{H_i/H_{i-1}}})/(H_i/H_{i-1}) \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M^{H_i/H_{i-1}} & \end{array}$$

and let π_* denote integration over the fibers of π. Then $\lambda_{H_i/H_{i-1}}$ is the composition:

$$H^*_{T/H_{i-1}}(M^{H_{i-1}}) \xrightarrow{i^*} H^*_{T/H_{i-1}}(M^{H_i/H_{i-1}}) \xrightarrow{p^*} H^*_{T/H_{i-1}}(S(\nu_{M^{H_i/H_{i-1}}})) \xrightarrow[\cong]{/(H_i/H_{i-1})}$$

$$\to H^*_{(T/H_{i-1})/(H_i/H_{i-1})}(S(\nu_{M^{H_i/H_{i-1}}})/(H_i/H_{i-1})) \xrightarrow{\pi_*} H^*_{(T/H_{i-1})/(H_i/H_{i-1})}(M^{H_i/H_{i-1}}),$$

where $i : M^{H_i/H_{i-1}} \hookrightarrow M^{H_{i-1}}$ is the inclusion map.

Using the localization maps, Martin proves the following result.

Theorem 7.12. ([Ma1, Theorem D]) Suppose that

$$c[\Theta,\xi] = \sum_{i\in I} c_i[\Theta_i,\xi_i] \in \mathcal{A}/\mathcal{R}, \quad c_i \in \mathbb{Z},$$

where Θ is a $\hat{T}$-flag of subtori and Θ_i is a T_i-flag of subtori.

Then for any $a \in H^*_T(M)$, we have

$$c\int_{M^{\hat{T}}/\!\!/_\xi \hat{T}} k(\lambda_\Theta(a)) = \sum_{i\in I} c_i \int_{M^{T_i}/\!\!/_{\xi_i} T_i} k(\lambda_{\Theta_i}(a)),$$

where k is the Kirwan map from the equivariant cohomology of a manifold to the ordinary cohomology of the symplectic quotient.

Remark 7.13. In order to compute the symplectic volume, we have to compute the integral on the left hand side with $[(1\subset T),0]$ and the cohomology class a defined in (7.1.3).

Let us now obtain a formula for $\lambda_\Theta : H^*_T(M) \to H^*(M^T)$ when Θ is a T-flag of subtori $\Theta = (1 = H_0 \subset \cdots \subset H_n = T)$. In this case, we can choose a decomposition

$$T = T_1 \times \cdots \times T_n \tag{7.2.1}$$

such that $T_i \cong H_i/H_{i-1} \cong S^1$.

Now the action of T on each connected component $F \subset M^T$ of the fixed point set is trivial and hence

$$H^*_T(F) \cong H^*(F) \otimes H^*_T(pt).$$

Moreover, from the decomposition of T in (7.2.1), we get a set of generators $\{u_1,\ldots,u_n\}$ of $H^*_T(pt) = \mathbb{Q}[u_1,\ldots,u_n]$, where u_i can be seen as the equivariant first Chern class of the representation on $\mathbb{C}$ of T, where T_i acts with weight 1 and the other T_j act trivially.

Consider the action of T on ν_F, the normal bundle of F in M and the decomposition associated to Θ

$$F \cong \nu_F^{H_n} \subset \cdots \subset \nu_F^{H_0} = \nu_F,$$

where $\nu_F^{H_k}$ is the subbundle of ν_F of points fixed by H_k. Let us denote by $V_i \to F$ the orthogonal complement of $\nu_F^{H_i}$ in $\nu_F^{H_{i-1}}$. In particular, $V_i \cong \nu_F^{H_{i-1}}/\nu_F^{H_i}$. Now we can define the map

$$\begin{aligned} l_i : \mathbb{Q}[u_i] &\to H^*(F) \times \mathbb{Q}[u_{i+1}, \ldots, u_n] \\ u_i^{j+k_i} &\mapsto s_j^{T_{i+1}\times\cdots\times T_n}(V_i, T_i), \end{aligned}$$

where $k_i + 1 = \text{rank } V_i$ and $s_j^{T_{i+1}\times\cdots\times T_n}(V_i, T_i)$ is the $T_{i+1}\times\cdots\times T_n$-equivariant weighted Segre class of the bundle $V_i \to F$.

Then l_i extends to a map

$$\widetilde{l_i} : H^*(F) \otimes \mathbb{Q}[u_1, \ldots, u_n] \to H^*(F) \otimes \mathbb{Q}[u_{i+1}, \ldots, u_n]$$

by tensoring with the identity map on the complement of $\mathbb{Q}[u_i]$. This extension $\widetilde{l_i}$ is a homomorphism of $H^*(F) \otimes \mathbb{Q}[u_{i+1}, \ldots, u_n]$-modules.

Proposition 7.14. For every $a \in H_T^*(M)$,

$$\lambda_\Theta(a) = o_T(M)\, \widetilde{l_n} \circ \cdots \circ \widetilde{l_1}(a|_F),$$

where $o_T(M)$ is the order of the stabilizer of M for the action of T.

7.3 Symplectic volume of $Pol_3(\underline{\alpha})$

From the results of Section 3.1.2 we have that

$$Pol_3(\underline{\alpha}) = \left(\mathbb{CP}^2, 2\alpha_1\omega_{FS}\right) \times \cdots \times \left(\mathbb{CP}^2, 2\alpha_n\omega_{FS}\right) /\!\!/_0\, PU(3).$$

Let us assume for simplicity that $\alpha_i \equiv 0 \mod 3$ for $i = 1, \ldots, n$. We can consider the maximal torus T^2 of $PU(3)$ acting on each $\mathbb{CP}^2$ as

$$(\lambda_1,\lambda_2)\cdot[z_1:z_2:z_3]=[\lambda_1 z_1:\lambda_2 z_2:z_3],\quad (\lambda_1,\lambda_2)\in T$$

and a moment map $\phi_T:(\mathbb{CP}^2,2\alpha_i)\to\mathbb{R}^2$ given by

$$[z_1:z_2:z_3]\mapsto\left(\frac{\alpha_i}{3},\frac{\alpha_i}{3}\right)-\alpha_i\left(\frac{|z_1|^2}{|z_1|^2+|z_2|^2+|z_3|^2},\frac{|z_2|^2}{|z_1|^2+|z_2|^2+|z_3|^2}\right).$$

It has three isolated fixed points $F_1=[1:0:0]$, $F_2=[0:1:0]$ and $F_3=[0:0:1]$ and the image $\mu_T(\mathbb{CP}^2)$ is the convex hull of the points

$$\mu_T(F_1)=\left(\frac{\alpha_i}{3},\frac{\alpha_i}{3}\right)-\alpha_i(1,0)=\left(-\frac{2\alpha_i}{3},\frac{\alpha_i}{3}\right)$$
$$\mu_T(F_2)=\left(\frac{\alpha_i}{3},\frac{\alpha_i}{3}\right)-\alpha_i(0,1)=\left(\frac{\alpha_i}{3},-\frac{2\alpha_i}{3}\right)$$
$$\mu_T(F_3)=\left(\frac{\alpha_i}{3},\frac{\alpha_i}{3}\right).$$

The T-action on $(\mathbb{CP}^2)^n$ has 3^n fixed points $F_{i_1}\times\cdots\times F_{i_n}$ with $i_j\in\{1,2,3\}$.

Let $\mathcal{I}$ be the set of 3-partitions of $\{1,\ldots,n\}$

$$I_1\sqcup I_2\sqcup I_3=\{1,\ldots,n\}.$$

Then the fixed point set is in one-to-one correspondence with the set $\mathcal{I}$, i.e.

$$\left((\mathbb{CP}^2)^n\right)^T=\{F_I\in(\mathbb{CP}^2)^n : I\in\mathcal{I}\},$$

where F_I is the n-tuple of points of $\mathbb{CP}^2$ such that $(F_I)_j$ is equal to F_1,F_2 or F_3 according to whether j is in I_1,I_2 or I_3.

Then if $\{e_1,e_2\}$ is the standard basis of $\mathfrak{t}^*\cong\mathbb{R}^2$, the moment map of the torus action at those points is

$$\begin{aligned}\mu_T(F_I) &= \frac{1}{3}\left(\sum_{i\in I_1}\alpha_i(-2e_1+e_2)+\sum_{i\in I_2}\alpha_i(e_1-2e_2)+\sum_{i\in I_3}\alpha_i(e_1+e_2)\right)\\ &= \frac{1}{3}\left(-2\sum_{I_1}\alpha_i+\sum_{I_1^c}\alpha_i,\ -2\sum_{I_2}\alpha_i+\sum_{I_2^c}\alpha_i\right).\end{aligned}\tag{7.3.1}$$

Example 7.15. Let $n = 5$ and consider the partition $I = \{1,3\} \sqcup \{2\} \sqcup \{4,5\}$, then

$$F_I = ([1:0:0],[0:1:0],[1:0:0],[0:0:1],[0:0:1])$$

and

$$\mu_{F_I} = \frac{1}{3}\left(-2(\alpha_1+\alpha_3)+\alpha_2+\alpha_4+\alpha_5, -2\alpha_2+\alpha_1+\alpha_3+\alpha_4+\alpha_5\right).$$

Remark 7.16. Note that since $\alpha_i \equiv 0 \mod 3$ and $\underline{\alpha}$ is 3-generic, zero is always a regular value of μ_T.

Example 7.17. Let $\underline{\alpha} = (6,6,9,9)$, the image of the moment map μ_T is the convex hull of the points $(10,10), (-20,10)$ and $(10,-20)$.

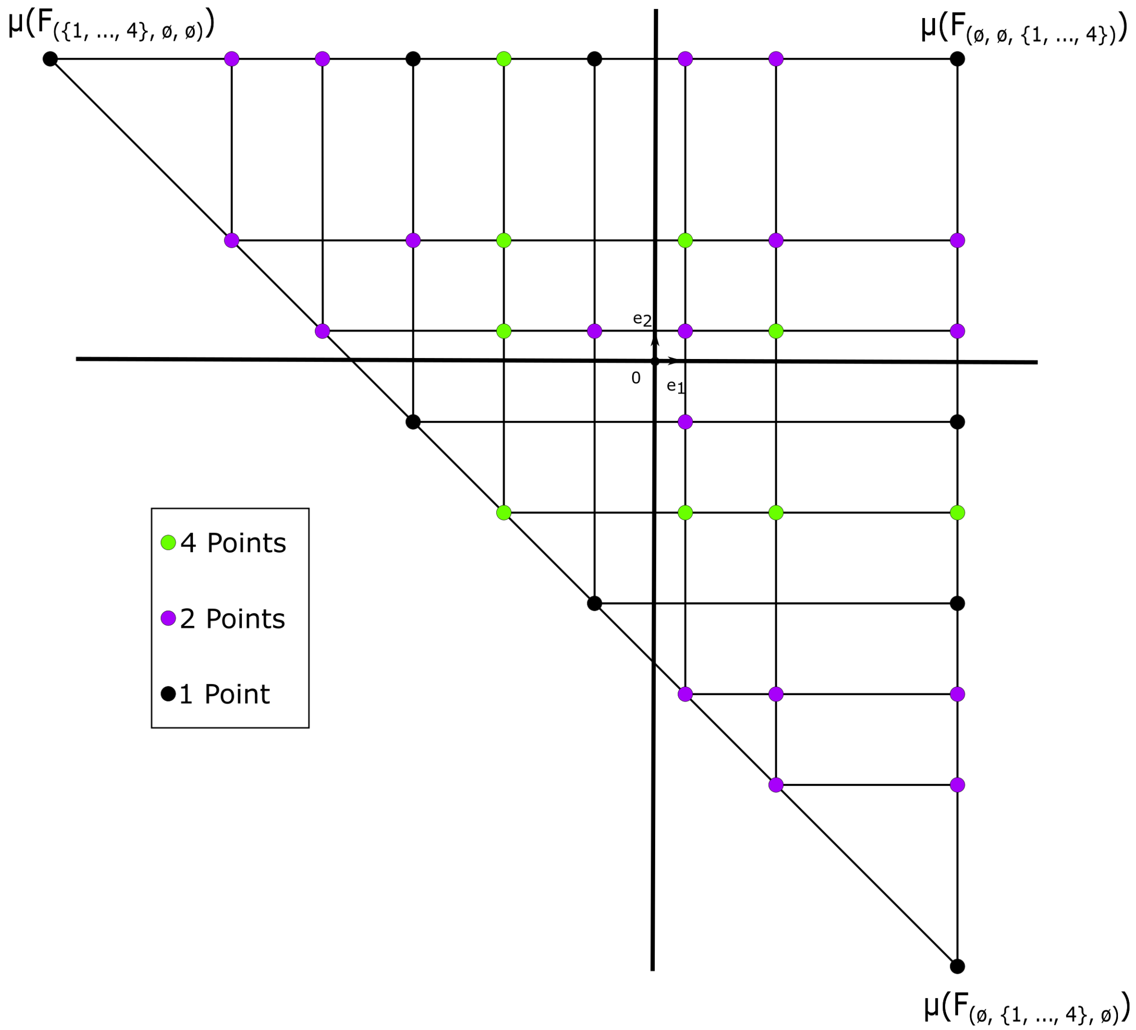

Figure 7.3.1: Moment map image for $\underline{\alpha} = (6,6,9,9)$.

The T action on $Pol_3(\underline{\alpha})$ has $3^4 = 81$ fixed points, but many of these points have the same image under μ_T, for example,

$$\mu_T(F_{\{1,3\},\emptyset,\{2,4\}}) = \mu_T(F_{\{1,4\},\emptyset,\{2,3\}}) = \mu_T(F_{\{2,3\},\emptyset,\{1,4\}}) = \mu_T(F_{\{2,4\},\emptyset,\{1,3\}}) = (-5,10)$$
$$\mu_T(F_{\{1,3\},\{2\},\{4\}}) = \mu_T(F_{\{1,4\},\{2\},\{3\}}) = \mu_T(F_{\{2,3\},\{1\},\{4\}}) = \mu_T(F_{\{2,4\},\{1\},\{3\}}) = (-5,4)$$
$$\mu_T(F_{\{3\},\{4\},\{1,2\}}) = \mu_T(F_{\{4\},\{3\},\{1,2\}}) = (1,1).$$

In general, the image of the moment map is always a (right) triangle with vertices

$$\begin{aligned}
\mu_T(F_{\emptyset,\emptyset,\{1,\dots,n\}}) &= \frac{1}{3}\left(\sum_{i=1}^n \alpha_i,\ \sum_{i=1}^n \alpha_i\right)\\
\mu_T(F_{\emptyset,\{1,\dots,n\},\emptyset}) &= \frac{1}{3}\left(\sum_{i=1}^n \alpha_i,\ -2\sum_{i=1}^n \alpha_i\right)\\
\mu_T(F_{\{1,\dots,n\},\emptyset,\emptyset}) &= \frac{1}{3}\left(-2\sum_{i=1}^n \alpha_i,\ \sum_{i=1}^n \alpha_i\right).
\end{aligned}$$

7.3.1 Decomposition of $[(1 \subset T), 0]$

Let us start with the regular point $p_0 = 0$ (see Remark 7.16) and obtain the decomposition of $[(1 \subset T), 0]$

Let H_1 and H_2 be the subcircles of T given by $H_1 := \{(\lambda,1)\}$ and $H_2 := \{(1,\lambda)\}$. Note that these correspond to vertical and horizontal walls respectively.

Proposition 7.18. Consider on H_2 the usual orientation and on H_1 the reversed orientation, denoted by $\overline{H}_1$. Let

$$\Theta_1 = (1 \subset H_2 \subset H_2 \times \overline{H_1} = T)$$

and

$$\Theta_2 = (1 \subset \overline{H_1} \subset \overline{H_1} \times H_2 = T)$$

be two T-flags of subtori.

Then

$$[(1 \subset T), 0] = \sum_{(I_1,I_2,I_3)\in R_1} [\Theta_1, F_{I_1,I_2,I_3}] + \sum_{(I_1,I_2,I_3)\in R_2} [\Theta_2, F_{I_1,I_2,I_3}],$$

where $R_1 \subset \mathcal{I}$ is the set of 3-partitions of $\{1,\dots,n\}$ for which I_2 and I_3 are 1-short for $\underline{\alpha}$ and $R_2 \subset \mathcal{I}$ is the set of 3-partitions of $\{1,\dots,n\}$ for which I_1 and I_3 are 1-long for $\underline{\alpha}$.

Proof. Since the line of equation $y = -x$ does not contain any image of a fixed point, we can assume our path Z to go from 0 to the exterior of the moment polytope along this line.

Indeed, note that if $\mu_T(F_I)$ were on the line $y = -x$, we would have

$$\sum_{I_2^c} \alpha_i - 2\sum_{I_2} \alpha_i = 2\sum_{I_1} \alpha_i - \sum_{I_1^c} \alpha_i$$

and so

$$2\sum_{I_3} \alpha_i = \sum_{I_1 \cup I_2} \alpha_i$$

and $\underline{\alpha}$ would not be 3-generic.

We can assume without loss of generality that Z is contained in the second quadrant.

The path starts at 0 and has its endpoint outside the moment polytope. Therefore, the direction of Z defines an orientation on the subgroups associated to the walls. We will always cross the walls in directions $-e_1, e_2$ so that we consider the standard orientation on H_2 and the reversed orientation on H_1.

Therefore if, to go from 0 to a point $q_0 \in Z$, we have to cross exactly one horizontal wall and r_0 is the crossing point of Z at this wall, we have

$$[(1 \subset T), 0] = [(1 \subset T), q_0] + [(1 \subset H_2 \subset T), r_0].$$

Consider now the path Z' that starts at r_0 and goes along the horizontal wall through r_0. It will eventually pass through an image of a fixed point p_1 until it reaches a point r_1 (see Figure 7.3.2). The direction of this path is determined by the one on Z and is according to the orientation of H_2.

Therefore

$$[(1 \subset H_2 \subset T), r_0] = [(1 \subset H_2 \subset T), r_1] + [\Theta_1, p_1].$$

We can repeat this procedure crossing k vertical lines until we reach a point r_k outside the polytope.

By Definition 7.10, $[(1 \subset H_2 \subset T), r_k] = 0$ and thus we can write

$$[(1 \subset H_2 \subset T), r_0] = \sum_{i=1}^{k} [\Theta_1, p_i].$$

Analogously, we can write

$$[(1 \subset T), q_0] = [(1 \subset T), q_1] + \sum_{i=1}^{k'} [\Theta_2, p'_i]$$

where $q_1, p'_1, \ldots, p'_{k'}$ are the images of the fixed points that lie on the first vertical wall crossed by Z. Repeating this procedure we get

$$[(1 \subset T), 0] = \sum_{i \in R_1} [\Theta_1, p_i] + \sum_{i \in R_2} [\Theta_2, p'_i]$$

for some index sets R_1, R_2.

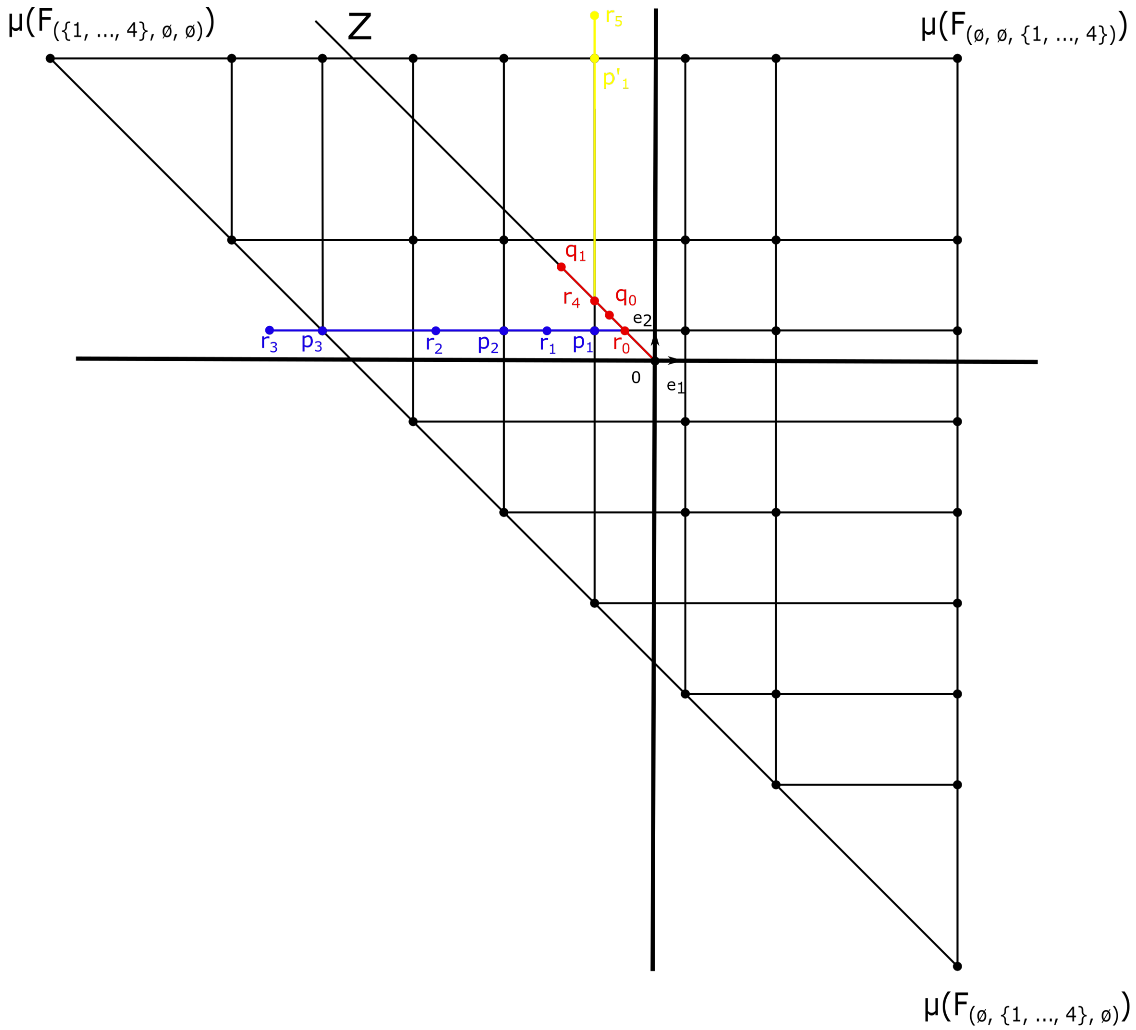

Figure 7.3.2: Decomposition for $\underline{\alpha} = (6, 6, 9, 9)$.

Note that the points p_i and p'_i are fixed by T and therefore can be written as F_{I_1, I_2, I_3} for some $(I_1, I_2, I_3) \in \mathcal{I}$.

The fixed points F_{I_1,I_2,I_3} of the upper semispace are those for which

$$2\sum_{I_2}\alpha_i < \sum_{I_2^c}\alpha_i,$$

(cf. (7.3.1)) i.e. such that I_2 is 1-short for $\underline{\alpha}$. Moreover the fixed points below the path Z are those satisfying

$$2\sum_{I_3}\alpha_i < \sum_{I_3^c}\alpha_i,$$

i.e. such that I_3 is 1-short for $\underline{\alpha}$. Indeed we need

$$\begin{aligned}
&2\sum_{I_1}\alpha_i - \sum_{I_1^c}\alpha_i > \sum_{I_2^c}\alpha_i - 2\sum_{I_2}\alpha_i \\
&\Leftrightarrow \sum_{I_1}\alpha_i - \sum_{I_3}\alpha_i > \sum_{I_3}\alpha_i - \sum_{I_2}\alpha_i \\
&\Leftrightarrow 2\sum_{I_3}\alpha_i < \sum_{I_3^c}\alpha_i.
\end{aligned}$$

The set R_2 is obtained analogously.

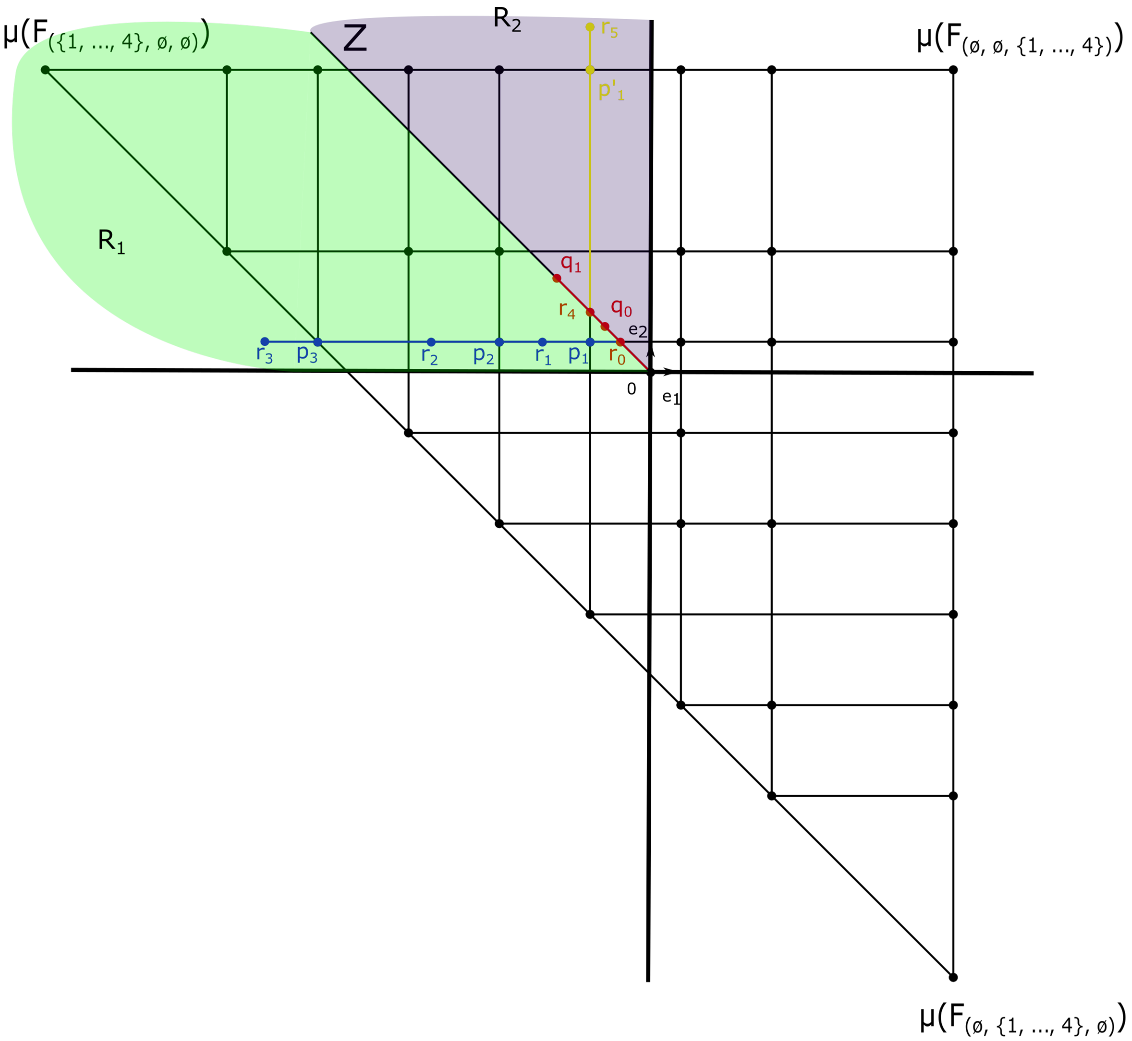

Figure 7.3.3: Decomposition and index sets for $\underline{\alpha} = (6,6,9,9)$.

□

7.3.2 The Localization maps λ_{Θ_i}

Proposition 7.19. The localization maps $\lambda_{\Theta_i} : H_T^*(\{pt\}) = \mathbb{Q}[u_1, u_2] \rightarrow H^*(\{pt\}) = \mathbb{Q}$, where u_1 and u_2 are the generators of H_1 and H_2, are given by

$$\lambda_{\Theta_1}(u_1^{j_1} u_2^{j_2}) = \begin{cases} (-1)^{|I_1|+1} \begin{pmatrix} j_2 - |I_2| - |I_3| \\ |I_1| + |I_2| - 1 \end{pmatrix} & \text{if } j_2 \geq n + |I_2| - 1 \\ & \text{and } j_1 + j_2 = 2n - 2 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\lambda_{\Theta_2}(u_1^{j_1}u_2^{j_2}) = \begin{cases} (-1)^{|I_2|+1}\begin{pmatrix} j_1-|I_1|-|I_3| \\ |I_1|+|I_2|-1 \end{pmatrix} & \text{if } j_1 \geq n+|I_1|-1 \\ & \text{and } j_1+j_2 = 2n-2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us fix the set $\{e_1-e_2, e_2\}$ of simple roots of $PU(3)$. Now the normal bundle of the fixed point F_{I_1,I_2,I_3} is

$$V := \nu_{F_{I_1,I_2,I_3}} \cong \mathbb{C}^{|I_1|}_{(-e_1+e_2)} \oplus \mathbb{C}^{|I_1|}_{(-e_1)} \oplus \mathbb{C}^{|I_2|}_{(e_1-e_2)} \oplus \mathbb{C}^{|I_2|}_{(-e_2)} \oplus \mathbb{C}^{|I_3|}_{(e_1)} \oplus \mathbb{C}^{|I_3|}_{(e_2)}.$$

The subbundle stabilized by H_2 is

$$V^{H_2} = \mathbb{C}^{|I_1|}_{(-e_1)} \oplus \mathbb{C}^{|I_3|}_{(e_1)}$$

and therefore

$$V/V^{H_2} \cong \mathbb{C}^{|I_1|}_{(-e_1+e_2)} \oplus \mathbb{C}^{|I_2|}_{(e_1-e_2)} \oplus \mathbb{C}^{|I_2|}_{(-e_2)} \oplus \mathbb{C}^{|I_3|}_{(e_2)}.$$

In particular, $\mathrm{rk}(V/V^{H_2}) = n+|I_2|$. Since $c_1^{H_1}(\mathbb{C}_{(k_1e_1+k_2e_2)}) = k_1u_1$, the weighted Segre class is given by

$$s^{\omega}_{H_1}(\mathbb{C}_{(k_1e_1+k_2e_2)}) = (k_2+k_1u_1)^{-1}.$$

Thus

$$\begin{aligned} s^{\omega}_{H_1}(V/V^{H_2}) &= (1-u_1)^{-|I_1|}(u_1-1)^{-|I_2|}(-1)^{-|I_2|}(1)^{-|I_3|} \\ &= (1-u_1)^{-(|I_1|+|I_2|)}. \end{aligned}$$

Therefore, for the flag $\Theta = (1 \subset H_2 \subset H_2 \times \overline{H}_1 = T)$ of subtori, we have

$$\widetilde{l_1}(u_2^j) = \begin{cases} 0, & \text{if } j < n+|I_2|-1 \\ \begin{pmatrix} j-|I_2|-|I_3| \\ |I_1|+|I_2|-1 \end{pmatrix} u_1^{j-(n+|I_2|-1)}, & \text{if } j \geq n+|I_2|-1 \end{cases}$$

and

$$\widetilde{l_2}(u_1^j) = \begin{cases} (-1)^{|I_1|+1}, & \text{if } j = n - |I_2| - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that we can compute $\widetilde{l_2}$ as if the flag were $\Theta = (1 \subset H_2 \subset H_2 \times H_1 = T)$, but then we have to multiply by -1 because of the reversed orientation.

Since the order of the stabilizer of $\mu^{-1}(0)$ is zero, we get

$$\lambda_{\Theta_1}(u_1^{j_1} u_2^{j_2}) = \begin{cases} (-1)^{|I_1|+1} \begin{pmatrix} j_2 - |I_2| - |I_3| \\ |I_1| + |I_2| - 1 \end{pmatrix} & \begin{array}{c} \text{if } j_2 \geq n + |I_2| - 1 \\ \text{and } j_1 + j_2 = 2n - 2 \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

The computations for the flag $\Theta = (1 \subset \overline{H}_1 \subset \overline{H}_1 \times H_2 = T)$ are analogous and the proposition follows. □

Remark 7.20. Since the set of roots is

$$\Delta = \{e_1 - e_2, e_1, e_2, -e_1 + e_2, -e_1, -e_2\},$$

from Section 7.2.1 is easy to compute $\prod_{\gamma \in \Delta} c_1^T(\mathbb{C}_{(\gamma)})$.

We have

$$\prod_{\gamma \in \Delta} c_1^T(\mathbb{C}_{(\gamma)}) = (-1)^3 u_1^2 u_2^2 (u_1 - u_2)^2 = 2u_1^3 u_2^3 - u_1^4 u_2^2 - u_1^2 u_2^4.$$

Hence $a|_{F_{I_1,I_2,I_3}}$ has always degree $2n-2$ and therefore the condition $j_1 + j_2 = 2n-2$ of λ_{Θ_i} in the nontrivial case is redundant for our computations.

7.3.3 Symplectic volume of $Pol_3(\underline{\alpha})$

Using the results of the previous sections we can finally obtain a formula for the symplectic volume of $Pol_3(\underline{\alpha})$.

Theorem 7.21. Let $\underline{\alpha} \in (\mathbb{R}^+)^n$ be 3-generic with $n \geq 5$. Then,

$$Vol(Pol_3(\underline{\alpha})) = \left(\frac{2\pi}{3}\right)^{2n-8} \cdot \frac{1}{6(2n-8)!} \cdot$$
$$\left\{ \sum_{(I_1,I_2,I_3)\in R_1} (-1)^{|I_1|} \sum_{k=0}^{n-|I_2|-3} \binom{2n-8}{k} \binom{n+|I_1|-k-6}{n-|I_3|-3} S_1^k S_2^{2n-8-k} + \right.$$
$$\left. + \sum_{(I_1,I_2,I_3)\in R_2} (-1)^{|I_2|} \sum_{k=0}^{n-|I_1|-3} \binom{2n-8}{k} \binom{n+|I_2|-k-6}{n-|I_3|-3} S_1^{2n-8-k} S_2^k \right\},$$

where

1. R_1 is the set of 3-partitions (I_1, I_2, I_3) of $\{1, \ldots, n\}$ with I_2 and I_3 1-short for $\underline{\alpha}$,
2. R_2 is the set of 3-partitions (I_1, I_2, I_3) of $\{1, \ldots, n\}$ with I_1 and I_3 1-long for $\underline{\alpha}$,
3. S_1 and S_2 are defined by

$$\begin{aligned} S_1(I_1, I_2, I_3) &= -2\sum_{I_1} \alpha_i + \sum_{I_1^c} \alpha_i \\ S_2(I_1, I_2, I_3) &= -2\sum_{I_2} \alpha_i + \sum_{I_2^c} \alpha_i. \end{aligned}$$

Proof. From the previous sections, we only need to determine $\left(c_1^T(\mathcal{L})|_{F_{I_1,I_2,I_3}}\right)^{2n-8}$ for every fixed point F_{I_1,I_2,I_3} and compute the image under λ_{Θ_i} of its product with $2u_1^3u_2^3 - u_1^4u_2^2 - u_1^2u_2^4$ for $i = 1, 2$.

Since

$$\mu_T(F_{I_1,I_2,I_3}) = \frac{1}{3}\left(-2\sum_{I_1} \alpha_i + \sum_{I_1^c} \alpha_i, -2\sum_{I_2} \alpha_i \sum_{I_2^c} \alpha_i\right) = \frac{1}{3}(S_1, S_2),$$

we have

$$\mathcal{L}|_{F_{I_1,I_2,I_3}} \cong \mathbb{C}_{(\mu_T(F_{I_1,I_2,I_3}))} \cong \mathbb{C}_{\frac{1}{3}(S_1,S_2)}$$

and therefore

$$c_1^T(\mathcal{L})|_{F_{I_1,I_2,I_3}} = \frac{1}{3}S_1u_1 + \frac{1}{3}S_2u_2.$$

Using Proposition 7.18 we have

$$\lambda_{\Theta_1}\left(\left(\frac{1}{3}S_1u_1+\frac{1}{3}S_2u_2\right)^{2n-8}\left(2u_1^3u_2^3-u_1^4u_2^2-u_1^2u_2^4\right)\right)=$$

$$=\frac{1}{3^{2n-8}}\sum_{k=0}^{2n-8}\binom{2n-8}{k}S_1^kS_2^{2n-8-k}\lambda_{\Theta_1}(u_1^ku_2^{2n-8-k})$$

$$=\frac{(-1)^{|I_1|}}{3^{2n-8}}\sum_{k=0}^{n-|I_3|-3}\binom{2n-8}{k}\binom{n+|I_1|-k-6}{n-|I_3|-3}S_1^k\,S_2^{2n-8-k}$$

and

$$\lambda_{\Theta_2}\left(\left(\frac{1}{3}S_1u_1+\frac{1}{3}S_2u_2\right)^{2n-8}\left(2u_1^3u_2^3-u_1^4u_2^2-u_1^2u_2^4\right)\right)$$

$$=\frac{1}{3^{2n-8}}\sum_{k=0}^{2n-8}\binom{2n-8}{k}S_1^kS_2^{2n-8-k}\lambda_{\Theta_2}(u_1^ku_2^{2n-8-k})$$

$$=\frac{(-1)^{|I_2|}}{3^{2n-8}}\sum_{k=0}^{n-|I_1|-3}\binom{2n-8}{k}\binom{n+|I_2|-k-6}{n-|I_3|-3}S_1^{2n-8-k}\,S_2^k,$$

where we used the equality

$$-2\binom{r+1}{s}+\binom{r+2}{s}+\binom{r}{s}=\binom{r}{s-2}.$$

The result follows from Theorems 7.1 and 7.12 (see Remark 7.2). □

Remark 7.22. The above volume formula is computed in [Ma1] for the equilateral case, i.e. $\alpha_i = k$ for some fixed value k.

A general formula was computed by Suzuki and Takakura in [ST] using a completely different method. They study the dimension of the trivial part in a tensor product of several irreducible representations for $SU(3)$ and its asymptotic behavior. This is equivalent to compute the symplectic volume.

Example 7.23. Let $0 < \varepsilon$ be small enough and $\underline{\alpha} = 3(1-\varepsilon, 2, 2, 2, 2, 2)$. As we have seen in

Chapter 6, the subsets $I \subset \{1, \ldots, 5\}$ that are 1-short for $\underline{\alpha}$ are those with $|I| < 2$ or with $|I| = 2$ and such that $1 \in I$. Likewise the subsets that are 1-long for $\underline{\alpha}$ are those with $|I| > 2$ or with $|I| = 2$ and such that $1 \notin I$.

Therefore the set of partitions R_1 is given by

$$R_1 = \{(I_1, I_2, I_3) : |I_2|, |I_3| < 2\} \cup \{(I_1, I_2, I_3) : |I_2| < |I_3| = 2 \text{ and } 1 \in I_3\}$$
$$\cup \{(I_1, I_2, I_3) : |I_3| < |I_2| = 2 \text{ and } 1 \in I_2\}.$$

Analogously we have

$$R_2 = \{(I_1, I_2, I_3) : \text{ with } I_2 = \emptyset \text{ or } I_2 = \{1\}\}.$$

Now using the formula from Theorem 7.21, we have

$$Vol\,(Pol_3(3 - 3\varepsilon, 6, 6, 6, 6)) = \frac{\pi^2}{3}(561 + 704\varepsilon - 71\varepsilon^2).$$

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Appendices

Appendix A

```
n=Input["What is the number of steps of the polygons?"];

alpha=List[]; (* alpha will be the weight vector *)
For[j=1,j<n+1,j++,alpha=Append[alpha,Input
["Please insert the weights of the polygon
(one by one in increasing order)"]]];
Print["Weight vector alpha= ",alpha];
Print[""];

(* a is a weight vector on the outer wall W_{1,...,n-1}} *)
a=List[];

For[k=1, k<n-1,k++,a=Append[a,alpha[[k]]]];
a=Append[a,Sum[alpha[[i]], {i, n-2}]-alpha[[1]]/10];
a=Append[a,Sum[alpha[[i]], {i, n-2}]-(2*alpha[[1]])/25];
Print["Initial weight vector on the outer wall: a= ",a];
Print[""];

TotalSum=Total[alpha]; (* Sum of all the weights *)

l=List[]; (* l is the list {1,...,n} *)
For[i=1,i<n+1,i++,l=Append[l,i]];

SS= Subsets[l]; (* SS is the list of all subsets of {1,...,n} *)
```

```
ShortSetsa=List[];
(* ShortSetsa will be the list of Short Sets for a;
NOTE: S is short if sum_S < 2 sum_Scomp *)
k=1;
TotalSuma=Total[a];
While[k< Length[SS]+1,
Listauxa=SS[[k]];
n=Length[Listauxa];
SumSa=0;
(* SumS will give the sum of the weights of a in the kth subset of SS *)
For[i=1,i<n+1,i++, If[Listauxa != {}, SumSa=SumSa+ a[[Listauxa[[i]]]]]];
If[3 * SumSa< 2 * TotalSuma, ShortSetsa=Append[ShortSetsa,Listauxa]];
k++];

Print["Short Sets for a= ", ShortSetsa];
Print[""];

ShortSetsalpha=List[];
k=1;TotalSumalpha=Total[alpha];
alphaGeneric = True;
(* In the end alphaGeneric will be False if alpha is not generic *)
Walls=List[];
(* Wall will list the walls that contain alpha if nongeneric *)
While[k< Length[SS]+1,
Listauxalpha=SS[[k]];
n=Length[Listauxalpha];
SumSalpha=0;
(* SumS will give the sum of the weights alpha in the kth subset in SS *)
For[i=1,i<n+1,i++, If[Listauxalpha != {},
SumSalpha= SumSalpha + alpha[[Listauxalpha[[i]]]]]];
If[3 * SumSalpha< 2 * TotalSumalpha,
```

```
ShortSetsalpha=Append[ShortSetsalpha,Listauxalpha],
If[3*SumSalpha == 2 * TotalSumalpha, alphaGeneric = False;
Walls=Append[Walls,Listauxalpha]]]; k++];

If[alphaGeneric == True,
Print["Short Sets for alpha= ", ShortSetsalpha];
Print[""];

(* short sets of a that are not short for alpha and vice versa *)

ShortSetsalphanota=Select[ShortSetsalpha, MemberQ[ShortSetsa,#]==False &] ;
ShortSetsanotalpha=Select[ShortSetsa, MemberQ[ShortSetsalpha,#]==False &] ;
Print["Short sets of alpha that
are not short for a= ",ShortSetsalphanota];
Print[""];

Print["Short sets of a that are not short for alpha= ",ShortSetsanotalpha];
Print[""];

i=1;
PoincarePol=((Sum[t^{2(u-1)},{u,n-4+1}])^2);
(* part of the Poincaré polynomial corresponding a chamber adjacent to
the outer wall W_{1,...,n-1} that does not involve any other crossing. *)
Print["Initial Poincaré Polynomial: ", PoincarePol[[1]]];
Print[""];

While[i< Length[ShortSetsalphanota]+1,

SetI=ShortSetsalphanota[[i]];
SetIcomp=Delete[l,Partition[SetI,1]];

SumIa=0;
```

```
SumIalpha=0;
k=1;
For[k=1, k<Length[SetI]+1, k++, SumIa=SumIa+a[[SetI[[k]]]]];
For[k=1, k<Length[SetI]+1,k++,SumIalpha=SumIalpha+alpha[[SetI[[k]]]]];

EpsilonIa=3 SumIa-2Total[a];
EpsilonIalpha=3 SumIalpha-2Total[alpha];

 Print["Wall corresponding to the set ",SetI];
 Print["EpsilonIa= ", EpsilonIa];
 Print["EpsilonIalpha= ", EpsilonIalpha];

NewalphaWI=List[];
For[s=1,s < n + 1, s++, NewalphaWI=
Append[NewalphaWI, Abs[EpsilonIa *alpha[[s]]-EpsilonIalpha * a[[s]]]]];
Print["Critical weight vector on the wall W_I: ",NewalphaWI];
(* critical weights at the wall W_I *)

NewalphaInI=Extract[NewalphaWI,Partition[SetI,1]];
(* critical weights at the wall W_I that are in the positions of I *)
NewalphaNotInI=Select[NewalphaWI, MemberQ[NewalphaInI,#]==False &];
(* critical weights at the wall W_I that
are in the positions complementary of I *)

If[(2*Last[NewalphaInI]<Total[NewalphaInI] && Length[SetI]>=3 ),

SS2I=Select[Subsets[SetI], 2*Total[NewalphaWI[[#]]]< Total[NewalphaInI] &];

PoincarePol = PoincarePol +
Simplify[1/(t^2(t^2-1))*((1+t^2)^(Length[SetI]-1)
-Sum[t^(2*Length[SS2I[[u]]]),{u,Length[SS2I]}])]
(Sum[t^(2u),{u,2(n-Length[SetI])-3}]-Sum[t^(2u),{u,Length[SetI]-3}])];
```

```
Print["Change in the Poincaré polynomial by crossing the wall W_I: Add ",
Simplify[1/(t^2(t^2-1))*((1+t^2)^(Length[SetI]-1)
-Sum[t^(2*Length[SS2I[[u]]]),{u,Length[SS2I]}])]
(Sum[t^(2u),{u,2(n-Length[SetI])-3}]-Sum[t^(2u),{u,Length[SetI]-3}])];
Print[""];

i++];

(* ---------------------- *)

i=1;
While[i< Length[ShortSetsanotalpha]+1,

SetI=ShortSetsanotalpha[[i]];
SetIcomp=Delete[l,Partition[SetI,1]];

SumIa=0;
SumIalpha=0;
k=1;
For[k=1, k<Length[SetI]+1, k++, SumIa=SumIa+a[[SetI[[k]]]]];
For[k=1, k<Length[SetI]+1,k++,SumIalpha=SumIalpha+alpha[[SetI[[k]]]]];

EpsilonIa=3 SumIa-2Total[a];

EpsilonIalpha=3 SumIalpha-2Total[alpha];

 Print["Wall corresponding to the set ",SetI];
 Print["EpsilonIa= ", EpsilonIa];
 Print["EpsilonIalpha= ", EpsilonIalpha];

NewalphaWI=List[];
```

```
For[s=1,s < n + 1, s++, NewalphaWI=Append[NewalphaWI,
Abs[EpsilonIa *alpha[[s]]-EpsilonIalpha * a[[s]]]]];
Print["Critical weight vector on the wall W_I: ",NewalphaWI];
(* critical weights at the wall W_I *)

NewalphaInI=Extract[NewalphaWI,Partition[SetI,1]];
(* critical weights at the wall W_I that are in the positions of I *)
NewalphaNotInI=Select[NewalphaWI, MemberQ[NewalphaInI,#]==False &];
(* critical weights at the wall W_I that are in the positions
complementary of I *)

If[(2*Last[NewalphaInI]<Total[NewalphaInI] && Length[SetI]>=3 ),

SS2I=Select[Subsets[SetI], 2*Total[NewalphaWI[[#]]]< Total[NewalphaInI] &];

PoincarePol = Simplify[PoincarePol +
Simplify[1/(t^2(t^2-1))*((1+t^2)^(Length[SetI]-1)
-Sum[t^(2*Length[SS2I[[u]]]),{u,Length[SS2I]}])]
(Sum[t^(2u),{u,Length[SetI]-3}]-Sum[t^(2u),{u,2(n-Length[SetI])-3}])]];

Print["Change in the Poincaré polynomial by crossing the wall W_I: Add ",
Simplify[1/(t^2(t^2-1))*((1+t^2)^(Length[SetI]-1)
-Sum[t^(2*Length[SS2I[[u]]]),{u,Length[SS2I]}])]
(Sum[t^(2u),{u,Length[SetI]-3}]-Sum[t^(2u),{u,2(n-Length[SetI])-3}])];
Print[""];

i++];
Print["PoincarePol= ",Expand[PoincarePol[[1]]]]
,
Print["The weight vector alpha is not Generic."];
Print["It is on the walls corresponding to the sets: ",Walls]];
```

Appendix B

```
(* Construir a lista de short sets *)

m=Input["What is the number of steps of the polygons?"];

alpha=List[]; (* alpha will be the weight vector *)
For[j=1,j<m+1,j++,alpha=Append[alpha,Input["Please insert
the weights of the polygon (one by one)"]]];
TotalSum=Total[alpha]; (* Sum of all the weights *)

l=List[]; (* l is the list {1,...,m} *)
For[i=1,i<m+1,i++,l=Append[l,i]];

SS= Subsets[l]; (* SS is the list of subsets of {1,...,m} *)

ShortSets=List[]; (* ShortSets will be the list of Short Sets *)
k=1; alphaGeneric = True;
(* In the end alphaGeneric will be False if alpha is not generic *)
While[k< Length[SS]+1,
Listaux=SS[[k]];
n=Length[Listaux];
SumS=0; (* SumS will give the sum of the weights in the kth subset in SS *)
For[i=1,i<n+1,i++, If[Listaux != {}, SumS=SumS+ alpha[[Listaux[[i]]]]]];
If[3 * SumS< 2 * TotalSum, ShortSets=Append[ShortSets,Listaux],
If[3 * SumS == 2 * TotalSum, alphaGeneric=False]];
```

```
k++];

SetsNotShort= Select[SS,Not[MemberQ[ShortSets,#]]&];
(* SetsNotShort will be the list of Sets that are not short *)

(* GoodPairsSets is the List of pairs (S,T) such that S and T
are subsets of {1,...,m} and S\cup T= {1,...,m} *)

PairsSets=Tuples[SS,2];
(* PairsSets is the list of all possible pairs of subsets of {1,...,m} *)
GoodPairsSets=Select[PairsSets,Sort
[DeleteDuplicates[Join[#[[1]],#[[2]]]]]==l&];

f=Function[{l1,l2}, If[(MemberQ[ShortSets,l1] && MemberQ[ShortSets,l2]) ||
(Not[MemberQ[ShortSets,l1]] && Not[MemberQ[ShortSets,l2]]), -1 ,
If[(MemberQ[ShortSets,l1] && Not[MemberQ[ShortSets,l2]]),-3,1]]];



(* Polinómio de Poincaré *)

(* Part 1 *)

QQ1=(1-a^6)^(m-1)/((1-a^2)^(m+3)(1+a^2)^2);

(* Part 2 *)

PP2=Expand[Sum[(1 + a^2)^Length[ShortSets[[i]]]
(a^(2*(Length[ShortSets[[i]]]+1))+a^(4*(m-Length[ShortSets[[i]]]-1))),
{i,1,Length[ShortSets]}]+Sum[(1 + a^2)^Length[SetsNotShort[[i]]]
(a^(2*(Length[SetsNotShort[[i]]]-2))+a^(4*(m-Length[SetsNotShort[[i]]])+2)),
{i,1,Length[SetsNotShort]}]];
QQ2=Simplify[PP2/((a^6-1)(1-a^2)^3(1+a^2))];
```

```
(* Part 3 *)

PP3= Expand[a^(2m) * Sum[a^(2*(Length[GoodPairsSets[[i]][[2]]]
-Length[GoodPairsSets[[i]][[1]]]+f[GoodPairsSets[[i]][[1]],
GoodPairsSets[[i]][[2]]])),{i,1,Length[GoodPairsSets]}]];
QQ3=Simplify[PP3/((1-a^2)^4(1+a^2)^2)];

(* Polinómio Final *)
QQ=Simplify[QQ1+QQ2+QQ3];

If[alphaGeneric == False, Print["alpha is not generic."],
Print["alpha=",alpha];
Print["ShortSets=",ShortSets];
Print["The Poincaré Polynomial is ",QQ]]
```