UNIVERSIDADE DE LISBOA
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Towards a new spacetime paradigm:
Gauge symmetries and post-Riemann geometries in gravitation

"Documento Definitivo"

Doutoramento em Astronomia e Astrofísica

Francisco Cabral

Tese orientada por:
Francisco Sabélio Nobrega Lobo
Diego Rubiera-Garcia

Documento especialmente elaborado para a obtenção do grau de doutor

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O trabalho é o amor tornado visível
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Towards a new spacetime paradigm:
Gauge symmetries and post-Riemann geometries in gravitation

With Applications to Particle Physics, Cosmology, Astrophysics and Gravitational Waves

Abstract

In this thesis the geometrical methods and symmetry principles in gravitation are explored motivating a new perspective into the spacetime paradigm. The effects of post-Riemann spacetime geometries with torsion are studied in applications to fundamental fermionic and bosonic fields, cosmology, astrophysics and gravitational waves. The physical implications and related phenomenological considerations of this study are addressed, and the fundamental ideas related to spacetime physics, motivated by geometrical methods and symmetry principles, are also discussed in the context of the possible routes towards a new spacetime paradigm in gravitation and unified field theories.

We explore the analogies between the gauge approach to gravity and the pre-metric approach to electrodynamics, within the exterior calculus of forms. These analogies are developed, reinforcing the hypothesis of the primacy of the conformal structure over the metric structure. Since the conformal symmetries seem to be broken symmetries in nature, the Poincaré gauge theories of gravity (PGTG) are taken into consideration. These presuppose a Riemann-Cartan (RC) spacetime geometry with curvature and torsion and motivate the search for torsion effects in physical systems. We study both minimal and non-minimal couplings of fermionic spinors to the background torsion and find changes in the energy levels (in the flat spacetime limit), including parity breaking effects. The Einstein-Cartan-Dirac-Maxwell theory is explored including its cosmological applications. The presence of minimal couplings to torsion induces non-linearities and non-minimal couplings in the matter fields dynamics and the resulting cosmological model is non-singular, including early and future bounces, early acceleration and torsion induced dark-energy due to fermionic vacuum condensates. Some potential astrophysical applications due to the torsion interaction with fermionic and bosonic fields are also considered as well as the effects of curvature in electromagnetic fields, including the extensions with inhomogeneous and anisotropic constitutive electromagnetic relations that respect the local isometries. In this context, the Parametrized Post-Newtonian (PPN) formalism is also implemented, making a bridge with the testing of different gravity theories. The effects of torsion are also analysed in gravitational wave (GW) physics, following the perturbations of a RC spacetime and in the field equations of a specific quadratic PGTG. The gravitational wave effects into electromagnetic fields are also studied with potential applications for non-standard detectors, which in principle could be extended for theories beyond GR, searches of extra polarizations and extra degrees of freedom.

Keywords: Gauge theories of gravity, post-Riemann geometries, spacetime symmetries, conformal structure, torsion, Einstein-Cartan theory, cosmology, gravitational waves.

Resumo

As teorias de gauge da gravidade baseiam-se num formalismo matemático auto-consistente que revela uma conexão fundamental entre as simetrias do espaçotempo e a sua geometria. Neste contexto, a abordagem metrica-afim é formulada de forma rigorosa e clara no âmbito das geometrias não Riemannianas com curvatura, torção e não-metricidade. Este formalismo pressupõe ainda o grupo afim para as simetrias fundamentais do espaçotempo, um conjunto bem determinado de potenciais de gauge (as "tetradas" e a conexão linear), as correntes conservadas de Noether que correspondem às fontes de campo e ainda uma métrica para o espaçotempo.

Nesta tese exploram-se os princípios de simetria e os métodos geométricos aplicados ao espaçotempo e à gravitação. A partir de uma elaboração fundamental matemática investigam-se diversas aplicações sobre os campos fundamentais, na astrofísica, cosmologia e ondas gravitacionais, de um subconjunto das teorias de gravidade de gauge, nomeadamente as teorias cujo grupo de simetria (local) é o grupo de Poincaré. A base fundamental matemática dos primeiros capítulos motiva ainda uma discussão clara em torno de uma nova perspectiva sobre o paradigma do espaçotempo e sobre o papel dos princípos de simetria e métodos geométricos na elaboração das teorias unificadas do campo.

Ao nível das aplicações, os efeitos das geometrias não Riemannianas com torção são estudados no âmbito dos campos fundamentais fermiónicos ($s = 1/2$) e bosónicos ($s = 1$), nas cosmologias de Friedmann-Robertson-Walker-Lemaitre, nas ondas gravitacionais e no contexto de potenciais aplicações astrofísicas. Analisa-se a fenomenologia subjacente a este estudo, bem como as suas implicações físicas e algumas considerações observacionais. Estas aplicações permitem principalmente aprofundar a investigação sobre os testes às teorias modificadas da gravidade que generalizam a Relatividade Geral (RG) de Einstein, indagando-se assim a natureza do espaçotempo e da gravitação. As ideias fundamentais sobre a física do espaçotempo (motivadas pelos métodos geométricos e princípios de simetria, na base deste estudo) são discutidas no contexto de possíveis vias para um novo paradigma do espaçotempo e para as teorias de campo unificadas. Ao nível mais fundamental, nesta tese exploram-se as analogias entre a abordagem pré-metrica da electrodinâmica formulada no cálculo exterior de formas e a abordagem de gauge à gravitação. As relações constitutivas entre os campos e as excitações constituem um postulado e estão subjacentes na formulação lagrangiana. No vácuo estas relações revelam uma conexão profunda entre a estrutura conforme (causal) e a electrodinâmica e podem ser interpretadas como relações constitutivas electromagnéticas para a variedade física de

espaçotempo. As referidas analogias são desenvolvidas reforçando-se a hipótese da primacia da estrutura causal (conforme) do espaçotempo relativamente à estrutura métrica. A conexão fundamental entre a electrodinâmica e a estrutura conforme de espaçotempo, juntamente com o princípio básico de simetrias locais (de gauge) na gravitação (incluindo grupos de simetria mais gerais que o grupo de Poincaré), reforçam a ideia de que a estrutura métrica não é tão fundamental como a estrutura conforme. Por outro lado, a própria estrutura causal/conforme pode ser derivada a partir do postulado das relações constitutivas da electrodinâmica. A abordagem pré-metrica na electrodinâmica e as relações constitutivas têm uma expressão análoga nos campos de Yang-Mills e nas teorias de gauge da gravitação. Estas analogias tornam claras a importância ao nível fundamental das simetrias e geometrias conformes locais pressupondo a existencia de conexões lineares não Lorentzianas, a quebra da invariancia de Lorentz e a existência de não-metricidade na geometria de espaçotempo. Para além disso, as analogias entre as relações constitutivas que conectam os campos ("field strengths") e as excitações (momentos conjugados), na electrodinâmica (e teorias de Yang-Mills) e na gravidade, permitem suportar a ideia de que as constantes de acoplamento que entram nestas relações reflectem as propriedades físicas do espaçotempo próprias da estrutura conforme. Estas propriedades electromagnéticas e gravitacionais que determinam as caracterísitacas de propagação dos campos nos referidos cones de causalidade, devem respeitar a simetria conforme local mas não são necessáriamente isotrópicas ou homogéneas. Esta é uma das hipóteses elaboradas ao longo da tese.

Tendo em conta que a simetria conforme parece ser uma simetria quebrada na Natureza, tomamos as teorias de gauge de Poincaré (TGP) para a gravitação. Estas teorias pressupõem uma geometria de Riemann-Cartan (RC) para o espaçotempo, com curvatura e torção, generalizando a RG em vários aspectos. Na formulação de gauge análoga à dos campos de Yang-Mills consideram-se Lagrangianas quadráticas nos invariantes de curvatura e torção que podem incluir termos que quebram a simetria de paridade. As fontes de campo são o tensor canónico de energia-momento (não simétrico) e o tensor de spin. No âmbito das aplicações exploradas nesta tese consideramos diversos efeitos devidos ao acoplamento com a torção, sobre os campos de matéria (fermiões e bosões), na cosmologia e nas ondas gravitacionais, passando também por algumas aplicações astrofísicas. Consideramos acoplamentos mínimos e não mínimos dos campos spinoriais à torção da geometria de fundo ("background"), explorando soluções e respectivos níveis de energia na aproximação de planura do espaçotempo (zero curvatura). Estes resultados revelam uma estrutura fina (hiperfina) semelhante à do efeito de Zeeman e incluem o caso mais geral de interacções que não preservam a paridade. Tais termos têm uma importância basilar no âmbito dos modelos para a emergência de uma assimetria entre partículas e anti-partículas no Universo primitivo. Os acoplamentos mínimos da torção aos campos fermiónicos e bosónicos são também investigados no âmbito da teoria de Einstein-Cartan-Dirac, incluindo a quebra de simetria $U(1)$ induzida pelo acoplamento à torção no sector bosónico. As equações dinâmicas neste modelo incluem não linearidades e interacções não mínimas nos campos de matéria, induzidas pela torção do espaçotempo. A teoia de Dirac generalizada subjacente quebra a invariacia face às transfromações C (de conjugação de carga), também vitais para a análise dos modelos de quebra de simetria entre partículas e antipartículas. O modelo cosmológico resultante é não singular e inclui soluções com um período de aceleração inicial após um "bounce" primordial, a

possibilidade de energia-escura efectiva induzida pela torção (na hipótese de condensados fermiónicos de vácuo) e ainda futuros colapsos não-singulares ("bounces") e modelos cíclicos. No âmbito das interacções entre os campos fermiónicos e bosónicos com a torção, são brevemente esboçadas algumas potencias aplicações astrofísicas, incluindo assinaturas da torção devidas às transições entre níveis de estrutura fina induzidos nos sistemas fermiónicos (estas podem incluir vestígios da quebra de paridade) e aplicações pertinentes no interior de corpos compactos como as estrelas de neutrões com ligação à astronomia com ondas gravitacionais. Para além destas considerações são ainda analisados diversos efeitos devidos à curvatura do espaçotempo sobre a electrodinâmica, no âmbito de cenários astrofísicos simples. Estas incluem generalizações às referidas relações constitutivas homogéneas e isotrópicas para os casos não homogéneos e anisotrópicos em concordância com as isometrias dos espaços com simetria esférica e axial. Neste contexto, o formalismo de parameterização pós-Newtoniana (PPN) é implementado, fazendo uma ponte com os testes às extensões da RG.

Dada a importância da nova janela da astronomia com ondas gravitacionais, é dada alguma importância ao seu estudo. Inclui-se uma breve análise das implicações e efeitos dos modos de torção, no contexto cosmológico, a partir de uma análise da teoria de perturbações na geometria de RC e nas equações de onda de modelos específicos quadráticos das TGP. Os efeitos das ondas gravitacionais nos campos electromagnéticos no contexto da RG são cuidadosamente investigados, assim como as possíveis aplicações fenomenológicas e observacionais no âmbito de detectores não *standard*. Estes estudos em princípio podem ser generalizados nas teorias mais gerais que a teoria de Einstein, havendo uma conexão relevante com a possibilidade de se detectarem modos de polarização e graus de liberdade extras.

Finalmente, são analisadas e discutidas diversas considerações sobre o paradigma de espaçotempo, os métodos geométricos e os princípios de simetria, no âmbito das teorias unificadas, e motivadas pelas ideias fundamentais exploradas na tese. Estas considerações incluem várias questões em aberto tais como: a primacia da estrutura causal-conforme face à estrutura métrica ao nível fundamental e o abandono do paradigma de espaçotempo absoluto; o espaçotempo com propriedades físicas (electromagnéticas, energia-momento, etc) e a identificação do vácuo físico clássico com a variedade de espaçotempo; as geometrias métricas-afins e a possibilidade de inter-conversão entre torção, curvatura e não-metricidade; as simetrias unificadas e as quebras de simetria (e transições de fase cosmológicas), incluindo a quebra de simetria de escala (ou conforme) e a emergência de constantes (de Lorentz-Poincaré) e respectivas escalas naturais presentes nas teorias efectivas; a hipótese da unificação do espaçotempo com os campos físicos (matéria-energia) usando métodos geométricos e princípios de simetria; assim como como diversas considerações inspiradas pela teoria quântica e termodinâmica, no âmbito de um paradigma de espaçotempo unificado com a matéria-energia e apresentando uma simetria conforme ao nível fundamental.

Palavras-chave: Relatividade Geral, teorias de gauge da gravidade, simetrias do espaçotempo, geometrias não Riemannianas, torção, não-metricidade, estrutura conforme, cosmologia, ondas gravitacionais, teorias de campo, unificação.

Preface

The research included in this thesis has been carried out at the Instituto de Astrofísica e Ciências do Espaço in the Faculty of Sciences of the University of Lisbon. Most of the material that can be found in chapters 2-7 have been published, while some parts are under review and others are in preparation.

A list of the works published included in this thesis ([1, 2, 3, 4, 5, 6, 7]) is given below:

1. "Electrodynamics and Spacetime Geometry: Foundations"
   Francisco Cabral (Lisbon U.), Francisco S. N. Lobo (Lisbon U.)
   1602.01492 [gr-qc]
   DOI: 10.1007/s10701-016-0051-6
   Found.Phys. 47 (2017) 2, 208-228.

2. "Electrodynamics and spacetime geometry: Astrophysical applications"
   Francisco Cabral (Lisbon U.), Francisco S. N. Lobo (Lisbon U.)
   1603.08180 [gr-qc]
   DOI: 10.1140/epjp/i2017-11618-2
   Eur.Phys.J.Plus 132 (2017) 7, 304

3. "Gravitational waves and electrodynamics: New perspectives"
   Francisco Cabral (Lisbon U.), Francisco S. N. Lobo (Lisbon U.)
   1603.08157 [gr-qc]
   DOI: 10.1140/epjc/s10052-017-4791-z
   Eur.Phys.J.C 77 (2017) 4, 237

4. "Einstein-Cartan-Dirac gravity with $U(1)$ symmetry breaking"
   Francisco Cabral (Lisbon U.), Francisco S.N. Lobo (Lisbon U.), Diego Rubiera-Garcia (Lisbon U.)
   1902.02222 [gr-qc]
   DOI: 10.1140/epjc/s10052-019-7536-3
   Eur.Phys.J.C 79 (2019) 12, 1023

5. "Cosmological bounces, cyclic universes, and effective cosmological constant in Einstein-Cartan-Dirac-Maxwell theory"
   Francisco Cabral (Lisbon U.), Francisco S.N. Lobo (Lisbon U.), Diego Rubiera-Garcia (Madrid U.)
   2003.07463 [gr-qc]
   DOI: 10.1103/PhysRevD.102.083509
   Phys.Rev.D 102 (2020) 8, 083509

6. "The cosmological principle in theories with torsion: The case of Einstein-Cartan-Dirac-Maxwell gravity"
   Francisco Cabral (Lisbon U.), Francisco S.N. Lobo (Lisbon U.), Diego Rubiera-Garcia (Madrid U.)
   2004.13693 [gr-qc]
   DOI: 10.1088/1475-7516/2020/10/057
   JCAP 10 (2020), 057

7. "Fundamental Symmetries and Spacetime Geometries in Gauge Theories of Gravity: Prospects for Unified Field Theories"
   Francisco Cabral (Lisbon U.), Francisco S.N. Lobo (Lisbon U.), Diego Rubiera-Garcia (Madrid U.)
   2012.06356 [gr-qc]
   DOI: 10.3390/universe6120238
   Universe 6 (2020), 238

The following paper [8] was under final revision and submission process, has been published after the thesis discussion.

1. "Imprints from a Riemann-Cartan space-time on the energy levels of Dirac spinors"
   Francisco Cabral (Lisbon U.), Francisco S.N. Lobo (Lisbon U.), Diego Rubiera-Garcia (Madrid U.)
   2102.02048 [gr-qc]
   DOI: 10.1088/1361-6382/ac1cca
   Class.Quant.Grav.38 (2021) 19.

Plan of the thesis

This thesis is organized in seven main chapters following the introduction. In the first chapter the introduction is presented, clarifying general and specific motivations that include mathematical, conceptual and physical considerations. The remaining chapters are divided into three parts. In the first part (chapters 2-3), the fundamental role of the conformal-causal structure of spacetime and the gauge theories of gravity are presented in detail, in connection to the main ideas defended in this thesis. The second part develops the applications, in particular the physical effects due to the torsion of Riemann-Cartan spacetime, for fermionic and bosonic fields, in cosmology, astrophysics, and gravitational waves (chapters 4-7, respectively). Finally, the third part (chapter 8) summarizes the relevant ideas that were presented in the first part and that motivate new perspectives into the spacetime paradigm. It includes also several considerations on open questions for future research. The diagram below illustrates the structure of the thesis according to the inputs from mathematical models, phenomenological considerations and philosophical/conceptual discussions.

Notation

Throughout this work, unless stated otherwise, indices with Greek letters represent spacetime indices ranging from 0 to 3, while Latin indices are spatial from 1 to 3. In some sections Latin indices are also used as symmetry indices ranging from 0 to 3. Unless stated otherwise, repeated indices imply Einstein's summation convention and we will be adopting a $(+---)$ signature for the spacetime metric. In most of the thesis the

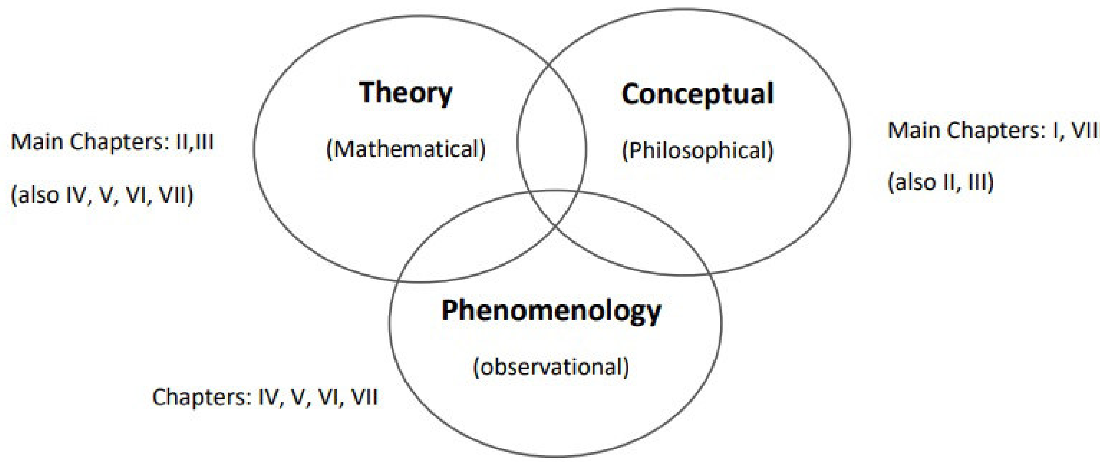

Figure 1: This diagram is a roadmap to the structure of the thesis in terms of three main areas: 1) Theory and mathematical modelling; 2) Interpretations at the conceptual level, including the exploration of open questions regarding the nature of spacetime and the role of geometrical methods and symmetry principles in unifying theories, and 3) Physical implications, phenomenology and related observational considerations (testing torsion effects in particle physics, cosmology, astrophysics and with gravitational wave probes)

$c = 1$ units are adopted, where c is the speed of light in vacuum, but in some parts this constant is explicitly introduced.

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Chapter 1

Introduction

Science is truly a remarkable adventure. Following its ancient routes one can find the astonishing endeavour of human beings throughout the millennia. The anthropological perspective reveals how science is deeply routed into our very nature, our first impulses and needs. We see humans exploring the physical environment, forming mental maps of the natural condition, developing the proto structures of articulated language and thought and using and expanding the imagination, memory and creativity. We see them making tools, experimenting and testing different strategies for survival and adaptation, learning from experience while developing at the same time a sense for aesthetics, for logical and causal thinking. Curiosity is perhaps one of the vital factors for the flowering of the scientific mind, together with deep observation, creative thinking, experimentation, the abandoning of ideas that are incompatible with the empirical information, a passion for exploration and discovery, and a relentless search for truth or at least a meaningful knowledge, one that is validated by experience and that is useful for the understanding of Nature and of our place within.

The physical environments on planet Earth constitute a marvellous diversity of geophysical conditions and ecosystems that are habitats for a spectacular biodiversity. Similarly, the human family manifests an equally remarkable and beautiful diversity, that is dynamical in time and in space. Indeed, the human geography reveals a profound and intimate relation between humans and the land since many thousands of years. Any group of human beings, from small tribes, to larger ethnic groups, develops a common culture, a collective memory and patterns of social bounding, that usually also encompasses a certain cosmogony, a narrative about the world, its origins and nature. This is often manifested in the form of myths and symbolic language reflecting also aspects of psycho-sociological nature (fears, aspirations, dreams, hopes, social structures, etc), as well as those from the human geography. In this sense, the intimate connection between human beings and the physical environment has always been represented within the collective narratives of all cultures.

The modern scientific paradigms of the various scientific fields have its routes in the great adventure of science that resembles an old tree, one that is growing into the light of understanding, through the process of learning, trial and error, as it liberates from illusions, prejudice and superstitions. It is a great adventure, that encompasses both the crystallization of ideas, models and procedures into paradigms, as well as the creative movement of exploring the frontiers of paradigms, breaking the limitations of pre-conditioning and leading to dramatic changes of paradigms. It is within this scientific stream that physics finds its history, encompassing a remarkable flow of ideas about the nature of matter, the physical forces, energy, the types of movements and changes, space and time, etc, that led to the tremendous diversity of specific areas of modern physics.

The routes of physics bring us inevitably to the legacy from the Hellenistic culture, with the contributions from various thinkers that supported the idea that the Universe was understandable, that its order, the Cosmos ($\kappa\acute{o}\sigma\mu o\zeta$) could be approached by the human mind in a meaningful way.

The history of physics and mathematics is fascinating and vast. Instead of developing it here, it is worthwhile to recall that while Aristotle systematized the ancient knowledge and gave a special emphasis into the study of the types of motion that can exist, the legacy from the ancient Greeks into natural philosophy was resurrected during the renaissance, with a profound transformation from the Aristotelian, Ptolomaic word-view into the Copernican heliocentric perspective. The contributions from Copernicus, Tycho Brahe, Kepler, Galileo, and many others revolutionized our views of the solar system, strengthening the scientific method based on observations and mathematics. The invention of calculus by Leibniz and Newton, the Newtonian mechanics and later the Lagrangian and Hamiltonian formulations, together with the Maxwell's equations, greatly deepened the mathematical modelling of physical phenomena, while astronomy continued to make tremendous advances. With the advances in the nineteenth century and the astonishing scientific revolutions of the twentieth century, our current paradigms in science provide a breathtaking narrative about the nature of matter, of space and time, of quantum fields and vacuum, including the inspiring modern astrophysical and cosmological perspectives.

The diversity of all scientific fields of knowledge have also reached a high degree of specialization and sophistication. The complexity of any object of study and indeed of any system on planet earth, reveals the strong need for interdisciplinary research, and of an epistemology of complexity, in order to address the systemic challenges, interdependences and non-linearities, typical of such complex systems. The planetary sciences, together with the complex systems sciences, face the modern challenges of understanding the profound interdependences and dynamics of natural ecosystems with all its bio-geo-chemical interactions involving the biosphere, lithosphere, hydrosphere and atmosphere, on one hand, and the complexity of human ecosystems, on the other hand. Science has here a vital role in the construction of a scientific culture that nurtures a sense of planetary awareness and responsibility. In this respect, astronomy, astrophysics and cosmology (together with planetary science) can also provide a valuable perspective onto the human condition and planetary civilisation. This perspective reveals the appropriate time scales for the understanding of Earth dynamics, stellar evolution, galactic evolution and cosmological dynamics. Indeed, the modern scientific narrative clarifies that the history of life on Earth is measured in thousands of millions of years, which is also the appropriate time scale for the geological history and for the evolution of the Solar system, the large scale structure of clusters of galaxies and of the cosmic expansion, while the spatial scales involved span the light-minutes, to light-years and thousands of millions of light years (or Gigaparsec). These notions of space and time can have profound consequences into the way we see the Universe and into our planetary perspective of civilisation.

Indeed the notions of time and space are at the most fundamental basis of physics which has evolved from a science of particles and fields in space and in time into a science of spacetime (and particles and fields). This evolution was accompanied by conceptual revolutions and mathematical developments. The deep connection between physics and mathematics is part of the wider open question concerning the intimate link between Nature, mathematics and the human capacity to explore such a link. The current understanding of this reveals a profound connection between the fundamental laws of Nature, the geometrical structures (and patterns), and fundamental symmetries [9].

In modern theoretical physics, mathematical symmetries have a deep and fundamental significance. In strict connection to conservation laws, these symmetries are at the foundations of our current understanding of physical nature. On one hand, the internal symmetries of (gauge) field theories determine the properties of the known fundamental interactions such as the electromagnetic, the weak and strong forces, therefore determining the properties of the mediating bosons, and on the other hand, spacetime symmetries determine the properties, such as mass/spin of all particles/fields, fermions and bosons. For example, the Minkowski spacetime symmetries are expressed in the invariances with respect to the Poincaré group $P(1,3)$ and the irreducible representations of it, via the Casimir operators allows us to find the sets of possible values for the masses and spins of particles, Wigner's mass/spin "universal" classification. Moreover, the internal symmetries also determine the Noether conserved charges of any set of fermionic/bosonic particles (multiplets), interacting via a specific interaction. One could call these "charges", "colours", "flavours", etc., internal properties, while those related to the spacetime concept as "external" properties - more rigorously, the (Noether) conserved current linked to spacetime translations, canonical energy-momentum, which give the mass-energy of particles. External, because these properties are linked to external symmetries (under spacetime translations) rather than internal and the notion of inertia is fundamentally related to the very nature of spacetime. Inertia and the nature of spacetime are two fundamental mysteries and its understanding remains an open question.

The spacetime paradigm in special relativity and relativistic quantum field theories is that of Minkowski. Therefore, it is based on the absoluteness of the spacetime metric and corresponding line elements. The fundamental spacetime distance element and all Lorentz scalars are invariant under global transformations of coordinates between inertial observers and the causal structure is also globally invariant. It is this absolute spacetime that serves as a rigid stage and reference concept relative to which inertial frames can be defined and therefore, the inertial properties of matter are defined with respect to this conceptual construction. Consequently, mass-energy and spin are seen as intrinsic properties to particles, not depending on anything else to exist other than the spacetime itself. This more philosophical interpretation is complementary and underlying to the mathematical notion that these "intrinsic properties" come from the Noether conserved currents associated to the Poincaré symmetries of Minkowski spacetime, as expressed through the Casimir operators.

Besides these important connections between internal symmetries, spacetime symmetries and the properties (internal, external) of particles and interactions (fermions and bosons), there is another fundamental role played by symmetries in deep relation to the very nature of spacetime and gravity. This is the profound link existing between spacetime symmetries and non-Euclidean spacetime geometries.

Following Einstein's path, gravity is intimately related to spacetime and in particular to its geometry. Einstein's remarkable legacy has been consistently passing numerous tests, in the solar system [10], in the orbits of stars around the Galactic center [11], in the Hulse-Taylor Pulsar binary (see [12]), the detection of various gravitational wave events from coalescing compact objects of stellar origin (black holes, neutron stars)[13, 14, 15, 16], in the recent direct observational evidence for a supermassive black hole in the center of M87 [17] and also, to some extent, in the context of the standard cosmological big-bang paradigm [18]. For this last case, the open issues related to the extra ingredients such as inflation (initial conditions), dark matter, dark energy (cosmic acceleration), present a huge challenge for physics. These challenges together with the problems of astrophysical and cosmological singularities and the search for a consistent theory of quantum gravity are a few of the well established motivations to explore extensions to Einstein's gravity, already at the classical level. Such extensions come in a wide range of modified theories of gravity, more or less motivated from first principles. In Einstein's General Relativity, the energy-momentum of matter fields become intimately linked to the differential geometry of the spacetime continuum. But Einstein has chosen a spacetime manifold with (pseudo)

Riemann geometry, just for a matter of convenience and simplicity. His theory opened the door for a dynamical role of spacetime geometry in physics but the specific nature of this link is still an evolving field of research and it is valid to consider extensions to the assumption of Riemann geometry.

Both the success of this remarkable geometrical principle due to Einstein and the incredible success of symmetries (and the gauge approach) in the physics of interactions can be combined, because spacetime geometries and spacetime symmetries turnout to be deeply and fundamentally connected. The extension of the gauge principle to gravity means that this fundamental "interaction" of nature can be described by localizing (gauging) the spacetime symmetries. Once this is done, the gauge potentials related to the symmetries generators are also identified as dynamical geometrical degrees of freedom and the corresponding field strengths are well-known mathematical objects from differential geometry, such as curvature, torsion and also non-metricity [19, 20, 21].

Gauging different groups of spacetime transformations leads to different spacetime paradigms and meta-classes of theories of gravity. In each spacetime paradigm, different classes of theories of gravity can be constructed, via the geometrical invariants (constructed from the gauge field strengths), for example, quadratic models of Yang-Mills type. One should therefore look carefully at the different relevant groups of spacetime transformations, and the messages they bring to physics: the spacetime paradigm, the physical invariants, the properties of particles, cosmological applications, symmetry breaking and phase transitions, astrophysical applications, gravitational waves and the nature of gravity, space and time.

From gauge symmetries to spacetime geometries

Recently, several papers have clarified that a new perspective into gravity and the nature of spacetime is open due a surprising result: the phenomenology of General Relativity (GR) can be formulated in terms of curvature (Riemann geometry), or torsion (Weitzenböck geometry) [22, 23, 24, 25], or non-metricity (generalized Weyl spacetime) [27, 28, 29, 30, 31]. Differences at the phenomenological level are expected in the couplings with fermions and for extensions of these three apparently equivalent descriptions of gravity.

Following this, the question arises on whether there are any fundamental guiding principles that can clarify the role of non-Riemann geometries in the formulation of gravity. In turns out that the gauge formulation of gravity can consistently provide that fundamental principle, according to which specific non-Euclidean spacetime geometries arise naturally from the localization (gauging) of specific spacetime symmetry groups. Accordingly, the gravitational coupling to fermions and the fundamental properties of extended objects as test-matter with/without hypermomentum (see below), results in different phenomenological predictions

In the context of the Gauge theory of Gravity, the Metric-Affine formalism, according to which the metric and the affine connection are both independent degrees of freedom, becomes clearly formulated. In fact, the Metric-Affine formalism includes a general (metric-affine) geometry with curvature, torsion and non-metricity, that arises naturally from the localization (gauging) of the Affine group (and considering also a metric structure, which is not really as "fundamental" as the linear connection[1]). The Affine group A(4,R) is the (semi-direct) product of the group of 4-translations T(4) with the "General Linear group" GL(4,R). The gauge potentials of T(4) are the vector valued 1-form tetrads (or covector frames), with the torsion (a vector valued 2-form) as the field strength, while

[1]We will develop this notion more rigorously in this thesis.

the gauge potentials of GL(4,R) are the 2-tensor valued linear connection 1-form, whose field strength is the curvature (a 2-tensor valued 2-form). In the most general case, the linear connection (many times called affine connection) is not "Lorentzian" (as in the Riemann-Cartan geometry) and therefore, it is not metric compatible (once a metric structure is considered) and the non-metricity (tensor valued 1-form) is different than zero. The GL(4,R) group is extremely rich and relevant for unification field theories. A map (isomorphism) of a part of it can be done to the internal symmetries of the standard model of particles and interactions, which is a gauge theory. Therefore it is of great relevance to consider the consequences of a gauge theory of gravity of GL(4,R) (Pure affine gravity)[2] or that of the Metric affine gravity (MAG). In the gauge formulation, the Noether currents associated to the spacetime symmetries are the "sources of gravity". In MAG the Noether currents (the sources of gravity) are the canonical energy-momentum (T(4)) and the hyper-momentum (GL(4,R)) which includes a spin part, a term related to dilatations and another connected to shear type of "deformations".

If one restricts the symmetry group to the Poincaré group P(1,3), the gauge approach leads to Poincaré Gauge Theories of Gravity (PGTG). The Poincaré group P(1,3) includes the 4-translations T(4) and the Lorentz transformations, usually represented by SO(1,3). The Riemann-Cartan geometry (with torsion and curvature) comes naturally from such formalism and the tetrads and (Lorentzian) metric compatible connection (sometimes called spin connection) are the gauge potentials representing the degrees of freedom of gravity. The corresponding Noether currents are the canonical energy-momentum and the spin densities. The most general quadratic Poincaré gauge gravity model was obtained (see [19, 21]) and includes the ECSK plus the Holst term and terms quadratic in torsion and curvature. The Holst term and some quadratic terms break the parity invariance, which can be very relevant in connection to particle physics and the early Universe. In this theory torsion propagates, therefore torsion GW modes are predicted.

In this thesis we will look carefully at the fundamental geometrical objects and its relation to spacetime symmetries in the gauge formulation of gravity. We will consider the structure of gauge theories of gravity and have a special focus on PGTG, in particular we will consider some of its predictions and implications including interactions with fermions and bosons, cosmological and gravitational wave applications, as well as some remarks on potential astrophysical applications.

From geometrical methods and symmetry principles towards a new spacetime paradigm and unified field theory

Extending the gauge symmetry group of spacetime coordinate transformations leads to extended geometries with curvature, torsion and non-metricity. Minkowski merged space and time into a single 4-dimensional manifold with pseudo-Euclidean geometry and global Poincaré symmetries. Then, Einstein changed the spacetime paradigm of a flat background into a dynamical curved 4-dimensional manifold with pseudo-Riemann geometry and local Lorentz invariance. With the maturation of gauge theories of gravity the possibility of a new classical spacetime paradigm emerges naturally from the fundamental link between spacetime symmetries and non-Euclidean (post-Riemann) geometries. Torsion, non-metricity and curvature endow spacetime with a much richer gravitational dynamics and phenomenology of physical fields minimally or non-minimally coupled to the spacetime geometry, astrophysical and cosmological applications, as well as relevant new windows to gravitational wave astronomy. Moreover, it seems that these three geo-

[2]Notice that the GL(4,R) group preserves the origin of the coordinates, resembling in some analogical sense the "internal transformations".

metrical objects might be fundamentally linked via specific duality (or correspondence) relations and it is plausible to postulate that they are inter-convertible[3] (we will return to this hypothesis in this thesis).

General gauge theories of gravity (beyond the PGTG), formulated in metric-affine spacetime geometries, lead naturally to the invalidity of spacetime metric absoluteness and are compatible with the idea that the metric structure is less fundamental than the conformally invariant structure of spacetime. This idea is also implicit in the pre-metric approaches to electrodynamics or general Yang-Mills fields, in the formalism of the exterior calculus of forms. Part of the work developed in the first chapter of the present thesis is related to a pre-metric approach to electrodynamics and the coupling of electromagnetism to the conformal-causal structure of spacetime via the constitutive relations.

Pre-metric electrodynamics (as well as pre-metric Yang-Mills interactions) and the constitutive relations, put forward the interpretation that the causal structure is more fundamental than the metric structure, and that the physical spacetime can be endowed with electromagnetic properties (electro vacuum), not necessarily spatially homogeneous and isotropic, but rather following the local isometries, without breaking the local causal structure. This primacy of the spacetime causal structure reinforces the importance of conformal symmetries in physics and in gravity.

In these lines, of particular interest is the conformal gauge theory of gravity due to the important role of the conformal group C(1,3) in both the causal structure of spacetime and in the context of the AdS/CFT correspondence[4] [32, 33]. It seems nonetheless that the conformal symmetry is not a perfect symmetry in Nature and some early Universe symmetry breaking (phase transition) into the Poincaré Group is a plausible scenario [19, 20, 21].

Several considerations concerning the role of symmetries and non-Riemann geometries in the description of classical gravitation can motivate a renewed perspective on space-time itself. Going beyond the classical approach, geometrical methods (non-Riemann geometries and exterior calculus of forms) and symmetry principles might play a fundamental role within the possible routes into the quantum gravity challenge and unification in physics.

A fundamental unified description of both matter-energy ("physical fields") and space-time into a single construct (just like Minkowski unified space and time) leading to a new spacetime paradigm with thermodynamic and electromagnetic properties and possibly a quantum nature, is a relevant hypothesis. The deformations of such "physical spacetime" describing the gravity phenomena in the classical limit would obey metric affine geometry.

This hypothesis and the primacy of the causal (conformal) structure with respect to the metric structure are some of the topics to be addressed in this thesis (particularly in the last part) at the conceptual level and motivated by different perspectives of modern physics and mathematical methods.

The thesis is divided into three related main parts: The first part (chapters 2 and 3) addresses fundamental topics on the conformal structure of spacetime, gauge theories of gravity and metric affine formalism. Here the basic ideas supporting and motivating a new spacetime paradigm are explored. In the second part (chapters 4 to 7) fundamental applications to fermionic and bosonic fields, cosmology, astrophysics, and gravitational waves are studied. Finally, in the last part (chapter 8), we present an extended discussion on the role of the geometrical methods and symmetry principles, explored in the

[3]As suggested by the analogies between the Bianchi identities in electrodynamics and the generalized Bianchi identities of Metric affine-geometries.

[4]An example of the so called Gauge-Gravity dualities

first part, towards a new spacetime paradigm and unified field theories. Several topics previously addressed are considered in depth, and various fundamental open questions are mentioned and discussed.

Part I

Symmetries and geometry of spacetime: Towards a new paradigm

Chapter 2

Towards a new spacetime paradigm I: The primacy of the conformal structure

In this chapter we explore the intimate connection between spacetime geometry and electrodynamics. This link is already implicit in the constitutive relations between the field strengths and excitations, which are an essential part of the axiomatic structure of electromagnetism, clearly formulated via integration theory and differential forms. We review the foundations of classical electromagnetism based on charge and magnetic flux conservation, the Lorentz force and the constitutive relations. These relations introduce the conformal part of the metric and allow the study of electrodynamics for specific spacetime geometries. At the foundational level, we discuss the possibility of generalizing the vacuum constitutive relations, by relaxing the fixed conditions of homogeneity and isotropy, and by assuming that the symmetry properties of the electro-vacuum follow the spacetime isometries. The implications of this extension are briefly discussed in the context of the intimate connection between electromagnetism and the geometry (and causal structure) of spacetime. In relation to this, we also revisit Mach's principle and a generalized principle of relativity with respect to the conformal group. The section 2.1 is inspired by the work in [1].

2.1 From pre-metric electrodynamics to the conformal (causal) structure of spacetime

The classical field theory of electromagnetism lies at the very heart of profound developments in our understanding of physics. Before the conceptual revolutions of special relativity, general relativity and quantum theory, the seeds planted by the works of Faraday and Maxwell led to the establishment of the important concept of a physical field while preparing at the same time the conditions for the advent of both special relativity and quantum field theory. Indeed, it was the electrodynamics of moving objects that inspired Einstein's work on special relativity and it was the form of Maxwell's equations that motivated and guided Lorentz, Poincaré and Einstein to derive the (Lorentz-Poincaré) spacetime transformations. This in turn led to the revolutionary spacetime unification of Minkowski. In fact, Maxwell's equations were successfully incorporated within quantum electrodynamics and played a key role in the development of general

Yang-Mills gauge models for the fundamental interactions in quantum field theories. It is well known that in this context, the Maxwell fields served as a prototype for understanding the deep relation between (gauge) symmetries and the dynamics of fundamental physical fields and interactions. On the other hand, the remarkable work of Noether allowed the understanding of the link between these so-called internal symmetries and conserved quantities.

Nevertheless, it is worth recalling that the development of gauge symmetries had its routes in the influential work of Weyl (1918) on gravity [34], soon after the formulation of General Relativity in 1915 (see [35, 36] for historical reviews). Weyl generalized the gravitational theory by assuming that the light cones have the principal relevance, while abandoning the absoluteness of spacetime distances. Accordingly, in his theory the conformal (or causal) structure of spacetime is invariant while the metric g is only fixed up to a proportionality leading to a (gauge) freedom, $g \to \lambda g$. A given choice provides a certain gauge that allows spatial and time intervals to be determined. With this idea, Weyl was able to incorporate Maxwell's equations in the spacetime geometry by introducing an additional structure besides the conformal: the gauge connection (or bundle connection). The set of all possible Lorentzian metrics (related by a conformal gauge transformation) sharing the same local light cone provided the local fibres of a gauge bundle over the base spacetime manifold and a bundle connection was required. The electromagnetic potential played the role of this connection which was incorporated in the covariant derivative and the electromagnetic field tensor was the curvature of the gauge connection. Therefore, this was not only one of the first early serious attempts to intimately link the electromagnetic field with spacetime geometry, in search for a unified field theory [35], but it also represented the very birth of gauge theories in the physics of interactions.

Weyl's emphasis on the light cone and therefore on the casual structure of spacetime echoes in some sense in modern ideas of gauge theories of gravitation (see [19]), since the local conformal group is more general than the Lorentz, the Poincaré or even the so called Weyl group. The 15 parametric conformal group includes the Poincaré sub group plus dilatations and special conformal transformations, where the last two break the line element invariance while preserving the light cone. The original ideas of Weyl changed, but the fundamental link between gauge symmetries and the dynamics, established through geometrical reasoning, namely, through gauge or bundle connections, remained in modern Yang-Mills theories and in gauge theories of gravity.

The intimate link between electromagnetism and spacetime geometry, and therefore gravity, is one of the most relevant topics of classical field theories. On the one hand, since electromagnetic fields have energy-momentum they gravitate, affecting spacetime geometry. On the other hand, light rays propagate along null geodesics, which express an important link between the causal structure of spacetime and the propagation of electromagnetic fields. The notion of causality is fundamental in physics and the idea that it is profoundly associated to electrodynamics gives this classical field theory a special relevance. Such a relation seems to be unique, since photons (and gravitons) are now viewed as the only massless particles of the standard model of elementary particles[1], and therefore are the only ones that can provide an experimental study of the null cones. Although the light cone first appeared within Minkowski spacetime, and Maxwell's equations were the first relativistic field equations, these can be shown to have a pre-metric formulation

[1]We are not considering here the gluons trapped within hadronic structures.

[37, 38, 39, 41, 42, 43, 44], while the light cone can be derivable from electrodynamics [39, 45, 46, 47]. In fact, in the spacetime framework, Maxwell's equations developed naturally into Cartan's exterior calculus of differential forms and in this formalism the field equations are indeed fully general, coordinate-free, covariant equations without any dependence on a metric or affine structure of spacetime.

2.1.1 Pre-metric foundations of electrodynamics and the coupling to conformal geometry

The pre-metric approach to electrodynamics[2] can be exclusively derivable from the empirically based postulates of charge and magnetic flux conservation and the Lorentz force (see [37, 41, 38, 39] for a clear axiomatizing of electrodynamics). Accordingly, the inhomogeneous equations can be derived from charge conservation and the homogeneous equations express magnetic flux conservation. In the exterior calculus formalism, the geometrical and physical interpretations become very simplified and clear. Assuming a 3+1 spacetime splitting (foliation), the Faraday 2-form F can be decomposed into an electric part E, which is a 1-form related to lines, and a magnetic part B, a 2-form related to surfaces. Similarly, the excitation 2-form G contains the electric 2-form and magnetic 1-form excitations, D and H, related to surfaces and lines, respectively. In order for the theory to be complete and to have a predictive power, some form for the constitutive relations between the field strengths $[F = (E, B)]$ and the excitations $[\mathcal{H} = (D, H)]$ is required, which constitutes a separate independent postulate in its own. In vacuum, these relations can be viewed as constitutive relations for the spacetime itself and its form will determine the electromagnetic theory that results and its physical predictions (see for example [41]). While the field equations rest on empirically well-established postulates, the constitutive relations usually assumed to be local, linear, homogeneous and isotropic have a not so well empirical basis. When considered in vacuum, these relations require the metric structure of spacetime or more specifically, the conformally invariant part of the metric [41].

One concludes that at the very foundations of electromagnetism the field equations are completely general, without the need of any metric or affine structure of spacetime, but its realization in spacetime via the constitutive relations, reveal a fundamental connection between electrodynamics and the causal (conformal) structure. In fact, Friedrich Hehl and Yuri Obukhov starting from pre-metric electrodynamics and assuming local and linear constitutive relations, were able to derive the light cone structure and therefore, the conformally invariant part of the metric, provided that there is no birefringence (double refraction) in vacuum [39]. The axiomatic approach developed by Hehl and Obukhov is complementary and compatible with the more traditional Lagrangian formulation [37]. Indeed, the constitutive relations are assumed in the action and the form of these therefore determines the resulting differential field equations. Having in mind the simplifying power of the pre-metric formalism of electrodynamics plus spacetime constitutive relations, in any case, the tensor or components formalism provides a realization of the field equations in spacetime (assuming specific constitutive relations), requiring the metric and affine structure.

For the case of (pseudo) Riemann geometry, one can then explore the effects of spacetime curvature on electromagnetic phenomena, derived from generalized Gauss and Maxwell-Ampère laws and wave equations. Accordingly, the effects of gravity on Maxwell fields, due to the curvature of the background spacetime, provide a window for the study and testing of theories of gravity (with Riemann geometry) with potential astrophysical

[2]Actually it would be more appropriate to use the term electromagnetodynamics, but we use the electrodynamics designation because it became the most common term for this area of physics.

applications related to black holes, pulsars, relativistic stars and gravitational waves (GW), for example. In this concern, it has been shown that gravitational waves affect the polarization of light (see for example [48]) but it was done in the geometric optics limit and deserves further research. With the advanced LIGO and VIRGO GW detectors [13] (see also [49, 50]), a gravitational signal was detected and this achievement was celebrated as an important mark of a future new window for astronomy, astrophysics and cosmology (see [49] for a review on the physics of GW and detectors). Therefore, the intimate relation between gravity, electromagnetism and spacetime geometry should be profoundly explored as it may reveal new alternative approaches for GW detection and also for the study of GW emission by astrophysical sources. In chapter 6 we briefly consider some astrophysical applications also in connection to testing extended theories beyond GR and Maxwell electrodynamics (with non-homogeneous/isotropic constitutive relations) [2] and in chapter 7 we will explore in depth the GW effects in electromagnetic fields [3].

A more consistent study of the coupling between gravity and electromagnetic fields can be achieved through the Einstein-Maxwell coupled equations or similar systems of equations for alternative theories of gravity coupled to electromagnetism. The Einstein-Maxwell system is able to successfully explain many phenomena such as the deflection of light, the gravitational redshift and the Shapiro time delay (see [51, 52], for example), but the geometrical optics limit is usually assumed. The exploration of the gravity-electromagnetic coupling beyond that limit continues to be of the utmost importance.

In this chapter we will explore the intimate relation between electromagnetism and spacetime geometry on a foundational level. In the following sections we review the foundations of electromagnetism based on charge and magnetic flux conservations, the Lorentz force and constitutive relations. The theory follows essentially the approach developed by F. Hehl *et al.* [37, 38, 39, 41]. By revisiting the 3-dimensional formulation using integration theory within a 3+1 spacetime splitting (foliation) [37], this allows to clarify the geometrical interpretations of the electromagnetic quantities and their relations. We then expose the same axiomatic theory in the language of forms in the 3-dimensional and also in the more general 4-dimensional formalism [38, 39, 41]. We assume local, linear constitutive relations and briefly address the topic on how different forms for these relations can affect the electromagnetic theory and related physical predictions. Finally, we summarize and discuss several physical and conceptual implications, following the fundamental connection between electrodynamics and the conformal structure of spacetime.

Foundations of electromagnetism

Electrodynamics relies on conservation laws and symmetry principles, also known from elementary particle physics. These symmetries are incorporated in the gauge theory and related action principle. Nevertheless, the variational principle is not the unique way to formally derive the electromagnetic theory. In the classical framework, we review the axiomatic approach developed by Friedrich W. Hehl and collaborators [37, 38, 39, 41] that use specific physical postulates and mathematical methods, namely, the calculus of differential forms but also integration theory, the Poincaré lemma and the Stokes theorem in the context of tensor analysis in 3-d space. There are two related ways of deriving Maxwell's theory with these tools: the first is based on integration theory and the second on the exterior calculus of differential forms. These approaches make clearer the geometrical significance of the fundamental electromagnetic quantities and their relations. Both methods rely on four basic physical principles or postulates. In the language of forms, the first three axioms enable to express electrodynamics in a pre-metric way. We will also present the 4-dimensional formalism using forms in which the most general field equations are completely pre-metric, coordinate-free and covariant. The 3-dimensional representation is based on a foliation of the spacetime manifold, requiring a certain choice for a (3+1) splitting in spatial hyper surfaces and an orthogonal time direction.

The starting point for a formal derivation of Maxwell's theory comes in the form of the following four main axioms (postulates):

- Axiom 1: Charge conservation;
- Axiom 2: Magnetic flux conservation;
- Axiom 3: Lorentz force;
- Axiom 4: Linear constitutive (spacetime) relations.

These axioms allow us to obtain the principal aspects of the theory (here the ordering of the axioms of magnetic flux conservation and Lorentz force is interchanged, with respect to the one used by Hehl *et al.*). Charge conservation alone is the foundation for the inhomogeneous equations, the Gauss and Maxwell-Ampère laws. The homogeneous equations are derivable from magnetic flux conservation and the Lorentz force. The fourth axiom brings in the metric of spacetime, exposing clearly that there is an intimate connection between electromagnetism and spacetime geometry already at the foundational basis of classical electrodynamics.

Two additional axioms which we will not consider in the present work, related to the energy-momentum distribution of the electromagnetic field [42], are required for a macroscopic description of electromagnetism (in matter). These are the following: The specification of the energy-momentum distribution of the electromagnetic field by means of the energy-momentum tensor and the splitting of the total electric charge and currents in a bound or material component which is conserved and a free or external component.

Axiomatic structure of Maxwell theory

Preliminary methods in the 3d formalism: Integration theory and differential forms. An important link between integration theory and geometrical considerations follows from the fact that integration is required to yield invariant quantities under arbitrary coordinate transformations. In 3-dimensional space there are three basic geometrical possibilities for integration, i.e., along lines, 2-surfaces and volumes. Taking into consideration the way in which line, surface and volume elements transform under general coordinate transformations, it is possible to find the correct or natural line, surface and volume integrands which transform in a complementary way in order to give (geometrically) invariant results. One concludes that:

- Covectors (1-forms), α_i, are natural line integrands;
- Vector densities, β^j, are natural surface integrands;
- Scalar densities, ϱ, are natural volume integrands.

These quantities transform under arbitrary changes of coordinates $x^i \to x^{j'}$ according to

$$\alpha_{k'} = \alpha_i \partial_{k'} x^i, \quad \beta^{j'} = |J|^{-1} \beta^i \partial_i x^{j'}, \quad \varrho' = |J|^{-1} \varrho, \tag{2.1}$$

where $J = \det\left[\partial x^{k'}/\partial x^j\right]$ is the Jacobian of the coordinate transformation and $|J| = \sqrt{|g^{3d}|}$, where g^{3d} is the determinant of the 3-dimensional metric. Note that if one considers the Jacobian J in the above expressions instead of its modulus, the scalar and vector densities can change sign under parity transformations. In this case they are sometimes designated by pseudo-tensor densities.

The Poincaré lemma states under which conditions certain mathematical objects can be expressed in terms of derivatives of other objects (potentials). Consider the natural integrands α_i, β^j, ϱ of line, surface and volume integrals respectively. Let us assume that they are defined in open connected regions of 3-dimensional space. Then:

1. If α_i is curl free, it can be written as the gradient of a scalar function f,

$$\epsilon^{ijk}\partial_j\alpha_k = 0 \quad \Rightarrow \quad \alpha_k = \partial_k f. \tag{2.2}$$

2. If β^j is divergence free, it can be written as the curl of the integrand α_i of a line integral,

$$\partial_i\beta^i = 0 \quad \Rightarrow \quad \beta^i = \epsilon^{ijk}\partial_j\alpha_k. \tag{2.3}$$

3. The integrand ϱ of a volume integral (scalar density) can be written as the divergence of an integrand β^i of a surface integral (vector density),

$$\varrho = \partial_i\beta^i. \tag{2.4}$$

In the expressions above, ϵ^{ijk} corresponds to the completely antisymmetric Levi-Civita (pseudo) tensor. We are now ready to proceed with the physical postulates underlying the electromagnetic theory.

Charge conservation and Maxwell's inhomogeneous equations. With these mathematical methods it follows that charge densities as natural volume integrands are scalar densities ϱ and current densities as natural surface integrands are vector densities $\jmath^k$ and the postulate of charge conservation

$$\partial_t\varrho + \partial_i\jmath^i = 0, \tag{2.5}$$

allows us to define the electric and magnetic excitations as natural surface and line integrands respectively, obeying the inhomogeneous Maxwell equations. In fact, using the Poincaré lemma it follows that

$$\varrho = \partial_i D^i, \tag{2.6}$$

where D^i is a vector density, and therefore related to surfaces, thus deriving the Gauss law. On the other hand, substituting the above expression on charge conservation and using the Poincaré lemma we deduce

$$\partial_i\left(\partial_t D^i + \jmath^i\right) = 0 \quad \Rightarrow \quad \partial_t D^i + \jmath^i = \epsilon^{ijk}\partial_j H_k, \tag{2.7}$$

where H_k is a line integrand and the Maxwell-Ampère law is thus derived.

It should be clear that we have obtained the inhomogeneous equations and the electric and magnetic excitations from charge conservation and the Poincaré lemma without introducing the concept of force. Notice that since charge conservation is valid in microscopic physics, the same is true for the inhomogeneous equations and for the excitations.

Magnetic flux conservation, Lorentz force and Maxwell's homogeneous equations. There is some analogy between vortex lines in hydrodynamics and magnetic flux lines. Helmholtz' works on hydrodynamics enabled to conclude that vortex lines are conserved. Vortex lines that span a 2-surface can be integrated over to originate a scalar called circulation. Circulation in a perfect fluid is constant provided the loop enclosing the surface moves along with the fluid. Analogously, there is good experimental evidence that magnetic flux is conserved. In fact, it seems that at the microscopic level magnetic flux occurs in quanta and the corresponding magnetic flux unit is called flux quantum or fluxon. One fluxon carries $\Phi_0 = h/2e \approx 2,7 \times 10^{-15} Wb$, where e is the elementary charge and h is Planck's constant [44]. Single quantized magnetic flux lines have been observed in the interior of type II superconductors when exposed to sufficiently strong magnetic fields ([44], p. 131) and they can be counted. We therefore assume that magnetic flux, defined as

$$\Phi_{mag} \equiv \int\int B^i da_i, \tag{2.8}$$

is conserved, where the magnetic field B^i is a natural surface integrand and therefore, a vector density. The corresponding global continuity equation (analogous to charge conservation)

$$\partial_t \Phi_{mag} + \oint j_i^\Phi dx^i = 0, \tag{2.9}$$

allows us to define the magnetic flux current density j_i^Φ as a natural line integrand (covector). Applying Stokes' theorem

$$\oint j_i^\Phi dx^i = \int\int \epsilon^{ijk} \partial_j j_k^\Phi da_i, \tag{2.10}$$

locally we get,

$$\partial_t B^i + \epsilon^{ijk} \partial_j j_k^\Phi = 0. \tag{2.11}$$

On the other hand, force is integrated through lines to yield work, therefore force is a natural line integrand, i.e., a covector f_i. The Lorentz force postulate

$$f_i = q(E_i + \epsilon_{ijk} u^j B^k), \tag{2.12}$$

implies that the electric field is also a line integrand, i.e., a covector E_k. Now, since j_k^Φ and the electric field have the same physical dimensions[3] and geometrical properties, it is plausible to make the identification $j_k^\Phi = E_k$ (in accordance to the Lenz rule) and recover the Faraday law, which expresses magnetic flux conservation

$$\partial_t B^i + \epsilon^{ijk} \partial_j E_k = 0. \tag{2.13}$$

Taking the divergence of this expression (remembering that the divergence of a curl is zero), we obtain

$$\partial_i (\partial_t B^i) = 0, \tag{2.14}$$

and taking into account the Poincaré lemma, we can define the magnetic charge scalar density $\varrho_{mag} \equiv \partial_i B^i$, and therefore conclude that

$$\partial_t \varrho_{mag} = 0, \tag{2.15}$$

[3]In S.I. units, flux/(time×length) correspond to V/m.

which rigorously states that the magnetic charge must be static in accordance to magnetic flux conservation. Now, since ϱ_{mag} is a scalar density, under a general coordinate transformation $\{x, t\} \to \{x', t'\}$, it transforms according to $\varrho'_{mag} = |J^{-1}|\varrho_{mag}$. Therefore, in general, $\partial_{t'}\varrho'_{mag} \neq 0$ and so we set it to zero, i.e.,

$$\varrho_{mag} = 0 \quad \Rightarrow \quad \partial_i B^i = 0, \tag{2.16}$$

which expresses the Gauss law for magnetism and the absence of magnetic monopoles. In other words, magnetic flux conservation is incompatible with magnetic monopoles.

We conclude that the electric field E and magnetic excitation H are both related to lines while the magnetic field B and electric excitation D are both related to surfaces, i.e.,

- Electric field: 1-form (covector); line integrand, E_i;
- Magnetic field: vector density; surface integrand, B^j;
- Electric excitation: vector density; surface integrand, D^j;
- Magnetic excitation: 1-form (covector); line integrand, H_i.

Constitutive relations. Maxwell's equations constitute 8 equations (6 of which are dynamical) with 12 unknown quantities. In order to solve these equations we need to postulate a form for the so-called constitutive relations, $D = D(E, B)$ and $H = H(E, B)$, between the excitations and the electric and magnetic fields. With these relations, Maxwell's equations reveal two independent electromagnetic degrees of freedom. The solutions to these equations and all the associated electromagnetic phenomena depend crucially on our assumption regarding the nature of the constitutive relations.

In order to relate the electromagnetic fields and their excitations there are two points to consider. One is geometrical and the other is physical. First, the physical consideration has to do with the dimensions involved and it is at this level that the electric permittivity and magnetic permeability tensors are introduced, characterizing the material medium or empty space. These so-called electromagnetic properties of vacuum are usually assumed to be homogeneous and isotropic, represented by diagonal matrices with equal constant diagonal components. This assumption is not necessarily the unique choice and one could argue that the permittivity and permeability tensors reflect electromagnetic properties of spacetime (or of electro-vacuum) and should reflect the spacetime symmetries. We will come back to this point. In second place, geometrically, the constitutive relations imply that one needs to relate 1-forms (co-vectors) to vector densities (which can be mapped to 2-forms). It follows that the spacetime metric enables to realize the required link. In fact, $g^{ij}\sqrt{|g|}$ transforms like a density and maps a covector (1-form) to a vector density.

With these considerations, we will assume local, linear, homogeneous and isotropic constitutive relations in vacuum without mixing electric and magnetic properties, through the following expressions

$$D^j = \varepsilon_0\sqrt{|h|}h^{ij}E_i, \quad E_i = \frac{1}{\varepsilon_0\sqrt{|h|}}D^j h_{ij}, \quad B^j = \mu_0\sqrt{|h|}h^{ij}H_i, \quad H_i = \frac{1}{\mu_0\sqrt{|h|}}B^j h_{ij}, \tag{2.17}$$

where h_{ij} is the 3-dimensional metric induced on the 3-dimensional hyper surfaces, and ε_0,μ_0 are the electric permittivity and magnetic permeability of classical vacuum, respectively.

Relation to differential forms (3d formalism). The postulates of charge conservation, magnetic flux conservation, Lorentz force and constitutive relations can be clearly expressed using forms leading to the same fundamental geometrical and physical conclusions. In this formalism, charge density ϱ is a 3-form, current density j a 2-form, electric fields E are 1-forms (related to lines), magnetic fields B are 2-forms (related to surfaces), the electric excitation D is a 2-form and the magnetic excitation H is a 1-form.

Considering a (3+1) foliation of spacetime, the electromagnetic quantities are defined on the 3-dimensional hyper surfaces. For a given foliation, the set of Maxwell's equations

$$dD = \varrho, \qquad dH = j + \partial_t D, \qquad dE + \partial_t B = 0, \qquad dB = 0, \tag{2.18}$$

are fully general pre-metric, covariant equations, coming directly from charge conservation, magnetic flux conservation and the Lorentz force. As previously mentioned, to solve this set of equations one requires the (spacetime) constitutive relations relating the electric and magnetic fields to the excitations. Assuming linear, homogeneous and isotropic constitutive relations, without mixing electric and magnetic properties, these relations, in the language of forms are achieved via the Hodge star operator in 3-dimensional space which maps k-forms to $(d-k)$-forms, where d is the dimension of the manifold under consideration (see appendix A.1), and are given by

$$D = \varepsilon_0 \star E, \qquad B = \mu_0 \star H, \tag{2.19}$$

which introduces the spacetime metric

$$D_{jk} = \varepsilon_0 \sqrt{|h|}\epsilon_{ijk} h^{im} E_m, \quad E_i = \frac{1}{2\varepsilon_0\sqrt{|h|}}\epsilon_{ijk} D_{mn} h^{mj} h^{nk}, \tag{2.20}$$

$$B_{jk} = \mu_0 \sqrt{|h|}\epsilon_{ijk} h^{im} H_m, \quad H_i = \frac{1}{2\mu_0\sqrt{|h|}}\epsilon_{ijk} B_{mn} h^{mj} h^{nk}. \tag{2.21}$$

We will come back to these important relations in the 4-dimensional formalism and in the final section of this work since, as previously mentioned, they reveal a fundamental connection between electromagnetic fields, the electromagnetic properties of vacuum and the conformal (causal) structure of spacetime.

The 3-dimensional formalism presented here using integration theory and linear forms is completely self-compatible, revealing in a clear way the geometrical meanings implicit to the electromagnetic quantities and their relations. In particular, we can map the electromagnetic 2-forms to the associated vector densities according to

$$D^a = \frac{1}{2}\epsilon^{abc} D_{bc}, \quad D_{ab} = \epsilon_{abc} D^c, \quad B^a = \frac{1}{2}\epsilon^{abc} B_{bc}, \quad B_{ab} = \epsilon_{abc} B^c. \tag{2.22}$$

With the introduction of the constitutive relations, the axiomatic approach to classical electrodynamics is completed.

4-dimensional formalism in differential forms

In the 4-dimensional formalism we will be using the constant c in some expressions involving the components of tensor quantities. This constant is required from dimensional analysis. As will be discussed the identification of this constant with the constant velocity of electromagnetic wave propagation, $c = 1/\sqrt{\varepsilon_0 \mu_0}$, according to Maxwell theory, presuppose the assumption of linear, local, homogeneous and isotropic electromagnetic-spacetime constitutive relations. Under this hypothesis or postulate, the spacetime conformal and metric structures associated to electrodynamics are those of Minkowski-Lorentz-Poincaré spacetime, and special relativity follows. A more general postulate for the constitutive relations does not imply that spacetime paradigm. At this point we will not be concerned with this issue and will come back to more general constitutive relations further on.

Charge conservation and the inhomogeneous equations. In the 4-dimensional formalism, charge conservation can be expressed by saying that the total (net) flux of electric charge through any closed 3-surface is zero. In order to integrate along a 3-surface we then require a 3-form electric charge current density J, which is related to the usual 4-current vector j^λ via the Hodge star product of the corresponding 1-form $j = j_\alpha dx^\alpha$

$$J = \star j, \qquad J = \frac{1}{3!}\epsilon_{\alpha\beta\gamma\lambda} j^\lambda \sqrt{-g} dx^\alpha \wedge dx^\beta \wedge dx^\gamma, \tag{2.23}$$

therefore

$$j_{123} = j^0\sqrt{-g} = \rho c\sqrt{-g}, \qquad j_{230} = j^1\sqrt{-g}, \qquad j_{301} = j^2\sqrt{-g}, \qquad j_{012} = j^3\sqrt{-g},$$

and $\jmath^\lambda \equiv \sqrt{-g} j^\lambda$ is a vector density. We can then write

$$\oint_{3d} J = \int_{4d} dJ = 0, \tag{2.24}$$

where we have applied the fundamental theorem of the exterior calculus of forms, namely, the Stokes theorem. The second equality is valid for any compact 4-dimensional volume enclosed by the 3-surface. Therefore we arrive at $dJ = 0$, which expresses charge conservation locally. Now, since J is a 3-form and $dd = 0$, it can be expressed by the exterior differential of a 2-form

$$dJ = 0 \quad \Rightarrow \quad d\mathcal{H} = J. \tag{2.25}$$

Therefore, in the language of forms, it is clear that charge conservation is at the foundation of Maxwell's inhomogeneous equations which in this formalism are fully general, coordinate-free, pre-metric and covariant equations. In components we have

$$\partial_{[\alpha}\mathcal{H}_{\beta\gamma]} = \epsilon_{\alpha\beta\gamma\lambda}\jmath^\lambda, \tag{2.26}$$

Therefore, with the following definitions

$$\mathcal{H}_{0i} \equiv -H_i, \qquad \mathcal{H}_{ij} \equiv \epsilon_{ijk}D^k c = D_{ij}c, \tag{2.27}$$

the most general expressions for the Gauss and Maxwell-Ampère laws in component form are

$$\partial_i D^i = \varrho, \qquad \partial_0 D^k c + \epsilon^{ijk}\partial_i H_j = \jmath^k, \tag{2.28}$$

where $\varrho \equiv \sqrt{-g}\rho$ and ρ is the charge density. Indeed, assuming the validity of a local foliation of spacetime, the electromagnetic excitation 2-form can be written in terms of its spatial and temporal parts establishing a link with the 3-dimensional formalism previously discussed

$$\mathcal{H} = H \wedge dx^0 + Dc. \tag{2.29}$$

Magnetic flux conservation and the homogeneous equations. Magnetic flux conservation can be expressed by

$$\oint_{surface} F = 0 \quad \Rightarrow \quad \int_{volume} dF = 0, \tag{2.30}$$

where F is the Faraday 2-form, obeying the homogeneous equations

$$dF = 0 \quad \Rightarrow \quad F = dA, \tag{2.31}$$

and A is the electromagnetic potential 1-form. The magnetic flux conservation is at the foundation of the homogeneous equations which also naturally follow as a Bianchi identity, resulting from the derivation of the potential twice. Using the spacetime foliation we can write

$$F = dx^0 \wedge E\frac{1}{c} - B. \tag{2.32}$$

Homogeneous and isotropic constitutive relations. In this formalism the linear, local, homogeneous and isotropic constitutive relations can be expressed through the Hodge star operator

$$\mathcal{H} = \mu_0^{-1} \star F. \tag{2.33}$$

With this assumption or postulate the inhomogeneous equations can then be written by

$$d(\star F) = \mu_0 J, \tag{2.34}$$

or in terms of the potential by

$$d \star dA = \mu_0 J. \tag{2.35}$$

In component form the above constitutive relations are

$$\mathcal{H}_{\mu\nu} = \frac{1}{2\mu_0}\sqrt{-g}g^{\alpha\lambda}g^{\beta\gamma}\epsilon_{\mu\nu\lambda\gamma}F_{\alpha\beta}. \tag{2.36}$$

The factor $\sqrt{-g}g^{\alpha\lambda}g^{\beta\gamma}$ is conformally invariant. Therefore, one arrives at

$$D^j = \sqrt{-g}\Big[\varepsilon_0 E_k \left(g^{0j}g^{k0} - g^{kj}g^{00}\right) - c^{-1}\mu_0^{-1}\frac{1}{2}B_{mn}(g^{mj}g^{n0} - g^{m0}g^{nj})\Big], \tag{2.37}$$

$$H_k = \sqrt{-g}\mu_0^{-1}\left[\frac{1}{2}B_{ij}\epsilon_{krs}g^{ir}g^{js} - c^{-1}E_j\epsilon_{krs}g^{0r}g^{js}\right], \tag{2.38}$$

again with $c = \sqrt{1/\varepsilon_0\mu_0}$. It is clear that assuming linear constitutive relations of the form (2.33), we are not excluding a mixing between electric and magnetic quantities, in contrast to the expressions in Eq. (2.19). In fact, according to the expressions above, this mixing will occur whenever the metric has off-diagonal elements involving the time-space components (as in gravitomagnetic astrophysical applications with axial symmetry, see [2]). For a diagonal metric, we have

$$D^j \sim \sqrt{-g}\varepsilon_0 E_j g^{jj}g^{00}, \quad H_k \sim \sqrt{-g}\mu_0^{-1}B^k g_{kk}, \tag{2.39}$$

where no contraction (summation rule) is assumed in the expressions above and we have used the fact that $\epsilon_{krs}\epsilon^{rsf} = 2\delta_k^f$. In Minkowski spacetime we get the familiar relations in vacuum, which assume homogeneity and isotropy.

Action principle. Maxwell's equations for the fields E and B can be derived from the following 4-form

$$S = \int F \wedge \mathcal{H} + \int J \wedge A, \tag{2.40}$$

assuming a specific set of constitutive relations between $\mathcal{H} = (H, D)$ and $F = (E, B)$. For the homogeneous and isotropic linear constitutive relations in Eq. (2.33) we get the usual free field action of electromagnetism,

$$S_{free} = \frac{1}{\mu_0} \int F \wedge \star F, \tag{2.41}$$

normally presented in relation to the gauge approach. It is clear that the constitutive relations (which imply the conformally invariant part of the metric) are implicit in the usual gauge approach to the (inhomogeneous) field equations.

More general linear constitutive relations

The Maxwell equations together with the spacetime relations, constitute the foundations of classical electrodynamics. These laws, in the classical domain, are assumed to be of universal validity. Only if vacuum polarization effects of quantum electrodynamics are taken into account or hypothetical non-local terms should emerge from huge accelerations, axiom 4 can pick up corrections yielding a non-linear law (Heisenberg-Euler electrodynamics [53]) or a non-local law (Volterra-Mashhoon electrodynamics [54]), respectively. In this sense, the field equations are more general than the constitutive spacetime relations, however, the latter are not completely untouchable. We may consider them as constitutive relations for spacetime itself, as discussed below.

As previously mentioned, the constitutive relations in vacuum not only introduce the spacetime metric but also the vacuum electromagnetic properties, via the electric permittivity and magnetic permeability tensors for e.g. The assumption of homogeneous and isotropic relations is based on the assumption that these vacuum electromagnetic properties are spatially homogeneous and isotropic. Can these assumptions be relaxed? It is clear that if only homogeneity is abandoned, then the velocity of light in vacuum can in principle vary with the spacetime point. In general, the principle of (local) conformal invariance, which guarantees the invariance of the casual structure of spacetime (locally), does not require homogeneity or isotropy for the speed of light in vacuum. As previously said, one argument in favour of letting go of the assumption of homogeneity and isotropy for the electromagnetic properties of vacuum is that these quantities could characterize the "electro-vacuum" which can be intimately related to (or identified with) the physical spacetime geometry. Therefore, these quantities should be related to the symmetry properties of spacetime. *In this sense, it seems more natural to assume that the symmetry properties of the tensors ε_{ij} and μ_{km} in vacuum follow the spacetime isometries.*

This reasoning could come from a self-compatible interpretation of the coupled Einstein-Maxwell equations. Electromagnetic fields affect spacetime geometry and this geometry affects the propagation of the fields. In fact, in the spirit of general relativity, the metric is not *a priori* given, it depends on the local energy-momentum content of physical fields. Therefore, spacetime symmetries are also not *a priori* given, they must be considered locally for each physical scenario. Why should the properties of vacuum, such as the electric permittivity and magnetic permeability be *a priori* given, in particular, why should these be homogeneous and isotropic for axially or spherically symmetric spacetime, for e.g.? For example, in spherical symmetric cases like the Schwarzschild solution, according to the interpretation here proposed, the speed of light in vacuum could have a dependence with the radial coordinate and this result could be tested experimentally (see chapter 6). A very simple expression for the linear constitutive relations (in vacuum) assuming a local (3+1) foliation can be given by

$$D^i = \sqrt{|h|}(\varepsilon_0)^{ij} E_j, \qquad B^i = \sqrt{|h|}(\mu_0)^{ij} H_j. \tag{2.42}$$

With these expressions we are assuming locality, linearity and a non-mixing between electric and magnetic components but without forcing the assumptions of homogeneity and isotropy. These relations will affect the inhomogeneous equations (2.28). In particular, for physical conditions where the spacetime metric has spherical symmetry, according to the interpretation here suggested, the permittivity and permeability tensors follow the spacetime isometries and therefore become diagonal with equal components (isotropy) but with a radial dependence on position (inhomogeneity and spherical symmetry), i.e., $(\varepsilon_0)^j_k = \varepsilon_0(r)\delta^j_k$, $(\mu_0)^j_k = \mu_0(r)\delta^j_k$. When $(\varepsilon_0)^j_k = \varepsilon_0\delta^j_k$ and $(\mu_0)^j_k = \mu_0\delta^j_k$ we recover the homogeneous and isotropic relations.

Following the approach of Hehl and Obukhov [41, 44], the most general expression for linear (local) relations in the 4-dimensional formalism is the following

$$\mathcal{H}_{\mu\nu} = \chi^{\alpha\beta}_{\mu\nu} F_{\alpha\beta}, \tag{2.43}$$

where the tensor $\chi^{\alpha\beta}_{\mu\nu}$ has in general, 36 independent components and it can be decomposed into its irreducible pieces, where the principal part is related to the relations in Eq. (2.36). The constitutive equations in matter are more complicated (see [44, 55, 56]) and it would be appropriate to derive them, using an averaging procedure, from a microscopic model of matter. For instance, this lies within the subject of solid state or plasma physics. Hehl and Obukhov arrived at the relations for a general linear magnetoelectric medium [39]. Such type of vacuum constitutive relations can also be applied to vacuum with linear electromagnetic properties. *This topic requires further investigation since these relations can be viewed as relations for spacetime itself implying a deep connection between physical properties of (classical) vacuum and spacetime (suggesting or reinforcing the idea of spacetime physicality, i.e., spacetime with well defined physical ontology).* From a variational point of view the equations for the permittivity and permeability tensors in Eq. (2.42) or for the tensor $\chi^{\alpha\beta}_{\mu\nu}$ in Eq. (2.43), could in principle be obtained from an appropriate action corresponding to a tensor-vector electromagnetic theory, as long as dissipation effects are disregarded.

To conclude this section we emphasize the fact that from a foundational point of view, the electromagnetic field equations derive from conservation laws and are more fundamental than a metric or affine structure of spacetime. In order to study electromagnetic phenomena in spacetime one needs to introduce the constitutive relations in the theory which require the (conformal part of the) spacetime metric. By fixing the spacetime geometry to be pseudo-Riemann, one arrives at electrodynamics in curved spacetime and by fixing the constitutive relations to be linear, local, homogeneous and isotropic one obtains Maxwell's field theory.

2.1.2 Classical electrovacuum: A first glimpse on the primacy of the conformal structure

The electromagnetic field equations based on the well-established postulates of charge and magnetic flux conservation and compatible with the Lorentz force are completely general and coordinate free, without requiring the metric or even the affine structure of the spacetime manifold [37, 38, 39, 40, 41]. What particularizes these equations for a given spacetime geometry are the constitutive relations between the field strengths and the excitations. Since, as mentioned, these relations in vacuum can be viewed as constitutive relations for the spacetime itself and necessarily introduce the conformal part of the metric [39], the causal structure of spacetime is intertwined with electrodynamics at the very foundational level. If one modifies the constitutive relations new field equations, and consequently, new predictions for the electromagnetic phenomena follow. These relations are implicit in the action (or gauge) approach, therefore different relations imply different actions as in the cases of Heisenberg-Euler non-linear electrodynamics [53] or Volterra-Mashhoon non-local electromagnetism [54]. Assuming linear and local relations do not necessarily require homogeneity and isotropy. The electric permittivity and magnetic permeability tensors in vacuum are required inside the constitutive relations, in order to relate the physical dimensions of the field strengths and the excitations. Now, whereas in the laboratory the homogeneous and isotropic constitutive relations might seem to be valid (by measuring electric and magnetic fields through their effects on charges and testing the usual expressions for the inhomogeneous equations), it is not proven that such relations remain unchanged in the presence of strong gravitational fields. We suggest that the assumptions of homogeneity and isotropy might be inappropriate for physical situations in which the spacetime isometries transgress spatial homogeneity and/or isotropy [1].

In the first section, we briefly addressed the issue that since these relations can be viewed as constitutive relations for the spacetime itself (or vacuum), these tensors can be interpreted to characterize the electromagnetic properties of spacetime [39, 45, 46, 47], or of what can be called the electro-vacuum. In this sense, two different, although related, issues deserve some debate. The first has a geometrical tone coming from the idea that if spacetime isometries are not *a priori* given, but must be considered locally for each astrophysical or cosmological scenario, then the same is expected for the symmetry properties of the permittivity and permeability tensors. The spacetime symmetries should be reflected in the components of these tensors, which in general, depend on the spacetime coordinates. This goes along with the line of reasoning of general relativity according to which, electromagnetic fields gravitate, affecting spacetime geometry, and propagate according to a law that depends on the local causal structure of spacetime. In this sense, without abandoning local conformal invariance, the *a priori* assumption of homogeneity and isotropy for the permittivity and permeability tensors can be abandoned. Consequently, according to these ideas, the velocity of light, determined by these electromagnetic properties of spacetime (or electrovacuum) is predicted to be isotropic but inhomogeneous for spherically symmetric geometries (having a radial dependence), and inhomogeneous and anisotropic for axially symmetric cases (such as the cases of rotating relativistic stars or black holes). These predictions might be tested experimentally. In chapter 6 we will come back to these hypothesis in the context of possible applications in relativistic astrophysics.

The second consideration that deserves a careful analysis is related to the idea that the physical properties of vacuum, spacetime geometry and electromagnetism seem to be deeply related as expressed in the constitutive relations. Operationally, these relations are required for the system of field equations to be complete and solvable, in principle. From the point of view of physical ontology these relations reinforce the idea of spacetime endowed with well-defined physical properties. Technically, the excitations are potentials for the charge and current distributions, as can be inferred using differential forms, and can be viewed as some sort of extended version of the so called sources, a fact that is clear from dimensional analysis. Therefore, physically the only way the fields can be causally linked to the charge and current distributions, according to the dynamical (inhomogeneous) equations, is via the constitutive relations that introduce the conformal spacetime structure and the electromagnetic permittivity and permeability tensors, for e.g. The link between fields and sources is achieved via physical spacetime and to introduce the notion of vacuum here is somehow unnecessary if one accepts the idea that the spacetime manifold has a well-defined physical ontology. This notion is somehow reinforced according to the idea that in GR spacetime is causally linked to mass-energy fields, becoming curved and affecting the propagation of physical fields, and it is possible to define the energy-momentum of the gravitational spacetime geometrodynamics. On the other hand, according to the constitutive relations, strictly speaking it is the conformal part of spacetime geometry that might be said to have electromagnetic properties (we will come back in this topic, in chapters 3 and 8).

One concludes that, in a very deep sense, the constitutive relations, are not only a technical detail of the electromagnetic theory (where the non-trivial cases are expected only for extraordinary non-linear or non-local effects, for example). These relations bring forward the debate on the very nature of space and time, of physical vacuum and its relations with electrodynamics.

2.2 Principle of Relativity revisited

2.2.1 Extending the (rigid) Lorentz symmetries to the conformal group.

One of the prime reasons for the interest in the conformal group is that it is perhaps the most important of the larger groups containing the Poincaré group

(A. O. Barut, 1985)

In the previous section we briefly explored the idea that the electromagnetic properties of physical vacuum should follow the spacetime isometries. This idea reinforces the debate and research about the deep relation between spacetime, electromagnetic fields, gravity and the nature of vacuum. In particular, the conformal (casual) structure of spacetime is fundamentally connected to electromagnetism in the constitutive relations. The electromagnetic properties of vacuum and the conformal part of the metric appear in these relations between the fields and excitations. Therefore, in principle the electromagnetic properties of vacuum can be connected to the spacetime isometries and more specifically to the local conformal structure, as it is these properties of vacuum that determine the propagation properties of electromagnetic fields in spacetime. Following this reasoning one can choose to reduce the number of concepts in an attempt to unify different approaches to the same problem, by making the identification between the classical (electro) vacuum and physical spacetime. In this sense one can speak about the electromagnetic properties of the "spacetime medium".

According to the ideas on pre-metric electrodynamics reviewed in the previous section, the conformal structure of spacetime becomes a more fundamental concept than the metric structure. This can suggest the change of paradigm in spacetime physics according to which spacetime is not absolute but rather it is the local causal structure that is invariant. Within this conformal symmetry principle, all observers agree on the (local) light cone related to some spacetime point but spacetime distances between any two events can be different for different observers. This change of paradigm naturally implies, through the gauge approach (chapter 3), theories of gravity in which the (local) gauge symmetries are not Poincaré symmetries but rather those of the 15 parametric conformal group $C(1,3)$ which includes the Weyl group $W(1,3)$ and the special conformal transformations. Dilatations (which belong to the Weyl group, together with the Poincaré transformations) and special conformal transformations change the spacetime line element. As will be explained in chapter 3, the gauging (localizing) of this wider group naturally invites non-Riemann geometries with curvature, torsion and non-metricity tensors.

These ideas put forward the hypothesis that in the limit of global symmetries and in analogy to the special relativity limit of GR i.e, in the absence of gravity, instead of a Minkowski spacetime paradigm with global $P(1,3)$ symmetries, there should be rather an extended conformal geometry with global conformal symmetries. This would lead to a generalization of the principle of (special) relativity: *The principle of conformal relativity.*

Let us consider a spacetime with global conformal symmetry. All transformations within the C(1,3) group including spacetime translations T(4) and Lorentz transformations SO(1,3) (within the 10 parameter P(1,3) group), dilatations (1 parameter) and special conformal transformations (4 parameters), preserve the causal cone structure $ds^2 = 0$. The Weyl group (11 parameters) includes P(1,3) plus dilatations and as previously mentioned, both dilatations and special conformal transformations change the line element. Therefore, in this spacetime paradigm (with conformal symmetries) spacetime distances are not absolute. A principle of relativity on a spacetime with global conformal symmetry can be formulated:

The principle of conformal relativity: *The laws of physics are to be the same for all observers linked together by general conformal transformations.*

This class of observers and this principle of relativity already includes non-inertial (accelerated) observers, since two reference frames linked by a special conformal transformation are not inertial frames (see below)

The theory does not include gravity. To move from a principle of relativity towards a theory that includes gravity just like Einstein, one needs to gauge, i.e, localize the spacetime symmetry group, in this case the conformal group. What happens when we go from Minkowski to (pseudo) Riemann or Post-Riemann geometries, such as metric-affine? What are the assumptions related to causality or the metric structure? We will analyse these possibilities in chapter 3.

Let us consider then the group of (rigid) spacetime conformal transformations. For infinitesimal transformations one gets

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x), \qquad \xi^{\mu} = \varepsilon^{\mu} + w^{\mu}{}_{\nu}x^{\nu} + \rho x^{\mu} + c^{\mu}x^2 - 2x^{\mu}c \cdot x \tag{2.44}$$

where $x^2 = x^{\mu}x^{\nu}\eta_{\mu\nu}$ and $c \cdot x = c^{\mu}x^{\nu}\eta_{\mu\nu}$ and where the constant 15 parameters ε^{μ}, $w_{\mu\nu} = -w_{\nu\mu}$, ρ and c^{μ} correspond to translations, Lorentz transformations, dilatations and special conformal transformations, respectively. As it is clear, the special conformal transformations in Minkowski spacetime (M_4) are non-linear.

For a general field ϕ we can introduce the generators that act on the field

$$\delta\phi \equiv \phi'(x) - \phi(x) = \left(\frac{1}{2}M_{\mu\nu}w^{\mu\nu} + P_{\mu}\varepsilon^{\mu} + D\rho + K_{\mu}c^{\mu}\right)\phi(x), \tag{2.45}$$

and in the representation space of scalar fields, the generators are

$$M_{\mu\nu} = x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} \equiv L_{\mu\nu}, \quad P_{\mu} = -\partial_{\mu}, \quad D = -x \cdot \partial \tag{2.46}$$

corresponding to Lorentz transformations, 4-translations and dilatations of the Weyl group, with $x \cdot \partial = x^\mu \partial^\nu \eta_{\mu\nu}$, and

$$K_\mu = 2x_\mu x \cdot \partial - x^2 \partial_\mu, \tag{2.47}$$

corresponding to special conformal transformations. With these generators one easily proves the (non-Abelian) Lie algebra of $C(1,3)$, including the Lie algebra of the Weyl subgroup

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - (\lambda \leftrightarrow \rho), \quad [M_{\mu\nu}, P_\lambda] = \eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu, \tag{2.48}$$

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, D] = 0, \quad [P_\lambda, D] = -P_\lambda, \quad [D, D] = 0 \tag{2.49}$$

together with the algebra

$$[M_{\mu\nu}, K_\lambda] = \eta_{\nu\lambda} K_\mu - \eta_{\mu\lambda} K_\nu, \quad [P_\mu, K_\nu] = 2(M_{\mu\nu} + \eta_{\mu\nu} D) \quad [D, K_\mu] = -K_\mu, \quad [K_\mu K_\nu] = 0, \tag{2.50}$$

due to the presence of special conformal transformations. For the general case of an arbitrary field ϕ the generators in (2.45) have the form

$$M_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}, \quad P_\mu = -\partial_\mu, \quad D = -x \cdot \partial + \Delta \tag{2.51}$$

and

$$K_\mu = 2x_\mu x \cdot \partial - x^2 \partial_\mu + 2(x^\nu \Sigma_{\mu\nu} - x_\mu \Delta) + \kappa_\mu, \tag{2.52}$$

where $\Sigma_{\mu\nu}$, Δ and κ_μ are matrix representations of $M_{\mu\nu}$, D and K_μ. Notice that $\Sigma_{\mu\nu}$ is the spin part of $M_{\mu\nu}$ (relevant for spinors) while $L_{\mu\nu}$ is the orbital part.

The finite conformal transformations for fields are

$$\phi' = G(w, a, \rho, c)\phi, \quad G(w, a, \rho, c) = e^{\frac{1}{2} w \cdot M + a \cdot P + \rho \Delta + c \cdot \kappa}, \tag{2.53}$$

while the finite conformal transformations of spacetime coordinates are given by

$$T(a) x^\mu = x^\mu + a^\mu \quad \Lambda(w) x^\mu = \Lambda^\mu_{\ \nu}(w) x^\nu, \quad D(\rho) x^\mu = e^\rho x^\mu, \tag{2.54}$$

and

$$K(c) x^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2c \cdot x + c^2 x^2}. \tag{2.55}$$

One can show that the special conformal transformations above can be understood as a composition of an inversion followed by a translation and another inversion, i.e, $K(c) x^\mu = (I \cdot T(-c) \cdot I)\, x^\mu$, where inversions are defined as $x'^\mu = -x^\mu / x^2$. In fact, the special conformal transformation can be written as $x'^\mu / x'^2 = x^\mu / x^2 - c^\mu$.

Let us recall now also the action of the Lorentz group on Dirac spinors ψ. In this case we obtain $\psi'(x') = S(w)\psi(x)$, where w represents the parameters of Lorentz transformations and S is a matrix with $S^{-1} \gamma^\mu S = \Lambda^\mu_{\ \nu}(w) \gamma^\nu$. Here γ^μ are the Dirac-Pauli matrices and the matrix for Lorentz transformations in the infinitesimal case is $\Lambda^\mu_{\ \nu} = \delta^\mu_\nu + w^\mu_{\ \nu}$, leading to $S(w) = 1 + \frac{1}{8} w^{\mu\nu} [\gamma_\mu, \gamma_\nu]$. For finite transformations one can write $\psi'(x') = S(w)\psi(x)$ with $S(w) = e^{w \cdot \Sigma / 2}$, with $\sigma_{\mu\nu} \equiv \frac{1}{4} [\gamma_\mu, \gamma_\nu] = \Sigma_{\mu\nu}$ identified as the generators of the Lorentz group in the spinorial representation. More generally, the finite Poincaré transformations on fields can be expressed as

$$\phi' = G(w,a)\phi, \quad G(w,a) = e^{\frac{1}{2}w\cdot M + a\cdot P}. \tag{2.56}$$

We also recall that in special relativity the Lorentz transformations allow us to arrive at the basic relation for time dilatation $\delta\tau = \gamma^{-1}\delta t$, where τ is the proper time and $\gamma \equiv (1 - v^2/c^2)^{-1/2}$. Then from this and the spacetime coordinates of a point particle $x = (c\tau, \vec{x}(\tau))$ moving along a curve parametrized by the proper time, the basic kinematic and dynamical quantities of relativistic mechanics are derived

$$u = (\gamma c, \gamma\vec{v}), \quad p = (\gamma mc, \gamma m\vec{v}), \quad f = (\dot{\gamma}\gamma mc, \gamma(\dot{\gamma}m\vec{v} + \gamma\vec{a})) = \left(\frac{\gamma}{c}\dot{E}, \gamma\dot{\vec{p}}\right), \quad E = \gamma mc^2 \tag{2.57}$$

where the dot stands for d/dt, while $\vec{v} = \dot{\vec{x}}$, $\vec{a} = \dot{\vec{v}}$ and $\vec{p} \equiv \gamma m\vec{v}$. From these quantities one constructs Lorentz invariant scalars, easily calculated in the proper frame of the particle, giving for instance the important relation $E^2 = p^2c^2 + m^2c^4$. Similarly, for conformal transformations, taking $x'^0 \equiv t'$, and the proper time $x^0 \equiv \tau$ (with $\vec{x} = 0$) one gets $t'(\tau) = \alpha^{-1}(x)\,(\tau + c^0\tau^2\eta_{00})$, with $\alpha(x) \equiv 1 + c\cdot x + c^2x^2$, therefore $dt'/d\tau = \alpha^{-1}(x)\,(1 + 2c^0\tau) - \alpha^{-1}(x)t'(\tau)\beta(x,u)$, where $\beta(x,u) \equiv d\alpha(x)/d\tau = c\cdot u + 2c^2u\cdot x$. With these expression and using $d/d\tau = (dt'/d)\tau d/dt$, one can obtain the generalized mechanics associated to the basic kinematical and dynamical objects (x'^μ; u'^μ; p'^μ; f'^μ). Moreover, for a field theory with the equations of motion $\delta\mathcal{L}/\delta\phi = 0$, the conserved currents for P(1,3) are

$$J^\mu = \frac{1}{2}w^{\alpha\beta}M^\mu_{\ \alpha\beta} - \varepsilon^\beta\tau^\mu_{\ \beta}, \tag{2.58}$$

where

$$\tau^\mu_{\ \beta} = \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} - \delta^\mu_\nu, \quad M^\mu_{\ \alpha\beta} = \left(x_\alpha\tau^\mu_{\ \beta} - x_\beta\tau^\mu_{\ \alpha}\right) - S^\mu_{\ \alpha\beta}, \tag{2.59}$$

are the canonical energy-momentum and angular momentum tensors and

$$S^\mu_{\ \alpha\beta} = -\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\Sigma_{\alpha\beta}\phi, \tag{2.60}$$

is the canonical spin tensor. Since the parameters of the transformation are constant, the conservation condition $\partial_\mu J^\mu = 0$ implies that

$$\partial_\mu\tau^\mu_{\ \beta} = 0, \quad \partial_\mu M^\mu_{\ \alpha\beta} = 0 \Leftrightarrow \partial_\mu S^\mu_{\ \alpha\beta} = \tau_{\alpha\beta} - \tau_{\beta\alpha}. \tag{2.61}$$

Therefore, and similarly, one expects a generalized relativistic mechanics to be derived from a spacetime paradigm with global conformal symmetries, and generalized (Noether) conserved currents including dilatation and special conformal currents. The conformally invariant current is given by

$$J^\mu = \frac{1}{2}w^{\alpha\beta}M^\mu_{\ \alpha\beta} - \varepsilon^\beta\tau^\mu_{\ \beta} - \rho D^\mu + c^\beta K^\mu_{\ \beta}, \tag{2.62}$$

where the canonical dilatation and special conformal currents are

$$D^\mu = x_\nu\tau^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}d\phi, \tag{2.63}$$

and

$$K^{\mu}_{\ \nu} = (2x_{\nu}x^{\lambda} - \delta^{\lambda}_{\nu}x^2)\tau^{\mu}_{\ \lambda} + 2\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}x^{\lambda}(\Sigma_{\nu\lambda} - \eta_{\nu\lambda}d)\phi - 2\sigma^{\mu}_{\ \nu}, \tag{2.64}$$

respectively, obeying $\partial_{\mu}D^{\mu} = 0$ and $\partial_{\mu}K^{\mu}_{\ \nu} = 0$. In the expressions above $\sigma^{\mu}_{\ \nu}$ is defined such that

$$V^{\mu} = \partial_{\lambda}\sigma^{\lambda\mu} \tag{2.65}$$

is verified, condition required for the invariance of the Lagrangian with respect to conformal transformations, with

$$V_{\mu} \equiv \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}}(\Sigma_{\mu\nu} - \eta_{\mu\nu}d)\phi. \tag{2.66}$$

The quantity d entering in the expressions above is called the scale dimension (of the field). It is related to the fact that if ϕ belongs to an irreducible representation of the Lorentz group, then the generators of dilatations acting in that space are given by

$$D = dI, \tag{2.67}$$

where I is the identity matrix. The scale dimension (which has some connection to the so-called Weyl dimension of Weyl rescaling) is a real number and the following definitions are usually taken [20]

$$d(fermions) = -3/2, \quad d(bosons) = -1. \tag{2.68}$$

Another requirement for conformal invariance of a field theory is that the scale dimension of the Lagrangian[4] is given by $d(\mathcal{L}) = -4$.

The algebra of the conformal group is isomorphic to $SO(2,4)$ which can be viewed as the group of special (pseudo) orthogonal transformations in a six dimensional pseudo-euclidean space M_6 with metric $\eta^{(6)} = (\eta_{\mu\nu}, -1, 1)$. The coordinate transformations of $SO(2,4)$ in M_6 are linear but the projection of these transformations into Minkowski spacetime M_4 give a non-linear realization of $SO(2,4)$ and therefore of $C(1,4)$.

Let us recall that the conformal group preserves angles and it is the most general group that preserves the causal structure of spacetime provided by the light cones. All conformal transformations preserve de light cone but dilatations and special conformal transformations change the line element according to

$$ds^2 \to \rho^2 ds^2, \quad ds^2 \to \sigma^2 ds^2, \quad \sigma^{-1} \equiv 1 + c^{\mu}x_{\mu} + c^{\mu}c_{\mu}x^2 \tag{2.69}$$

respectively.

In Einstein's special relativity, the principle of relativity and the constancy of the speed of light lead to Minkowski spacetime, Lorentz-Poincaré transformations and relativistic mechanics. Both these principles have been motivated by Maxwell's equations and the requirement of its invariance under transformations of coordinates between inertial observers. In fact, the Maxwell electromagnetic theory which strictly speaking presuppose linear, local, homogeneous and isotropic constitutive relations imply the constancy of the speed of light in vacuum, and this notion that the properties of classical vaccum are the same for all observers then leads to Minkowski spacetime and special relativity. At the foundational level however, as we discussed, the field equations do

[4]In the context of Weyl gauge theories of gravity, this condition excludes simple Lagrangians linear in the curvature scalar, unless an appropriate scalar field ϕ is introduced such that $d(\phi R) = -4$ [20]

not imply special relativity (with its restricted class of inertial observers), nor Lorentz-Poincaré symmetries nor Minkowski spacetime. In fact, the field equations call for a principal of conformal relativity.

One may argue that the total conformal invariance is not an actual complete symmetry in the physical world, but rather a broken symmetry into the Poincaré group. This can also motivate the construction of a unified gauge field theory including gravity and spacetime symmetries which incorporates a Higgs-like mechanism for breaking the complete conformal symmetry, for example and in particular this has to include the breaking of scale invariance. The exploration of gauge theories of gravity then provide a natural framework for symmetry breaking phase transitions in the early Universe. Such breaking of scale invariance is plausibly connected to the emergence of well-defined physical scales for mass-energy and physical constants (Lorentz- Poincaré invariants). Nevertheless, even if conformal symmetry (and scale invariance) is not an exact symmetry in nature, the conformal structure of spacetime can be considered to be more fundamental than the full metric structure and the notion of spacetime (metric) absoluteness, being the result of a broken symmetry of cosmological nature.

2.3 Mach's principle revisited

2.3.1 Inertia, spin and spacetime with global conformal symmetry.

In Newton's spacetime space is absolute and separate from absolute time. Inertia is defined with respect to an hypothetical absolute reference frame which allows to define the class of inertial observers. The *ether* was the natural candidate for such absolute frame. In Minkowski-Poincaré-Einstein spacetime, space is relative, time is relative, but unified 4-dimensional spacetime is absolute, the line element is an invariant quantity under the rigid (global) Poincaré group of coordinate transformations. The inertial property of matter, i.e, inertial mass is defined with respect to absolute spacetime. In this spacetime paradigm, one can tell if a reference frame is inertial or non-inertial by comparing it with an (hypothetical) absolute spacetime frame. Therefore, even in vacuum Newton's rotating bucket of water would still manifest a curvature on the surface of water since inertial effects are defined with respect to absolute classical spacetime-vacuum. With an absolute concept that allows the definition of inertial frames, inertial mass is then seen as an intrinsic property of physical bodies. According to Mach's ideas of having physics described by relative notions, on the other hand, inertial properties of any physical system can only be defined with respect to the distribution of all matter in the Universe. In this Mach's principle, inertia is not an intrinsic property of physical particles, bodies, systems, but rather, a relative concept, manifesting through interdependence between all matter. Accordingly, by removing all the remaining matter in the Universe one cannot define the inertial properties of an object (in vacuum). It is even meaningless to consider an isolated system, because the so called properties of any physical system arise through interdependence with all matter. In this line of reasoning the idea of spacetime absoluteness is naturally questioned. Following Mach's principle one is led to abandon what is usually called the physical properties of matter as being intrinsic and therefore spacetime absoluteness as well. In fact it also goes the other way around, *if one lets go absolute spacetime one easily abandons the idea that inertia is intrinsic to objects*. In this non-absolute spacetime Newton's rotating bucket of water would manifest no curvature on the surface of its water, if it was in vacuum. If there is no matter surrounding and no absolute frame, how could one even define inertial effects?

A spacetime with global conformal symmetries is one which has no absolute metric structure and is compatible therefore with Mach's ideas. On the other hand one does not completely abandon the idea of invariances in physics, on the contrary, one follows the spirit of Einstein by enquiring not "what is relative?", but rather "what is invariant?". The answer, in this case is not, the spacetime metric and all Lorentz invariant quantities, but rather, causality, i.e, the conformal structure and all conformally invariant quantities. Let us recall that Mach's ideas deeply influenced Einstein in the construction of General Relativity. In fact, the interpretation of his theory when it reached its final stage, moved away from Mach's ideas but not completely. Indeed, the post-Riemannian spacetime of GR inherits from Minkowski spacetime the absoluteness of the metric, therefore the inertial frame is locally defined with respect to absolute spacetime. But on the other hand, the local spacetime metric depends on the local content of matter-energy, therefore there is a reminiscence of Mach's principle. One says the theory of GR is partially Machian.

Coming back to the physical properties of matter, in more technical terms, Wigner's "universally valid" mass-spin classification of particles of the standard model rests deeply on the Poincaré symmetry properties of Minkowski's spacetime, through the Casimir operators. Therefore, in a fundamental way, the mass-spin classification assumes absolute spacetime. The irreducible unitary representations of the Poincaré group can be characterized by the eigenvalues of the Lorentz-invariant Casimir operators, P^2, W^2 where P is the 4-momentum operator of quantum theory and W is the Pauli-Lubanski pseudovector $W^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}J_{\nu\alpha}P_\beta$, with $J^{\mu\nu}$ being the relativistic angular momentum tensor operator $J^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$. The eigen values of the Casimir operators can characterize the mass and spins of all the possible states in a representation space of a field theory in Minkowski spacetime.

The ideas discussed in section 2.1 suggest that the electromagnetic properties of spacetime-vaccum, the electric permittivity and magnetic permeability are not *a priori* constants, but rather tensor fields. In fact, the premetric approach in electrodynamics allows to clarify a simple classification of the so called fundamental constants into two classes: Lorentz-Poincaré constants and diffeomorphism invariant constants [57]. The velocity of light follows into the first class which is less general. In the context of the axiomatic program to electrodynamics, it follows from a specific choice for local, linear, homogeneous and isotropic constitutive relations. Now, at the fundamental level these electromagnetic quantities, or so called coupling constants, are not constants but fields and the description of coupling constants in field theories as being (scalar, tensor, vector) fields brings forward a revitalization of some of Mach's considerations. Mach's principle can be compatible with electrodynamics being conformally invariant, the spacetime having fundamental conformal symmetries, the abandoning of metric at the fundamental level and of spacetime absoluteness, the scalar-tensor theories and its generalizations in modified theories of gravity and gauge theories of gravity beyond the Poincaré group. Again, the idea that beyond Poincaré symmetries are not exact in nature, or that metric absoluteness of the Minkowski paradigm seem to be locally validated experimentally and the idea of empirically established (Lorentz-Poincaré invariant) physical constants and natural mass-energy scales, seem to suggest symmetry breaking mechanisms, possibly of cosmological origin.

Chapter 3

Towards a new spacetime paradigm II: Gauge theories of Gravity

Gravity can be formulated as a gauge theory by combining symmetry principles and geometrical methods in a consistent mathematical framework. The gauge approach to gravity leads directly to non-Euclidean, post-Riemannian spacetime geometries, providing the adequate formalism for metric-affine theories of gravity with curvature, torsion and non-metricity. In this chapter, we analyse the structure of gauge theories of gravity and consider the relation between fundamental geometrical objects and symmetry principles as well as different spacetime paradigms. We review the MAG formalism and give a special attention to Poincaré gauge theories of gravity, their field equations and the Noether conserved currents which are the sources of gravity. We then discuss several topics of the gauge approach to gravitational phenomena, namely, quadratic Poincaré gauge models, the Einstein-Cartan-Sciama-Kibble theory, the teleparallel equivalent of general relativity, quadratic metric-affine Lagrangians, non-Lorentzian connections, and the breaking of Lorentz invariance in the presence of non-metricity. We also highlight the probing of post-Riemannian geometries with test matter. Finally, we briefly discuss some perspectives regarding the pre-metric approach to gravity and its relation to the thesis of the primacy of the conformal structure over the metric structure, establishing a bridge with the ideas discussed in chapter 2. This chapter is inspired by the work in [7].

3.1 Gauge theories of Gravity and Post-Riemann geometries

The success of Einstein's General Theory of Relativity (GR) to describe the gravitational interaction is quite remarkable. As mentioned in the introduction, GR has passed all tests performed so far: Solar System observations and binary pulsars [10], stellar orbits around the central galactic black hole [11], gravitational waves (GWs) from coalescing compact objects (black holes and neutron stars) [13, 15, 16, 14], or the indirect observation of the black hole horizon with the Event Horizon Telescope [17], among others. At the same time, it provides us with the observationally valid framework for the standard cosmological paradigm when supplemented with the (cold) dark matter and dark energy hypothesis [18].

Soon after General Relativity (GR) theory, Weyl (1918) introduced the notion of gauge transformations, in an attempt to unify gravity and electromagnetism [34]. By extending the Local Lorentz group to include scale transformations (dilatations) he was led to assume what we call a Riemann-Weyl spacetime geometry, a post-Riemann geometry with (the trace-vector part of) non-metricity beyond curvature. The theory was then abandoned and Weyl (1929) clarified that the electromagnetic field is intimately related to local internal symmetries, under the U(1) group that act on the 4-spinor fields of charged matter [58]. In the mid 1950's Yang (1954) and Mills (1954) [59] further explored the notion of gauge symmetries in field theories going beyond the $U(1)$ group to include non-abelian Lie groups ($SU(2)$), in order to address nuclear physics, while Utiyama (1956) [60] extended the gauge principle to all semi-simple Lie groups including the Lorentz group.The gauge principle is based on the localization of the rigid, global symmetry group of a field theory, introducing a new interaction described by the gauge potential. The latter is a compensating field that makes it possible for the matter Lagrangian to be locally invariant under the symmetry group and is included in the covariant derivative of the theory. There is a clear geometrical interpretation of the gauge potential as the connection of the fiber bundle, which is the manifold obtained from the base spacetime manifold and the set of all fibers. These are attached at each spacetime point and are the (vector, tensor or spinor) spaces of representation of the local symmetries. In the geometrical interpretation, the imposition of local symmetries implies that the geometry of the fiber bundle is non-Euclidean, and the gauge field strengths are the curvatures of such a manifold.

The gauge formulation of gravity was resumed through the works of Kibble (1961) [61] and Sciama [62] (1964), who gauged the (rigid) Poincaré group of Minkowski spacetime symmetries. This can be viewed as the starting point of a self-consistent gauge theory of gravity. They arrived at what is now known as a Riemann-Cartan (RC) geometry, and to the corresponding Einstein-Cartan-Sciama-Kibble (ECSK) gravity, with non-vanishing torsion and curvature. This is a natural extension of GR, which is able to successfully incorporate the intrinsic spin of fermions as a source of gravity, while passing all weak-field limit tests. The theory has no free parameters but introduces a new scale given by Cartan's density, which yields many relevant applications in cosmology and astrophysics [63, 64, 65, 66, 67, 68, 69, 70, 71]. The ECSK is the simplest of all Poincaré gauge theories of gravity (PGTG) in RC spacetime. Beyond the Poincaré group we have, for instance, the Weyl and the conformal groups, which live on a subset of a general metric-affine geometry, with non-vanishing curvature, torsion and non-metricity (for a detailed analysis and reviews on several topics of the gauge approach to gravity see the remarkable works in [19, 20, 21]). *By extending the gauge symmetry group of gravity, one is naturally led to extend the spacetime geometry paradigm as well.* There are other extensions of the PGTG, by localization of wider groups, as the (Anti)deSitter gauge theories or the metric-affine gravity (MAG). Metric-Affine geometries have a correspondence, in the continuum limit to the geometries appearing in crystals with defects [72]. Line defects or dislocations are related to torsion and point defects to non-metricity, for example. These defects can be seen as deformations of the regular crystalline structure in elasticity/condensed matter theory. This is an indication of an important role of post-Riemann geometries and the gauge methods in gravity towards a thermodynamical approach to gravity, a quantization of spacetime or to a coherent quantum gravity theory. In fact, the Riemann-Cartan geometry, linked to the gauging of the Poincaré group is also important in Super gravity (SUGRA) and post-Riemann geometries in general might be inevitably required for a coherent quantum description of gravity and spacetime.

Just as quantum field theories of the standard model are gauge theories with (local) internal symmetries, similarly, classical theories of gravity can indeed be formulated as gauge theories, leading to non-Euclidean geometries. *Therefore, the gauge principle (with its geometrical methods) is a convenient formalism in order to address the possible avenues towards unified field theories of matter, gravity and spacetime geometry. The spacetime paradigm changes when we extend the gauge group of gravity and the history* *of physics have shown that changing the fundamental ideas about space and time is at the basis of major breakthroughs.* Gauge methods in gravity and post-Riemann geometries deserve further developments with potential applications for astrophysical and cosmological conditions with very strong gravitational fields. Above all, they are a vital part of the effort to understand the nature of spacetime and gravitation.

3.1.1 Fundamental geometrical structures of spacetime and its relation to symmetry groups

Consider a 4-dimensional differential manifold $\mathcal{M}$ as an approximate representation of physical spacetime. We will now introduce the fundamental geometrical objects and its relation to group theory of spacetime symmetries.

Linear frames and co-frames

At each point P of $\mathcal{M}$ we introduce the set of four linearly independent vectors which constitute the coordinate (holonomic) vector basis $\{\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$, with $\bar{e}_0 \equiv \partial_0$, $\bar{e}_1 \equiv \partial_1$, $\bar{e}_2 \equiv \partial_2$, $\bar{e}_3 \equiv \partial_3$, where each vector is tangent to a coordinate line. This is called a linear frame basis. Similarly, at the same point we introduce the dual co-frame basis $\{\bar{\theta}^0, \bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3\}$, with $\bar{\theta}^0 \equiv dx^0$, $\bar{\theta}^1 \equiv dx^1$, $\bar{\theta}^2 \equiv dx^2$, $\bar{\theta}^3 \equiv dx^3$. This dual basis $(\bar{e}_b \rfloor \bar{\theta}^a = \delta^a_b)$[1] is the linear co-frame basis. Any of these basis can be called the natural (coordinate/holonomic) frame/co-frame. This geometrical structure comes naturally with the notion of coordinates on the spacetime manifold, it is intrinsic to the coordinates structure. One can choose any set of linearly independent vectors/co-vectors to form arbitrary linear frames and co-frames. To do so we consider the independent combinations $e_b = e_b{}^\nu \partial_\nu$ and $\theta^a = \theta^a{}_\mu dx^\mu$. The indices $a, b = 0, 1, 2, 3$ are called anholonomic indices, sometimes called symmetry or group indices and play a fundamental role in the gauge approach to gravity due to its connection to spacetime symmetries. It is clear that for the natural frame/co-frame we have $\bar{e}_b = \delta_b{}^\nu \partial_\nu$ and $\bar{\theta}^a = \delta^a{}_\mu dx^\mu$. For arbitrary (non-coordinate) anholonomic vector basis, the Lie brackets is non-vanishing $[U, V] \equiv \pounds_U V \neq 0$ (where $\pounds_U V$ is the Lie derivative of V with respect to U) for any two vectors U, V in the basis. In relation to this one can show, using the definitions and duality relations already introduced, the following algebra $[e_a, e_b] = f_{ab}{}^c e_c$ where the objects $f_{ab}{}^c$ are sometimes called the (group) structure constants. This algebra can be used to characterize the local spacetime symmetries of the tangent/cotangent spaces.

Since the linear co-frame is a set of four linearly independent 1-forms and potentials in gauge theory are always 1-forms (since they enter in the definition of the gauge covariant derivative) we select this set.

The linear coframe and spacetime symmetries. In four dimensions, the set of vector valued 1-forms θ^a constitute 16 independent components $\theta^a{}_\mu$ (the tetrads) and are the potentials for the group of (local) spacetime translations T(4). This group has four generators, therefore we have four potentials and four field strengths T^a. The field strength $T^a = \frac{1}{2} T^a{}_{\mu\nu} dx^\mu \wedge dx^\nu$ is a vector valued 2-form field, corresponding to the torsion

[1] The symbol $\rfloor$ stands for the interior product, also called contraction operator, which gives a contraction between a p-form and a vector, resulting in a (p-1)-form (see appendix A.1).

of the spacetime manifold and is given by

$$T^a = D\theta^a = d\theta^a + \Gamma^a{}_b \wedge \theta^b. \tag{3.1}$$

Here, D is the (gauge) covariant exterior derivative, d is the exterior derivative, a kind of curl operator that raises the degree of any p-form, $\wedge$ is the wedge product (see appendix A.1), and the second term on the right-hand side, sometimes called the non-trivial part, includes the linear connection $\Gamma^a{}_b$ 1-form. The presence of this term is of high significance in the context of the Poincaré gauge gravity and MAG, as will be discussed in this section. The torsion 2-form has $4 \times 6 = 24$ independent components

$$T^a{}_{\mu\nu} = 2\partial_{[\mu}\theta^a{}_{\nu]} + 2\Gamma^a{}_{c[\mu}\theta^c{}_{\nu]}. \tag{3.2}$$

We will introduce the linear connection and the torsion below.

Linear connection

The linear connection (sometimes called affine connection) $\Gamma^a{}_b$ is a tensor valued 1-form that connects neighbouring points of the manifold. Accordingly if $v = v^a e_a$ is a vector, then under a parallel transport by an infinitesimal displacement δx, the difference between the parallel displaced vector and the original vector is given by

$$v^a_{\|}(x+\delta x) - v^a(x) = \delta_{\|} v^a = -\Gamma^a{}_b v^b, \qquad \Gamma^a{}_b = \Gamma^a{}_{b\mu} dx^\mu. \tag{3.3}$$

Accordingly under an arbitrary infinitesimal displacement from a given point in $\mathcal{M}$, the linear frame is transformed (for e.g., under a Lorentz rotation or some more generic linear transformation) and the connection gives a measure of such change in the linear frame. Before the introduction of an affine structure of the spacetime manifold $\mathcal{M}$, i.e, before an affine connection $\Gamma^\alpha_{\lambda\mu}$, there is no well-defined comparison between tensor quantities in different spacetime points. The affine connection allows the definition of a covariant derivative and in this way it establishes a rule for the parallel transport of tensors along curves of $\mathcal{M}$. It allows therefore to determine the affine geodesics (straightest lines) which do not necessarily coincide with extremal geodesics (”shortest” paths).

The connection has an interesting property, relevant for unified field theories, which is related to the following facts: i) The difference of two connections is a tensor and ii) under the transformation $\Gamma \rightarrow \Gamma + \tau$, where τ is a tensor, the covariant derivative of tensors retains its covariance. The components of the covariant derivative of a tensor still transform as a tensor. This opens perspectives for unifying field theories, since one can incorporate new degrees of freedom in the geometrical (affine) structure of spacetime while preserving the covariance of the equations. Naturally such extensions of the connection presuppose an extended spacetime geometry, while the inclusion of extra (gauge) degrees of freedom in a field theory, require extending the local symmetry group. These two facts are inevitably interrelated in gauge theories of gravity. Due to this arbitrariness of the affine connection, it is natural to consider the Levi-Civita as a sort of reference in the space of linear connections, as we will see in this section.

The linear connection and spacetime symmetries. In four dimensions, the set of tensor valued 1-forms $\Gamma^a{}_b$ constitute 64 independent components $\Gamma^a{}_{b\mu}$ and are the potentials for the 4-dimensional group of general linear transformations GL(4,$\Re$). With the definition of the linear frame/coframe the arbitrary (general) non-degenerate linear transformations of spacetime coordinates can be defined and $\Gamma^a{}_b$ turn out to be the generators of such group of transformations. This group has 16 generators, therefore we have 16 potentials which are analogous to the Yang-Mills potentials of the SU(3).The corresponding field strength $R^a{}_b$ is a tensor valued 2-form field $R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$, corresponding to the curvature of the spacetime manifold and is given by

$$R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b. \tag{3.4}$$

The curvature 2-form has $16 \times 6 = 96$ independent components

$$R^a{}_{b\mu\nu} = 2\partial_{[\mu} \Gamma^a{}_{b|\nu]} + 2\Gamma^a{}_{c[\mu} \Gamma^c{}_{b|\nu]}. \tag{3.5}$$

Metric

Although one can argue that the metric is not as fundamental as the previously introduced structures (which is the perspective taken in this thesis) one can introduce the spacetime metric in order to measure time and space intervals as well as angles. Accordingly we introduce the Lorentzian metric as the (0,2) tensor $g = g_{ab}\theta^a \otimes \theta^b$, where $a, b = 0, 1, 2, 3$ are anholonomic indices and the spacetime metric components in the coordinate frame is given by $g_{\mu\nu} = \theta^a{}_\mu \theta^a{}_\nu g_{ab}$. The Lorentzian metric g_{ab} is the metric of the tangent space required to compute the inner product between vectors and making a map between tangent space vectors and the corresponding dual co-vectors $v_b = g_{ab}v^a$. The spacetime metric $g_{\mu\nu}$ establishes maps between the contravariant components of vectors and the covariant components of the corresponding dual covectors in the coordinate (holonomic) basis and gives the inner products $g(u, v) = g_{\alpha\beta}u^\alpha v^\beta = u_\alpha v^\beta$. Therefore, the spacetime metric can be seen as the deformation of the Lorentzian (tangent space) metric according to $g_{\mu\nu} = \Omega^{ab}{}_{\mu\nu} g_{ab}$, with $\Omega^{ab}{}_{\mu\nu} \equiv \theta^a{}_\mu \theta^a{}_\nu$ being a deformation tensor. The tangent space has a pseudo-Euclidean geometry, therefore if the spacetime manifold has non-Euclidean geometry, the deformation tensor has to vary from point to point and it is the linear connection that gives a measure of how the linear frame and coframe changes under displacements in $\mathcal{M}$. We will be considering geometries where the spacetime metric is symmetric $g_{\alpha\beta} = g_{\beta\alpha}$, non degenerate $g = det(g_{\mu\nu}) \neq 0$ and determines the local line element $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ with a Lorentzian signature (± 2).

Besides the linear co-frame (tetrads) and the linear connection, we therefore introduce the 0-form metric g_{ab} as a kind of potential and the corresponding field strength $Q_{ab} = Q_{ab\mu}dx^\mu$ as the tensor valued 1-form given by $Q_{ab} = Dg_{ab} = dg_{ab} + \Gamma^c_a \wedge g_{cb} + \Gamma^c_b \wedge g_{ac} = 2\Gamma_{(ab)}$[2] with $10 \times 4 = 40$ independent components $Q_{ab\mu}$. This field strength corresponds to the Non-metricity tensor valued 1-form and one concludes that if the connection is non-Lorentzian ($\Gamma^{ab} \neq -\Gamma^{ba}$) then, the non-metricity is non-vanishing.

Revisiting curvature, torsion and non-metricity

In (pseudo) Euclidean geometries such as that of Minkowsi spacetime in special relativity, there is always a coordinate system where the components of the connection (Christoffel symbols) and its derivatives vanish, whereas in the (pseudo) Riemann geometry of GR with non-vanishing curvature, one can find a local geodesic system of coordinates where the connection vanishes and the metric is given by the Minkowski metric (a freely falling frame), but the derivatives of the connection cannot be set to zero. In such geometries

[2]The components of metric of the tangent space are constant.

the so called Levi-Civita connection and the metric are fundamentally related and are not independent. The metricity condition (vanishing of the covariant derivatives of the metric) implies that the connection is proportional to the first derivatives of the metric and as such, in GR, the presence of a physical gravitational field is traced to the non-vanishing of the second derivatives of the metric. The Weyl part of the Riemannian curvature is not absent in a freely falling frame, which translates into tidal effects. Recall that the Levi-Civita connection

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right), \tag{3.6}$$

is the only symmetric connection that obeys the metricity condition. But both the symmetry of a connection and the metricity condition can be relaxed leading to more general geometries with torsion and non-metricity respectively.

The linear connection 1-form can be decomposed according to

$$\Gamma_{ab} = \tilde{\Gamma}_{ab} + N_{ab} = \tilde{\Gamma}_{ab} + N_{[ab]} + \frac{1}{2} Q_{ab} \ , \tag{3.7}$$

where the Levi-Civita part of the connection, $\tilde{\Gamma}_{ab}$, obeys the Cartan structure equation

$$d\theta^a + \tilde{\Gamma}^a{}_b \wedge \theta^b = 0, \tag{3.8}$$

and $N^a{}_b$ is the so-called distortion 1-form characterizing the post-Riemannian geometries. In particular, one finds that $Q_{ab} = 2N_{(ab)}$ and $T^a = N^a{}_b \wedge \theta^b$. If the linear connection obeys the condition $\Gamma^{ab} = -\Gamma^{ba}$, then it is called a Lorentzian connection (or spin connection) and corresponds to 24 independent components. This is the case when the linear connection is the potential for the Lorentz group $SO(1,3)$ (not the full linear group GL(4,$\Re$)). Accordingly, as we shall see later, the Lorentzian connection is the linear connection of PGTG.

In the tensor formalism, any affine connection can be decomposed into three independent pieces. In holonomic (coordinate) components we have

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} \ , \tag{3.9}$$

where $\tilde{\Gamma}^{\lambda}{}_{\mu\nu}$ is the Levi-Civita connection associated to the Riemannian curvature $\tilde{R}^{\alpha}{}_{\beta\mu\nu} = 2\partial_{[\mu}\tilde{\Gamma}^{\alpha}{}_{\beta|\nu]} + 2\tilde{\Gamma}^{\alpha}{}_{[\mu|\lambda}\tilde{\Gamma}^{\lambda}{}_{\beta|\nu]}$, the second term is associated to the torsion tensor $T^{\lambda}{}_{\alpha\beta} \equiv \Gamma^{\lambda}_{[\alpha\beta]}$ and is denoted contortion[3]

$$K^{\lambda}{}_{\mu\nu} \equiv T^{\lambda}{}_{\mu\nu} - 2T_{(\mu}{}^{\lambda}{}_{\nu)} = T^{\lambda}{}_{\mu\nu} + 2T_{(\mu\nu)}{}^{\lambda}, \tag{3.10}$$

while the third term is associated to the non-metricity tensor $Q_{\rho\mu\nu} \equiv \nabla_\rho g_{\mu\nu}$ and is called disformation,

$$L^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\beta} \left(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu} \right). \tag{3.11}$$

We will revisit the curvature, torsion and non-metricity tensors below.

[3]By construction, contortion is antisymmetric on its first two indices, $K_{\alpha\beta\gamma} = -K_{\beta\alpha\gamma}$.

Curvature of a connection. In holonomic basis the curvature of a connection has the 96 independent components

$$R^{\alpha}_{\ \beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\ \beta\nu} - \partial_\nu \Gamma^{\alpha}_{\ \beta\mu} + \Gamma^{\alpha}_{\ \lambda\mu}\Gamma^{\lambda}_{\ \beta\nu} - \Gamma^{\alpha}_{\ \lambda\nu}\Gamma^{\lambda}_{\ \beta\mu}. \tag{3.12}$$

Consider at some point P of $\mathcal{M}$ the vectors $U = d/d\lambda$ and $V = d/d\sigma$ tangent to two curves intersecting at P, where λ and σ are some respective affine parameters that parametrize the curves. Since the curvature tensor is of (1,3) type, it can be applied (contracted) to U and V and then to some vector field Z, giving the resulting vector field

$$R(U,V)Z = \nabla_U \nabla_V Z - \nabla_V \nabla_U Z - \nabla_{[U,V]} Z, \tag{3.13}$$

where $[U,V] = \pounds_U V$ represents the Lie derivative (of V with respect to U).

If we consider an infinitesimal closed loop, with $ds^{\mu\nu}$ being the surface element spanned by such loop, then after a parallel transport of some vector v along the loop, the initial and final vectors do not coincide. There is a rotation with a difference vector

$$\delta v^\alpha \approx R^{\alpha}_{\ \beta\mu\nu} v^\beta ds^{\mu\nu}. \tag{3.14}$$

Parallel transport of a vector $\Longrightarrow$ a Rotation.

We define also the Homothetic curvature tensor,

$$g^{\alpha\beta} R_{\alpha\beta\mu\nu} = R^{\alpha}_{\ \alpha\mu\nu} \equiv \Omega_{\mu\nu}, \tag{3.15}$$

that shall be useful once we revisit non-metricity and (Riemann) Weyl geometry.

Torsion of a connection. The torsion tensor can be introduced in holonomic coordinates as the antisymmetric part of the affine connection $\Gamma^{\alpha}{}_{[\beta\gamma]}$. It has 24 independent components.

$$T^{\alpha}{}_{\beta\gamma} \equiv \Gamma^{\alpha}{}_{[\beta\gamma]}. \tag{3.16}$$

If we aply to the vectors U and V we obtain the vector field

$$T(U,V) = \nabla_U V - \nabla_V U - [U,V]. \tag{3.17}$$

This expression allows to visualize the geometrical effect of torsion. Indeed, starting from a given point on $\mathcal{M}$, where both vector fields have their corresponding realizations and parallel transporting V along the (integral curve of) U through an infinitesimal distance, and doing the complementary with U, parallel transporting it along V then, if the spacetime has a non-vanishing torsion, the expected parallelogram does not close. The two end points are separated to each other by a (spacetime) translation, which is given by the norm of the vector $T(U,V)$.

As previously, if we consider an infinitesimal closed loop, after parallel transporting of some vector v along the loop, the initial and final vectors do not coincide. There is a translation vector between both and given by

$$\xi^\alpha \approx 2T^{\alpha}{}_{\mu\nu} ds^{\mu\nu} \tag{3.18}$$

Parallel transport of a vector $\Longrightarrow$ Translation.

The torsion field has three irreducible pieces completing the 24 independent components,

$$T^{\lambda}_{\ \mu\nu} = \bar{T}^{\lambda}_{\ \mu\nu} + \tfrac{2}{3}\delta^{\lambda}_{[\nu} T_{\mu]} + g^{\lambda\sigma}\epsilon_{\mu\nu\sigma\rho}\breve{T}^{\rho}, \tag{3.19}$$

where the traceless tensor (16 components) obeys $\bar{T}^{\lambda}_{\ \mu\lambda} = 0$ and $\epsilon^{\lambda\mu\nu\rho}\bar{T}_{\mu\nu\rho} = 0$, while $T_\mu = T_{\mu\lambda}{}^{\lambda}$ is the trace vector and $\breve{T}^{\lambda} \equiv \frac{1}{6}\epsilon^{\lambda\alpha\beta\gamma} T_{\alpha\beta\gamma}$ the pseudo-trace (axial) vector.

Non-metricity. In holonomic coordinates the non-metricity tensor can be defined as the covariant derivative of the spacetime metric $g_{\beta\gamma}$. It has 40 independent components.

$$Q_{\alpha\beta\gamma} = \nabla_\alpha g_{\beta\gamma}. \tag{3.20}$$

The trace-vector $Q_\mu \equiv Q_{\mu\alpha}{}^{\alpha}$ is the only non-vanishing part of the non-metricity in Weyl geometry (Riemann-Weyl or Cartan-Weyl, for e.g), also known as the Weyl co-vector. It is related to the homothetic curvature in (3.15) via[4]

$$\Omega_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu Q_\nu - \partial_\nu Q_\mu\right). \tag{3.21}$$

After the parallel transport of a vector along an infinitesimal closed loop, there is a change in length given by

$$\delta l \approx l(v)\Omega_{\mu\nu} ds^{\mu\nu}. \tag{3.22}$$

If Q_μ is the only non-vanishing part of $Q_{\alpha\beta\gamma}$, then $\nabla_\alpha g_{\beta\gamma} \sim Q_\alpha g_{\beta\gamma}$ and the norm of vectors can change according to

$$\nabla_\nu(V^2) \sim Q_\nu V^2, \qquad V^2 = g_{\beta\gamma} V^\beta V^\gamma. \tag{3.23}$$

Parallel transport of a vector $\Longrightarrow$ Change in length.

From the definition of the non-metricity tensor, one can derive the following relation, which corresponds to a Bianchi identity (obtained from deriving a potential twice)

$$\nabla_{[\mu} Q_{\nu]\alpha\beta} = -R_{(\alpha\beta)\mu\nu} + Q_{\lambda\alpha\beta} T^{\lambda}{}_{\mu\nu}, \tag{3.24}$$

therefore, if $Q_{\alpha\beta\gamma} \neq 0 \Rightarrow R_{(\alpha\beta)\mu\nu} \neq 0$. The quantity $R_{(\alpha\beta)\mu\nu}$ could be called a non-riemannian part of the curvature and its related to a non-Lorentzian linear connection and the breaking of Lorentz invariance. This becomes clear in the gauge approach to gravity since such connection is the potential of a symmetry group more general than the Poincaré group. Finally, non-metricity can be decomposed into its trace-vector part and the traceless tensor part $\bar{Q}_{\alpha\beta\gamma}$, according to $Q_{\alpha\beta\gamma} = \bar{Q}_{\alpha\beta\gamma} + \frac{1}{4} g_{\beta\gamma} Q_\alpha$. In the language of forms, the latter can be further decomposed into a shear co-vector and a shear 2-form part in such a way that the tensor valued non-metricity 1-form ends up having four irreducible pieces with respect to the Lorentz group.

3.1.2 A brief outlook on metric-affine gravity

Let us now summarize the fundamental structures of spacetime and their relation to symmetry groups. We have the fundamental 1-forms, the linear co-frame θ^a and the linear connection $\Gamma^a{}_b$, which are the potentials for the 4-dimensional translations $T(4)$ and the general linear $GL(4,\Re)$ groups, respectively. The corresponding field strengths correspond to well known objects from differential geometry, namely, the torsion $T^a = D\theta^a = d\theta^a + \Gamma^a{}_b \wedge \theta^b$ and curvature $R^a{}_b = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b$ 2-forms, respectively. The metric is introduced as the 0-form potential with the corresponding field strength being the non-metricity 1-form $Q_{ab} = Dg_{ab}$. The linear frame establishes a link between the (symmetries on the) local tangent fibers and the spacetime manifold, while the linear connection can be viewed as a guidance field reflecting the inertial character for matter fields propagating on the spacetime manifold. Finally, the metric allows the

[4] It was this curvature that Weyl identified as the electromagnetic tensor, while Q_μ represented the electromagnetic potential.

determination of spatial and temporal distances and angles. The spacetime geometry with all these structures is called a *metric-affine geometry*, with non-vanishing torsion, curvature and non-metricity, and its fundamental local group of spacetime symmetries is the affine group $A(4,\Re) = T(4) \rtimes GL(4,\Re)$, which is the semi-direct product of the group of translations and the general linear group.

A truly independent linear connection is given by the decomposition in (3.7) that is useful to analyse the relation between non-Lorentzian metrics, the breaking of Lorentz invariance and the presence of non-metricity. Since $Q_{ab} = 2N_{(ab)}$, if non-metricity is zero, then the connection is Lorentzian and the spacetime geometry is the RC one with curvature and torsion. Such a spacetime is fundamentally linked to the local symmetry group of Poincaré transformations $P(1,3)$, which is the semi-direct product between the translations $T(4)$ and the Lorentz group $SO(1,3)$, i.e, $P(1,3) = T(4) \rtimes SO(1,3)$. As one can see from the expressions for the field strengths, torsion and curvature, in (3.1) and (3.4), respectively, the connection potential also enters in the expression for the field strength of the co-frame potential. This term is unavoidably present and is due to the semi-direct product[5] structure of the Poincaré group (or the affine group) and, therefore, curvature and torsion are somehow intertwined.

In the self-consistent metric-affine formalism, the gravitational interaction is described as a gauge theory of the affine group $A(4,\Re)$, together with the assumption of a metric, with the potentials (θ^a, Γ^{ab}) coupled to the corresponding Noether currents $(\tau^a, \Delta^a{}_b)$. The latter are the vector-valued canonical energy-momentum $\tau^a = \delta\mathcal{L}_{mat}/\delta\theta_a$ and the tensor-valued hypermomentum $\Delta^a_{\ b} = \delta\mathcal{L}_{mat}/\delta\Gamma_a^{\ b}$ 3-form currents[6], respectively, while the metric g_{ab} couples to the symmetric (Hilbert) energy momentum $T_{ab} = 2\delta\mathcal{L}_{mat}/\delta g^{ab}$. The hypermomentum can be decomposed according to the expression

$$\Delta_{ab} = s_{ab} + \frac{1}{4} g_{ab}\Delta^c{}_c + \bar{\Delta}_{ab} \ , \tag{3.25}$$

including the spin $s_{ab} = -s_{ba}$, the dilatation $\Delta^c{}_c$, and the shear $\bar{\Delta}_{ab}$ currents. The Noether currents are the fundamental sources of gravity in MAG. A truly independent connection in MAG can be written as

$$\Gamma^{ab} = \Gamma^{[ab]} + \frac{1}{4} g^{ab}\Gamma^c{}_c + \left(\Gamma^{(ab)} - \frac{1}{4} g^{ab}\Gamma^c{}_c\right) , \tag{3.26}$$

including the Lorentzian piece $\Gamma^{[ab]}$, the trace part $\frac{1}{4} g^{ab}\Gamma^c{}_c$ and the shear part $\Gamma^{(ab)} - \frac{1}{4} g^{ab}\Gamma^c{}_c$. The Lorentzian connection couples to the spin current and the trace and shear parts couple to the dilatation and shear currents, respectively. As previously said it is the non-Lorentzian part of the connection that imply the non-vanishing of non-metricity, therefore, the dilatation and shear hypermomentum currents are intimately related to the non-metricity of metric affine spacetime geometry.

The variational principle is applied to the action of this theory (including the gravitational part and the matter Lagrangian) by varying it with respect to the gauge potentials

[5]The semi direct product implies that the generators of $T(4)$ and $GL(4,\Re)$ (or $SO(1,3)$) do not commute.

[6]These currents can be represented as 1-forms or as 3-forms. In fact, in the gauge approach to gravity they emerge naturally from Noether equations as 3-forms, being natural objects for integration over volumes. As 1-forms one can write $\tau^a = \tau^a_\mu dx^\mu$ and $\Delta^a{}_b = \Delta^a{}_{b\mu} dx^\mu$. This map between 3-form or 1-form representation is related to the fact that in d-dimension a k-form has $\frac{d!}{k!(d-k)!}$ independent components. In four dimensions both 3-forms and 1-forms have four independent components.

of the affine group, (θ^a, Γ^{ab}). This leads to two sets of dynamical equations, while a third set of equations is obtained by varying the action with respect to the metric potential g_{ab}. At the end, the dynamics is described only via two sets of equations, since the gravitational equation obtained by varying with respect to the metric potential or the one obtained by variation with respect to the translational potential can be dropped out, as long as the other gravitational equation (derived from variation with respect to the linear connection) is fulfilled (for details we recommend the reader to see [20]). This procedure and the fundamental quantities and relations here exposed summarizes the basics of the MAG formalism.

3.1.3 A survey on classical spacetime geometries

We will now briefly revise some important classical spacetime paradigms, with a special attention to the Riemann-Cartan spacetime geometry.

Minkowski spacetime M_4. It is a pseudo-Euclidean geometry with vanishing curvature, torsion and non-metricity. It has global/rigid Poincaré symmetries and a globally absolute causal structure. For inertial observers the connection vanishes. The inertial frame and the inertial properties of matter are defined with respect to this absolute spacetime. In particular, the mass and spin of particles can be considered to be intrinsic to particles and are classified with the help of the Casimir operators mentioned in section 2.3, using the irreducible representations of the Poincaré group.

(Pseudo)Riemann geometry of GR V_4. A post-Euclidean geometry with curvature and vanishing torsion and non-metricity. It obeys local Lorentz symmetries and in local geodesic frames the (Levi-Civita) connection vanishes but (the Weyl part of) curvature is non-zero. The causal structure $ds^2 = 0$ is locally invariant under local Lorentz symmetries. The Inertial properties of matter are locally defined with respect to absolute spacetime.

Riemann-Cartan geometry U_4. The RC spacetime is the appropriate geometry for the PGTG. It has curvature and torsion but zero non-metricity, therefore the connection is Lorentzian (spin connection). It obeys local Poincaré P(1,3) symmetries and the causal structure is locally invariant under the P(1,3) group. The inertial properties of matter are locally defined with respect to absolute spacetime. In holonomic coordinates the RC (spin) Lorentzian connection can be written as $\Gamma^{\alpha}_{\ \beta\nu} = \tilde{\Gamma}^{\alpha}_{\ \beta\nu} + K^{\alpha}_{\ \beta\nu}$, where the contorsion tensor $K^{\alpha}_{\ \beta\nu}$ is given by (3.10). From the definition of curvature and the RC connection we obtain the expressions for the RC curvature

$$R^{\alpha}_{\ \beta\mu\nu} = \tilde{R}^{\alpha}_{\ \beta\mu\nu} + \tilde{\nabla}_{\mu} K^{\alpha}_{\ \beta\nu} - \tilde{\nabla}_{\nu} K^{\alpha}_{\ \beta\mu} + K^{\alpha}_{\ \lambda\mu} K^{\lambda}_{\ \beta\nu} - K^{\alpha}_{\ \lambda\nu} K^{\lambda}_{\ \beta\mu}, \tag{3.27}$$

the generalized Ricci tensor

$$R_{\beta\nu} = \tilde{R}_{\beta\nu} + \tilde{\nabla}_{\alpha} K^{\alpha}_{\ \beta\nu} - \tilde{\nabla}_{\nu} K^{\alpha}_{\ \beta\alpha} + K^{\alpha}_{\ \lambda\alpha} K^{\lambda}_{\ \beta\nu} - K^{\alpha}_{\ \lambda\nu} K^{\lambda}_{\ \beta\alpha}, \tag{3.28}$$

and the generalized Ricci curvature

$$R = \tilde{R} - 2\tilde{\nabla}^{\lambda} K^{\alpha}_{\ \lambda\alpha} + g^{\beta\nu}(K^{\alpha}_{\ \lambda\alpha} K^{\lambda}_{\ \beta\nu} - K^{\alpha}_{\ \lambda\nu} K^{\lambda}_{\ \beta\alpha}), \tag{3.29}$$

respectively. The curvature tensor obeys the following first and second Bianchi identities

$$\nabla_{[\gamma}R^{\alpha}_{\ \beta|\mu\nu]} = 2R^{\alpha}_{\ \beta\lambda[\mu}T^{\lambda}_{\ \nu\gamma]}\,, \tag{3.30}$$

$$R^{\alpha}_{\ [\beta\mu\nu]} = -2\nabla_{[\nu}T^{\alpha}_{\ \beta\mu]} + 4T^{\alpha}_{\ \lambda[\beta}T^{\lambda}_{\ \mu\nu]}\,. \tag{3.31}$$

These relations can be deduced from the corresponding expressions in terms of the curvature and torsion 2-forms, namely[7]

$$DR^{a}{}_{b} = 0 \qquad DT^{c} = R^{c}{}_{d}\wedge\theta^{d}, \tag{3.32}$$

that are a direct consequence of deriving the potentials twice, and are intrinsic to the RC geometry. The Lorentz indices are also called symmetry indices since they are concerned to the tangent fibers where the local spacetime symmetries are characterized. Spin connections are related to rotations (two Lorentz indices) and the tetrads or co-frames are related to translations (one Lorentz index). The same is valid for the corresponding field strengths. The symmetry indices have a direct link to geometrical interpretations via the strong relation between group theory and geometry. Two symmetry indices for curvature means that it is related to rotations, and one symmetry index for torsion means that it is related to translations. On the other hand, since both curvature and torsion are represented by 2-forms, as geometrical objects these are therefore connected to 2-surfaces. One can say that Cartan (1922) pictured a RC geometry by associating to each infinitesimal surface element a rotation and a translation (see the note below). The components of the curvature and torsion 2-forms in terms of the tetrads and spin connection are

$$R^{a}_{\ b\mu\nu} = \partial_{\mu}\Gamma^{a}_{\ b\nu} - \partial_{\nu}\Gamma^{a}_{\ b\mu} + \Gamma^{a}_{\ c\mu}\Gamma^{c}_{\ b\nu} - \Gamma^{a}_{\ d\nu}\Gamma^{d}_{\ b\mu}. \tag{3.33}$$

$$T^{a}_{\ \mu\nu} = \partial_{\nu}\theta^{a}_{\ \mu} - \partial_{\mu}\theta^{a}_{\ \nu} + \Gamma^{a}_{\ b\mu}\theta^{b}_{\ \nu} - \Gamma^{a}_{\ b\nu}\theta^{b}_{\ \mu}. \tag{3.34}$$

The components of the 1-form spin connection

$$\Gamma^{a}_{\ b\nu} = \tilde{\Gamma}^{a}_{\ b\nu} + K^{a}_{\ b\nu}, \tag{3.35}$$

where $K^{a}_{\ b\mu}$ are the components of the contorsion 1-form, are related to the holonomic (spacetime) components of the affine connection through the relations

$$\Gamma^{a}_{\ b\nu} = \theta^{a}_{\ \mu}\partial_{\nu}e_{b}^{\ \mu} + \theta^{a}_{\ \mu}\Gamma^{\mu}_{\ \beta\nu}e_{b}^{\ \beta} \qquad \Gamma^{\lambda}_{\ \nu\mu} = e_{a}^{\ \lambda}\partial_{\mu}\theta^{a}_{\ \nu} + e_{a}^{\ \lambda}\Gamma^{a}_{\ b\mu}\theta^{b}_{\ \nu}. \tag{3.36}$$

The connection characterizes the way the linear frame/co-frame changes from point to point. Accordingly, the relations above can be deduced from the equation

$$\partial_{\nu}e_{b}^{\ \mu} + \Gamma^{\mu}_{\ \beta\nu}e_{b}^{\ \beta} \equiv \Gamma^{c}_{\ b\nu}e_{c}^{\ \mu}, \tag{3.37}$$

which expresses the fact that the total covariant derivative of the tetrads with respect to both holonomic and anholonomic (Lorentz) indices is vanishing, i.e $\partial_{\nu}e_{b}^{\ \mu} + \Gamma^{\mu}_{\ \beta\nu}e_{b}^{\ \beta} - \Gamma^{c}_{\ b\nu}e_{c}^{\ \mu} = 0$ and $\partial_{\mu}\theta^{a}_{\ \nu} - \Gamma^{\beta}_{\ \nu\mu}\theta^{a}_{\ \beta} + \Gamma^{a}_{\ b\mu}\theta^{b}_{\ \nu} = 0$. By changing from one point on $\mathcal{M}$ to another, the frame/coframe, i.e, the tetrads also change. The new tetrads $\bar{e}_{b}^{\ \mu}$ can be expressed in terms of the original ones by $\bar{e}_{b}^{\ \mu} = \Lambda^{a}_{b}e_{a}^{\ \mu}$, and using the relations

$$\eta_{ab} = e_{a}^{\ \mu}e_{b}^{\ \nu}g_{\mu\nu}, \qquad g_{\mu\nu} = \theta^{a}_{\ \mu}\theta^{b}_{\ \nu}\eta_{ab}, \qquad \sqrt{-g} = det(\theta^{a}_{\ \mu}) \tag{3.38}$$

[7]Here $DR^{a}{}_{b} = dR^{a}{}_{b} + \Gamma^{a}{}_{c}\wedge R^{c}{}_{b} + \Gamma^{c}{}_{b}\wedge R^{a}{}_{c}$ and $DT^{c} = dT^{c} + \Gamma^{c}{}_{b}\wedge T^{b}$.

one can show that the matrices Λ^a_b are Lorentz matrices. The tetrads undergo a Lorentz rotation under the motion from one spacetime point to another. For that reason, the linear connection, which characterizes the change in the frame/co-frame is called a Lorentzian or spin connection. As previously said the six Lorentzian connections are the potentials associated to the generators of the Lorentz group.

The matrices $e_a^{\ \mu}$ and its inverses $\theta^b_{\ \nu}$, which establish a correspondence between the local spacetime metric and the Minkowski metric on the tangent/cotangent planes (3.38), constitute a map between holonomic and anholonomic basis. Accordingly, for example, for vectors we have

$$v^\mu = v^a e_a^{\ \mu}, \quad v^b = v^\nu \theta^b_{\ \nu}, \quad h_\mu = h_a \theta^a_{\ \mu}, \quad h_b = h_\nu e_b^{\ \nu}. \tag{3.39}$$

Since the tangent and co-tangent spaces $T_p(M)$ e $T^*_p(M)$ change while moving from one point on $\mathcal{M}$ to another, the notion of covariant derivative is extended for quantities with Lorentz indices. This is done with the spin connection. For (Lorentz) vectors and co-vectors we have

$$^s\nabla_\mu v^a = \partial_\mu v^a + \Gamma^a_{\ b\mu} v^b, \qquad ^s\nabla_\nu h_c = \partial_\nu h_c - \Gamma^d_{\ c\nu} h_d, \tag{3.40}$$

where $^s\nabla$ represents the covariant derivative with respect to the Lorentz or spin connection. As previously mentioned, the anholonomic basis e_a and θ^b also characterize the local spacetime symmetries in the tangent fibers, since the local symmetry group algebra is implicit in the relations

$$[e_a, e_b] = f^c_{\ ab} e_c, \qquad f^c_{\ ab} = e_a^{\ \mu} e_b^{\ \nu} (\partial_\nu \theta^c_{\ \mu} - \partial_\mu \theta^c_{\ \nu}), \tag{3.41}$$

where $f^a_{\ bc}$ are the (group) structure constants, also known as Ricci rotation coefficients.

In RC geometry, the affine (self-parallel) geodesics in general differ from the extremal geodesics and are given by

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\ \beta\gamma} u^\beta u^\gamma = 0 \quad \Leftrightarrow \quad \frac{du^\alpha}{ds} + \tilde{\Gamma}^\alpha_{\ \beta\gamma} u^\beta u^\gamma = -K^\alpha_{\ (\beta\gamma)} u^\beta u^\gamma, \tag{3.42}$$

where u^α are the components of the (4-velocity) tangent vector to the curve. In the self-consistent framework of PGTG in the RC spacetime, matter particles with vanishing intrinsic spin are insensitive to the non-riemannian part of the geometry, therefore to the torsion. Such particles follow the extremal paths computed from the Levi-Civitta connection. Moreover if torsion is completely antisymmetric, as in the case of ECSK theory, then $K^\alpha_{\ (\beta\gamma)} = 0$ and the extremal and self-parallel geodesics coincide. In any case, the appropriate evaluation of the motion of particles with intrinsic spin (fermions) in a RC spacetime should be performed from an analysis of the corresponding Dirac equation and then proceeding with a classical approximation, for instance, using a WKB method. In fact there are strong arguments suggesting that the very notion of a point particle is incompatible with the spacetime paradigm of RC geometry since the generalized Bianchi identities of such geometry are incompatible with the point-like configuration usualy obtained within a multipolar expansion of the continuity equations for the conserved currents (see [99, 73]).

To complete this review of RC spacetime, let us emphasize again that since curvature and torsion are 2-forms, one can imagine the RC geometry as having at each point an associated infinitesimal surface element with a rotational (curvature) and translational (torsion) transformation. This geometrical fact motivates the following note on spacetime geometry quantization:

The picture of a RC manifold with a discrete structure can naturally emerge from imposing a finite minimum surface element (rater than infinitesimal elements), which would imply, to some extent, that torsion and curvature become quantized. Indeed, as we will see, the mathematical methods of the exterior calculus of Cartan (differential forms), within gauge theories of gravity, contribute to a clarification of the appropriate physical degrees of freedom and mathematical objects that should be quantized in a quantum (Yang-Mills) gauge theory of gravity (see table 3.2, for e.g.). This procedure, in this formalism of forms can be done in a metric-free (pre-metric) way, avoiding the difficulties often found in perturbative approaches that require some well-behaved background (spacetime-vaccum), with respect to which the perturbations can be defined. Since gravity theory, following Einstein's path, should be (metric) background independent, this is usually a huge problem, since the background is also the very thing one would like to quantize and should come as a solution of the dynamical equations. Gravity in exterior forms can shine light on this challenge, as we will see, at least by establishing a metric-independent framework and a well identified set of canonical conjugate variables to be quantized. As we will see, in Yang-Mills type of gauge theories of gravity this will not require the full metric structure but only the conformally invariant part of the metric, that is the conformal-causal structure implicit in the constitutive relations.

We will briefly revisit this topic at the end of this chapter.

Riemann-Weyl geometry W_4. This is the spacetime geometry implicit in Weyl's gauge theory for unifying gravity and electromagnetism, by extending the local Lorentz symmetries to include dilatations. It has curvature and non-metricity and vanishing torsion and it obeys local symmetries under the Weyl group W(1,3) which includes the P(1,3) and dilatations. The causal structure is locally invariant under the W(1,3) group, but the spacetime (metric) is not absolute, it changes under dilatation type of coordinate transformations. Accordingly, the inertial properties of matter cannot be defined with respect to an absolute metric structure. One may postulate that matter is endowed with conformally-invariant physical properties as we briefly explored in section 2.2.1. The Weyl connection is given by $\Gamma^{\alpha}_{\ \beta\nu} = \tilde{\Gamma}^{\alpha}_{\ \beta\nu} + N^{\alpha}_{\ \beta\nu}$, where the distortion tensor $N^{\alpha}_{\ \beta\nu} \equiv q^{\alpha}_{\ \beta\nu}$ in this case is related to the Weyl co-vector via

$$q^{\alpha}_{\ \beta\nu} \equiv \frac{1}{2}(\delta^{\alpha}_{\beta} Q_{\gamma} + \delta^{\alpha}_{\gamma} Q_{\beta} - Q^{\alpha} g_{\beta\gamma}). \tag{3.43}$$

The curvature of the Weyl connection is

$$R^{\alpha}_{\ \beta\mu\nu} = \tilde{R}^{\alpha}_{\ \beta\mu\nu} + \tilde{\nabla}_{\mu} q^{\alpha}_{\ \beta\nu} - \tilde{\nabla}_{\nu} q^{\alpha}_{\ \beta\mu} + q^{\alpha}_{\ \lambda\mu} q^{\lambda}_{\ \beta\nu} - q^{\alpha}_{\ \lambda\nu} q^{\lambda}_{\ \beta\mu}. \tag{3.44}$$

The Weyl co-vector $Q_{\mu} \equiv Q_{\mu\lambda}{}^{\lambda}$ is the trace vector part of the non-metricity, therefore there are scale (dilatations) type of distortions to the geometry, implicit in the relation $\nabla_{\alpha} g_{\beta\gamma} \sim Q_{\alpha} g_{\beta\gamma}$ and, as previously mentioned, the length of vectors change in this post-Riemann Weyl geometry according to (3.23). Recall that in the very birth of gauge theories, Weyl identified the Weyl co-vector as the electromagnetic 4-potential and the homothetic curvature in (3.15) as the electromagnetic Faraday tensor. Also previously mentioned, the pre-metric foundations of electrodynamics indeed show the deep connection between electromagnetism and conformal geometry and conformal symmetries, which presuppose a natural framework for the breaking of Lorentz symmetry, therefore for non-Lorentzian connections and non-metricity. One can say that in this respect, Weyl was on solid grounds (The Weyl group is part of the conformal group and Weyl rescalings are sometimes called conformal transformations).

The Riemann-Cartan-Weyl spacetime Y_4 geometry (or simply Cartan-Weyl) is a generalization of this Weyl geometry by including torsion. The Cartan-Weyl spacetime is the appropriate manifold for the Weyl Gauge theories of Gravity (WGTG). Only matter with hypermomentum, that include the spin current and the dilatations current can be sensitive to the torsion and to the (Weyl-covector part of) non-metricity.

Metric-affine geometry (L_4, g). This richer geometry has curvature, torsion and non-metricity. The natural symmetry group is the affine group A(4,$\Re$). The affine connection can be written as $\Gamma^{\alpha}_{\beta\nu} = \tilde{\Gamma}^{\alpha}_{\beta\nu} + N^{\alpha}_{\beta\nu}$, where the distortion tensor $N^{\alpha}_{\beta\nu} = K^{\alpha}_{\beta\nu} + L^{\alpha}_{\beta\nu}$ includes contorsion and the disformation tensor $L^{\alpha}_{\beta\nu} \equiv \frac{1}{2}(Q^{\alpha}_{\beta\gamma} + Q_{\beta\gamma}{}^{\alpha} - Q_{\gamma}{}^{\alpha}{}_{\beta})$, related to non-metricity. The curvature can be written as

$$R^{\alpha}_{\beta\mu\nu} = \tilde{R}^{\alpha}_{\beta\mu\nu} + \tilde{\nabla}_{\mu} N^{\alpha}_{\beta\nu} - \tilde{\nabla}_{\nu} N^{\alpha}_{\beta\mu} + N^{\alpha}_{\lambda\mu} N^{\lambda}_{\beta\nu} - N^{\alpha}_{\lambda\nu} N^{\lambda}_{\beta\mu}, \tag{3.45}$$

or, alternatively

$$R^{\alpha}_{\beta\mu\nu} = \bar{R}^{\alpha}_{\beta\mu\nu} + \bar{\nabla}_{\mu} L^{\alpha}_{\beta\nu} - \bar{\nabla}_{\nu} L^{\alpha}_{\beta\mu} + L^{\alpha}_{\lambda\mu} L^{\lambda}_{\beta\nu} - L^{\alpha}_{\lambda\nu} L^{\lambda}_{\beta\mu}, \tag{3.46}$$

where $\bar{R}^{\alpha}_{\beta\mu\nu}$ and $\bar{\nabla}_{\mu}$ are the curvature and covariante derivative of a Riemann-Cartan connection.

3.1.4 Gauge Theories of Gravity

The Gauge approach to gravity broadens our study of the deep relation between symmetry principles (group theory) and geometrical methods. Particularly relevant is of course the Poincaré Gauge theory of Gravity which constitutes a valid and quite promising class of models for an appropriate description of classical gravity including post-Einstein strong gravity predictions.

The Weyl-Yang-Mills procedure

In order to present the structure of the gauge approach to gravity, such as the PGTG, it is useful to revisit the Weyl-Yang-Mills formalism for gauge fields. It follows from two major steps, the rigid (global) symmetries of a physical system described by a matter fields Lagrangian, and the localization (gauging) of those symmetries.

In the first step one considers rigid (global) symmetries as follows:

- Start with a field theory: $\mathcal{L}_m = \mathcal{L}_m(\Psi, d\Psi)$ of some matter fields Ψ.
- The matter Lagrangian is invariant under some internal symmetries described by a (semi-simple) Lie group with generators T_a.
- Noether's first theorem implies a conserved current: $dJ = 0$.

In the second step the localization (gauging) of the symmetries is performed according to the following procedure:

- The symmetries are described on each spacetime point introducing the compensating (gauge) field $A = A^a_{\mu} T_a dx^{\mu}$, with $A_{\mu} \equiv A^a_{\mu} T_a$.

- This is a new field that couples minimally to matter and represents a new interaction.
- To preserve the symmetries this gauge potential A transforms in a suitable way allowing to construct a (gauge) covariant derivation $d\Psi \longrightarrow D\Psi = (d + A)\Psi$.
- The Lagrangian includes this minimal coupling between the matter fields and the gauge potential $\mathcal{L}(\Psi, d\Psi) \longrightarrow \mathcal{L}(\Psi, D\Psi)$.
- The gauge potential acts on the components of the matter fields defined with respect to some frame. Geometrically, it is the connection of the frame bundle (fiber bundle) related to the symmetry group.
- The conservation equation is generalized as $dJ = 0 \rightarrow DJ = 0$.

In order for A to represent a true dynamical variable with its own degrees of freedom, the Lagrangian of the theory has to include a kinetic term, representing the new interaction $\mathcal{L} = \mathcal{L}_m + \mathcal{L}_A$. The invariance of $\mathcal{L}_A$ is secured by constructing it with the gauge invariant field strength $F = DA = dA + A \wedge A$ which, geometrically, can be interpreted as the curvature 2-form of the fiber bundle. Note that to have second order (on A) inhomogeneous Yang-Mills field equations, one must choose $\mathcal{L}_A = \mathcal{L}_A(F)$, without dependences on derivatives of F, for example. Written in terms of exterior forms, the inhomogeneous Yang-Mills equations for the gauge potential are

$$DH = J \quad \Leftrightarrow \quad dH = J - A \wedge H \ , \tag{3.47}$$

where $H = \partial\mathcal{L}/\partial F$ is the excitation 2-form, and $J = \partial\mathcal{L}_m/\partial A$ is the conserved Noether current, which can be interpreted as a source in the field equations.

The homogeneous field equation corresponds to a Bianchi identity, obtained from the derivation of the potential twice, namely

$$DF = 0 \quad \Leftrightarrow \quad dF = -A \wedge F \ . \tag{3.48}$$

The equation for the conservation of the Noether current is generalized via the gauge covariant exterior derivative, therefore,

$$DJ = 0 \quad \Leftrightarrow \quad dJ = -A \wedge J \ . \tag{3.49}$$

For non-abelian groups the gauge field contributes with an associated ("isospin") current, $-A \wedge H$. In such a case $dJ \neq 0$, and the (gauge) interaction field is charged[8], unlike the case of abelian groups, such as the $U(1)$ group of electromagnetism.

In order to have a wave-like inhomogeneous Yang-Mills (quasi-linear) equation, and paralleling the case of electromagnetism, $\mathcal{L}$ can depend quadratically with F at most and, therefore, H must depend linearly, for instance as $H = H(F) = \alpha \star F$. The Yang-Mills inhomogeneous equations then turn into

$$D \star F = d \star F + A \wedge \star F = \alpha^{-1} J \Leftrightarrow d \star F = \alpha^{-1}(J + J^A) \tag{3.50}$$

where $J^A \equiv -\alpha A \wedge \star F$. In this formalism one can see the clear analogies between classical mechanics and Yang-Mills field theory, which we summarize in Table 3.1. In particular, it is clear that the field strength F is the generalized velocity while the

[8]As it is well known, because of this, electron, muon and tau leptons can be transformed to the respective neutrinos via the charged mediating W bosons.

	Yang-Mills $\mathcal{L} = \mathcal{L}(A, DA)$	Classical Mechanics $\mathcal{L} = \mathcal{L}(q, \dot{q})$
Configuration variables	A	q
Generalized velocities	$F \equiv DA$	$\dot{q}$
Lagrange equations	$D\left(\frac{\partial \mathcal{L}}{\partial F}\right) = J$ $J = \frac{\partial \mathcal{L}_m}{\partial A}$	$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$
Conjugate momenta	$H = \frac{\partial \mathcal{L}}{\partial DA} = \frac{\partial \mathcal{L}}{\partial F}$	$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$
Constitutive relations	$H = H(F)$ $H \sim \star F$ (linear)	$p = p(\dot{q})$
Canonical variables	(A, H)	(q, p)
Hamiltonian	$\mathcal{H} \equiv F \wedge H - \mathcal{L}$ $F = F(H)$	$\mathcal{H} \equiv \dot{q}p - \mathcal{L}(q, \dot{q})$ $\dot{q} = \dot{q}(p)$
Hamilton equations	$DA = F = \frac{\partial \mathcal{H}}{\partial H}$ $DH = -\frac{\partial \mathcal{H}^m}{\partial A} = J$	$\frac{d}{dt}q = \frac{\partial \mathcal{H}}{\partial p}$ $\frac{d}{dt}p = -\frac{\partial \mathcal{H}}{\partial q}$

Table 3.1: The analogies between classical mechanics and Yang-Mills fields.

	Gravity Yang-Mills $\hat{\mathcal{H}} = \hat{\mathcal{H}}(\hat{\Gamma}^{ab}, \hat{\vartheta}^a, \hat{H}^{ab}, \hat{H}^a)$	Yang-Mills $\hat{\mathcal{H}} = \hat{\mathcal{H}}(\hat{A}, \hat{H})$	Quantum Mec. $\hat{\mathcal{H}} = \hat{\mathcal{H}}(\hat{q}, \hat{p})$
Quantum operators	$\Gamma^{ab} \to \hat{\Gamma}^{ab}$ $\vartheta^a \to \hat{\vartheta}^a$ $H^{ab} \to \hat{H}^{ab}$ $H^a \to \hat{H}^a$	$A \to \hat{A}$ $H \to \hat{H} \sim -i\frac{\partial}{\partial A}$	$q \to \hat{q}$ $p \to \hat{p} = -i\hbar\frac{d}{dq}$
Commutation relations	$[\hat{\Gamma}^{ab}, \hat{H}^{ab}] \neq 0$ $[\hat{\vartheta}^a, \hat{H}^a] \neq 0$	$[\hat{A}, \hat{H}] \neq 0$	$[\hat{q}, \hat{p}] = -i\hbar$

Table 3.2: The analogies between canonical quantization in quantum mechanics and in the exterior calculus approach to Yang-Mills theories and gauge theories of gravity (*à la* Yang-Mills).

excitation H is the conjugate momentum. The constitutive relation $H(F)$ is implicit in the Lagrangian formulation and corresponds in perfect analogy to the functional relation between generalized velocities and conjugate momenta of classical mechanics. As an extension of these analogies, one is naturally led to identify the appropriate pairs of canonically conjugate variables that should be quantized in the corresponding (canonical) quantum field theory (see Table 3.2).

The gauge approach to gravity

The question that arises now is whether we can apply the same procedure to gravity. The approach of Yang, Mills and Utiyama went beyond the first ideas on gauge invariance

	Gravity Yang-Mills $\mathcal{L}_g = \mathcal{L}_g(g_{ab}, \vartheta^a, D\Gamma^a_b, D\vartheta^a)$	Yang-Mills $\mathcal{L} = \mathcal{L}(A, DA)$
Configuration variables	$(\Gamma^a_b, \vartheta^a)$	A
Generalized velocities	$R^a_b = D\Gamma^a_b$ $T^a = D\vartheta^a$	$F \equiv DA$
Lagrange equations	$D\left(\frac{\partial \mathcal{L}}{\partial R^{ab}}\right) - \varsigma_{ab} = s_{ab}$ $D\left(\frac{\partial \mathcal{L}}{\partial T^a}\right) - \pi_a = \tau_a$	$D\left(\frac{\partial \mathcal{L}}{\partial F}\right) = J$ $J = \frac{\partial \mathcal{L}_m}{\partial A}$
Conjugate momenta	$H_{ab} = -\frac{\partial \mathcal{L}}{\partial R^{ab}}$ $H_a = -\frac{\partial \mathcal{L}}{\partial T^a}$	$H = \frac{\partial \mathcal{L}}{\partial DA} = \frac{\partial \mathcal{L}}{\partial F}$
Constitutive relations	$H^{ab} = H^{ab}(R^{ab}, T^a) \quad H^a = H^a(R^{ab}, T^a)$ $H^{ab} \sim \star R^{ab} \quad H^a \sim \star T^a$ (linear)	$H = H(F)$ $H \sim \star F$ (linear)
Canonical variables	$(\Gamma^a_b, H^a_b) \quad (\vartheta^a, H^a)$	(A, H)
Hamiltonian	$\mathcal{H} \equiv R^{ab} \wedge H_{ab} + T^a \wedge H_a - \mathcal{L}(\Gamma, \vartheta, R, T)$ $R^{ab} = R^{ab}(H^{ab}, H^a) \quad T^a = T^a(H^{ab}, H^a)$	$\mathcal{H} \equiv F \wedge H - \mathcal{L}$ $F = F(H)$
Hamilton equations	$D\Gamma^{ab} = \frac{\partial \mathcal{H}}{\partial H_{ab}} \quad DH_{ab} = -\frac{\partial \mathcal{H}^{eff}}{\partial \Gamma^{ab}}$ $D\vartheta^a = \frac{\partial \mathcal{H}}{\partial H_a} \quad DH_a = -\frac{\partial \mathcal{H}^{eff}}{\partial \vartheta^a}$	$DA = F = \frac{\partial \mathcal{H}}{\partial H}$ $DH = -\frac{\partial \mathcal{H}^m}{\partial A} = J$

Table 3.3: The analogies between Yang-Mills field and gravity-Yang-Mills theory in the language of exterior forms.

introduced by Weyl. In fact, while Yang and Mills [59] extended Weyl's gauge principle to the $SU(2)$ isospin rotations in an attempt to describe nuclear interactions, Utiyama [60] extended the gauge principle to all semi-simple Lie groups including the Lorentz group and tried to derive GR from the gauging of the Lorentz group. Although there is some validity in his approach and an undoubtedly importance of the Lorentz group in GR, his derivation is not fully self-consistent to the formal structure of a gauge field theory. This is mainly because the Noether conserved current of the Lorentz group is not the energy-momentum, which Utiyama forced to be the source of gravity in order to obtain GR.[9] Although these efforts did not include gravity consistently, it revealed that, on a fundamental level, gauge symmetries lie at the heart of modern field theories of physical interactions. Nevertheless, this gauge formalism eventually returned to gravity when in the 60's Sciama and Kibble localized the Poincaré group of spacetime symmetries and in this way managed to show that gravity can also be consistently described as a gauge theory. Indeed, the analogies with gauge Yang-Mills theories can be easily established, as summarized in tables 3.3 and 3.4.

One of the most remarkable features of the gauge approach to gravity is the intimate

[9]From a gauge theoretical perspective, the Lorentz group is not the symmetry group of Einstein's gravity. It became clear some years later that the appropriate way to derive GR from a gauge principle is to consider it as a translational gauge theory of gravity. This class of theories lives on a Weitzenböck spacetime geometry with torsion and vanishing curvature and non-metricity. These geometries and its use in attempts for a unified classical field theory were worked out by Weitzenböck, Cartan and Einstein, for example, during the first period of the so-called teleparallel formulation gravity (up to 1938). A second period in the 60's by Möller and others revitalised the interest in such theories which have more recently re-gained much attention, particularly via its $f(T)$ extensions (see e.g. [151]).

link between group considerations and spacetime geometry. Non-rigid (local) spacetime symmetries require non-rigid (non-Euclidian) geometries. Moreover, as previously mentioned, by extending the symmetry group one is led to extend the spacetime geometry as well and, in this way, post-Riemann geometries have a natural place within gauge theories of gravity. For instance, while the translational gauge theories (TGTG) include non-vanishing torsion but zero curvature and non-metricity, Poincaré gauge gravity requires a RC geometry, and both Weyl(-Cartan) gauge gravity (WGTG) and conformal gauge gravity (CGTG) live on subsets of the more general metric-affine geometry with curvature, torsion and non-metricity. In particular, in WGTG the traceless part of the non-metricity vanishes. Both the 10-parametric Poincaré group and the 11-parametric Weyl group are non-simple, meaning that they can be divided into two smaller groups (a non-trivial normal sub-group and the corresponding quotient group) and the natural extension from the corresponding theories of gravity into one with a simple group leads to CGTG. The 15-parametric conformal group $C(1,3)$ is simple (its only normal sub-groups are the trivial group and the group itself), but this extension requires a generalization of Kibble's gauge procedure, due to the fact that although locally $C(1,3)$ is isomorphic to $SO(2,4)$ its realization in M_4 (Minkowski spacetime) is non-linear [74].

The PGTG can further be extended into the de Sitter or anti-de Sitter (A)dS gauge theories of gravity by localizing the $SO(1,4)$ or $SO(2,3)$ groups, respectively. Due to the fact that the (A)dS space is a maximally symmetric space which can be embedded into 5-dimensional Minkowski space (with two or one time coordinates for AdS or dS, respectively), its isometries obey Lorentz type of algebra. Under a specific limit (by setting $l \to \infty$, where l is a parameter of the group algebra), the group goes into the Poincaré algebra. Depending on the choices of the Lagrangian, one can then have explicit [75] or spontaneous [76] symmetry breaking from $SO(2,3)$ to $SO(1,3)$, for instance.

Another important class of extensions requires going beyond the Lie algebra by considering the algebra with anticommutators, in order to arrive at the super-Poincaré group [77] containing the usual Poincaré generators and proper supersymmetry (SUSY) transformations. These are generated by a Majorana spinor which acts as the (anticommuting) generator of the transformations between fermions and bosons. The simple (with one supersymmetry generator) AdS supersymmetry generalizes the simple super-Poincaré algebra although it has the same generators, and it goes back to the super-Poincaré algebra under the same limit as the AdS group goes back into the PGTG. Further extensions include the consideration of a number $1 < N < 8$ of supersymmetry generators [20]. The gauging of these super-algebras lead directly to the bosonic gravity sector and therefore, supergravity (SUGRA) is an important class of supersymmetric gauge theories of gravity, extremely relevant for unification methods of bosons and fermion by the link it establishes between external (spacetime) symmetries and internal symmetries. In the self-consistent gauge approach, this class of theories needs to take into account post-Riemannian spacetime geometries, although many of the approaches have been done within the Riemannian geometry [19]. For more details on the history, mathematics and philosophical aspects of gauge theories see [36].

To illustrate the structure of the gauge approach to gravity we next consider the PGTG in more detail.

The gravity Yang-Mills equations of PGTG

By applying to gravity a similar procedure as that of the Yang-Mills approach to gauge fields, one arrives at the mathematical structure of gauge theories of gravity. One starts with the (rigid) global symmetries of a matter Lagrangian with respect to a specific group of spacetime coordinate transformations, and the conserved Noether currents are identified. Then, by localizing (gauging) the symmetry group, the gauge gravitational

	Gravity Yang-Mills $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m$ $\mathcal{L}_g = \mathcal{L}_g(g_{ab}, \vartheta^a, D\Gamma^a_{\ b}, D\vartheta^a)$	Yang-Mills $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_m$ $\mathcal{L} = \mathcal{L}(A, DA)$
gauge potentials	$(\Gamma^a_{\ b}, \vartheta^a)$ 1-forms	A 1-form
Field strengths	$R^a_{\ b} = D\Gamma^a_{\ b}$ $T^a = D\vartheta^a$	$F \equiv DA$
Symmetry group	$PGTG : SO(1,3) \rtimes$ $T(4)$ $MAG : GL(4, \Re) \rtimes$ $T(4)$	$SU(N)$
Noether currents (sources)	$PGTG:$ $s^{ab}_{\ \ \mu}$ spin , $s_{ab} \equiv \delta\mathcal{L}_m/\delta\Gamma^{ab}$ $\tau^a_{\ \mu}$ energy-momentum , $\tau_a \equiv \delta\mathcal{L}_m/\delta\vartheta^a$ $MAG:$ $\Delta^{ab}_{\ \ \mu}$ Hypermomentum, $\Delta_{ab} \equiv \delta\mathcal{L}_m/\delta\Gamma^{ab}$ $\tau^a_{\ \mu}$ energy-momentum , $\tau_a \equiv \delta\mathcal{L}_m/\delta\vartheta^a$	$J = \frac{\partial\mathcal{L}_m}{\partial A}$ (charge, isospin, etc)
Excitations	$PGTG:$ $H_{ab} = -\delta\mathcal{L}_A/\delta R^{ab}$ $H_a = -\delta\mathcal{L}_A/\delta T^a$	$H = \delta\mathcal{L}_A/\delta F$
Field equations	$PGTG:$ $DH_{ab} - \varsigma_{ab} = s_{ab}$ $DH_a - \pi_a = \tau_a$	$DH = J$
Bianchi identities	$PGTG:$ $dR^a_{\ b} + \Gamma^a_{\ c} \wedge R^c_{\ b} = -R^a_{\ c} \wedge \Gamma^c_{\ b} \quad (DR^a_{\ b} = 0)$ $DT^a = R^a_{\ c} \wedge \vartheta^c$	$dF = -F \wedge A$ $(DF = 0)$

Table 3.4: The analogies between Yang-Mills fields and gravity-Yang-Mills theory in the language of exterior forms.

potentials are introduced as well as the gauge covariant derivative and the respective field strengths, which are well known objects from differential geometry. Indeed, the gauge potentials represent the generators of the local symmetry group and couple to the respective conserved Noether currents, which act as sources of gravity. In practical terms, the identification of the appropriate gauge field potentials comes from the requirement of covariance of $D\Psi$.

In PGTG the tetrads and the spin connection 1-forms are the gauge potentials, associated with translations, $T(4)$, and Lorentz rotations, $SO(1,3)$, respectively. Torsion and the curvature 2-forms are the respective field strengths. Torsion can be decomposed into 3 irreducible parts $T^a = T^a_{(1)} + T^a_{(2)} + T^a_{(3)}$, made of a tensor part with 16 independent components, a vector part and an axial (pseudo) vector, both with 4 independent components[10]. As for the curvature, it has 36 independent components which can be decomposed into 6 irreducible parts: Weyl (10), Paircom (9), Ricsymf (9), Ricanti (6), scalar (1), and pseudoscalar (1). In addition, there are 6 generators in the Lorentz

[10] In the minimal coupling to fermions, only the axial vector torsion is involved.

group with 6 potentials ($\Gamma^{ab} = -\Gamma^{ba}$) and 6 spin (Noether) currents ($s^{ab} = -s^{ba}$). Analogously, there are 4 generators in the group of spacetime translations, which entail 4 gauge potentials θ^a and 4 conserved Noether currents τ^a. By constructing the gravitational Lagrangian with the curvature and torsion invariants, the potentials are coupled to the Noether currents via $24 + 16 = 40$ second order field equations. The PGTG formalism is summarized below

Matter. We consider a physical system, represented by the matter fields Lagrangian $L_m = L_m(g_{ab}, \theta^c, D\Psi)$.

- The covariant derivative with respect to the Riemann-Cartan connection $D\Psi$ allows the Lagrangian to be invariant under Local Poincaré spacetime transformations.
- The Noether conserved currents are: The canonical energy-momentum tensor density which is equivalent to the dynamical tetrad energy-momentum density $\tau_a \equiv \delta L_m/\delta\theta^a$ and the canonical spin density, which is equivalent to the dynamical spin density $s_{ab} \equiv \delta L_m/\delta\Gamma^{ab}$
- These currents couple to the gravitational potentials, acting as sources of gravity and obey generalized conservation equations

Inhomogeneous gravity Yang-Mills equations The gravity sector in the action is constructed with the gauge-invariant gravitational field strengths in the kinetic part associated with the dynamics of the gravitational degrees of freedom.

The total Lagrangian density thus reads

$$\mathcal{L} = \mathcal{L}_G(g_{ab}, \theta^a, T^a, R^{ab}) + \mathcal{L}_m(g_{ab}, \theta^a, D\Psi) \ . \tag{3.51}$$

By varying this action with respect to the gauge fields of gravity (θ^a, Γ^{ab}) and the matter fields Ψ, we get the corresponding field equations. For the fermionic matter fields, the variational principle $\delta L_m/\delta\Psi = 0$ leads to a generalization of the Dirac equation. As for the bosonic sector (gravity), the inhomogeneous Yang-Mills equations in PGTG are

$$DH_a - \pi_a = \tau_a, \qquad DH_{ab} - \varsigma_{ab} = s_{ab}, \tag{3.52}$$

where $H_a = -\partial\mathcal{L}_G/\partial T^a$ and $H_{ab} = -\partial\mathcal{L}_G/\partial R^{ab}$ are the 2-form excitations (field momenta) associated to torsion and curvature, and $\tau_a \equiv \delta\mathcal{L}_m/\delta\theta^a$ and $s_{ab} \equiv \delta\mathcal{L}_m/\delta\Gamma^{ab}$ are the 3-form canonical energy-momentum and spin currents. The 3-forms π_a and ς_{ab} can be interpreted as the energy-momentum and spin of the gravitational gauge fields, respectively, defined as

$$\pi_a \equiv e_a \rfloor L_G + (e_a \rfloor T^b) \wedge H_b + (e_a \rfloor R^{cd}) \wedge H_{cd}, \quad \varsigma_{ab} \equiv -\theta_{[a} \wedge H_{b]} \ . \tag{3.53}$$

For a given theory, one only needs to compute the excitations, the source currents and the gravitational currents from the Lagrangian density (3.51) and substitute directly in the inhomogeneous equations (3.52). Note that in this formalism of exterior forms these field equations are completely metric-free, fully general and coordinate-free, with well defined gravitational energy-momentum and spin currents. The PGTG have two sets of Bianchi identities, previously introduced as $DR^a_b = 0$ and $DT^c = R^c_d \wedge \theta^d$, which are intrinsic to the geometrical structure of RC spacetime. Via the field equations, these can be related to the generalized conservation equations for the energy-momentum and the spin currents.

Constitutive relations. The 2-form excitations expressed in terms of the field strengths (torsion and curvature), $H_a = H_a(T^c, R^{bd})$ and $H_{ab} = H_{ab}(T^c, R^{bd})$, represent two sets of constitutive relations and are implicit in the Lagrangian formulation. These excitations are in exact analogy to the canonically conjugate momenta of classical mechanics, while torsion and curvature are the generalized velocities for the gravitational degrees of freedom represented by the gauge potentials. *These constitutive relations in vacuum can be interpreted as describing the gravitational propagation properties of the spacetime physical manifold. It is via these relations that the conformally invariant part of the metric is introduced, via the Hodge star operator.* Moreover, in these relations the coupling constants of the theory are required in order to adequately convert the dimensions of the field strengths (field velocities) to the excitations (field momenta). *As constitutive relations for the spacetime vacuum itself, one can postulate that such coupling constants characterize physical properties of the spacetime manifold endowed with gravitational geometrodynamics.* This hypothesis is in clear analogy to the similar interpretation for the electromagnetic properties entering in the corresponding constitutive relations as we saw in section 2 [1].

Quadratic Poincaré gauge gravity (including parity breaking terms)

Poincaré gauge theories of gravity (PGTG) have been investigated with special interest on Lagrangians quadratic in the curvature and torsion invariants, and in applications regarding cosmology, gravitational waves, and spherical solutions, see e.g. [78, 79, 80, 81, 82, 83]. The PGTG class is fundamental given the importance of the Poincaré symmetries in relativistic field theories and the most general quadratic Lagrangian (*à la* Yang-Mills) contains parity breaking terms induced by the richer Riemann-Cartan (RC) geometry with curvature and torsion [78]. It can be written as

$$\begin{aligned} L &= \frac{1}{2\kappa^2}\Big[(a_0\eta_{ab} + \bar{a}_0\theta_a \wedge \theta_b) \wedge R^{ab} - 2\Lambda\eta - T^a \wedge \sum_{I=1}^{3}\left(a_I(\star T_a^{(I)}) + \bar{a}_I T_a^{(I)}\right)\Big] \\ & \quad -\frac{1}{2\rho}R^{ab} \wedge \sum_{I=1}^{6}\left(b_I(\star R_{ab}^{(I)}) + \bar{b}_I R_{ab}^{(I)}\right). \end{aligned} \tag{3.54}$$

In this expression $\kappa^2 = 8\pi G$, where G is Newton's coupling constant. The first term on the first line corresponds to the ECSK theory plus the Holst term $\sim (\theta_a \wedge \theta_b) \wedge R^{ab}$, where $\eta_{ab} \equiv e_b \rfloor \eta_a = \star(\theta_a \wedge \theta_b)$.[11]. The second term corresponds to a cosmological constant. The remaining terms on the first line contain the terms quadratic in the torsion field strength and the index $I = 1, 2, 3$ runs over the three irreducible pieces of the torsion. In the third line we have the curvature quadratic terms and the index $I = 1, ..., 6$ runs over the six irreducible pieces of curvature. The free parameters include the $2 + 6 + 12 = 20$ (a, b) coefficients, plus the cosmological constant and the some times called "strong gravity" parameter ρ. In this Lagrangian, all the terms with the coefficients with a bar $\bar{a}_0, \bar{a}_I, \bar{b}_I$ break the symmetry under Parity transformations. For specific choices and assumptions this Lagrangian includes GR, the Teleparallel equivalent to GR (TEGR), or the ECSK theory, for example. In the latter, Dirac fermions have axial-axial contact interactions (the Hehl-Datta term) with a repulsive character, while in general quadratic PGTG this contact spin-spin interaction is generalized by predicting a propagating interaction. In particular, intermediating gauge bosons with spins $s = 0, 1, 2$ are predicted, which correspond to massive or massless scalar, vector[12] and tensor propagating modes, respectively.

[11] Here $\eta_a = e_a \rfloor \eta = \star\theta_a$ is a 3-form and $\eta = \star 1$ is the natural volume 4-form (see appendix A.1).

[12] This is actually a torsion axial vector which couples to elementary particles.

In these GW fields there are odd parity (parity breaking) modes which could manifest themselves as signatures of chirality in the GW cosmological backgrounds from the early Universe[13] (see [84] for excelent review on cosmological GW backgrounds). Let us also note that in PGTG it is possible to identify ghost-free Lagrangians which can also be quantized [85, 86][14].

The Teleparallel equivalent to GR. Choosing the Weitzenböck spacetime geometry, i.e, retaining only torsion and setting curvature to zero and under specific restrictions (appropriate choice for the Weitzenböck connection), a Lagrangian quadratic in torsion can be chosen which gives a dynamics formally equivalent to that of GR for spinless matter. This is more consistently approached from a translational gauge theory of gravity perspective. Heuristically, one gets [19, 20]

$$D_\alpha T_c{}^{\beta\gamma} + (...) \sim \kappa^2 \tau_c{}^{\gamma}, \tag{3.55}$$

or in terms of the tetrads

$$\Box\theta^a_{\ \mu} + (...) \sim \kappa^2 \tau^a_{\ \mu}, \tag{3.56}$$

where the missing terms on the left-hand side are non-linear terms, and $\Box$ is a d'Alembertian operator. The equation (3.56) resembles Einstein's GR equation $\Box g_{\mu\nu} + ... \sim \kappa^2 T_{\mu\nu}$,, and it turns out that both equations yield the same gravitational phenomenology for matter described by fundamental scalar fields or Maxwell fields, where the canonical $\tau_{\mu\nu}$ and the dynamical (Hilbert-Einstein) $T_{\mu\nu}$ energy-momentum tensors coincide. For fermions the theories are not fully equivalent. It is interesting to point out that the quadratic (in torsion) Lagrangian of this teleparallel equivalent of GR (TEGR) is locally Lorentz invariant and equivalent to the Hilbert-Einstein Lagrangian [22], but it results from a translational gauge approach. Therefore, if one wants to formulate GR as a gauge theory then one must gauge the translational group instead of the Lorentz group. This provides yet another motivation to go beyond GR, since it is plausible to consider the whole Poincaré symmetries in Nature to be valid, not only the translational group

The Einstein-Cartan-Sciama-Kibble. Although Cartan introduced the theory almost one century ago, it continues to trigger interest due to its non-singular solutions (in black holes and cosmology), by the bridge it establishes between fermionic spinors and gravity, and by its elegance and simplicity, as it possesses no free parameters besides Newton's constant. Within its many applications we underline bouncing cosmologies [64, 71, 87], inflation, cosmological constant and dark energy [88, 68, 89], perturbations and cosmic microwave background radiation [90, 91], phase and signature transitions [66, 92], or compact objects [69, 70, 67]. For EC gravity coupled to Dirac fields, one also finds applications in particle physics, see e.g. [93, 94, 95, 96, 97, 98, 99]. The need to go beyond EC theory was recognized long ago, mainly due to the fact that the theory is still non-renormalizable [100], but also because quadratic Lagrangians present a natural and theoretically preferable extension [101, 102, 103] (see also [104, 105]).

In the formalism of exterior forms, the EC Lagrangian can be written as

$$L = \frac{1}{2\kappa^2}\eta_{ab} \wedge R^{ab}. \tag{3.57}$$

[13]For that, one needs non-planar detectors, 3-point correlation functions analysis and sufficient signal/noise ratio, besides a clear distinction from other possible GW sources with a similar power spectrum signature.

[14]It has been understood that this is possible at the linear perturbation level. At higher orders non-linearities tipically introduce ghosts and dynamical instabilities.

The field equations are then obtained by varying this action with respect to the tetrads and the (Lorentzian) spin connection. In the more common tensor formalism this Lagrangian corresponds to the linear Lagrangian in the curvature scalar, yielding the action

$$S_{\rm EC} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R(\Gamma) + \int d^4x\sqrt{-g}\,\mathcal{L}_m\,. \tag{3.58}$$

In the RC spacetime the curvature scalar (3.29) includes terms quadratic in torsion, while the matter Lagrangian, $\mathcal{L}_m = \mathcal{L}_m(g_{\mu\nu}, \Gamma, \psi_m)$ depends on the metric and the matter fields, ψ_m and also on the contortion via the covariant derivatives. The Cartan equations can be obtained by varying the action (3.58) with respect to the contortion tensor $K^{\alpha}_{\ \beta\gamma}$ (or to the spin connection), which can be written as

$$T^{\alpha}_{\ \beta\gamma} + T_\gamma\delta^\alpha_\beta - T_\beta\delta^\alpha_\gamma = \kappa^2 s^{\alpha}_{\ \beta\gamma}, \tag{3.59}$$

where

$$s^{\gamma\alpha\beta} \equiv \frac{\delta\mathcal{L}_m}{\delta K_{\alpha\beta\gamma}}\,, \tag{3.60}$$

is the spin density tensor, $T_\beta \equiv T^{\gamma}_{\ \beta\gamma}$ and $s_\beta \equiv s^{\gamma}_{\ \beta\gamma}$ are the torsion and spin (trace) vectors, respectively. Cartan's equations (3.59) imply that torsion is related to the spin density of matter fields via linear and algebraic relations therefore, in the absence of spin (as in vacuum) torsion vanishes. Variation of the action (3.58) with respect to the spacetime metric $g_{\mu\nu}$ (or the tetrads) yields the generalized Einstein equations which can be written as

$$G_{\mu\nu} = \kappa^2\tau_{\mu\nu}, \tag{3.61}$$

where $G_{\mu\nu}$ is the generalized Einstein tensor of the RC geometry and $\tau_{\mu\nu}$ is the canonical energy-momentum. These equations can also be suitably written as

$$\tilde{G}_{\mu\nu} = \kappa^2(T_{\mu\nu} + U_{\mu\nu}), \tag{3.62}$$

where $\tilde{G}_{\mu\nu}$ is the Einstein tensor computed with the Levi-Civita connection. The effective stress-energy tensor

$$T^{\rm eff}_{\mu\nu} = T_{\mu\nu} + U_{\mu\nu}\,, \tag{3.63}$$

includes the (dynamical) metric energy-momentum tensor of the matter fields $T_{\mu\nu}$ and the tensor, $U_{\mu\nu}$, where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}, \quad U_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}C)}{\delta g^{\mu\nu}}, \tag{3.64}$$

with

$$C \equiv -\frac{1}{2\kappa^2}\Big(K^{\gamma}_{\ \beta\gamma}K^{\alpha\beta}_{\ \ \ \alpha} + K^{\alpha\lambda\beta}K_{\lambda\beta\alpha}\Big), \tag{3.65}$$

containing corrections quadratic in torsion $U \sim \kappa^{-2}T^2$. These corrections can also be expressed in terms of the spin tensor using Cartan's equations (3.59), i.e, $U \sim \kappa^2 s^2$. In general, torsion also contributes to the tensor $T_{\mu\nu}$, since the covariant derivatives present in the kinetic part of $\mathcal{L}_m$ introduce new terms depending on torsion via minimal or non-minimal couplings. Since $U \sim \kappa^2 s^2$, Eq. (3.62) defines a typical density[15] $\rho_C \sim$

[15] In cosmological applications the critical density can be written as $\rho_{crit} \sim m/\lambda_{Comp} l_{Pl}^2$, where l_{Pl} and λ_{Comp} are Planck's length and Compton wavelength, respectively. For electrons we get $\rho_{crit} \sim 10^{52}{\rm g/cm}^3$, corresponding to $T_{crit} \sim 10^{24}K$ and around $t \sim 10^{-34}$s after BigBang.

10^{54}g/cm^3 (if one considers nuclear matter), known as Cartan's density. Therefore, in principle, EC theory can only introduce significant physical effects via equation (3.62), in environments of very large spin densities, which might arise in the early universe or in the innermost regions of black holes.

To conclude this part, let us mention that, once the coupling to fermions is considered, as we will see in chapter 4, the generalized Dirac (Hehl-Datta) equation for spinors in RC spacetime coupled to the ECSK gravity, is cubic in the spinors and includes a torsion induced spin-spin (axial-axial) contact interaction. This type of interactions have been searched in particle physics experiments including studies at HERA, LEP and Tevatron in electron-proton scattering [106, 107].

Quadratic models in MAG

One can generalize the PGTG by considering the affine group as the gauge symmetry group for gravity. If this is performed *à la* la Yang-Mills, then one gets quadratic models as in quadratic PGTG (qPGTG), where the quadratic terms are sometimes referred to as (hypothetical) "strong gravity" terms. This idea has been recovered from time to time, as in Yang [108], in the tensor dominance model [109] or in chromogravity [110]. Depending on the choice of the Lagrangian, the strong gravity (bosonic sector) can be very massive or massless. In some respect these gauge bosonic gravity fields are similar to the Yang-Mills bosons, and if they are massive it is typically assumed that the masses are of the order of the Planck mass or even above. As in quadratic PGTG, in quadratic metric-affine models there are intermediate gauge bosons with spins ($s = 0, 1, 2$) corresponding to scalar, vector or tensor modes (massive or massless). In this respect, the quadratic model in Eq.(3.54) for the PGTG can be extended to metric-affine gravity by including terms with Q_{ab} and $R_{(ab)}$ which are both zero in PGTG, in accordance with the choice of a Lorentzian connection and, therefore, of zero non-metricity (see [111, 112, 113]).

As an example of a quadratic metric-affine model, one can consider a Lagrangian density (for the bosonic-gravity sector) of the form (schematically)

$$\mathcal{L}_{MAG} \sim \frac{1}{\kappa^2}\left(R + T^2 + TQ + Q^2\right) + \frac{1}{\rho}\left(W^2 + Z^2\right), \tag{3.66}$$

where $W_{ab} \equiv R_{[ab]}$, and $Z_{ab} \equiv R_{(ab)}$ are known as the "rotational curvature" and the "strain curvature", respectively. The terms proportional to the coupling constant ρ are referred to as strong gravity terms, in contrast to the terms that are proportional to the "weak" gravity coupling constant κ^2. Note that in this model the connection is non-Lorentzian and the non-metricity 1-form (represented by Q) is non-vanishing. The bosonic sector of metric-affine gravity was analyzed, for instance, in [111, 112, 113, 114], while the fermionic part is more delicate (see for instance [115, 116, 117, 118]). In this respect, there is no finite dimensional spinor representation of the $GL(4, \Re)$ group, which leads to the introduction of the "world spinors" (with infinite components) of Ne'eman, and to the corresponding generalization of Dirac equation. The world spinor formalism is related to Regge trajectories, which are themselves related to spin-2 excitations of hadrons (see [119]). Non-minimal couplings in metric-affine extensions to $f(R)$ gravity can be found in [120].

For an interesting review on exact solutions in MAG see [121]. Further research in this context involves cosmological scenarios [122], and one should also mention exact spherically symmetric solutions with $Q \sim 1/r^d$ [123], which suggests the existence of massless modes.

3.1.5 Probing non-Riemannian geometry with test matter

It is known that torsion can give rise to precession effects in systems with intrinsic spin, for example, elementary particles such as the electron, or baryons such as the neutron [124, 125]. This is a model-independent result which can be obtained from a (WKB) semi-classical approximation of the Dirac fermionic dynamics in a RC spacetime. In principle this prediction can be used to distinguish between the spacetime paradigms of GR and TEGR. If v is the polarization vector of the (intrinsic) spin then one can deduce the simple expression

$$\dot{v} = 3\breve{t}v \ , \tag{3.67}$$

where the axial vector $\breve{t}$ is given by $\breve{t}^{\alpha} \equiv -\epsilon^{\alpha\beta\gamma\delta}T_{\beta\gamma\delta}$. In this sense it is plausible that experiments similar to the Gravity Probe-B [126] but using gyroscopes with intrinsic (macroscopic) spin can be used to constrain or detect the effects of the (hypothetical) torsion around the Earth [127]. There have been quite a number of studies on spin precession effects induced by torsion (see also [128]). On the other hand, Lammerzahl have set experimental limits for detecting torsion ($|T| \sim 10^{-15}m^{-1}$) using Hughes-Drever (spectroscopic) type of experiments [129].

Moreover, as we will see in chapter 4, from the Dirac Lagrangian of a fermion minimally coupled to the background RC geometry, one can predict torsion effects on the energy levels of quantum systems [8]. Similarly, if non-minimal couplings between the torsion trace vector and Dirac axial vector and/or between torsion axial vector and Dirac vector (see chapter 4) are present, then the parity symmetry is broken and the corresponding energy level corrections due to torsion will contain the signatures of those parity breaking interactions. Therefore, tests with advanced spectrographs might be able to probe torsion effects on quantum systems. Also, the previously mentioned spin-spin contact interactions of the ECSK theory, or the propagating spin-spin interactions mediated by gravitational gauge ($s = 0, 1, 2$) bosons might be tested/constrained in laboratory experiments and cosmological GW probes.

Regarding non-metricity, the gauge approach to gravity clearly shows that the hypermomentum currents, such as the dilatations or the shear currents, couple to the trace and shear parts of the connection (3.26), respectively and therefore to non-metricity. This coupling is evident since, as we have seen, the potential for the $GL(4, \Re)$, the (non-Lorentzian) connection, couples to the Noether conserved hypermomentum and a non-Lorentzian connection implies non-metricity. Further developments have shown that, if torsion can be measured by the spin precession of test matter with intrinsic spin, then the non-metricity of spacetime can be measured by pulsations (mass quadrupole excitations) of test matter with (non-trivial) hypermomentum currents. In order to be "sensitive" to non-Riemann geometries, test matter should carry dilatation, shear, or spin currents, whether macroscopic or at the level of fundamental fields/particles. In the latter case and regarding the shear currents, the Regge trajectories provide an adequate mathematical illustration of test matter as Ne'eman's world spinors with shear.

Let us also point out that Obukhov and Puetzfeld [130] have derived the equations of motion for matter fields in metric-affine gravity. By making use of the Bianchi identities one can arrive at the following expression for the translational Noether current τ^a:

$$\tilde{D}\Big[\tau_a + \Delta^{bc}\left(e_a \rfloor N_{bc}\right)\Big] + \Delta^{bc} \wedge (\pounds_{e_a} N_{bc}) = s^{bc} \wedge (e_a \rfloor \tilde{R}_{cb}), \tag{3.68}$$

where, as usual, the tilde refers to quantities defined in the Riemann geometry, while τ_a, Δ^{bc}, s^{bc} are the canonical energy-momentum, hypermomentum current, and spin current, respectively, and $N_{bc} = \Gamma_{bc} - \tilde{\Gamma}_{bc}$ gives the non-Riemannian piece of the connection 1-form, that is, the distortion 1-form. Note that in the right-hand side of this equation we can identify the Mathisson-Papapetrou force density for matter with spin. In standard GR, one obtains $\tilde{D}\tau_a = 0$, which gives the geodesic equation for spinless matter with energy-momentum, while for $N_{bc} = 0$ we get $\tilde{D}\tau_a = s^{bc} \wedge (e_a \rfloor \tilde{R}_{cb})$, which is the Mathisson-Papapetrou equation in GR for matter with spin. In this equation (3.68), if matter has neither (intrinsic) spin, nor dilatation/shear currents, then it follows the Riemannian (extremal length) geodesics, regardless of the geometry of spacetime or of the form of the Lagrangian in metric-affine geometry.

3.1.6 Metric-Affine geometry and the breaking of Lorentz symmetry

Regarding the analogous law for the $GL(4, \Re)$ Noether current, $\Delta^a_{\ b}$, we have

$$D\Delta^a_{\ b} + \theta^a \wedge \tau_b = \tau^a_{\ b} \ . \tag{3.69}$$

This gives the evolution of hypermomentum and the general relativistic limit of this expression gives the Mathisson-Papapetrou equation for the evolution of spin in GR (its predictions served as basic starting point for the Gravity probe-B experiment [126]). In PGTG the connection is Lorentzian $\Gamma^{ab} = -\Gamma^{ba}$, while in the WGTG the trace part of the connection $\frac{1}{4}g^{ab}\Gamma^c_{\ c}$ is also non-vanishing. A fully independent connection in metric-affine theory is given by the expression in (3.26). It is this linear connection that couples to the (intrinsic) hypermomentum current (see Eq.(3.25)). As mentioned, the Lorentzian connection couples to spin s_{ab} carrying $SO(1,3)$ charges, while the trace part couples to the dilatation current $\Delta^c_{\ c}$, and the shear part of the connection couples to the shear current $\bar{\Delta}_{ab}$ carrying $SL(4,\Re)/SO(1,3)$ (intrinsic) shear charges. The shear current seems to be related to the Regge trajectories [118] and these represent spin-2 excitations of hadrons with the same internal quantum numbers. In fact, The Regge trajectories can be described by the group $SL(3,\Re)$ and this can be embedded in $SL(4,\Re)$. The relation between the Regge trajectories and the hypermomentum shear charges, remains an open question under study and its validity seems to point to a quite remarkable and promising connection between the strong interaction of hadrons and spacetime post-Riemann geometries. The shear charge is actually a measure of the breaking of Lorentz invariance. The bottom line is that, in order to get Lorentz symmetry breaking, one does not require the introduction of extra particle degrees of freedom, but this can be obtained solely via the geometrical structures of spacetime, namely a non-Lorentzian connection. The presence of a non-Lorentzian connection implies the non-vanishing non-metricity and the non-vanishing strain curvature $R_{(ab)}$.

3.1.7 Formulations of GR: Einstein, Teleparalell, Symmetric Teleparalell - Open questions

The (canonical) metric formulation of GR requires a pseudo-Riemannian manifold with a symmetric, $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$, and metric-compatible, $\nabla^\Gamma_\alpha g_{\mu\nu} = 0$, affine connection (the Levi-Civita one). However, it is well known nowadays that the teleparallel equivalent of GR, formulated in the Weitzenböck spacetime, yields a dynamically equivalent theory to metric GR, with exactly the same predictions (for spinless matter) [22, 24, 23, 25]. Besides this approach from a translational gauge principle under specific assumptions, there is yet another formulation equivalent to GR based on zero curvature and torsion but non-zero non-metricity, called symmetric teleparallel gravity, whose properties have begun to be unravelled very recently [26, 27, 28, 29, 30, 31].

Since these are, to some extent, equivalent gravitational models under different spacetime paradigms, one may ask if there is any guiding principle which could determine what spacetime geometry and degrees of freedom can represent gravity at its most fundamental level. The application of the gauge approach to gravity shows clearly that GR can be formulated as a translational gauge theory and, therefore, lives on a subset of the RC spacetime. On the other hand the generalization of the translational symmetry to the Poincaré symmetries points towards the direction of modifications to GR within the PGTG formulated in the RC spacetime geometry. This is a natural path to follow since the empirical approaches show that Nature reveals not only translational symmetries but rather at least Poincaré symmetries. Also, from a conceptual point of view and even if under certain physical conditions Lorentz symmetries can be broken, there is no fundamental reason for considering that spacetime and field theories don't have symmetries under Lorentz rotations at least for a wide range of phenomena. It is relevant to underline (once again) that the three mentioned approaches to GR have different assumptions regarding the spacetime geometrical paradigm, but their equivalence breaks as soon as one considers fermionic (spinor) fields. As we saw, in order to probe the post-Riemann geometrical sructures of spacetime, test matter with hypermomentum currents is required. Indeed, if one could measure the different effects of non-Riemannian geometries upon matter, one might be able to distinguish between these spacetime paradigms.

The existence of formal maps between these equivalent descriptions of GR under different spacetime paradigms, together with the generalized Bianchi identities in (3.31) and (3.24), motivate the following hypothesis:

The curvature, torsion and non-metricity of general metric-affine spacetime geometries might be interconvertible.

This is reminiscient of the Weyl hypothesis in GR according to which the Ricci part of curvature dominates the very early Universe, and the Weyl curvature dominates the late-Universe around self-gravitating objects (and assymptotically, around the last remaining BHs before final evaporation). In this hypothesis, the Ricci part of spacetime curvature is converted into the Weyl part. Analogously, as one can see from (3.45), both torsion and non-metricity tensors, together with the Levi-Civita Riemannian curvature are encompassed inside the full curvature. The Bianchi identities are analogous to the Bianchi identities of electromagnetism (and Yang-Mill fields), which represent flux conservation and an electric-magnetic dynamical interconversion according to Faraday law. If a similar dynamical interconversion between curvature, torsion and non-metricity is physically possible, this can open very interesting perspectives regarding relativistic astrophysics, cosmology and gravitational wave physics. We will come back to these topics in chapter 8.

3.2 Metric-Affine geometry and the conformal symmetry

3.2.1 The primacy of the causal structure revisited

In accordance with important developments in geometrical methods in field theories, the spacetime metric is no longer to be considered as a fundamental field. One can say that both the pre-metric foundations explored in chapter 2 and the gauge approach to gravity point in the same direction. Let us establish a bridge between the ideas explored in chapter 2 and those of the present chapter. We start by recall an important idea that was mentioned in chapter 2, the notion that the spacetime metric, up to a conformal factor, can be derived from local and linear electrodynamics [39, 40, 41, 45, 46, 47]. We recall that the pre-metric approach to electrodynamics as expressed in differential forms give completely general, coordinate-free covariant inhomogeneous and homogeneous equations from charge conservation and magnetic flux conservation respectively. No metric is involved and therefore, electrodynamics is not linked to Minkowski spacetime at a fundamental level. The postulate on the constitutive relations $H = H(F)$ in vacuum, interpreted as constitutive relations for the spacetime itself, introduce the conformally invariant part of the spacetime metric. Then, once the propagation of electromagnetic fields is considered and the geometrical optics limit is taken, one finds a quartic Fresnel surface which becomes a quadratic surface under the imposition of zero birefringence (double refraction) in vacuum. This quadratic surface defines the light-cone. Therefore, pre-metric electrodynamics, together with linear, local, homogeneous constitutive spacetime-electromagnetic relations and zero birefringence, gives the spacetime metric up to a conformal factor, i.e, the causal structure of spacetime.

Now, the resulting lightcone or causal-electromagnetic structure is a conformal geometry with local conformal symmetries associated to the lightcone at each spacetime point. In such geometry, under the assumption of locality, the parallel transport of a light cone from a given point to a neighbouring point gives rise to a deformation of the local causal/light-cone structure according to the non-metricity tensor. And since, as we saw in this chapter, the tensor-valued non-metricity 1-form is linked to the existence of a non-Lorentzian linear connection, from a gauge point of view, this alone inevitably leads to Lorentz symmetry breaking. Therefore, on one hand, and in spite of the historical reasons relating Maxwell's theory with special relativity and Minkowski spacetime, electrodynamics is fundamentally connected to the conformal geometrical structure and the conformal group, and not to Minkowski spacetime, nor the Poincaré or Lorentz group. On the other hand, the notion of local symmetries, the basis of gauge theories, as applied to spacetime leads to gravitational theories with post-Riemann geometries, and non-metricity is directly related to non-Lorentzian connections, i.e, to Lorentz symmetry breaking. *If one takes at the same time the gauge theories of gravity and the premetric formulation of electrodynamics and its spacetime constitutive relations, we are led to consider a primacy of the conformal structure, post-Riemann geometries with non-metricity and conformal gauge theories of gravity. Symmetry breaking mechanisms are plausible scenarios for bringing both gravity and electrodynamics (light) into the phenomenological regime of Poincaré symmetries and related spacetime paradigm.* Finally, one can postulate that there should be some fundamental relation between the localization of the conformal group and the $U(1)$ gauge symmetry of electrodynamics. This alone motivates the exploration of super-conformal algebras that can relate the internal and external (spacetime) symmetries of field theories.

3.2.2 On the route to pre-metric gravity

The analogies explored in this chapter between Yang-Mills fields and gauge theories of gravity in the exterior calculus of forms, clearly show that gravitational dynamics can be described also in a pre-metric way, just as electrodynamics. The field excitations, H^a and H^{ab}, related to the field strengths (which are analogous to the generalized velocities of the gravitational gauge potentials), torsion T^a and curvature R^{ab}, correspond to field momenta in the Hamiltonian formalism. These relations between the field momenta (excitations) and the field strengths can be understood as gravitational spacetime constitutive relations and similar ideas and conclusions as those explored in chapter 2 can be adapted here. In particular, since these gravitational constitutive relations are implicitly assumed in the choice of a given Lagrangian model, the form of these relations determines the gravitational theory and related phenomenology, just as one can get linear/non-linear, local/non-local electrodynamics according to the postulate on the electromagnetice-spacetime constitutive relations. Different classes of theories of gravity can be encompassed within choices for these constitutive relations. These can be linear or non-linear, local or non-local, include a mixing between the field strengths or not, etc. Moreover, since the Hodge star operator is naturally involved in these relations, the gravitational propagation properties and the conformal spacetime structure are presupposed in such relations. The assumptions for these gravitational-spacetime constitutive relations will therefore determine different predictions for the propagation properties of GWs. In connection to this, and still following the analogies with the considerations in chapter 2, the choice of coupling constants inside these relations points towards the assumption of spacial homogeneity and isotropy. More general non-homogeneous and anisotropic relations could be postulated within linear type of expressions such as $H^a \sim \chi^a{}_b T^b$ and $H^{ab} \sim \Theta^{ab}{}_{cd} R^{cd}$, in the case without the curvature torsion mixing. Again, this wider class of models could be expressed as tensor-tensor theories, where the propagation properties of the gravitational modes are encompassed inside the objects $\chi^a{}_b$ and $\Theta^{ab}{}_{cd}$. The corresponding tensor quantities might change from point to point in space (non-homogeneity) and be anisotropic, following the local isometries, while preserving conformal symmetries, i.e, the local causal gravitational-cone structure of spacetime.

The exploration of the gauge theories of gravity including its connection to spacetime symmetries and post-Riemann geometries can lead to a vast number of predictions relevant for relativistic astrophysics, cosmology and GWs and also to interesting avenues towards unified gauge field theories. Moreover the connection between the gauge formalism, the Lagrangian and Hamiltonian formalisms, the premetric foundations, the constitutive relations and the causal structure of spacetime, might also contribute towards relevant non-perturbative, background-independent methods for quantum field theory and quantum gravity. The gauge approach can clarify which gravitational geometrical degrees of freedom should be taken as canonical conjugate variables for quantization (see table 3.2, for e.g.). In its turn, this can relax the traditional emphasis that is usually put into the metric degrees of freedom, for historical reasons, within attempts at reconciling GR (or more general metric theories) with quantum theory. Of particular relevance, and in relation to a premetric program in gravity, is the possibility of what could be called a semi-perturbative method, according to the following:

Local conformal symmetry is taken at the basic order of approximation, and the sets of conformally related Lorentzian metrics sharing the same causal-cone at each point constitute the local fibers. Quantum superposition of different metrics at each point in spacetime can then be performed within the local fibers, without the need for a well-behaved background (metric) spacetime-vacuum. A similar geometrical construction can be applied to the 4-dimensional energy-momentum space (that can be seen as an internal space) at each spacetime point. Therefore different metrics $\bar{g}(p^\mu)$ in this energy-momentum manifold also share a local conformal structure and conformal geometry, and fluctuations, i.e, energy-momentum superposition can be defined without any inconvenience due to the absence of a background metric structure. The geometrical merging of these two spaces, the extended phase-space manifold should encompass in a consistent way the indeterminacy principle such that for any physical system containing gravity one has $\delta x^\mu \delta p^\mu \neq 0$.

These are open questions to which we will return in chapter 8 of this thesis.

Part II

Applications: Particle physics, cosmology, astrophysics and GWs

Chapter 4

Classical fields in RC geometry: theory and physical effects

Non-riemann spacetime geometries apear naturaly in gravitation in strict relation to the gauging (localizing) of spacetime symmetries. Classical fields propagating in this geometry, could in principle manifest the effects of torsion and provide physical tests for the underlying theories of gravity and the nature of spacetime. In this chapter we explore the dynamics of fermionic and bosonic ($s = 1$) fields in a RC spacetime background. In section 4.1 we consider fermions minimally and non-minimally coupled to the background torsion and in section 4.2 we take the dynamics of bosonic fields minimally coupled to torsion. Finally in section 4.3 we include the gravitational dynamics by considering both fermions and bosons minimally coupled to RC geometry within Einstein-Cartan (EC) gravity, as well as non-minimal couplings of fermions to torsion within the EC and EC+Holst gravity.

In section 4.1, we investigate the basic interactions between torsion and fermions by considering torsion effects in the dynamics of spinors. Besides the case of fermions minimally coupled to the background torsion we also address non-minimal extensions (including parity breaking and parity preserving couplings between torsion and the vector and axial fermionic currents). The main physical implications are qualitatively analysed and studied in simplifying regimes (such as spacetime flatness). In particular, we make an estimate of a Zeeman-like effect on energy levels (splitting) and corresponding transitions (fine-structure). In the limit of zero-curvature, and for the cases of constant and spherically symmetric torsion, we find (Zeeman-like) changes to the energy levels of fermions/anti-fermions depending on whether the spin is aligned or anti-aligned with the background axial vector torsion, and determine the corresponding fine-structure energy transitions. We further elaborate on the detection of torsion effects via the splitting of energy levels in astrophysics and cosmology using current capabilities, as wel as other physical implications. Spacetime torsion, if consistently detected will provide a major breakthrough in gravitational physics. Whether it can propagate in vacuum in strong gravity environments or only within regions permeated by matter fields with high spin densities, as in the ECSK theory, it is an open question. In section 4.2 we briefly explore some physical consequences of couplings between electromagnetic fields and the spacetime torsion of a Riemann-Cartan geometry. In chapters 6 and 7 we will also consider briefly possible astrophysical implications for compact objects (such as neutron stars, BHs and supermassive BH), with potential applications to gravitational wave physics

In section 4.3 we also explore the Einstein-Cartan-Dirac theory with an electromagnetic (Maxwell) contribution minimally coupled to torsion. This contribution breaks the

U(1) gauge symmetry, which is suggested by the possibility of a torsion-induced phase transition in the early Universe, yielding new physics in extreme (spin) density regimes. We obtain the generalized gravitational, electromagnetic and fermionic field equations for this theory, including torsion induced non-linearities and non-minimal couplings in the matter fields, we estimate the strength of the corrections and discuss the corresponding phenomenology. In particular, we briefly address some astrophysical considerations regarding the relevance of the effects which might take place inside ultra-dense neutron stars with strong magnetic fields (magnetars). Finally, we also briefly study the coupled gravity-fermionic dynamics within the Einstein-Cartan-Dirac theory with couplings of torsion to the axial and trace Dirac vectors, and in the case of Einstein-Cartan-Dirac with the Holst term.

The section 4.1 is inspired by the work in [8], while section 4.3 by the work in [4].

4.1 Fermionic fields in RC spacetime

Let us recall that the Dirac equation in Minkowski spacetime provided significant corrections to the non-relativistic Schrödinger equation. Accordingly, consider first the Dirac Lagrangian in curved spacetime and minimally coupled to the electromagnetic field

$$\tilde{\mathcal{L}}_{\text{Dirac}} = \frac{i\hbar}{2}\left(\bar{\psi}\gamma^{\mu}\tilde{D}_{\mu}\psi - (\tilde{D}_{\mu}\bar{\psi})\gamma^{\mu}\psi\right) - m\bar{\psi}\psi + j^{\lambda}A_{\lambda}, \tag{4.1}$$

where

$$\tilde{D}_{\mu}\psi = \partial_{\mu}\psi + \frac{1}{2}\tilde{\Gamma}_{ab\mu}\sigma^{ab}\psi, \quad \tilde{D}_{\mu}\bar{\psi} = \partial_{\mu}\bar{\psi} - \frac{1}{2}\tilde{\Gamma}_{ab\mu}\bar{\psi}\sigma^{ab}, \tag{4.2}$$

are the Fock-Ivanenko co-variant derivatives of spinors and its adjoints $\bar{\psi} = \psi^{+}\gamma^{0}$ in curved spacetime, the matrices $\sigma^{ab} \equiv \frac{1}{4}\left[\gamma^{a}, \gamma^{b}\right] = \frac{1}{2}\gamma^{[a}\gamma^{b]}$ are the Lorentz group generators in the spinorial representation, and $j^{\lambda} = q\bar{\psi}\gamma^{\lambda}\psi$ is the U(1) charge current density vector. The effect of spacetime curvature is encoded in the Levi-Civita "spin connection" $\tilde{\Gamma}_{ab\mu}$ 1-form, of the Riemann geometry. The corresponding Dirac equation

$$i\hbar\gamma^{\mu}\tilde{D}_{\mu}\psi + (q\gamma^{\mu}A_{\mu} - m)\psi = 0, \tag{4.3}$$

in the flat spacetime (Minkowski) and quasi non-relativistic limit (leaving terms up to $(v/c)^2$, reinserting the speed of light c), gives the time independent equation in the static external electromagnetic potential $A = (\phi, A_j)$,

$$\left[\frac{1}{2m}\left(\hat{\vec{p}} - q\vec{A}\right)^{2} - \frac{\hat{p}^{4}}{8m^{3}c^{2}} + q\phi + \frac{q\hbar^{2}}{4m^{2}c^{2}}\frac{1}{r}\partial_{r}\phi\hat{\vec{S}}\cdot\hat{\vec{L}} - \frac{q\hbar}{m}\hat{\vec{S}}\cdot\hat{\vec{B}} - \frac{q\hbar^{2}}{4m^{2}c^{2}}\partial_{r}\phi\partial_{r} - E\right]\psi(\vec{r}) = 0, \tag{4.4}$$

with $E \ll mc^2$ and $q\phi \ll mc^2$, and the spherical symmetry $\phi = \phi(r)$ is assumed. The solution to these equations gives the 4-spinor $\psi = \psi(\vec{r})e^{-\frac{i}{\hbar}Et}$, eigen function of the Hamiltonian corresponding to the energy E. In the expression above, $\hat{\vec{S}} = \hbar\vec{\sigma}/2$ is the intrinsic angular momentum (spin) operator naturally included within the gamma matrices (multiplied by $\hbar$), and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ is the Pauli matrices spatial vector. As it is well known, the second term on the left hand side is a relativistic correction to the 3-momentum, the forth and the fifth terms give the spin-orbit and Zeeman-effect magnetic energy, respectively and the sixth term is the so called Darwin term correction.

If we consider the case of an electron in the Coulomb potential $\phi = -Ze/r$, then the corresponding energy levels are

$$E = mc^2\Big(1 - \frac{Z^2\alpha^2}{2n^2} - \frac{Z^4\alpha^4}{2n^4}\left(\frac{n}{j+1/2} - \frac{3}{4}\right) + \mathcal{O}(Z^6\alpha^6)\Big), \tag{4.5}$$

where n is the principle quantum number and j is the total angular momentum quantum number, and α is the fine structure constant. The first term is the relativistic correction of the energy associated to the mass of the electron, the second term in the brackets corresponds to the Bohr energy levels while the next term is the fine structure (spin-orbit) corrections. As an example, the fine-structure between the energy levels (nl_j) $2P_{3/2}$ and $2P_{1/2}$ corresponds to the energy difference of $|\triangle E| = mc^2Z^2\alpha^4/32$.

If one considers instead of the Minkowski limit (Dirac theory), a curved background spacetime, then there will be gravitational metric-induced corrections to the energy levels. The message from this short introduction and review is very clear: By changing the spacetime paradigm from the Newtonian picture into the Minkowski 4-dimensional continuum, Dirac leads us to his relativistic theory of the electron with energy corrections to the non-relativistic Schrödinger equation and by changing again the spacetime paradigm by assuming a Riemann curvature (non-flatness) one is led again to energy-level corrections relevant for fermionic fields propagating in strong gravitational fields. The spacetime paradigm is thus very important at a fundamental level in the dynamics of fermions. What happens if one generalizes the Riemann geometry, to include the torsion of a Riemann-Cartan spacetime? As we will see, in the minimal coupling scenario a clear analogy with the Zeeman-effect term can be recognized.

4.1.1 Minimal coupling to torsion

We now consider the minimal coupling of fermions to the Cartan connection of RC spacetimes. For a more detailed analysis of fermions in RC and metric-affine geometries see for example [125, 131, 132, 133, 134]. Consider a free Dirac fermionic field minimally coupled to the RC spacetime geometry. The Lagrangian reads

$$\mathcal{L}_{\text{Dirac}} = \frac{i\hbar}{2}\left(\bar{\psi}\gamma^\mu D_\mu\psi - (D_\mu\bar{\psi})\gamma^\mu\psi\right) - m\bar{\psi}\psi, \tag{4.6}$$

for spinors ψ and their adjoints, and the generalized Fock-Ivanenko covariant derivatives of spinors are

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{2}\Gamma_{ab\mu}\sigma^{ab}\psi = \tilde{D}_\mu\psi + \frac{1}{4}K_{ab\mu}\gamma^{[a}\gamma^{b]}\psi, \tag{4.7}$$

$$D_\mu\bar{\psi} = \partial_\mu\bar{\psi} - \frac{1}{2}\Gamma_{ab\mu}\bar{\psi}\sigma^{ab} = \tilde{D}_\mu\bar{\psi} - \frac{1}{4}K_{ab\mu}\bar{\psi}\gamma^{[a}\gamma^{b]}, \tag{4.8}$$

and we recall that the Lorentzian spin connection ($\Gamma_{ab\nu} = -\Gamma_{ba\nu}$) of RC spacetime can be written as the spin connection of Riemann geometry plus the contorsion $\Gamma_{ab\mu} = \tilde{\Gamma}_{ab\mu} +$

$K_{ab\mu}$. The matrices γ^μ are the induced Dirac-Pauli matrices[1] obeying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I$, where I is the 4×4 unit matrix and $g_{\mu\nu}$ is the space-time metric. Replacing the covariant derivatives in the Lagrangian, one gets

$$\mathcal{L}_{\text{Dirac}} = \tilde{\mathcal{L}}_{\text{Dirac}} + \frac{i\hbar}{8} K_{ab\mu} \bar{\psi} \{\gamma^\mu, \gamma^a \gamma^b\} \psi \,, \tag{4.9}$$

where $\tilde{\mathcal{L}}_{\text{Dirac}} = \frac{i\hbar}{2}\left(\bar{\psi}\gamma^\mu \tilde{D}_\mu \psi - (\tilde{D}_\mu \bar{\psi})\gamma^\mu \psi\right) - m\bar{\psi}\psi$ and $\{\gamma^\mu, \gamma^a\gamma^b\} = \gamma^\mu\gamma^a\gamma^b + \gamma^a\gamma^b\gamma^\mu$. Using the tetrads, $K_{ab\mu} = \theta^c{}_\mu K_{abc}$, and $\gamma^\mu = e_d^{\ \mu}\gamma^d$, so that the Lagrangian can also be written as

$$\mathcal{L}_{\text{Dirac}} = \tilde{\mathcal{L}}_{\text{Dirac}} + \frac{i\hbar}{8} K_{abc} \bar{\psi} \{\gamma^c, \gamma^a \gamma^b\} \psi \,. \tag{4.10}$$

Then, by using the identities $\{\gamma^c, \gamma^a\gamma^b\} = 2\gamma^{[c}\gamma^a\gamma^{b]} = 2i\epsilon^{cabd}\gamma_d\gamma^5$, we get

$$\mathcal{L}_{\text{Dirac}} = \tilde{\mathcal{L}}_{\text{Dirac}} + \frac{i\hbar}{4} K_{abc} \bar{\psi} \gamma^{[c}\gamma^a\gamma^{b]} \psi, \tag{4.11}$$

and recalling from (3.10) that $K_{[\alpha\beta\gamma]} = T_{\alpha\beta\gamma}$, we arrive at the final expression

$$\mathcal{L}_{\text{Dirac}} = \tilde{\mathcal{L}}_{\text{Dirac}} + 3\breve{T}^\lambda \breve{s}_\lambda, \tag{4.12}$$

where

$$\breve{s}^\lambda \equiv \frac{\hbar}{2}\bar{\psi}\gamma^\lambda\gamma^5\psi \tag{4.13}$$

is the Dirac axial spin vector current and

$$\breve{T}^\lambda \equiv \frac{1}{6}\epsilon^{\lambda\alpha\beta\gamma}T_{\alpha\beta\gamma} \tag{4.14}$$

is the axial vector part of torsion, and we have reinserted the spacetime (holonomic) indices. This simple expression means that, in the minimal coupling, Dirac fermionic fields only interact with the axial vector part of torsion. This expression is valid for any Dirac field minimally coupled to a RC spacetime geometry, regardless of the gravitational theory.

The magnitude of the axial vector $\breve{s}^\lambda$ represents the density of fermionic spin (spin/volume or energy/area, in $c = 1$ units).[2] To see this more explicitly consider the $\gamma^a\gamma^5$ matrices,

$$\gamma^a\gamma^5 = \left\{ \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \right\}, \tag{4.15}$$

(with $i = 1, 2, 3$), respectively, with I representing the 2×2 identity matrix and we recall that the usual spin operator is $\hat{\vec{S}} = \hbar\vec{\sigma}/2$. The eigen values of the Pauli matrices are

[1]The usual constant Pauli-Dirac matrices γ^c, which obey $\{\gamma^a, \gamma^b\} = 2\eta^{ab}I$, are related to the γ^μ matrices via $\gamma^\mu\theta^a{}_\mu = \gamma^a$, where $\theta^a{}_\mu$ are the tetrads satisfying the relations in eq. (3.38)and η_{ab} is the Minkowski metric.

[2]By reintroducing the speed of light c, we get $\breve{s}^\lambda \equiv \frac{\hbar c}{2}\bar{\psi}\gamma^\lambda\gamma^5\psi$, representing the flux of spin (spin/area$\times$time).

$\lambda = \pm 1$, for the spin up/down configurations, and in the usual Pauli-Dirac representation the σ^3 is already diagonal, therefore, we can use this direction as the chosen direction relative to which we define the up and down spin states. Then one can show that

$$|\breve{s}^3| \sim \frac{\hbar}{2} n, \tag{4.16}$$

where n is a normalization constant giving the number of particles (or anti-particles) per volume.

The torsion-spin interaction term resembles a Zeeman-like effect with the axial torsion vector playing the role of an external magnetic field

$$\mathcal{L}_{\text{torsion-spin}} \sim \breve{T} \cdot \breve{s} \quad \Longrightarrow Zeeman - like\ effect. \tag{4.17}$$

The Dirac equation corresponding to the Lagrangian above is

$$i\hbar\gamma^\mu \tilde{D}_\mu \psi - m\psi = -\frac{3\hbar}{2} \breve{T}^\lambda \gamma_\lambda \gamma^5 \psi \,, \tag{4.18}$$

for spinors, and

$$i\hbar(\tilde{D}_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = -\frac{3\hbar}{2} \breve{T}^\lambda \bar{\psi} \gamma_\lambda \gamma^5 \tag{4.19}$$

for the adjoint spinors.

Imprints of torsion upon fermion/anti-fermion energy levels

In order to study the effects of torsion we will take the spacetime flatness limit of Dirac equation, search for solutions and obtain the energy levels. Some possible *ansatze* for the axial torsion are

i) Constant background axial torsion
ii) Static, spherically symmetric torsion (for e.g. with a Coulomb-type of law $T \sim 1/r^2$)
iii) Non-static, harmonic (GW) propagating axial torsion

Here we consider the cases i) and ii). We start by taking the zero-curvature (spacetime flatness) limit and consider an axial torsion vector along one specific direction (for example the z axis of a Cartesian coordinate system). The Dirac equation in this limit is

$$i\hbar\gamma^\alpha \partial_\alpha \psi = m\psi - \frac{3\hbar}{2} \breve{T}^3 \gamma_3 \gamma^5 \psi \,. \tag{4.20}$$

More explicitly, taking into account the Dirac-Pauli (gamma) matrices and recalling that $\gamma^3 = -\gamma_3$, since the 4-spinor ψ can be represented as the set of two 2-spinors,[3]

[3] These 2-spinors $\psi^I = (\psi_1^I, \psi_2^I)$, $\psi^{II} = (\psi_1^{II}, \psi_2^{II})$ can be interpreted as representing the particle and anti-particle (2-spinor) fields.

$\psi = (\psi^I, \psi^{II})$, equation (4.20) corresponds to the following dynamical system

$$i\hbar\Big(\partial_t\psi^I + \sigma^k\partial_k\psi^{II}\Big) = \left(m + \frac{3\hbar}{2}\breve{T}^3\sigma^3\right)\psi^I, \tag{4.21}$$

$$-i\hbar\Big(\partial_t\psi^{II} + \sigma^k\partial_k\psi^{I}\Big) = \left(m - \frac{3\hbar}{2}\breve{T}^3\sigma^3\right)\psi^{II}, \tag{4.22}$$

with $k = 1, 2, 3$ and the 2×2 identity matrix I is implicit in the first terms of the left-hand side and in the first (mass) terms on the right-hand side. In terms of the 4-spinor components $\psi = (\psi^1, \psi^2, \psi^3, \psi^4)$ this can be explicitly written as

$$\begin{aligned}
i\hbar\Big(\partial_t\psi^1 + \partial_1\psi^4 - i\partial_2\psi^4 + \partial_3\psi^3\Big) &= \left(m + \frac{3\hbar}{2}\breve{T}^3\right)\psi^1,\\
i\hbar\Big(\partial_t\psi^2 + \partial_1\psi^3 + i\partial_2\psi^3 - \partial_3\psi^4\Big) &= \left(m - \frac{3\hbar}{2}\breve{T}^3\right)\psi^2,\\
i\hbar\Big(-\partial_t\psi^3 - \partial_1\psi^2 + i\partial_2\psi^2 - \partial_3\psi^1\Big) &= \left(m - \frac{3\hbar}{2}\breve{T}^3\right)\psi^3,\\
i\hbar\Big(-\partial_t\psi^4 - \partial_1\psi^1 - i\partial_2\psi^1 + \partial_3\psi^2\Big) &= \left(m + \frac{3\hbar}{2}\breve{T}^3\right)\psi^4,
\end{aligned}$$

respectively. From these equations one can see that the axial-axial interaction between the fermionic spin density and the background spacetime torsion gives an energy correction (that can also be seen as an effective mass correction) which is spin state dependent. The energy of the fermionic fields is different depending on the spin up/down configuration along the direction of the axial torsion vector. So, in principle an electron or any massive free fermion in a well defined momentum (eigen) state will have two possible energy levels depending on the alignment or anti-alignment between its spin and the spacetime axial torsion vector. This is analogous to the Zeeman effect. Moreover, suppose that $\breve{T}^3 > 0$, then, as will become more clear, the anti-alignment is preferred for the fermion/anti-fermion as it corresponds to the lower energy level.

Constant background axial torsion. Consider a 4-spinor $\psi = \psi(\vec{r})e^{-iEt/\hbar}$, corresponding to the eigen function of a well defined energy state. After substitution in Eq. (4.20), we obtain the time-independent equation

$$-i\hbar\gamma^k\partial_k\psi + \Big(m - \frac{3\hbar}{2}\breve{T}^3\gamma_3\gamma^5\Big)\psi(\vec{r}) = \gamma^0 E\psi(\vec{r})\ . \tag{4.23}$$

In terms of their components this equation reads

$$-i\hbar\sigma^k\partial_k\psi^{II} = \left(E - m - \frac{3\hbar}{2}\breve{T}^3\sigma^3\right)\psi^I, \tag{4.24}$$

$$-i\hbar\sigma^k\partial_k\psi^{I} = \left(E + m - \frac{3\hbar}{2}\breve{T}^3\sigma^3\right)\psi^{II}, \tag{4.25}$$

or, more explicitly:

$$\begin{aligned}
-i\hbar\Big(\partial_1\psi^4 - i\partial_2\psi^4 + \partial_3\psi^3\Big) &= \left(E - m - \frac{3\hbar}{2}\breve{T}^3\right)\psi^1, \\
-i\hbar\Big(\partial_1\psi^3 + i\partial_2\psi^3 - \partial_3\psi^4\Big) &= \left(E - m + \frac{3\hbar}{2}\breve{T}^3\right)\psi^2, \\
i\hbar\Big(-\partial_1\psi^2 + i\partial_2\psi^2 - \partial_3\psi^1\Big) &= \left(E + m - \frac{3\hbar}{2}\breve{T}^3\right)\psi^3, \\
i\hbar\Big(-\partial_1\psi^1 - i\partial_2\psi^1 + \partial_3\psi^2\Big) &= \left(E + m + \frac{3\hbar}{2}\breve{T}^3\right)\psi^4.
\end{aligned}$$

Moreover, taking into account the harmonic solution $\psi(\vec{r}) = \chi e^{i\vec{k}\cdot\vec{r}} = \chi e^{i\vec{p}\cdot\vec{r}/\hbar}$, corresponding to a well defined momentum state, where χ is a constant 4-spinor, we get the system of equations for the χ components as

$$\begin{aligned}
p_1\chi^4 - ip_2\chi^4 + p_3\chi^3 &= \left(E - m - \frac{3\hbar}{2}\breve{T}^3\right)\chi^1, \\
p_1\chi^3 + ip_2\chi^3 - p_3\chi^4 &= \left(E - m + \frac{3\hbar}{2}\breve{T}^3\right)\chi^2, \\
-\left(-p_1\chi^2 + ip_2\chi^2 - p_3\chi^1\right) &= \left(E + m - \frac{3\hbar}{2}\breve{T}^3\right)\chi^3, \\
-\left(-p_1\chi^1 - ip_2\chi^1 + p_3\chi^2\right) &= \left(E + m + \frac{3\hbar}{2}\breve{T}^3\right)\chi^4,
\end{aligned}$$

respectively. Since χ is assumed to have constant components, the background torsion has to be constant also. In this static and constant background axial torsion regime, assuming again that torsion is positively oriented, $\breve{T}^3 > 0$, there are two independent solutions for the spinor $\psi(\vec{r}, t) = \chi e^{i(\vec{p}\cdot\vec{r} - Et)/\hbar}$, corresponding to the free particle momentum eigenstates with spin up and spin down but, as opposed to Dirac theory in Minkowski spacetime, in this case the presence of torsion breaks the degeneracy in energy. The two spinor states have different (positive) energy values. As an example, consider the case of motion along the p_1 direction for which we get

$$\begin{aligned}
p_1\chi^4 &= \left(E - m - \frac{3\hbar}{2}\breve{T}^3\right)\chi^1, \\
p_1\chi^3 &= \left(E - m + \frac{3\hbar}{2}\breve{T}^3\right)\chi^2, \\
p_1\chi^2 &= \left(E + m - \frac{3\hbar}{2}\breve{T}^3\right)\chi^3, \\
p_1\chi^1 &= \left(E + m + \frac{3\hbar}{2}\breve{T}^3\right)\chi^4.
\end{aligned}$$

The two possible energy solutions for the particle are then given by

$$E_{\pm}^2 = p^2 + \left(m \pm \frac{3\hbar}{2}\breve{T}^3\right)^2, \tag{4.26}$$

for the spin up/down, respectively. The two independent solutions for the spin up (aligned) state and the spin down (anti-aligned) state are given by

$$N\begin{pmatrix} 1 \\ 0 \\ 0 \\ \dfrac{p}{E+\left(m+\dfrac{3\hbar}{2}\breve{T}^3\right)} \end{pmatrix}, \quad N\begin{pmatrix} 0 \\ 1 \\ \dfrac{p}{E+\left(m-\dfrac{3\hbar}{2}\breve{T}^3\right)} \\ 0 \end{pmatrix},$$

respectively, where N is a normalization constant (typically chosen to satisfy $\psi^\dagger\psi = 2E$) given in this case by $N = \sqrt{E + (m \pm 3\hbar\breve{T}^3/2)}$ for the spin up/down (aligned/anti-aligned) states, and $p = p_1$. The physical interpretation is quite interesting, since not only the axial-axial torsion-spin interaction is analogous to a Zeeman effect, but also the equations reveal that one could think of the fermion state with the spin aligned with torsion as being slightly more massive than the fermion state with the spin anti-aligned to the axial torsion. In the coupling to the spacetime structure, the torsion is providing an effective mass to fermions but distinguishes between spin states.

In this regime of static constant background torsion there are two more independent solutions for the spinor $\psi(\vec{r}) = \chi e^{-i(\vec{p}\cdot\vec{r}-Et)/\hbar}$, corresponding to the free anti-particle momentum eigenstates with spin down or spin up. In this case, we obtain

$$N\begin{pmatrix} 0 \\ \dfrac{p}{E+\left(m-\dfrac{3\hbar}{2}\breve{T}^3\right)} \\ 1 \\ 0 \end{pmatrix}, \quad N\begin{pmatrix} \dfrac{p}{E+\left(m+\dfrac{3\hbar}{2}\breve{T}^3\right)} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4.27}$$

respectively, with $N = \sqrt{E + (m \mp 3\hbar\breve{T}^3/2)}$ for the spin down/up (anti-aligned/aligned) states, respectively. The two corresponding energy levels are

$$E^2 = p^2 + \left(m \mp \frac{3\hbar}{2}\breve{T}^3\right)^2 , \tag{4.28}$$

for the spin down/up states.

For completeness, let us also mention that in the general case of motion along any direction, with $\vec{p} = (p_1, p_2, p_3)$, then we would reach similar conclusions with the spin up and spin down solutions for particles:

$$N\begin{pmatrix} 1 \\ 0 \\ \dfrac{p_3}{E+\left(m+\dfrac{3\hbar}{2}\breve{T}^3\right)} \\ \dfrac{p_1+ip_2}{E+\left(m+\dfrac{3\hbar}{2}\breve{T}^3\right)} \end{pmatrix}, \quad N\begin{pmatrix} 0 \\ 1 \\ \dfrac{p_1-ip_2}{E+\left(m-\dfrac{3\hbar}{2}\breve{T}^3\right)} \\ \dfrac{-p_3}{E+\left(m-\dfrac{3\hbar}{2}\breve{T}^3\right)} \end{pmatrix}, \tag{4.29}$$

and those for anti-particles:

$$N\begin{pmatrix} \dfrac{p_1 - ip_2}{E + \left(m + \dfrac{3\hbar}{2}\breve{T}^3\right)} \\ \dfrac{-p_3}{E + \left(m + \dfrac{3\hbar}{2}\breve{T}^3\right)} \\ 0 \\ 1 \end{pmatrix}, \quad N\begin{pmatrix} \dfrac{p_3}{E + \left(m - \dfrac{3\hbar}{2}\breve{T}^3\right)} \\ \dfrac{p_1 + ip_2}{E + \left(m - \dfrac{3\hbar}{2}\breve{T}^3\right)} \\ 1 \\ 0 \end{pmatrix}, \tag{4.30}$$

respectively. In all cases the energy of the anti-aligned state is lower than in the aligned state.

Let us denote by $m_{\breve{T}}$ the mass correction due to the spin-torsion interaction, and consider the two possible energy levels E_1 and E_2, with $E_2 > E_1$. We therefore get the expression for the energy transition

$$h\nu = E_2 - E_1 = \frac{4mm_{\breve{T}}}{\tilde{p}_+ + \tilde{p}_-}\,, \tag{4.31}$$

where $\tilde{p}_\pm^2 = p^2 + (m \pm m_{\breve{T}})^2$, and in the reference frame of the particle we obtain

$$h\nu = \frac{1}{2}m_{\breve{T}} = \frac{3\hbar}{4}\breve{T}\,. \tag{4.32}$$

Therefore, reinserting the speed of light in vacuum, we get

$$\nu = \frac{3c}{8\pi}\breve{T} \tag{4.33}$$

If we consider, for instance, $\breve{T} \sim 10^{-16}m^{-1}$, then we end up with the prediction of a transition in the $\nu \sim$nHz regime.

Spherically symmetric torsion background. We will now analyse the case of a static, spherically symmetric torsion background, which is relevant for astrophysical applications. The Dirac equation in (4.18) in this case becomes

$$i\hbar\gamma^\mu\tilde{D}_\mu\psi - m\psi = -\frac{3\hbar}{2}\breve{T}^\lambda(r)\gamma_\lambda\gamma^5\psi\,. \tag{4.34}$$

To estimate the effect of a spherically symmetric background torsion on the energy levels we can consider the following axial torsion around some astrophysical source and neglect at the moment the effect of curvature

$$\breve{T}^\mu(r) = b^\mu f(r), \tag{4.35}$$

where b^μ is constant (axial) 4-vector. The zero-curvature limit of the generalized Dirac equation above is

$$i\hbar\gamma^\alpha\partial_\alpha\psi = m\psi - \frac{3\hbar}{2}\breve{T}^\lambda(r)\gamma_\lambda\gamma^5\psi\,. \tag{4.36}$$

The torsion-spin interaction can be seen as a small perturbation to the (unperturbed) time-independent Hamiltonian. Using perturbation theory to first order, we have

$$E \simeq E_{(0)} + < \psi_{(0)} | \hat{U}_{torsion-spin} | \psi_{(0)} >, \tag{4.37}$$

where $\psi_{(0)}$ are the eigen states of the unperturbed Hamiltonian associated to the eigen value $E_{(0)}$. Again, taking the 4-spinor $\psi = \psi(\vec{r})e^{-iEt/\hbar}$, corresponding to the eigen function of a well defined energy state, we obtain the time independent equation

$$-i\hbar\gamma^k \partial_k \psi + \left(m - \frac{3\hbar}{2}\breve{T}^\lambda(r)\gamma_\lambda\gamma^5\right)\psi = \gamma^0 E\psi\,. \tag{4.38}$$

such that

$$\hat{U}_{torsion-spin} = -\frac{3\hbar}{2}\hat{\breve{T}}^\lambda(r)\gamma_\lambda\gamma^5. \tag{4.39}$$

Now, consider the 4-spinor state

$$\left|\psi_{(0)}\right\rangle = \left|\psi^1_{(0)}\right\rangle \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \left|\psi^2_{(0)}\right\rangle \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \left|\psi^3_{(0)}\right\rangle \begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \left|\psi^4_{(0)}\right\rangle \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{4.40}$$

solution to the unperturbed Hamiltonian. In configuration space, this expression becomes

$$\left\langle \vec{r}\,|\psi_{(0)}\right\rangle = \psi^1_{(0)}(r) \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \psi^2_{(0)}(r) \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \psi^3_{(0)}(r) \begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \psi^4_{(0)}(r) \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{4.41}$$

If we assume the motion to take place along a specific direction, then there are four independent solutions, two for the particle states (up/down):

$$N \begin{pmatrix}1\\0\\0\\ \dfrac{p}{E_{(0)}+m}\end{pmatrix} e^{i\vec{p}\cdot\vec{r}/\hbar}, \quad N \begin{pmatrix}0\\1\\ \dfrac{p}{E_{(0)}+m}\\0\end{pmatrix} e^{i\vec{p}\cdot\vec{r}/\hbar}, \tag{4.42}$$

and two for the anti-particle states

$$N \begin{pmatrix}0\\ \dfrac{p}{E_{(0)}+m}\\1\\0\end{pmatrix} e^{-i\vec{p}\cdot\vec{r}/\hbar}, \quad N \begin{pmatrix}\dfrac{p}{E_{(0)}+m}\\0\\0\\1\end{pmatrix} e^{-i\vec{p}\cdot\vec{r}/\hbar}, \tag{4.43}$$

corresponding to the cases of spin down and spin up, respectively, with $N = \sqrt{E_{(0)} + m}$ and $E_{(0)}^2 = p^2 + m^2$. Next, we need to compute from Eq. (4.39) the following quantity

$$\begin{aligned} \langle \psi_{(0)} | \hat{U}_{ts} | \psi_{(0)} \rangle &= -\frac{3\hbar}{2} \int \psi_{(0)}^\dagger(\vec{r}) \hat{\breve{T}}^\lambda(r) \gamma_\lambda \gamma^5 \psi_{(0)}(\vec{r}) d^3r = -\frac{3\hbar b^0}{2} \int f(r) \psi_{(0)}^\dagger(\vec{r}) \gamma_0 \gamma^5 \psi_{(0)}(\vec{r}) d^3r \\ &\quad + \frac{3\hbar}{2} \sum_{i=1}^{3} b^i \int f(r) \psi_{(0)}^\dagger(\vec{r}) \gamma^i \gamma^5 \psi_{(0)}(\vec{r}) d^3r \ , \end{aligned} \tag{4.44}$$

which, taking into account the quite useful general relations

$$\gamma_0\gamma^5 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_3 \\ z_4 \\ -z_1 \\ -z_2 \end{pmatrix}, \quad \gamma^1\gamma^5 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_1 \\ -z_4 \\ -z_3 \end{pmatrix}, \quad \gamma^2\gamma^5 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} -iz_2 \\ iz_1 \\ iz_4 \\ -iz_3 \end{pmatrix},$$

$$\gamma^3\gamma^5 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_1 \\ -z_2 \\ -z_3 \\ z_4 \end{pmatrix},$$

and assuming the particle with the spin up configuration in Eq. (4.42), with $z_1 = Ne^{i\vec{p}\cdot\vec{r}/\hbar}$, $z_4 = N\frac{p}{E+m}e^{i\vec{p}\cdot\vec{r}/\hbar}$ and $z_2 = z_3 = 0$, we arrive at

$$\langle \psi_{(0)} | \hat{U}_{ts} | \psi_{(0)} \rangle = \frac{3\hbar b^3}{2} N^2 \left[1 + \frac{p^2}{(E_{(0)} + m)^2} \right] F(r) \ , \tag{4.45}$$

where $F(r) = \int f(r) d^3r$ represents the geometrical factor coming from a spherically symmetric torsion function integrated over the relevant volume of the spatial 3-dimensional hypersurfaces, for a specific spacetime foliation. Doing the same exercise with the spin down state with $z_2 = Ne^{i\vec{p}\cdot\vec{r}/\hbar}$, $z_3 = N\frac{p}{E_{(0)}+m}e^{i\vec{p}\cdot\vec{r}/\hbar}$ and $z_1 = z_4 = 0$, we get

$$\langle \psi_{(0)} | \hat{U}_{ts} | \psi_{(0)} \rangle = -\frac{3\hbar b^3}{2} N^2 \left[1 + \frac{p^2}{(E_{(0)} + m)^2} \right] F(r) \ . \tag{4.46}$$

Finally, the energy difference between these two states is given by

$$\delta E = 3\hbar b^3 N^2 \left(1 + \frac{p^2}{(E_{(0)} + m)^2} \right) F(r) \ , \tag{4.47}$$

corresponding to the frequency

$$\nu = \frac{\delta E}{h} = \frac{3E_{(0)}}{\pi} F(r) \ , \tag{4.48}$$

where we have used again the conventional normalization $\psi_0^\dagger \psi_0 = 2E_0$. Note that the same result for the energy levels would have been obtained if we had considered all the components of the 3-momentum.

The effects so far analysed result from the interaction of fermions and anti-fermions with the background spacetime torsion. The corresponding Lagrangian and (extended) Dirac equation are invariant under the C (charge conjugation) and P (Parity) transformations. If the Lagrangian is generalized to include parity breaking terms, interesting effects might be implied with relevant applications for the very early universe (GUT scales and beyond). In the following subsection we will consider these extended cases, specifically by including parity breaking and parity preserving non-minimal couplings with the background geometry.

4.1.2 Non-minimal couplings to torsion including parity breaking

Let us study fermionic Lagrangians non-minimally coupled to the RC spacetime geometry. Consider first the vector and axial vector fermionic currents $j^\lambda \equiv \bar{\psi}\gamma^\lambda\psi$ and $a^\lambda \equiv \bar{\psi}\gamma^\lambda\gamma^5\psi$, and its couplings to torsion as in the following Lagrangian

$$\mathcal{L}_{\text{fermions}} = \tilde{\mathcal{L}}_{\text{Dirac}} + \alpha_1 T \cdot j + \alpha_2 \breve{T} \cdot a, \tag{4.49}$$

where $T^\lambda \equiv T^{\lambda\nu}{}_\nu$ is the trace vector part of torsion. For $\alpha_2 = \dfrac{3\hbar}{2}$ and $\alpha_1 = 0$ we recover the minimal coupling to torsion case, previously analysed. The corresponding extended Dirac equation is

$$i\hbar\gamma^\mu \tilde{D}_\mu\psi - m\psi = -\alpha_1 T^\lambda\gamma_\lambda\psi - \alpha_2 \breve{T}^\lambda\gamma_\lambda\gamma^5\psi\,. \tag{4.50}$$

This model for the free fermion in a RC spacetime geometry is symmetric under parity transformations, and it can be seen as a particular case of a more general model with parity breaking effects.

Parity breaking fermionic Lagrangian in RC spacetime. To include parity breaking terms in the Lagrangian[4], we consider also the couplings $\mathbf{T}\cdot\mathbf{a}$ and $\breve{\mathbf{T}}\cdot\mathbf{j}$ in the Lagrangian

$$\mathcal{L}_{\text{fermions}} = \tilde{\mathcal{L}}_{\text{Dirac}} + (\alpha_1 T + \beta_2 \breve{T})\cdot j + (\alpha_2 \breve{T} + \beta_1 T)\cdot a. \tag{4.51}$$

The terms $\beta_1 T^\lambda a_\lambda$ and $\beta_2 \breve{T}^\mu j_\mu$ break the parity invariance. The corresponding extended Dirac equation is given by

$$i\hbar\gamma^\mu \tilde{D}_\mu\psi - m\psi = -\left(\alpha_1 T^\lambda + \beta_2 \breve{T}^\lambda\right)\gamma_\lambda\psi - \left(\alpha_2 \breve{T}^\lambda + \beta_1 T^\lambda\right)\gamma_\lambda\gamma^5\psi\,, \tag{4.52}$$

and for the adjoint spinor

$$i\hbar(\tilde{D}_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} = -\left(\alpha_1 T^\lambda + \beta_2 \breve{T}^\lambda\right)\bar{\psi}\gamma_\lambda - \left(\alpha_2 \breve{T}^\lambda + \beta_1 T^\lambda\right)\bar{\psi}\gamma_\lambda\gamma^5\,. \tag{4.53}$$

[4]For a more detailed account of parity violation in the general framework of Poincaré theories of gravity see the recent work [79].

To estimate the new physics involved in this model we will take again the zero-curvature (spacetime flatness) limit in order to identify the effects of torsion and have a qualitative notion of its consequences in the context of beyond standard-model particle physics interactions. In flat spacetime we get

$$i\hbar\gamma^\mu\partial_\mu\psi - m\psi = -\left(\alpha_1 T^\lambda + \beta_2\breve{T}^\lambda\right)\gamma_\lambda\psi - \left(\alpha_2\breve{T}^\lambda + \beta_1 T^\lambda\right)\gamma_\lambda\gamma^5\psi\,. \tag{4.54}$$

One can consider different relevant hypothesis for the background torsion, such as, static, dynamical, homogeneous or non-homogeneous torsion. The dynamical harmonic torsion case is relevant for GW studies of modified gravity, while the non-homogeneous spherically symmetric torsion background is appropriate in astrophysical applications, for simple models of the RC geometry around spherical compact objects. For free fermionic spinors we will consider again the harmonic solutions $\psi = \psi(\vec{r})e^{-\frac{i}{\hbar}Et}$, with $\psi(\vec{r}) = \chi e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$, where as a first approach χ is a 4-spinor with constant components. Accordingly, we have the following time-independent Dirac equation

$$-i\hbar\gamma^k\partial_k\psi + \left[m - \left(\alpha_1 T^\lambda + \beta_2\breve{T}^\lambda\right)\gamma_\lambda - \left(\alpha_2\breve{T}^\lambda + \beta_1 T^\lambda\right)\gamma_\lambda\gamma^5\right]\psi(\vec{r}) = \gamma^0 E\psi(\vec{r}), \tag{4.55}$$

Taking into account the matrices $\gamma_k = -\gamma^k$, $\gamma_k\gamma^5$ and $\gamma0$, we can also write this equation via the Hamiltonian matrix,

$$\hat{H}\psi(\vec{r}) = E\psi(\vec{r})\ , \tag{4.56}$$

which explicitly reads as

$$\begin{pmatrix} m-(t^0-\vec{\tau}\cdot\vec{\sigma}) & \vec{\sigma}\cdot\hat{\vec{p}}-(\tau^0-\vec{t}\cdot\vec{\sigma}) \\ -\vec{\sigma}\cdot\hat{\vec{p}}+(\tau^0-\vec{t}\cdot\vec{\sigma}) & m+(t^0-\vec{\tau}\cdot\vec{\sigma}) \end{pmatrix}\begin{pmatrix}\psi_I\\ \psi_{II}\end{pmatrix} = \begin{pmatrix} E & 0\\ 0 & -E\end{pmatrix}\begin{pmatrix}\psi_I\\ \psi_{II}\end{pmatrix}, \tag{4.57}$$

where $\hat{\vec{p}} = -i\hbar\vec{\nabla}$ is the 3-momentum operator, and we have introduced the following notation for the torsion quantities

$$t^\nu \equiv \alpha_1 T^\nu + \beta_2\breve{T}^\nu, \qquad \tau^\lambda \equiv \alpha_2\breve{T}^\lambda + \beta_1 T^\lambda\ . \tag{4.58}$$

Alternatively, this system can also be written in the (perhaps) more convenient way

$$\begin{pmatrix} m-E & \vec{\sigma}\cdot\hat{\vec{p}} \\ -\vec{\sigma}\cdot\hat{\vec{p}} & m+E\end{pmatrix}\begin{pmatrix}\psi_I\\ \psi_{II}\end{pmatrix} = \begin{pmatrix} t^0-\vec{\tau}\cdot\vec{\sigma} & \tau^0-\vec{t}\cdot\vec{\sigma} \\ -\tau^0+\vec{t}\cdot\vec{\sigma} & -t^0+\vec{\tau}\cdot\vec{\sigma}\end{pmatrix}\begin{pmatrix}\psi_I\\ \psi_{II}\end{pmatrix}\ , \tag{4.59}$$

which highlights the fact that the matrix on the right-hand side contains the geometrical effects due to torsion, including spin-torsion interactions of both parity breaking and parity-preserving types. The eigenvalue problem above is a system of two coupled equations for the 2-spinors ψ_I and ψ_{II}. To solve it we use the general form of the spinor $\psi(\vec{r}) = \chi e^{i\vec{p}\cdot\vec{r}/\hbar}$ and the properties of Pauli matrices, so that the first of these equations can be written as

$$\begin{aligned}&\begin{pmatrix} p_3+t^3-\tau^0 & p_1+t^1-i(p_2+t^2) \\ p_1+t^1+i(p_2+t^2) & -p_3-t^3-\tau^0\end{pmatrix}\begin{pmatrix}\chi_1^{II}\\ \chi_2^{II}\end{pmatrix}\\ &= \begin{pmatrix} E-m+t^0-\tau^3 & -(\tau^1-i\tau^2) \\ -(\tau^1+i\tau^2) & E-m+t^0+\tau^3\end{pmatrix}\begin{pmatrix}\chi_1^{I}\\ \chi_2^{I}\end{pmatrix}.\end{aligned} \tag{4.60}$$

Now let us consider the two orthogonal spin up/down solutions for the particle: $\chi^I = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi^I = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, to obtain the corresponding 4-spinor solutions. In the first (spin up) case, we get the system of equations

$$(p_3 + t^3 - \tau^0)\chi_1^{II} + (p_1 + t^1 - i(p_2 + t^2))\chi_2^{II} = E - m + t^0 - \tau^3, \tag{4.61}$$

$$(p_1 + t^1 + i(p_2 + t^2))\chi_1^{II} + (-p_3 - t^3 - \tau^0)\chi_2^{II} = -(\tau^1 + i\tau^2), \tag{4.62}$$

and therefore we find the solution $\chi = \begin{pmatrix} 1 \\ 0 \\ \chi_1^{II} \\ \chi_2^{II} \end{pmatrix}$, with

$$\chi_1^{II} = \frac{(E - m + t^0 - \tau^3)(-p_3 - t^3 - \tau^0) + (p_1 + t^1 - i(p_2 + t^2))(\tau^1 + i\tau^2)}{(p_3 + t^3 - \tau^0)(-p_3 - t^3 - \tau^0) - (p_1 + t^1 - i(p_2 + t^2))(p_1 + t^1 + i(p_2 + t^2))}, \tag{4.63}$$

$$\chi_2^{II} = \frac{-(\tau^1 + i\tau^2)(p_3 + t^3 - \tau^0) - (p_1 + t^1 + i(p_2 + t^2))(E - m + t^0 - \tau^3)}{(p_3 + t^3 - \tau^0)(-p_3 - t^3 - \tau^0) - (p_1 + t^1 - i(p_2 + t^2))(p_1 + t^1 + i(p_2 + t^2))}. \tag{4.64}$$

Note that, in the vanishing-torsion (Minkowski) limit we obtain $\chi_1^{II} = \frac{p_3}{E+m}$, $\chi_2^{II} = \frac{p_1+ip_2}{E+m}$, which is exactly the 4-spinor solution corresponding to the free fermion, spin up state, with $E^2 = p^2 + m^2$. As for the second (spin down) case, we obtain the system

$$(p_3 + t^3 - \tau^0)\chi_1^{II} + (p_1 + t^1 - i(p_2 + t^2))\chi_2^{II} = -(\tau^1 - i\tau^2), \tag{4.65}$$

$$(p_1 + t^1 + i(p_2 + t^2))\chi_1^{II} + (-p_3 - t^3 - \tau^0)\chi_2^{II} = E - m + t^0 + \tau^3 , \tag{4.66}$$

and therefore we find the solution $\chi = \begin{pmatrix} 0 \\ 1 \\ \chi_1^{II} \\ \chi_2^{II} \end{pmatrix}$, with

$$\chi_1^{II} = \frac{-(\tau^1 - i\tau^2)(-p_3 - t^3 - \tau^0) - (p_1 + t^1 - i(p_2 + t^2))(E - m + t^0 + \tau^3)}{(p_3 + t^3 - \tau^0)(-p_3 - t^3 - \tau^0) - (p_1 + t^1 - i(p_2 + t^2))(p_1 + t^1 + i(p_2 + t^2))} \tag{4.67}$$

$$\chi_2^{II} = \frac{(E - m + t^0 + \tau^3)(p_3 + t^3 - \tau^0) + (p_1 + t^1 + i(p_2 + t^2))(\tau^1 - i\tau^2)}{(p_3 + t^3 - \tau^0)(-p_3 - t^3 - \tau^0) - (p_1 + t^1 - i(p_2 + t^2))(p_1 + t^1 + i(p_2 + t^2))}. \tag{4.68}$$

Again we have the correct Minkowski limit, $\chi_1^{II} = \frac{p_1-ip_2}{E+m}$, and $\chi_2^{II} = \frac{-p_3}{E+m}$, describing the free particle, spin down state. Note that, proceeding in a similar manner, we could derive the corresponding expressions for the 4-spinor solutions associated to the anti-fermion in the spin up/down states.

To simplify further our analysis let us consider the *ansatz* for the torsion components $t^\mu = (0, t^1, t^2, t^3)$, and $\tau^\mu = (0, 0, 0, \tau)$ in Eq. (4.58). The spin up particle solution is then given by

$$\psi = \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_3^{\text{eff}}}{E + m_{\text{eff}}} \\ \dfrac{p_1^{\text{eff}} + i p_2^{\text{eff}}}{E + m_{\text{eff}}} \end{pmatrix} e^{i(\vec{p}\cdot\vec{r} - Et)/\hbar} , \tag{4.69}$$

where

$$E^2 = p_{\text{eff}}^2 + m_{\text{eff}}^2 , \tag{4.70}$$

with the definitions

$$p_k^{\text{eff}} \equiv p_k + t^k \tag{4.71}$$

$$p_{\text{eff}}^2 = (p_1 + t^1)^2 + (p_2 + t^2)^2 + (p_3 + t^3)^2 \tag{4.72}$$

$$m_{\text{eff}} = m + \tau . \tag{4.73}$$

Analogously, for the spin down particle we get the solution

$$\psi = \begin{pmatrix} 0 \\ 1 \\ \dfrac{p_1^{\text{eff}} + i p_2^{\text{eff}}}{E + m_{\text{eff}}} \\ \dfrac{-p_3^{\text{eff}}}{E + m_{\text{eff}}} \end{pmatrix} e^{i(\vec{p}\cdot\vec{r} - Et)/\hbar} , \tag{4.74}$$

where Eqs. (4.70)–(4.72) still hold but now the effective mass in Eq. (4.73) becomes $m_{\text{eff}} \equiv m - \tau$. Therefore, two different energy levels are obtained for the spin up and spin down states. For the particle, the energy of the anti-aligned state with respect to the direction of $\vec{\tau}$ is lower than the aligned state. These two possible energy states, $E_2^2 = p_{\text{eff}}^2 + (m + \tau)^2$ and $E_1^2 = p_{\text{eff}}^2 + (m - \tau)^2$, correspond to the energy transition

$$h\nu = E_2 - E_1 = \frac{4m\tau}{E_1 + E_2} , \tag{4.75}$$

which in the reference frame of the particle reads

$$h\nu = \frac{4m\tau}{\left[\vec{t}^2 + (m+\tau)^2\right]^{1/2} + \left[\vec{t}^2 + (m-\tau)^2\right]^{1/2}} , \tag{4.76}$$

where $\vec{t}^2 \equiv (t^1)^2 + (t^2)^2 + (t^3)^2$ can be written simply as t^2 assuming that $\vec{t}$ is aligned in any of the spatial axis directions of the reference system of coordinates. We recall that the torsion functions t^μ and τ^λ are constructed from the torsion trace vector and axial vectors and depend on the (parity-preserving) (α_1, α_2) and (parity-breaking) (β_1, β_2) coupling parameters.

The parity symmetry breaking is one of the conditions usually considered in order to explain some early Universe particle physics that can provide a matter/anti-matter asymmetry. The torsion of the RC spacetime can induce parity breaking effects in the fermionic sector via the non-minimal couplings explored here. These effects include the prediction of well-defined frequencies that a free fermion can absorb or emit in order to make transitions between the predicted two energy levels that arise depending on the spin orientation with respect to external torsion quantities. The signature of parity breaking (chirality) might also be present in the radiated field itself, in the form of polarized light.

4.1.3 Physical implications and observational effects

We now summarize some possible effects which might have interesting applications:

i) First of all one expects a generalization of these results for the case of bound states of electrons within atoms and molecules and also for the bound states of nucleons within atomic nuclei. Therefore, the possibility of detecting the effects of torsion by measuring spectral lines and searching for new fine and hyper-fine structures is plausible, within strong gravity regimes and using ultra-sensitive spectrographs. There could be astrophysical spectral signatures of torsion waiting to be discovered around intense gravitational fields of neutron stars or even in X-ray binaries where one of the objects is a blackhole candidate surrounded by an accretion disk. For a given bound system, a specific initial energy level could be chosen such that transitions from this level into the two (Zeeman-like) lower levels due to the torsion-spin interaction could be searched for, using advanced spectrographs. This could be done for different values of the predicted torsion in the emission region/s, according to different relevant gravitational theories.

ii) Another quite intriguing possibility is related to the fact that the Hawking radiation is due to fermion/anti-fermion pair production within quantum field theory in the curved spacetime around blackholes. If the geometry around these object is non-Riemann but instead Riemann-Cartan, then the out-going and in-going energy flux through the event horizon will be spin state dependent. In particular, one can say that the (out flux) energy loss for particles (or anti-particles) aligned with the background torsion is more efficient.

iii) Another quite relevant fact is the notion that the energy flux due to particle/anti-particle pair annihilation, in the form of photons will have different energy or (intensity) depending on the relative spin alignment/anti-alignment of the pair. Indeed, non-standard mechanisms (in the presence of external electromagnetic fields for example) can generate polarized pair production, with relative spins aligned rather than anti-aligned. Moreover, for a positively oriented torsion, the pair configuration with both spins aligned with the background torsion is more energetic (effectively more massive) than the case with both spins anti-aligned with respect to the axial torsion. There are three distinct classes of energy configurations. These correspond to the cases where the particle and anti-particle are anti-aligned with respect to each other and the two remaining configurations, where both particles have the same spin orientation (aligned or anti-aligned with respect to torsion). Moreover, the expected quantum field theory vacuum fluctuations due to virtual particle/anti-particle pairs, in the presence of a well-defined background torsion, will also depend on the homogeneous/inhomogeneous and dynamical/static character of the axial torsion.

iv) Also related to the emission of photons due to pair annihilation, let us suppose high energy astrophysics environments where particle/anti-particle annihilation takes place with the resulting emission of gamma rays, and suppose there are strong gravitational fields involved. Then, an observable signature of the background torsion could be obtained by comparing the measured flux spectra with the detailed theoretical prediction of the emission curves. In the most general case (with the three possible energy outcomes for the pair configuration), the theory suggests that the radiated flux should result from the superposition of three curves peaked at the characteristic nearby frequencies, corresponding to the three values of the effective mass of the pair. Disregarding complex environmental effects and significant changes in the background torsion within the typical scales of the emission region, these emission lines (broadened due to environmental effects and natural broadening), would be detected very close together, resembling a kind of hyperfine structure. In unconventional case of pair production with the fermion and anti-fermion aligned with respect to each other, the theory predicts a specific characteristic frequency (determined by the particle's mass, the torsion and possibly the temperature of the emission region), with a value slightly different from the corresponding predicted frequency in the absence of torsion.

v) For astrophysical applications, if one takes into account the spatial changes in the intensity of torsion, then the photons due to particle/anti-particle pair annihilation, will have different energy or (intensity) depending on how strong the torsion field is, or in other words, on the specific region where the particle annihilation takes place, in certain strong gravity astrophysical scenarios. This could be used to probe different regions of the high energy astrophysics with strong gravitational fields as in compact objects with accretions disks. A similar reasoning also applies to the case of emission due to transitions between the energy levels of fermions induced by torsion.

vi) Coming back to the assumption that there might exist non-standard mechanisms that can generate polarized pair production, the photons generated through the subsequent pair annihilation will have different energies depending if both spins are aligned or anti-aligned with respect to the background torsion. In simple terms there are three different possible energies for the fermion/anti-fermion pair in the background torsion: 1) particle and anti-particle anti-aligned with each other; 2) pair with both spins up (aligned to torsion); 3) pair with both spins down. In terms of pair production, these correspond to three energy/mass levels that the photons can generate where the first case is intermediate, the second is the highest and the third is less massive/energetic outcome. In the very early universe, within a period where effects from non-Riemann geometries can affect the particle physics, these values also can be translated into the temperature of the quark-gluon-lepton plasma. In principle, at a specific critical temperature the second case (non-standard) of pair production from the thermal photons is no longer possible and at another critical value the first case (standard) is also impossible, and finally, at another lower temperature, the last case (non-standard) is also impossible. The difference between these three channels are only significant while the background torsion is above a certain minimum value that can be estimated for different gravity models with torsion.

vii) Also another interesting possibility is the interaction of a Dirac fermion with the torsion of a sea of vacuum fermion-condensates. In other words, if the vacuum has spin due to non-zero expectation values of fermionic-condensates, then by invoking the ECSK theory, it produces a background (vacuum) torsion. So, fermions in vacuum would have a different effective mass according to the relative orientation of the fermionic spin with respect to the background vacuum axial torsion (see section 4.3). This could be tested in laboratories, putting bounds on the predicted effects, with the advantage that these tests due not require strong gravitational fields, but with the challenging requirement of reproducing the conditions of fermionic vacuum condensates in the laboratory.

viii) Another physical prediction comes from the fact that according to the general and quite simple scheme presented here, there should be a continuous and smooth phase transition for a Bose-Einstein system as a superconducting fluid/material in a spacetime background with torsion. Cooper pairs of anti-aligned fermions in bound states are required in the Bardeen-Cooper-Schrieffer (BCS) model of superconductivity and in general Bose-Einstein condensates. Since the effective mass of the pair depends on the interaction with the torsion background, the effective spin-zero bosonic field due to the ensemble of Cooper pairs, will have a differential energy increasing for higher values of torsion, $\partial\varepsilon/\partial m_{\check{T}} = 4m > 0$, where $\varepsilon \equiv E_1^2 + E_2^2$. A similar effect could be relevant within the exotic physics of Bose-Einstein fermion pairs of superconducting and superfluid phases of the interiors of neutron stars and in hypothetical quark and (strange quark) stars. Accordingly, the ground state energies of the BCS bosonic field of the superfluid or super conducting layers increase with depth, as one approaches the core of the star and it is reasonable to assume two relevant predictions: 1) This increase of the energy of the bosonic field with decreasing radius should affect the effective macroscopic equation of state, which in principle can be tested against GW observations of NS-NS mergers (and more rigorously if detectors are built with sensitivity curves optimized to the high frequencies of the GW spectrum, which in this way enables to better probe observationally the physics of the NS interior); and 2) This effect should affect the estimates for the lower limit of NS masses and possibly also the higher mass limit respecting the equilibrium configuration condition. Moreover, the stability of Cooper pairs might be strongly perturbed as the torsion increases above a certain threshold, since the background torsion axial vector along a well defined direction can act exactly as an external magnetic field does in paramagnetic materials i.e, above a certain critical value of the external field, a significant number of large clusters of "aligned" spins develop (and percolating the whole system) and the material is magnetized. The spin-spin interaction that naturally exists in a system with spins is analogous to a thermal-like interaction (increasing temperature tends to rise the entropy, and generate a random distribution of spins), while the external field tends to counter act the random distribution of spins, by establishing gradually a more ordered state. Therefore, torsion can also act as an external field driving a phase transition in a macroscopic system of microphysical components with spin, magnetizing the material, with the emergence of a macroscopic (intrinsic) spin. In that sense, the superfluid/superconducting phase of the BCS models could suffer a phase transition for sufficiently strong "external" spacetime torsion, inside ultra-dense compact objects. These topics deserve a much more careful analysis, since they evolve very complicated physics of the interiors of neutron stars and related objects.

xix) Finally, these simple results open another window for laboratory tests of the spacetime torsion near Earth. In perfect analogy with the magnetic spin resonance, one could design a torsion-spin resonance. In the first case an external magnetic field generates the splitting of energy levels (Zeeman-effect) in an appropriate material sample and a time varying current produces an electromagnetic wave which suffers a measurable absorption, once the resonance frequency is achieved matching the energy gap. In the torsion-spin resonance, the minute torsion around the Earth if it exists, is predicted to give rise to a hyper-fine Zeeman-like effect. The indirect detection of this would be achieved by verification of a measurable absorption (decrease in intensity) of the electromagnetic wave interacting with the material sample, once the resonance frequency is achieved. For free fermions we saw that the predicted frequency (in the particle's frame) does not depend on the fermionic mass, only on the background torsion. If this torsion has a magnitude of about $10^{-16} - 10^{-15} m^{-1}$ we get a resonance frequency around $1 - 10 nHz$. This corresponds to resonance frequencies in the radio band.

The basic torsion-spin interaction underlying these results might suggest a more fundamental question related to the structure of quantum physics in the presence of gravity and non-Riemann spacetime geometries:

If the spin-torsion interaction is lower for one of the spin states, is this reinforcing the idea (often defended by Roger Penrose) that gravity plays a role in quantum state reduction, and more specifically is this requiring that the torsion forces the fermionic spin mixture of states to collapse into its lowest energy configuration, by breaking the symmetry of spin states, that is, its energy degeneracy in the absence of gravity?

This is an open question that deserves further careful attention as it seems that the torsion of Riemann-Cartan spacetimes might indeed lead to a re-analysis of the measurement problem with regards to fermionic fields.

4.2 Bosonic (s=1) fields in RC spacetime

The coupling of electromagnetic fields to the spacetime torsion can be expressed through the minimal procedure (partial derivatives are changed to covariant ones). It is also possible to introduce non-minimal couplings directly in the field equations or couplings that come from exploring the constitutive relations between the field strengths $F = (E, B)$ and the excitations $\mathcal{H} = (D, H)$ (see [135, 136]). Not all of these possibilities respect the charge conservation postulate, which has passed the observational tests at the physical scales where it was tested. In fact, in the context of modified theories of gravity it is often the case where the particle number can change due to intense gravitational fields. It is a matter of interpretation to say that the energy-momentum (of a particle system) is not conserved or that the generalized energy-momentum is conserved with contributions coming from the geometrical part of gravity, i.e, from the spacetime geometrodynamics, (which acquires physical properties, such as energy-momentum). The same can be said for gravity theories where the coupling with electromagnetism induces the non-conservation of charge currents. This is the case of minimal couplings with torsion.

It is usually assumed that torsion does not minimally couple to the electromagnetic field, since it breaks the $U(1)$ gauge invariance (for details see e.g. [100]). However, the physics of phase transitions in condensed matter systems, superconductivity and early universe is permeated by processes that lead to spontaneous symmetry breaking, and in high density environments (early Universe, ultra-compact objects) torsion could provide a physical mechanism to induce such a symmetry breaking. The electromagnetic field equations are thus generalized and interesting new physics comes into play, with analogies with Proca fields. Since in the ECSK theory torsion vanishes outside the (spin) matter sources and is negligible at low densities, in this case the Maxwell equations remain valid for all phenomena that we can presently probe directly.

One could maintain the foundational connection between the electromagnetic field and the U(1) gauge symmetry and simply consider some bosonic vector ($s = 1$) field minimally or non-minimally coupled to torsion. Then, strictly speaking this would not be the photon-electromagnetic field although it behaves exactly as the $U(1)$ gauge field once torsion vanishes. The vanishing of torsion occurs for example in the vacuum environments according to the ECSK theory, or in the cosmological context after the torsion dominated *era* in a transition similar to symmetry breaking phase transitions, with a typical critical density and temperature ($10^{24}K$ for electrons).

The main aim of this subsection is to study the (minimal) coupling of torsion with electromagnetism and some of its physical implications[5].

4.2.1 Minimal coupling to torsion: U(1) symmetry breaking.

Let us consider the electromagnetic theory from an action variational principle and look at the coupling with torsion via the minimal procedure. Our theory is expressed in the following Lagrangian

$$\mathcal{L}_{\text{Max}} = \frac{\lambda}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu, \tag{4.77}$$

[5] Although we use the term electromagnetism here, it should be clear that we are studying the coupling of torsion to a ($s = 1$) bosonic field that does not necessarily correspond to the Maxwell field.

where λ is a coupling parameter, $F_{\mu\nu}$ is the field strength defined as

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu = \tilde{F}_{\mu\nu} + 2K^\lambda{}_{[\mu\nu]}A_\lambda \tag{4.78}$$

which generalizes the usual Faraday tensor $\tilde{F}_{\mu\nu}$, ∇ is the covariant derivative in RC space-time, A_μ is the 4-potential and j^μ is the current 4-vector. The equation (4.77) is the generalized Maxwell Lagrangian in a RC geometry. It satisfies the local Poincaré invariance of RC spacetime but explicitly breaks the U(1) gauge symmetry. It is equivalent to the following effective Lagrangian

$$\mathcal{L}_{\text{Max}} = \tilde{\mathcal{L}}_{\text{Max}} + \lambda\left(K^{\lambda[\mu\nu]}K^\gamma{}_{[\mu\nu]}A_\gamma + K^{\lambda[\mu\nu]}\tilde{F}_{\mu\nu}\right)A_\lambda. \tag{4.79}$$

This expression is valid for any theory of gravity in RC spacetime. It only assumes a minimal coupling between the 4-potential and the torsion. For a specific gravity model, the torsion can be replaced using the appropriate field equations. For example, for the Einstein-Cartan (ECSK) theory, the torsion corrections are proportional to the spin density of Dirac fermions. Varying the action in eq. (4.77) with respect to the vector potential A_μ yields

$$\nabla_\mu F^{\mu\nu} = \lambda^{-1}j^\nu, \tag{4.80}$$

which can be conveniently rewritten as

$$\tilde{\nabla}_\mu\tilde{F}^{\mu\nu} = \lambda^{-1}(j^\nu + J^\nu), \tag{4.81}$$

where we have defined the torsion-induced four-current

$$J^\nu = -\lambda\Big(2(K^\nu{}_{\lambda\mu}K^{\gamma[\mu\lambda]} + K_\lambda K^{\gamma[\lambda\nu]})A_\gamma + K^\nu{}_{\lambda\mu}\tilde{F}^{\mu\lambda} + K_\lambda\tilde{F}^{\lambda\nu} + 2\tilde{\nabla}_\mu(K^{\gamma[\mu\nu]}A_\gamma)\Big), \tag{4.82}$$

with $K_\lambda \equiv K^\alpha{}_{\lambda\alpha}$, and encompassing the new physics coming from the minimal coupling with the spacetime torsion of a RC spacetime. The generalized current can also be written as

$$J^\nu = -\lambda\Big[2(T^\nu{}_{\lambda\mu}T^{\gamma\mu\lambda} + 2T_\lambda T^{\gamma\lambda\nu})A_\gamma + T^\nu{}_{\lambda\mu}\tilde{F}^{\mu\lambda} + 2T_\lambda\tilde{F}^{\lambda\nu} + 2\tilde{\nabla}_\mu(T^{\gamma\mu\nu}A_\gamma)\Big]\ , \tag{4.83}$$

where we have used the fact that contortion is antisymmetric in the first two indices and also that $K^\nu{}_{[\lambda\mu]} = T^\nu{}_{\lambda\mu}$ and $K_\lambda = 2T_\lambda$. Due to this coupling, as it is apparent from Eq.(4.78), only the antisymmetric part of the contortion tensor enters in the generalized Faraday tensor and in the Lagrangian. But in the field equations, notice that the first contorsion factors in the first four terms, come from the covariant derivative, therefore the antisymmetrization is not there. Still, for the terms where both indices are contracting with an antisymmetric object, only the antisymmetric part survives the contraction. The terms proportional to the 4-potential on the right-hand side, resemble the Proca field, where the coupling to the spacetime torsion (squared) is providing an effective mass to the bosonic vector field. It is a natural analogy, since Proca fields also break the $U(1)$gauge invariance. We can also rewrite the field equations in (4.80), in the following way

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = \lambda^{-1}(j^\nu + \jmath^\nu), \tag{4.84}$$

where in this case the new current correction is given by

$$j^{\nu} = -\lambda \left(K_{\lambda} F^{\lambda\nu} + K^{\nu}{}_{[\beta\alpha]} F^{\alpha\beta} \right), \tag{4.85}$$

or, in terms of torsion

$$j^{\nu} = -\lambda \left(2T_{\lambda} F^{\lambda\nu} + T^{\nu}{}_{\beta\alpha} F^{\alpha\beta} \right). \tag{4.86}$$

One can clearly understand that if the electromagnetic field itself contributes to torsion via Cartan-like equations (in ECSK gravity or its generalizations), then, upon substitution in the field equations above, the result is an effective non-linear electrodynamics[6] induced by the Riemann-Cartan geometry. Indeed, the singular behaviour of Maxwell electrostatic equations near charge particles is removed way by the resulting (classical) non-linear theory of electrodynamics (see [21]). The Einstein theory of gravity with non-linear electrodynamics can be equivalent to the Einstein-Cartan theory of usual electrodynamics minimally coupled to gravity. Similarly, as we will see in the next section, GR with a classical non-linear spinor (obeying an Heisenberg-type cubic equation) can be equivalent to the ECSK theory of a Dirac spinor (with minimal coupling).

Finally, from the field equations one sees that the generalized conservation equation can be written as

$$\tilde{\nabla}_{\nu} j^{\nu} = -\tilde{\nabla}_{\nu} J^{\nu}. \tag{4.87}$$

or alternatively

$$\nabla_{\nu} j^{\nu} = \frac{\lambda}{2} \left[\nabla_{\nu}, \nabla_{\mu} \right] F^{\mu\nu} , \tag{4.88}$$

where

$$\left[\nabla_{\nu}, \nabla_{\mu} \right] F^{\mu\nu} = R^{\mu}{}_{\varepsilon\nu\mu} F^{\varepsilon\nu} + R^{\nu}{}_{\varepsilon\nu\mu} F^{\mu\varepsilon} + 2T^{\gamma}{}_{\nu\mu} \nabla_{\gamma} F^{\mu\nu}, \tag{4.89}$$

is the commutator of covariant derivation of an antisymmetric $(0,2)$-tensor in RC spacetime. This expression is valid for all gravity theories with this RC space-time geometry. We see that the 4-current is not conserved, which suggests the possibility that the particle number can change due to intense gravitational fields. In other words, spacetime geometrodynamics can generate particle creation/annihilation. On the other hand , as will be discussed, one can redefine the notion of charge current by including the contribution from the torsion coupling and it is this extended current that is conserved.

Generalized Maxwell dynamics revisited: Exploring the field equations

The field equations (4.80) can be expressed as

$$\partial_{\mu} \tilde{F}^{\mu\nu} + \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g}) \tilde{F}^{\mu\nu} = \lambda^{-1}(j^{\nu} + J^{\nu}), \tag{4.90}$$

Then, with the following re-definitions

$$\bar{F}^{\mu\nu} \equiv \sqrt{-g} \tilde{F}^{\mu\nu}, \quad \bar{j}^{\mu} \equiv \sqrt{-g} j^{\mu}, \quad \bar{J}^{\nu} \equiv \sqrt{-g} J^{\nu} \tag{4.91}$$

we get the suitable form for the field equations

$$\partial_{\mu} \bar{F}^{\mu\nu} = \lambda^{-1}(\bar{j}^{\nu} + \bar{J}^{\nu}). \tag{4.92}$$

[6] More rigorously we get a non-linear massive bosonic field dynamics.

These are the generalized inhomogeneous equations in RC spacetime in a form appropriate to compare with the usual electric Gauss and Maxwell-Ampere laws in Minkowski spacetime. We will consider the usual electromagnetic fields related to the Faraday tensor, defined by the expressions

$$\tilde{E}_k/c \equiv \tilde{F}_{0k} \quad \tilde{F}_{ij} \equiv -\epsilon_{ijk}\tilde{B}^k. \tag{4.93}$$

One can further present the field equations (4.92) in this way

$$\partial_\mu(\bar{F}_{\alpha\beta})g^{\mu\alpha}g^{\nu\beta} + \bar{F}_{\alpha\beta}\Omega_\mu^{\ \mu\alpha\nu\beta} = \lambda^{-1}(\bar{j}^\nu + \bar{J}^\nu), \tag{4.94}$$

where $\Omega_\mu^{\ \mu\alpha\nu\beta} \equiv \partial_\mu(g^{\mu\alpha}g^{\nu\beta})$. Then, it is easy to verify that the field equations can be expressed in the following familiar manner

$$\vec{\nabla}\cdot\vec{\hat{E}} = \frac{c^2}{\lambda}(\bar{\varrho} + \bar{J}^0), \quad \vec{\nabla}\times\vec{\hat{B}} = \lambda^{-1}(\vec{\bar{j}} + \vec{\bar{J}}) + \frac{1}{c^2}\frac{\partial\vec{\hat{E}}}{\partial t} \tag{4.95}$$

with

$$\hat{E}^k/c \equiv -\bar{F}^{0k} = -\sqrt{-g}\tilde{F}^{0k} \quad \bar{F}^{ij} \equiv -\epsilon^{ijk}\hat{B}_k. \tag{4.96}$$

Notice that for diagonal metrics we get $\hat{E}^k/c = -\sqrt{-g}g^{00}g^{kk}\tilde{F}_{0k}$ (no contraction here), therefore in Minkowski spacetime $\hat{E}^k = \tilde{E}_k$. In principle, given the charge 4-current, the background torsion and metric, one can solve these equations for the fields $(\hat{E}, \hat{B})$, and from these obtain the original fields $(\tilde{E}_k, \tilde{B}^j)$. Regarding the usual homogeneous equations, these remain the same

$$\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow \partial_\alpha\tilde{F}_{\beta\gamma} + cicl.perm = 0. \tag{4.97}$$

On the other hand, the same is not true for the new or generalized fields obtained from $F_{\mu\nu}$. We easily deduce that

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} \neq 0,$$

with

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = -\left(K^\lambda{}_{\beta\alpha}F_{\lambda\gamma} + K^\lambda{}_{\gamma\alpha}F_{\beta\lambda} + 2\tilde{\nabla}_\alpha(K^\lambda{}_{[\beta\gamma]}A_\lambda) + cicl.perm.\right).$$

Alternatively, one can write

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = [\nabla_\alpha, \nabla_\beta]A_\gamma + cicl.perm., \tag{4.98}$$

with

$$[\nabla_\mu, \nabla_\nu]A_\varepsilon = R^\lambda{}_{\varepsilon\mu\nu}A_\lambda + T^\lambda{}_{\mu\nu}\nabla_\lambda A_\varepsilon. \tag{4.99}$$

In the expression above, in the right hand side we have the curvature and torsion of RC spacetime U4. Therefore, the "generalized" magnetic flux is not conserved, which can be interpreted as the appearance of magnetic monopoles, induced by spacetime torsion. Finally, the electromagnetic equations in vacuum are

$$\vec{\nabla}\cdot\vec{\hat{E}} = \frac{c^2}{\lambda}\bar{J}^0, \quad \vec{\nabla}\times\vec{\hat{B}} = \lambda^{-1}\vec{\bar{J}} + \frac{1}{c^2}\frac{\partial\vec{\hat{E}}}{\partial t}, \quad \vec{\nabla}\cdot\vec{\tilde{B}} = 0, \quad \vec{\nabla}\times\vec{\tilde{E}} = -\frac{\partial\vec{\tilde{B}}}{\partial t}, \tag{4.100}$$

therefore, for plane waves we see that $\vec{k} \cdot \hat{\vec{E}} \neq 0$ and $\vec{k} \times \hat{\vec{B}} \sim \frac{w}{c^2}\hat{\vec{E}} + \lambda^{-1}\hat{\vec{J}}$, which means that the electric vector field acquires a longitudinal component (in the direction of the wave vector). This correction will only significantly manifest itself in physical environments where torsion is non-negligible, i.e in very strong gravity regimes.

In principle, different *ansatze* for torsion can be taken in order to explore potential effects in the electromagnetic field dynamics:

i) Constant (background torsion) $T_{\alpha\beta\gamma} \sim const.$

ii) Spherically symmetric torsion field $\breve{T}^{\mu}(r) = b^{\mu} f(r)$.

iii) Harmonic wave $T_{\alpha\beta\gamma} \sim A_{\alpha\beta\gamma} e^{-ik^{\mu}x_{\mu}}$, with $A_{\alpha\beta\gamma}$ being constants.

In all applications it is also useful to consider, for simplification, that torsion is only given by one of its irreducible components: The tensor, trace-vector or the pseudo (axial) vector part. The second case is relevant for relativistic astrophysics and the testing of PGTG, while the third case is the natural *ansatz* to consider GW applications and also tests of gravity theories with propagating torsion modes.

4.3 Einstein-Cartan theory with fermions and bosons

4.3.1 ECSK gravity with fermions and bosons minimally coupled to RC geometry: U(1) and C symmetry breaking

We now return our attention to the ECSK theory [137]. In general torsion in ECSK theory becomes important in scenarios where high-spin densities are present.

A path recently explored to extend this theory is the analysis of new (non-minimal) couplings between torsion and the matter fields [138, 139, 140, 141, 142]. The coupling between torsion and electromagnetism have also been carefully analysed in the literature with the result that, in general this can be achieved by either changing the field equations with minimal/non-minimal couplings, or via the constitutive relations between the field strengths $F = (E, B)$ and the excitations $\mathcal{H} = (D, H)$ (see [135, 136]). Though it is usually assumed that torsion does not minimally couple to the electromagnetic field, since this breaks the $U(1)$ gauge invariance (for details see e.g. [100, 135]), we explore here such coupling. Since in EC theory torsion vanishes outside the matter sources and is negligible at low densities, the Maxwell equations remain valid for all phenomena that we can presently probe directly.

The main aim of this section, inspired by the work in [4], is to study the (minimal) coupling of torsion with fermions and bosonic fields, in ECSK gravity. Through the Cartan equations relating the space-time torsion to the matter fields, this coupling induces non-minimal and self-interactions in the matter fields, and provides a physical mechanism to generate a $U(1)$ symmetry breaking for high densities and fields. In the broken phase, torsion provides an effective mass for the photon, with the electromagnetic potential obeying an extended Proca-like equation. The minimal coupling to torsion have also been analysed in [143].In our approach we consider first the regime in which torsion is only sourced by fermions, and extend it later to the general case where both fermionic and bosonic fields contribute to torsion via the corresponding spin energy densities. The first case is a simplifying *ansatz*, where the electromagnetic fields are influenced by the (background) spacetime torsion without backreacting on it. This case serves to illustrate some of the effects in the new dynamics due to the minimal coupling of bosons and torsion. The second case encodes the full dynamics with the bosonic spin contribution to torsion, which induces new non-linearities. Therefore, in this section we address the most relevant features of EC-Dirac-Maxwell model with $U(1)$ symmetry breaking as well as some of its physical implications.

In this section we start by reviewing the EC gravity, focusing on the case where fermions are represented by a Dirac field. Then we extend the EC theory by introducing a vector bosonic field minimally coupled to torsion, and find the corresponding dynamics for gravitational, electromagnetic and fermionic sectors in two cases: (i) fermionic background torsion and (ii) the full case, including the bosonic backreaction to torsion via its spin tensor. We conclude with a broad discussion of the phenomenological implications of these results, including some future perspectives.

Einstein-Cartan-Dirac theory

Let us consider, as the matter sector in the action (3.58), a free Dirac fermionic field with mass m, minimally coupled to torsion. The corresponding Lagrangian density can be expressed as in equation (4.6). We have

$$\mathcal{L}_{\text{Dirac}} = \tilde{\mathcal{L}}_{\text{Dirac}} + K^{\alpha\lambda\beta} s_{\lambda\beta\alpha}, \tag{4.101}$$

given the definition of the spin tensor in (3.60). This expression is valid for any Dirac field minimally coupled to the RC spacetime geometry, it does not depend on any particular theory of gravity.

For this matter source the spin tensor is totally antisymmetric, i.e.,

$$s^{\mu\nu\varepsilon} = \frac{1}{2}\epsilon^{\mu\nu\varepsilon\alpha}\breve{s}_\alpha, \tag{4.102}$$

and is expressed in terms of the previously introduced Dirac (axial) spin vector as (see e.g. [21, 64, 95, 93, 94])

$$\breve{s}^\beta = \frac{\hbar}{2}\bar{\psi}\gamma^\beta\gamma^5\psi. \tag{4.103}$$

This Dirac pseudo-vector field will play a crucial role later. Accordingly, the Cartan equations simplify since torsion is completely antisymmetric and Eq. (3.59) becomes

$$T_{\alpha\beta\gamma} = K_{\alpha\beta\gamma} = \frac{\kappa^2}{2}\epsilon_{\alpha\beta\gamma\lambda}\breve{s}^\lambda, \tag{4.104}$$

therefore, the above Lagrangian introduces an effective spin-spin interaction induced by torsion. Using the Cartan equations we then have

$$\mathcal{L}_{\text{Dirac}} = \tilde{\mathcal{L}}_{\text{Dirac}} + \kappa^2 s^{\alpha\lambda\beta} s_{\lambda\beta\alpha} = \tilde{\mathcal{L}}_{\text{Dirac}} - \frac{3\kappa^2}{2}\breve{s}^\lambda\breve{s}_\lambda. \tag{4.105}$$

The first equality is valid in EC theory and the second equality is specific of the Einstein-Cartan-Dirac (ECD) model. Although in this section we will not use the Einstein-like equations (leaving this for the exploration of the cosmological applications in chapter 5), we can compute the form of the torsion-induced corrections on the right-hand side of the Einstein equations (3.62), with $U_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(C\sqrt{-g})}{\delta g^{\mu}} = -2\frac{\delta C}{\delta g^{\mu\nu}} + Cg_{\mu\nu}$, and in this case C becomes simplified

$$C \equiv -\frac{1}{2\kappa^2}\left(K^{\alpha\lambda}{}_{\alpha}K_{\gamma\lambda}{}^{\gamma} + K^{\alpha\lambda\beta}K_{\lambda\beta\alpha}\right) = -\frac{1}{2\kappa^2}K^{\alpha\lambda\beta}K_{\lambda\beta\alpha} = -\frac{\kappa^2}{2}s^{\alpha\lambda\beta}s_{\lambda\beta\alpha}, \tag{4.106}$$

due to the fact that for Dirac spinors contortion is completely antisymmetric. Using (4.101) and (4.106), the effective matter Lagrangian becomes

$$\mathcal{L}_m^{\text{eff}} = \tilde{\mathcal{L}}_m + \frac{\kappa^2}{2} s^{\alpha\beta\gamma} s_{\beta\gamma\alpha},$$

where the second term, corresponds to the energy density of a well-known spin-spin (contact) interaction.

Einstein-Cartan-Dirac-Maxwell theory

Let us now generalize the action (3.58) to incorporate a minimal coupling between torsion and the electromagnetic field (for recent works on torsion-matter couplings see e.g. [138, 139, 140, 141, 142]). This can be directly implemented at the level of the matter fields by assuming the matter Lagrangian density

$$L_m = \mathcal{L}_{\text{D}} + \mathcal{L}_{\text{M}} + j^{\mu} A_{\mu}, \tag{4.107}$$

where the Dirac Lagrangian is the same as in Eq. (4.6) therefore including a minimal coupling to the RC geometry, while A_μ is the electromagnetic four-potential and j^ν the electric charge current density of fermions. On the other hand, we now have the generalized Maxwell Lagrangian in a RC space-time (satisfying local Poincaré invariance) written as

$$\mathcal{L}_{\text{Max}} = \frac{\lambda}{4} F_{\mu\nu} F^{\mu\nu} \ , \tag{4.108}$$

where λ is a coupling parameter setting the system of units, and the generalized field strength tensor is defined in (4.78) where the second term breaks the $U(1)$ local symmetry. More explicitly, the Lagrangian density in Eq. (4.108) is given by (4.79). We will next proceed with the derivation of the electromagnetic and fermionic field equations corresponding to the action (3.58) with the $U(1)$-breaking term just introduced. The gravitational equations will be studied in the next chapter.

Cartan equations and torsion effects: Fermionic background torsion. Let us assume a background torsion resulting from the spin density of fermionic fields. Variation of the action with respect to the metric for the above matter sources yields the gravitational equations. The torsion corrections to the Einstein-Cartan-Dirac model are derived from the U(1) breaking Lagrangian term in (4.79), yielding the corresponding energy-momentum tensor. We will explore the consequences of these corrections in the gravitational sector in the next chapter (cosmological applications).

In general, the (minimal) coupling of torsion with fermions and bosons give rise to non-minimal interactions between the matter fields, as well as self-interactions, once the Cartan equations are used to replace the torsion components by the matter field variables. Under the *ansatz* that torsion is exclusively resulting from matter fields with half-integer intrinsic spin (fermions), the self-interactions in the bosonic sector vanish. In this torsion due to fermionic spin *ansatz*, the torsion tensor becomes completely antisymmetric and the Cartan equations in this case are given by Eq.(4.104). Under such an *ansatz*, the interaction part in the effective stress-energy (see chapter five) tensor introduces terms both linear and quadratic in the spin density, all of which depend on the electromagnetic quantities. Accordingly, the value of the coupling constant λ determines the scale at which the electromagnetic contribution becomes non-negligible. Thus, in principle, one could test torsion effects at spin densities smaller than the Cartan density threshold, for sufficiently high electromagnetic fields, which suggests that new gravitational (metric) effects could be present in the core of magnetars [144] and (hypothetical) quark stars [145]. Significant effects are expected for polarized matter because the linear terms will not average to zero and can introduce stronger torsion (spin) contributions to the Einstein equations at lower densities, given a sufficiently high electromagnetic potential. In summary, as we will see more explicitly, the torsion contributions to the metric field equations scale with $\kappa^4 s^2$ for the pure EC spin density correction, and $\kappa^4\lambda s\tilde{F}A$ and $\kappa^6\lambda s^2A^2$ for the linear and quadratic $U(1)$ symmetry breaking terms, respectively.

In general, the macroscopic description of a physical system is achieved through an averaging procedure. For simplicity, we will consider physical systems where the spin density obeys an approximate random distribution. This simplification (which is not valid in the presence of sufficiently intense magnetic fields that tend to align the spins) allows us to neglect the terms linear in the spin density and consider only the quadratic ones. For Dirac fermions, if we consider only the terms quadratic in torsion, we obtain after some algebraic manipulations, the energy momentum corresponding to the $U(1)$ symmetry breaking Lagrangian (second term in (4.79))

$$\Pi^{U(1)\text{break}}_{\mu\nu} = -\lambda\kappa^4\left[\frac{g_{\mu\nu}}{2}\Big(A^2\breve{s}^2-(\breve{s}\cdot A)^2\Big)-\breve{s}^2A_\mu A_\nu - A^2\breve{s}_\mu\breve{s}_\nu+4(\breve{s}\cdot A)\breve{s}_{(\mu}A_{\nu)}\right]. \tag{4.109}$$

Here, as usual $\breve{s}^2\equiv\breve{s}^\lambda\breve{s}_\lambda$, $A^2\equiv A^\lambda A_\lambda$ and $\breve{s}\cdot A\equiv\breve{s}^\lambda A_\lambda$.

Cartan equations and torsion effects: Including the spin tensor from the bosonic sector. If we also consider, besides the fermionic spin, the contribution from the generalized electromagnetic Lagrangian (4.79) to the (total) spin tensor, $s_{\lambda\alpha\beta} = s^M_{\lambda\alpha\beta}+s^D_{\lambda\alpha\beta}$, we obtain

$$s_{\lambda\alpha\beta} = \lambda\Big(A_{[\alpha}\tilde{F}_{\beta]\lambda}+2A_{[\alpha}T^\gamma{}_{\beta]\lambda}A_\gamma\Big)+s^D_{\lambda\alpha\beta}, \tag{4.110}$$

where $s^{D}{}^{\alpha}_{\ \beta\gamma}$ is Dirac's spin tensor and

$$s^M_{\lambda\mu\nu} = \lambda A_{[\mu}F_{\nu]\lambda} = \lambda\left(A_{[\mu}\tilde{F}_{\nu]\lambda}+2A_{[\mu}T^\alpha{}_{\nu]\lambda}A_\alpha\right), \tag{4.111}$$

represents the electromagnetic contribution to the spin tensor, i.e,

$$s^M_{\lambda\mu\nu} = \delta\mathcal{L}^{U(1)break}_M/\delta K^{\mu\nu\lambda}, \tag{4.112}$$

which also depends on torsion due to the minimal coupling previously introduced. The new Cartan equations can be written as

$$T^\alpha{}_{\beta\gamma} = \kappa^2\Big(s^{D}{}^{\alpha}_{\ \beta\gamma}+s^{M}{}^{\alpha}_{\ \beta\gamma}+\delta^\lambda_{[\beta}s^M_{\gamma]}\Big), \tag{4.113}$$

since Dirac's (completely antisymmetric) spin tensor has zero trace vector. With a bit of algebra we get

$$\tau^\alpha{}_{\beta\gamma}-2\lambda\kappa^2A_\lambda A_{[\beta}T^\lambda{}_{\gamma]}{}^\alpha = \kappa^2\left(\frac{1}{2}\epsilon^\alpha{}_{\beta\gamma\lambda}\breve{s}^\lambda+\lambda A_{[\beta}\tilde{F}_{\gamma]}{}^\alpha\right),$$

where $\tau^{\alpha}{}_{\beta\gamma} \equiv T^{\alpha}{}_{\beta\gamma} + T_{\gamma}\delta^{\alpha}_{\beta} - T_{\beta}\delta^{\alpha}_{\gamma}$ is the modified torsion tensor and the first term on the right-hand side corresponds to Dirac's spin tensor previously introduced. These expressions show that it is not trivial to separate the purely geometric torsion functions from the matter fields.

Let us contract the indices α and γ to obtain

$$-2T_{\beta} + \lambda\kappa^2 A_{\lambda} T^{\lambda}{}_{\beta\gamma} A^{\gamma} = -\frac{\lambda\kappa^2}{2}\tilde{F}_{\beta\gamma}A^{\gamma}. \tag{4.114}$$

which we shall use to find an expression for the torsion trace vector. From the equation above one gets the result $T_{\beta}A^{\beta} = 0$ and this can be used after contracting Eq. (4.114) with $A_{\alpha}A^{\gamma}$ to arrive at

$$-A^2 T_{\beta} + (1 + \lambda\kappa^2 A^2) A_{\lambda} T^{\lambda}{}_{\beta\gamma} A^{\gamma} = -\frac{\lambda\kappa^2}{2} A^2 \tilde{F}_{\beta\gamma} A^{\gamma}. \tag{4.115}$$

Therefore, from this system of two equations we easily solve for T_{β}

$$T_{\beta} = -\frac{\lambda\kappa^2}{2(2 + \lambda\kappa^2 A^2)}\tilde{F}_{\beta\gamma}A^{\gamma}. \tag{4.116}$$

Proceeding in a similar manner, by contracting Eq. (4.114) with A_{α}, after some algebra it is finally possible to transform the Cartan equations into a form in which the geometric torsion is separated from the matter fields, that is

$$T^{\alpha}{}_{\beta\gamma} = \kappa^2 \left[\tilde{s}^{M\alpha}{}_{\beta\gamma} + s^{D\alpha}{}_{\beta\gamma} - 2\lambda\kappa^2 A_{\lambda} A_{[\beta} s^{D\lambda\alpha}{}_{\gamma} + \frac{2}{2+\lambda\kappa^2 A^2}(\delta^{\alpha}_{[\beta}\tilde{s}^{M}_{\gamma]} - \lambda\kappa^2 A^{\alpha} A_{[\beta}\tilde{s}^{M}_{\gamma]}) \right], \tag{4.117}$$

which can be further simplified down to

$$T^{\alpha}{}_{\beta\gamma} = \kappa^2 \left[\tilde{s}^{M\alpha}{}_{\beta\gamma} + \rho^{\sigma}_{\beta}\rho^{\rho}_{\gamma} s^{D\alpha}{}_{\sigma\rho} + \frac{2}{2+\lambda\kappa^2 A^2}(\delta^{\alpha}_{[\beta}\tilde{s}^{M}_{\gamma]} - \lambda\kappa^2 A^{\alpha} A_{[\beta}\tilde{s}^{M}_{\gamma]}) \right], \tag{4.118}$$

with

$$\rho^{\alpha}_{\beta} \equiv \delta^{\alpha}_{\beta} + \lambda\kappa^2 A^{\alpha} A_{\beta}\,, \tag{4.119}$$

and we denote

$$\tilde{s}^{M\alpha}{}_{\beta\gamma} \equiv \lambda A_{[\beta}\tilde{F}_{\gamma]}{}^{\alpha}\,, \tag{4.120}$$

the torsion-free part of the (generalized) Maxwell spin tensor (therefore $\tilde{s}^{M}_{\beta} = -\frac{\lambda}{2}\tilde{F}_{\beta\gamma}A^{\gamma}$). A further simplification turns the Cartan equations into the final form

$$T^{\alpha}{}_{\beta\gamma} = \kappa^2 \left[\tilde{s}^{M\alpha}{}_{\beta\gamma} + s^{D\alpha}{}_{\beta\gamma} + 2\lambda\kappa^2 s^{D\alpha}{}_{[\beta}{}^{\rho} A_{\gamma]} A_{\rho} + \frac{2}{2+\lambda\kappa^2 A^2}(\delta^{\alpha}_{[\beta}\tilde{s}^{M}_{\gamma]} - \lambda\kappa^2 A^{\alpha} A_{[\beta}\tilde{s}^{M}_{\gamma]}) \right], \tag{4.121}$$

This expression for torsion as a function of the matter fields can then be replaced in the matter Lagrangian (4.107), i.e, in the bosonic sector,

$$\mathcal{L}_{\text{Max}} = \tilde{\mathcal{L}}_{\text{Max}} + \lambda\left(T^{\lambda\mu\nu}T^{\gamma}{}_{\mu\nu}A_{\gamma} + T^{\lambda\mu\nu}\tilde{F}_{\mu\nu}\right)A_{\lambda}, \tag{4.122}$$

and in the Dirac Lagrangian,

$$\mathcal{L}_{\rm Dirac} = \tilde{\mathcal{L}}_{\rm Dirac} + \frac{i\hbar}{4} K_{\alpha\beta\mu}\bar{\psi}\gamma^{[\mu}\gamma^{\alpha}\gamma^{\beta]}\psi, \tag{4.123}$$

and we recall that this can be expressed as $\mathcal{L}_{\rm Dirac} = \tilde{\mathcal{L}}_{\rm Dirac} + 3\breve{T}^{\lambda}\breve{s}_{\lambda}^{Dirac}$. Since bosons are also contributing to the torsion with the corresponding spin tensor, the axial torsion vector $\breve{T}^{\lambda}$ has now a new contribution (besides that of the Dirac axial spin vector). From Eq. (4.118), we obtain

$$\breve{T}^{\lambda} = \kappa^2 \left[-\frac{\breve{s}^{\lambda}}{2} + \frac{\lambda}{6}\epsilon^{\mu\beta\gamma\lambda}\left(2\kappa^2 s^{D}_{\rho[\mu\beta}A_{\gamma]}A^{\rho} + A_{[\mu}\tilde{F}_{\beta\gamma]} \right) \right], \tag{4.124}$$

where we omit the D symbol in the Dirac axial spin vector. Substituting in the Dirac Lagrangian above we get, after some algebra

$$\mathcal{L}_{\rm D} \;=\; \tilde{\mathcal{L}}_{\rm D} - \breve{s}_{\lambda}\breve{s}^{\lambda}\left(\frac{3\kappa^2}{2} + \lambda\kappa^4 A^2 \right) + \lambda\kappa^4 (A\cdot\breve{s})^2 + \frac{\lambda\kappa^2}{2}\epsilon^{\mu\beta\gamma\lambda}\breve{s}_{\lambda}A_{[\mu}\tilde{F}_{\beta\gamma]}. \tag{4.125}$$

The first term is Dirac's Lagrangian on a (pseudo) Riemann space-time, while the other terms come from the corrections of a RC geometry where torsion (given by the Cartan equations) is due to the spin tensors of fermionic spinors and electromagnetic fields. The first term inside the parenthesis corresponds to the well known spin-spin (axial-axial) contact interaction. Due to the presence of new fermionic-electromagnetic interactions induced by torsion, we see that the spin-spin contact interaction is now modulated at each point by the strength of the electromagnetic 4-potential (squared). The spin-spin effect is therefore affected locally by the electromagnetic potential at very high densities and fields, due to the κ^4 factor. The other two terms represent further (fermionic) spin-electromagnetic interactions. In the first of these, significant at very high densities and fields, the relative orientation (alignment) between the spin vector and the electromagnetic potential is relevant, which might suggest that this interaction could involve precession effects and possibly generate anisotropies in the spin distribution, for example via a macroscopic (averaged) alignment of the fermionic spin.

Assuming random fermionic spin distributions, we can compute the correction to the bosonic Lagrangian as

$$\begin{aligned}\mathcal{L}^{M}_{\rm corr} \;\approx\; & \lambda^2\kappa^2 A^{[\mu}\tilde{F}^{\nu]\lambda}\tilde{F}_{\mu\nu}A_{\lambda} + \frac{2\lambda\kappa^2\tilde{F}_{\mu\nu}}{2+\lambda\kappa^2A^2}A^{[\mu}\tilde{s}^{\nu]}(1-\lambda\kappa^2A^2) + \lambda^3\kappa^4 A^{[\mu}\tilde{F}^{\nu]\lambda}A_{[\mu}\tilde{F}_{\nu]\gamma}A_{\lambda}A^{\gamma} \\ & + \frac{4\lambda\kappa^4 A^{[\mu}\tilde{s}^{\nu]}A_{[\mu}\tilde{s}_{\nu]}}{(2+\lambda\kappa^2A^2)^2}(1-\lambda\kappa^2A^2(2-\lambda\kappa^2A^2)) - \frac{\lambda\kappa^4}{2}(A^2\breve{s}^2 - (A\cdot\breve{s})^2). \end{aligned} \tag{4.126}$$

The last term here depends on the spinors via Dirac axial vector $\breve{s}^{\lambda}$ and represents non-minimal boson-fermion interactions, while every other term in that expression corresponds to self-interactions[7]. Note that we have dropped the M from the trace vector of what we called the torsionless part of the bosonic spin tensor $\tilde{s}^{M\alpha}{}_{\beta\gamma} \equiv \lambda A_{[\beta}\tilde{F}_{\gamma]}{}^{\alpha}$. To

[7]The third and fourth terms in the Lagrangian above can be re-written as $\frac{\lambda^3}{2}\kappa^4 A^2\tilde{F}^{\nu}{}_{\lambda}\tilde{F}_{\nu\gamma}A^{\lambda}A^{\gamma}$ and $\frac{2\lambda\kappa^4(\tilde{s}^2A^2-(A\cdot\tilde{s})^2)}{(2+\lambda\kappa^2A^2)^2}(1-\lambda\kappa^2A^2(2-\lambda\kappa^2A^2))$, respectively

simplify things we will assume spatial homogeneity and isotropy, with $A = (\phi, 0, 0, 0)$, and $\tilde{s} = 0 = \tilde{F}$, which allows us to write the torsion tensor

$$T_{\alpha\beta\gamma} = \kappa^2 \left(s^D_{\alpha\beta\gamma} + 2\lambda\kappa^2 s^D_{\alpha[\beta}{}^{\rho} A_{\gamma]} A_\rho \right), \tag{4.127}$$

therefore we find

$$s^M_\mu = \tilde{s}^M_\mu + \lambda T_{(\alpha\beta)\mu} A^\alpha A^\beta = 0, \tag{4.128}$$

and

$$s^M_{\alpha\beta\gamma} = 2\lambda\kappa^2 A_{[\beta} s^{D\,\lambda}{}_{\gamma]\alpha} A_\lambda. \tag{4.129}$$

The total matter Lagrangian will give rise to extended Dirac and electromagnetic equations. To compute this we will now analyse the bosonic and fermionic field dynamics.

Electromagnetic sector

The electromagnetic field equations are those in (4.81). As can be seen in the expression for the Lagrangian in Eq. (4.79), or in the field equations (4.81), the terms quadratic in the contortion or, equivalently, in the spin density, resemble Proca-like terms. From this analogy, the coupling between the electromagnetic four-potential and the space-time torsion provides an effective mass for the photon $m_\gamma^2 \sim \lambda T^2$ in physical environments where the $U(1)$-breaking phase transition takes place. The terms linear in torsion, on the other hand, reveal new physical effects due to the coupling between electromagnetism and torsion, which in this framework become significant for spin densities much lower than the Cartan density. That is, way before the manifestation of torsion-induced metric effects, that will be considered in the cosmological applications in chapter 5, the torsion (spin) of fermions start interacting significantly with bosonic fields, affecting the generalized Maxwell dynamics. This is another motivation to consider physical effects of the full dynamics in astrophysical and cosmological environments with spin densities below the Cartan threshold, as in the core of neutron stars and in the early Universe.

Fermionic background torsion. Assuming the ansatz of a completely antisymmetric background torsion, as in the case where torsion comes from the background Dirac fermionic fields, we get the same form of the field equations but the torsion-induced current gets simplified

$$J^\nu = -\lambda \left[2K^\nu{}_{\lambda\mu} K^{\gamma\mu\lambda} A_\gamma - K^\nu{}_{\lambda\mu} \tilde{F}^{\lambda\mu} + 2\tilde{\nabla}_\mu (K^{\gamma\mu\nu} A_\gamma) \right]. \tag{4.130}$$

According to the minimal coupling between torsion and electromagnetic fields, as it is apparent from Eq. (4.78), only the antisymmetric part of the contortion tensor enters the electromagnetic sector in a RC space-time (at the Lagrangian level). However, for fermions both torsion and contortion are totally antisymmetric, so we have dropped out the brackets for antisymmetrization. In that case it is useful to express the Maxwell Lagrangian with torsion contributions, Eq. (4.79), as

$$\mathcal{L}_{\text{Max}} = \tilde{\mathcal{L}}_{\text{Max}} - \lambda \left[\frac{\kappa^4}{2} \left(\breve{s}^2 A^2 - (\breve{s} \cdot A)^2 \right) - \frac{\kappa^2}{2} f^\nu \breve{s}_\nu \right], \tag{4.131}$$

where we have introduced the (axial) vector

$$f^\rho \equiv \epsilon^{\lambda\mu\nu\rho} A_\lambda \tilde{F}_{\mu\nu}. \tag{4.132}$$

Under the assumption of the random spin distribution, from Eq. (4.130) and using the Cartan equations (4.104), we obtain the following (spin) torsion-induced four-current

$$J^{\nu} = -\kappa^4\lambda\left(\breve{s}^2 A^{\nu} - (\breve{s}\cdot A)\breve{s}^{\nu}\right), \tag{4.133}$$

which arises from the interaction between the fermionic axial vector field and the electromagnetic 4-potential.

Full approach: including the spin tensor from the bosonic sector. In this case, the Cartan equations are given by Eq. (4.118). We will consider for convenience the generalized current as in (4.83). Now, given the fact that the total matter Lagrangian can be written as in (4.107) where $\mathcal{L}_{\mathrm{D}}$ includes bosonic-fermionic interactions and is given by Eq. (4.125) and $\mathcal{L}_{\mathrm{M}}$ is given in Eq. (4.79), upon applying the variational principle with respect to the electromagnetic potential, we get a new generalized Maxwell equation in Eq. (4.81) given by

$$\tilde{\nabla}_{\mu}\tilde{F}^{\mu\nu} = \lambda^{-1}(j^{\nu} + J^{\nu} + \xi^{\nu})\ , \tag{4.134}$$

where

$$\xi^{\nu} = \lambda\Big(2\kappa^4\left[\breve{s}^{\lambda}(A\cdot\breve{s}) - \breve{s}_{\lambda}\breve{s}^{\lambda}A^{\nu}\right] + \frac{\lambda\kappa^2}{2}\left[\epsilon^{\nu\beta\gamma\lambda}\tilde{F}_{\beta\gamma}\breve{s}_{\lambda} - 2\epsilon^{\rho\mu\nu\lambda}\tilde{\nabla}_{\mu}(A_{\rho}\breve{s}_{\lambda})\right]\Big), \tag{4.135}$$

comes from the (effective) Dirac Lagrangian (4.125) as

$$\xi^{\nu} \equiv \frac{\partial\mathcal{L}_{\mathrm{D}}}{\partial A_{\nu}} - \tilde{\nabla}_{\mu}\left(\frac{\partial\mathcal{L}_{\mathrm{D}}}{\partial(\tilde{\nabla}_{\mu}A_{\nu})}\right).$$

Using now Eq. (4.118) we obtain a long expression for the torsion-induced current J^{ν} in (4.83) with non-linear terms. One can also use the effective Maxwell Lagrangian in Eq. (4.126) to obtain

$$J^{\nu} \equiv \frac{\partial\mathcal{L}_{\mathrm{M}}^{corr}}{\partial A_{\nu}} - \tilde{\nabla}_{\mu}\left(\frac{\partial\mathcal{L}_{\mathrm{M}}^{corr}}{\partial(\tilde{\nabla}_{\mu}A_{\nu})}\right),$$

as

$$\begin{aligned} J^{\nu} = {}& \lambda\kappa^2\Big[\tilde{F}_{\alpha\beta}\left(\lambda A^{[\alpha}\tilde{F}^{\beta]\nu} + 2A^{[\alpha}\tilde{s}^{\beta]}A^{\nu}X(A)\right) + 2\tilde{F}^{\nu}{}_{\beta}\left(F^{\beta}{}_{\lambda}A^{\lambda} + 2s^{\beta}Y(A)\right) \\ & + \lambda^2\kappa^2\left(A^{\nu}\tilde{F}^{\alpha\lambda}A_{\lambda} + A^2\tilde{F}^{\alpha\nu}\right)\tilde{F}_{\alpha\gamma}A^{\gamma} + (A^{\nu}\tilde{s}^2 - 2\tilde{s}^{\nu}(A\cdot\tilde{s}))Z(A) \\ & + A^{\nu}(A^2\tilde{s}^2 - (A\cdot\tilde{s})^2)W(A) - \kappa^2(A^{\nu}\breve{s}^2 - \breve{s}^{\nu}(A\cdot\breve{s}))\Big] - \tilde{\nabla}_{\mu}\left(\frac{\partial\mathcal{L}_{\mathrm{M}}^{corr}}{\partial(\tilde{\nabla}_{\mu}A_{\nu})}\right), \end{aligned} \tag{4.136}$$

where the last term is computed as

$$\begin{aligned} \frac{\partial\mathcal{L}_{\mathrm{M}}^{corr}}{\partial(\tilde{\nabla}_{\mu}A_{\nu})} = {}& 2\lambda^2\kappa^2\Big(A^{[\mu}\tilde{F}^{\nu]\lambda}A_{\lambda} + \tilde{F}^{\alpha[\mu}A^{\nu]}A_{\alpha} - \tilde{F}^{[\mu}{}_{\beta}A^{\nu]}A^{\beta}\Big) + 4\lambda\kappa^2A^{[\mu}\tilde{s}^{\nu]}\frac{1-\lambda\kappa^2A^2}{2+\lambda\kappa^2A^2} \\ & + \lambda^3\kappa^4A^2\tilde{F}^{[\mu}{}_{\gamma}A^{\nu]}A^{\gamma}\,, \end{aligned} \tag{4.137}$$

and we have introduced the definitions

$$\begin{aligned} X(A) &\equiv -\frac{6\lambda\kappa^2}{(2+\lambda\kappa^2A^2)^2}\,, \\ Y(A) &\equiv -\frac{1-\lambda\kappa^2A^2}{2+\lambda\kappa^2A^2}\,, \\ Z(A) &\equiv \frac{2\kappa^2(1-(2-\lambda\kappa^2A^2))}{2+\lambda\kappa^2A^2}\,, \\ W(A) &\equiv \Big[4\lambda\kappa^2\Big((2+\lambda\kappa^2A^2)(\lambda\kappa^2A^2-1)\,, \\ &\quad -\big(1-\lambda\kappa^2A^2(2-\lambda\kappa^2A^2)\big)\Big)\Big]/(2+\lambda\kappa^2A^2)^3\,. \end{aligned}$$

These complicated expressions can be interpreted as non-linear electrodynamics with non-minimal couplings between fermionic matter (spinors) and electromagnetic fields induced by the RC space-time geometry. These equations are simplified in two cases: (i) matter with a random distribution of fermionic spins, where we neglect all quantities linear in the Dirac spin, leaving only the quadratic ones which do not vanish after macroscopic averaging and (ii) the case of homogeneity and isotropy, with $A=(\phi,0,0,0)$, and $\tilde{s}=0=\tilde{F}$. In the first case we obtain

$$\xi^\nu \approx 2\lambda\kappa^4\left[\breve{s}^\nu(A\cdot\breve{s})-\breve{s}_\lambda\breve{s}^\lambda A^\nu\right], \tag{4.138}$$

and in the second case, the simplified J^ν is simply

$$J^\nu = -\lambda\kappa^4\left[A^\nu\breve{s}^2-\breve{s}^\nu(A\cdot\breve{s})\right], \tag{4.139}$$

where the non-linearities (in the electromagnetic quantities) disappear and the equation above corresponds exactly to what we had in the first approach in Eq. (4.133). If we had varied the action with respect to the bosonic vector field and only replaced the torsion components (using Cartan equations) after the variational principle, i.e, in the field equations, we would have arrived to a simpler generalized Maxwell equation. In that case we would have obtained (4.134) whith $\xi^\nu=0$.

Fermions

Fermionic background torsion. Let us consider first the case in which the matter fields are fermionic spinors. The variation of the Dirac action in a RC space-time (given by the Lagrangian density in Eq.(4.6)) with respect to fermionic fields yields the Fock-Ivanenko-Heisenberg-Hehl-Datta equation

$$i\hbar\gamma^\mu\tilde{D}_\mu\psi-m\psi=\frac{3\kappa^2\hbar^2}{8}(\bar{\psi}\gamma^\nu\gamma^5\psi)\gamma_\nu\gamma^5\psi\,, \tag{4.140}$$

where torsion was substituted by its source, the spin density of Dirac fermions, via the Cartan equations. Now we introduce electromagnetic fields minimally coupled to torsion, but without backreacting on it. In this case, the variation of the action (3.58) with the new Lagrangian $\mathcal{L}_m=\mathcal{L}_\Psi+\mathcal{L}_{\rm Max}$ includes non-minimal couplings of fermions with the four-potential, in the generalized Hehl-Datta equation of EC-Dirac theory. For charged fermions, the new Hehl-Datta equation reads

$$i\hbar\gamma^\mu\tilde{D}_\mu\psi + \left(q\gamma^\mu A_\mu - \frac{\kappa^2\lambda\hbar}{4}f^\rho\gamma_\rho\gamma^5\right)\psi - m\psi = \left(\frac{\kappa^4\lambda\hbar^2}{2}A^2 + \frac{3\kappa^2\hbar^2}{4}\right)(\bar{\psi}\gamma^\nu\gamma^5\psi)\gamma_\nu\gamma^5\psi - \frac{\kappa^4\lambda\hbar^2}{2}(\bar{\psi}\gamma^\beta\gamma^5\psi)\gamma_\lambda\gamma^5\psi A_\beta A^\lambda. \quad (4.141)$$

The Hehl-Datta term, $\sim \kappa^2\hbar^2(\bar{\psi}\gamma^\nu\gamma^5\psi)\gamma_\nu\gamma^5\psi$, which is cubic in the spinors, is already present in the usual EC-Dirac theory. This term represents a spin-spin contact interaction inside fermionic matter. For charged anti-fermions, after performing the charge conjugation operation ($\psi \to -i\gamma^2\psi^* \equiv \psi^{ch}$) we have instead

$$i\hbar\gamma^\mu\tilde{D}_\mu\psi^{ch} - \left(q\gamma^\mu A_\mu + \frac{\kappa^2\lambda\hbar}{4}f^\rho\gamma_\rho\gamma^5\right)\psi^{ch} - m\psi^{ch} = -\left(\frac{\kappa^4\lambda\hbar^2}{2}A^2 + \frac{3\kappa^2\hbar^2}{4}\right)(\bar{\psi^{ch}}\gamma^\nu\gamma^5\psi^{ch})\gamma_\nu\gamma^5\psi^{ch} + \frac{\kappa^4\lambda\hbar^2}{2}(\bar{\psi^{ch}}\gamma^\beta\gamma^5\psi^{ch})\gamma_\lambda\gamma^5\psi^{ch}A_\beta A^\lambda. \quad (4.142)$$

All cubic terms, similarly to the term having the fermionic charge, have flipped sign after the C-transformation relative to the mass term. This behaviour is connected to the fact that the corresponding effective Lagrangian terms behave in an opposite manner under a C-transformation in relation to the rest of the terms in the Lagrangian [97].

It has been shown that the Hehl-Datta term, which corresponds to an effective axial-axial spinor interaction of repulsive nature, can provide important physical effects in the particle domain [94, 93, 96, 97, 98, 99, 146], including a valid mechanism for generating a residual matter/anti-matter asymmetry in the context of baryogenesis in cosmology, and has been shown to posses other applications, such as an effective cosmological constant [98] and non-singular configurations [99]. Such a term can be derived from an effective interaction Lagrangian of the form $L^{\rm int}_{\rm Hehl-Datta} \sim \kappa^2\breve{s}^\mu\breve{s}_\mu$. Analogously, the new cubic terms we have derived also come from similar effective Lagrangian terms quadratic in Dirac's axial (spin) vector $L_{\rm eff} \sim \kappa^4\lambda\breve{s}^2A^2$, and are induced from the coupling between torsion and the electromagnetic potential. These terms correspond to the quadratic ones appearing in Eq.(4.79). Therefore, the axial-axial or spin-spin contact interaction effect is potentially enhanced (at very high densities) by the presence of the electromagnetic four-potential. Moreover, in general the four-potential propagates, therefore a richer dynamics is induced in the effective spin-spin interaction. This scenario is of course compatible with the fact that we have broken the local (gauge) $U(1)$ invariance under a phase transition above a certain critical value of the spin density. Accordingly, the vector potential that appears explicitly in the dynamical equations can be thought as representing physical degrees of freedom[8].

Full approach: including the spin tensor from the bosonic sector. Previously, using Eq. (4.118) we arrived at the fermionic Lagrangian given in (4.125). If we

[8]In some sense, there are good empirical motivations to consider the electromagnetic potential to represent physical degrees of freedom which come from some interpretations given to the Aharonov-Bohm effect, namely the observed change in the phase of an electron wave function in the presence of negligible electromagnetic fields, due to the interaction between the fermion and the electromagnetic four-potential

consider the total matter Lagrangian $L_m = L_D + L_M + j^\mu A_\mu$, including the contribution from the bosonic side (4.126), we obtain the following extended Dirac (cubic) equation

$$\begin{aligned} i\hbar\gamma^\mu \tilde{D}_\mu \psi + (q\gamma^\mu A_\mu - m)\,\psi \;&=\; f(A)(\bar{\psi}\gamma^\nu\gamma^5\psi)\gamma_\nu\gamma^5\psi + \alpha^{\beta\lambda}(\bar{\psi}\gamma_\beta\gamma^5\psi)\gamma_\lambda\gamma^5\psi \\ &\quad +\beta^\alpha(A,\tilde{F})\gamma_\alpha\gamma^5\psi, \end{aligned} \tag{4.143}$$

where

$$\begin{aligned} f(A) \;&\equiv\; \frac{3\kappa^2\hbar^2}{4} + \frac{\lambda 3\kappa^4\hbar^2}{2}A^2 \\ \alpha^{\sigma\varepsilon} \;&\equiv\; -\lambda\hbar^2\kappa^2\Big(\frac{\kappa^2}{2}A^\sigma A^\varepsilon + \frac{\kappa^2}{2}\Theta^{\lambda}{}_{\mu\nu}{}^{\varepsilon}(\epsilon^{\gamma\mu\nu\sigma} \\ &\quad +2\lambda\kappa^2\epsilon^{\gamma[\mu|\rho\sigma}A^{|\nu]}A_\rho)A_\gamma A_\lambda\Big) \\ \beta^\alpha \;&\equiv\; -\lambda\Big(A_\lambda(2A_\gamma\Theta^{(\lambda|}{}_{\mu\nu}{}^{\alpha}T_M^{\gamma)\mu\nu} + \tilde{F}_{\mu\nu}\Theta^{\lambda\mu\nu\alpha}) + \frac{\kappa^2\hbar}{2}\epsilon^{\mu\beta\gamma\alpha}A_{[\mu}\tilde{F}_{\beta\gamma]}\Big)\,, \end{aligned} \tag{4.144}$$

and we have

$$\Theta^{\lambda\mu\nu\alpha} \equiv \frac{\kappa^2\hbar}{4}\left(\epsilon^{\lambda\mu\nu\alpha} + 2\lambda\kappa^2\epsilon^{\lambda[\mu|\rho\alpha}A^{\nu]}A_\rho\right), \tag{4.145}$$

while $T_M^{\gamma\mu\nu}$ is the purely bosonic part of the torsion tensor in (4.121). This equation can be considered in the approximation of space-time flatness and also in the non-relativistic limit. One can then solve the energy levels problem which is expected to reveal a kind of hyperfine structure that could be used to probe for the existence of torsion with high resolution spectrography. In fact, the correction terms in (4.125) can be interpreted as effective interaction potentials

$$\mathcal{L}_{\rm D} = \tilde{\mathcal{L}}_{\rm D} + U(\varphi,\chi,\zeta)\;, \tag{4.146}$$

with $\varphi \equiv \breve{s}^2$, $\chi \equiv A^2$, $\zeta \equiv A\cdot\breve{s}$ and we neglected the term linear in $\breve{s}$, for simplicity. To close this section, let us mention that for anti-particles we have:

$$\begin{aligned} i\hbar\gamma^\mu \tilde{D}_\mu \psi^{ch} - (q\gamma^\mu A_\mu + m)\,\psi^{ch} \;&=\; -f(A)(\bar{\psi^{ch}}\gamma^\nu\gamma^5\psi^{ch})\gamma_\nu\gamma^5\psi^{ch} \\ &\quad -\alpha^{\beta\lambda}(\bar{\psi^{ch}}\gamma_\beta\gamma^5\psi^{ch})\gamma_\lambda\gamma^5\psi^{ch} + \beta^\alpha(A,\tilde{F})\gamma_\alpha\gamma^5\psi^{ch}. \end{aligned} \tag{4.147}$$

which is not exactly the same dynamics, suggesting possible applications for asymmetries and baryogenesis.

We obtained this Dirac equation by varying the effective matter Lagrangian after the torsion components have been replaced as functions of the matter fields, using Cartan equations. Now, if instead, we only replace the torsion for the matter fields after varying the action with respect to spinors, we get a simpler equation, yielding essentially the same physics. To see this we start from the general expression of the Dirac equation minimally coupled to the RC geometry, which we write again here

$$i\hbar\gamma^\mu \tilde{D}_\mu \psi + (q\gamma^\mu A_\mu - m)\,\psi = -\frac{3\hbar}{2}\breve{T}^\lambda\gamma_\lambda\gamma^5\psi. \tag{4.148}$$

We now simply substitute the axial torsion vector in (4.124), derived from the full Cartan equations (4.121). After some algebra, we obtain the following extended Dirac (cubic) equation

$$\begin{aligned} i\hbar\gamma^{\mu}\tilde{D}_{\mu}\psi + (q\gamma^{\mu}A_{\mu} - m)\,\psi &= f(A)(\bar{\psi}\gamma^{\nu}\gamma^{5}\psi)\gamma_{\nu}\gamma^{5}\psi + \alpha^{\lambda}_{\ \alpha}(A)(\bar{\psi}\gamma^{\alpha}\gamma^{5}\psi)\gamma_{\lambda}\gamma^{5}\psi \\ &\quad + \beta^{\lambda}(A,\tilde{F})\gamma_{\lambda}\gamma^{5}\psi\ , \end{aligned} \tag{4.149}$$

where we have defined

$$\begin{aligned} f(A) &\equiv \frac{3\kappa^2\hbar^2}{8} + \frac{\lambda\kappa^4\hbar^2}{4}A^2, \\ \alpha^{\sigma\varepsilon}(A) &\equiv -\lambda\kappa^4\hbar^2 A^{\sigma}A^{\varepsilon}, \\ \beta^{\lambda}(A,\tilde{F}) &\equiv -\frac{\lambda\kappa^2\hbar}{2}\epsilon^{\lambda\alpha\beta\gamma}A_{[\alpha}\tilde{F}_{\beta\gamma]}\ . \end{aligned} \tag{4.150}$$

Due to the presence of the cubic terms, this equation also changes under the actions of C (charge conjugation) transformations, as in the previous case.

4.3.2 Fermions non-minimally coupled to torsion in ECSK and in ECSK+Holst: Parity and C symmetry breaking

Spinors in Einstein-Cartan with coupligs to vector and axial vector fermionic currents. We now briefly consider non-minimal couplings as those explored in section 4.1, within the ECSK theory. Using torsion irreducible components in Eq. (3.19), we get for the Ricci scalar

$$R \sim \tilde{R} - 4\tilde{\nabla}_{\alpha}T^{\alpha} - \frac{1}{3}T^{\lambda}T_{\lambda} + \frac{1}{24}\breve{T}^{\lambda}\breve{T}_{\lambda} + \frac{1}{2}\bar{T}_{\mu\nu\rho}\bar{T}^{\mu\nu\rho}. \tag{4.151}$$

Inserting this in the action

$$S_{\rm EC} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R(\Gamma) + \int d^4x\sqrt{-g}\,\mathcal{L}_{fermions}\,, \tag{4.152}$$

with

$$\mathcal{L}_{\rm fermions} = \tilde{\mathcal{L}}_{\rm Dirac} + \alpha_1 T\cdot j + \alpha_2\breve{T}\cdot a, \tag{4.153}$$

the Cartan equations become

$$T^{\mu} \sim \kappa^2\alpha_1 j^{\mu} \quad \breve{T}^{\mu} \sim \kappa^2\alpha_2 a^{\mu}. \tag{4.154}$$

Re-inserting these expressions in the previous Lagrangian we obtain effective vector-vector contact interactions besides the usual well-known axial-axial (spin-sin) interaction (Hehl-Datta term)

$$\mathcal{L}_{\rm fermions} \sim \tilde{\mathcal{L}}_{\rm Dirac} + \frac{\kappa^2}{3}(\alpha_1)^2 j\cdot j - \frac{\kappa^2}{24}(\alpha_2)^2 a\cdot a. \tag{4.155}$$

The corresponding Dirac equation can be written as

$$i\hbar\gamma^{\mu}\tilde{D}_{\mu}\psi - m\psi = \frac{\kappa^2\alpha_2^2}{12}(\bar{\psi}\gamma^{\lambda}\gamma^{5}\psi)\gamma_{\lambda}\gamma^{5}\psi - \kappa^2\frac{2\alpha_1}{3}(\bar{\psi}\gamma^{\lambda}\psi)\gamma_{\lambda}\psi\,. \tag{4.156}$$

As in the usual Dirac-Hehl-Datta equation, under charge conjugation operation $\psi \to \psi^{ch}$ one obtains different dynamics for the ψ^{ch} representing fermions. If we use instead the Lagrangian in (4.51), then the Cartan eqs are

$$T^\mu \sim \kappa^2(\zeta_1 j^\mu + \zeta_2 a^\mu) \quad \breve{T}^\mu \sim \kappa^2(\theta_1 j^\mu + \theta_2 a^\mu), \tag{4.157}$$

where ζ_i, θ_i $(i = 1, 2)$ are constants, and the resulting Dirac equation, after substitution in (4.52), includes parity breaking and C breaking cubic terms.

Spinors in Einstein-Cartan plus Holst term

In the case of Einstein-Cartan theory plus the so-called Holst term, the action is

$$S_{\rm EC} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R(\Gamma) + \frac{1}{2\gamma\kappa^2}\int d^4x\sqrt{-g}\epsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} + \int d^4x\sqrt{-g}\,\mathcal{L}_{fermions}\,, \tag{4.158}$$

where γ is the Barbero-Immirzi parameter and the Holst term $\epsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ is a parity breaking term and can be expressed as

$$\epsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} \sim -\tilde{\nabla}_\alpha \breve{T}^\alpha - \frac{1}{3}\breve{T}^\lambda T_\lambda + \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\bar{T}^\lambda{}_{\alpha\beta}\bar{T}_{\lambda\mu\nu}. \tag{4.159}$$

The generalized Cartan equations, using (4.49), become

$$T^\mu \sim \kappa^2\frac{3\gamma}{1+\gamma^2}(\alpha_1\gamma j^\mu + a^\mu) \quad \breve{T}^\mu \sim \kappa^2\frac{3\gamma}{1+\gamma^2}(\alpha_1 j^\mu - \alpha_2\gamma a^\mu). \tag{4.160}$$

One obtains generalized Dirac equation and Lagrangian with vector-vector, axial-axial and parity breaking vector-axial (contact) self-interactions.

4.3.3 Discussion and summary

In this section we have studied the ECDM model with $U(1)$ symmetry breaking and discussed its physical relevance. We also considered non-minimal couplings to torsion in ECSK and in the generalization to ECSK theory to include the Holst term.

ECDM theory with U(1) symmetry breaking. Regarding the first case, we considered a Dirac field and an electromagnetic field minimally coupled to torsion, which induces rich gravitational dynamics and non-linear fermionic and bosonic dynamical equations, including non-minimal and self-interactions. We considered two regimes: i) one in which torsion is sourced by fermions and ii) the full case with the contribution from both fermions and bosons to the total spin tensor entering in Cartan's equations.

In general, the effects for the space-time metric only become important at very high (spin) densities, as in the usual EC theory. For example, in the first approach with torsion generated by fermionic spin, torsion (or spin) contributions to the metric field equations scale with $\kappa^4\breve{s}^2$ for the pure EC correction, while the model with the $U(1)$ symmetry breaking studied here introduces terms both linear and quadratic with torsion, that scale as $\kappa^4\lambda\breve{s}\tilde{F}A$ and $\kappa^6\lambda\breve{s}^2A^2$, respectively. This has to be compared with the

$\kappa^2\tilde{T}_{\mu\nu}$ contribution from the usual stress-energy tensor in GR. Thus, for very strong electromagnetic fields/potential, the term linear in the spin density could become important (in polarized matter) at densities slightly (but not significantly) below Cartan's typical density. On the other hand, the effects of torsion in the electromagnetic and fermionic sectors require a more careful analysis.

Let us discuss the electromagnetic dynamics. The generalized Maxwell theory include terms linear in torsion (also in the spin density) that become significant at densities much lower than Cartan's density, which should be taken into account in strong gravity regimes such as in the interior of astrophysical compact objects (neutron stars, magnetars, quark stars) and in the early Universe. These terms are non-negligible for polarized matter, i.e., for non-random spin distributions and, consequently, the presence of strong magnetic fields provide the adequate physical conditions for the study of the phenomenology associated with these corrections. For approximately random spin distributions, i.e., for unpolarized matter, only the quadratic terms (in torsion or in the spin density) are non-vanishing with its phenomenology being related to much higher densities. In any case, the presence of strong electromagnetic fields (potential) tend to enhance such effects.

When the $U(1)$ symmetry is broken the corresponding (Noether) charge current is not conserved. Although the fermionic charge density and number density of the fermions is not conserved locally in this model, the equations suggest interpreting the terms of geometric origin as effective charge currents that compensate and balance the non-conservation of the usual charge current. In other words, by following this interpretation the space-time geometrodynamics would gain physical features, such as effective mass, spin or charge currents, when it couples to matter. When these terms are considered, then a new conserved quantity is clear. Another way to see this is to deduce the phenomenology associated to such an interpretation and search for possible observational tests of the predictions. In this context, this type of models where the stress-energy tensor or the charge current is not conserved in the usual sense, predict the creation of particles from the energy available in the space-time geometrodynamics, in strong gravity environments.

When the contribution from the bosonic sector to the spin tensor is taken into account, then the bosonic field propagates on a RC spacetime and backreacts on its geometry. Since torsion in EC theory is given by an algebraic expression of the matter fields, one then gets non-minimal couplings between these but also self-interactions. Therefore, we obtain effectively non-linear dynamical equations for the bosonic fields. In fact, just as in the case of fermions where a linear Dirac field in RC space-time of the EC theory is equivalent to a non-linear spinor in GR, also here the linear electromagnetic Lagrangian in the RC space-time leads to an effective non-linear electrodynamics in GR. Non-linear dynamics in the matter fields can emerge naturally from the (minimal) couplings of these fields with the extended space-time geometries of gauge theories of gravity.

In the case of fermionic fields in EC theory, torsion effects can also become significant in environments where the density is lower than Cartan's density. This is not so commonly mentioned in the literature, on the contrary, much emphasis is put on the fact that in EC theory the effects of torsion in Einstein's equations, i.e., for the metric, are only significant at extremely high densities such as those found in the very early Universe or inside black holes. Since the Cartan equations imply $K \sim \kappa^2 \check{s}$, after its substitution in the Dirac equation $i\hbar\gamma^\lambda D_\lambda\psi - m\psi = 0$, one obtains the (cubic) Hehl-Datta equation where the torsion-induced term will become significant at (spin) densities comparable to any strong-gravity regime where GR effects become important.

Let us stress that the Hehl-Datta term, which is related to an effective axial-axial (spin-spin) repulsive interaction, has been studied in connection to different physical mechanisms important for particle physics and cosmology, such as non-singular black holes, matter/anti-matter asymmetry and energy-levels, etc. Analogously, in our $U(1)$

symmetry-breaking model similar cubic terms are present that are quadratic in the electromagnetic four-potential. In this case, these torsion-induced corrections scale with κ^4, which means that the corresponding physical effects (on the energy levels, generalized effective Feynmann diagrams, etc) will only become relevant at extremely high densities (Cartan's density or above, but still lower than Planck density). In this model, the minimal coupling between the electromagnetic potential and torsion induce, at the dynamical equation level, a non-minimal coupling between fermions and electromagnetic potential/fields, in the generalized Dirac equation. Formally, this follows after the substitution of torsion by its corresponding spin density source via Cartan's equations. The new terms are both linear and cubic in the spinors. The former introduces effects that will become relevant around the same densities as for the original Hehl-Datta term. These considerations motivate further study on the full EC-Dirac-Maxwell dynamics inside astrophysical compact objects.

In our view, there are good motivations to consider gravitational models where non-Riemannian geometries, fermionic spin densities, and symmetry breaking phase transitions become important, which can be tested with astrophysical, cosmological and gravitational wave observations.

Non-minimal couplings in ECSK and ECSK+Holst. In these cases one obtains generalized Dirac equation and Lagrangian with vector-vector, axial-axial and parity breaking vector-axial (contact) self-interactions. These might be relevant inside compact objects like neutron stars, quark stars, strange (quark) stars and also in the early Universe. The Holst term is the simplest parity breaking extension to the Einstein-Cartan gravitational action. If the coupling constants are taken to be dynamical scalar fields, then this scenario leads naturally to the idea of parity breaking phase transitions for matter under extreme conditions, induced by the torsion-fermion currents couplings. We also see that the Einstein-Cartan plus Holst with $T \cdot j$ and $\breve{T} \cdot a$ couplings can be made equivalent to the usual Einstein-Cartan theory with $T \cdot j$ and $\breve{T} \cdot a$ plus (parity breaking) $T \cdot a$ and $\breve{T} \cdot j$ couplings.

Chapter 5

Cosmological applications

The Einstein-Cartan theory is an extension of the standard formulation of General Relativity characterized by a non-vanishing torsion. The latter is sourced by the matter fields via the spin tensor, and its effects are expected to be important at very high spin densities. In this chapter, we analyse in detail the cosmology of the Einstein-Cartan theory with Dirac and Maxwell fields minimally coupled to the spacetime torsion, in section 5.1. We also include the implementation of the cosmological principle, that is, the assumption of homogeneity and isotropy in the spatial distribution of matter in the Universe, within the context of Einstein-Cartan theory including minimal couplings of both Dirac and Maxwell fields to torsion.

The minimal couplings of the matter fields to torsion in the ECSK with fermions and bosons breaks the U(1) gauge symmetry, which is suggested by the possibility of a torsion-induced phase transition in the early Universe. The resulting Dirac-like and Maxwell-like equations are non-linear with self-interactions as well as having fermion-boson non-minimal couplings. We discuss several cosmological aspects of this theory under the assumption of randomly oriented spin densities (unpolarized matter), including bounces, acceleration phases and matter-antimatter asymmetry in the torsion *era*, as well as late-time effects such as the generation of an effective cosmological constant, dark energy, and future bounces within cyclic solutions. In the last part of section 5.1 we take a similar but different approach, since we impose the cosmological principle from the onset to the geometrical degrees of freedom (metric and torsion functions), which constrains the torsion components and the corresponding correction terms in the Friedmann-like equations and in the resulting fermionic and bosonic (non-linear) dynamics. We derive the corresponding cosmological dynamics for the geometrical and matter degrees of freedom and discuss the validity of this approach. The section 5.1 is inspired by the works in [5, 6].

5.1 The Einstein-Cartan-Dirac-Maxwell cosmology

The standard model of particles and interactions is an extremely successful theoretical construction, being able to describe the phenomena that we observe with current detectors in particle accelerator collisions and in cosmic rays. It rests deeply on i) (quantum) gauge field theories, which reveal a fundamental role of symmetry principles in the physics of interactions, and ii) on the rigid four-dimensional flat (Minkowski) spacetime background of special relativity, and as such it does not include gravity. The physics of particles and interactions of the early Universe is extrapolated from the success of this paradigm to describing the phenomena up to very high densities and temperatures at the electroweak scale. In the very early Universe one should incorporate strong-field gravitational effects, which requires new ideas in order to unveil the nature of the gravitational interaction on such scales.

As analysed in chapter 3, both the amazing successes of symmetry principles in the physics of interactions and the geometrical methods in gravity can be consistently combined by extending the gauge principle to gravity. It is reasonable to assume that classical gauge theories of gravity such as the metric-affine theories or PGTG are effective, low-energy limits of a more fundamental quantum gravity theory. The simplest PGTG is the Einstein-Cartan-Sciama-Kibble (ECSK) theory [19, 20, 21], which predicts torsion effects at very high energy densities via an algebraic relation between the spin density of matter fields and spacetime torsion. As seen in the previous chapters, the latter affects the Einstein-like equations for the metric but also the dynamics of fermions and bosons coupled to gravity. Although the effects upon the metric are expected to be relevant only at extreme densities, such as those found in the early universe or inside black holes, the effects on the matter fields can be important in the deep interior of compact objects such as magnetars or hypothetical quark stars. In cosmology, these effects are relevant for the physics around the Grand Unification phase transition scale and beyond, and one speaks of a torsion *era*, where the corresponding energy density is expected to scale with $\sim a^{-6}$. Indeed, theories with torsion in cosmological scenarios have been thoroughly studied in the literature for decades, with a large pool of applications [64, 66, 68, 71, 87, 88, 89, 90, 91, 92], as well as for their $f(T)$ extensions [138, 139, 140, 147, 148, 149, 150, 151, 152].

In the previous chapter (section 4.3) we considered the ECSK theory with fermionic (Dirac) and bosonic (Maxwell) fields coupled to the RC geometry. The resulting Einstein-Cartan-Dirac-Maxwell (ECDM) model contains new non-linear generalized Dirac-Hehl-Datta and electromagnetic equations with non-minimal interactions between fermionic and bosonic fields. While the coupling to Dirac fields has been considered previously in the literature, for instance within particle physics [94, 93, 95, 96, 98, 99], the minimal coupling of Maxwell fields to torsion breaks the electromagnetic $U(1)$ gauge symmetry, which in the cosmological context can be understood as a valid physical mechanism to generate a phase transitions during the torsion *era* [4].

Let us mention several cosmological, astrophysical and particle physics applications that can be worked out from the theory considered in this work. In Cosmology one expects the possibility of non-singular models as in the usual EC model, and new physics during the torsion-dominated era. One should also expect the production of gravitational waves from the transitions between primordial phases: from the $U(1)$-broken phase to the $U(1)$-restored phase, and from the usual torsion-dominated phase of EC to the radiation phase. These transitions can contribute to a stochastic gravitational wave background of cosmological origin, with possible imprints from the physics beyond the standard model.

The main aim of this section is to derive the cosmological equations governing the geometry and the matter fields within ECDM theory, and to study thoroughly their consequences for cosmological bounces, acceleration/desacceleration phases, matter-antimatter asymmetry, and late-time effects including the generation of an effective cosmological constant and the existence of cyclic cosmologies. We will also consider the implications to torsion from the assumption of the cosmological principle, in the presence of fermions and bosons minimally coupled to the RC spacetime of the ECSK model. After considering the generalized Lagrangian and the torsion as function of the matter fields, the modified Friedman equations and related phenomenology of interest for different cosmological regimes is studied. We also address the dynamics of bosonic (Maxwell) and fermionic (Dirac) fields in the cosmological framework.

5.1.1 Matter Lagrangian and torsion from fermionic and bosonic fields

As in chapter 4 let us consider the minimal coupling between torsion and matter, the latter represented by classical bosonic (four-vector) and fermionic (four-spinor) fields. This can be directly implemented at the level of the matter Lagrangian in (4.107), with the previously introduced Dirac Lagrangian with minimal coupling to the geometry of RC spacetime

$$\mathcal{L}_{\rm D} = \tilde{\mathcal{L}}_{\rm D} + K_{\alpha\beta\mu} s_D^{\mu\alpha\beta} \ , \tag{5.1}$$

and the Maxwell Lagrangian in a RC spacetime $\mathcal{L}_{\rm M} = \frac{\lambda}{4} F_{\mu\nu} F^{\mu\nu}$. For this matter Lagrangian, we saw that the Cartan equations (3.59) yield torsion as a function of the fermionic and bosonic fields according to (4.118). A particular case of this expression is the one of fermionic torsion

$$T^\alpha{}_{\beta\gamma} = \kappa^2 s^{D\alpha}{}_{\beta\gamma}, \qquad s^D{}_{\alpha\beta\gamma} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\lambda} \breve{s}^\lambda \ , \tag{5.2}$$

that is, torsion being exclusively the result of fermionic spin, neglecting the contribution from bosonic fields to the spin tensor. Under this condition, we simply have $T^\alpha{}_{\beta\gamma} = K^\alpha{}_{\beta\gamma}$. We will consider this simplified regime later in some applications.

As we saw, in the Dirac Lagrangian (4.105) only the (completely) antisymmetric part couples minimally to torsion giving $\mathcal{L}_{\rm Dirac} = \tilde{\mathcal{L}}_{\rm Dirac} + 3\breve{T}^\lambda \breve{s}^D_\lambda$, where the axial vector part of torsion, in the full regime is given by (4.124), so that the Dirac Lagrangian becomes as in (4.125). The first term of Eq.(4.125) is Dirac's Lagrangian on a (pseudo) Riemann spacetime, while the other terms come from the corrections of a RC geometry, including spin-spin self-interactions and non-minimal couplings with the bosonic fields. As for the generalized Maxwell Lagrangian, in the regime of random fermionic spin distributions (zero average, macroscopic spin), where we retain only the terms quadratic with the Dirac spin quantities, we obtained the equation (4.126). The first four terms in that equation correspond to self-interactions while the last one depends on the spinors via the Dirac axial vector $\breve{s}^\lambda$, and represents non-minimal boson-fermion interactions.

Let us note that in the simplified regime of fermionic torsion the Dirac Lagrangian boils down to (4.105) and the electromagnetic one to (4.131), where the term linear in the spin pseudo-vector can be neglected under the assumption of random spin distribution.

5.1.2 Gravitational field equations

We now investigate the effective Einstein equations in (3.62). The right-hand side yields quadratic corrections in torsion, $U \sim \kappa^{-2} T^2$, or in the spin variables, $U \sim \kappa^2 s^2$, via Cartan's equations. On the other hand, the dynamical energy-momentum tensor, $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$, is computed as usual from the matter Lagrangian, which yields the explicit result

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} - 4 j_{(\mu} A_{\nu)} + j^\lambda A_\lambda g_{\mu\nu} + \Pi^{M{\rm int}}_{\mu\nu} + \Xi^{D{\rm int}}_{\mu\nu} \ , \tag{5.3}$$

where $\tilde{T}_{\mu\nu} = \tilde{T}^{\rm Dirac}_{\mu\nu} + \tilde{T}^{\rm Max}_{\mu\nu}$, is the energy-momentum tensor for the matter fields in a Riemannian spacetime. In the above expression, the term $\Pi^{M{\rm int}}_{\mu\nu} = -\dfrac{2}{\sqrt{-g}} \dfrac{\partial \mathcal{L}^{\rm M}_{\rm corr}}{\partial g^{\mu\nu}}$ arises from the (second term in the) bosonic Lagrangian (4.79) and includes non-minimal boson-fermion interactions (induced by torsion) and also bosonic self-interactions. Similarly, $\Xi^{D\text{int}}_{\mu\nu}$ comes from the Dirac Lagrangian, i.e, $\Xi^{D\text{int}}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial \mathcal{L}^{\mathrm{D}}_{\mathrm{corr}}}{\partial g^{\mu\nu}}$, where $L^{\mathrm{D}}_{\mathrm{corr}} \equiv 3\breve{T}^{\lambda}\breve{s}^{\mathrm{D}}_{\lambda}$, and it corresponds to non-minimal fermion-boson interactions (induced by torsion) and also spin-spin fermionic self-interactions.

To illustrate these expressions, let us consider the *ansatz* $\tilde{F}_{\mu\nu} = 0$, corresponding to a spacetime with $A_\mu = (\phi(t), 0, 0, 0))$. In this case we find $\tilde{s}^{\alpha\beta\gamma} = 0$, and the last two terms in Eq. (5.3) read

$$\begin{aligned}\Pi^{M\text{int}}_{\mu\nu} &+ \Xi^{D\text{int}}_{\mu\nu} = 6\left(\kappa^2 + \lambda\kappa^4 A^2\right)\breve{s}_\mu\breve{s}_\nu + 6\lambda\kappa^4\breve{s}^2 A_\mu A_\nu - 16\lambda\kappa^4(A\cdot\breve{s})A_{(\mu}\breve{s}_{\nu)} \\ &+ \frac{1}{2}\left[\lambda\kappa^4(A\cdot\breve{s})^2 - \breve{s}^2\left(3\kappa^2 + \lambda\kappa^4 A^2\right)\right]g_{\mu\nu}\ . \end{aligned} \tag{5.4}$$

while Eq. (3.65) yields (by substituting the torsion components by spin quantities using Cartan's equations)

$$C = -\frac{\kappa^2}{2}\left[s^\lambda s_\lambda + s^{\mu\nu\lambda}\left(s_{\nu\lambda\mu} + s_{\lambda\mu\nu} + s_{\mu\lambda\nu}\right)\right]\ . \tag{5.5}$$

From this expression we can compute the torsion-induced contribution to the energy-momentum tensor (due to the Ricci scalar in a RC spacetime), using $s_{\lambda\alpha\beta} = s^M_{\lambda\alpha\beta} + s^D_{\lambda\alpha\beta}$ and(4.129), we get

$$\begin{aligned} U_{\mu\nu} &= \kappa^2\Big\{A_\mu A_\nu 2\lambda\kappa^2\Big[\breve{s}^2(2-\lambda\kappa^2A^2) - \lambda\kappa^2\left[A^2\breve{s}^2 - (A\cdot\breve{s})^2\right]\Big] \\ &+\breve{s}_\mu\breve{s}_\nu\left[2\lambda\kappa^2A^2(2-\lambda\kappa^2A^2) - 3\right]\Big\} + 4\lambda\kappa^4(2-\lambda\kappa^2A^2)(A\cdot\breve{s})A_{(\mu}\breve{s}_{\nu)} + Cg_{\mu\nu}\ , \end{aligned} \tag{5.6}$$

with

$$C = -\frac{\kappa^2}{2}\left[\lambda\kappa^2(2-\lambda\kappa^2A^2)(A^2\breve{s}^2 - (A\cdot\breve{s})^2) - \frac{3}{2}\breve{s}^2\right]\ . \tag{5.7}$$

The effective energy-momentum tensor in (3.63) is therefore given by the expression

$$\begin{aligned} T^{\text{eff}}_{\mu\nu} &= \tilde{T}_{\mu\nu} - 4j_{(\mu}A_{\nu)} + j^\lambda A_\lambda g_{\mu\nu} - \breve{s}_\mu\breve{s}_\nu\kappa^2\left[-3 - \lambda\kappa^2A^2(1 + 2(\lambda\kappa^2A^2))\right] \\ &+A_\mu A_\nu\lambda\kappa^4\Big[\breve{s}^2[6 + 2(2-\lambda\kappa^2A^2)] - 2\lambda\kappa^2[A^2\breve{s}^2 - (A\cdot\breve{s})^2]\Big] \\ &+A_{(\mu}\breve{s}_{\nu)}(A\cdot\breve{s})\lambda\kappa^4\left(4(2-\lambda\kappa^2A^2) - 16\right) \\ &-\Big[\kappa^2\breve{s}^2\Big(\frac{3}{4} + \frac{\lambda\kappa^2A^2}{2}\Big) + \frac{\lambda\kappa^4}{2}\big[(2-\lambda\kappa^2A^2)(A^2\breve{s}^2 - (A\cdot\breve{s})^2) - (A\cdot\breve{s})^2\big]\Big]g_{\mu\nu}\ , \end{aligned} \tag{5.8}$$

and for vanishing A^μ the Einstein-Cartan-Dirac model is re-obtained. Another simplifying scenario is when torsion exclusively results from the spin tensor of fermions, neglecting the bosonic contribution to the Cartan equations. Furthermore, let us keep only terms quadratic in the Dirac spin variables, with the linear ones vanishing upon averaging for random distributions of spin. Under these assumptions, using Cartan's equations (5.2) we get

$$U_{\mu\nu} = \kappa^2\left(\frac{3}{4}\breve{s}^2 g_{\mu\nu} - 3\breve{s}_\mu\breve{s}_\nu\right)\ , \tag{5.9}$$

a particular case of (5.6), and

$$\Xi^D_{\mu\nu} = \kappa^2 \left(6\breve{s}_\mu \breve{s}_\nu - \frac{3}{2}\breve{s}^2 g_{\mu\nu} \right) , \tag{5.10}$$

(derived from Eq.(4.105)) and

$$\Pi^M_{\mu\nu} = \lambda\kappa^4 \Big[\breve{s}^2 A_\mu A_\nu + A^2 \breve{s}_\mu \breve{s}_\nu - 4(\breve{s}\cdot A)\breve{s}_{(\mu}A_{\nu)} - \frac{g_{\mu\nu}}{2}\Big(A^2\breve{s}^2 - (A\cdot\breve{s})^2 \Big) \Big] , \tag{5.11}$$

(derived from (4.131)). Thus, the final result reads

$$\begin{aligned} T^{\rm eff}_{\mu\nu} &= \tilde{T}_{\mu\nu} - 4j_{(\mu}A_{\nu)} + j^\lambda A_\lambda g_{\mu\nu} - \kappa^2 \breve{s}_\mu\breve{s}_\nu \left(-3 + \lambda\kappa^2 A^2\right) \\ &\quad + \lambda\kappa^4 \left[A_\mu A_\nu \breve{s}^2 - 4A_{(\mu}\breve{s}_{\nu)}(A\cdot\breve{s}) \right] - \left[\frac{3\kappa^2}{4}\breve{s}^2 + \frac{\lambda\kappa^4}{2}\left(A^2\breve{s}^2 - (A\cdot\breve{s})^2 \right) \right] g_{\mu\nu} . \end{aligned} \tag{5.12}$$

In rigour, if we take the non-linear (self-interactions) in the bosonic Lagrangian of (4.126), then a lengthy and tedious process allows to compute the total stress-energy contribution from non-minimal and self-interactions as

$$\begin{aligned} \Pi^{M{\rm int}}_{\mu\nu} + \Xi^{D{\rm int}}_{\mu\nu} &= -2\lambda^2\kappa^2 \tilde{F}_{\alpha\beta} A^{[\alpha}\tilde{F}^{\beta]}{}_\mu A_\nu - \frac{4\lambda\kappa^2}{2+\lambda\kappa^2A^2}\Big(\tilde{s}^\beta \tilde{F}_{\mu\beta} - \lambda\kappa^2 A^{[\alpha}\tilde{s}^{\beta]}\tilde{F}_{\alpha\beta}A_\mu \Big) A_\nu \\ &\quad -4\lambda^3\kappa^4 A^2 A_\lambda A_{[\mu}\tilde{F}_{\gamma]}{}^\lambda \tilde{F}_\nu{}^\gamma - \frac{4\lambda\kappa^4 A_{[\mu}\tilde{s}_{\gamma]}\tilde{F}_\nu{}^\gamma}{2+\lambda\kappa^2A^2}(1-\lambda\kappa^2A^2) + (\mu\leftrightarrow\nu) \\ &\quad -2\lambda^3\kappa^4\Big(A^2 \tilde{F}_\mu{}^\lambda \tilde{F}_\nu{}^\gamma + A_\mu A_\nu \tilde{F}_\alpha{}^\lambda \tilde{F}^{\alpha\gamma} - 2A_{(\mu}\tilde{F}_{\nu)}{}^\gamma \tilde{F}_\alpha{}^\lambda A^\alpha \Big) A_\lambda A_\gamma \\ &\quad -16\lambda\kappa^4 A^2 \tilde{F}^\alpha{}_\mu A_\nu \tilde{F}_{\alpha\gamma}A^\gamma - A_\mu A_\nu \left(\tilde{s}^2 h(A) - (A^2\tilde{s}^2 - (A\cdot\tilde{s})^2)t(A) \right) \\ &\quad -\Big(A^2\tilde{s}_\mu\tilde{s}_\nu - (A\cdot\tilde{s})\tilde{s}_{(\mu}A_{\nu)} \Big) h(A) + \mathcal{L}^M_{\rm self} g_{\mu\nu} + 6\left(\kappa^2 + \lambda\kappa^4A^2\right)\breve{s}_\mu\breve{s}_\nu \\ &\quad +6\lambda\kappa^4\breve{s}^2 A_\mu A_\nu - 16\lambda\kappa^4(A\cdot\breve{s})A_{(\mu}\breve{s}_{\nu)} \\ &\quad +\frac{1}{2}\Big(\lambda\kappa^4(A\cdot\breve{s})^2 - \breve{s}^2\left(3\kappa^2 + \lambda\kappa^4A^2\right) \Big) g_{\mu\nu} , \end{aligned} \tag{5.13}$$

which includes (5.4) as a particular case, and the functions $h(A)$ and $t(A)$ are given by

$$\begin{aligned} h(A) &= \frac{4\lambda\kappa^4}{(2+\lambda\kappa^2A^2)^2}(1-\lambda^2\kappa^2A^2(2-\kappa^2A^2)), \\ t(A) &= \frac{8\lambda^2\kappa^6(\lambda\kappa^2A^2+1)}{(2+\lambda\kappa^2A^2)^2} + \frac{16\lambda\kappa^2}{(2+\lambda\kappa^2A^2)^3}(1-\lambda^2\kappa^2A^2(2-\lambda\kappa^2A^2)) , \end{aligned}$$

respectively. In the expression above for the torsion-induced corrections to the matter energy-momentum, only the last four terms depend on the spinors via the Dirac spin axial vector. The term $\mathcal{L}^M_{\rm self}g_{\mu\nu}$ corresponds to the purely electromagnetic terms of Eq.(4.126), i.e, the self-interactions. In absence of electromagnetic potentials we recover the ECD model.

5.1.3 Cosmological dynamics

The set of gravitational, electromagnetic and fermionic equations for the ECDM model derived from (3.58), and (4.107), implement the minimal coupling between torsion and matter fields, which results in non-minimal couplings between fermions and bosons and also in self-interactions. We shall study the cosmological dynamics associated to this framework, under the assumption of randomly oriented spin densities (unpolarized matter).

Fluid description and Friedman equations

Let us assume a homogeneous and isotropic Universe, which is described by the Friedman-Lemaître-Robertson-Walker (FLRW) metric, given by the line element

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) , \tag{5.14}$$

where $a(t)$ is the scale factor and k denotes the curvature of space. As usual, matter is described by a perfect fluid with an energy-momentum tensor $T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)$, where ρ and p are the energy density and pressure, respectively. For the sake of this section we shall consider both (relativistic) fermionic matter and radiation coupled to spacetime torsion.

One of the most common approaches to cosmology with spin is to consider the Weyssenhof spin fluid (see Ref. [21] for details), which can be seen as the classical approximation of a fluid of fermionic matter with macroscopic spin effects. In this section, however, we shall take instead the approach from fundamental Dirac spinors. To this end, it is usual to consider that for comoving observers the spin (axial) vector is spatial, i.e., $\breve{s}^{\lambda}u_{\lambda} = 0$, where $u^{\alpha} = (1, 0, 0, 0)$ is the fluid's unit four-velocity field. Nevertheless, since fermionic fields are appropriately represented by four-spinors, which can be regarded as fundamental quantum fields, in order to establish a (macroscopic) fluid description we will adopt the correspondence principle approach, through the definitions[1]

$$\breve{s}^2 \equiv \frac{\hbar^2}{4}\left\langle \bar{\psi}\gamma^{\nu}\gamma^5\psi(\bar{\psi}\gamma_{\nu}\gamma^5\psi)\right\rangle . \tag{5.15}$$

Accordingly, in the expressions for the effective energy densities and pressures all (fermionic) spin quantities should be regarded as expectation values. Moreover, throughout the rest of this paper we shall assume that the cosmological fluid has vanishing macroscopic intrinsic spin on average ($\overline{\breve{s}} \simeq 0$), under the unpolarized matter assumption. However, quantities quadratic in spin do not average to zero. Thus, by taking $\overline{\breve{s}^2} = g^{kk}\overline{\breve{s}_k\breve{s}_k}$ and $\overline{\breve{s}^i\breve{s}_j} \approx \mathrm{diag}(\overline{\breve{s}_1\breve{s}_1}g^{11}, \overline{\breve{s}_2\breve{s}_2}g^{22}, \overline{\breve{s}_3\breve{s}_3}g^{33}) \approx \delta^i_j\overline{\breve{s}^2}/3$, invoking isotropy, we assume that for fermions we have (on average)

$$\overline{\breve{s}^2} = \beta_s n^2(t), \qquad \overline{\breve{s}^i\breve{s}_j} = \frac{\overline{\breve{s}^2}}{3}\delta^i_j \sim \frac{n^2(t)}{3}\delta^i_j , \tag{5.16}$$

[1] One can see that

$$\breve{s}^2 \equiv \frac{\hbar^2}{4}\left\langle \bar{\psi}\gamma^{\nu}\gamma^5\psi(\bar{\psi}\gamma_{\nu}\gamma^5\psi)\right\rangle = \frac{\hbar^2}{4}\left\langle \bar{\psi}\gamma^{a}\gamma^5\psi(\bar{\psi}\gamma_{a}\gamma^5\psi)\right\rangle ,$$

should scale as $\breve{s}^2 \sim \left\langle(\bar{\psi}\psi)^2\right\rangle \sim n^2(t)$, where n is the number density of fermions. We recall that $a = 0, 1, 2, 3$ and the usual constant Pauli-Dirac matrices γ^c, which obey $\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}I$, are related to the γ^{μ} matrices via $\gamma^{\mu}\theta^a{}_{\mu} = \gamma^a$ (chapter 4), where $\theta^a{}_{\mu}$ are the tetrads (chapter 3).

where $n(t) \sim a^{-3}$ and $|\beta_s| \sim \hbar^2$. We therefore neglect possible anisotropic pressure contributions from the $\breve{s}^i \breve{s}_j$ terms.

As for the bosonic vector potential, we use two different *ansatze*: i) $A_\mu = (\phi(t), 0, 0, 0)$, therefore $\tilde{F}_{\mu\nu} = 0$ and $\tilde{s}^{\alpha\beta\gamma} = 0$, and ii) $A = (0, \vec{A}(t))$ (with its orientation randomly distributed to respect isotropy), therefore $\overline{\vec{A}} \simeq 0$ and we have $\overline{A^k A_k} = \overline{\vec{A}^2} \neq 0$ and take $\overline{A^i A_j} \approx \delta^i_j \overline{\vec{A}^2}/3$, again invoking isotropy.

Friedman equations

The generalized Einstein equations (3.62) can be written, in the FLRW background (5.14), for isotropic pressure, as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\left(\rho + \rho^{\rm corr}\right) - \frac{k}{a^2}, \tag{5.17}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\left[3(p + p^{\rm corr}) + (\rho + \rho^{\rm corr})\right], \tag{5.18}$$

where as usual dots over functions mean time derivatives. Here $\rho^{\rm eff} = \rho + \rho^{\rm corr}$ and $\rho^{\rm corr} = \rho^s + \rho^{s-A} + \rho^A$ with the corrections to GR corresponding to the spin-spin interaction energy, the non-minimal interactions between fermionic spin and the bosonic four-potential, and self-interactions in the bosonic sector, respectively The same apply to the pressure contributions. We shall split now our analysis in four different cases.

• CASE I: Under the ansatz of random fermionic spin and $A_\mu = (\phi(t), 0, 0, 0)$, using Eq.(5.8) we get for the correction densities and pressures:

$$\rho^{\rm corr} \simeq -\kappa^2 \breve{s}^2 \left[\frac{3}{4} + \lambda\kappa^2\phi^2\left(\frac{7}{2}\lambda\kappa^2\phi^2 - \frac{17}{2}\right)\right], \tag{5.19}$$

$$p^{\rm corr}\delta^i_j \simeq -\kappa^2 \breve{s}^2 \left[\frac{1}{4} + \lambda\kappa^2\phi^2\left(\frac{1}{6} - \frac{1}{6}\lambda\kappa^2\phi^2\right)\right]\delta^i_j\ , \tag{5.20}$$

respectively.

• CASE II: Under the *ansatz* of A_μ to be spatial and randomly oriented, $A = (0, \vec{A}(t))$, the Maxwell Lagrangian in (4.126) can be written as

$$\mathcal{L}^{\rm M}_{\rm corr} = f(A)\tilde{F}^0_{\ k}\tilde{F}_{0j}A^k A^j - \frac{\lambda\kappa^4}{2}\left[A^2\breve{s}^2 - (A\cdot\breve{s})^2\right]\ . \tag{5.21}$$

Using also Eq.(4.125) and neglecting the term linear in $\breve{s}$, we obtain

$$\rho^{\rm corr} = -\kappa^2\breve{s}^2\left(\frac{3}{2} + \lambda\kappa^2A^2\right) - f(A)\tilde{F}_{0k}\tilde{F}_{0j}A^kA^j + \rho^{\rm corr}_U, \tag{5.22}$$

with $f(A)$ given by

$$f(A) = \frac{\lambda^2\kappa^2}{2}\Big[\frac{\lambda\kappa^2A^2(4 + \lambda\kappa^2A^2(\lambda\kappa^2A^2 - 1)) + 2}{(2 + \lambda\kappa^2A^2)^2} + \lambda\kappa^2A^2 - 2\Big]\ . \tag{5.23}$$

Here $\rho_U^{\rm corr}$ is the contribution coming from the $U_{\mu\nu}$ tensor, which gives a quite cumbersome and far from illuminating expression, and the other terms come from the corrections to the $T_{\mu\nu}$ tensor. Since the relevant torsion-induced corrections coming from the tensor $U_{\mu\nu}$ are quadratic in the contortion, by taking into account Eq.(4.121) and neglecting terms that scale linearly with $\breve{s}$, we obtain (approximately) a similar expression as that resulting from the first two terms in (5.22),

$$\rho^{\rm corr} \approx \breve{s}^2 \left(C + (h + bA^2)A^2\right) + h(A)\dot{A}_j\dot{A}_k A^j A^k \ , \tag{5.24}$$

with $h(A)$ some expression of A with dimensions of $\lambda^2\kappa^2$. The first term includes spin-spin fermion self-interactions and fermion-boson non-minimal couplings, while the last term represents the energy density from bosonic self-interactions, although other self-interactions of the form $\sim \lambda^2\kappa^4 x(A)\tilde{F}^2 A^6$ can also be present.

As for the pressure corrections, p^{s-A}, neglecting anisotropic stresses we arrive at a similar (approximate) expression

$$p^{\rm corr}\delta^k_m \approx \left[\breve{s}^2 \left(D + (q + cA^2)A^2\right) + t(A)\dot{A}_j\dot{A}_k A^j A^k\right]\delta^k_m \ . \tag{5.25}$$

The anisotropic stresses are present, in general, coming for instance from a term of the form $\tilde{F}_{0k}A^k\tilde{F}_{0(i}A_{j)}$ in the effective energy momentum tensor, which can be written as $\sim \tilde{F}_{0i}\tilde{F}_{0j}A^2$ using $\overline{A^iA_j} \approx \vec{A}^2\delta^i_j/3$. The corresponding stresses T^k_j can be recast into the (averaged) isotropic form $\sim \dot{A}^m\dot{A}_m A^2\delta^k_j$, by making the approximation $\overline{\dot{A}^k\dot{A}_j} \approx \overline{\dot{A}^m\dot{A}_m}\delta^k_j/3$. In this case, the final expression would be approximately isotropic, having exactly the same functional form as in the equation above. The second term can be simplified, using again $\overline{A^iA_j} \approx \vec{A}^2\delta^i_j/3$ and $\overline{\dot{A}^k\dot{A}_j} \approx \overline{\dot{A}^m\dot{A}_m}\delta^k_j/3$, which yields

$$\rho^{s-A} \approx \breve{s}^2 \left(C + (h + bA^2)A^2\right) + \frac{1}{9}h(A)\dot{A}^2A^2 \ , \tag{5.26}$$

and

$$p^{\rm corr}\delta^k_m \approx \left[\breve{s}^2 \left(D + (q + cA^2)A^2\right) + \frac{1}{9}t(A)\dot{A}^2A^2\right]\delta^k_m \ . \tag{5.27}$$

- <u>CASE III</u> (fermionic torsion): Let us now consider the regime in which the bosonic spin tensor does not contribute to torsion, i.e., bosonic fields are influenced by spacetime torsion and affect the cosmological dynamics but do not back-react on torsion. In this case, using the *ansatz* $A_\mu = (\phi(t), 0, 0, 0)$, from Eq.(5.12) we find

$$\rho^{\rm corr} \simeq -\kappa^2\breve{s}^2\left(\frac{3}{4} + \frac{\lambda\kappa^2}{2}\phi^2\right) \ , \tag{5.28}$$

$$p^{\rm corr}\delta^i_j \simeq -\kappa^2\breve{s}^2\left(\frac{1}{4} - \frac{5\lambda\kappa^2}{6}\phi^2\right)\delta^i_j \ . \tag{5.29}$$

- <u>CASE IV</u> (fermionic torsion): Under the *ansatz* $A_\mu = (0, \vec{A}(t))$ we get

$$\rho^{\rm corr} \simeq -\kappa^2\breve{s}^2\left(\frac{3}{4} + \frac{\lambda\kappa^2}{3}\vec{A}^2\right) \ , \tag{5.30}$$

$$p^{\rm corr}\delta^i_j \simeq -\kappa^2\breve{s}^2\left(\frac{1}{4} - \frac{2\lambda\kappa^2}{3}\vec{A}^2\right)\delta^i_j \ . \tag{5.31}$$

A slight modification of this case occurs when, instead of the approximations $\overline{A^k A_j} \approx \overline{A^2}\delta^k_j/3$, and $\overline{\breve{s}^k \breve{s}_j} \approx \overline{\breve{s}^2}\delta^k_j/3$, we consider $\overline{A^k A_j} \approx \overline{A^2}\delta^k_j$, and $\overline{\breve{s}^k \breve{s}_j} \approx \overline{\breve{s}^2}\delta^k_j$. This way we arrive at the following expressions:

$$\rho^{\text{corr}} \simeq -\kappa^2 \breve{s}^2 \left(\frac{3}{4} - \lambda\kappa^2 \vec{A}^2\right) , \quad (5.32)$$

$$p^{\text{corr}}\delta^i_j \simeq -\kappa^2 \breve{s}^2 \left(\frac{9}{4} + \lambda\kappa^2 \vec{A}^2\right)\delta^i_j . \quad (5.33)$$

In all these cases we consider $\breve{s}^2 = \beta_s n^2(t) = \alpha_s a^{-6}$, which means that in the very early Universe the spin-spin effects start to strongly dominate over the usual energy density and pressure of the relativistic fluid. The $\breve{s}^2 \sim a^{-6}$ behaviour is usually considered in cosmological applications of ECSK theory for fluids with spin. It follows directly from a conserved fluid component corresponding to the spin-spin interaction, with an effective stiff-like equation of state, $w^s = p^s/\rho^s = 1$. It is also a natural result from the theory of fermionic Dirac spinors. In ECSK theory it is the negative value of ρ_s that acts as a repulsive effect. In the present ECDM model the other contributions (ρ_{s-A} for e.g.) may affect the early Universe dynamics by reinforcing or counter-acting this repulsive phenomena, depending on the sign and strength of these extra terms.

In order to explore the solutions of the dynamics in this torsion *era* we need to evaluate the time dependence of the bosonic four-potential, or equivalently its behaviour with the cosmological scale factor. Besides the Friedman equations we have at our disposal also the effective energy-momentum conservation equation, $\tilde{\nabla}_\mu T^{\mu\nu}_{\text{eff}} = 0$, the generalized electromagnetic equations and the corresponding effective charge conservation. Beyond the fluid approach, one needs to consider the dynamics of fundamental fermionic degrees of freedom, that is, the Dirac equation in the FLRW cosmological framework.

Effective conservation equation

Let us thus consider the generalized energy-momentum conservation:

$$\dot{\rho}_{\text{eff}} + 3H(\rho_{\text{eff}} + p_{\text{eff}}) = 0 , \quad (5.34)$$

with $\rho_{\text{eff}} = \rho + \rho_{\text{corr}}$ and $p_{\text{eff}} = p + p_{\text{corr}}$. For simplicity, we shall consider the different contributions to the effective energy density as different fluid components which are independently conserved. These components correspond to the usual relativistic fluid ("radiation") term, the spin-spin interaction, an additional term representing the non-minimal interaction between fermionic spin and the bosonic potential (both induced by the spin-torsion Cartan relation), as well as torsion-induced bosonic self-interactions, i.e.,

$$\rho_{\text{eff}} = \rho + \rho^{s-s} + \rho^{s-A} + \rho^A , \quad (5.35)$$

and analogously for the pressures. From now on we will focus our attention in cases I, III and IV, neglecting in this way the bosonic self interactions ρ^A. Therefore, independent conservation implies

$$\dot{\rho}_{s-A} + 3H(\rho_{s-A} + p_{s-A}) = 0 . \quad (5.36)$$

This can be solved in order to provide the $A_\mu(t)$ dependence or, alternatively, to get the dependence with the scale factor $A(a)$ as

$$\frac{d\rho_{s-A}}{da} + \frac{3}{a}(w_{s-A} + 1)\rho_{s-A} = 0 , \quad (5.37)$$

which yields the solution $\rho_{s-A} \sim a^{-3(w_{s-A}+1)}$ for constant w_{s-A}.

Fermionic torsion. As a specific example let us consider the Cases III and IV above. We have $w^{s-A} = -5/3$ and $w^{s-A} = -2$, respectively and, therefore, $\rho^{s-A} \sim a^2$ and $\rho^{s-A} \sim a^3$, respectively, which in turn implies that $\phi \sim a^4$ and $A_j A^j = (A_j)^2 g^{jj} \sim a^9$, respectively. In the last case, since $g^{jj} \sim a^{-2}$ we get $A_j \sim a^{11/2}$. More rigorously, for $\rho_{s-A} = C\breve{s}^2\phi^2$, (as in Case III) with C a constant and $p_{s-A}/\rho_{s-A} = w_{s-A}$ also constant, we obtain

$$\breve{s}^2 \frac{d\phi^2}{da} + \frac{3}{a}\left(w_{s-A} + 1 + \frac{a}{3\breve{s}^2}\frac{d\breve{s}^2}{da}\right)\breve{s}^2\phi^2 = 0 \; , \tag{5.38}$$

which yields the solution

$$\phi(a) \sim a^{-3(w_{s-A}-1)/2} \; , \tag{5.39}$$

This is compatible with the previous conclusion that for $w_{s-A} = -5/3$ we get $\phi \sim a^4$. Analogously, for $\rho_{s-A} = C\breve{s}^2\vec{A}^2$ (as in Case IV) with $p_{s-A}/\rho_{s-A} = w_{s-A}$ constant, we obtain

$$\vec{A}^2 \sim a^{-3(w_{s-A}-1)} \; , \tag{5.40}$$

and therefore

$$A_j^2 \sim a^{-3(w_{s-A}+1-2)+2} \; , \tag{5.41}$$

which for $w_{s-A} = -2$, provides $A_j \sim a^{11/2}$.

Let us summarize the main conclusions so far. Under the simplifying assumption that the various energy contributions due to relativistic fermions and bosons, including the spin-spin interaction and the fermion-boson non-minimal interactions, are separately conserved, with no energy exchanges between them, the terms representing the non-minimal interactions scale with $\rho^{s-A} \sim -\lambda\kappa^4\hbar^2 a^2$ ($w^{s-A} = -5/3$) or $\rho^{s-A} \sim \lambda\kappa^4\hbar^2 a^3$ ($w^{s-A} = -2$) depending on the *ansatz* for the bosonic four-potential. In the alternative derivation of Case IV we get instead $\rho^{s-A} \sim -\lambda\kappa^4\hbar^2 a^0$ ($w^{s-A} = -1$). This means that at least when torsion is exclusively due to fermionic spin, the non-minimal couplings induced by the $U(1)$ symmetry breaking should not introduce major deviations from the usual ECSK theory in the torsion *era* of the early Universe. This follows from the $\rho^s \sim a^{-6}$ behaviour that dominates the early-Universe dynamics. However interesting late-time effects can occur, as we shall see.

Full approach. In this scenario, for Case I we have

$$\rho^{s-A} = C\breve{s}^2\phi^2(h + b\phi^2) \; , \tag{5.42}$$

$$p^{s-A} = C\breve{s}^2\phi^2(d + c\phi^2) \; . \tag{5.43}$$

Assuming that $w_{s-A}(a) = p^{s-A}/\rho^{s-A} \simeq$ constant we get

$$\rho^{s-A} \sim a^{-3(w_{s-A}+1)} \; . \tag{5.44}$$

Moreover, in this case we can take the approximation $w_{s-A}(a) \simeq c/b = -1/24$, that gets progressively more accurate for larger values of ϕ, and we have

$$\rho^{s-A} \sim O(a^{-2,88}) \; , \tag{5.45}$$

again not competing with the a^{-6} behaviour of the spin-spin energy density. The evolution for $\phi(a)$ can be then inferred from

$$\phi^2(h + b\phi^2) \sim O\left(a^{-3(w_{s-A}-1)}\right) \; , \tag{5.46}$$

which implies that $\phi \sim O(a^{0,78})$ or, alternatively, from the conservation equation, leading to

$$\frac{d\phi^2}{da} + \frac{3}{a}\left(w_{s-A} + 1 + \frac{a}{3\breve{s}^2}\frac{d\breve{s}^2}{da}\right)\phi^2 + \frac{b\phi^2}{(h+b\phi^2)}\frac{d\phi^2}{da} = 0, \tag{5.47}$$

which yields the solution

$$a(\phi) \sim \exp\left[\frac{1}{3\bar{w}\phi} + \sqrt{\frac{b}{h}}\tan^{-1}\left(\sqrt{\frac{b}{h}}\,\phi\right)\right] \ , \tag{5.48}$$

with $\bar{w} \equiv w_{s-A} - 1$.

More rigorously, if we do not assume $w_{s-A}(a) = p^{s-A}/\rho^{s-A}$ to be constant then we get

$$\frac{d\phi^2}{da}\left[1 + \frac{b\phi^2}{(h+b\phi^2)}\right] + \frac{3}{a}\left(\frac{d+c\phi^2}{h+b\phi^2}\right)\phi^2 = -\frac{3}{a}\left(1 + \frac{a}{3\breve{s}^2}\frac{d\breve{s}^2}{da}\right)\phi^2 \ , \tag{5.49}$$

which yields the following solution

$$a(\phi) \sim \exp\left\{-\frac{h}{3(h-d)\phi} + \frac{[b(h-2d)+ch]\tan^{-1}[(\sqrt{b-c}\phi)/\sqrt{h-d}]}{3\sqrt{b-c}(h-d)^{3/2}}\right\} \tag{5.50}$$

For the values of h, b, d, c given in the expression of Case I, we obtain a specific bi-parametric family of curves (depending on the parameter λ and an integration constant), which show ϕ increasing with increasing scale factor in the domains where the function is invertible. We obtain a similar solution for Case II, with $\rho^{s-A} \approx C\breve{s}^2 A^2(h + bA^2)$ if we neglect the $\sim (\dot{A}_j A^j)^2$ term by replacing $\phi \to A$ and h, b by the corresponding coefficients.

As a final comment, let us mention that, as usual, the cosmological solutions for the evolution of the scale factor can be derived from the expression ($da/d\eta = a^2 H$)

$$\int \frac{da}{a^2\left(\kappa^2\rho^{\text{eff}}(a)/3 - k/a^2\right)^{1/2}} = \int d\eta + C, \tag{5.51}$$

where η is the usual conformal time, $dt = a d\eta$.

Bouncing Cosmology

i. Non-singular solutions. In principle, the minimum of the scale factor, which is present in the ECSK theory, should change in the ECDM model presented here. The Friedman equations can be combined as

$$H^2(a) = \frac{\kappa^2}{3}\left[\alpha_{\text{rad}}a^{-4} - \kappa^2\alpha_s a^{-6} + \rho^{s-A}(a)\right] - ka^{-2} \ , \tag{5.52}$$

with $\rho^s(a) = -\kappa^2\alpha_s a^{-6}$, $\alpha_s > 0$ and $\rho^{s-A}(a) = \lambda\rho^s f(A(a))$. By simplicity let us take the choice $k = 0$, and by looking for the zeroes of $H^2(a) = 0$ we get the equation

$$a^2 - \frac{\kappa^2\alpha_s}{\alpha_{\text{rad}}} + \frac{a^6\rho^{\text{s-A}}(a)}{\alpha_{\text{rad}}} = 0 \ . \tag{5.53}$$

In the standard ECSK theory ($\rho^{\mathrm{s-A}}$ switched off) we obtain the value of the scale factor at the bounce:

$$a_{\mathrm{b}} = \sqrt{\frac{\kappa^2\alpha_s}{\alpha_{\mathrm{rad}}}}\ . \tag{5.54}$$

For the ECDM model considered in this work, the exact value for the scale factor at the bounce will depend on the parameters α_r, α_s as well as on the parameter λ and on the value of $\rho^{\mathrm{s-A}}$ at some reference time. We can take the general case with $\rho^{\mathrm{s-A}} = \alpha_{s-A}a^b$ and for the cases we have seen above (for instance $b = -2.88$, $b = 2$ and $b = 0$), the corresponding expressions for the scale factor at the bounce can be obtained.

To this end, let us consider first the Friedman equation without the (dust) matter term, which can be written as

$$H^2(x) = H_0^2\left(\Omega_0^{rad}x^{-4} + \Omega_0^s x^{-6} + \Omega_0^{s-A}x^b + \Omega_0^k x^{-2}\right)\ , \tag{5.55}$$

with $x \equiv a/a_0$ and the parameters

$$\begin{aligned}\alpha^s &= -3\kappa^{-4}H_0^2\Omega_0^s a_0^6, \quad \alpha^{rad} = 3\kappa^{-2}H_0^2\Omega_0^{rad}a_0^4,\\ \alpha^{s-A} &= 3\kappa^{-2}H_0^2\Omega_0^{s-A}a_0^{-b}, \quad \Omega_0^{s-A} = \lambda f(A)\Omega_0^s(\frac{a}{a_0})^{-6-b}\ ,\end{aligned}$$

and $|\Omega_0^s| \sim \kappa^2\hbar^2\Omega_0^{mat}n_0$, with $n_0 \sim (n_0/n_0^{\gamma CMB})n_0^{\gamma CMB}$ being the present fermion density number as a function of the ratio of fermions to CMB photons. In the expression $\Omega_0^{s-A} = \lambda f(A)\Omega_0^s(\frac{a}{a_0})^{-6-b}$ one can see that $f(A) \sim a^{6+b}$, which is compatible with $\rho^{s-A}(a) = \lambda\rho^s f(A) = \alpha^{s-A}a^b$. If we include now the matter term, for different values of b (positive or negative) one gets a bounce in the early universe just like in the usual ECSK cosmology, where the scale factor and the energy densities remain finite. To illustrate this idea, in the case $\rho^{\mathrm{s-A}}(a) \sim a^{-4}$ ($b = -4$) with $k = 0$ we get

$$a_{\mathrm{b}} = \sqrt{\frac{\kappa^2\alpha_s}{\alpha_{\mathrm{rad}} + |\alpha_{s-A}|}}\ . \tag{5.56}$$

For the specific case of spherical spatial hypersurfaces of constant cosmic time, $k = 1$, we have the following two solutions

$$a_{\mathrm{b}} = \left[\frac{\kappa^2}{6}(\alpha_{\mathrm{rad}} + |\alpha_{s-A}|) \mp \frac{1}{6}\sqrt{-12\kappa^2\alpha_s + \kappa^4(-|\alpha_{s-A}| - \alpha_{\mathrm{rad}})^2}\right]^{1/2}\ . \tag{5.57}$$

Finally, for hyperbolic spatial hypersurfaces of constant cosmic time, $k = -1$, we arrive at the following two solutions

$$a_{\mathrm{b}} = \left[\frac{-\kappa^2}{6}(|\alpha_{s-A}| + \alpha_{\mathrm{rad}}) \mp \frac{1}{6}\sqrt{12\kappa^2\alpha_s + \kappa^4(\alpha_{\mathrm{rad}} + |\alpha_{s-A}|)^2}\right]^{1/2}\ . \tag{5.58}$$

These expressions can be compared with the corresponding solutions for the ECSK model: for $k = 1$ we have

$$a_{\mathrm{b}} = \sqrt{\frac{\kappa^2\alpha_{\mathrm{rad}} \mp \sqrt{\kappa^4\alpha_{\mathrm{rad}}^2 - 12\kappa^2\alpha_s}}{6}}\ , \tag{5.59}$$

and for $k = -1$:

$$a_{\rm b} = \sqrt{\frac{-\kappa^2\alpha_{\rm rad} \mp \sqrt{\kappa^4\alpha_{\rm rad}^2 + 12\kappa^2\alpha_s}}{6}} \ . \qquad (5.60)$$

For ECDM theory with $b = -2$ we get similar expressions, for flat geometries, $k = 0$:

$$a_{\rm b} = \sqrt{\frac{\alpha_{\rm rad} \pm \sqrt{\alpha_{\rm rad}^2 - 4\kappa^2\alpha_s|\alpha_{s-A}|}}{2|\alpha_{s-A}|}} \ ,$$

two solutions for spherical geometries, $k = 1$:

$$a_{\rm b} = \sqrt{\frac{\kappa^2\alpha_{\rm rad} \pm \sqrt{\kappa^4\alpha_{\rm rad}^2 - 12\kappa^4\alpha_s - 4\kappa^6\alpha_s|\alpha_{s-A}|}}{6 + 2\kappa^2|\alpha_{s-A}|}} \ , \qquad (5.61)$$

and also two solutions for the hyperbolic geometries, $k = -1$:

$$a_{\rm b} = \sqrt{\frac{\kappa^2\alpha_{\rm rad} \pm \sqrt{\kappa^4\alpha_{\rm rad}^2 + 12\kappa^4\alpha_s - 4\kappa^6\alpha_s|\alpha_{s-A}|}}{-6 + 2\kappa^2|\alpha_{s-A}|}} \ . \qquad (5.62)$$

For the other values of b one gets similar results, although the expressions are quite more cumbersome. We emphasize the fact that the presence of a minimum value of the scale factor in the early hot Big Bang implies the finiteness of geometrical quantities at the bounce, such as the Ricci curvature and torsion of the RC spacetime. For instance, in the *ansatz* $A_\mu = (\phi(t), 0, 0, 0)$ one has

$$T_{\alpha\beta\gamma} \ = \ \kappa^2(s^D_{\alpha\beta\gamma} + 2\lambda\kappa^2 s^D_{\alpha[\beta|\rho}A_{\gamma]}A^\rho) \ . \qquad (5.63)$$

Since $s^D_{\alpha\beta\gamma}$ is totally antisymmetric, then $K^\lambda = 2T^\lambda = 0$, so that using Eq. (5.7) we have

$$R \ = \ \tilde{R} - \kappa^4\left[\lambda\kappa^2(2 - \lambda\kappa^2\phi^2)\phi^2\breve{s}^2 - \frac{3}{2}\breve{s}^2\right] \ . \qquad (5.64)$$

In Case III, $\phi(a) \sim a^4$ and in general $s^D \sim \breve{s} \sim n(t) \sim a^{-3}$, therefore,

$$R(a_b) \sim \tilde{R}(a_b) - 2\alpha\lambda\kappa^6 a_b^2 + \beta\lambda^2\kappa^8 a_b^{10} + \gamma a_b^{-6} \ , \qquad (5.65)$$

where α, β, γ are constants and a_b is the scale factor at the bounce. Similarly the torsion components also remain finite. Let us note that the second and third terms in the correction to the usual Ricci scalar of GR scale with $\sim a^2$ and $\sim a^{10}$, respectively, which could imply a cosmological future singularity, occurring asymptotically when the scale factor goes to infinity. We will also briefly analyse the late-time dynamics of the ECDM model.

ii. Early acceleration and cyclic cosmology. One can show that, for any $\lambda \neq 0$, in the cases studied above for $b = -2.88$, $b = 2$, $b = 3$ and $b = 0$ (variation of Case IV), besides the minimum of the scale factor at the Big Bang there is a period of acceleration where the Hubble parameter increases until it reaches a maximum and starts decreasing (period of deceleration). This is valid for the spherical, flat, and hyperbolic spatial

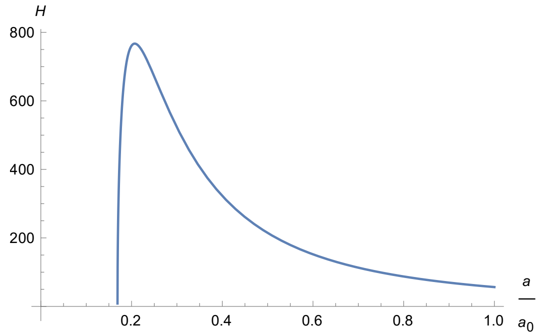


Figure 5.1: Evolution of the Hubble parameter $H(a)$ with the scale factor a/a_0 for the ECDM model without (ECSK model, left) and with (right) the non-minimal couplings in the matter fields induced by torsion. These corrections to the effective energy density $\rho^{s-A} \sim \rho^s f(A) \sim a^b$ give raise to late-time effects, whereas $\rho^s \sim -\kappa^2 \breve{s}^2$ is the spin-spin interaction term that is responsible for the non-singular behaviour in the early Universe. The plot on the right shows a typical solution with a future bounce, a non-singular behaviour at the minimum of the scale factor, and a period of early accelerated expansion. All models we analysed, except case IV ($b = 3$, $\rho^{s-A} > 0$), show a typical cosmological behaviour as illustrated on the right plot, for the three spatial geometries $k = -1, 0, 1$. The parameters used are: $\Omega_r = 0.7$, $\Omega_m = 0.32$, $\Omega_s = -0.02$, $H_0 = 68$, $\Omega_k = 0.01$, $\alpha_{s-A} = -0.08$, $b = 2$.

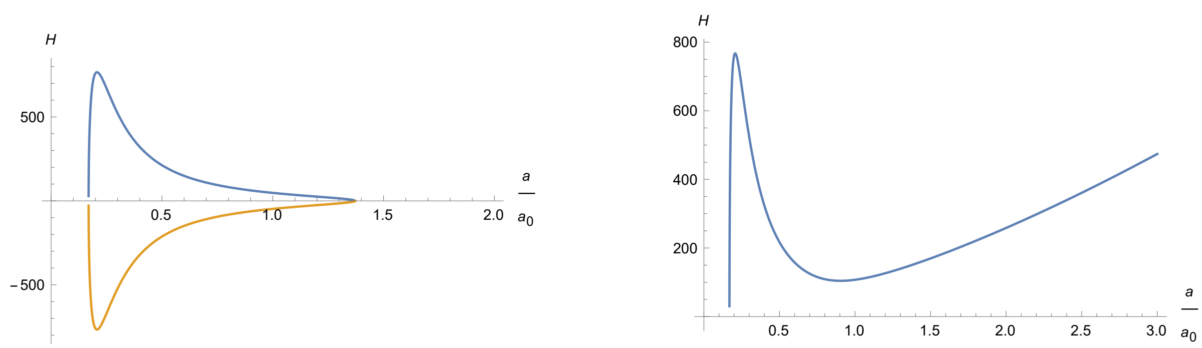


Figure 5.2: On the left figure we can see the cyclic behaviour of ECDM model explicitly, with the two branches in $H(a) = \pm\sqrt{\rho^{\text{eff}}(a) - k/a^2}$ smoothly joined together at the bounces. A period of early accelerated expansion is followed by decelerated expansion, bounce and accelerated contraction, decelerated contraction and again the bounce at the minimum of the scale factor, with the repetition of the cosmological cycle. On the right we have the relevant Case IV ($b = 3$, $\rho^{s-A} > 0$), where a late-time accelerated phase is also present. The parameters used are (left): $\Omega_r = 0.7$, $\Omega_m = 0$, $\Omega_s = -0.02$, $H_0 = 68$, $\Omega_k = -0.01$, $\alpha_{s-A} = -0.08$, $b = 0$; (right): $\Omega_r = 0.7$, $\Omega_m = 0$, $\Omega_s = -0.02$, $H_0 = 68$, $\Omega_k = -0.01$, $\alpha_{s-A} = 1.8$, $b = 3$.

geometries. The effect of increasing the strength of the corrections to progressively higher values of λ are different. For $b = -2.88$ (Case I) and for the three spatial geometries, both the value of the scale factor at the bounce and the "instant" of transition from positive acceleration towards deceleration tend to move into later times. On the other hand, in

	Case I $b=-2.88$	Case III $b=2$	Case IV $b=3$	Case IV (var) $b=0$
A_μ	$\phi \sim a^{0.78}$	$\phi \sim a^4$	$A_j \sim a^{11/2}$	$A_j \sim a^4$
ρ^{s-A}	< 0	< 0	> 0	< 0
w^{s-A}	$\approx -1/24$	-5/3	-2	-1
Early bounce (a_{min})	yes	yes	yes	yes
Early acceleration	yes	yes	yes	yes
Future bounce (a_{max})	yes	yes	-	yes
Late-time acceleration	-	-	yes	-

Table 5.1: In this table one can see the main dynamical features of various cosmological scenarios studied in this section. The cosmological dynamics is determined by the Friedman equations with spin-spin and non-minimal couplings effects (in the matter fields) induced by torsion. The late-time effects are dominated by the non-minimal interactions $\rho^{s-A} \sim a^b$.

	Case I $b=-2.88$	Case III $b=2$
Torsion	$T \sim \kappa^2 s^D + \lambda\kappa^4 s^D \phi^2$ $\to 0$	$T \sim \kappa^2 s^D$ $\to 0$
$U(1)$ - Lagrangian	$\mathcal{L}_{U1} \sim \lambda\kappa^4 \breve{s}^2 \phi^2$ $\to 0$	$\mathcal{L}_{U1} \sim \lambda\kappa^4 \breve{s}^2 \phi^2$ $\sim a^2$

Table 5.2: In this table we illustrate that even though torsion is expected to decay, the $U(1)$-breaking Lagrangian does not necessarily decay too (see Case III above). Note that, as explained in the text, in the ansatz $A_\mu = (\phi, 0, 0, 0)$, from Eq.(4.121), one can see that $T \sim \kappa^2 s^D + \lambda\kappa^4 s^D \phi^2$ and while the first term always decays, the second might not if $\phi \sim a^m$ with $m \geqslant 3/2$. This is what happened in the alternative version of Case I, where from charge current arguments it was found that $\phi \sim a^3$, therefore implying a non-zero constant background torsion in homogeneous cosmologies.

the cases $b = 2$ (Case III) and $b = 0$ (variation of Case IV) and also for $b = -2.88$, an increasing λ also reveals the relevance of a negative contribution to the energy density at later times. Indeed, for a critical value of such a contribution there will be a value of the scale factor for which the Hubble parameter vanishes (the deceleration and the expansion itself stops) and above that value it becomes imaginary, $H^2(a) < 0$.

The case of a constant energy density contribution ($b = 0$) is particularly illuminating on this issue. From the Friedman equations (5.52), the late-time cosmology of a positive constant energy density dominating asymptotically leads to the convergence of the Hubble parameter into a constant value of $H(a)$, but if the contribution from a negative energy density component starts to dominate, then the Hubble parameter is not well defined from the Friedman equations, as it becomes imaginary. This transition (when $H = 0$) could be interpreted as a future bounce, and it is compatible with the idea of nature obeying, at least, the dominant energy condition $\rho \geqslant |p|$ (which implies the weak condition $\rho \geqslant 0$, $\rho + p \geqslant 0$), an interpretation that becomes quite clear in the flat case, $k = 0$. Furthermore, due to the symmetry of the underlying Friedman equations this future bounce would be followed by a contraction, $H(a) < 0$, gradually accelerated, then the contraction would move towards a decelerated contraction phase (since $H(a)$ has a local minimum) until finally reaching the minimum of the scale factor. At that point, the energy conditions and the requirement of a non-imaginary (real) Hubble parameter imply a non-singular behaviour and the new cycle of accelerated expansion followed by decelerated expansion would start. This contracting behaviour is a natural path for the solution at the future bounce since there are two real solutions, $H(a) = \pm\sqrt{\rho^{\rm eff}(a) - k/a^2}$, corresponding to two branches of the possible cosmic history, in this case joined together at the two bounces. In both the early accelerated expansion (in branch 1) and in the sudden halt of the accelerated contraction (in branch 2) into a period of decelerated contraction, the effects due to the contribution of the dominant spin-spin (torsion induced) interaction will prevent a cosmic singularity. This cyclic behaviour is what happens in Cases I and III. We summarize this discussion in Table 5.1, and depict these behaviours in Figs. 5.1 and 5.2. We point out that this dynamics could be further explored by explicitly introducing a positive cosmological constant, though we shall deal in the next section with an effective cosmological constant out of the spin-spin interaction of fermionic vacuum condensates.

Effective cosmological constant and dark-energy

Let us now present three different results relevant for the cosmological constant/dynamical dark energy problem [68] within ECDM theory. We begin by noting that one can easily show that if instead of $\overline{\breve{s}^i\breve{s}_j} \sim \overline{\breve{s}^2}\delta^i_j/3$ and $\overline{A^iA_j} \sim \overline{\vec{A}^2}\delta^i_j/3$ we take $\overline{\breve{s}^i\breve{s}_j} \sim \overline{\breve{s}^2}\delta^i_j$ and $\overline{A^iA_j} \sim \overline{\vec{A}^2}\delta^i_j$, then Case IV corresponds to $w_{s-A} = -1$, and $A_j \sim a^4$, with $\rho_{s-A} \sim$ constant ($b = 0$). This yields an effective cosmological constant with an energy density scale set by $\lambda\kappa^4\hbar^2 n^2_{\rm ref}A^2_{\rm ref}$, where $n_{\rm ref}$ is the fermion number density at some reference cosmic time. Indeed, in this case we have

$$\rho^{\rm corr} \simeq -\kappa^2\breve{s}^2\left(\frac{3}{4} - \lambda\kappa^2\vec{A}^2\right) , \tag{5.66}$$

where

$$\rho^{s-A} = \rho^{\rm eff}_{\Lambda} = \frac{\lambda\kappa^4}{2}\beta_s n^2\vec{A}^2 = \text{const} , \tag{5.67}$$

with $\beta_s \sim \hbar^2$. As we saw previously, since $\vec{A}^2 < 0$, instead of having a positive cosmological constant effect and the resulting late-time acceleration one gets a future bounce with a transition from decelerated expansion into a period of accelerated contraction, in the cyclic scenario discussed above.

The second interesting solution corresponds to $b = 3$ in the first version of Case IV. Here we have $\rho^{\rm s-A} \simeq -\kappa^2\breve{s}^2\dfrac{\lambda\kappa^2}{3}\vec{A}^2 > 0$ and

$$\rho^{\rm corr} \simeq -\alpha_s a^{-6} + \alpha_{s-A}a^3, \qquad \alpha_{s-A} > 0 , \tag{5.68}$$

representing a non-singular cosmology with early acceleration (as in the other cases) but it also predicts a late-time accelerated expansion phase. This behaviour is driven by an effective dark energy effect supported by the term $\rho \sim a^3$ and arising from a non-minimal coupling in the matter fields induced by torsion, which starts dominating at later times.

The third result is motivated by the possibility of quark condensates in vacuum predicted by QCD, i.e., the effects of non-zero vacuum expectation values $\langle 0|\bar{\psi}\psi|0\rangle$. Indeed, in ECDM theory we can generalize the effective cosmological constant obtained in the literature of ECSK theory [98], arriving at

$$\rho_\Lambda^{\text{eff}} \sim \frac{3\kappa^2}{4}\left\langle 0|\breve{s}^2|0\right\rangle + \lambda\kappa^4\Big[(\alpha+\zeta\lambda\kappa^2A^2)\left\langle 0|\breve{s}^2|0\right\rangle A^2 + (\beta+\varepsilon\lambda\kappa^2A^2)\left\langle 0|\breve{s}^\mu\breve{s}^\nu|0\right\rangle A_\mu A_\nu\Big] , \tag{5.69}$$

with $\alpha, \beta, \varsigma, \varepsilon$ constants, which depend on the above spin density vacuum expectation values and on the electromagnetic four-potential. Since we are considering fermions, we will assume that these can form a condensate in vacuum and use the Shifman-Vainshtein-Zakharov vacuum state approximation, as in Ref. [98]. In such an approximation, the following expression is valid

$$\left\langle 0|\bar{\psi}\Gamma_1\psi\bar{\psi}\Gamma_2\psi|0\right\rangle = \frac{1}{12^2}\left(\text{tr}\,\Gamma_1\text{tr}\,\Gamma_2 - \text{tr}(\Gamma_1\Gamma_2)\right)\times\left(\left\langle 0|\bar{\psi}\psi|0\right\rangle\right)^2 ,$$

where Γ_1, Γ_2 are any matrix from the set $\{I, \gamma^i, \gamma^{[i}\gamma^{j]}, \gamma^5, \gamma^5\gamma^i\}$. Then, for quarks, QCD predicts a non-zero expectation value of $\bar{\psi}\psi$ in vacuum

$$\left\langle 0|\bar{\psi}\psi|0\right\rangle \approx \lambda_{\text{QCD}}^3 \approx -(230\,\text{MeV})^3 , \tag{5.70}$$

in geometrical system of units. We then get the general result

$$\rho_\Lambda^{\text{eff}} \sim (54\,\text{meV})^4 + f(A)\left(\left\langle 0|\bar{\psi}\psi|0\right\rangle\right)^2 , \tag{5.71}$$

where the second term is the modification in the prediction of the ECSK theory of fermions.

Fermionic torsion. From the expression $T_{\mu\nu}^{\text{eff}} = T_{\mu\nu} + U_{\mu\nu}$, for the case of fermionic torsion, from (5.12) we have

$$T_{\mu\nu}^\Lambda = -\left[\frac{\kappa^4\lambda}{2}\left(A^2\breve{s}^2 - (\breve{s}\cdot A)^2\right) + \frac{3}{4}\kappa^2\breve{s}^\lambda\breve{s}_\lambda\right]g_{\mu\nu} , \tag{5.72}$$

where we recall that $\breve{s}^\mu = \frac{\hbar}{2}\bar{\psi}\gamma^\mu\gamma^5\psi$. Therefore, we get two additional terms contributing to an effective cosmological constant beyond the usual one coming from the spin-spin interaction already present in the ECSK model. We can then compute the expression for dark energy, in the ansatz $A_\mu = (0, \vec{A})$, as

$$\rho_\Lambda^{\text{eff}} = \rho_\Lambda^{\text{ECD}} - \frac{\kappa^4\lambda}{2}\Big(\left\langle 0|\breve{s}_j\breve{s}^j|0\right\rangle A^2 - \left\langle 0|\breve{s}^k\breve{s}^j|0\right\rangle A_k A_j\Big) , \tag{5.73}$$

with $|\rho_\Lambda^{\text{ECD}}| \sim (54\,\text{meV})^4$, and after some algebra, we obtain

$$\rho_\Lambda^{\text{eff}} \approx \rho_\Lambda^{\text{ECD}} + \frac{\kappa^4\lambda\hbar^2}{3}\left(\left\langle 0|\bar{\psi}\psi|0\right\rangle\right)^2\left[\frac{2}{3}A^2 - \frac{1}{96}\left[(A_1)^2+(A_2)^2+(A_3)^2\right]\right] . \tag{5.74}$$

In the ansatz $A_\mu = (\phi, 0, 0, 0)$ we get instead

$$\rho_\Lambda^{\rm eff} \;=\; \rho_\Lambda^{\rm ECD} + \frac{\kappa^4\lambda}{2}\left(\langle 0|\breve{s}_j \breve{s}^j|0\rangle\, \phi^2\right) \;, \tag{5.75}$$

therefore

$$\rho_\Lambda^{\rm eff} \;\approx\; \rho_\Lambda^{\rm ECD} - \kappa^4\lambda\hbar^2\frac{2}{9}\phi^2 \times \left(\langle 0|\bar{\psi}\psi|0\rangle\right)^2 \;. \tag{5.76}$$

These expressions extend the results from the standard ECSK theory [98] by adding a dynamical dark energy term which depends on the four-potential ($\phi \sim a^4$ in Case III and $A_j \sim a^{11/2}$ in Case IV), during the $U(1)$-breaking symmetry phase induced by torsion. Let us point out that, as long as the minimal coupling between torsion and the bosonic four-potential takes place, the dynamical dark energy term is present. In other words, in the regimes in which the $U(1)$ breaking term in the bosonic Lagrangian (4.79) is non-negligible the four-potential will evolve with the scale factor as it is explored in this work. Note that should λ be considered as a scalar field then it would govern the transition for a symmetry breaking regime, rather than having an explicit symmetry breaking as in the case where λ is considered to be a constant coupling factor.

In absolute value, the result from the simple ECSK theory is much better than the $\sim$ 120 order of magnitude discrepancy from observations (assuming GR with cosmological constant) with respect to the predictions from quantum field theory. In the ECDM model, and from the expressions above, in principle this result could be further improved depending on the ansatz taken for the four-potential.

Full approach. Let us now consider the most general case in which torsion not only couples to the bosonic sector but it is also a result of the contribution from the total spin density including the spin density of bosons. Indeed, in such a case one has to consider Eqs. (4.79) and (4.121). Let us begin by isolating the following piece of the energy-momentum tensor (5.8)

$$T^{\rm eff\Lambda}_{\mu\nu} = -\left[\kappa^2\breve{s}^2\left(\frac{3}{4}+\frac{\lambda\kappa^2A^2}{2}\right)+\frac{\lambda\kappa^4}{2}\times\left[(2-\lambda\kappa^2A^2)(A^2\breve{s}^2-(A\cdot\breve{s})^2)-(A\cdot\breve{s})^2\right]\right]g_{\mu\nu},$$

which was derived in the ansatz $A_\mu = (\phi, 0, 0, 0)$. So, we can write

$$T^{\rm eff\Lambda}_{\mu\nu} = -\Big[\kappa^2\breve{s}^2\left(\frac{3}{4}+\frac{\lambda\kappa^2\phi^2}{2}\right)+\frac{\lambda\kappa^4}{2}(2-\lambda\kappa^2\phi^2)\phi^2\breve{s}^2\Big]g_{\mu\nu}, \tag{5.77}$$

and therefore

$$\rho_\Lambda^{\rm eff} \approx \rho_\Lambda^{\rm ECD} - \frac{\kappa^4\lambda}{2}\phi^2\left[1+(2-\lambda\kappa^2\phi^2)\right]\langle 0|\breve{s}^2|0\rangle\,,$$

leading to

$$\rho_\Lambda^{\rm eff} \;\approx\; \rho_\Lambda^{\rm ECD} - \kappa^4\lambda\hbar^2\frac{2}{9}\phi^2\left[1+(2-\lambda\kappa^2\phi^2)\right]\left(\langle 0|\bar{\psi}\psi|0\rangle\right)^2 \;.$$

Moreover, from Eq.(4.121) we see that $T \sim \kappa^2 s^D + \lambda\kappa^4 s^D\phi^2$, (since $\tilde{s} = 0$). While the first term always decays, the second might decay or not (if $\phi \sim a^m$ with $m \geqslant 3/2$).

However, the predicted behaviour for ϕ rests on the validity of the extended Maxwell Lagrangian. As one can see in Eq.(4.79) the first term in the $U(1)$-breaking term scales as $\lambda T^2\phi^2$, and therefore torsion does not decay if $m \geqslant 3$, in the case of fermionic torsion, and if $m \geqslant 3/2$, in the general case.

In Case I we obtained the approximate solution $\phi \sim a^{0.78}$, so that the dark energy effect above is valid only during the transient $U(1)$ broken phase since in this case the $U(1)$-breaking term in Eq.(4.79), $\lambda T^2\phi^2$, decays with the increasing scale factor (see Table II). Note that the $\phi \sim a^{0.78}$ behaviour was deduced from a simplified and not very robust approximation and, as we shall show below, the generalized charge conservation equation seems to suggest that $\phi \sim a^3$ also in this case. If so, then interestingly the torsion tensor $T \sim \kappa^2 s^D + \lambda\kappa^4 s^D\phi^2$ does not decay to zero, leaving a constant torsion background. Moreover, as can be seen in Eq.(4.79) the first term in the $U(1)$-breaking term $\lambda T^2\phi^2$ also remains constant.

Coupling to Maxwell dynamics

The electromagnetic field equations in the ECDM model are those in 4.81 where the induced four-current correction term J^ν is due to the presence of non-minimal couplings between A_μ and the spinors $\psi, \bar{\psi}$ and bosonic self-interactions, both effects induced by torsion. It can be obtained by substituting the Cartan equations in (4.83), or by direct variation of the effective Maxwell Lagrangian (4.126). This torsion-induced current J^ν is given by (4.136) As mentioned, these highly involved expressions can be interpreted as non-linear electrodynamics with non-minimal couplings between fermionic matter (spinors) and electromagnetic fields induced by the RC spacetime geometry.

Fermionic torsion. In the case of fermionic torsion (neglecting the contribution from the spin tensor of the bosonic field), the bosonic Lagrangian is simplified to (4.131). Under the assumption of the random spin distribution we obtain

$$J^\nu = -\kappa^4\lambda\left(\breve{s}^2 A^\nu - (\breve{s}\cdot A)\breve{s}^\nu\right) . \tag{5.78}$$

Since we take $\breve{s}^\lambda$ to be spatial, we then have $J^0 = -\kappa^4\lambda\breve{s}^2\phi$ and $J^i = 0$ for $A_\mu = (\phi, 0, 0, 0)$, while $J^0 = 0$ and $J^i = -\kappa^4\lambda\left(\breve{s}^2 A^i - (\breve{s}\cdot A)\breve{s}^i\right)$ for $A^\mu = (0, \vec{A})$. In the last expression, using the previous assumptions after an average procedure, i.e., $\breve{s}^i\breve{s}_j = \breve{s}^2\delta^i_j/3$, we obtain $J^i = -\dfrac{2}{3}\kappa^4\lambda\breve{s}^2 A^i$.

Full approach. In this case we will again consider matter with a random distribution of fermionic spins, where we neglect all quantities linear in the Dirac spin, leaving only the quadratic ones which do not vanish after macroscopic averaging. Taking the *ansatz* $A_\mu = (\phi, 0, 0, 0)$ we find

$$J^\nu = -\lambda\kappa^4\left[A^\nu\breve{s}^2 - \breve{s}^\nu(A\cdot\breve{s})\right] , \tag{5.79}$$

with $J^0 = -\kappa^4\lambda\breve{s}^2\phi$ and $J^i = 0$, just as we had in the case of a background fermionic torsion. Then, using the conservation equation (4.87) we are led to $J^0 \sim a^{-3}$, with $\phi \sim a^3$. Since this seems to be a more robust result than the $\phi \sim a^{0.78}$ previously used, if we go back to the fluid description in Case I, we then get $\rho^{s-A} \sim B + Ca^6$ with $B > 0$ and $C < 0$ constants. This fluid component manifests its effects in the evolution of the

Hubble rate at late times implying an anticipation of the future bounce into earlier times, in comparison with the other cosmological solutions with future bounce.

On the other hand, taking into account the *ansatz* $A_\mu = (0, \vec{A})$, Maxwell's equations can be written as

$$\ddot{A}_i + H\dot{A}_i = \lambda^{-1}(j^i + J^i) \ . \tag{5.80}$$

and we can take $j^i \simeq 0$, on average. This equation together with the Friedman equation

$$H^2(a) = \frac{\kappa^2}{3}\left(\alpha_{\rm rad}a^{-4} - \kappa^2\alpha_s a^{-6} + \rho^{s-A}(a)\right) - ka^{-2} \ , \tag{5.81}$$

determine the dynamics for the relevant degrees of freedom in the early Universe.

Spinors in a cosmological context and matter/anti-matter asymmetry

Fermionic torsion. The full cosmological dynamics is contained in the Friedman equations (5.17) and (5.18), the equation for the four-potential (**??**), and the Dirac equation in a FLRW background. To derive such dynamics consider first the Dirac action in a RC spacetime given by the Lagrangian density in Eq.(4.105), for the case of fermionic torsion (5.2). This yields the Fock-Ivanenko-Heisenberg-Hehl-Datta equation in (4.140). For cosmological applications it is useful to consider the conformal time variable $d\eta = dt/a(\eta)$, and the FLRW metric in its conformally flat expression

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \ . \tag{5.82}$$

Then, we can use the identity

$$\gamma^\mu \tilde{D}_\mu \psi = a^{-\frac{5}{2}}(\eta)\gamma^b\partial_b\left(a^{\frac{3}{2}}(\eta)\psi\right) \ , \tag{5.83}$$

with $b = 0, 1, 2, 3$, to arrive at the Hehl-Datta (Dirac) equation in a FLRW background

$$i\hbar\gamma^0\chi' = ma\chi + \frac{3\kappa^2\hbar^2}{8}a^{-2}(\bar{\chi}\gamma^\nu\gamma^5\chi)\gamma_\nu\gamma^5\chi,$$

where

$$\chi(\eta) \equiv a^{\frac{3}{2}}(\eta)\psi \, , \qquad \bar{\chi}(\eta) \equiv a^{\frac{3}{2}}(\eta)\bar{\psi} \, , \tag{5.84}$$

and the derivative is now performed with respect to the conformal time η.

Analogously, the generalized Hehl-Datta (Dirac) equation, including the non-minimal interaction with the electromagnetic four-potential in the case of fermionic torsion, can be easily derived from equations (4.105) and (4.131) and is given by[2]

$$\begin{aligned} i\hbar\gamma^\mu\tilde{D}_\mu\psi + \left(q\gamma^\mu A_\mu - \frac{\kappa^2\lambda\hbar}{4}f^\rho\gamma_\rho\gamma^5 - m\right)\psi \ &= \ \left(\frac{\kappa^4\lambda\hbar^2}{2}A^2 + \frac{3\kappa^2\hbar^2}{8}\right)(\bar{\psi}\gamma^\nu\gamma^5\psi)\gamma_\nu\gamma^5\psi \\ &\quad -\frac{\kappa^4\lambda\hbar^2}{2}(\bar{\psi}\gamma^\beta\gamma^5\psi)\gamma_\lambda\gamma^5\psi A_\beta A^\lambda \ , \end{aligned} \tag{5.85}$$

[2]Note, however, that if one performs the variational principle from $\mathcal{L}_m$ without substituting the torsion tensor via Cartan relations (5.2) and only make such a replacement after the derivation of the dynamical equations, then in this case of fermionic torsion one arrives again at the usual Hehl-Datta equation.

and in the background of a FLRW cosmological metric it becomes

$$\begin{aligned} i\hbar\gamma^0\chi' + \left(q\gamma^\mu A_\mu - \frac{\kappa^2\lambda\hbar}{4} f^\rho\gamma_\rho\gamma^5 - m\right) a\chi &= \left(\frac{\kappa^4\lambda\hbar^2}{2}A^2 + \frac{3\kappa^2\hbar^2}{8}\right) a^{-2}(\bar{\chi}\gamma^\nu\gamma^5\chi)\gamma_\nu\gamma^5\chi \\ &\quad -\frac{\kappa^4\lambda\hbar^2}{2} a^{-2}(\bar{\chi}\gamma^\beta\gamma^5\chi)\gamma_\lambda\gamma^5\chi A_\beta A^\lambda \ , \end{aligned} \tag{5.86}$$

with a similar dynamical (diffusion-like) cubic equation for $\bar{\chi}$. We have assumed homogeneous fields, so that each variable depends only on the conformal time. Accordingly, f^ν is given by Eq.(4.132), where the only non-vanishing components of the Faraday tensor in this system of coordinates are $\tilde{F}_{0j}(\eta) = \partial_\eta A_j = a(\eta)\dot{A}_j$.

In the *ansatz* of $A_\mu = (\phi(t), 0, 0, 0)$ we get

$$\begin{aligned} i\hbar\gamma^0\chi' + \left(q\gamma^0\phi - m\right) a\chi &= \left(\frac{\kappa^4\lambda\hbar^2}{2}\phi^2 + \frac{3\kappa^2\hbar^2}{8}\right) a^{-2}(\bar{\chi}\gamma^\nu\gamma^5\chi)\gamma_\nu\gamma^5\chi \\ &\quad -\frac{\kappa^4\lambda\hbar^2}{2} a^{-2}\phi^2(\bar{\chi}\gamma^0\gamma^5\chi)\gamma_0\gamma^5\chi, \end{aligned} \tag{5.87}$$

with $\phi \sim a^4$ (Case III), which yields

$$i\hbar\gamma^0\chi' + \left(q\gamma^0 C a^4 - m\right) a\chi = \frac{3\kappa^2\hbar^2}{8} a^{-2}(\bar{\chi}\gamma^\nu\gamma^5\chi)\gamma_\nu\gamma^5\chi + \frac{\kappa^4\lambda\hbar^2}{2} C a^6(\bar{\chi}\gamma^k\gamma^5\chi)\gamma_k\gamma^5\chi \tag{5.88}$$

where C is an integration constant.

Full approach. To consider the general case, i.e, taking into account the bosonic contribution to the spin tensor and therefore to torsion, as we did before, we start from the general expression of the Dirac equation minimally coupled to the RC geometry

$$i\hbar\gamma^\mu\tilde{D}_\mu\psi + (q\gamma^\mu A_\mu - m)\,\psi = -\frac{3\hbar}{2}\breve{T}^\lambda\gamma_\lambda\gamma^5\psi, \tag{5.89}$$

and replace the axial torsion vector in (4.124), derived from (4.121). We obtained an extended Dirac (cubic) equation

$$\begin{aligned} i\hbar\gamma^\mu\tilde{D}_\mu\psi + (q\gamma^\mu A_\mu - m)\,\psi &= f(A)(\bar{\psi}\gamma^\nu\gamma^5\psi)\gamma_\nu\gamma^5\psi + \alpha^\lambda{}_\alpha(A)(\bar{\psi}\gamma^\alpha\gamma^5\psi)\gamma_\lambda\gamma^5\psi \\ &\quad +\beta^\lambda(A,\tilde{F})\gamma_\lambda\gamma^5\psi. \end{aligned} \tag{5.90}$$

Therefore, in the context of FRLW cosmology

$$\begin{aligned} i\hbar\gamma^0\chi' + \left[q\gamma^\mu A_\mu - \beta^\rho(A,\tilde{F})\gamma_\rho\gamma^5 - m\right] a\chi &= f(A)a^{-2}(\bar{\chi}\gamma^\nu\gamma^5\chi)\gamma_\nu\gamma^5\chi \\ &\quad +\alpha^{\beta\lambda}(A)a^{-2}(\bar{\chi}\gamma_\beta\gamma^5\chi)\gamma_\lambda\gamma^5\chi, \end{aligned} \tag{5.91}$$

and in the ansatz $A_\mu = (\phi(t), 0, 0, 0)$ we have

$$\begin{aligned} f(\phi) &\equiv \frac{3\kappa^2\hbar^2}{8} + \frac{\lambda\kappa^4\hbar^2}{2}\phi^2 \,, \\ \alpha^{00}(\phi) &= -\frac{\lambda\hbar^2\kappa^4}{4}\phi^2 \,, \\ \beta^\alpha &= 0 \,, \end{aligned} \tag{5.92}$$

which yields the result

$$i\hbar\gamma^0\chi' + \left(q\gamma^0\phi - m\right)a\chi = f(\phi)a^{-2}(\bar{\chi}\gamma^\nu\gamma^5\chi)\gamma_\nu\gamma^5\chi + \alpha^{00}(\phi)a^{-2}(\bar{\chi}\gamma_0\gamma^5\chi)\gamma_0\gamma^5\chi \,. \tag{5.93}$$

Using the result derived from the generalized charge conservation, $\phi(a) \sim a^3$, we then get $f(\phi) \sim \text{const} + a^6$, and $\alpha^{00} \sim -a^6$. This is coupled to the equation for the adjoint spinors

$$i\hbar\bar{\chi}'\gamma^0 - a\bar{\chi}\left(q\gamma^0\phi - m\right) = -f(\phi)a^{-2}(\bar{\chi}\gamma^\nu\gamma^5\chi)\gamma_\nu\bar{\chi}\gamma^5 - \alpha^{00}(\phi)a^{-2}(\bar{\chi}\gamma_0\gamma^5\chi)\gamma_0\bar{\chi}\gamma^5 \,. \tag{5.94}$$

Under a charge conjugation (C) operation $\psi \to -i\gamma^2\psi^* \equiv \psi^{ch}$, corresponding to the Dirac equation for antiparticles, we have instead

$$\begin{aligned} i\hbar\gamma^0(\chi^{ch})' - \left(q\gamma^0\phi + m\right)a\chi^{ch} &= -f(\phi)a^{-2}(\bar{\chi^{ch}}\gamma^\nu\gamma^5\chi^{ch})\gamma_\nu\gamma^5\chi^{ch} \\ &\quad -\alpha^{00}(\phi)a^{-2}(\bar{\chi^{ch}}\gamma_0\gamma^5\chi^{ch})\gamma_0\gamma^5\chi^{ch} \,. \end{aligned} \tag{5.95}$$

Since the dynamics for (homogeneous) spinors representing fermions and anti-fermions are different and are therefore related to different decay laws, this is highly relevant for the topic of matter/anti-matter asymmetry in the early Universe. To illustrate this idea qualitatively one could simply consider two different orbits in the space (y',y) for different values of η in the following dynamical scenario $y'(y;\eta) = y\left[B\eta^{1/2} \pm (C\eta^2 + D\eta^{-1})y^2\right]$, which is motivated from the above equations. Such a simplified but quite general behaviour can be obtained by considering $a \sim t^{2/(3+3w^{\rm dom})}$ and $w^{\rm dom} = 1$ for the dominant fluid in the early Universe, leading to $\rho^{\rm dom} \sim a^{-6}$, $a \sim t^{1/3}$, $\eta \sim t^{2/3}$, $t \sim \eta^{3/2}$, and therefore $a \sim \eta^{1/2}$. Of course in our model things are more complicated since we have four component spinors and so on, but the trajectories associated to the + and − sign above (corresponding to fermions and anti-fermions, respectively) already illustrate how a matter/anti-matter asymmetry could be generated in the torsion *era* of the early Universe. Although there are no parity-breaking terms in our model (which is one of the Zakharov requisites, together with C breaking, for a successful mechanism generating matter/anti-matter asymmetry), our model does include an explicit C-symmetry breaking. One could go beyond the minimal coupling of fermions and torsion to include such parity-breaking terms, as these appear naturally in some quadratic models of Poincaré gauge theory of gravity.

5.1.4 Cosmological principle and the $U(1)$ symmetry breaking

To implement the cosmological principle from the onset, one needs to consider the six Killing vectors ξ related to the isometries of the maximally symmetric spatial hypersurfaces, which imply the following Lie derivatives along the directions of such vectors (see

[19] and references therein)

$$L_\xi g_{\mu\nu} = 0, \qquad L_\xi T^{\mu}_{\alpha\beta} = 0 \ . \tag{5.96}$$

Therefore the metric is of Friedmann-Lemaitre-Robertson-Walker (FLRW) type, Eq. (5.14), and the only non-vanishing components of the torsion tensor obey

$$T_{abc} = f(t)\epsilon_{abc}, \qquad T^{a}{}_{b0} = h(t)\delta^a_b \ , \tag{5.97}$$

where $f(t)$ and $h(t)$ are arbitrary functions of time, while ϵ_{abc} and δ^a_b are the three-dimensional Levi-Civita and Kronecker symbols, respectively. As a consequence, from the definition of the contortion tensor (3.10), we obtain

$$K_{abc} = f(t)\epsilon_{abc} \, , \quad K_{0ab} = -K_{a0b} = 2h(t) g_{ab} \, , \tag{5.98}$$

while any other component vanishes. This is valid for any gravity theory with a RC spacetime if one imposes the cosmological principle to the torsion tensor. Let us now consider the bosonic and fermionic sectors in this context.

Bosonic (vector) fields

Writing the electromagnetic piece of the Lagrangian density in (4.107) as $\mathcal{L}^{\rm EM} = \mathcal{L}^{\rm M} + j^\lambda A_\lambda$, with $\mathcal{L}^{\rm M}$ given by (4.79), the correction term (implementing the $U(1)$-symmetry breaking) with the ansatz (5.97) becomes

$$\mathcal{L}^{\rm M}_{\rm corr} = \lambda \Big[2\left(f^2(t) + h^2(t)\right) A^a A_a + f(t)\epsilon^{abc}\tilde{F}_{bc}A_a - 2h(t)\tilde{F}_{0b}A^b \Big] \, , \tag{5.99}$$

with the four-potential $A_\mu = (\phi, \vec{A})$. We can now replace the expressions for the contortion components (5.97) in the expression (4.83) or derive this induced four-current correction directly from the (effective) previous Lagrangian in (5.99), to arrive at the result

$$J^k = \lambda \left[4(f^2 + h^2)A^k(t) + f(t)\tilde{F}_{bc}\epsilon^{kbc} - 2h(t)\tilde{F}^{0k} \right] . \tag{5.100}$$

Then, if $\vec{A}(t) \neq 0$, it obeys the Maxwell-like equations in the cosmological (FLRW) context as in (5.80) that corresponds to an extended Proca-like equation with torsion contributing to an effective mass for the photon. We point out that for homogeneous fields we have $\tilde{F}_{bc} = 0$ and we can furthermore set, for simplicity, $j_k = 0$ (on average) for comoving observers, in agreement with the cosmological principle.

Fermionic (spinor) fields

As for the Dirac sector, re-writing equation (4.11) as

$$\mathcal{L}_{\rm Dirac} = \tilde{\mathcal{L}}_{\rm Dirac} + \frac{\hbar}{4} K_{abc}\epsilon^{cabd}\bar{\psi}\gamma_d\gamma^5\psi \tag{5.101}$$

and using the constraints on the contortion tensor (5.98), we arrive at

$$\mathcal{L}_{\rm D} = \tilde{\mathcal{L}}_{\rm D} + \frac{\hbar}{4}K_{abc}(\epsilon^{cab0}\bar{\psi}\gamma_0\gamma^5\psi + \epsilon^{cabd}\bar{\psi}\gamma_d\gamma^5\psi) + \frac{\hbar}{4}K_{0bc}(\epsilon^{c0bd}\bar{\psi}\gamma_d\gamma^5\psi - \epsilon^{cb0d}\bar{\psi}\gamma_d\gamma^5\psi), \tag{5.102}$$

which reads explicitly

$$\mathcal{L}_{\rm D} = \tilde{\mathcal{L}}_{\rm D} + \frac{\hbar}{4} f(t)\epsilon_{abc}(\epsilon^{cab0}\bar{\psi}\gamma_0\gamma^5\psi + \epsilon^{cabd}\bar{\psi}\gamma_d\gamma^5\psi) \; . \tag{5.103}$$

Alternatively, one could consider the axial vector part of torsion (4.14) in the ansatz (5.97):

$$\breve{T}^\lambda = \frac{1}{6} f(t)\epsilon^{\lambda abc}\epsilon_{abc} \; , \tag{5.104}$$

and combine it with Eq. (4.12) to arrive at

$$\mathcal{L}_{\rm D} = \tilde{\mathcal{L}}_{\rm D} + \frac{\hbar}{4} f(t)\epsilon^{\lambda abc}\epsilon_{abc}\bar{\psi}\gamma_\lambda\gamma^5\psi \; . \tag{5.105}$$

Since here $a, b, c = 1, 2, 3$, we get

$$\mathcal{L}_{\rm D} = \tilde{\mathcal{L}}_{\rm D} + \frac{3\hbar}{2} f(t)\bar{\psi}\gamma_0\gamma^5\psi \; . \tag{5.106}$$

The corresponding Dirac equation for this Lagrangian is

$$i\hbar\gamma^\mu\tilde{D}_\mu\psi - m\psi = -\frac{3\hbar}{2} f(t)\gamma_0\gamma^5\psi \; . \tag{5.107}$$

In the cosmological context, by performing a conformal transformation $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$, one can use the identity $\gamma^\mu\tilde{D}_\mu\psi = a^{-5/2}(\eta)\gamma^\mu\partial_\mu(a^{3/2}(\eta)\psi)$, to arrive at the Dirac equation in an FLRW cosmological background

$$i\hbar\gamma^0\chi' - a(\eta)\left(m - \frac{3\hbar}{2} f(\eta)\gamma_0\gamma^5\right)\chi = 0 \; , \tag{5.108}$$

where $\chi(\eta) \equiv a^{\frac{3}{2}}(\eta)\psi$ and $\bar{\chi}(\eta) \equiv a^{\frac{3}{2}}(\eta)\bar{\psi}$, while now time derivatives are performed with respect to the conformal time η. We can also write the above equation as the linear non-autonomous dynamical system

$$\vec{\chi}' = \mathbf{A}(\eta)\vec{\chi} \; , \tag{5.109}$$

where $\vec{\chi}$ is a four-spinor and $\mathbf{A}$ is the matrix with components

$$(\mathbf{A})_{CD} = -\frac{i}{\hbar} a(\eta)\left(m\gamma^0_{CD} - \frac{3\hbar}{2} f(\eta)\gamma^5_{CD}\right) \; , \tag{5.110}$$

with $C, D = 1, 2, 3, 4$. This equation can be solved, in principle, upon specification of the torsion function and the evolution of the scale factor.

Gravitational field equations

The equations above governing the dynamics of homogeneous fermionic and bosonic fields are coupled to the gravitational (Friedmann) equations whose explicit form we now derive. These equations can be conveniently written in the following form

$$\tilde{G}_{\mu\nu} = \kappa^2\left(T^{\rm pfluid}_{\mu\nu} + U^{\rm tor}_{\mu\nu} + \Pi^{\rm M}_{\mu\nu} + \Sigma^{D}_{\mu\nu}\right) , \tag{5.111}$$

Let us analyze each piece on the right-hand side of these equations separately. Note that the first term $T^{\rm pfluid}_{\mu\nu} = {\rm diag}(-\rho, p, p, p)$ is just the standard energy-momentum tensor of a perfect fluid. The second term is related to contortion is defined in (3.64), with C given by (3.65) in agreement with the Einstein-Hilbert action in a RC spacetime (3.58), where the curvature scalars are related as in (3.29). In the expression for C, as before we neglected the total derivative term $2\tilde{\nabla}^\lambda K^\alpha{}_{\lambda\alpha}$, since it does not contribute to the field equations. After some algebra, this leads to the explicit expression

$$\begin{aligned} U^{\rm tor}_{\mu\nu} &= 2\kappa^{-2}\Big[K_{(\nu}{}^{0a}K_{0a|\mu)} + K_{(\nu}{}^{ab}K_{ab|\mu)} + K^a{}_{(\nu}{}^{b}K_{\mu)ba} + K^{a0}{}_{(\nu}K_{0|\mu)a} + K^{ab}{}_{(\nu}K_{b|\mu)a} \\ &\quad +2K_{(\mu}{}^{\lambda}{}_{\nu)}K_\lambda + K_\mu K_\nu\Big] + Cg_{\mu\nu} \ . \end{aligned} \tag{5.112}$$

Using now the ansatz (5.97) we have $K^\lambda K_\lambda = 64h^2(t)$ and $K^{\alpha\lambda\beta}K_{\lambda\beta\alpha} = 6f^2(t) - 16h^2(t)$, which turns Eq. (3.65) into

$$C = -\frac{1}{2\kappa^2}\Big(6f^2(t) + 48h^2(t)\Big) \ . \tag{5.113}$$

Associating energy density and pressure contents to the temporal and spatial components of the tensor $U^{\rm tor}_{\mu\nu}$, like in a standard perfect fluid, one finds an energy density

$$\rho^{\rm tor} = -\kappa^{-2}\left(3f^2(t) - 160h^2(t)\right) \ , \tag{5.114}$$

while the pressure terms ($p\delta^a_b = -U^a_b$) are

$$(U^{\rm tor})^a_b = -\kappa^{-2}\left(104h^2 - 9f^2(t)\right)\delta^a_b \ . \tag{5.115}$$

As the simplest case, setting $h(t) = 0$ we get the equation of state $w^{\rm tor} \equiv p^{\rm tor}/\rho^{\rm tor} = 3$.

Regarding the two last contributions in Eq. (5.111), they are defined as

$$\Pi^{\rm M}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}^{\rm M}_{\rm corr})}{\delta g^{\mu\nu}} \ , \tag{5.116}$$

$$\Sigma^{\rm D}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}^{\rm D}_{\rm corr})}{\delta g^{\mu\nu}} \ , \tag{5.117}$$

respectively. Let us deal first with $\Pi^{\rm M}_{\mu\nu}$. It represents the effective energy-momentum contribution from the coupling between torsion and the electromagnetic (radiation) field. By keeping only terms quadratic in the torsion functions, we arrive at

$$\begin{aligned} \Pi^{\rm M}_{\mu\nu} &= -4\lambda\Big[(T^\lambda{}_{(\mu}{}^{\beta}T_{\gamma|\nu)\beta} + T^{\lambda\alpha}{}_{(\mu}T_{\gamma\alpha|\nu)})A_\lambda A^\gamma + T_{(\mu}{}^{\alpha\beta}T_{\gamma\alpha\beta}A_{\nu)}A^\gamma + T^{\lambda\alpha\beta}T_{(\mu|\alpha\beta}A_\lambda A_{\nu)}\Big] \\ &\quad +\mathcal{L}^{\rm M}_{\rm corr}g_{\mu\nu} \ . \end{aligned} \tag{5.118}$$

The energy density and pressure terms associated to this tensor read

$$\rho^{\rm M} = -\lambda\left[A^aA_a(6h^2(t) - 2f^2(t)) - 2h(t)\tilde{F}_{0b}A^b\right] , \tag{5.119}$$

$$\begin{aligned} (\Pi^M)_{ij} &= -4\lambda\left[(2f^2(t) + 6h^2(t))A_iA_j - 4h(t)\tilde{F}_{0(i}A_{j)}\right] \\ &\quad +\lambda\left[(h^2(t) - 6f^2(t))A^kA_k - 2h(t)\tilde{F}_{0c}A^c\right]g_{ij} \ , \end{aligned}$$

respectively, where we already considered homogeneous fields ($\tilde{F}_{bc}=0$). The second term in stress tensor above is compatible with isotropy, whereas the first term can introduce an anisotropic pressure/stress. If we set $h(t)=0$ we get

$$\rho^{\rm M} = 2\lambda f^2(t)\vec{A}^2, \tag{5.120}$$

$$(\Pi^M)^k_m = -8\lambda f^2(t)A^kA_m - 6\lambda f^2(t)\vec{A}^2\delta^k_m, \tag{5.121}$$

and if we neglect the anisotropic term in the equation above we obtain the isotropic pressure $p^M = 6\lambda f^2(t)\vec{A}^2$ and the equation of state $w^{\rm M}\equiv p^{\rm M}/\rho^{\rm M}=3$. For the Friedmann equations we will be neglecting the energy density contribution from the minimal coupling of electromagnetism to fermions, assuming that on average $j^\mu\approx 0$, and take only the above energy density due to the coupling of the bosonic field to torsion.

As for the $\Sigma^{\rm D}_{\mu\nu}$ piece, which represents the contribution from Dirac fields, it can be computed as

$$\Sigma^{\rm D}_{\mu\nu} = 3\breve{T}^\lambda\breve{s}_\lambda g_{\mu\nu} - 12\breve{T}_{(\mu}\breve{s}_{\nu)}\ , \tag{5.122}$$

and since from Eq. (5.104) one has that $\breve{T}^0=f(t)$ ($\breve{T}^m=0$), then $\breve{T}^\lambda$ is necessarily time-like, as a consequence of the cosmological principle, and we obtain the associated densities and pressures as

$$\begin{aligned}\rho^{\rm D} &= -12\breve{T}_0\breve{s}_0 + 3\breve{T}^0\breve{s}_0 = -9f(t)\breve{s}_0, &(5.123)\\ (\Sigma^{\rm D})^a_b &= 3\breve{T}^0\breve{s}_0\delta^a_b = 3f(t)\breve{s}_0\delta^a_b = -p^{\rm D}\delta^a_b, &(5.124)\end{aligned}$$

for the energy and pressure terms, respectively. In this case we have $w^{\rm D}\equiv p^{\rm D}/\rho^{\rm D}=1/3$.

A few remarks on these derivations are in order. First, we are not considering any condition on $\breve{s}^\lambda$ to be space-like (and therefore $u^\alpha\breve{s}_\alpha=0$ for comoving observers), but we are rather focusing on the cosmological principle restriction for the torsion components. Second, we have not used Cartan's equations yet. Should we use them, then in the simplest case where torsion is due to the spin tensor of fermions one would have $\breve{T}^\lambda = -\kappa^2\breve{s}^\lambda/2$ and therefore, if the cosmological principle is also taken into consideration, one is led to conclude that $\breve{s}^\lambda$ has to be time-like as well. Finally, we emphasize that in this subsection we are considering fermionic and bosonic fields propagating in the cosmological torsion background. Under a suitable averaging procedure these fields can then contribute as perfect fluids to the cosmological Friedmann equations but torsion can be seen here as an extra (external) homogeneous and isotropic tensor field that enriches the background spacetime geometry. We are neglecting the source of torsion, for simplicity.

Friedmann equations

We have all the elements ready to discuss the Friedmann equations for the ECDM model according to the implementation of the cosmological principle. The first such equation reads

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa^2}{3}(\rho+\rho^{\rm corr}) - \frac{k}{a^2}\ , \tag{5.125}$$

where the correction to the usual energy density of the perfect fluid ρ is spelled out as $\rho^{\rm corr}=\rho^{\rm tor}+\rho^{\rm M}+\rho^{\rm D}$. It can be cast, as a function of the background (external) torsion and the fundamental matter fields $(\vec{A}, \psi, \bar{\psi})$ as

$$\rho^{\rm corr} = -f^2\left(\frac{3}{\kappa^2} - 2\lambda\vec{A}^2\right) - h^2\left(6\lambda\vec{A}^2 - \frac{160}{\kappa^2}\right) - 9f(t)\breve{s}_0 + 2h\tilde{F}_{0b}A^b \ , \quad (5.126)$$

where $\vec{A}^2 \equiv A^c A_c$. The Friedmann equation above, together with the Maxwell equation in (5.80) and the Dirac equation (5.108) form the resulting dynamical system.

The generalized continuity equation, $\tilde{\nabla}^\mu T^{\rm total}_{\mu\nu} = 0$, in the FLRW background, reads

$$\dot{\rho} + 3H\rho(1 + w^{\rm rad}) = -\left[\dot{\rho}^{\rm corr} + 3H\left(\rho^{\rm corr} + p^{\rm corr}\right)\right], \quad (5.127)$$

where $w^{\rm rad} = 1/3$ and ρ refer to the usual relativistic matter (radiation) term in the early Universe, and one can set the right-hand side equal to zero (independent conserved fluid components), with the total pressure correction term (neglecting any anisotropic contributions) given by

$$p^{\rm corr}\delta^k_m = -f^2\left(\frac{9}{\kappa^2} - 6\lambda\vec{A}^2\right)\delta^k_m - 3f(t)\breve{s}_0\delta^k_m + h^2\left(\frac{104}{\kappa^2} - \lambda\vec{A}^2\right)\delta^k_m + 2h\tilde{F}_{0b}A^b\delta^k_m \, .$$

To solve the full dynamics some simplifications must be made. For instance, one can take the case in which the torsion functions are constrained as $h(t) = 0$ and $f(t) \neq 0$, which, as we will show later, follows naturally from the Cartan equations. Then, in that case where we take $\rho^{\rm tor}$, $\rho^{\rm D}$ and $\rho^{\rm M}$ as independent fluid components, representing the energy density of torsion self-interactions, torsion-fermions and torsion-bosons interactions, respectively, we have: $w^{\rm tor} = 3$, $w^{\rm D} = 1/3$, $w^{\rm M} = 3$, respectively. Now, should we consider the hypothesis that these fluid components are independently conserved, then as expected for conserved barotropic fluids we would have $\rho^{\rm tor} \sim a^{-12}$ (with $\rho^{\rm tor} < 0$), $\rho^{\rm D} \sim a^{-4}$ and $\rho^{\rm M} \sim a^{-12}$ (with $\rho^{\rm M} < 0$), together with the usual radiation term $\rho \sim a^{-4}$.

In these conditions, the solution to the dynamics is inferred from the Friedmann equation (5.125), which allows to compute the family of possible trajectories in the plane $(a, \dot{a})$. If we take also $\rho^{\rm D} < 0$ (which will turn out to be valid, as we will see after using the Cartan equations), these trajectories show that the dynamics is that of a non-singular Universe, with very strong torsion effects avoiding the singularity, and replacing it by a minimal value of the scale factor, from which the Universe undergoes a period of early accelerated expansion followed by a period of decelerated expansion. Nevertheless, the correct treatment should come from considering the torsion tensor as function of the matter fields, using the Cartan equations. This is what we will do in the last part of this chapter.

Before going that way it should be pointed out that, regardless of what the source of torsion is, if the gravity theory introduces an effective (torsion) fluid correction to the Friedmann equations, and assuming that this can be decomposed as $\rho^{\rm corr} \sim \rho_s + \rho_{s-A}$, where ρ_s is due to a spin-spin interaction of fermions and ρ_{s-A} is the interaction between fermions and bosons (for instance $\rho_{s-A} \sim \rho_s A^2$), both effects induced by torsion, then assuming that $\rho_{s-A} \approx \Sigma$ is constant, the Friedmann equation (5.125) becomes quite simple:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}(\rho + \rho^s(a) + \Sigma) - \frac{k}{a^2} \ . \quad (5.128)$$

Furthermore, assuming that $\rho^s \sim a^{-6}$ (which is a reasonable assumption if $\rho^s \sim n^2$, where n is the number density of fermions), with $\rho^s < 0$, we get

$$H^2(a) = H_0^2\left[\Omega_0^{\rm rad}\left(\frac{a_0}{a}\right)^4 + \Omega_0^{\rm s}\left(\frac{a_0}{a}\right)^6 + \frac{\kappa^2\Sigma}{3H_0^2}\right] - \frac{k}{a^2H_0^2}, \quad (5.129)$$

with $\Omega_0^{\rm s} < 0$ and as usual the subscript 0 refers to the present value for the corresponding cosmological parameter, and $\Omega_0^j = 8\pi G\rho_0^j/3H_0$, for the fluid species j. Then, as shown in this chapter, the resulting dynamics yields a bounce in the early Universe [5], for the three cases $k = -1, 0, 1$. Moreover, if $\Sigma < 0$, a future cosmological bounce is also found, as given by the zeroes of $H(a) = \pm\sqrt{\rho^{\rm eff}(a) - k/a^2}$, and since $H(a)$ cannot become imaginary, the solution can transit from the positive branch to the negative one, corresponding to a transition from decelerated expansion into accelerated contraction. In the positive branch, the solutions show an early accelerated expansion phase from a bounce at the minimal value of the scale factor, until $H(a)$ reaches a local maximum, followed by the decelerated expansion phase until the future bounce. In the negative branch, i.e, the contracting phase, after a period of accelerated contraction the Hubble parameter reaches a local minimum and starts increasing in a decelerated contraction phase due to the spin-spin repulsive effects preventing the singularity to occur, and a new bounce and subsequent accelerated expansion establish a cyclic cosmology.

On the other hand, if $\Sigma > 0$ the dynamical system describes a non-singular cosmology again with a period of early accelerated expansion followed by a period of decelerated expansion, until the Hubble parameter stabilizes (asymptotically) at a fixed constant value. This is valid for all geometries, and the inclusion of the term $\rho_{s-A} \approx \Sigma$ leads therefore to an effective (positive) cosmological constant and the corresponding period of late-time accelerated expansion. The addition to this scenario of the (dust) matter term representing baryonic and dark matter fluids does not change this general dynamics. This example illustrates well how a simple model as the ECDM with minimal couplings between torsion and the matter fields can give rise to such remarkable solutions without the need for any inflaton, quantum gravity or dark energy, yielding in fact a unique classical solution that is non-singular, includes an abrupt early acceleration period, and a late-time acceleration period.

Cartan relations and the cosmological principle

Writing the effective matter Lagrangian as $\mathcal{L} = C + \mathcal{L}^{\rm M}_{\rm corr} + \mathcal{L}^{\rm D}_{\rm corr}$, where C is given by Eq. (5.113) and the other two terms by the expressions

$$\mathcal{L}^{\rm M}_{\rm corr} = \lambda\Big[2\left(f^2(t) + h^2(t)\right) A^a A_a + f(t)\epsilon^{abc}\tilde{F}_{bc}A_a - 2h(t)\tilde{F}_{0b}A^b\Big]\,, \tag{5.130}$$

and

$$\mathcal{L}^{\rm D}_{\rm corr} = \frac{3\hbar}{2} f(t)\bar{\psi}\gamma_0\gamma^5\psi\,, \tag{5.131}$$

and varying it with respect to $f(t)$ and $h(t)$ we find the Cartan equations in this case as

$$2f(t)\left(2\lambda\vec{A}^2 - \frac{3}{\kappa^2}\right) + \frac{3\hbar}{2}\bar{\psi}\gamma_0\gamma^5\psi + \lambda\tilde{F}_{bc}\epsilon^{abc}A_a = 0, \tag{5.132}$$

and

$$h(t)\left(4\lambda\vec{A}^2 - 2\lambda\tilde{F}_{0b}A^b - \frac{48}{\kappa^2}\right) = 0\,, \tag{5.133}$$

respectively, and we can set $h(t) = 0$. Since for homogeneous fields one has that $\tilde{F}_{ab} = 0$ then Eq. (5.132) allows us to solve for $f(t)$ as

$$f(t) = -\frac{3\kappa^2\hbar}{2}\bar{\psi}\gamma_0\gamma^5\psi\left(4\lambda\kappa^2\vec{A}^2 - 6\right)^{-1}\,. \tag{5.134}$$

If we now substitute this torsion function back into the cosmological equations for the Maxwell (5.80) and Dirac fields (5.108) and in the Friedmann equation (5.125), we arrive at the final system of equations for the matter fields $(A_\mu, \psi, \bar{\psi})$ and the scale factor a. By doing this one obtains non-linear equations for both Maxwell and Dirac fields with explicit non-minimal couplings between fermions and bosons. Some specific scenarios can be now considered:

i) For instance, in the ansatz of $A_\mu = (\phi, 0, 0, 0)$, Eq.(5.134) becomes

$$f(t) = \frac{\kappa^2 \hbar}{4} \bar{\psi}\gamma_0\gamma^5\psi \ , \tag{5.135}$$

therefore in terms of the conformal time this torsion function reads

$$f(\eta) = \frac{\kappa^2 \hbar}{4} (\bar{\chi}\gamma_0\gamma^5\chi) a^{-3}(\eta) \ , \tag{5.136}$$

which yields the (non-linear) Dirac equation (5.108)

$$i\hbar\gamma^0\chi' - a(\eta)m\chi + \frac{3\kappa^2\hbar^2}{8} a^{-2}(\eta) \left(\bar{\chi}\gamma_0\gamma^5\chi\right) \gamma_0\gamma^5\chi = 0 \ , \tag{5.137}$$

and the correction (5.126) to the Friedmann equations becomes

$$\rho^{\rm corr} = -\frac{3}{\kappa^2} f^2(t) - 9f(t)\breve{s}_0 = -\frac{21\kappa^2\hbar^2}{16} (\bar{\psi}\gamma_0\gamma^5\psi)\bar{\psi}\gamma_0\gamma^5\psi < 0 \ . \tag{5.138}$$

In this case the bosonic field does not contribute with extra terms to the dynamics, besides the usual radiation term. Moreover, assuming that $(\bar{\psi}\gamma_0\gamma^5\psi)^2 \sim n^2 \sim a^{-6}$, with n being the fermionic number density[3], this correction can be written as

$$\rho^{\rm corr} = -\frac{21\kappa^2\hbar^2}{16} \beta_s a^{-6}, \tag{5.139}$$

where β_s is a constant.

The cosmological solutions in this case are easy to obtain, and correspond to a non-singular bouncing behaviour due to the strong (repulsive) spin-spin effects induced by torsion in the very early-Universe. This non-singular behaviour is present in all possible spatial geometries $(k = -1, 0, 1)$ including a period of early acceleration followed by the usual deceleration expansion phase of Friedmann models (without dark energy/cosmological constant).

ii) On the other hand, if we set $\vec{A} \neq 0$, then Eq.(5.134) replaced in (5.126) yields the correction to the Friedmann equations

$$\rho^{\rm corr} = -\frac{63\kappa^2\hbar^2}{16} (\bar{\psi}\gamma_0\gamma^5\psi)^2 \left(3 - 2\lambda\kappa^2\vec{A}^2\right)^{-1} < 0 \ , \tag{5.140}$$

and using the same assumption as in the previous example it becomes

$$\rho^{\rm corr} = -\frac{63\kappa^2\hbar^2}{16} \left(3 - 2\lambda\kappa^2\vec{A}^2\right)^{-1} \beta_s a^{-6} \ , \tag{5.141}$$

[3]More rigorously, one should consider the corresponding expectation value $\langle(\bar{\psi}\gamma_0\gamma^5\psi)^2\rangle \sim a^{-6}$, as we did before.

where we recall that $\vec{A}^2 \equiv A^j A_j < 0$. The corresponding Dirac-Hehl-Datta equation (5.108) becomes

$$i\hbar\gamma^0\chi' - a(\eta)m\chi = -\frac{3\kappa^2\hbar^2}{8\left(4\lambda\kappa^2\vec{A}^2 - 6\right)} \times a^{-2}(\eta)(\bar{\chi}\gamma_0\gamma^5\chi)\gamma_0\gamma^5\chi \ . \tag{5.142}$$

Finding solutions in this case is less trivial as it involves a non-minimal coupling between the fermionic spin-spin energy interaction and the bosonic field. Due to this, it is not straightforward to derive the evolution of the bosonic potential from the Maxwell equations (5.80), using Eq. (5.134), which are non-linear and coupled to Friedmann equations. From the general expression for the pressure functions obtained above and neglecting anisotropic stresses one can prove that this "fluid" component obeys

$$p^{\text{corr}} \approx -\frac{45\kappa^2\hbar^2}{16}\frac{(\bar{\psi}\gamma_0\gamma^5\psi)^2}{3 - 2\lambda\kappa^2\vec{A}^2} \ , \tag{5.143}$$

so therefore the equation of state of this fluid is given by $w^{\text{corr}} \equiv p^{\text{corr}}/\rho^{\text{corr}} = 45/63$.

In principle, this can be used in the corresponding (fluid component) conservation (5.127), given in this case by

$$\frac{d\rho^{\text{corr}}}{da} + \frac{3}{a}\left(w^{\text{corr}}(A) + 1\right)\rho^{\text{corr}} = 0 \ , \tag{5.144}$$

to obtain $\rho^{\text{corr}} \sim a^{-3(w^{\text{corr}}+1)} \approx a^{-5,14}$. Nevertheless, one should take into account that we have neglected anisotropic (off-diagonal) components of the energy-momentum tensors describing the torsion corrections. One can also take a more general approach, considering that $\rho^{\text{corr}} < 0$ and assuming that its relevant effect on the dynamics can be qualitatively modeled by a power-law

$$\rho^{\text{corr}} \sim \xi + \varsigma a^{-n} + \mu a^{-m} \ , \tag{5.145}$$

with the exponents m, n being real numbers and the constants $\xi, \varsigma < 0$ (to fulfil the condition $\rho^{\text{corr}} < 0$ at all times), then the resulting dynamics is quite rich. Indeed, as long as one of the exponents is larger than the corresponding exponent of the usual radiation fluid, i.e, $m > 4$ or $n > 4$, we have a non-singular early cosmology. Moreover, if the other term survives in the late-time dynamics, decaying slower than a^{-2}, then it will give rise to a future bounce, where $H(a) = 0$, and a similar cyclic behaviour as that found in Ref. [5] is expected, as we saw in this chapter before.

Let us finally point out that, for both of the *ansatze* for the bosonic four-vector sector above, the resulting non-linear Dirac equations are of the Hehl-Datta-Heisenberg type with cubic terms that change sign upon a charge conjugation transformation. Accordingly, the cosmological dynamics for fermions and anti-fermions is different, with interesting consequences to the matter/anti-matter asymmetries.

Ricci scalar and the cosmological principle

Taking into account the general expression (3.29) for the Ricci scalar of a RC spacetime geometry and using the expression

$$\tilde{\nabla}_\lambda K^\lambda = \partial_0 K^0 + \frac{1}{\sqrt{-g}}\partial_0(\sqrt{-g})K^0 = \dot{K^0} + 3H(t)K^0 \ , \tag{5.146}$$

where in the last equality we have implemented a FLRW geometry, we arrive at

$$R = \tilde{R} + 8\left[\dot{h}(t) + 3H(t)h(t)\right] - 6f^2(t) - 48h^2(t) \ , \tag{5.147}$$

which for $h(t) = 0$ reads

$$R = \tilde{R} - 6f^2(t) \ . \tag{5.148}$$

This way, if one considers that torsion scales as $T \sim a^{-3}$ (which is a reasonable assumption in the case of fermionic torsion, i.e, when $T \sim \breve{s}$), then we get

$$R = \tilde{R} - \frac{\alpha}{a^6} \ , \tag{5.149}$$

where α is a constant. Therefore, in the asymptotic limit $a \to \infty$ we find that both scalars coincide, while for a certain a_b corresponding to the minimum value of the scale factor at the (early) cosmological bounce, a non-singular description of curvature is obtained, where

$$R \to \tilde{R}(a_b) - \frac{\alpha}{a_b^6}, \qquad (a \to a_b) \ , \tag{5.150}$$

is finite. Let us point out that only for $\rho^{\rm corr} < 0$ it is possible to get an early bounce (where $H(a_b) = 0$), that is to say, to prevent a singular behaviour. It should be possible to prove in this type of models the geodesic completeness of the full set of solutions, which in principle suggests the interpretation of a previous contracting Universe phase.

5.1.5 Discussion and summary

Einstein-Cartan-Dirac-Maxwell cosmology. In this section we have studied the ECDM model implementing the $U(1)$-symmetry breaking and discussed its cosmological applications. The theoretical foundations of this model rely on fermionic (spinors) and bosonic (vector) fields minimally coupled to torsion of a Riemann-Cartan spacetime geometry. In this framework one is led to the Cartan equations relating the torsion tensor to the fundamental matter fields via the total matter spin tensor. Substituting torsion as a function of the matter field variables one obtains generalized Einstein-like, Dirac-like and Maxwell-like equations. This induces non-linear Dirac and electromagnetic dynamics with self-interactions (fermion-fermion, and boson-boson) and non-minimal fermion-boson interactions, and the resulting energy-momentum contributions for the gravitational equations.

Regarding cosmology, the ECDM model presented here gives rise to generalized Friedman dynamics coupled to bosonic and fermionic fields. The model is simplified if one takes an effective fluid description without needing to solve for the (generalized) Hehl-Datta-Dirac equation on a FLRW background. The resulting model predicts non-singular cosmologies with a bounce, similarly as in the original ECSK theory. In the $U(1)$-broken phase and neglecting bosonic self-interactions, there is an effective fluid component with energy density scaling as $\rho^{\rm corr} \sim \rho^s + \rho^{s-A}$, where $\rho^s \sim \kappa^2 \breve{s}^2 \sim n^2 \sim a^{-6}$ is a (negative) contribution from the spin-spin self interaction, and $\rho^{s-A} \sim \kappa^2 \breve{s}^2 f(A)$ comes from the non-minimal interactions (induced by torsion) between fermionic and bosonic fields. The latter can also introduce a negative contribution to the energy density (depending on the $f(A)$ contribution, and therefore to the evolution of the bosonic 4-vector $A(a)$). We considered two different *ansatze* for the four-potential, namely $A_\mu = (\phi, 0, 0, 0)$ and $A_\mu = (0, \vec{A})$. A typical example is $f(A) \sim \lambda \kappa^2 A^2$, in the approximation where torsion is exclusively due to the spin tensor of fermions (Cases III and IV), although $\lambda^2 \kappa^4 A^4$ terms can also be present in the case where the bosonic spin tensor also contributes to torsion (Cases I and II). In all cases, we get a non-singular early Universe description in terms of a minimum value for the scale factor at which $H(a) = 0$ for all possible spatial curvature values $k = -1, 0, 1$, due to the (negative) contribution from the spin-spin interaction. Moreover, these solutions show an accelerated expansion period after the bounce until $H(a)$ reaches a maximum value, followed by a decelerated expansion.

Regarding the effects of the non-minimal interactions induced by torsion, in the variation of Case IV we get $\rho^{s-A} \simeq \text{constant} < 0$, which has no significant effect on the early dynamics, but it can give rise to a halt of the decelerated expansion period at some future value of the scale factor, that is, $H(a_{max}) = 0$. In fact, most cases present this late-time behaviour for non-negligible values of λ. By considering the two branches of the family of solutions for the Hubble parameter, $H(a) = \pm\sqrt{\kappa^2/3(\rho + \rho^{\text{corr}}) - k/a^2}$, and the physical requirement of matter obeying the weak or dominant energy conditions, one is naturally led to interpret such future behaviour as a bounce (continuously) bridging a decelerated expansion phase to an accelerated contraction phase. Then, following this negative solution of the square root above, the accelerated contraction also reaches a maximum (absolute) value when the Hubble parameter reaches a (negative) minimum and the contraction progresses in a decelerated manner until it reaches another minimum of the scale factor. At that instant, again the Hubble parameter $H(a)$ vanishes and the solution transits from the negative root to the positive root branch, in accordance with the physical energy (weak) conditions. This is another bounce, linking a decelerated contraction phase to an accelerated expansion and the cycle repeats over and over (see Fig. 5.2). This cyclic behaviour depends on both the existence of the strong spin-spin (negative energy) effect and on the (negative) energy contribution from the non-minimal couplings, which only becomes relevant in the late-time decelerated expansion phase. The strength of such a term depends on the single free parameter of the model, λ. The cyclic Universes are more intuitive for models where $f(A) \sim a^n$ with $n \geqslant 6$ (but are not exclusive to these), with $\rho^{\text{corr}} < 0$.

One of the solutions found (Case IV) is particularly interesting as it is a non-singular cosmology with an early acceleration period followed by a decelerated expansion and finally by a late-time accelerated epoch (see Fig. 5.2). In general, all these late-time effects seem surprising, since usually one takes the torsion effects on the metric to be only significant at or above Cartan's density 10^{54}g cm^{-3}. Although this is true for the (axial-axial) four-fermion spin-spin self-interaction effects induced by torsion, the effects due to the non-minimal couplings in the matter fields induced by torsion can be relevant for late-time cosmology. The emergence in the same solution of bouncing early-time behaviour, an early period of accelerated expansion, a deceleration phase, and a late-time period of acceleration, is a fantastic example on the richness of the cosmological dynamics of an extremely simple theory as the Einstein-Cartan theory with matter fields minimally coupled to the RC spacetime geometry.

The ECDM models predicts a negative cosmological constant in the variation of Case IV with an energy density scale set by $\lambda\kappa^4\hbar^2 n_{\text{ref}}^2 A_{\text{ref}}^2$. Such a constant is responsible for a cyclic cosmological behaviour as described above. On the other hand, if one takes a semi-quantum approach in the quark-gluon plasma and consider the presence of quark condensates in vacuum as predicted by QCD, i.e., the non-vanishing vacuum expectation value of $\langle 0|\bar{\psi}\psi|0\rangle$, then the model predicts the existence of an effective cosmological constant and a dynamical dark energy contribution. The first term comes from the spin-spin energy interaction of vacuum (of fermionic quark fields) which enters the ECSK equations, and the second term is due to the (non-minimal) interaction between this vacuum term and the bosonic fields, taken here as classical fields. These results extend those of standard ECSK theory [98], by adding a dynamical dark energy term which depends on the four-potential, and which cannot be neglected during the $U(1)$-broken symmetry phase induced by torsion. As long as the minimal coupling between torsion and the bosonic four-potential takes place, the dynamical dark energy term will be there. In other words, in the regimes in which the $U(1)$-breaking term in the bosonic Lagrangian (4.79) is non-negligible the four-potential will evolve with the scale factor as derived from the corresponding Maxwell-like equations, or from the generalized continuity equations.

It is pertinent to ask when does the torsion ceases to be important and becomes negligible. The answer depends on the case: for instance, in the usual ECSK theory with torsion coupled only to fermions one gets $T_{\alpha\beta\gamma} \sim \kappa^2 s^D_{\alpha\beta\gamma} \sim a^{-3}$ and the metric torsion effects scale with $\sim a^{-6}$, leading to a torsion *era* in the very early Universe. Now, when torsion couples also to vector bosonic fields, but it is only sourced by fermion spin density, then the $U(1)$-symmetry breaking Lagrangian term in (4.79) can in principle decrease until it becomes negligible or not (see Table 5.2). In the most general case, when torsion not only couples to the bosonic sector but it is also a result of the contribution from the total spin density including the spin density of bosons, the situation is similar to the case where torsion is due to fermionic spin densities, but there are situations in which a non-vanishing constant torsion background is predicted (variation of Case I). This topic requires further research since it needs to be carefully addressed in a quantum field theory context within a RC spacetime and strong-gravity regime.

One should also mention that the energy-momentum tensor terms derived from the non-minimal couplings in the matter Lagrangian could give rise to an effective fluid description which introduces anisotropic stresses. This should affect the dynamics via the Raychaudhuri equation and/or the conservation equation. We did not take into account such effects in the present work since we used the assumptions $\bar{A^i} A_j \sim A^2 \delta^i_j$ and $\bar{\breve{s}^i} \breve{s}_j \sim \breve{s}^2 \delta^i_j$ to simplify the analysis. Again, this is reminiscent of the studies of the Weyssenhof fluid, which is not fully compatible with the cosmological principle, but can still be considered in the context of FLRW models by invoking macroscopic averaging arguments [21]. Similarly, by exploring this idea, our model calls for a more self-consistent cosmological approach, for instance within Bianchi spacetimes. Alternatively, if one maintains the FLRW models at the background level, the perturbations should incorporate the anisotropic stresses, which might be important for the generation of cosmological GWs induced by spin density fluctuations (with non-zero, time varying quadrupole moment) in the early universe. One should also expect the production of GWs from the transitions between the primordial phases: from the $U(1)$-broken phase to the $U(1)$-restored phase (in particular, if this symmetry breaking is spontaneously induced rather than explicit), and possibly also from the usual torsion-dominated phase to the radiation phase. These transitions can contribute to a stochastic GW background of cosmological origin, with possible imprints from the physics beyond the standard model.

ECDM model and the cosmological principle. We have addressed the implementation of the cosmological principle in the ECDM model. In this gravity theory one finds generalized Dirac-like and Maxwell-like equations coupled to the background torsion, and by using the Cartan equations relating the torsion tensor to the matter fields, one explicitly obtains both self-interactions (fermion-fermion and boson-boson) and non-minimal (fermion-boson) interactions. The correct application of the cosmological principle to theories with torsion and, in particular, to the ECDM model, requires a careful and critical analysis.

The cosmological principle is motivated by the fact that in observational cosmology we observe a high degree of spatial isotropy both in the maps of the CMB radiation and in the observations of the distributions of clusters of galaxies at large scales. Then, assuming the laws of physics to be the same for any observer and that we do not constitute a preferred class of observers, we extrapolate these observations to the idea of the cosmological principle (spatial homogeneity and isotropy), valid for all comoving observers. In GR this gives rise to FLRW metrics representing the metric structure of the possible spatially maximally symmetric spacetimes and the perfect fluid form for the energy-momentum tensor.

On the other hand, since matter is known to be more accurately described by fundamental field theories, then typically it is quite reasonable to assume (spatially) homogeneous and isotropic vector, tensor or spinor fields in a cosmological context upon some appropriate average procedure. However, the imposition of the maximally symmetric spaces directly on the geometrical side is a much stronger requirement since it is imposing global spatial symmetries into the metric tensor that characterizes the local geometry (metric structure). Nevertheless, and although it is also reasonable to take the FLRW metric as a zeroth-order approximation and consider the metric fluctuations only at the level of perturbation theory, it can be argued that the imposition of the cosmological principle to the torsion tensor right from the start has some advantages and some limitations. This follows from a careful analysis of the paradigm changes that are required to consistently interpret gauge theories of gravity with non-Riemann geometries. In particular, the EC theory is a simple example of a PGTG with a RC spacetime and, as such, due to the richer RC geometry several consequences are unavoidable, given the nature of the physical fields representing matter, and also as far as the connection between these matter fields and geometrical structures (torsion) is concerned. In particular, the approximation of a point-like particle is no longer valid, because its description from a multipolar expansion approach of matter fluids is not compatible with the generalized Bianchi identities in RC geometry [99]. Therefore, non-Riemann geometries seem to suggest strong limitations in the classical picture of matter and are suitable to provide a more consistent description of microscopic gravity [21] at classical and semi-classical regimes. Indeed, instead of considering geodesic equations (valid for point-like particles) to study the effects of non-Riemannian geometries [69], one should consider instead the fermionic field equations (generalized Dirac) and the bosonic field equations (generalized Maxwell or Proca-like), propagating in such geometries, or the equations of motion of extended test objects having the appropriate (Noether) current properties [153]. This supports the approach of taking Cartan's equations as valid also in the microscopic realm.

In view of the discussion above, for cosmological applications it seems more reasonable to impose the condition that the torsion tensor should be given as a function of the fundamental matter fields (via the spin tensor in Cartan equations) and take these fields to depend on time (to respect spatial homogeneity). Then a macroscopic averaging procedure should guarantee the isotropy requirement, as explicitly implemented in the first subsections of this chapter, by considering random fermionic spin distributions and random bosonic three-vector fields. Due to these considerations, we favour the cosmological models with Cartan's equations as microscopically meaningful, for example using fundamental fermionic spinor fields instead of a Weyssenhof fluid. Moreover, we should take the imposition of the cosmological principle to the torsion tensor as a zeroth-order approximation, having its limitations, that can be improved via perturbation theory or, more consistently, in fully non-homogeneous and anisotropic models. In any case, in the present chapter we obtained the restrictions from the implementation of the cosmological principle to the torsion tensor components. We considered fermionic and bosonic fields propagating in the cosmological torsion background. These fields, under a suitable average procedure, contribute to the usual perfect fluid components in the cosmological Friedmann equations and also "feel" the torsion which can be considered as an extra (external) homogeneous and isotropic tensor field that enriches the background spacetime geometry. We derived the generalized Maxwell-like, Dirac-like and the Friedmann equations in this scenario. While the matter field equations are linear at this level of the analysis, when substituting the torsion function $f(t)$ as a function of the matter fields, using Cartan's equations (the latter derived from the effective total Lagrangian via variation with respect to the torsion functions), one arrives at a non-linear dynamics with non-minimal couplings in the matter sectors. From these, rich non-singular cosmological scenarios emerge which deserve more investigation, particularly with regards to small perturbations and the corresponding imprints in the CMB or in gravitational wave cosmological backgrounds from phase transitions in the early Universe. In both the ECSK and the ECDM models the dominating torsion effects in the early Universe prevent an initial singularity and a simple interpretation of the solutions is that of a bounce from a previous contracting phase. At the bounce the curvature scalar is finite and one can respect geodesic completeness.

A rigorous implementation of the cosmological principle in theories with torsion introduces some subtle modifications in their cosmological dynamics, which has a relevant impact for their predictions, such as the appearance of non-minimal couplings, non-singular cosmologies, baryon-antibaryon asymmetries, etc. Taking the viewpoint that Cartan's equations should be approximately valid at a microscopic level our model would call for a more self-consistent cosmological approach, for instance within Bianchi spacetimes. More work is thus necessary upon these theories to investigate their cosmological viability. To conclude, in our view there are good motivations to keep with the analysis of gravitational models where non-Riemannian geometries, fermionic spin densities, and phase transitions become important, which can be tested with astrophysical and cosmological GW observations in the near future.

Chapter 6

Astrophysical applications

In this chapter we address some possible astrophysical implications from theories of gravity in a RC spacetime. This includes both effects due to curvature and torsion. We briefly address some possible connections to GW astronomy. More specific GW applications will be explored in chapter 7 and we emphasize here also that several potential astrophysical application related to the interaction between the gravitational torsion and fermionic systems were also briefly discussed in chapter 4, section 4.1.

In section 6.1 we start by exploring some physical consequences of electrodynamics in curved spacetime. In general, new electromagnetic couplings and related phenomena are induced by the spacetime curvature. The applications of astrophysical interest considered here correspond essentially to the following geometries: the Schwarzschild spacetime and the spacetime around a rotating spherical mass in the weak field and slow rotation regime. In the latter, we use the Parametrised Post-Newtonian (PPN) formalism. We also explore the hypothesis that the electric and magnetic properties of vacuum reflect the spacetime isometries (see chapter 2). Therefore, the permittivity and permeability tensors should not be considered homogeneous and isotropic *a priori*. For spherical geometries we consider the effect of relaxing the homogeneity assumption in the constitutive relations between the fields and excitations. This affects the generalized Gauss and Maxwell-Ampére laws where the electric permittivity and magnetic permeability in vacuum depend on the radial coordinate in accordance with the local isometries of space. For the axially symmetric geometries we relax both the assumptions of homogeneity and isotropy. We explore simple solutions and discuss the physical implications related to different phenomena such as: the decay of electromagnetic fields in the presence of gravity, magnetic terms in Gauss law due to the gravitomagnetism of the spacetime around rotating objects, a frame-dragging effect on electric fields and the possibility of a spatial (radial) variability of the velocity of light in vacuum around spherical astrophysical objects for strong gravitational fields. Then, in the last parts we generalize this approach by briefly considering possible astrophysical implications from electrodynamics coupled to RC geometry due to astrophysical sources. This chapter contains material inspired by the work in [2].

6.1 The astrophysics of electromagnetism in RC spacetime

Both gravitational and electromagnetic fields are fundamental at astrophysical scales governing the dynamics and driving the complex processes of structure formation (stars, galaxies, clusters, etc) as well as creating the conditions for extreme thermodynamical states of matter and nuclear reactions. Many highly energetic phenomena in astrophysics involve very strong gravity and electromagnetic fields interacting with very hot relativistic plasmas (see [154]). Therefore, the study of the coupling between gravity and electromagnetic fields is fundamental to a deeper understanding of many observed phenomena in high energy astrophysics. It is also relevant for the processes behind the formation and evolution of Active Galactic Nuclei (AGN) and to the physics of compact objects such as pulsars and black holes. For example, it is well know that gravitomagnetic fields play an important role in models for the collimation of astrophysical jets along a well established axis [155, 156, 157, 158, 159, 160, 161], but such models should include the coupling of electromagnetic fields with gravity. Such couplings, in particular, with the gravitomagnetic term, might be useful to deepen the understanding of astrophysical jets in radio (active) galaxies. Magnetic fields are particularly relevant, since they pervade the physical Cosmos at planetary, stellar, galactic and extragalactic scales with different magnitudes (see [162] and references therein). These fields also play a vital role in the complex interaction between protostars and the environment of the hosting molecular giant clouds, which drives the processes of star formation and constitute a fundamental ingredient for the understanding of (phenomenological) stellar formation rates in galaxies [163].

The applications in the following subsections can have some astrophysical relevance but the main purpose here is to illustrate the effect of spacetime geometry in electric and magnetic fields motivated by relatively realistic astrophysical scenarios. An appropriate treatment of magnetic fields in relativistic astrophysical situations (with strong gravity) would require the equations governing fluids and fields of magnetohydrodynamics in the background of an appropriate spacetime metric. Furthermore, for cases with very strong magnetic fields, which can contribute to the gravitational field, the coupled Einstein-Maxwell equations (or its generalizations within qPGTG for e.g.) need also to be included in the analysis. Such procedures require appropriate numerical methods and simulations.

6.1.1 Curvature effects from GR and extensions

To search for observable effects of spacetime curvature (gravity) in electrodynamics we explore the field equations in a general pseudo-Riemann background spacetime [2]. In this framework, electrostatics and magnetostatics are no longer separated, instead they become coupled due to the presence of curved geometry. Furthermore, new terms in the wave equations can be derived [2].

It can be easily shown that the coupling to spacetime geometry, in particular to the gravitomagnetic (time-space off-diagonal) terms, gives magnetic corrections to Gauss' law [2]. Therefore, the Gauss law in the background geometry of a rotating spherical mass necessarily includes magnetic terms, becoming electromagnetic. This coupling gives rise to the possibility of having even static magnetic fields as sources of electric fields. Accordingly, in astrophysical scenarios such as neutron stars with strong gravity, an induced electric field might arise due to the coupling between the magnetic field and the gravitomagnetism of the surrounding spacetime. This is an illustration of a gravitomagnetic effect affecting electromagnetism directly through the very nature of the field equations in curved spacetime. The presence of magnetic fields in Gauss' law disappear for vanishing off-diagonal time-space metric components, but in general there are extra electromagnetic couplings in Maxwell's equations due to the gravitomagnetism of spacetime. For a diagonal metric such couplings are no longer present, but other gravity effects appear. In general the electric field in vacuum will necessarily be non-uniform, i.e., spacetime curvature will introduce a spatial variability in the electric field.

As in the case of Gauss' law, new effects emerge in the Maxwell-Ampère law due to the curvature of spacetime geometry [2]. For vanishing currents the presence of an electric field can be a source of magnetic fields, with an extra contribution to Maxwell's displacement current induced by spacetime dynamics. These functions vanish for stationary spacetimes but might have an important contribution for strongly varying gravitational waves (high frequencies) [2]. While in Gauss' law the electromagnetic coupling disappears for a diagonal metric, in the Maxwell-Ampère law this coupling is always present. In particular, non-stationary spacetime will necessarily induce a time varying electric field via the Gauss law. Accordingly, gravitational waves are expected to produce a direct effect in magnetic fields as well.

The physics of electromagnetic waves in the presence of a background gravitational field has been studied in the past, but it remains an extremely relevant topic for relativistic astrophysics. It is natural to expect an effect in the polarization of electromagnetic waves induced by the curvature of (pseudo) Riemann spacetimes (see also [48]). One can show that for stationary geometries, the gauge invariant wave equations have no electromagnetic couplings [2]. Nevertheless, a coupling between the dynamics of the different field components appears, which suggests polarization effects induced by spacetime geometry. The theory also seems to suggest longitudinal modes induced by curved spacetime geometry. These longitudinal modes in vector (a^k) wave equations appear whenever $\partial_k a^k \neq 0$, and in fact, in usual electromagnetism such terms are absent because the fields in vacuum have zero divergence. In curved spacetime, we have $\partial_k E^k \neq 0$ and this manifests in extra terms containing the first derivatives of the electric field in the wave equation [2], even for Cartesian coordinates. These longitudinal modes seem to be a prediction of electromagnetic wave dynamics in curved spacetime.

The inhomogeneous equations, the Coulomb-Gauss and Oersted-Ampère laws and the wave equations, constitute the basis for the physical applications we explore in the following subsections. The equations will be shown explicitly and some physical consequences explored for two different geometrical backgrounds: 1) the Schwarzschild metric and 2) the geometry outside a stationary spherical gravitational mass with slow rotation in the weak field limit. The second case will be analysed using the Parametrized Post-Newtonian (PPN) formalism, first developed for solar system tests, which allows to parameterize different theories of gravity (see [51, 52, 164, 165, 166, 167, 168, 169, 170, 171]).

Maxwell fields in curved spacetime: brief review

The set of Maxwell's equations in curved spacetime (pseudo-Riemann geometry) are the well known expressions[1]

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})F^{\mu\nu} = \mu_0 j^\nu, \quad \partial_{[\alpha}F_{\beta\gamma]} = 0. \tag{6.1}$$

[1]Here we are working in curved (pseudo-Riemann) spacetime geometry, therefore we will be omitting the tilde in the equations.

where we have used the general expression for the divergence of anti-symmetric tensors in pseudo-Riemann geometry, $\nabla_\mu \Theta^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\Theta^{\mu\nu}\right)$. With the definitions

$$F_{0k} \equiv \frac{E_k}{c}, \qquad F_{jk} \equiv -B_{jk} = -\epsilon_{ijk}B^i, \tag{6.2}$$

the homogeneous equations are the usual Faraday and magnetic Gauss laws

$$\partial_t B^i + \epsilon^{ijk}\partial_j E_k = 0, \qquad \partial_j B^j = 0, \tag{6.3}$$

while the inhomogeneous equations can be separated into the generalized Coulomb-Gauss and Oersted-Ampère laws. These are, respectively

$$\alpha^{kj}\partial_k E_j + \gamma^j E_j - cg^{m\mu}g^{n0}\epsilon_{kmn}\partial_\mu B^k - c\sigma^{mn0}\epsilon_{kmn}B^k = \frac{\rho}{\varepsilon_0}, \tag{6.4}$$

and

$$\frac{1}{c}\alpha^{\mu ji}\partial_\mu E_j + \frac{1}{c}E_j\xi^{ji} - \epsilon_{kmn}\partial_\mu B^k g^{m\mu}g^{ni} - B^k\epsilon_{kmn}\sigma^{mni} = \mu_0 j^i, \tag{6.5}$$

with the following geometric coefficients

$$\alpha^{kj} \equiv \left(g^{0k}g^{j0} - g^{jk}g^{00}\right), \quad \alpha^{\mu ji} \equiv \left(g^{0\mu}g^{ji} - g^{j\mu}g^{0i}\right),$$

$$\gamma^j \equiv \left[\partial_k\left(g^{0k}g^{j0} - g^{jk}g^{00}\right) + \frac{1}{\sqrt{-g}}\partial_k(\sqrt{-g})\left(g^{0k}g^{j0} - g^{jk}g^{00}\right)\right],$$

$$\sigma^{mn\beta} \equiv \left[\partial_\mu(g^{m\mu}g^{n\beta}) + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})(g^{m\mu}g^{n\beta})\right],$$

$$\xi^{ji} \equiv \left[\partial_\mu\left(g^{0\mu}g^{ji} - g^{j\mu}g^{0i}\right) + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g})\left(g^{0\mu}g^{ji} - g^{j\mu}g^{0i}\right)\right].$$

We will refer to these as simply the Gauss and Maxwell-Ampère laws. One sees clearly, that new electromagnetic phenomena is expected due to the presence of extra electromagnetic couplings induced by spacetime curvature. In particular, the magnetic terms in the Gauss law are only present for non-vanishing off-diagonal time-space components g^{0j}, which in linearised gravity correspond to the gravitomagnetic potentials. These terms are typical of axially symmetric geometries (see [172, 173]). For diagonal metrics, the inhomogeneous equations, i.e., the Gauss and Maxwell-Ampère laws, can be recast into the familiar forms

$$\partial_k \bar{E}^k = \frac{\varrho}{\varepsilon_0}, \tag{6.6}$$

$$\epsilon_{ijk}\partial_j \bar{B}^{iijjk} = \mu_0(\jmath^i + \jmath^i_D), \tag{6.7}$$

where

$$\bar{E}^j \equiv -g^{jj}g^{00}\sqrt{-g}E_j, \qquad \varrho \equiv \sqrt{-g}\rho, \tag{6.8}$$

and

$$\bar{B}^{iijjk} \equiv g^{ii}g^{jj}\sqrt{-g}B^k, \qquad \jmath^i \equiv \sqrt{-g}j^i, \quad \jmath^i_D \equiv -\varepsilon_0\sqrt{-g}\left(g^{00}g^{ii}\partial_t E_i + c^2 E_i\xi^{ii}\right). \tag{6.9}$$

These equations are special cases of the more general expression which can be obtained from (6.1), namely $\partial_\mu \bar{F}^{\mu\nu} = \jmath^\nu$, where $\bar{F}^{\mu\nu} \equiv \sqrt{-g} F^{\mu\nu}$ and $\jmath^\nu \equiv \sqrt{-g} j^\nu$. New electromagnetic effects induced by spacetime geometry include an inevitable spatial variability (non-uniformity) of electric fields whenever we have non-vanishing geometric functions γ^k, electromagnetic oscillations (therefore waves) induced by gravitational radiation and also additional electric contributions to Maxwell's displacement current in the generalized Maxwell-Ampère law. Notice that the functions ξ^{ii} vanish for stationary spacetimes but might have an important contribution for strongly varying gravitational waves (high frequencies), since they depend on the time derivatives of the metric. Moreover, besides these predictions, as previously said, for axially symmetric spacetimes gravitomagnetic effects induce magnetic contributions to the Gauss law, with even static magnetic fields as possible sources of electric fields. These are physical, observable effects of spacetime geometry in electromagnetic fields expressed in terms of the extended Gauss and Maxwell-Ampère laws which help the comparison with the usual inhomogeneous equations in Minkowski spacetime, making clearer the physical interpretations of such effects.

Finally, we review the field equations in terms of the electromagnetic 4-potential which in vacuum are also useful for the issue of electromagnetic waves. From (6.1), we get

$$\nabla_\mu \nabla^\mu A^\nu - g^{\lambda\nu} R_{\varepsilon\lambda} A^\varepsilon - \nabla^\nu \left(\nabla_\mu A^\mu\right) = \mu_0 j^\nu, \tag{6.10}$$

where $R_{\varepsilon\lambda}$ is the Ricci tensor. Using the expression for the (generalized) Laplacian in pseudo-Riemann manifolds, $\nabla_\mu \nabla^\mu \psi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\lambda} \partial_\lambda \psi\right)$, and considering vacuum we arrive at

$$\partial_\mu \partial^\mu A^\nu + \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\lambda}\right) \partial_\lambda A^\nu - g^{\lambda\nu} R_{\varepsilon\lambda} A^\varepsilon - \nabla^\nu \left(\nabla_\mu A^\mu\right) = 0, \tag{6.11}$$

which is a generalized Proca-like equation with variable (spacetime dependent) effective mass induced by the curved geometry. The second term in Eq. (6.11) can also be written in terms of the Levi-Civita connection, through the formula $g^{\alpha\beta} \Gamma^\lambda_{\alpha\beta} = -\frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\lambda}\right)$, valid in pseudo-Riemann geometry. In usual Proca-like wave equations there is no such a term dependent on the first derivative of the (massive) vector field. Similar terms appear for wave phenomena with longitudinal modes. For a diagonal metric in vacuum, we get

$$\partial_\mu \partial^\mu A^\nu + \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\mu}\right) \partial_\mu A^\nu - g^{\nu\nu} R_{\varepsilon\nu} A^\varepsilon - \nabla^\nu \left(\nabla_\mu A^\mu\right) = 0, \tag{6.12}$$

with no contraction assumed in ν. In general, and contrary to electromagnetism in Minkowski spacetime, the equations for the components of the electromagnetic 4-potential are coupled even in the (generalized) Lorenz gauge ($\nabla_\mu A^\mu = 0$). On the other hand, for Ricci-flat spacetimes, the term containing the Ricci tensor vanishes. Naturally, the vacuum solutions of GR are examples of such cases. New electromagnetic phenomena are expected to be measurable, for gravitational fields where the geometry dependent terms in Eq. (6.11) are significant.

The spherically symmetric spacetime geometry

We consider the spherically symmetric spacetime around a spherical mass M, given by the Schwarzschild solution of Einstein's equations

$$ds^2 = \left[1 - \Psi(r)\right] c^2 dt^2 - \left[1 - \Psi(r)\right]^{-1} dr^2 - r^2 \left(d\theta^2 + \sin\theta^2 d\varphi^2\right), \tag{6.13}$$

where $\Psi(r) \equiv 2GM/c^2r$. This is the metric representing the geometry outside a (non-rotating) spherical mass due to a star for example, or outside a spherical (non-rotating) blackhole. When we consider the above metric in the next applications we assume that $r > 2GM/c^2$. We recall that the coordinate r in general has no direct relation to the physical (proper) distance. Rather, it was chosen such that given the line element $dl^2 = f(t, r')d\Omega^2$ (where $d\Omega^2 = d\theta^2 + \sin\theta^2 d\varphi^2$) for the 2-spheres of constant t and r', under the appropriate coordinate change $r' \longrightarrow r$ (such that $f(t, r) = r^2$), the perimeter and area of the 2-spheres with constant t and r, are given by the expressions $2\pi r$ and πr^2, respectively [165].

Inside stars with radius R_*, imposing the continuity for the metric functions at $r = R_*$, we get

$$ds^2 = e^{2\Phi(r)}c^2dt^2 - [1 - \psi(r)]^{-1} dr^2 - r^2 \left(d\theta^2 + \sin\theta^2 d\varphi^2\right), \tag{6.14}$$

where $\psi(r) \equiv 2Gm(r)/c^2r$, $m(R_*) \equiv M$ and the surface of the star is defined such that $p(R_*) = 0$, (p is the pressure inside the star). The potential $\Phi(r)$ is obtained from the r component of the energy-momentum conservation laws $\nabla_\mu T^{\mu\nu} = 0$, which gives [52, 165]

$$(\rho_m c^2 + p)\frac{d\Phi}{dr} = -\frac{dp}{dr}, \tag{6.15}$$

where ρ_m is the mass density. To solve this equation one needs the equation of state describing the thermodynamical properties of the interior of the star, as well as the other two remaining differential equations describing the inner structure of spherical relativistic stars, which are derived from Einstein's equations. These correspond to the equations for $\rho_m(r)$ and $m(r)$. The coordinate pathology happening for $r = 2GM/c^2$ in the black hole case, has no similar problem here because a careful analysis of the interior solution for ordinary stars shows that $r > 2Gm(r)/c^2$ always [165]. A simple pedagogical model which is not realistic (in fact it predicts an infinite speed of sound inside the star) is that of a star with constant density. Other more realistic well known exact solution is that of Buchdahl (1981) [165], which assumes an equation of state of the form $\rho_m/c^2 = 12\sqrt{\bar{p}p} - 5p$, where $\bar{p}$ is an arbitrary constant, which can be made causal (with a speed of sound lower than that of light) and which for low pressures reduces to the equation of state of a $n = 1$ polytrope in Newtonian theory of stellar structure [165].

Beyond ordinary stars, some old neutron stars can have a negligible rotation and have an approximately spherical gravitational metric field both inside and outside the matter. Nevertheless, the theoretical research on the equation of state for neutron stars is more delicate and it still has many open questions due to our lack of understanding of the properties of matter at supranuclear densities characteristic of the inner cores of such compact objects. We hope to clarify many of the physical issues involved with the advent of precision gravitational wave astronomy. For simplicity, we will not consider interior solutions.

We now proceed with the analysis of electrodynamics in the background of the spherical geometry in Eq. (6.13). Let us start by observing that in general, the first term in Maxwell's inhomogeneous equations in Eq. (6.1) for any fixed ν, corresponds to the usual divergence, while the factors in the second term which are being contracted with (the contravariant components of) the Faraday tensor correspond to the components of a gradient. These terms must be computed using the appropriate expressions for a given system of coordinates. From these considerations in spherical coordinates, with the metric in Eq. (6.13), the following expressions are obtained for the geometric functions that enter in the Gauss law (6.4)

$$\gamma^r(\mathbf{x}) = \frac{4}{r}, \qquad \gamma^\theta(\mathbf{x}) = \frac{2\cot\theta}{r^3\left[1 - \Psi(r)\right]}, \qquad \gamma^\varphi(\mathbf{x}) = 0, \tag{6.16}$$

and in the Maxwell-Ampère law (6.5)

$$\sigma^{r\varphi\varphi}(\mathbf{x}) = \frac{1}{r^3 \sin^2\theta}\left[2-\Psi(r)\right], \qquad \sigma^{r\theta\theta}(\mathbf{x}) = \frac{2-\Psi(r)}{r^3},$$
$$\sigma^{\theta rr}(\mathbf{x}) = \frac{2\cot\theta}{r^3}\left[1-\Psi(r)\right], \qquad \sigma^{\theta\varphi\varphi}(\mathbf{x}) = 0,$$
$$\sigma^{\varphi\theta\theta}(\mathbf{x}) = 0, \qquad \sigma^{\varphi rr}(\mathbf{x}) = 0, \qquad \xi^{ii}(\mathbf{x}) = 0,$$

respectively.

Electrostatics in the Schwarzschild geometry . The Gauss law (6.4) in the Schwarzschild geometry (6.13), for the case of a static radial field with spherical symmetry becomes

$$\frac{dE_r}{dr} + \frac{4}{r}E_r = \frac{\rho}{\varepsilon_0}. \tag{6.17}$$

The charge distribution can be that of a spherical charge or a spherical shell. In any case, a characteristic radius can be defined, which we simply represent by R. This solution can be solved in vacuum, outside the gravitational source, be it a star or a black hole (in which case we consider $r > 2GM/c^2$). Near the surface of the massive body the deviation of the solution with respect to the case without gravity is largest. In particular, such effects can be considered for charged stars or charged black holes, by identifying R with R_* in the case of stars or with $2GM/c^2$ for the black hole case.

In particular in vacuum, we get

$$\frac{E_r^{flat}(r) - E_r^{curved}(r)}{E_r^{flat}(r)} = 1 - \alpha\frac{R^4}{r^4}, \qquad r \geq R. \tag{6.18}$$

Electric field due to charged plates in equatorial orbit in the (gravitational) weak field limit. Previously, we assumed that the electric field had spherical symmetry just like the metric. Now we consider the simplest case in which the electric field has a single component E_x (in a certain fixed system of coordinates). In this case, outside the charge distribution, the Gauss law (6.4) provides

$$\partial_x E_x = \frac{\gamma^x}{g^{xx}g^{00}}E_x. \tag{6.19}$$

Let us consider the weak field limit of the Schwarzschild solution (expanding the metric in powers of $2GM/c^2r$ up to first order). In isotropic Cartesian coordinates the metric is then given by

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2\tilde{r}}\right)dt^2 - \left(1 + \frac{2GM}{c^2\tilde{r}}\right)\left(dx^2 + dy^2 + dz^2\right), \tag{6.20}$$

where, $\tilde{r} \equiv \sqrt{x^2+y^2+z^2}$. We then have the following solution

$$E_x = E_{0x}e^{\int_{x_0}^{x}\gamma^x g_{xx}g_{00}dx} = E_{0x}\sqrt{\frac{\tilde{r}/R_{sch}-1}{\tilde{r}/R_{sch}+1}}. \tag{6.21}$$

In flat spacetime, the electric field is uniform with magnitude given by E_{0x}. Sufficiently far away from the gravitational source (star, planet or black hole) the electric field is uniform. The effect of gravity on the electric field can then be characterized by the dimensionless quantity

$$|E_x - E_{0x}|/E_{0x} = 1 - \sqrt{\frac{\tilde{r}/R_{sch} - 1}{\tilde{r}/R_{sch} + 1}}, \tag{6.22}$$

which vanishes when $\tilde{r} \to \infty$. Recall that electric fields with quasi parallel field lines can result from charged plates. In particular, this result means that in the vicinity of a spherical mass, the electric field created by a single charged plate or by a plane capacitor (oriented along the x axis in this case) is no longer uniform. The field changes its magnitude due to spacetime curvature. Far from the gravitational source we recover the uniform electric field.

An interesting application of this result is the following thought experiment: Consider a plane capacitor with charge Q, an area given by $A = L^2$ and a distance between plates D. Suppose this system is put in equatorial orbit around a certain quasi-spherical astrophysical mass M with negligible rotation. Further, suppose that the line perpendicular to the capacitor's plates is always aligned with a certain reference distant Quasar, in the x axis (of the Cartesian system centred in the gravitational source). Once the system is in orbit, that direction will be aligned with the radial direction relative to the central mass, twice per cycle. Alternatively this direction will be perpendicular to the radial line, also twice a cycle (when the system is crossing the y axis). The electric field, which is always aligned with the x axis, has a magnitude which varies with the distance to the center and therefore, twice a cycle the change in magnitude is either along the line connecting the two plates or along the perpendicular to that direction. The maximum change in magnitude for both cases is equal if $D = L$. On the other hand, if $D \gg L$ and L is sufficiently small, then the field is approximately uniform whenever the system is crossing the y axis and non-uniform in the rest of the orbit. If, whenever the capacitor is crossing the x axis the maximum change in magnitude is $|\Delta E_x|$, then the dimensionless quantity of experimental relevance is $|\Delta E_x|/E_{0x}$, which measures the strength of the effect of gravity in the weak field regime. The value $E_{0x} = Q/\varepsilon_0 L^2$ is the usual value of the uniform field inside the capacitor in the absence of gravity.

The strength of this effect when the capacitor is crossing the x axis, with one plate at position x and the other at $x + D$, is given by

$$\frac{|E_x(x) - E_x(x+D)|}{E_{0x}} = \sqrt{1 - \frac{2}{1 + \tilde{X} + \tilde{D}}} - \sqrt{\frac{\tilde{X} - 1}{\tilde{X} + 1}}, \tag{6.23}$$

where the distances are in units of R_{sch}, $\tilde{X} \equiv \tilde{r}/R_{sch}$, $\tilde{D} \equiv D/R_{sch}$. This dimensionless quantity measures the maximum change of the magnitude of the electric field inside the capacitor due to the gravitational field of the astrophysical spherical mass. In principle, this effect could be used to test GR and modified theories of gravity, providing another complementary (weak field) test in the Solar System.

Since the values of the Schwarzschild radius for the Sun and Jupiter are approximately 2.95×10^3m and 2.2m, respectively, the effect should be very small unless one gets extremely close to their surfaces. For example, for a laboratory in orbit around Jupiter at a distance approximately equal to three times Jupiter's radius ($\tilde{X} \sim 9 \times 10^7$) we have the following values

$$\begin{aligned} |\Delta E_x|/E_{0x} &\sim 10^{-14} \quad (\tilde{D} = 100)\,, \\ |\Delta E_x|/E_{0x} &\sim 10^{-13} \quad (\tilde{D} = 1000). \end{aligned} \tag{6.24}$$

Naturally, these tiny values represent an enormous experimental challenge in terms of the sensitivities and noise control requirements. The fact that we are not using test masses, but electric fields instead to study gravity, complicates further the experiment due to various possible environmental effects related to space weather, in particular, solar and planetary magnetospheres. In any case, it is always better to measure voltage differences than to measure the electric field itself, since better sensitivities can be obtained. The Voltage drop between the two plates will be less than in the absence of gravity. In principle, since different points inside the capacitor will be at different radial coordinates which change with time, we also expect the generation of electromagnetic radiation with a frequency related to the orbital frequency of the spacecraft. This electromagnetic radiation should be linearly polarized and is completely induced by the effect of gravity in the electric field inside the capacitor and the orbital motion of the spacecraft.

Electric field in the gravitational field of a massive spherical object – Case of non-homogeneous, isotropic constitutive relations. In chapter 2 we considered the possibility of changing the constitutive relations between the electromagnetic fields and excitations by relaxing the assumptions of homogeneity and isotropy. The idea behind this suggestion comes from the very deep relation between spacetime geometry and electrodynamics, already present at the foundations of electromagnetism, and so well explained by Hehl and Obukhov. But, it also comes from the notion that the physical properties of vacuum (or electrovacuum) should not be a priori given. This follows the spirit of GR which is a background independent theory and therefore the local geometry of spacetime is not a priori given, rather it has to be considered for each physical system as a solution of the dynamical equations. Likewise, we postulate that the electric permittivity and magnetic permeability tensors for vacuum should reflect the local symmetries of spacetime geometry. Let us then consider spherical symmetry. Relaxing the assumption of homogeneity around spherical bodies, the linear, local, isotropic and non-homogeneous constitutive relation between the electric field and electric excitation is (see chapter 2)

$$D^i = -g^{00} g^{ij} \sqrt{-g}\, \varepsilon_0(r) E_j. \tag{6.25}$$

The Gauss law can be written as

$$\partial_i D^i = \rho\sqrt{-g}. \tag{6.26}$$

We will use the following Ansatz

$$\varepsilon_0(r) = \varepsilon_0 \left(1 + \bar{\gamma}\frac{2GM}{c^2 r}\right). \tag{6.27}$$

which can be seen as the linear approximation of a Taylor expansion in powers of $2GM/(c^2 r)$. Here $\bar{\gamma}$ is a dimensionless parameter. Then, considering the Schwarzschild geometry, and choosing $\bar{\gamma} = 1$ we find the strength of the effect is given by

$$|E_r^{\varepsilon_0(r)} - E_r^{\varepsilon_0}|/E_r^{\varepsilon_0} = \frac{R_{sch}/r}{1 + R_{sch}/r}, \tag{6.28}$$

where $E_r^{\varepsilon_0}$ corresponds to the solution without the contribution from $\varepsilon_0(r)$.

According to the hypothesis we explore here in the curved geometry around a massive object, the electric permittivity of vacuum is not homogeneous, but has a radial dependence instead. For black holes, this effect influences electric fields more strongly near the horizon. The strength of the effect is maximum at the horizon where the magnitude of the electric field is 0.5 times weaker than the case with a constant permittivity tensor.

If we now repeat the analysis for the case of the electric field inside a plane capacitor in orbit around a spherical mass, we arrive at the result

$$E_{\tilde{r}} = E_{0x}\frac{\sqrt{1-R_{sch}^2/\tilde{r}^2}}{(1-R_{sch}/\tilde{r})^2}. \tag{6.29}$$

The strength of the effect can be quantified again by comparison with the case where the electric permittivity is constant, where we get

$$|E_{\tilde{r}}^{\varepsilon_0(\tilde{r})} - E_{\tilde{r}}^{\varepsilon_0}|/E_{\tilde{r}}^{\varepsilon_0} = 1 - \frac{\tilde{r}/R_{sch}}{1+\tilde{r}/R_{sch}}. \tag{6.30}$$

In principle, this effect could be tested using electric fields inside capacitors and these expressions can be generalized to include the PPN approach.

Magnetostatics in the Schwarzschild geometry - Astrophysical applications. Suppose that in some reference frame we have a static magnetic field due to a (stationary) current and no electric field. Then, the generalized Maxwell-Ampère law in equation (2.28) is given by the following equations

$$\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta H_\varphi\right) - \frac{1}{r\sin\theta}\partial_\varphi H_\theta = \sqrt{-g}j^r, \tag{6.31}$$

$$\frac{1}{r\sin\theta}\partial_\varphi H_r - \frac{1}{r}\partial_r(rH_\varphi) = \sqrt{-g}j^\theta, \tag{6.32}$$

$$\frac{1}{r}\partial_r(rH_\theta) - \frac{1}{r}\partial_\theta H_r = \sqrt{-g}j^\varphi. \tag{6.33}$$

We can solve this for the homogeneous and isotropic constitutive relations, that can be derived from Eq. (2.36):

$$H_k = \mu_0^{-1}\sqrt{-g}g_{jk}B^j\,. \tag{6.34}$$

The following applications can have some astrophysical relevance although, as previously mentioned, the main purpose here is to illustrate the effect of spacetime geometry in magnetostatics motivated from minimally realistic astrophysical scenarios. An appropriate treatment of magnetic fields in relativistic astrophysical situations (with strong gravity) would require the equations governing fluids and fields of magnetohydrodynamics in the background of an appropriate spacetime metric. Furthermore, for cases with very strong magnetic fields, which can contribute to the gravitational field, the coupled Einstein-Maxwell (or its generalizations) also need to be included in the analysis.

Magnetic field created by a ring of circulating plasma around a spherical mass. Consider a spherical astrophysical mass, which could be a black hole, with a stationary current loop around it on the equatorial plane ($\theta = \pi/2$). This current distribution, which could be due to a disk of very hot plasma, would create an axially symmetric magnetic field of the general form

$$\mathbf{B} = B^r(r,\theta)\mathbf{e_r} + B^\theta(r,\theta)\mathbf{e}_\theta\,, \tag{6.35}$$

which implies

$$\mathbf{H} = H_r(r,\theta)\mathbf{e^r} + H_\theta(r,\theta)\mathbf{e}^\theta. \tag{6.36}$$

Then, the relevant Maxwell equation is

$$\frac{1}{r}\partial_r(rH_\theta) - \frac{1}{r}\partial_\theta H_r = \sqrt{-g}j^\varphi, \tag{6.37}$$

which can be solved outside the current distribution by setting the right hand side to zero. We will consider this equation in the equatorial plane and outside the disk (or ring) of currents, i.e., for $r > R$ where R is a mean representative of the radius of the circular current distribution. Setting $\theta = \pi/2$ the magnetic field will only have the θ component

$$\mathbf{B} = \pm B^\theta(r, \pi/2)\mathbf{e}_\theta, \tag{6.38}$$

where the $\pm$ refers to the cases in which the circulating current is moving in the direction of positive or negative φ, respectively. We then get $H_\theta(r, \pi/2) = f(\pi/2)/r$, which implies

$$B^\theta(r, \pi/2) = \frac{\mu_0}{\sqrt{-g}\; g_{\theta\theta}} H_\theta \propto \frac{1}{r^5}. \tag{6.39}$$

We conclude that the magnetic field (on the equatorial plane) due to the circular current distribution decays faster with the radial distance, in the curved spacetime of Schwarzschild geometry, than in the flat (Minkowski) case (although, strictly speaking the radial coordinate here does not correspond to a physical distance).

We recall that at this stage we are neglecting rotation, therefore these calculations can be viewed as having an approximate validity around quasi-static spherical masses with an electric current due to a highly ionized gas in the orbiting accretion disk. Since the electromagnetic equations were considered and solved outside the current distribution creating the magnetic field and in the exterior (Schwarzschild) spacetime, another astrophysical scenario compatible with the approach taken here is that of a (quasi) spherical mass with negligible rotation with a magnetic field generated by electric currents in its interior, as long as the magnetic field energy density has a negligible effect in the gravitational field (no back reaction).

Magnetic field created by an astrophysical jet. Consider now another idealized astrophysical scenario in which the spherical body emits a stationary jet of charged particles vertically, defining an axis. We set this to be the z axis. In this case, a magnetic field will arise with axial symmetry and along the φ direction with symmetrical configurations above or below the $\theta = \pi/2$ plane

$$\mathbf{B} = \pm B^\varphi(r, \theta)\mathbf{e}_\varphi. \tag{6.40}$$

The relevant Maxwell-Ampère equations outside the current distribution are

$$\frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta H_\varphi\right) = 0, \qquad -\frac{1}{r}\partial_r\left(rH_\varphi\right) = 0, \tag{6.41}$$

and therefore $H_\varphi(r, \theta) \propto (r\sin\theta)^{-1}$, which implies

$$B^\theta(r, \theta) = \frac{\mu_0}{\sqrt{-g}\; g_{\varphi\varphi}} H_\varphi \propto \frac{1}{r^5\sin^4\theta}. \tag{6.42}$$

Here we considered a constant magnetic permeability tensor for vacuum, but following our hypothesis that the properties of this tensor should reflect the local spacetime isometries it should be interesting to compute the changes to these results if the magnetic permeability has a radial dependence. This is what we do next.

Magnetic field around a spherical gravitational field, with non-homogeneous constitutive relations. Let us consider the effect of relaxing the condition of homogeneity in the constitutive relations, assuming a radial dependence of the permeability tensor in vacuum, i.e.,

$$H_k = \mu_0^{-1}(r)\sqrt{-g}g_{jk}B^j \,. \tag{6.43}$$

We introduce the ansatz

$$\mu_0(r) = \mu_0\left(1 + \bar{\bar{\gamma}}\frac{r_{Schw}}{r}\right), \tag{6.44}$$

and choose $\bar{\bar{\gamma}} = 1$. The above results are therefore generalized into

$$\begin{aligned} B^\theta(r,\pi/2) &\propto \frac{\mu_0}{r^5}\left(1+\frac{r_{Schw}}{r}\right), \\ B^\theta(r,\theta) &\propto \frac{\mu_0}{r^5\sin^4\theta}\left(1+\frac{r_{Schw}}{r}\right), \end{aligned}$$

for the magnetic field due to a ring of current (in the equatorial plane) and to the astrophysical jet, respectively. The strength of this effect in comparison with the case with a homogeneous magnetic permeability tensor, is stronger near the horizon

$$\frac{|\delta\mathbf{B}|}{\mathbf{B}} = \frac{r_{Schw}}{r}. \tag{6.45}$$

Contrary to what happened with the electric field case, the radial dependence of the permeability tensor enhances the magnetic field in comparison with the case of magnetic homogeneity of vacuum. In the limit, at the horizon of black holes, the magnetic field is stronger by a factor of 2. This fact comes from the very nature of the constitutive relations and the physical dimensions of the magnetic field. Indeed, the magnetic permeability appears in the denominator in (6.43) while the electric permittivity appears in the numerator in (6.25). Like in the case of the electric field, sufficiently far way from the source, the effect becomes neglibible.

Maxwell fields around slowly rotating objects in the weak field limit within the PPN formalism

The appropriate metric describing the local geometry outside a stationary rotating mass is the Kerr metric [174]. Expanding this geometry to first order in the angular momentum J, we get the geometry outside the source in the limit of slow rotation [52]

$$ds^2_{Kerr} \approx ds^2_{Schawrzschild} + \frac{4GJ}{c^2 r}\sin^2\theta d\varphi dt. \tag{6.46}$$

Recall that the deformations of the object's spherical symmetry depend quadratically with angular momentum J, while the metric already changes at the linear level. Therefore, this geometry is a good approximation to that of a slowly rotating (quasi) spherical gravitational mass.

In the weak field limit (non-relativistic weak gravitational sources) the above metric is given by the following expressions in spherical and (isotropic) Cartesian coordinates, respectively [52]

$$ds^2 = c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 - \left(1+\frac{2GM}{c^2r}\right)dr^2 - r^2\left(d\theta^2+\sin\theta^2 d\varphi^2\right) + \frac{4GJ}{c^2r}\sin^2\theta d\varphi dt, \tag{6.47}$$

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2\tilde{r}}\right)dt^2 - \left(1 + \frac{2GM}{c^2\tilde{r}}\right)\left(dx^2 + dy^2 + dz^2\right) + \frac{4GJ}{c^2\tilde{r}^3}dt(xdy - ydx), \tag{6.48}$$

where $\tilde{r} \equiv \sqrt{x^2 + y^2 + z^2}$, as before. The first three terms correspond to the Schwarzschild geometry in the first order approximation. In fact, these expressions are a particular case of the general line-element for non-relativistic stationary sources, understood as a linear perturbation of Minkowski background spacetime [51]

$$ds^2 = c^2\left(1 - \frac{2\Phi^{(g)}}{c^2}\right)dt^2 - \left(1 + \frac{2\Phi^{(g)}}{c^2}\right)\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + 2A_i^{(g)}dx^i dt, \tag{6.49}$$

with $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, where $\Phi_{(g)}$ and $A^i_{(g)}$ are the gravitoelectric and gravitomagnetic potentials respectively, which can be defined through the expressions below [51]

$$\bar{h}^{00} \equiv \frac{4\Phi_{(g)}}{c^2}, \qquad \bar{h}^{0i} \equiv \frac{A^i_{(g)}}{c}, \tag{6.50}$$

and

$$A_i dx^i = \eta_{ij}A^i dx^j = -\vec{A}\cdot\vec{dx}, \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \qquad h = \eta^{\mu\nu}h_{\mu\nu}. \tag{6.51}$$

For a brief review on gravitolelectromagnetism in the perturbative as well as geometric approaches, see [175]. In this perspective, we can clearly see that the influence of the object's angular momentum in the surrounding spacetime express the gravitomagnetic effect (measured around Earth in the Gravity Probe B experiment [126]).

The PPN (Parametrized Post-Newtonian) formalism allows one to parameterize geometrical gravitational theories within the Solar System, or more generally around spherically symmetric stationary (possibly rotating) gravitational sources. Many dimensionless parameters appearing in Taylor expansions of the metric are thus defined describing deviations from GR. For example, two of the most important ones, β and γ, can arise from an expansion in powers of GM/c^2r of the most general static spherically symmetric spacetime [52]

$$\begin{aligned} ds^2 &= D(r)c^2dt^2 - B(r)dr^2 - r^2\left(d\theta^2 + \sin\theta^2 d\varphi^2\right) \\ &= \text{weak field (Newtonian)} + \text{post} - \text{Newt.}, \end{aligned} \tag{6.52}$$

obtaining

$$\begin{aligned} D(r) &= 1 - \Psi(r) + (\beta - \gamma)\Psi(r)^2 + ..., \\ B(r) &= 1 + \gamma\Psi(r) + ..., \end{aligned} \tag{6.53}$$

where γ is a measure of the spatial curvature produced by a unit rest mass, while β is a measure of how much non-linearity is present in the superposition law for gravity. The higher order terms in the expansion give the so-called Post-Newtonian corrections while the expansion up to first order represent the weak field limit and is sometimes referred to as the Newtonian limit. However, strictly speaking there can be corrections to Newton gravity even in the first order (weak field) limit, as in the case of slow rotation where the gravitomagnetic potential and corresponding vector field allows non-Newtonian predictions, such as frame dragging or Lens-Thirring effects [51]. In GR, we have $\beta = \gamma = 1$.

Many of the so called gravitational classical tests can be expressed in terms of these parameters [51, 52]. For example, the deflection of light due to the Sun's gravitational field, the precession of the perihelion of planetary orbits and the Shapiro time delay of light signals in the Sun's field (see [52], for example). Will [166, 167], Ni [168, 169] and Misner et al [170] used ten parameters in the Beta-Delta notation and later, a different set of 10 parameters was used in the Alpha-Zeta notation (see for example [171]), but γ and β coincide in both notations. In the first notation Δ_1 and Δ_2 are intimately related to the off-diagonal elements characteristic of a Kerr-like metric, since Δ_1 measures how much dragging of inertial frames is produced by unit linear momentum and Δ_2 measures the difference between radial and transverse momentum on dragging of inertial frames.

All parameters are potentially useful to constrain alternative or extended geometric theories of gravity. For the slowly rotating object of our interest, when the expansions on the dimensionless quantities GM/c^2r and GJ/c^3r^2 are taken up to first order (in the most general axially symmetric metric), one arrives at the following expressions in spherical and (isotropic) Cartesian coordinates

$$ds^2 = c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 - \left(1+\gamma\frac{2GM}{c^2r}\right)dr^2 - r^2d\Omega^2 + \left(1+\gamma+\frac{\alpha_1}{4}\right)\frac{2GJ}{c^2r}\sin^2\theta d\varphi dt, \tag{6.54}$$

$$\begin{aligned} ds^2 &= c^2\left(1-\frac{2GM}{c^2\tilde{r}}\right)dt^2 - \left(1+\gamma\frac{2GM}{c^2\tilde{r}}\right)\left(dx^2+dy^2+dz^2\right) \\ &\quad + \left(1+\gamma+\frac{\alpha_1}{4}\right)\frac{2GJ}{c^2\tilde{r}^3}dt(xdy-ydx), \end{aligned} \tag{6.55}$$

respectively, with $d\Omega^2 \equiv \left(d\theta^2+\sin^2\theta d\varphi^2\right)$ and $\alpha_1 \equiv 7\Delta_1+\Delta_2-4\gamma-4$, measures the extent of preferred frame effects and is equal to zero in GR. The following bounds taken from Will [176] due to local (Solar System) tests should be taken under consideration by any gravitational theory

$$|\gamma-1| = 2.3\times10^{-5}, \quad |\beta-1| = 3\times10^{-3}, \quad \alpha_1 < 10^{-4},$$

where the first result was obtained through light deflection and time delay, the second is due to perihelion shift and the last from orbit polarization in Lunar Laser Ranging (LLR).

Gravitomagnetic coupling between electric and magnetic fields

We now consider Maxwell's inhomogeneous equations in the background spacetime around a spherical mass with slow rotation in the weak field (first order) limit (using Eq (6.54)). New gravitomagnetic terms appear in the Gauss and Maxwell-Ampère laws due to the off-diagonal time-space metric component. The generalized Gauss law, Eq. (6.4), has now a mixture of electric and magnetic components. The coupling to spacetime geometry and the gravitomagnetic effect provides magnetic corrections, which vanish in the non-rotating regime. This puts forward the interesting possibility of having even static magnetic fields as sources of electric fields.

A possible application concerns the magnetic field around rotating neutron stars. This field feels the presence of very strong gravity where the curvature of spacetime should not be neglected. In such astrophysical conditions the theory here exposed suggests an induced electric field component due to the coupling between the magnetic field and the gravitomagnetic character of gravity.

Another application comes from supermassive rotating black holes in the center of disk-shaped galaxies. It is well know that the frame-dragging character of the gravitomagnetic field around the rotating mass can be understood as a differential rotation of the curved spacetime around the rotation axis of the object (as seen from an observer far away from the source). Such a rotation is analogous to what happens in tornadoes where, in contrast to rigid bodies, the angular velocity is higher towards the center and decays with radial distance. Accordingly, from the coupling between gravitomagnetism and electromagnetic fields within Maxwell's equations, it is natural to expect a frame-dragging effect on these fields (see also [160, 161]). Therefore, an electric field around a super-massive rotating black hole would feel the differential rotation of spacetime resulting in a spiral pattern for the electric field lines along the galactic equator. If this relativistic (non-Newtonian) effect might provide some light into the understanding of the formation processes of spiral structures in galaxies, remains up to this moment a challenging and open question.

Changing the Cartesian (isotropic) coordinates in Eq. (6.55) to (axi-symmetric) cylindrical coordinates $(t, \tilde{R}, \phi, z)$, the metric becomes

$$\begin{aligned} ds^2 &= c^2\left(1-\frac{2GM}{c^2\left(\tilde{R}^2+z^2\right)^{1/2}}\right)dt^2-\left(1+\gamma\frac{2GM}{c^2\left(\tilde{R}^2+z^2\right)^{1/2}}\right)\left(d\tilde{R}^2+\tilde{R}^2d\phi^2+dz^2\right) \\ &\quad+\left(1+\gamma+\frac{\alpha_1}{4}\right)\frac{2GJ}{c^2\left(\tilde{R}^2+z^2\right)^{3/2}}\tilde{R}^2dtd\phi, \end{aligned} \tag{6.56}$$

where $\tilde{R}$ is a radial coordinate related to the physical distance to the rotation axis. The Gauss law, in vacuum, for this case is

$$\left[\frac{1}{\tilde{R}}\partial_{\tilde{R}}(\tilde{R}D^{\tilde{R}})+\partial_z D^z+\frac{1}{\tilde{R}}\partial_\phi D^\phi\right]=0. \tag{6.57}$$

We will assume for simplicity no magnetic fields. Then we have the following linear (homogeneous and isotropic) constitutive relations (see chapter 2)

$$D^k=\varepsilon_0\sqrt{-g}(g^{0i}g^{0k}-g^{00}g^{ik})E_i\,. \tag{6.58}$$

Let us consider an electric field with axial symmetry $[\mathbf{E}=E_{\tilde{R}}(\tilde{R},z)\mathbf{e}^{\tilde{R}}+E_\phi(\tilde{R},z)\mathbf{e}^\phi]$, where we have assumed that the E_z component is negligible near the equatorial plane $(z=0)$. Therefore, for this approximation, we have $[\mathbf{D}=D^{\tilde{R}}(\tilde{R},z)\mathbf{e}_{\tilde{R}}+D^\phi(\tilde{R},z)\mathbf{e}_\phi]$. Thus, the Gauss law provides the solution $D^{\tilde{R}}(\tilde{R},z)=f(z)/\tilde{R}$.

In the equatorial plane, setting $z=0$, the radial component of the electric field in the spacetime around the rotating massive object is given by

$$E_{\tilde{R}}=\frac{f(0)}{4\varepsilon_0 mc\tilde{R}^2}\frac{\left[r_{Sch}^2J^2(4+\alpha+4\gamma)^2+16c^2m^2(1-r_{Sch}/\tilde{R})(1+\gamma r_{Sch}/\tilde{R})\tilde{R}^2\right]^{1/2}}{(1+\gamma r_{Sch}/\tilde{R})}, \tag{6.59}$$

and for $J=0$

$$E_{\tilde{R}}=\frac{C}{\tilde{R}}\sqrt{\frac{(1-r_{Sch}/\tilde{R})}{(1+\gamma r_{Sch}/\tilde{R})}}, \tag{6.60}$$

where C is a constant. On the other hand, the Faraday law gives $\partial_z E_\phi = \partial_{\tilde{R}} E_\phi = 0$, therefore E_ϕ is an arbitrary constant that can be set to zero without a significant loss of generality. We see that the gravitomagnetic term affects the gravitationally induced decay of the radial component of the electric field.

These astrophysical scenarios intend to illustrate possible gravitomagnetic effects, affecting electromagnetism directly through the very nature of the field equations in curved spacetime. In fact, in the most general case it becomes clear that electrostatics and magnetostatics are no longer separated, but instead become coupled in the presence of gravitomagnetism. We assumed for simplicity a vanishing magnetic field, but as previously mentioned, in general, even a static magnetic field will contribute to the electric field via this gravitomagnetic coupling induced by astrophysical sources with rotation.

Non-homogeneous and anisotropic constitutive relations.

Following our hypothesis that the electromagnetic properties of vacuum should reflect the local spacetime isometries, the constitutive relation (6.58) in this case (neglecting magnetic fields for simplicity) is generalized into

$$D^{\tilde{R}} = -\varepsilon_0^{\tilde{R}}(\tilde{R}, z)\sqrt{-g}g^{00}g^{\tilde{R}\tilde{R}}E_{\tilde{R}}, \tag{6.61}$$

$$D^{\phi} = \varepsilon_0^{\phi}(\tilde{R}, z)\sqrt{-g}\left[(g^{0\phi})^2 - g^{00}g^{\phi\phi}\right]E_\phi. \tag{6.62}$$

The important physical idea here is that the electric permitivity should have axial symmetry and depend on the direction, i.e., $(\varepsilon_0)^{ij} = \mathrm{diag}[\varepsilon_0^{\tilde{R}}(\tilde{R}, z), \varepsilon_0^{\phi}(\tilde{R}, z), \varepsilon_0^{z}(\tilde{R}, z)]$.

Let us assume as an Ansatz, the following expression

$$\varepsilon_0^{\tilde{R}} = \varepsilon_0 \left[1 + \bar{\gamma}\frac{2GM}{c^2\tilde{R}} + \bar{\Delta}\frac{2GJ\tilde{R}}{c^3\left(\tilde{R}^2 + z^2\right)^{3/2}}\right], \tag{6.63}$$

where $\bar{\gamma}$ and $\bar{\Delta}$ are dimensionless parameters. This corresponds to the linear approximation of a Taylor expansion in powers of the relevant dimensionless quantities related to mass and angular momentum. By considering for simplicity the equatorial plane, i.e $z = 0$, then the result in Eq. (6.59) is generalized by replacing ε_0 with $\varepsilon_0^{\tilde{R}}(\tilde{R}, z = 0)$. As a consequence, in this case the electric field has a radial dependence which includes both the contribution from spacetime curvature (gravity) and a variable electric permittivity of vacuum. This is analogous to what we had in the spherical symmetric case generalized for the axial symmetric geometries. By turning off the angular momentum, we recover the spherical case. Further research is required to understand how the effects of having non-homogeneous (and anisotropic) permittivity and permeability tensors can have an impact on physical observables. Electric and magnetic fields interact with astrophysical plasmas in black hole accretion disks and around neutron stars. These interactions depend on the coupling constants which are basically the electric and magnetic properties of the medium. Such interactions need to be carefully taken care of, for instance, using magnetohydrodynamical computations. On the other hand, by changing the permittivity and permeability tensors, the propagation properties of electromagnetic waves is affected. We briefly discuss these issues in the conclusions. However, testing these hypotheses via observations need further analysis using the wave equations and also taking into account environmental effects, although this is not considered in the present work.

6.1.2 Discussion and summary

In this section, we explored the physical applications of electrodynamics in the background of a (pseudo) Riemann spacetime manifold. The main electromagnetic effects induced by spacetime curvature addressed here include the following: gravitational contributions for the decay of electric and magnetic fields in spherically symmetric spacetime, magnetic contributions to the Gauss law due to the gravitomagnetic character of the spacetime around rotating objects and the effects of relaxing the assumptions of homogeneity and isotropy of the electromagnetic properties of vacuum (the electromagnetic oscillations induced by gravitational waves were presented in another work [3]). In particular, the physical (possibly measurable) effects of spacetime geometry in electromagnetic fields, expressed in terms of the extended Gauss and Maxwell-Ampère laws, helps the comparison with the usual results obtained from electromagnetism in Minkowski spacetime, making clearer the physical interpretations of such effects. In the following, we briefly summarize the topics explored in this work:

In the spacetime around spherical sources, the results confirm that gravity induces a (geometric) contribution to the decay of electric and magnetic fields along any radial direction. In principle, even electric fields due to plane charged plates, which are uniform in the absence of gravity will manifest a spatial variability as is clear from Eq. (6.19). This effect could be tested in principle, under appropriate experimental conditions similar to those used in the GP-B experiment, namely with recourse to drag-free motion of satellites in polar or equatorial orbits around a spherical mass, housing the probe (in this case a capacitor) in vacuum under extreme low temperatures achieved by cryogenics.

According to the hypothesis we explore here, in the curved geometry around a massive object the electric permittivity of vacuum is not homogeneous, but has a radial dependence instead. For black holes, this effect influences electric fields more strongly near the horizon. The strength of the effect is maximum at the horizon where the magnitude of the electric field is 0.5 times weaker than the case with a constant permittivity tensor. In principle, this effect could be tested using electric fields inside capacitors. The expressions (6.28) and (6.30) we obtained can be generalized to include the PPN approach. Similar considerations apply to magnetic fields around massive spherical (non-rotating) objects.

The hypothesis for non-homogeneous permittivity and permeability (electromagnetic) properties of vaccum leads to the result that the speed of electromagnetic waves is not the same in every point around a massive object. Instead, it must have a radial dependence. Using the Ansatz considered in this work, Eqs. (6.27) and (6.44), in the first order approximation (with respect to Taylor expansions in powers of r_{Sch}/r), we get

$$c(r) = \frac{c_0}{(1 + r_{Sch}/r)}, \qquad c_0 \equiv \frac{1}{\varepsilon_0 \mu_0}.$$

As a consequence, local observers could still agree about the velocity of light and the local conformal structure of spacetime, i.e., the local light-cone, but the change in the (local) light cone structure from one point to another now has both the influence of the spacetime curvature and the fact that the permittivity and permeability tensors change. This prediction for a non-homogeneous (but isotropic) speed of light in vacuum, in the spherical gravitational fields around massive objects, should have observable consequences that need to be tested experimentally.

In the axisymmetric spacetime around rotating sources, electrostatics and magnetostatics are no longer separated, but instead become coupled due to the presence of off-diagonal time-space metric components. We considered the metric in Eq. (6.54) which corresponds to the spacetime around a rotating spherical mass in the weak field and slow rotation regime. This metric has the off diagonal component proportional to the angular momentum of the source. Such metric components, correspond in linearised gravity to the components of the gravitomagnetic potential characteristic of frame dragging (Lens-Thirring) effects around axisymmetric astrophysical systems in rotation. The coupling to spacetime geometry, in particular to gravitomagnetism, induces magnetic corrections to the Gauss law, i.e., Eq. (6.4). This coupling suggests that even static magnetic fields can act as sources of electric fields via the gravitomagnetic frame dragging character of spacetime around rotating objects. In fact, the magnetic field around rotating neutron stars feels the presence of very strong gravity and therefore spacetime curvature should not be neglected. In such astrophysical conditions, the theory here exposed, suggests an induced electric field component, generated by the coupling between the magnetic field and the geometrodynamical character of gravity. Some work has been done in the past related to these issues (see [160, 161]) but much more can be investigated. This is an illustration of a gravitomagnetic effect affecting electromagnetism directly through the very nature of the field equations in curved spacetime. It opens our perspectives in the way we imagine the astrophysical environment of such compact objects and also other sources of strong astromagnetic fields

Another possible application comes from supermassive rotating black holes in the center of disk-shaped galaxies. It is believed that the interaction of these supermassive black holes with the surrounding galactic environment is an important ingredient in the formation and evolution of the whole Galaxy and in the formation and evolution of AGNs and stellar formation bursts. On the other hand, it is also well known that the frame-dragging character of the gravitomagnetic field around a rotating mass can be understood as a differential rotation of curved spacetime around the rotation axis of the object. Such a rotation is analogous to what happens in tornadoes where, in contrast to rigid bodies, the angular velocity is higher towards the center and decays with radial distance. Accordingly, from the coupling between gravitomagnetism and electromagnetic fields within Maxwell's equations, it is natural to expect a frame-dragging effect on these fields. Therefore, an electric field produced by a supermassive charged, rotating black hole, would feel the differential rotation of spacetime resulting in a spiral pattern for the electric field lines along the galactic equator. This could induce currents and stationary charge density spiral patterns on the surrounding ionized gas. If this relativistic (non-Newtonian) effect might provide some light into the understanding of the formation processes of spiral structures in galaxies, remains a challenging and open question.

In any case, we found the radial component of an axially symmetric electric field, solution to the Gauss law, in the geometry given by Eq. (6.56). This solution again confirms that gravitational fields can decrease the magnitude of electromagnetic fields in vacuum and the gravitomagnetic term (proportional to the source's angular momentum) also contributes to this effect. It is also pedagogical to illustrate the role of the coupling between gravity and electromagnetism for testing different theories of gravity. We also generalized the constitutive relations in this case, to introduce non-homogeneity and anisotropy in the permittivity tensor, corresponding to a spacetime with axial symmetry.

Regarding electromagnetic waves in the presence of gravity, one can show that extra terms appear in the generalized wave equations which deserves further research. Indeed, going beyond the geometrical optics limit, light deflection (null geodesics) and gravitational redshift are not the only effects arising from the coupling between light and gravity. More generally, all electromagnetic waves can experience gravitational effects on the amplitudes, frequencies and polarizations [3] (see also [177]). Besides, electric and magnetic wave dynamics become coupled in general, even in the Lorenz gauge. The coupling between the dynamics of different components suggests polarization effects. Moreover, there are terms in the wave equations depending on the first derivatives of the electromagnetic fields and similar terms for the wave equations written in terms of the potentials (6.12). These terms might be responsible for a gravitational contribution to the decay of the oscillations, but formally, these are also compatible with the existence of longitudinal modes induced by spacetime curvature, since similar terms appear in wave equations for vector fields with longitudinal modes. The fundamental reason behind this is the fact that in the presence of gravity the electric field in vacuum is no longer divergent-free (in the sense that $\partial^k E_k \neq 0$).

One can easily show that in the spacetime around a rotating astrophysical object, the equation for the potential includes the influence of the gravitomagnetic term and therefore, one expects that the electromagnetic field will experience a frame-dragging (Lens-Thirring) effect due to the gravitomagnetism. In fact, one can show that the gauge invariant wave equation for the coupled electric and magnetic fields also includes similar gravitomagnetic terms. These terms not only contribute to the decay of the wave amplitude, but will also provide a geometrically induced coupling between the dynamics of the various electric and magnetic components which will most probably affect the polarization. Gravitomagnetic effects on electromagnetic waves deserve further research with potential applications for relativistic astrophysics related to Pulsars, AGNs and for the study of the electromagnetic counterpart of gravitational wave sources.

6.1.3 Torsion effects

If torsion is allowed to minimally couple to ($s = 1$) bosonic fields, it breaks the $U(1)$ gauge invariance. This could be the case in high density environments (early Universe, ultra-compact objects) where torsion could provide a physical mechanism to induce such a symmetry breaking.

In ECSK theory torsion vanishes outside the (spin) matter sources and is negligible at low densities. On the other hand, for theories with propagating torsion, electromagnetic fields linked to the plasma regions of acretion disks, around supermassive blackholes of Quasars (or around stellar blackholes), will interact with the spacetime torsion which is due to the central blackhole's gravity. In this strong gravity regime, physical imprints from this interaction are a natural consequence. For example, the sinchronton radiation in the radio band could provide a window for observational studies of torsion signatures of such intense gravity regions. These interactions can also be studied in the context of gravitational wave emission from stellar binary blackholes coalescence (and other compact object mergers) with accretion disks (LIGO-VIRGO-Kagra-LigoIndia), or from supermassive blackhole binaries, in galaxy interactions (LISA). For example, the dynamical and strong gravity regime near a neutron star binary system in close orbits, makes it a nice laboratory to test modified gravity by looking at the interaction between the GWs and the (ultra-strong) magnetic field of one of the companions, when it is a magnetar. It was shown for example, that an electromagnetic wave with the same frequency as the GW is induced that can be studied to test for extra-modes in the gravity sector [178]. We will now briefly look at much simpler cases, without going into a detailed analysis here.

Simple application: Gauss law in ECSK theory.

As a simple illustration of torsion effects in the electromagnetic fields that could have potential astrophysical applications, let us consider the theory developed in 4.2. We take the source of torsion from the spin density of fermions within the ECSK theory. As a simple application, we consider the Gauus law for a diagonal spacetime metric

$$\partial_j \hat{E}^j = \frac{c^2}{\lambda}(\bar{\varrho} + \bar{J}^0), \tag{6.64}$$

and the Cartan equations relating the torsion to the Dirac spin density of fermionic fields

$$T_{\alpha\beta\gamma} = K_{\alpha\beta\gamma} = \frac{\kappa^2}{2}\epsilon_{\alpha\beta\gamma\varepsilon}\breve{s}^{\varepsilon}, \quad \breve{s}^{\lambda} = \frac{\hbar}{2}\bar{\psi}\gamma^{\lambda}\gamma^{5}\psi. \tag{6.65}$$

The electromagnetic fields are in this case considered to be propagating in the background Riemann-Cartan geometry where fermions are the main contributors to the torsion field. As we saw in the previous chapters, in the Einstein-Cartan-Dirac model the contorsion is completely anti-symmetric and equal to torsion. In this case we get the extended 4-current

$$J^{\nu} = -\lambda\Big(2K^{\nu}{}_{\lambda\mu}K^{\gamma[\mu\lambda]}A_{\gamma} + K^{\nu}{}_{\lambda\mu}\tilde{F}^{\mu\lambda} + 2\tilde{\nabla}_{\mu}(K^{\gamma[\mu\nu]}A_{\gamma})\Big), \tag{6.66}$$

and we arrive at

$$J^{0} = -\lambda\Big(\kappa^4\left(\breve{s}^0(\breve{s}\cdot A) - A^0\breve{s}^2\right) + \kappa^2\left(\vec{B}\cdot\vec{\breve{s}} + c\tilde{u}rl\vec{A}\cdot\vec{\breve{s}} - c\tilde{u}rl\vec{\breve{s}}\cdot\vec{A}\right)\Big), \tag{6.67}$$

with the following definitions

$$\breve{s}^2 = \breve{s}^{\lambda}\breve{s}_{\lambda}, \quad \breve{s}\cdot A = \breve{s}^{\mu}A_{\mu}, \quad \vec{B}\cdot\vec{\breve{s}} = B^k\breve{s}_k, \tag{6.68}$$

and

$$c\tilde{u}rl\vec{A}\cdot\vec{\breve{s}} = \epsilon^{ijk}(\tilde{\nabla}_j A_k)\breve{s}_i, \quad c\tilde{u}rl\vec{\breve{s}}\cdot\vec{A} = \epsilon^{ijk}(\tilde{\nabla}_j\breve{s}_k)A_i. \tag{6.69}$$

Notice that, due to the symmetry of the Levi-Civita connection, we have

$$c\tilde{u}rl\vec{A}\cdot\vec{\breve{s}} = \vec{B}\cdot\vec{\breve{s}}, \quad c\tilde{u}rl\vec{\breve{s}}\cdot\vec{A} = (\vec{\nabla}\times\vec{\breve{s}})\cdot\vec{A}, \tag{6.70}$$

with $\vec{B} \equiv \vec{\nabla}\times\vec{A}$. The second term in (6.67) becomes important in comparatively lower spin densities but since it is linear with the spin, for macroscopic systems with nearly random spin distributions (unpolarized), it vanishes. On the other hand, for polarized matter, in particular inside neutron stars with ultra-intense magnetic fields, it can become very important. The following simplifications can then be explored:

i) Unpolarized matter

$$J^0 \approx -\lambda\kappa^4\left(\breve{s}^0(\breve{s}\cdot A) - A^0\breve{s}^2\right) \tag{6.71}$$

ii) Polarized matter

$$J^0 \approx -\lambda\kappa^2\left(2\vec{B}\cdot\vec{\breve{s}} - (\vec{\nabla}\times\vec{\breve{s}})\cdot\vec{A}\right) \tag{6.72}$$

One may then search for solutions of Gauss law using on the right-hand side

$$\bar{J}^0 = \sqrt{-g}J^0. \tag{6.73}$$

Even in the case with vanishing charge densities the spin-electromagnetic coupling induced by spacetime torsion acts as a source for electric fields. Further simplification can come from considering stactic fields and geometry and also the limiting case of flat spacetime with torsion. In the unpolarized matter case and considering only the spatial part of the spin vector, we get

$$\partial_j \hat{E}^j = \frac{c^2}{\lambda}(\bar{\varrho} + \kappa^4 \lambda \sqrt{-g} A^0 \breve{s}^2), \tag{6.74}$$

Consider the case for negligible charge density.

$$\partial_j \hat{E}^j = \kappa^4 c g^{00} \Phi \breve{s}^2, \tag{6.75}$$

with $\Phi \equiv \sqrt{-g}\phi$ and $g^{00}\phi/c = A^0$ for diagonal metrics. We could say that the (electric) scalar field coupled to the spin density of matter, at any given point of space, is acting as a monopole (positive or negative) for the electric field. To get an equation for ϕ, consider Gauss law in the form

$$(\partial_j \bar{E}_j)|g^{jj}g^{00}| + \bar{E}_j \Omega_j^{jj00} = \lambda^{-1} \bar{J}^0, \tag{6.76}$$

for diagonal metrics, with $\Omega_j^{jj00} = \partial_j(|g^{jj}g^{00}|)$. Defining $\bar{\phi}$ such that $\bar{E}_j \equiv -\partial_j \bar{\phi}$, we then get

$$- \quad (\partial_j \partial_j \Phi)|g^{jj}g^{00}| - (\partial_j \Phi)\Omega_j^{jj00} = \left(\kappa^4 c \breve{s}^2 g^{00} - \left(|g^{jj}g^{00}| + \Omega_j^{jj00}\right) \frac{\partial_j \sqrt{-g}}{\sqrt{-g}} \right) \Phi, \tag{6.77}$$

or

$$-(\partial_j \partial^j \Phi)|g^{00}| - (\partial_j \Phi)\Omega_j^{jj00} = \Sigma \Phi. \tag{6.78}$$

with $\Sigma \equiv \left(\kappa^4 c \breve{s}^2 g^{00} - \left(|g^{jj}g^{00}| + \Omega_j^{jj00}\right) \frac{\partial_j \sqrt{-g}}{\sqrt{-g}} \right)$. This expression corresponds to an extended Proca-like equation. The flat limit of the above equation can be solved for constant background $\breve{s}^2$ (a sea of fermions), or for fermionic localized configurations. We can also write

$$\nabla^2 \phi = -\kappa^4 c s^2 \phi. \tag{6.79}$$

Notice that one can introduce some *ansatz* for the spin density squared of the background fermions to illustrate the physical effects in the electric potential and resulting fields, but in principle the quantity $\breve{s}^2$ is derived from the solution of the Dirac-Hehl-Data equation (which has analytical solutions in the limit spacetime flatness). As an example, we consider a simple application of the gauss law in (6.64) and (6.79). One possibility is to consider the following non-flat geometries inside compact objects:

1) Spherical symmetry
2) Axial symmetry (in the slow rotation regime).

In the second case one needs to correct for the off diagonal (gravitomagnetic terms). In such calculations, it is convenient to consider also the polarized matter case, since neutron stars can have strong magnetic fields. As a simplification one could then consider the simplified rhs of Gauss law, such that

$$\partial_j \hat{E}^j \approx \frac{c^2}{\lambda}(\bar{\varrho} - 2\lambda\kappa^2 \vec{B} \cdot \vec{\breve{s}}). \tag{6.80}$$

All quantities will depend only on (R, θ), the magnetic and spin vector can be assumed to be approximately aligned along the zz axis, and the electric field pattern can be estimated from this equation.

Other potential applications

Here we briefly mention some potential applications of torsion interactions with electromagnetic fields (and matter fields), relevant for specific astrophysical scenarios

Spin density from the electromagnetic sector. Using the ECSK with the spin tensor computed from the electromagnetic field minimally coupled to torsion, one obtains an effective theory equivalent to non-linear electrodynamics in Riemann spacetime. As previously mentioned, this is similar to the equivalence between the ECSK theory with a minimal coupling to Dirac fermions, and the model of a non-linear spinor with contact spin-spin interactions in Riemann spacetime. When the electromagnetic sector is the source of torsion in the ECSK model, the coupling with torsion provides the source for the non-linearity. In this case, not only one can solve the Gauss law and prove that it is free of singularities (this was done in the flat limit, see [21]), but also, self gravitating solutions of geonic type, supported by the electromagnetic field, could be searched in ECSK and in more general quadratic PGTG.

Electromagnetism and propagating torsion. Other applications of the electromagnetic equations with torsion, include the case of electromagnetic fields around the strong gravity of supermassive blackholes or stellar blackholes (with accretion disks). The geometry can be considered approximately spherical, so that one can use the Baekler Riemann-Cartan solution in quadratic Poincaré Gauge gravity (see [20]), where torsion can propagate outside the sources.

Electromagnetic fields interacting with GWs with torsion modes. Other application is the case where the geometry is not static, but rather it corresponds to that of gravitational waves interacting with the electromagnetic fields. One can consider in the very early universe a stochastic spectrum of GWs or the emission of GWs from astrophysical binaries, which will interact with the electromagnetic fields from one of the objects in the pair. In the first case one can consider the ECSK theory in the early universe, while in the second case one should consider also the propagating torsion from quadratic Poincaré gravity for example. The model-independent interaction of torsion modes coupled to electromagnetic fields in detectors would be the natural generalization of the study presented in chapter 7 (section 7.4) of the present thesis.

Torsion interactions with matter fields: Astrophysical implications. Other effects due to the coupling of torsion with the matter fields, with potential astrophysical applications include the spectrographic signatures of energy transitions induced by torsion within the ECDM model (generalizing the discussion in section 4.1). A non-negligible torsion effect in the energy levels of fermionic structures within the outer crust of magnetars might be probed with future observations and advanced spectrographs. Going beyond the EC theory and considering more general qPGTG where torsion can propagate, the analysis in chapter 4 (section 4.1) of the physical implications with potential astrophysical applications, can in principle be generalized. Moreover, studies of the interior of highly compact objects within the ECDM theory might reveal the possibility that self-interactions and non-minimal couplings in the matter fields, induced by torsion, give rise to an effective repulsive pressure that could regulate the instabilities found in GR above a certain mass/density threshold. In particular, the fermionic self-interactions induced by torsion might allow for equilibrium configurations of highly compact astrophysical bodies, with densities greater than the denser neutron stars ever found. The evolving field of GW astronomy will probe the nature of compact objects associated to the sources of gravitational radiation, while testing GR and extended theories of gravity.

Chapter 7

Gravitational Wave probes

Given the recent direct measurement of gravitational waves (GWs) by the LIGO-VIRGO collaboration, a new window to observational astrophysics and cosmology is open. In this chapter we review several topics on multimessenger astronomy and highlight its importance, section 7.1, present the basics of GW physics, in section 7.2, and study its relevance for astrophysics and cosmology and also for testing of theories of gravity in 7.3. We include an exploration of GWs with torsion modes within the linear perturbation approach to a RC spacetime geometry and consider the physics of GW propagation in qPGTG. Finally, we also study GW effects in electromagnetic fields 7.4 which in principle could be extended for theories beyond GR to search for extra polarizations and the effects from extra propagating tensor degrees of freedom. The exploration of the coupling between electromagnetic fields and gravity have a special relevance since it opens new perspectives for future GW detectors and also potentially provides information on the physics of highly energetic GW sources where both gravitational and electromagnetic fields need to be considered. We explore such couplings using the field equations of electrodynamics on (pseudo) Riemann manifolds and apply it to the background of a GW, seen as a linear perturbation of Minkowski geometry. Electric and magnetic oscillations are induced that propagate as electromagnetic waves and contain information about the GW which generates them. The most relevant results are the presence of longitudinal modes and dynamical polarization patterns of electromagnetic radiation induced by GWs. These effects might be amplified using appropriate resonators, effectively improving the signal to noise ratio around a specific frequency. We also briefly address the generation of charge density fluctuations induced by GWs and the implications for astrophysics.

7.1 GWs in the era of multimessenger astronomy

We have recently witnessed the birth of GW astronomy, when the LIGO/VIRGO Collaboration reported compelling evidence on the detection of gravitational waves, which is compatible with a scenario of binary black hole mergers predicted by General Relativity (GR) [13]. This finding (and others that followed) has sparked the interest in probing the strong-field regime of GR via gravitational wave observations of compact objects [16, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187]. At the writing of this thesis around 48 compact object coalescences (COC) events were reported from the first two runs of the LIGO-Virgo collaboration together with around 39 GW candidates for COC from the first half of the third observing run (O3) [188, 189]. The first detection of a neutron star binary merger [14], GW170817, with its electromagnetic counterpart compatible with a short gamma ray burst, was extremely important, clearly showing the potential of multimessenger astronomy [15]. After these, two more other NS-NS events from the second observing run (O2) and at least one candidate from the first half of O3, have been reported. The majority of the events and event candidates are consistently interpreted as the result of binary blackhole mergers with high confidence, but there are also a few cases where the nature of one of the pair's objects is unknown and these might correspond to systems composed of BH-NS pairs. The BH population from GW signals seems complementary to those inferred from X-ray BH candidates, having typically higher masses, even including what might be possibly confirmed as the first detections of so called intermediate mass BHs [190].

As more GW events are being detected, the window is open for a deeper understanding of the physical nature of the sources, through systematic analysis of the astrophysical and kinematical parameters that allow statistical characterization of the source populations. Merger rates are also better estimated as new data arrives, and these estimates together with the population characterization drives valuable inputs into the astrophysical modelling of the formation and evolution of compact objects and of compact binaries.

The GW emission from the coalescence of highly compact sources provides the opportunity to probe astrophysical phenomena in the very strong gravity regime. This is a fascinating opportunity to study not only GR but also extended theories of gravity both classically [191] as well as those including "quantum corrections" from quantum field theory. These can predict a GW signature of the non-classical physics happening at or near the black hole's horizon (see [192, 193]). In the aftermath of the first observations, researchers have quickly gone to discuss how well their favourite gravitational models extending GR have fared against them [180, 181, 182, 183, 184, 185, 186, 187]. The dynamical, highly non-linear, strong-gravity regimes involved in the processes of GW emission allow for unprecedented tests of GR and constraints on extended theories beyond GR, and therefore allows to deepen the study of the nature of gravity and of spacetime. The new window of GW astronomy also enables to deepen the arena of precision observational cosmology, by studying the propagation properties of gravitational radiation over cosmic distances and through the unique probing of the early Universe, that it provides. We will briefly review the basics of GWs in this chapter in connection to its astrophysical and cosmological relevance and present also non-standard methods to study GWs in both GR and in theories with propagating torsion. For reviews on the physics of GWs see for e.g. [49, 194, 195, 196, 197].

The GW spectrum: brief outlook. The history of GWs is quite rich, although we will not expose here its details. Nevertheless, it is never enough to mention that while Einstein first predicted GWs in 1918, until the mid of the 20th century there was a strong debate on the physical nature of such mathematical wave solutions to the field equations of GR. The main issue was concerned with coordinate freedom and the notion of gravitational energy. In the second half of the 20th century strong arguments supported the idea that these waves should be interpreted as physical waves carrying energy and momentum, and could interact with detectors in a measurable way, in particular they could generate friction and heat transfer. Meanwhile the first detectors were put into operation by Weber in the sixties. These were cylinder bars, essentially test masses that were expected to generate measurable effects (via electro-mechanical correlations) due to the logitudinal oscillations driven by the passing of GWs. Moving forward in time, the first known binary pulsar, the famous Hulse-Taylor (PSRB1913+16) system was subjected to systematic measurements of the orbital parameters that allowed to characterize the orbital decay over the years, since 1976 up to 2005, which revealed a remarkable agreement with the prediction from Einstein's GR of the energy loss via GW emission. This was an incredible agreement between theory and observations that favoured GR and its prediction of GWs, but it was the first indirect detection nevertheless. Meanwhile in the period ranging from 1980 to 1994 there were some advances and performances in cryogenic bar detectors as well as interferometer type of detectors [194, 49, 196].

The terrestrial interferometric GW detectors such as the 4 Km long LIGO (Caltech, MIT), VIRGO (France-CNRS, Italy-INFN) with 3 Km, GEO600 (Germany-Max Planck Institute, UK-PPARC) with 600 m, the 80 m AIGO (Australia) and the 300 m TAMA300-KAGRA (Japan), cover the range of frequencies from 1 Hz to $10^4 Hz$. Within the next decade LIGO india is expected to join this network of terrestrial detectors. Besides these there are also approved and planned missions for space interferometers such as eLISA (0,1 mHz - 1Hz), ASTROD-GW (0,01 Hz - 100Hz) and DECIGO (100 nHz - 1mHz). Therefore, these space and terrestrial detectors jointly span about 11 orders of magnitude in the GW spectrum. Moreover, the GW spectrum is further explored via Pulsar Timming Arrays (PTA), through the International PTA consortium (IPTA) which is a network formed by the jointly collaboration of the european PTA (EPTA), the North American Nanohertz Observatory for Gravitational Waves (NANOGrav), and the Parkes Pulsar Timing Array (PPTA), and aims at characterizing the low-frequency GW sources using an array of approximately 100 mili-second pulsars and a global array of radio telescopes. Once the Square Kilometer Array (SKA) of radio telescopes joins the observations, the observing capabilities are expected to improve significantly.

The main sources of GWs that are expected are compact binaries of stellar origin, core-collapse massive stars (for e.g Gamma Ray bursts GRB), isolated deformed neutron stars, compact objects captured by supermassive BH in extreme mass ratio inpirals (EMRI), Binaries of supermassive BHs, as well as cosmological drivers of stochastic backgrounds (such as quantum fluctuations in inflationary models, first-order phase transitions and topological defects). To have a glimpse on how these sources are distributed along the GW spectrum it is usefull to consider simple predictions based on Einstein's GR and heuristic arguments. Let us start by stating that GR theory makes five relevant predictions: The existence of BHs, an upper limit on the compactness of self-gravitating objects endowed with a surface (such as neutron stars), a maximum mass for stable degenerate matter at nuclear densities (via the Volkoff-Oppenheimer equation), an upper mass for hypothetical super-massive stars dominated by radiation pressure and the emission of GWs by accelerated masses in non-spherical motion, or more generally, by physical sources with a non-vanishing time-varying quadrupole moment. All these predictions are under observational tests within GW astronomy. A rigorous treatment of GWs in GR requires both analytical methods and numerical relativity tools. In the analytical case the most common approach is to consider perturbative methods. From perturbation theory, one gets a simple relation between the amplitude of the GW metric perturbation and the second time derivative of the source quadrupole moment.

Without going into details here let us now consider in general terms the relation between the frequency and the characteristic mass of GW sources [195]. In fact there is a close link between the frequency, the mass and the compactness of GW emitters. Let f_{nat} designate the natural unit for frequency, such that

$$f_{nat} \equiv \frac{c^3}{GM} = \frac{c}{R_G} \simeq 2 \times 10^5 \frac{M_\odot}{M} \quad Hz, \tag{7.1}$$

with $R_G \equiv \dfrac{GM}{c^2}$. It turns out that for any astrophysical source that is a self-bound system of mass M and typical size R, the natural frequency of oscillation, rotation, orbital revolution and dynamical collapse is given by

$$f_{source} \sim \left(\frac{GM}{R^3}\right)^{1/2} = f_{nat} C^{3/2}, \quad C \equiv \frac{R_G}{R} \tag{7.2}$$

where $C \leq 1$ is the typical compactness of the source. As a simple but useful estimation, one can say that the typical frequency of the GWs generated by such sources corresponds to this expression, i.e,

$$f_{GW} \sim f_{source} = f_{nat} C^{3/2}. \tag{7.3}$$

Let us take BHs as an example. In this case, the compactness is maximum, $C = 1$, therefore $f_{GW} \sim f_{nat}$. More precisely, if we consider binaries and take the inner most stable circular orbit (ISCO) we get that $f_{isco} \equiv f_{nat}/(\pi 6)^{3/2}$ and for stellar BHs [195]

$$f_{isco} \simeq 0,44 \times 10^3 \frac{10 M_\odot}{M} \quad Hz, \tag{7.4}$$

while for supermassive BHs

$$f_{isco} \simeq 4,4 \times 10^{-3} \frac{10^6 M_\odot}{M} \quad Hz. \tag{7.5}$$

In compact binaries coalescences the frequency of the GWs rises until a peak f_{GW}^{peak} at the time of the merger. For BHs, in general we have

$$f_{isco} \gg f_{GW} \longrightarrow f_{GW}^{peak} \sim f_{isco}. \tag{7.6}$$

These expressions give a simple estimation of the order of magnitude of frequencies of about $10^2 Hz$ to $10 kHz$ for stellar BHs (between $1 M_\odot$ and $10^2 M_\odot$) and from $\sim nHz$ to $0,1 \quad mHz$ for super massive BHs (between $10^5 M_\odot$ and $10^9 M_\odot$), with the intermediate range of frequencies being produced by the so called intermediate mass BHs. In simple terms, low frequency GW sources include coalescing (super) massive BHs, core-collapse of supermassive stars (with associated Type Ib/Ic Supernovae and Long-Gamma Ray burst) and Extreme mass ratio inspirals (EMRIs), while the high frequency sources are stellar origin coalescing compact binaries (mainly BHs and neutron stars) and isolated asymmetrical neutron stars.

Any detector has a specific characteristic sensitivity curve within a range of frequencies $f_{min} \leq f \leq f_{max}$. Using the formulas presented above that relate the frequency to the mass, we can make simple estimates on the lowest and maximum masses of GW sources that can be detected. For low frequency sources, we get [195]

$$4,4 \times 10^4 \left(\frac{0,1 Hz}{f_{max}} \right) M_\odot \leq M \leq 4,4 \times 10^7 \left(\frac{10^{-4} Hz}{f_{min}} \right) M_\odot, \tag{7.7}$$

and for high frequency sources

$$2,2 \left(\frac{2000 Hz}{f_{max}} \right) M_\odot \leq M \leq 440 \left(\frac{10 Hz}{f_{min}} \right) M_\odot. \tag{7.8}$$

Terrestrial observatories such as LIGO/VIRGO are essentially sensitive to stellar origin compact binary coalescences, core-collapsing massive stars and isolated perturbed (deformed) neutron stars (magnetars, for instance). While LISA is sensitive to compact binaries (including WD, NS and BHs), supermassive blackholes in the centers of galaxies and active galactic nuclei (AGN)[198, 199], EMRIs and GWs of cosmological origin (inflation driven amplification of quantum perturbations, first-order phase transitions in the early universe, topological defects such as cosmic strings). The PTA with milisecond pulsars as high precision clocks are sensitive to supermassive BHs, EMRIs and GWS from the early universe. The study of GW signals from isolated NS, stellar origin COCs, core-collapse supernova (of massive stars) and EMRIs, will allow to complement the information gathered via electromagnetic radiation and improve the characterization of the distribution of stellar remnants and compact binary populations in terms of mass, spin and other relevant astrophysical parameters. Moreover, the models for the formation and evolution of compact objects and binary populations leading to compact binaries can be constrained via GW maps. On the other hand, GW maps from BHs at the centres of galaxies from EMRI sources and supermassive BH binaries, will allow to complement the information from the electromagnetic spectrum and provide valuable inputs into the models for the formation and evolution of Galaxies and large scale structure, as well on a better understanding of the deep correlation between super-massive BHs and the host galaxies.

Finally, the detection and analysis of GWs from physical processes in the early Universe have the potential to probe beyond-standard model fundamental physics way beyond the surface of last scattering and the electroweak energy scale, and up to the Planck scale. In this way, GWs of cosmological origin can probe the early universe beyond the electromagnetic limits set by the epoch of recombination when photon decouples from the thermal equilibrium of the plasma of nuclei and electrons. These signals also probe the much earlier Universe than that which might be possible with cosmological neutrino backgrounds (generated when neutrino decouples from the plasma, when $T \sim 10^{10} K$, $kT \sim 0,9 MeV$). Moreover, the energies that can be probed are much higher than those accessible by any particle or astroparticle accelerators.

In summary, one can say that the GW window is complementary to the other messengers in astronomy, namely electromagnetic radiation and astroparticles (such as neutrinos). The first joint detection of GWs with an electromagnetic counterpart from the same source was the NS merger GW170817, that provided an incredible amount of valuable data to constrain the models for short-gamma ray burst with synergies between the GW data, the electromagnetic data over the full spectrum and the models for jet generation and collimation, turbulence, scock waves, r-processes of nuclei reactions, and other relevant phenomena related to the associated kilonovae. On the other hand, the first solid synergy between electromagnetic observations and cosmic rays detection have been reported in connection to the Blazar (TX S 0506+056, IceCube-170922A neutrino event). The confirmation that cosmic neutrinos have its origin in Blazars is a relevant achievement to be possible also by multimessenger astronomy. The Blazar is believed to be an AGN (Quasar) with the supermassive BH and its magnetized accretion disk producing a highly collimated jet that is directly pointed along the line of sight. The colimated ultra-relativistic jet is somehow generated from the conversion of gravitational, rotational and magnetic energy (via the Blandford-Znajec mechanism, for instance) and acts as an astrophysical accelerator of particles. Hadronic interactions can then produce neutrinos and gamma rays, with the potential for jointly detections of gamma ray flares and neutrino fluxes from the same source. Similarly, it is expected that sources involving BHs (of stellar origin or in AGNs) driving high energy astrophysics phenomena such as jets, are plausible candidates for a joint detection of GWs and astroparticles with electromagnetic counterparts.

Astrophysics with GWs. Gravitational wave astronomy will enable researchers to address specific relevant topics related to astrophysics and relativistic astrophysics. Here we summarize a few of these astrophysical topics that brings the attention of the GW astronomy community [195].

GW observations will enable us to study what is the maximum mass for neutron stars and what are the minimum and maximum masses for stellar BHs. With the accumulation of many detected events from COC, we can study the mass function and redshift distribution of neutron stars and stellar BHs. GWs from binaries also provide an indirect study of the dependence of the masses and spins of neutron stars and BHs with the astrophysical environment and with cosmic time. The study of compact object binaries with GW might also help to clarify which are the main formation channels for these binaries, namely via binary stars or from dynamical interactions in dense star clusters. Another question that might be directly addressed from the statistics of the observations is related to the typical merger rate of compact objects in galaxies and how does it evolve with redshift. Moreover, if GW signals from type Ib/Ic Supernovae are detected this might help to probe the physical mechanisms behind these events, therefore constraining the models of explosions driven by core-collapsing massive stars, for instance on how asymmetric is the collapse. The possible detection of further GW events with electromagnetic counterparts associated to short gamma-ray bursts (SGRBs) and Kilonovae might confirm the association of these explosions to NS-NS mergers. Also, it is still an open question if an electromagnetic counterpart from BH-BH mergers can be detected.

Another relevant questions include: Can we estimate from GW observations how many compact binaries of stellar origin exist in the Milky Way Galaxy, and how can this information constrain the models for the star formation history? Are type Ia Supernovae driven by ultra-compact WD binaries with accretion disks and mass transfers? What are the main processes for the formation and growth of massive BHs, is it via accretion and/or merger of less massive BHs or via direct collapse of hypothetical very massive stars? Can we characterize the population of stellar remnants around the supermassive BHs at the centres of galaxies? Are intermediate mass BHs (with masses ranging from $10^2 M_\odot$ to $10^5 M_\odot$) in the centres of dwarf galaxies? How do the BH's in the centres of Galaxies form and evolve, is it through accretion of material and/ or BH mergers? What is the typical rate of such growth? And finally: What is the merger rate of supermassive BHs? Can we probe the mass and spin distributions of the first BH's in the oldest Galaxies or in pre-galactic halos? How does galaxy mergers influences galaxy formation and evolution?

It is also relevant to mention that a stochastic foreground due to GWs from unresolved WD binaries within the Milky way can be a source of noise for LISA, but it can also be used to probe the morphology of the binary distribution within the Galaxy (due to regular modulations in the signal induced by the concentration of sources in the Galactic center and to the characteristics of the e-LISA antenna pattern) [199]. The details of this unresolved foreground signal can be used, not only to study the distribution of such compact binary population within the different Galaxy components such as thin disk, thick disk and halo, but also to make an estimation of the total number of ultra-compact binaries.

This summary is just an illustrative compilation but there are many other fields of research of astrophysical nature that GW astronomy can help to improve, and the possibility of detecting new astrophysical objects with GWs is a stimulating plausible scenario.

GW cosmology. The information from GW maps due to localized events at different redshifts can be correlated with electromagnetic maps from large-scale structure surveys, at different redshifts in order to better understand the properties of Galaxy cluster distributions. These maps and correlations, in principle can also be used to make cosmological parameter constraints. Just as electromagnetic signals are subject to strong lensing by specific sources and weak lensing due to large scale structure, also GW signals are lensed in a similar way. The potential is there to complement the information from the electromagnetic spectrum in order to improve cosmological parameter estimation.

Moreover, GW emission from compact binaries can be used as standard sirens, giving a direct measure of cosmic distances. If electromagnetic counterparts are known, then redshifts can be obtained and the Hubble parameter estimated, and many detections provide a statistically robust probe of the expansion rate history. Note that as we will see, the distances are extracted directly from the amplitude information of the (noise-free) GW signal. From the propagation properties of GWs through cosmological backgrounds, the Hubble flow can then be observationally probed via calibrated standard sirens and the models for dark energy (the equation of state) can be constrained.

The detection of stochastic backgrounds of cosmological origin can constrain inflationary models, the physics beyond the standard model, symmetry breaking mechanisms and first-order phase transitions, models for cosmic string networks and cosmic super strings, and extended theories of gravity in the very early Universe. In principle phase transitions induced by symmetry breaking from gauge theories of gravity can be probed and constrained, including the models with parity breaking terms which could leave its signature as chiral/polarized backgrounds. In any case, the computation of the predicted stochastic GW power spectrum for different theories of cosmological GWs is vital in order to distinguish the models and to distinguish also from the GW background signal due to an assembly of unresolved (super-massive) BH binaries.

Testing GR and the nature of gravity. The signals arriving at the detectors corresponding to GW event candidates are compared with a template of theoretical GW waveforms through the process of match filtering. By extending such analysis to allow for deviations from GR via model-independent post-Einsteinian parameters or through direct tests with wave forms from specific models, the opportunity to test GR and its extensions in the dynamical, strong-field regime provides a valuable contribution for the study of the nature of gravity and of spacetime. The direct testing of GR via match filtering allow researchers to analyse to what extent the strong-field, dynamical, non-linear regime of gravity is in accordance with the theoretical predictions of GR, concerning the fundamental properties of GWs, such as the wave-forms, the polarizations, the speed of propagation, its tensor nature, etc. Moreover one can test or constrain possible couplings with other dynamical degrees of freedom, such as massive or massless scalars. The accumulation of many signals compatible with BH-BH mergers and also from EMRI gather fundamental information on several important questions such as: Is the spacetime around astrophysical BHs as the Kerr metric? Is gravitational collapse into Kerr-type BHs inevitable (uniqueness hypothesis)? Is the no-hair conjecture[1] observationally valid? Can we test the cosmic censorship, i.e, the exclusion of naked singularities? As well as other fundamental and related questions.

The signals from NS-NS and BH-NS mergers also have a great potential to adress fundamental physics questions such as the nature of the short-range interactions (hadronic physics) and the states of matter at supra-nuclear densities, the equation of state for neutron stars, as well as the lowest energy state of baryonic matter at supra-nuclear densities. With respect to this and as briefly mentioned in the previous chapter, one may explore the possibility of self-gravitating objects in stable (or meta-stable) equilibrium configurations that are denser than neutron stars, in theories beyond GR. We briefly addressed this issue in the previous chapter in the context of PGTG.

[1]The no-hair conjecture essentially states that any detailed information regarding the physical properties of the matter that collapses into a BH (including the information on fluctuations, inhomogeneities and asymmetric distributions), is observationally inaccessible with the only parameters describing a BH being its total mass M, angular momentum J and charge Q.

7.2 Linearized gravity and weak GWs in GR - Fundamentals

Let us review the basics of linearised GR and (weak) GW propagation (for excellent reviews on GWs see [49, 195, 196], for e.g.). The usual procedure is to consider the perturbation theory applied to a Minkowski background metric $\eta_{\alpha\beta}$, therefore we start with the spacetime metric as follows

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x), \quad |h_{\alpha\beta}| \ll 1, \tag{7.9}$$

where $h_{\alpha\beta}$ represents a small gravitational perturbation. This expression has an intrinsic gauge symmetry, in the sense that, under local coordinate transformations,

$$x^\mu \to x^\mu + \xi^\mu(x), \tag{7.10}$$

the perturbation transforms as

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \tag{7.11}$$

leading to

$$g'_{\alpha\beta}(x') = \eta_{\alpha\beta} + h'_{\alpha\beta}(x'), \quad |h'_{\alpha\beta}| \ll 1. \tag{7.12}$$

Therefore, the definition of the spacetime metric as a small perturbation of the Minkowski metric has a coordinate freedom under a specific subgroup of local diffeomorphisms[2].

For convenience we define the so called trace-reversed metric perturbation $\bar{h}_{\mu\nu}$ as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{7.13}$$

where $h \equiv h^\alpha_\alpha$ is the trace of the metric perturbation and it is straightforward to see that $h = -\bar{h}$. In perturbation theory applied to a Minkowski background, the gravitational perturbation can be seen as a tensor field propagating in such a background, therefore indices are raised or lowered using the Minkowski metric. At the background level the Levi-Civita connection vanishes and so does the Ricci tensor. From the perturbed metric $g_{\mu\nu}$ in (7.9), one can compute a Levi-civita connection $\delta\Gamma^\lambda_{\ \mu\nu}$ and Ricci tensor $\delta R_{\mu\nu}$, substitute into the Einstein tensor and simplify in terms of the redefined (trace reverse) metric perturbation, to arrive at

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\alpha \partial_\nu \bar{h}^\alpha_{\ \mu} + \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha \partial^\beta \bar{h}^\alpha_{\ \beta}\right). \tag{7.14}$$

Recall that this expression is valid for a small perturbation of the Minkowski background metric and therefore has the same (gauge) symmetry as (7.9). More specifically, under the transformation of coordinates (7.10), this expression remains invariant, while the redefined (trace-reversed) perturbation transforms as

$$\bar{h}'_{\mu\nu}(x') = \bar{h}_{\mu\nu}(x) + \left(\eta_{\mu\nu}\partial_\alpha \xi^\alpha - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu\right). \tag{7.15}$$

Using this freedom one chooses the Lorenz gauge,

$$\partial'^\mu \bar{h}'_{\mu\nu}(x') = 0, \tag{7.16}$$

[2]The expression in (7.12) is valid, provided certain conditions are respected. For example the Lorentz boosts Λ^α_μ only respect the symmetry with $|h'_{\alpha\beta}| << 1$ if $|\Lambda^\alpha_\mu \Lambda^\beta_\nu h_{\alpha\beta}| << 1$.

and since from (7.15) we get

$$\partial'^{\mu}\bar{h}'_{\mu\nu}(x') = \partial^{\mu}\bar{h}_{\mu\nu}(x) - \Box\xi_{\nu}, \tag{7.17}$$

choosing ξ such that $\Box\xi_{\nu} = \partial^{\mu}\bar{h}_{\mu\nu}(x)$, is enough to respect the Lorenz gauge. In the Lorenz gauge the Einstein tensor simplifies, i.e,

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}, \tag{7.18}$$

leading to the linearised Einstein equations

$$\Box\bar{h}_{\mu\nu} = -\frac{2}{m_p^2}T_{\mu\nu}. \tag{7.19}$$

In vacuum, $\Box\bar{h}_{\mu\nu} = 0$, and naturaly the superposition of harmonic solutions hold

$$\bar{h}_{\mu\nu} = \int d^3k\left(\bar{h}_{\mu\nu}(\vec{k})e^{ik^{\mu}x_{\mu}} + \bar{h}^{*}_{\mu\nu}(\vec{k})e^{-ik^{\mu}x_{\mu}}\right). \tag{7.20}$$

Since there is still some residual gauge freedom under coordinate transformations, such that

$$x^{\mu} \to x^{\mu} + \xi^{\mu}, \quad \Box\xi^{\nu} = 0, \tag{7.21}$$

preserves the Lorentz condition (7.16), it is common to choose the transverse-traceless (TT) gauge, such that there is no difference between the metric perturbation and the re-defined one, i.e $h_{\mu\nu} = \bar{h}_{\mu\nu}$, and

$$h_{\mu 0} = 0, \quad h^{i}_{i} = h = 0, \quad \partial^{i}h_{ij} = 0. \tag{7.22}$$

As a direct consequence, one has $h_{11} = -h_{22} \equiv h_{+}$ and $h_{12} = h_{21} \equiv h_{\times}$, i.e,

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The expressions in (7.22) represent 8 conditions, therefore leading to the prediction of solely 2 propagating physical tensor modes in GR. In this gauge it is straightforward to see that GW in GR are transverse. For instance for harmonic solutions $\bar{h}_{\mu\nu} = \alpha_{\mu\nu}e^{ik^{\mu}x_{\mu}}$, where $\alpha_{\mu\nu}$ is the polarization tensor, the Lorentz condition implies that $k^{\mu}\alpha_{\mu\nu} = 0$. Moreover, substitution of this harmonic solution into the vacuum Einstein equations $\Box\bar{h}_{\mu\nu} = 0$, leads to $k^{\mu}k_{\mu} = 0$, which implies that GW propagate at the speed of light following the same causal light cones as Maxwell electromagnetic waves in the geometrical optics limit.

Since, GW travel vast distances across galactic, extra-galactic and cosmological scales, the amplitude of the gravitational fluctuations are extremely small, which motivates the application of the linearised (perturbation) theory for simple calculations. A typical example is the consideration of the dynamical effect on the test-masses of detectors due to the passing of GWs. In the transverse-traceless gauge the only non-zero components of the Riemann tensor are given by

$$R_{i0j0} = -\frac{1}{2}\ddot{h}_{ij}. \tag{7.23}$$

After substitution into the geodesic deviation equation between test masses, this leads to the two well-known (+) and (×) motions (polarizations) for a set of test particles in an initial circular distribution in the plane perpendicular to the direction of propagation of the GW. In fact, it is the Weyl part of the Riemann curvature that propagates in vacuum, therefore the perturbations induced in any material system change shapes while preserving the volumes.

If one considers weak sources of GWs, Green functions methods can be used to find solutions to the Einstein equations with the source term (7.19). One finds the retarded solutions

$$\bar{h}_{ij}(t,\vec{x}) = \frac{4G}{c^4}\int d^3y \frac{T_{ij}(t_R,\vec{y})}{|\vec{x}-\vec{y}|}, \tag{7.24}$$

where,$i,j = 1,2,3$ and $t_R = t - |\vec{x}-\vec{y}|/c$. From this expression in the limit of a non-relativistic source (with typical velocities $v << c$), the GWs with wavelengths much larger than the characteristic size of the source, at a distance r far from the source, obey the following

$$\bar{h}_{jk} = \frac{2G}{c^4 r}\frac{d^2 I_{jk}}{dt^2}, \quad I_{jk} = \int \rho(t,\vec{y}) y_j y_k dy^3 \tag{7.25}$$

where ρ is a mass density and I_{jk} is the second moment of the mass distribution in the source. In the TT gauge we get

$$h^{TT}_{jk} = \frac{2G}{c^4 r}\left(\frac{d^2\mathcal{Q}_{mn}}{dt^2}\right) P^m_j P^n_k, \tag{7.26}$$

where the traceless tensor $\mathcal{Q}_{ik} = I_{ik} - \frac{1}{3}\delta_{ij} I$, can be defined as the (reduced) quadrupole moment, $I \equiv I^j_j$ is the trace of the second mass moment and $P_{ij} \equiv \delta_{ij} - n_i n_j$ is a projection tensor used to clear out vector components parallel to $\vec{n} = \vec{x}/r$, where $\vec{x}$ is the position vector of the source. Moreover, the luminosity emitted in GWs from a non-relativistic system with weak internal gravity is given by the famous quadrupole formula, originally derived by Einstein in 1918,

$$L = \frac{dE}{dt} = \frac{1}{5}\langle \dddot{\mathcal{Q}}_{ij} \dddot{\mathcal{Q}}^{ij} \rangle, \tag{7.27}$$

where the brackets denotes time averaging over a wave period. Applying this formula for a binary star system in circular orbits around the center of mass, with relative separation a and total mass M, one finds that

$$|L| = \frac{32G^4}{5c^5}\frac{M^3\mu^2}{a^5}, \tag{7.28}$$

where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass. The amplitude is proven to obey

$$\bar{h}_{ij} \sim 8GM\frac{w^2 a^2}{r}, \tag{7.29}$$

where w is the angular velocity. Both these expressions can be corrected to account for the eccentricities, but to illustrate the simple ideas we will not consider such corrections here. Using Kepler's third law $w = \left(\frac{GM}{4a^3}\right)^{1/2}$, therefore the frequency is given by

$$f = \frac{cR_S^{1/2}}{4\pi\sqrt{2}a^{3/2}}, \tag{7.30}$$

where $R_S = 2GM/c^2$ is Schwarzchild radius. The amplitude which we can term as h symbolically, can be expressed as $h \sim R_S^2/ra$. Therefore, if one considers for example a binary system with total mass of about $M \sim 10M\odot$, with a relative separation $a \sim 10R_S$, at a distance of 100 Mpc, one obtains $f \sim 10^2 Hz$ and $h \sim 10^{-21}$. By its turn, $h \sim \delta L/L$ gives a measure of the relative variation in the length L of the arms of interferometers. These simple expressions clearly illustrate the need for huge arm detectors, in order to detect the minute changes δL. Notice that as energy is radiated away from the binary system via GWs, the orbit shrinks and the amplitude and frequency of the GW increase.

It is expected that compact object coalescences are by far the most common sources of GWs. Typically, one characterizes three important regimes of the GW emission from binary mergers, using the Bh-BH mergers as paradigmatic examples: the inspiral, the merger and the ring-down. The ring-down consists of a brief period of relaxation of the system into the most stable and symmetrical equilibrium configuration that the system can take. During this period the GW amplitude decays and this signal can provide very important information about the final object after the merger and is very useful for testing GR and for searches of signatures from extended theories of gravity. In the merger and ring-down, the strong gravity regime presuppose non-linear effects making it impossible to use only analytical tools. In the inspiral phase the separation and velocities between the two objects in the binary are such that analytical methods can be applied using perturbation theory, while a complete treatment of the merger and the ring-down require numerical methods. Both the analytical part and the numerical part of the waveform are then matched together with a consistency regarding the kinematical and physical parameters of the binary that can be extracted from the GW signal. Finally, the propagation through cosmological distances have to be taken into account in the theoretical modelling in order to make plausible/realistic comparisons with the observed signals that where emitted quite often from sources at Mpc distance scales. Let us now look at the propagation of GW in a cosmological context.

7.3 GW cosmology

7.3.1 GWs in cosmological backgrounds: GR

To address the propagation in a cosmological background in GR we start by considering the propagation on a curved spacetime metric. In this case we take the following perturbation

$$g_{\mu\nu}(x) = \mathring{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \tag{7.31}$$

where $\mathring{g}_{\mu\nu}$ is the background metric of a curved spacetime. Without going into the details, the Riemann tensor, Ricci tensor and Ricci scalar are computed from the metric in (7.31) and then the energy-momentum tensor is also subject to a perturbative approach

$T_{\mu\nu} = \mathring{T}_{\mu\nu} + \delta T_{\mu\nu}$, leading to the Einstein equations

$$- \frac{1}{2}\mathring{\Box}\bar{h}_{\mu\nu} + \mathring{R}_{\lambda\mu\nu\sigma}\bar{h}^{\lambda\sigma} + \mathring{\nabla}_{(\nu}\mathring{\nabla}^{\sigma}\bar{h}_{\mu)\sigma} - \frac{1}{2}\mathring{g}_{\mu\nu}\mathring{\nabla}^{\alpha}\mathring{\nabla}^{\beta}\bar{h}_{\alpha\beta} \tag{7.32}$$

$$+\mathring{R}^{\alpha\beta}\left(\frac{1}{2}\mathring{g}_{\mu\nu}\bar{h}_{\alpha\beta} - \frac{1}{2}\mathring{g}_{\alpha\beta}\bar{h}_{\mu\nu} + \mathring{g}_{\beta(\mu}\bar{h}_{\nu)\alpha}\right) = 8\pi G\delta T_{\mu\nu}, \tag{7.33}$$

for the redefined trace-reverse metric perturbation $\bar{h}_{\mu\nu}$ and we are now using $c = 1$ units. At the background level it is assumed that the Einstein equations are also satisfied, i.e, $\mathring{G}_{\mu\nu} = 8\pi G\mathring{T}_{\mu\nu}$.

Making a coordinate transformation into the (generalized) Lorenz gauge

$$\mathring{\nabla}^{\mu}\bar{h}_{\mu\nu} = 0, \tag{7.34}$$

we get

$$-\mathring{\Box}\bar{h}_{\mu\nu} + 2\mathring{R}_{\lambda\mu\nu\sigma}\bar{h}^{\lambda\sigma} = 16\pi G\delta T_{\mu\nu}, \tag{7.35}$$

where $\Box \equiv \partial^{\alpha}\partial_{\alpha}$. For cosmological applications we take the perturbed background metric of FRWL models, in the flat case $k = 0$ for simplicity, and considering solely the tensor perturbations, i.e,

$$ds^2 = dt^2 - a^2(t)\left(\delta_{ij} + h_{ij}\right)dx^i dx^j. \tag{7.36}$$

In the transverse, traceless gauge, $\partial^i h_{ij} = 0$ and $h^i_i = 0$, and the Einstein equations (7.35) become

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = 16\pi G\Pi^{TT}_{ij}, \tag{7.37}$$

where $H(a)$ is the Hubble parameter, $\Pi^{TT}_{ij} = \Pi_{mn}P^m_i P^n_j$ is the transverse traceless part of the anisotropic stress Π_{ij}, with $a^2\Pi_{mn} = T_{mn} - pa^2(\delta_{mn} + h_{mn})$, which corresponds to the (physical) tensor perturbations in $\delta T_{\mu\nu}$, and p is the pressure of the source. Recall that in the TT gauge there is no distinction between the metric perturbation and its trace reverse. Considering the superposition of harmonic solutions

$$h_{ij}(\vec{x}, t) = \sum_{(p)=+,\times}\int \frac{d^3k}{(2\pi)^3}h_{(p)}(\vec{k}, t)e^{-i\vec{k}\cdot\vec{x}}e^{(p)}_{ij}(\vec{e}_k), \tag{7.38}$$

where the index p represents the contribution of the two polarizations $+, \times$, while $e^{(p)}_{ij}$ is the polarization tensor, $h_{(p)}$ is the amplitude for each polarization and Fourier mode, and $\vec{e}_k$ is the unit vector along the direction of propagation. We can then change to Fourier space and make the transformation into conformal time ($d\eta = dt/a(t)$), with

$$ds^2 = a^2(\eta)\left(d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j\right), \tag{7.39}$$

and the Einstein wave equations become

$$H''_{ij}(\vec{k}, \eta) + \left(k^2 - \frac{a''}{a}\right)H_{ij}(\vec{k}, \eta) = 16\pi G a^3\Pi^{TT}_{ij}(\vec{k}, \eta). \tag{7.40}$$

In this expression the prime denotes derivation with respect to conformal time, $k^2 = \|\vec{k}\|$ and $H_{ij}(\vec{k}, \eta) \equiv a h_{ij}(\vec{k}, \eta)$. This equation in vacuum and in terms of the amplitudes for each polarization and each Fourrier mode becomes

$$H''_{(p)}(\vec{k}, \eta) + \left(k^2 - \frac{a''}{a}\right)H_{(p)}(\vec{k}, \eta) = 0, \tag{7.41}$$

with $H_{(+,\times)}(\vec{k},\eta) = a h_{(+,\times)}(\vec{k},\eta)$. This equation can also be rewritten as

$$h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = 0, \tag{7.42}$$

where we dropped the $(\vec{k},\eta)$ dependences and $\mathcal{H} \equiv a'/a$. It can be shown that because of the second term induced by the cosmological expansion we get that the observed frequency is the redshifted emitted frequency and the amplitude decays with the (GW) luminosity distance, i.e,

$$f^{obs} = f^{emit}/(1+z), \quad h \sim 1/d_L^{GW}. \tag{7.43}$$

For a binary system with masses m_1 and m_2, in linear perturbation with a Minkowski background one gets the following general expression

$$h_{+,\times} \sim \frac{\mathcal{M}_c^{5/3} f^{obs}}{r} F_{+,\times} \cos\Phi(t), \tag{7.44}$$

with $\mathcal{M}_c \equiv \mu^{3/5} M^{2/5}$ being the shirp mass, where M is the total mass and μ is the reduced mass. The function $F_{+,\times}$ gives a geometric expression with the information on the binary configuration with respect to the detector while the time dependence is contained within the phase $\Phi(t)$. Now, for waves propagating over cosmological distances it is sufficient to take the solution in (7.44) and simply make the replacements

$$\mathcal{M}_c \to \mathcal{M}_c(1+z), \quad 1/r \to 1/d_L^{GW}, \tag{7.45}$$

where the luminosity distance is

$$d_L^{GW} = (1+z)\int_0^z \frac{c}{H(z')} dz'. \tag{7.46}$$

Therefore, directly from the amplitude of the detected GW signal one can obtain the luminosity distance. If the redshift of the source is also known from observations, one can then probe the expansion history, by comparing with the theoretical models for the Hubble parameter evolution as a function of the different mass-energy components in the Universe, such as dark matter, dark energy, radiation and (pressureless) baryonic matter. In this way, compact binary coalescences can be used as standard sirenes providing another independent observational tool for a cosmic distance ladder, beyond the conventional methods used with electromagnetic astronomy (such as Cepheid variable stars and type Ia supernovae). The redshift of the source can be obtained directly from observations if an electromagnetic counterpart is detected or indirectly via statistical methods applied to an ensemble of potential host galaxies, after an estimation of the sky location of the GW emitter is obtained from triangulation of the information from several detectors.

7.3.2 GWs in cosmological backgrounds: modified gravity

Model-independent corrections

In order to test extended theories of gravity with GW signals, one can take two approaches. The first case corresponds to model-dependent tests, while in the second case a model-independent formalism is developed for testing classes of theories of gravity [207].

In both cases one of the main goals is to construct theoretical wave forms for compact object coalescences to be compared with GW event candidates. As previously mentioned, the inspiral phase can be treated analytically and the Parametrized post-Newtonian formalism (PPN) is considerd, by considering Taylor expansions including first-order (weak field) plus higher-order (Post-Newtonian) contributions, where the parameters in the expansion can take different possible values from a parameter space that represents classes of theories of gravity including GR and its extensions. the first order terms are some times called Newtonian, but in fact, it corresponds to linearised gravity that already includes corrections to Newtonian physics (for example gravitomagnetic corrections [208][3]). As already mentioned in chapter 6 the PPN formalism was originally developed for solar system tests, while in the context of binaries sometimes one speaks of the (parametrized) post-Keplerian formalism. In the PPN formalism the perturbation theory is applied to the background spacetime. On the other hand, in order to address the strong gravity and non-linear effects fully, one can use the Parametrized post-Einsteinian (PPE) formalism which parameterizes deviations from general relativistic wave forms. The PPE is then applied to the complete regime (inspiral+merger+ring-down) of the GW emission using numerical methods and semi-analytical tools. In general, when modified gravity is taken into account, together with the propagation over cosmological backgrounds the principal types of possible beyond-GR effects are the following: additional polarizations, additional degrees of freedom, amplitude effects (leading to different luminosity distances), effects on the phase and different dispersion relations (including velocity of propagation different than c)[4].

One possible parametrization encompassing a wide class of theories of gravity is provided in the following wave equation in cosmological backgrounds

$$h''_{ij} + (2+\nu)\mathcal{H}h'_{ij} + (c_g^2 k^2 + m_g^2 a^2)h_{ij} = \tilde{\Pi}_{ij}. \tag{7.47}$$

In this equation the parameters (ν, c_g, m_g) and the source term represent deviations from GR, as can be seen by comparison with (7.42). The parameter ν represents an additional "friction" term, affecting the decay induced by the cosmological expansion. The effect of a propagation speed different than the speed of light is represented by c_g, while m_g stands for an effective mass for the graviton ($s = 2$) field. In GR $\nu = 0$, $c_g = 1$ and $m_g = 0$ and the right-hand side of (7.47) is equal to zero. The tensor $\tilde{\Pi}_{ij}$ represents an extra "source" term due to additional propagating tensor fields.

In the PGTG for example, there are two sets of field equations corresponding to the dynamics of the two gravitational potentials $(\theta^a_\mu, \Gamma^a_{b\nu})$, and this system of equations can be expressed in terms of the equivalent set of dynamical variables $(g_{\alpha,\beta}, K^\alpha_{\mu\nu})$. In quadratic PG models the torsion in general propagates and therefore, it is clear that in this case the contorsion is an extra propagating tensor field with respect to GR and therefore, the wave equation for the metric in the first-order perturbation theory in cosmological backgrounds will have a term as $\tilde{\Pi}_{ij}$ in (7.47) due to the propagating torsion (or contorsion) tensor degrees of freedom.

Let us start by considering the absence of extra propagating tensor fields.

a) Case with $\tilde{\Pi}_{ij} = 0$

[3]If torsion is present in the gravitational theory under consideration, the gravitomagnetic terms include torsion corrections with respect to GR.

[4]In the mathematical expressions used in this subsection, unless stated otherwise, we are taking $c = 1$.

It can be shown that the wave solution to (7.47) in this case can be written in the following way, simbolically

$$h_{GW} \sim h_{GR} e^{-\frac{1}{2}\int \nu \mathcal{H} d\eta} e^{ik\int \left(\alpha_T + a^2 m_g^2/k^2\right)^{1/2} d\eta}, \tag{7.48}$$

where h_{GR} represents the GR solution and $\alpha_T \equiv c_g^2 - 1$. The first exponential factor (depending on ν which is different for various gravity models) affects the amplitude, while last factor encompasses effects in the phase of the wave form. Moreover, due to the possibility of a different decay law and speed of propagation, the luminosity distance is different than in GR and also different to the luminosity distance from electromagnetic signals. The corresponding expression is given by

$$d_L^{new} = (1+z)\frac{c_g(z)}{c_g(0)} e^{\frac{1}{2}\int_0^z \frac{\nu}{1+z'}dz'} \int_0^z \frac{c_g(z')}{H(z')} dz' \tag{7.49}$$

Next we consider the presence of extra tensor propagating degrees of freedom.

b) Case with $\tilde{\Pi}_{ij} \neq 0$

In FRWL background, this "source" term is present whenever there are additional propagating tensor modes. One example is that of bi-gravity, where under a suitable (re)definition of the metric perturbations, the coupled wave equations can be written as:

$$\begin{pmatrix} h'' \\ \gamma'' \end{pmatrix} + (k^2 + m_g^2) \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} \begin{pmatrix} h \\ \gamma \end{pmatrix} = 0.$$

In this equation, the Hubble "friction" term was absorbed in the definition of the metric perturbations (h, γ) and θ is a mixing angle, such that the the system of the two coupled wave equations represent interactions between the two metric propagating (tensor) fields leading to the prediction of GW oscillations in analogy to the neutrino oscillations due to the mixing of the different species. Following this example, one expects a similar mixing effect for the case of metric and torsion coupled GW modes.

Linear perturbation theory in RC spacetime: GW with torsion modes

We will apply a perturbation method to the RC spacetime. The background spacetime in general is assumed to have the following metric and connection $(\mathring{g}_{\mu\nu}, \mathring{\Gamma}^\alpha_{\ \lambda\beta})$. The perturbed quantities are

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}, \quad \Gamma^\alpha_{\ \lambda\beta} = \mathring{\Gamma}^\alpha_{\ \lambda\beta} + \gamma^\alpha_{\ \lambda\beta}, \tag{7.50}$$

where the perturbations $h_{\mu\nu} \equiv \delta g_{\mu\nu}$ and $\gamma^\alpha_{\ \lambda\beta} \equiv \delta\Gamma^\alpha_{\ \lambda\beta}$, are assumed to be small. The RC connection is given by $\Gamma^\alpha_{\ \beta\nu} = \tilde{\Gamma}^\alpha_{\ \beta\nu} + K^\alpha_{\ \beta\nu}$, expression also valid at the background level. Therefore

$$\Gamma^\alpha_{\ \beta\mu} = \mathring{\Gamma}^\alpha_{\ \beta\mu} + \gamma^\alpha_{\ \beta\mu} = \left(\tilde{\mathring{\Gamma}}^\alpha_{\ \beta\mu} + \mathring{K}^\alpha_{\ \beta\mu}\right) + \gamma^\alpha_{\ \beta\mu} = \tilde{\Gamma}^\alpha_{\ \beta\mu} + K^\alpha_{\ \beta\mu}, \tag{7.51}$$

and since $\tilde{\Gamma}^\alpha_{\ \beta\mu}$ is the Levi-Civita connection corresponding to the metric in (7.50), then

$$\tilde{\Gamma}^\alpha_{\ \beta\mu} = \tilde{\mathring{\Gamma}}^\alpha_{\ \beta\mu} + \tilde{h}^\alpha_{\ \beta\mu}, \tag{7.52}$$

where

$$\tilde{h}^{\alpha}_{\ \beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left(\partial_\beta h_{\gamma\lambda} + \partial_\gamma h_{\beta\lambda} - \partial_\lambda h_{\beta\gamma} \right) + \frac{1}{2} h^{\alpha\lambda} \left(\partial_\beta \mathring{g}_{\gamma\lambda} + \partial_\gamma \mathring{g}_{\beta\lambda} - \partial_\lambda \mathring{g}_{\beta\gamma} \right) \tag{7.53}$$

and at first order

$$\frac{1}{2} g^{\alpha\lambda} \left(\partial_\beta h_{\gamma\lambda} + \partial_\gamma h_{\beta\lambda} - \partial_\lambda h_{\beta\gamma} \right) \simeq \frac{1}{2} \mathring{g}^{\alpha\lambda} \left(\partial_\beta h_{\gamma\lambda} + \partial_\gamma h_{\beta\lambda} - \partial_\lambda h_{\beta\gamma} \right) \tag{7.54}$$

and therefore, from (7.51), we have

$$K^{\alpha}_{\ \beta\mu} = \mathring{K}^{\alpha}_{\ \beta\mu} + \gamma^{\alpha}_{\ \beta\mu} - \tilde{h}^{\alpha}_{\ \beta\mu}. \tag{7.55}$$

with $\delta K^{\alpha}_{\ \beta\mu} \equiv K^{\alpha}_{\ \beta\mu} - \mathring{K}^{\alpha}_{\ \beta\mu}$ and $\delta\Gamma^{\alpha}_{\ \beta\mu} \equiv \Gamma^{\alpha}_{\ \beta\mu} - \mathring{\Gamma}^{\alpha}_{\ \beta\mu} = \gamma^{\alpha}_{\ \beta\mu}$, i.e

$$\delta\Gamma^{\alpha}_{\ \beta\mu} = \delta K^{\alpha}_{\ \beta\mu} + \tilde{h}^{\alpha}_{\ \beta\mu}. \tag{7.56}$$

The perturbation in the connection includes a perturbation in contorsion and another part related to the background metric and metric perturbation. In summary, the relevant quantities in this perturbative approach can be chosen to be

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}, \quad K^{\alpha}_{\ \lambda\beta} = \mathring{K}^{\alpha}_{\ \lambda\beta} + \kappa^{\alpha}_{\ \lambda\beta}, \tag{7.57}$$

where $\kappa^{\alpha}_{\ \beta\mu} \equiv \delta K^{\alpha}_{\ \beta\mu} = \gamma^{\alpha}_{\ \beta\mu} - \tilde{h}^{\alpha}_{\ \beta\mu}$. One might be interested in studying the case where the background geometry can be approximately described by a Riemann geometry, that is, $\mathring{K}^{\alpha}_{\ \beta\mu} \simeq 0$, and also in the flat background limit. In that case

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}, \quad \mathring{g}_{\mu\nu} = \eta_{\mu\nu} \quad , K^{\alpha}_{\ \lambda\beta} = \kappa^{\alpha}_{\ \lambda\beta} = \gamma^{\alpha}_{\ \beta\mu} - \frac{1}{2} \eta^{\alpha\lambda} \left(\partial_\beta h_{\gamma\lambda} + \partial_\gamma h_{\beta\lambda} - \partial_\lambda h_{\beta\gamma} \right). \tag{7.58}$$

In the most general case the background has a RC geometry. Consider the case of a torsionless curved background. Recall that the difference between two connections is a tensor, therefore the tensor $\gamma^{\alpha}_{\ \beta\mu}$, which is in that case is a perturbation of a Riemannian (LeviCivita) connection, has to have a non-vanishing antisymmetric part in the last two indices $\gamma^{\alpha}_{\ [\beta\mu]} \neq 0$ in order to induce a RC geometry, at the perturbative level. We now consider the curvature quantities in (3.27)-(3.29). The RC curvature can be then written as

$$\begin{aligned} R^{\alpha}_{\ \beta\mu\nu} \ &= \ \tilde{R}^{\alpha}_{\ \beta\mu\nu} + 2\tilde{\nabla}_{[\mu} \mathring{K}^{\alpha}_{\ \beta|\nu]} + 2\tilde{\nabla}_{[\mu} \kappa^{\alpha}_{\ \beta|\nu]} + \mathring{K}^{\alpha}_{\ \lambda\mu} \mathring{K}^{\lambda}_{\ \beta\nu} - \mathring{K}^{\alpha}_{\ \lambda\nu} \mathring{K}^{\lambda}_{\ \beta\mu} \\ &\quad + \mathring{K}^{\alpha}_{\ \lambda\mu} \kappa^{\lambda}_{\ \beta\nu} - \mathring{K}^{\alpha}_{\ \lambda\nu} \kappa^{\lambda}_{\ \beta\mu} + \kappa^{\alpha}_{\ \lambda\mu} \mathring{K}^{\lambda}_{\ \beta\nu} - \kappa^{\alpha}_{\ \lambda\nu} \mathring{K}^{\lambda}_{\ \beta\mu}, \end{aligned} \tag{7.59}$$

and for a torsionless background we obtain

$$R^{\alpha}_{\ \beta\mu\nu} \ = \ \tilde{R}^{\alpha}_{\ \beta\mu\nu} + 2\tilde{\nabla}_{[\mu} \kappa^{\alpha}_{\ \beta|\nu]}. \tag{7.60}$$

Then, from (7.52), at the first order in the metric and connection perturbations, we get

$$R^{\alpha}_{\ \beta\mu\nu} = \tilde{R}^{\alpha}_{\ \beta\mu\nu} + 2\mathring{\tilde{\nabla}}_{[\mu} \kappa^{\alpha}_{\ \beta|\nu]}. \tag{7.61}$$

Similarly, the generalized Ricci tensor is given by

$$\begin{aligned} R_{\beta\nu} &= \tilde{R}_{\beta\nu} + 2\tilde{\nabla}_{[\alpha}\mathring{K}^{\alpha}_{\ \beta|\nu]} + 2\tilde{\nabla}_{[\alpha}\kappa^{\alpha}_{\ \beta|\nu]} + \mathring{K}^{\alpha}_{\ \lambda\alpha}\mathring{K}^{\lambda}_{\ \beta\nu} - \mathring{K}^{\alpha}_{\ \lambda\nu}\mathring{K}^{\lambda}_{\ \beta\alpha} \\ &\quad + \mathring{K}^{\alpha}_{\ \lambda\alpha}\kappa^{\lambda}_{\ \beta\nu} - \mathring{K}^{\alpha}_{\ \lambda\nu}\kappa^{\lambda}_{\ \beta\alpha} + \kappa^{\alpha}_{\ \lambda\alpha}\mathring{K}^{\lambda}_{\ \beta\nu} - \kappa^{\alpha}_{\ \lambda\nu}\mathring{K}^{\lambda}_{\ \beta\alpha}, \end{aligned} \tag{7.62}$$

and for a torsionless background, to first order in the perturbations

$$R_{\beta\nu} = \tilde{R}_{\beta\nu} + 2\mathring{\tilde{\nabla}}_{[\alpha}\kappa^{\alpha}_{\ \beta|\nu]} \tag{7.63}$$

while the generalized Ricci curvature scalar

$$\begin{aligned} R &= \tilde{R} - 2\tilde{\nabla}^{\lambda}\mathring{K}^{\alpha}_{\ \lambda\alpha} - 2\tilde{\nabla}^{\lambda}\kappa^{\alpha}_{\ \lambda\alpha} + g^{\beta\nu}(\mathring{K}^{\alpha}_{\ \lambda\alpha}\mathring{K}^{\lambda}_{\ \beta\nu} - \mathring{K}^{\alpha}_{\ \lambda\nu}\mathring{K}^{\lambda}_{\ \beta\alpha}) \\ &\quad + g^{\beta\nu}(\mathring{K}^{\alpha}_{\ \lambda\alpha}\kappa^{\lambda}_{\ \beta\nu} - \mathring{K}^{\alpha}_{\ \lambda\nu}\kappa^{\lambda}_{\ \beta\alpha}) + g^{\beta\nu}(\kappa^{\alpha}_{\ \lambda\alpha}\mathring{K}^{\lambda}_{\ \beta\nu} - \kappa^{\alpha}_{\ \lambda\nu}\mathring{K}^{\lambda}_{\ \beta\alpha}), \end{aligned} \tag{7.64}$$

and for the torsionless background at first order

$$R = \tilde{R} - 2\mathring{\tilde{\nabla}}^{\lambda}\kappa^{\alpha}_{\ \lambda\alpha}., \tag{7.65}$$

We recall that in the generalized Lorenz gauge (7.34), the Einstein tensor in first order perturbation theory is given by

$$\tilde{G}_{\mu\nu} = -\frac{1}{2}\mathring{\Box}\bar{h}_{\mu\nu} + \mathring{R}_{\lambda\mu\nu\sigma}\bar{h}^{\lambda\sigma}, \tag{7.66}$$

and for a FRWL cosmological and torsionless background, we get the relevant components in the TT gauge given by

$$\tilde{G}_{ij} = \frac{a^2}{2}\left(\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij}\right). \tag{7.67}$$

In order to study the propagation of tensor perturbations at the linear level of perturbation theory, including torsion modes we will consider qPGTG. The torsionless FRWL geometry can be considered at the background level for simplicity.

7.3.3 GW in qPGTG

We now take into consideration the class of quadratic Poincaré Gauge theories of Gravity. A self-consistent non-perturbative analytical treatment of GW in PGTG can be seen in the literature (see for e.g. [80]). Those authors have derived harmonic solutions with propagating torsion modes, including $s = 0, 1, 2$ modes. Here we will take the perturbative approach. We choose a particular model, namely [21]

$$\mathcal{L} = \frac{1}{2\kappa^2}\left(R - 2\alpha T^{\alpha}_{\ \mu\nu}T_{\alpha}^{\ \mu\nu}\right) - \frac{1}{8\lambda}R^{\alpha\beta}_{\ \ \mu\nu}R_{\alpha\beta}^{\ \ \mu\nu} + \mathcal{L}_m, \tag{7.68}$$

where α and λ are free parameters. The field equations corresponding to this Lagrangian can be obtained upon variation with respect to the metric tensor and the contorsion. Let us choose $\alpha = 1$ and define $l_0 \equiv \kappa^2/\lambda$. The resulting system of coupled equations are

$$\frac{l_0^2}{2}\left(\tilde{\nabla}_{\nu}R_{\alpha\beta}^{\ \ \mu\nu} + K^{\lambda}_{\ \alpha\nu}R_{\beta\lambda}^{\ \ \mu\nu} + K^{\lambda}_{\ \beta\nu}R_{\lambda\alpha}^{\ \ \mu\nu}\right) + 3g^{\mu\nu}K_{[\alpha\beta\nu]} + \delta^{\mu}_{[\alpha}K_{\beta]} = \kappa^2 s^{\mu}_{\ \alpha\beta} \tag{7.69}$$

$$\tilde{G}_{\mu\nu} - \tilde{\nabla}_{(\nu}K_{\mu)} + g_{\mu\nu}\tilde{\nabla}_\lambda K^\lambda + K_\lambda K^\lambda_{\ (\mu\nu)} - \frac{3}{2}\left(K_{[\alpha\beta\mu]}K^{\alpha\beta}_{\ \ \ \nu} + K_{[\alpha\beta\nu]}K^{\alpha\beta}_{\ \ \ \mu}\right)$$
$$+\frac{1}{2}\left(K^\lambda K_\lambda + 3K_{[\lambda\alpha\beta]}K^{\lambda\alpha\beta}\right) - \frac{l_0^2}{2}\left(R^{\alpha\beta}_{\ \ \ \mu\lambda}R_{\alpha\beta\nu}^{\ \ \ \ \lambda} - \frac{1}{4}g_{\mu\nu}R^{\alpha\beta}_{\ \ \ \rho\sigma}R_{\alpha\beta}^{\ \ \ \rho\sigma}\right) = \kappa^2\tau_{(\mu\nu)}, \tag{7.70}$$

$$3\tilde{\nabla}^\lambda K_{[\mu\nu\lambda]} - \tilde{\nabla}_\nu K_\mu + K_\lambda K^\lambda_{\ (\mu\nu)} + K_\lambda K^\lambda_{\ [\mu\nu]} + 3K_{[\alpha\mu\nu]}K_\beta^{\ \mu\nu} - 3K_{[\beta\mu\nu]}K_\alpha^{\ \mu\nu} = \kappa^2\tau_{[\mu\nu]}, \tag{7.71}$$

where we have split the second field equation into its symmetric (7.70) and antisymmetric parts (7.71) respectively. For a torsionless background geometry, the system of equations for the metric and torsion perturbations in vacuum are, to first order,

$$\frac{l_0^2}{2}\left(\tilde{\nabla}_\nu R_{\alpha\beta}^{\ \ \ \mu\nu} + \kappa^\lambda_{\ \alpha\nu}R_{\beta\lambda}^{\ \ \ \mu\nu} + \kappa^\lambda_{\ \beta\nu}R_{\lambda\alpha}^{\ \ \ \mu\nu}\right) + 3\mathring{g}^{\mu\nu}\kappa_{[\alpha\beta\nu]} + \delta^\mu_{[\alpha}\kappa_{\beta]} = 0, \tag{7.72}$$

and

$$\tilde{G}_{\mu\nu} - \tilde{\mathring{\nabla}}_{(\nu}\kappa_{\mu)} + \mathring{g}_{\mu\nu}\tilde{\mathring{\nabla}}_\lambda\kappa^\lambda - \frac{l_0^2}{2}\left(R^{\alpha\beta}_{\ \ \ \mu\lambda}R_{\alpha\beta\nu}^{\ \ \ \ \lambda} - \frac{1}{4}g_{\mu\nu}R^{\alpha\beta}_{\ \ \ \rho\sigma}R_{\alpha\beta}^{\ \ \ \rho\sigma}\right) = 0, \tag{7.73}$$

together with the condition

$$3\tilde{\mathring{\nabla}}^\lambda\kappa_{[\mu\nu\lambda]} - \tilde{\mathring{\nabla}}_\nu\kappa_\mu = 0, \tag{7.74}$$

where $\tilde{G}_{\mu\nu}$ is the perturbed Einstein tensor, corresponding to the metric in (7.50). Using (7.61), the first two equations become

$$\frac{l_0^2}{2}\left(\tilde{\nabla}_\nu\tilde{R}_{\alpha\beta}^{\ \ \ \mu\nu} + 2\tilde{\mathring{\nabla}}_\nu\tilde{\mathring{\nabla}}^{[\mu}\kappa_{\alpha\beta}^{\ \ \ \nu]} + \kappa^\lambda_{\ \alpha\nu}\tilde{\mathring{R}}_{\beta\lambda}^{\ \ \ \mu\nu} + \kappa^\lambda_{\ \beta\nu}\tilde{\mathring{R}}_{\lambda\alpha}^{\ \ \ \mu\nu}\right) + 3\mathring{g}^{\mu\nu}\kappa_{[\alpha\beta\nu]} + \delta^\mu_{[\alpha}\kappa_{\beta]} = 0, \tag{7.75}$$

$$\tilde{G}_{\mu\nu} - \tilde{\mathring{\nabla}}_{(\nu}\kappa_{\mu)} + \mathring{g}_{\mu\nu}\tilde{\mathring{\nabla}}_\lambda\kappa^\lambda - \frac{l_0^2}{2}\left(\tilde{R}^{\alpha\beta}_{\ \ \ \mu\lambda}\tilde{R}_{\alpha\beta\nu}^{\ \ \ \ \lambda} - \frac{1}{4}g_{\mu\nu}\tilde{R}^{\alpha\beta}_{\ \ \ \rho\sigma}\tilde{R}_{\alpha\beta}^{\ \ \ \rho\sigma}\right) \tag{7.76}$$
$$-\frac{l_0^2}{2}\left(2\tilde{\mathring{R}}^{\alpha\beta}_{\ \ \ \mu\lambda}\tilde{\mathring{\nabla}}^{[\mu}\kappa_{\alpha\beta}^{\ \ \ \nu]} - \frac{2}{4}g_{\mu\nu}\tilde{\mathring{R}}^{\alpha\beta}_{\ \ \ \rho\sigma}\tilde{\mathring{\nabla}}^{[\mu}\kappa_{\alpha\beta}^{\ \ \ \nu]}\right) - \frac{l_0^2}{2}\left(2\tilde{\mathring{\nabla}}^{[\mu}\kappa_{\alpha\beta}^{\ \ \ \nu]}\tilde{\mathring{R}}_{\alpha\beta\nu}^{\ \ \ \ \lambda} - \frac{2}{4}g_{\mu\nu}\tilde{\mathring{\nabla}}^{[\mu}\kappa_{\alpha\beta}^{\ \ \ \nu]}\tilde{\mathring{R}}_{\alpha\beta}^{\ \ \ \rho\sigma}\right) = 0.$$

Taking into account the Levi-civita connection in (7.52), the Riemann tensor $\tilde{R}^\alpha_{\ \beta\mu\nu}$ becomes, to first order

$$\tilde{R}^\alpha_{\ \beta\mu\nu} = \tilde{\mathring{R}}^\alpha_{\ \beta\mu\nu} + \partial_\mu\tilde{h}^\alpha_{\ \beta\nu} - \partial_\nu\tilde{h}^\alpha_{\ \beta\mu} + \tilde{\mathring{\Gamma}}^\alpha_{\ \lambda\mu}\tilde{h}^\lambda_{\ \beta\nu} - \tilde{\mathring{\Gamma}}^\alpha_{\ \lambda\nu}\tilde{h}^\lambda_{\ \beta\mu} + \tilde{h}^\alpha_{\ \lambda\mu}\tilde{\mathring{\Gamma}}^\lambda_{\ \beta\nu} - \tilde{h}^\alpha_{\ \lambda\nu}\tilde{\mathring{\Gamma}}^\lambda_{\ \beta\mu}. \tag{7.77}$$

Moreover, we can write

$$\tilde{\nabla}_\nu\tilde{R}_{\alpha\beta}^{\ \ \ \mu\nu} = \tilde{\nabla}_\nu\tilde{\mathring{R}}_{\alpha\beta}^{\ \ \ \mu\nu} + \tilde{\mathring{\nabla}}_\nu\Theta_{\alpha\beta}^{\ \ \ \mu\nu}, \tag{7.78}$$

where

$$\tilde{\Theta}^\alpha_{\ \beta\mu\nu} \equiv \partial_\mu\tilde{h}^\alpha_{\ \beta\nu} - \partial_\nu\tilde{h}^\alpha_{\ \beta\mu} + \tilde{\mathring{\Gamma}}^\alpha_{\ \lambda\mu}\tilde{h}^\lambda_{\ \beta\nu} - \tilde{\mathring{\Gamma}}^\alpha_{\ \lambda\nu}\tilde{h}^\lambda_{\ \beta\mu} + \tilde{h}^\alpha_{\ \lambda\mu}\tilde{\mathring{\Gamma}}^\lambda_{\ \beta\nu} - \tilde{h}^\alpha_{\ \lambda\nu}\tilde{\mathring{\Gamma}}^\lambda_{\ \beta\mu}. \tag{7.79}$$

and therefore, to first order

$$\tilde{R}^{\alpha\beta}_{\ \ \ \rho\sigma}\tilde{R}_{\alpha\beta}^{\ \ \ \rho\sigma} = \tilde{\mathring{R}}^{\alpha\beta}_{\ \ \ \rho\sigma}\tilde{\mathring{R}}_{\alpha\beta}^{\ \ \ \rho\sigma} + \tilde{\mathring{R}}^{\alpha\beta}_{\ \ \ \rho\sigma}\tilde{\Theta}_{\alpha\beta}^{\ \ \ \rho\sigma} + \tilde{\Theta}^{\alpha\beta}_{\ \ \ \rho\sigma}\tilde{\mathring{R}}_{\alpha\beta}^{\ \ \ \rho\sigma} \tag{7.80}$$

and

$$\tilde{R}^{\alpha\beta}{}_{\mu\lambda}\tilde{R}_{\alpha\beta\nu}{}^{\lambda} = \tilde{\mathring{R}}^{\alpha\beta}{}_{\mu\lambda}\tilde{\mathring{R}}_{\alpha\beta\nu}{}^{\lambda} + \tilde{\mathring{R}}^{\alpha\beta}{}_{\mu\lambda}\tilde{\Theta}_{\alpha\beta\nu}{}^{\lambda} + \tilde{\Theta}^{\alpha\beta}{}_{\mu\lambda}\tilde{\mathring{R}}_{\alpha\beta\nu}{}^{\lambda}. \tag{7.81}$$

Instead of addressing possible effects due to the propagation of GW in cosmological backgrounds, for simplicity let us illustrate the changes with respect to GR by considering the propagation over a (torsionless) Minkowski background geometry. Taking, in this case, $\tilde{\mathring{\Gamma}}^{\alpha}_{\beta\mu} = 0$ and $\tilde{h}^{\alpha}_{\beta\gamma} \simeq \frac{1}{2}\eta^{\alpha\lambda}\left(\partial_\beta h_{\gamma\lambda} + \partial_\gamma h_{\beta\lambda} - \partial_\lambda h_{\beta\gamma}\right)$ we obtain, to first order

$$\tilde{\Theta}^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \tilde{h}^{\alpha}_{\beta\nu} - \partial_\nu \tilde{h}^{\alpha}_{\beta\mu}, \quad \tilde{R}^{\alpha\beta}{}_{\rho\sigma}\tilde{R}_{\alpha\beta}{}^{\rho\sigma} = 0, \quad \tilde{R}^{\alpha\beta}{}_{\mu\lambda}\tilde{R}_{\alpha\beta\nu}{}^{\lambda} = 0, \tag{7.82}$$

therefore, the field equations become

$$\frac{l_0^2}{2}\left(\partial_\nu \tilde{\Theta}_{\alpha\beta}{}^{\mu\nu} + 2\partial_\nu \partial^{[\mu}\kappa_{\alpha\beta}{}^{\nu]}\right) + 3\eta^{\mu\nu}\kappa_{[\alpha\beta\nu]} + \delta^{\mu}_{[\alpha}\kappa_{\beta]} = 0, \tag{7.83}$$

$$\tilde{G}_{\mu\nu} - \partial_{(\nu}\kappa_{\mu)} + \eta_{\mu\nu}\partial_\lambda \kappa^\lambda = 0, \tag{7.84}$$

and, from (7.71) one gets $\partial_\nu \kappa_\mu = 3\partial^\lambda \kappa_{[\mu\nu\lambda]}$. We recall that the perturbed Einstein tensor in terms of the redefined trace reversed metric perturbation is given by

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\left(\partial_\alpha \partial_\nu \bar{h}^{\alpha}_{\mu} + \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha \partial^\beta \bar{h}^{\alpha}_{\beta}\right). \tag{7.85}$$

The equation (7.84) can be written in the form

$$\tilde{G}_{\mu\nu} = \Lambda_{\mu\nu}, \tag{7.86}$$

or

$$\partial_\alpha \partial_\nu \bar{h}^{\alpha}_{\mu} + \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha \partial^\beta \bar{h}^{\alpha}_{\beta} = 2\Lambda_{\mu\nu}, \tag{7.87}$$

in the trace-reversed, re-defined metric perturbation, where

$$\Lambda_{\mu\nu} \equiv \partial_{(\nu}\kappa_{\mu)} - \eta_{\mu\nu}\partial_\lambda \kappa^\lambda \tag{7.88}$$

represents the contribution from the propagating contorsion tensor field seen as a perturbation to the background Minkowski spacetime geometry. The equation (7.87) represents therefore an extension to the wave equation in the linear perturbation approach to GR in Minkowski backgrounds, derived from qPGTG. The propagating torsion/contorsion tensor field (perturbation) acts as a source term to the usual Einstein equation in GR. If the gauge transformation (7.10) is performed, the Einstein tensor can be simplified, for example in the Lorenz gauge, and then another transformation can be performed into the TT gauge. In any of these transformations the metric perturbation transforms as in (7.11), the trace reverse metric perturbation as in (7.15) and the contorsion tensor perturbation according to

$$\kappa'^{\alpha}{}_{\mu\nu} = \kappa^{\alpha}{}_{\mu\nu} + \kappa^{\gamma}{}_{\mu\nu}\partial_\gamma \xi^\alpha - \kappa^{\alpha}{}_{\rho\nu}\partial_\mu \xi^\rho - \kappa^{\alpha}{}_{\mu\sigma}\partial_\nu \xi^\sigma, \tag{7.89}$$

while the tensor $\Lambda_{\mu\nu}$ transforms as

$$\Lambda'_{\mu\nu} = \Lambda_{\mu\nu} - \Lambda_{\alpha\nu}\partial_\mu \xi^\alpha - \Lambda_{\mu\beta}\partial_\nu \xi^\beta. \tag{7.90}$$

Focusing now on the other equation (7.83), from (7.82), we get

$$\tilde{\Theta}^{\alpha}_{\beta\mu\nu} = \partial_\beta \partial_{[\mu} h^{\alpha}_{\nu]} - \partial^\alpha \partial_{[\mu} h_{\beta|\nu]} \tag{7.91}$$

and this leads to

$$\partial_\nu \tilde{\Theta}_{\alpha\beta}^{\ \ \mu\nu} = \partial_\nu \partial^\nu \partial_{(\alpha} h_{\beta)}^{\ \ \mu}. \tag{7.92}$$

or in terms of the trace-reverse metric perturbation, after some algebra, one gets

$$\partial_\nu \tilde{\Theta}_{\alpha\beta}^{\ \ \mu\nu} = \Box \left(\partial_{(\alpha} \bar{h}_{\beta)}^{\ \ \mu} - \frac{1}{2} \delta^{\mu}_{(\alpha} \partial_{\beta)} \bar{h} \right). \tag{7.93}$$

The set of field equations are then

$$\frac{l_0^2}{2} \left(\Box \left(\partial_{(\alpha} \bar{h}_{\beta)}^{\ \ \mu} - \frac{1}{2} \delta^{\mu}_{(\alpha} \partial_{\beta)} \bar{h} \right) + 2 \partial_\nu \partial^{[\mu} \kappa_{\alpha\beta}^{\ \ \ \nu]} \right) + 3 \eta^{\mu\nu} \kappa_{[\alpha\beta\nu]} + \delta^{\mu}_{[\alpha} \kappa_{\beta]} = 0, \tag{7.94}$$

$$\partial_\alpha \partial_\nu \bar{h}^{\alpha}_{\ \mu} + \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} - \Box \bar{h}_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial^\beta \bar{h}^{\alpha}_{\ \beta} = 2 \Lambda_{\mu\nu}, \tag{7.95}$$

and in the TT (Lorentz) gauge $\bar{h}_{\mu\nu} = h_{\mu\nu}$, with $h = 0$ and we arrive at

$$\frac{l_0^2}{2} \left(\Box \partial_{(\alpha} \bar{h}_{\beta)}^{\ \ \mu} + 2 \partial_\nu \partial^{[\mu} \kappa_{\alpha\beta}^{\ \ \ \nu]} \right) + 3 \eta^{\mu\nu} \kappa_{[\alpha\beta\nu]} + \delta^{\mu}_{[\alpha} \kappa_{\beta]} = 0, \tag{7.96}$$

$$-\Box \bar{h}_{ij} = 2 \Lambda^{TT}_{ij}. \tag{7.97}$$

In principle one can apply Greens functions methods to get

$$\bar{h}_{ij}(t, \vec{x}) \sim \int d^3 y \frac{\Lambda^{TT}_{ij}(t_R, \vec{y})}{|\vec{x} - \vec{y}|}, \tag{7.98}$$

in analogy with (7.24). This reinforces the notion that metric and torsion perturbations can be interconvertible. Moreover, the equation (7.97) can be used to replace in (7.96) to obtain a second order partial diferential equation for the torsion modes

$$\frac{l_0^2}{2} \left(-2 \partial_{(\alpha} \Lambda_{\beta)}^{\ \ \mu} + 2 \partial_\nu \partial^{[\mu} \kappa_{\alpha\beta}^{\ \ \ \nu]} \right) + 3 \eta^{\mu\nu} \kappa_{[\alpha\beta\nu]} + \delta^{\mu}_{[\alpha} \kappa_{\beta]} = 0, \tag{7.99}$$

with all quantities calculated in the TT gauge. Now, taking into account the RC curvature tensor in (7.59) for a RC background and in (7.61) for a torsionless background (to first order), respectively, one expects generalizations to the geodesic deviation equations. This implies a generalization to the GR effects of the propagating GW in test-masses. For example, in Minkowski backgrounds, useful to compute the weak, linearised GW effects in detectors, we have

$$R_{\alpha\beta\mu\nu} = \tilde{R}_{\alpha\beta\mu\nu} + 2 \partial_{[\mu} \kappa_{\alpha\beta|\nu]}, \tag{7.100}$$

and using the TT gauge, the non-zero components of the Riemann part of the RC curvature are given by

$$\tilde{R}_{i0j0} = -\frac{1}{2} \ddot{h}_{ij}. \tag{7.101}$$

If we take the corresponig components in (7.100),

$$R_{i\beta j\nu} = -\frac{1}{2}\ddot{h}_{ij}\delta^0_\beta\delta^0_\nu + 2\partial_{[j}\kappa_{i\beta|\nu]}, \tag{7.102}$$

where δ^0_β are defined as kronecker symbols, we see explicit extra terms due to the presence of the propagating contorsion perturbation.

The FRWL torsionless background geometry can be taken to illustrate the effects of torsion GW modes propagating over cosmological distances. For simplicity the spatially flat FRWL metric background can be taken. For this purpose the equations (7.75)-(7.76) need to be considered, together with (7.77)-(7.81), in the FRWL backgrounds. In a more general approach the perturbed versions of the field equations in (7.69)-(7.71) would be computed over a RC backgroud geometry with a FRWL metric and a background contorion obeying the cosmological principle, as in (5.98). Notice also that in curved backgrounds, all the terms $\tilde{\mathring{R}}^2$, $\tilde{\mathring{R}}\kappa$ and $\tilde{\mathring{R}}\tilde{\mathring{\nabla}}\kappa$ in (7.75) and (7.76) do not vanish. Therefore, a far more richer phenomenology is expected than the one derived from a flat, Minkowski, torsionless background. In principle, it should be possible to obtain a set of two wave equations in cosmological backgrounds, similar to and generalizing the expression in (7.47). From such equations, the general beyond-GR effects can be recognized in the amplitudes, dispersion relation and due to the presence of extra degrees of freedom (for example contorsion d.o.f.). Then these direct consequences can be applied for the general expressions for the GW forms (in the weak source limit) due to binaries, in order to encompass the effects of a propagation over cosmological distances. We will not perform such procedure here, but emphasize the relevance of it in order to search for signatures of propagating torsion modes in the GW signals.

Besides the considerations presented here, it is relevant to recall that spacetime torsion in ECSK and in quadratic PGTG interacts with fermionic currents with interesting physics regarding self-interactions of the matter fields. These can include contact axial-axial, vector-vector and (parity-breaking) vector-axial spinor interactions (as seen in chapter 4), but also for the quadratic Lagrangians, interactions via (propagating) torsion boson mediators. The macroscopic averaged result of this rich particle physics in dense environments, can be translated in the fluid language as the emergence of effective pressures that can counter-act against the gravitational collapse of an ultra-dense neutron star. The standard EC theory can prevent black hole singularities and, therefore, the research on whether one can have equilibrium configurations in compact objects denser than neutron stars, before the appearance of an horizon, is of utmost relevance. In the ECDM model we analysed in chapters 4 and 5 there are physical mechanisms induced by torsion that act as an effective repulsive interaction, which could possibly provide the required pressure to balance the self-gravity of a newly born (unstable) neutron star. After the coalescence of two neutron stars in models of GW emission, it is usually assumed that the resulting object stabilizes to a neutron star or decays into a black hole (directly or after some relaxation time), due to GR instabilities, but in modified gravity, torsion/spin effects should allow for other equilibrium configurations, i.e, stable compact objects denser than neutron stars. This possibility of a stable compact spherical objects supported by these torsion induced pressure sources is very relevant for GW astronomy in connection to potential new discoveries of ultra-compact objects. Moreover, the interaction of propagating torsion modes with fermionic systems are expected to induce a dynamical Zeeman effect and the subsequent transitions that could be detected. This effect should be computed through the methods of time dependent perturbation theory, generalizing the analysis performed in chapter 4. It is also reasonable to consider dynamical spin precession effects in gyroscopes with macroscopic intrinsic spin, that might be possible to detect with the use of advanced magnetometers as SQUIDS (Super Conducting Interfering Devices). The derivation of such dynamical effect should result from a semi-classical approach (as in the WKB approximation) to the dynamics of Dirac fermions in the background of a GW geometry perturbation with torsion.

7.4 GW and electromagnetic fields: non-standard detectors

The celebrated measurement of GW emission was done using laser interferometry, but other methods such as pulsar timing arrays [200] will most probably provide positive detections in the near future. However, it is crucial to keep investigating diferent routes towards GW measurements (see [200, 162, 48, 49, 50]) and one such route lies at the very heart of this section. Instead of using test masses and measuring the minute changes of their relative distances, as it is done in Laser interferometry (used in LIGO, VIRGO, GEO600, TAMA300 and will be used in KAGRA, LIGOIndia and LISA), we can also explore the effects of GWs on electromagnetic fields. For this purpose, one needs to compute the electromagnetic field equations on the spacetime background of a GW perturbation. This might not only provide models and simulations which can test the viability of such GW-electromagnetic detectors, but it might also contribute to a deeper understanding of the physical properties of astrophysical and cosmological sources of GWs, since these waves interact with the electromagnetic fields and plasmas which are expected to be common in many highly energetic GW sources (see [154]).

Before approaching the GW effects in electrodynamics, let us mention very briefly other possible routes in the quest for GW measurements. Recall that linearised gravity is also the context in which gravitoelectric and gravitomagnetic fields can be defined [175]. In particular, gravitomagnetism is associated to spacetime metrics with time-space components. Similarly, the $(\times)$ polarization of GWs is related to space-space off-diagonal metric components. This analogy might provide a motivation to explore the dynamical effects of GWs on gyroscopes. In fact, an analogy with gravitomagnetism brings interesting perspectives. In particular, gravitomagnetic effects on gyroscopes are known to be fully analogous to magnetic effects on dipoles. Now, in the case of gravitational waves these analogous (off-diagonal) effects on gyroscopes will, in general, be time dependent. The tiny gravitomagnetic effect on gyroscopes due to Earth's rotation, was successfully measured during the Gravity Probe B experiment [126], where the extremely small geodetic and Lens-Thirring (gravitomagnetic) deviations of the gyro's axis were measured with the help of Super Conducting Quantum Interference Devices (SQUIDS). Analogous (time varying) effects on gyroscopes due to the passage of GWs, might be measured with SQUIDS. On the other hand, rotating superconducting matter seems to generate anomalous (stronger) gravitomagnetic fields (anomalous gravitomagnetic London moment) [201, 202] so, if these results are robustly confirmed then superconductivity and superfluidity might somehow amplify gravitational phenomena. This hypothesis deserves further theoretical and experimental research as it could contribute for future advanced GW detectors.

A promising route comes from the study of the coupling between electromagnetic fields and gravity, the topic of our concern in the present section. Are there measurable effects on electric and magnetic fields during the passage of a GW? Could these be used in practice to study the physics of GW production from astrophysical sources, or applied to GW detection? Although very important work has been done in the past (see for example [177, 154]), it seems reasonable to say that these routes are far from being fully explored. Regarding electromagnetic radiation, there are some studies related to the effects of GWs (see for example [203, 204]). It has been shown that gravitational waves have an important effect on the polarization of light [48]. On the other hand, lensing has been gradually more and more relevant in observational astrophysics and cosmology and it seems undoubtedly relevant to study the effects of GWs (from diferent types of sources) on lensing, since a GW should in principledynamically distort any lensed image. Could lensing provide a natural amplification of the gravitational perturbation signal due to the coupling between gravity and light? These topics need careful analysis for a better understanding of the possible routes (within the reach of present technology) for gravity wave astronomy and its applications to astrophysics and cosmology.

This section is outlined in the following manner: We start from the basic electromagnetic equations on a pseudo-Riemannian spacetime. Then, we explore the coupling between electromagnetic fields and gravitational waves, discuss our results and establish the conclusions.

The electromagnetic field equations on the background of a general (pseudo) Riemann spacetime manifold in the tensor formalism (6.1), with the definitions in (6.2)can be separated into the generalized Gauss and Maxwell-Ampère laws in (6.4) and (6.5), respectively. As previously mentioned, one can clearly verify that new electromagnetic phenomena are expected due to the presence of extra electromagnetic couplings induced by spacetime curvature. In particular, the magnetic terms in the Gauss law are only present for non-vanishing off-diagonal time-space components g^{0j}, which in linearised gravity correspond to the gravitomagnetic potentials. These terms are typical of axially symmetric geometries (see [173]) as we saw in chapter 6. For diagonal metrics, the inhomogeneous equations, the Gauss and Maxwell Ampère laws, can be recast into the following forms

$$-g^{kk}g^{00}\partial_k E_k + E_k\gamma^k = \frac{\rho}{\varepsilon_0}, \tag{7.103}$$

$$\epsilon_{ijk}g^{ii}g^{jj}\partial_j B^k + \frac{1}{c^2}g^{00}g^{ii}\partial_t E_i + \epsilon_{ijk}\sigma^{jii}B^k + E_i\xi^{ii} = \mu_0 j^i, \tag{7.104}$$

with

$$\gamma^k(\mathbf{x}) \equiv -\left[g^{kk}g^{00}\frac{1}{\sqrt{-g}}\partial_k(\sqrt{-g}) + \partial_k\left(g^{kk}g^{00}\right)\right], \tag{7.105}$$

and

$$\sigma^{jii}(\mathbf{x}) \equiv g^{jj}g^{ii}\frac{1}{\sqrt{-g}}\partial_j(\sqrt{-g}) + \partial_j(g^{jj}g^{ii}), \tag{7.106}$$

$$\xi^{ii}(\mathbf{x}) \equiv g^{00}g^{ii}\frac{1}{c^2}\frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}) + \frac{1}{c^2}\partial_t(g^{00}g^{ii}). \tag{7.107}$$

The Einstein summation convention is applied in Eq. (7.104) only for j and k while the index i is fixed by the right-hand side. Also, no contraction is assumed in Eq. (7.105) nor in the expression for σ^{jii}. Notice that the difference in signs with resect to the expression in (6.5) result from the different ordering of the indices in the 3-dimensional Levi Civita pseudo tensor. New electromagnetic effects induced by the spacetime geometry include an inevitable spatial variability (non-uniformity) of electric fields whenever we have non-vanishing geometric functions γ^k, electromagnetic oscillations (therefore waves) induced by gravitational radiation and also additional electric contributions to Maxwell's displacement current in the generalized Maxwell-Ampère law. This last example becomes clearer by re-writing Eq. (7.104) as in (6.7) and (6.9). In such expressions the functions ξ^{ii} vanish for stationary spacetimes but might have an important contribution for strongly varying gravitational waves (high frequencies), since they depend on the time derivatives of the metric. Analogously, Eq. (7.103) can be written as in (6.6) with (6.8). Finally, the field equations in terms of the electromagnetic 4-potential are given by (6.11).

7.4.1 GWs and electromagnetic fields

Due to the huge distances in the Cosmos, any GW reaching Earth should have an extremely low amplitude. Therefore, the linearisation of gravity is applied in a background dependent perturbative approach, as we revisited in section 7.2. In principle, the passage of a GW in a region with electromagnetic fields will have a measurable effect. To compute this we have to consider Maxwell's equations on the perturbed background of a GW. We shall consider a GW as a perturbation of Minkowski spacetime given by $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $|h_{\alpha\beta}| \ll 1$, so that

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 + h_{\alpha\beta}dx^\alpha dx^\beta, \tag{7.108}$$

where the perturbation corresponds to a wave travelling along the z axis, i.e.,

$$ds^2 \;=\; c^2dt^2 - dz^2 - [1 - f_+(z-ct)]dx^2 - [1 + f_+(z-ct)]dy^2 + 2f_\times(z-ct)dxdy, \tag{7.109}$$

and $(+)$ and $(\times)$ refer to the two independent polarizations characteristic of GWs in GR. This metric is a solution of Einstein's field equations in the linear approximation, in the TT (Transverse-Traceless) Lorenz Gauge. For this metric, we get

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}) = \frac{f_\times(\partial_z f_\times) + f_+(\partial_z f_+)}{f_\times^2 + f_+^2 - 1}, \tag{7.110}$$

$$\frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}) = \frac{f_\times(\partial_t f_\times) + f_+(\partial_t f_+)}{f_\times^2 + f_+^2 - 1}. \tag{7.111}$$

These quantities will be useful further on.

GW effects in electric and magnetic fields

Consider an electric field in the background of a GW travelling in the z direction. The general expression for Gauss' law (6.4), in vacuum, is now given by

$$[1 - f_+(z,t)]^{-1}\partial_x E_x + [1 + f_+(z,t)]^{-1}\partial_y E_y + \partial_z E_z - f_\times^{-1}(z,t)\left(\partial_y E_x + \partial_x E_y\right) + \left[\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g})\right]E_z = 0, \tag{7.112}$$

which clearly shows that physical (possibly observable) effects are induced by the propagation of GWs.

As for the Maxwell-Ampère law, in (7.104) or (6.5) provides the following relations in vacuum:

$$\begin{aligned}&\frac{1}{c^2}\left[f_\times^{-1}\partial_t E_y - (1-f_+)^{-1}\partial_t E_x\right] + E_x\xi^{xx} + E_y\xi^{yx}\\ &-(1-f_+)^{-1}\left[(1+f_+)^{-1}\partial_y B^z - \partial_z B^y\right] - B^y\sigma^{zxx}\\ &+B^x\sigma^{zyx} - f_\times^{-1}\left(f_\times^{-1}\partial_y B^z + \partial_z B^x\right) = 0\,,\end{aligned} \tag{7.113}$$

$$\begin{aligned}&\frac{1}{c^2}\left[f_\times^{-1}\partial_t E_x - (1+f_+)^{-1}\partial_t E_y\right] + E_y\xi^{yy} + E_x\xi^{xy}\\ &-(1+f_+)^{-1}\left[(1-f_+)^{-1}\partial_x B^z - \partial_z B^x\right] + B^x\sigma^{zyy}\\ &-B^y\sigma^{zxy} + f_\times^{-1}\left(f_\times^{-1}\partial_x B^z + \partial_z B^y\right) = 0\,,\end{aligned} \tag{7.114}$$

$$-\frac{1}{c^2}\partial_t E_z + E_z\xi^{zz} - f_\times^{-1}\left(\partial_y B^y - \partial_x B^x\right)$$
$$+\left[(1-f_+)^{-1}\partial_x B^y - (1+f_+)^{-1}\partial_y B^x\right] = 0\,, \tag{7.115}$$

with the non-vanishing geometric coefficients given by

$$\xi^{xx} = \frac{1}{c^2}\frac{f_\times(f_+-1)\partial_t f_\times - (f_\times^2+f_+-1)\partial_t f_+}{(f_+-1)^2(f_\times^2+f_+^2-1)},$$

$$\xi^{yx} = \xi^{xy} = \frac{1}{c^2}\frac{-(f_+^2-1)\partial_t f_\times + f_\times f_+\partial_t f_+}{f_\times^2(f_\times^2+f_+^2-1)},$$

$$\xi^{yy} = \frac{1}{c^2}\frac{-f_\times(f_++1)\partial_t f_\times + (f_\times^2-f_+-1)\partial_t f_+}{(f_++1)^2(f_\times^2+f_+^2-1)},$$

$$\xi^{zz} = -\frac{1}{c^2}\frac{f_\times\left(\partial_t f_\times\right) + f_+\left(\partial_t f_+\right)}{f_\times^2+f_+^2-1},$$

$$\sigma^{zxx} = -\frac{f_\times(f_+-1)\partial_z f_\times - (f_\times^2+f_+-1)\partial_z f_+}{(f_+-1)^2(f_\times^2+f_+^2-1)},$$

$$\sigma^{zyy} = -\frac{-f_\times(f_++1)\partial_z f_\times + (f_\times^2-f_+-1)\partial_z f_+}{(f_++1)^2(f_\times^2+f_+^2-1)},$$

$$\sigma^{zxy} = \sigma^{zyx} = -\frac{-(f_+^2-1)\partial_z f_\times + f_\times f_+\partial_z f_+}{f_\times^2(f_\times^2+f_+^2-1)}.$$

A natural consequence of these laws is the generation of electromagnetic waves induced by gravitational radiation. Initially static electric and magnetic fields become time dependent during the passage of GWs which might be detectable in this way. In general, the system of coupled Eqs. (6.4)-(6.5) and the homogeneous equations in (6.3) have to be taken as a whole. As we will see from Eq. (7.112), in some specific situations the electric field can be solved directly from Gauss' law. This electric field can in turn act as a source for magnetism via the Maxwell-Ampère relations in Eqs. (7.113)-(7.115), where the presence of the GW induces extra terms proportional to the electric field. In this work, we will explore relatively simple situations in order to illustrate the effects of GWs in electric and magnetic fields. Let us start by considering the effects of GWs in electric fields.

Electric field oscillations induced by GWs: ZZ axis alignment. An electric field along the z axis can easily be achieved by charged plane plates constituting a capacitor. In the absence of GWs the electric field thus produced is approximately uniform (neglecting boundary effects) for static uniform charge distributions. The field can also be time variable if there is an alternate current (as in the case of a RLC circuit with a variable voltage signal generator). With the passage of the GW, in general the electric field is perturbed by both the (+) and (×) modes. To see this let us look at Gauss' law when the electric field is aligned with the direction of the GW propagation. From Eq. (7.112), we have

$$\partial_z E_z + E_z\left[\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g})\right] = 0, \tag{7.116}$$

where $\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g})$ is given by the expression in Eq. (7.110). We can see that even if in the absence of any GW the electric field was static and uniform, during the passage of the spacetime disturbance, the field will be time varying and non-uniform, oscillating with the same frequency of the passing GW. In fact, the general solution is

$$E_z(x,y,z,t) = \frac{E_0}{\sqrt{-g}} = \frac{E_0}{\sqrt{1-f_+^2-f_\times^2}}, \tag{7.117}$$

where in the most general case, $E_0 = E_0(x,y;t)$. To get the full description of the electric field one has to consider also both the Maxwell-Ampère relations in Eqs. (7.113)-(7.115) and the Faraday law. Nevertheless it is already clear from Eq. (7.117) that GWs induce propagating electric oscillations. We will consider the most simple case in which E_0 is a constant (without any dependence on x,y or t). Indeed, one can easily verify that the fields $E = (0,0,E_z)$, $B = (0,0,0)$ constitute a (trivial) solution of the full Maxwell equations, namely Eqs. (7.112) and (7.113)-(7.115), together with the homogeneous equations in (6.3). Notice that for zero magnetic field the z Maxwell-Ampère equation (7.115) is

$$-\frac{1}{c^2}\partial_t E_z + E_z\xi^{zz} = 0, \tag{7.118}$$

which is verified by the solution in (7.117) for a constant E_0. This can easily be seen when one considers that

$$\xi^{zz} = -\frac{1}{c^2}\frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}), \tag{7.119}$$

in accordance with the expressions previously shown for ξ^{zz} and Eq. (7.111). In this case, the coupling between the electric field and the GW in the expression for the generalized Maxwell displacement current density, compensates the traditional term which depends on the time derivative of the electric field. In fact, by multiplying by c^2, then Eq. (7.118) can be interpreted as the conservation of the total electric flux density. This situation is thus compatible with the experimental scenario where there are no currents producing any magnetic field and the electric field, although changing in time, due to the coupling with gravity does not give rise to any magnetic field, since the total electric flux is conserved. Naturally, this is not the general case. For example in the presence of currents along the z axis, $B^x, B^y \neq 0$ and due to the gravitational factors in the equations (7.113)-(7.115) the magnetic field is dynamical (time dependent). Therefore, this field necessarily affects the electric field via the Faraday law,

$$\partial_y E_z = \frac{\partial_y E_0}{\sqrt{-g}} = -\partial_t B^x, \qquad \partial_x E_z = \frac{\partial_x E_0}{\sqrt{-g}} = \partial_t B^y, \tag{7.120}$$

which implies that in general $E_0 = E_0(x,y,t)$. Since E_0 is time dependent, in such a case the electric field contributes to the magnetic field via the (non-null) generalized Maxwell displacement current, in accordance with Eq. (7.113)-(7.115), where now

$$-\frac{1}{c^2}\partial_t E_z + E_z\xi^{zz} \neq 0. \tag{7.121}$$

As a practical application consider the following harmonic GW perturbation

$$f_+(z,t) = a\cos(kz - wt), \tag{7.122}$$
$$f_\times(z,t) = b\cos(kz - wt + \alpha). \tag{7.123}$$

In this case, we get the following electric oscillations

$$E_z(z,t) = \tilde{E}_0\left[1 - a^2\cos^2(kz - wt) - b^2\cos^2(kz - wt + \alpha)\right]^{-1/2}, \quad (7.124)$$

for $a^2 + b^2 \leq 1$, which is obeyed by the extremely low amplitude GWs reaching the Solar System. Here $\tilde{E}_0$ is an arbitrary fixed constant and α is the phase difference. These electric oscillations will show distinct features sensitive to the $(+)$ or $(\times)$ GW modes. It provides a window for detecting and analysing GW signals directly converted into electromagnetic information. Notice that the electric waves produced are longitudinal, since these are propagating along the same direction of the GW, even though the electric field is aligned with this direction. To grasp the physical interpretation behind this non-intuitive result, recall that the electric energy density depends quadratically on the field and therefore it is the energy density fluctuation induced by the GW which propagates along the direction of $k = k^z e_z$.

In order to have an approximate idea on the energy density u^{em} of the resulting electromagnetic wave we can use the usual expression (derived from Maxwell electrodynamics in Minkowski spacetime). We get

$$u^{em} \sim \varepsilon_0 E_z^2(z,t) = \varepsilon_0\tilde{E}_0^2\left[1 - a^2\cos^2(kz - wt) - b^2\cos^2(kz - wt + \alpha)\right]^{-1}, \quad (7.125)$$

and the energy per unit area and unit time through any surface (with a normal making an angle ϑ with the z axis) is approximately expressed by

$$\|\vec{S}\|\cos\vartheta = \varepsilon_0 c\tilde{E}_0^2\left[1 - a^2\cos^2(kz - wt) - b^2\cos^2(kz - wt + \alpha)\right]^{-1}\cos\vartheta, \quad (7.126)$$

where $\vec{S}$ is the Poynting vector, and $S \equiv u^{em}c$.

If $\tilde{E}_0$ is the electric field in the absence of GWs, then the relevant dimensionless quantity to be measured is given by the following expression

$$\left|\frac{E_z(z,t) - \tilde{E}_0}{\tilde{E}_0}\right| = \left|\left[1 - a^2\cos^2(kz - wt) - b^2\cos^2(kz - wt + \alpha)\right]^{-1/2} - 1\right|, \quad (7.127)$$

and in terms of the energy density, we get

$$\left|\frac{u^{em}(z,t) - u_0^{em}}{u_0^{em}}\right| = \left|\left[1 - a^2\cos^2(kz - wt) - b^2\cos^2(kz - wt + \alpha)\right]^{-1} - 1\right|, \quad (7.128)$$

with $u_0^{em} = \varepsilon_0\tilde{E}_0^2$. Substituting in these two expressions the typical amplitudes for GWs due to binaries (10^{-25}–10^{-21}), the induced electric field and corresponding energy density oscillations signal will be extremely small. Concerning GWs reaching the Solar System, the detectors which might have a response proportional to the electric field magnitude or rather to its energy (proportional to the square of the electric field magnitude), must be extremely sensitive. We emphasize the fact that, in principle, this electromagnetic wave can be confined in a cavity using very efficient reflectors for the frequency w. Then, under appropriate (resonant) geometric conditions, the signal can be amplified. This might have very important practical applications for future GW detectors.

Electric field oscillations induced by GWs: XY plane polarization Suppose we have an electric field in the x direction. The electric field could initially be uniform and confined within a plane capacitor. In these conditions, the Gauss law in vacuum becomes

$$[1 - f_+(z,t)]^{-1}\frac{\partial E_x}{\partial x} - (f_\times)^{-1}(z,t)\frac{\partial E_x}{\partial y} = 0. \tag{7.129}$$

A similar expression is obtained if the electric field is aligned with the y axis. Assuming a separation of variables $E_x(x,y,z,t) = F_1(x,z,t)F_2(y,z,t)$, where z and t are seen as external parameters, substituting in the above equation and dividing it by E_x, we obtain

$$(1 - f_+)^{-1}\frac{\partial_x F_1}{F_1} = f_\times^{-1}\frac{\partial_y F_2}{F_2}, \tag{7.130}$$

therefore, one arrives at the following expressions

$$F_1(x;z) = C_1(z,t)e^{-(1-f_+)x}, \quad F_2(x;z) = C_2(z,t)e^{-f_\times y}. \tag{7.131}$$

Since we can always add a constant to the solution obtained from $F_1(x,z,t)F_2(y,z,t)$, we can write

$$E_x(x,y,z,t) = E_{0x}\left[1 + \tilde{C}(z,t)e^{-[(1-f_+)x+f_\times y]}\right], \tag{7.132}$$

where in general $\tilde{C}(z,t) = \tilde{C}\left[f_+(z,t), f_\times(z,t)\right]$ can be obtained by taking into account the other Maxwell equations. The full solution should be compatible with the limit without gravity in which we recover the uniform field $E_x = E_{0x}$. Therefore

$$f_+ = f_\times = 0 \Rightarrow \tilde{C}(z,t) = 0. \tag{7.133}$$

A natural *anszatz* is

$$\tilde{C}(z,t) = \eta f_+^{\alpha_1} + \beta f_\times^{\alpha_2} + \mu f_+^{\alpha_3} f_\times^{\alpha_4}, \tag{7.134}$$

where η, β, μ and α_i $(i = 1,2,3,4)$ are constants. But as previously said the form of this function can be studied by considering compatibility with the remaining Maxwell equations.

For the harmonic GW introduced before, the second term in the solution above, Eq. (7.132) is given by the following expression

$$E_{0x}\tilde{C}(z,t)\exp\left\{-\left[(1 - a\cos(kz - wt))\,x + b\cos(kz - wt + \alpha)\,y\right]\right\}. \tag{7.135}$$

The solution obtained is also sensitive to the existence or not of two modes in the GW, to their amplitudes and phase difference. Although this solution obeys the Gauss law, it implies a non-zero dynamical magnetic field, according to Faraday's law. As mentioned, to get a full treatment one should then check the consistency with the other Maxwell equations, in order to derive restrictions on the mathematical form of $\tilde{C}(z,t)$.

Let us consider now the case where an electric field $E_1 = (E_x, 0)$ is generated by a plane capacitor oriented along the x axis and a second electric field $E_2 = (0, E_y)$ is generated by another similar capacitor oriented along the y axis. In this condition, the resulting electric field in the vacuum between the charged plates, $E = E_1 + E_2 = (E_x, E_y)$, obeys the equation

$$(1 - f_+)^{-1}\partial_x E_x - (f_\times)^{-1}\partial_y E_x + (1 + f_+)^{-1}\partial_y E_y - (f_\times)^{-1}\partial_x E_y = 0. \tag{7.136}$$

A possible solution to this equation is given by

$$E_x(x,y,z,t) = E_{0x}\left[1+\tilde{C}_1(z,t)e^{-[(1-f_+)x+f_\times y]}\right], \tag{7.137}$$

$$E_y(x,y,z,t) = E_{0y}\left[1+\tilde{C}_2(z,t)e^{-[f_\times x+(1+f_+)y]}\right], \tag{7.138}$$

where for $f_+ = f_\times = 0$ we get $\tilde{C}_1(z,t) = \tilde{C}_2(z,t) = 0$.

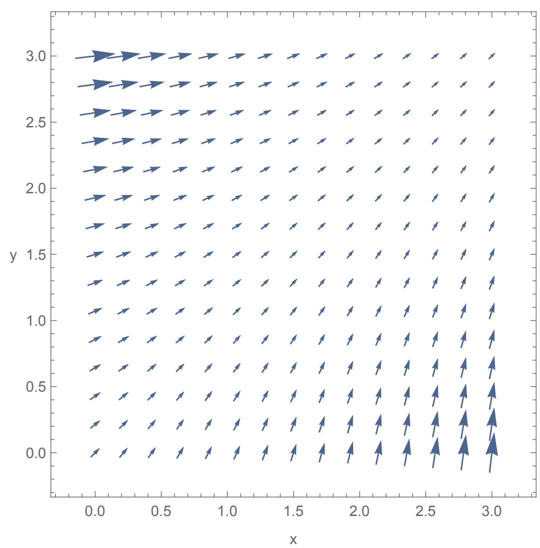


Figure 7.1: This vector plot illustrates the spatial (non-linear) polarization pattern on the (x, y) plane for the electric field at a given instant. This pattern is exclusively induced by the GW. The GW parameters are: $a = 0.036$, $b = 0.766$, $w/2\pi = 89.81Hz$, $\alpha = 0.11\pi$. We have used Eqs. (7.137) and (7.138), where for simplicity we assumed $\tilde{C}_1(z,t) = \tilde{C}_2(z,t) = 1$ and electric field magnitudes $E_{0x} = E_{0y} = 10^{-3}V/m$.

The resulting electric oscillations propagate along the z axis as an electromagnetic wave with non-linear polarization. This wave results from a linear gravitational perturbation of Minkowski spacetime and therefore (in this first order approximation) it can be thought of as an electromagnetic disturbance propagating in Minkowski background with a dynamical polarization. In fact, the angle between the resulting electric field and the x axis is then $\Theta \simeq \arctan(E_y/E_x)$ i.e., for $E_{0x} = E_{0y}$

$$\Theta(x,y,z,t) \simeq \arctan\left\{\frac{\left[1+\tilde{C}_2(z,t)e^{-[f_\times x+(1+f_+)y]}\right]}{\left[1+\tilde{C}_1(z,t)e^{-[(1-f_+)x+f_\times y]}\right]}\right\}.$$

Even if we had $\tilde{C}_1 = \tilde{C}_2$, we still necessarily get a non-linear, dynamical polarization. Such an oscillating polarization could in principle be another distinctive signature of the GW that is causing it. The solutions obtained already give sufficient information to

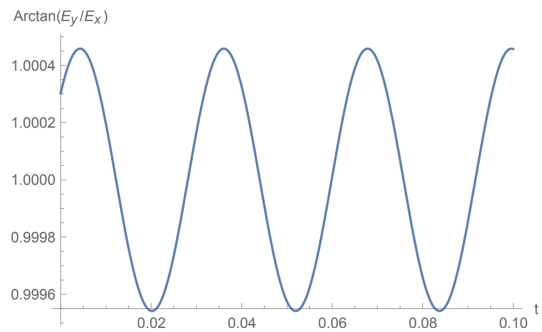


Figure 7.2: The dynamical polarization of the electric fluctuations generated by the GW is represented here for a fixed position in the (x, y) plane. On the vertical axis we have the $\arctan(E_y/E_x)$ normalized to the respective value in the absence of GW and in the horizontal axis we have time in seconds. The GW parameters are: $a = 3.6 \times 10^{-5}$, $b = 2.6 \times 10^{-5}$, $w/2\pi = 31.48Hz$, $\alpha = 0.26\pi$. We have used Eqs. (7.137) and (7.138), where for simplicity we assumed $\tilde{C}_1(z,t) = \tilde{C}_2(z,t) = 1$ and electric field magnitudes $E_{0x} = E_{0y} = 10^{-3}V/m$.

conclude that it is possible to obtain polarization fluctuations induced by GWs, where for $E_{0x} = E_{0y}$ the strength of the effect is given by $|\pi/2 - \Theta(x, y, z, t)|/(\pi/2)$. A dynamical spatial polarization pattern is therefore expected in our detector. This contrasts with the other cases where the resulting wave was linearly polarized. This effect is shown in Figs. 7.1 and 7.2.

Nevertheless, again, the Faraday law and the Maxwell-Ampère relations can provide constraints on the functions $\tilde{C}_1$ and $\tilde{C}_2$.

Electric field oscillations induced by GWs: GW with vanishing ($\times$) mode. If we consider solely the (+) GW mode, the spacetime metric (7.109) becomes diagonal and the Gauss and Maxwell-Ampère equations simplify to the expressions in (6.7)-(6.8) and the generalized Maxwell displacement current density is

$$j_D^i = \varepsilon_0 \partial_t \tilde{E}^i, . \tag{7.139}$$

Let us search for a trivial electric field solution which is fully compatible with the complete system of Maxwell equations. If we consider the field

$$\tilde{E} = (\tilde{E}_0^x(y, z, t), \tilde{E}_0^y(x, z, t), \tilde{E}_0^z(x, y, t)), \tag{7.140}$$

the Gauss law is trivially obeyed and the electric field is given by

$$E = \left(\frac{1 - f_+}{\sqrt{1 - f_+^2}} \tilde{E}_0^x, \frac{1 + f_+}{\sqrt{1 - f_+^2}} \tilde{E}_0^y, \frac{\tilde{E}_0^z}{\sqrt{1 - f_+^2}} \right). \tag{7.141}$$

Furthermore, if $\partial_t \tilde{E}_0^k = 0$ $(k = 1, 2, 3)$, the generalized Maxwell displacement current density j_D^i is zero, therefore effectively the electric field does not contribute to the Maxwell-Ampère equations. Consequently, in the absence of electric currents, such an electric field solution seems to be compatible with the condition $B = 0$. Let us assume that this is the case. Regarding the remaining Maxwell equations, the Magnetic Gauss law $\partial_i B^i = 0$ is trivially obeyed but what about Faraday's law? In this case, one can show that the condition $\partial_t B = -\mathrm{curl}\, E = 0$, leads to a field $\tilde{E}$ which necessarily depends on time which contradicts the hypothesis of zero magnetic field according to the Maxwell-Ampère relations in (6.7) and (6.9) and the expression (7.139). In fact, one arrives at the field.

$$\tilde{E} = (\tilde{E}_0^x(z,t), \tilde{E}_0^y(z,t), \tilde{E}_0^z), \tag{7.142}$$

where $\tilde{E}_0^z$ is a constant and $\tilde{E}_0^x(z,t), \tilde{E}_0^y(z,t)$ are given by

$$\tilde{E}_0^x = \tilde{C}_0^x \exp\left[-\int \partial_z \left(\frac{1-f_+}{\sqrt{-g}}\right)\frac{\sqrt{-g}}{1-f_+}\right], \quad \tilde{E}_0^y = \tilde{C}_0^y \exp\left[-\int \partial_z \left(\frac{1+f_+}{\sqrt{-g}}\right)\frac{\sqrt{-g}}{1+f_+}\right],$$

where $\tilde{C}_0^x$ and $\tilde{C}_0^y$ are constants of integration. These functions clearly depend on time and therefore the generalized Maxwell displacement current cannot be zero leading to a non-vanishing magnetic field. When considering a generic electric field with three components as in (7.141), one cannot assume that $\partial_t \tilde{E}_0^k = 0$ neither a zero magnetic field. Therefore in the general case one needs to consider the influence of the electric field in the magnetic field through the generalized Maxwell displacement current. An exception to this is the special case initially considered, where the electric field is aligned with the direction of the propagation of the GW.

Magnetic field oscillations induced by GWs

The passage of the GW can induce a non-vanishing time varying magnetic field, even for an initially static electric field. In general the full system of the Maxwell equations can be explored numerically to compute the resulting electric and magnetic oscillations. These magnetic fluctuations could be measured in principle using SQUIDS (Super Conducting Quantum Interference Devices) that are extremely sensitive to small magnetic field changes.

To get a glimpse of the gravitationally induced magnetic field fluctuations, we can consider for simplicity only the (+) GW mode and take the generalized Maxwell-Ampère law in the form of Eq. (6.7). We will be considering an electric field aligned with the z axis given by the following solution to the Gauss law

$$E = \left(0, 0, \frac{\tilde{E}_0^z(x,y,t)}{\sqrt{1-f_+^2}}\right), \qquad \partial_k \tilde{E}^k = 0\,. \tag{7.143}$$

We can also consider an electric current I along the z axis such that in principle, by symmetry we expect a magnetic field in the xy plane, $B = (B^x, B^y, 0)$. Then the Maxwell-Ampère equations (6.7) are

$$\nabla \times \bar{B} = \mu_0(\sqrt{-g}j + \varepsilon_0 \partial_t \tilde{E}_0), \tag{7.144}$$

where $\bar{B} \equiv (\bar{B}^{zzyy\mathbf{x}}, \bar{B}^{zzxx\mathbf{y}}, 0)$, while the Faraday law provides the equations

$$\partial_t B^x = -\frac{\partial_y \tilde{E}_0^z}{\sqrt{1-f_+^2}}, \qquad \partial_t B^y = \frac{\partial_x \tilde{E}_0^z}{\sqrt{1-f_+^2}}. \tag{7.145}$$

Then we can perform an integration over an "amperian" closed line coincident to a magnetic field line (in perfect analogy with the method taken in usual electromagnetism) to integrate the Maxwell-Ampère law, assuming axial symmetry, around the charge current distribution and electric flux (Maxwell displacement) current.

We obtain the following solution to Eq. (7.144)

$$\bar{B} = \frac{\mu_0 \tilde{I}_{tot}}{2\pi\sqrt{x^2+y^2}}\left(\cos\phi\, e_y - \sin\phi\, e_x\right), \tag{7.146}$$

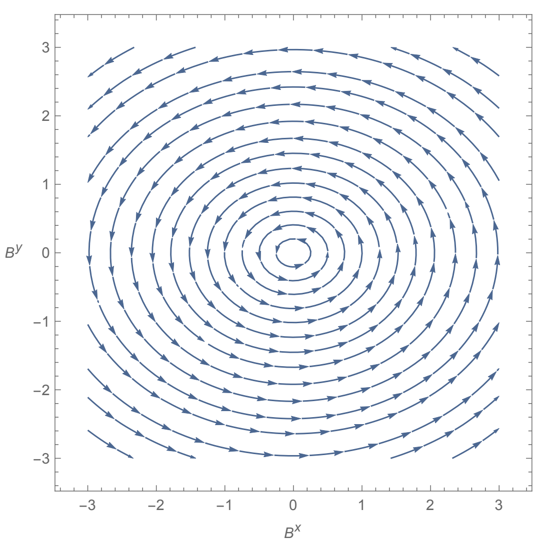
where $\tilde{I}_{tot}(x,y,z,t) = \sqrt{-g}I + I_D(x,y,t)$ and $I_D = \int\int j_D^z dxdy$. I is the (constant) electric current and $j_D^z = \varepsilon_0 \partial_t \tilde{E}_0^z$ is the Maxwell displacement current density. We then get the magnetic field components

$$B^x = -\frac{1+f_+}{\sqrt{1-f_+^2}}\left[\frac{\mu_0 \tilde{I}_{tot}(x,y,z,t)}{2\pi(x^2+y^2)}y\right], \tag{7.147}$$

and

$$B^y = \frac{1-f_+}{\sqrt{1-f_+^2}}\left[\frac{\mu_0 \tilde{I}_{tot}(x,y,z,t)}{2\pi(x^2+y^2)}x\right], \tag{7.148}$$

respectively.



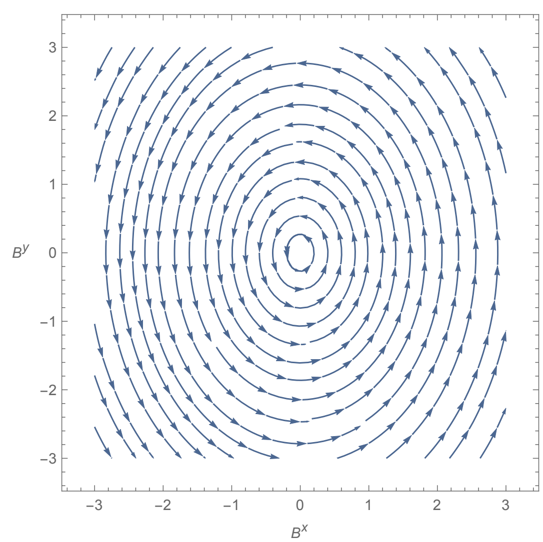


Figure 7.3: These vector plots illustrate the changes of the magnetic field lines on the (x,y) plane which follow the GW (+) mode. The two patterns are separated in time by $\tau/2$, where τ is the period. The GW parameters are: $a = 0.312$, $w/2\pi = 26.80Hz$. We have used expressions (7.151) and (7.152), with $I = 4.6A$

Making the Ansatz

$$\partial_x \tilde{E}_0^z = -\partial_y \tilde{E}_0^z, \tag{7.149}$$

the Faraday equations then imply that $\partial_t B^x = \partial_t B^y$, from which one derives an equation for $\tilde{I}(x,y,z,t)$ with the general solution

$$\tilde{I} = C\exp\left\{-\int \frac{\sqrt{1-f_+^2}}{(x-y)(1-f_+)} \times \left[x\partial_t\left(\frac{1-f_+}{\sqrt{1-f_+^2}}\right) + y\partial_t\left(\frac{1+f_+}{\sqrt{1-f_+^2}}\right)\right] dt\right\}, \tag{7.150}$$

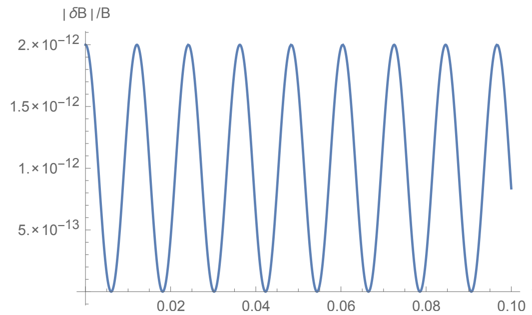


Figure 7.4: Here we see the strength of the magnetic fluctuation induced by the GW as a function of time (in seconds). In the vertical axis we have the non-dimensional quantity $|\|B_{GW}\| - \|B_0\||/\|B_0\|$, where $B_{GW} = (B^x, B^y)$ is the magnetic field in the presence of the passing GW, obtained from Eqs. (7.151) and (7.152) and B_0 is the control magnetic field in the case without GW. The GW parameters are: $a = 2.0 \times 10^{-6}$, $w/2\pi = 41.38 Hz$.

where C is an integration constant. In order to illustrate the general effect of the GW on the magnetic field, consider without great loss of generality that $\tilde{I} = I$ is a constant. Then the magnetic field in the background of the harmonic GW considered in (7.123) has the following fluctuations

$$B^x = -\frac{\mu_0 I y}{2\pi(x^2+y^2)}\left[1 + a\cos\left(kz - wt\right)\right], \tag{7.151}$$

$$B^y = \frac{\mu_0 I x}{2\pi(x^2+y^2)}\left[1 - a\cos\left(kz - wt\right)\right], \tag{7.152}$$

respectively.

We can easily see that for any point (x, y) fixed on any magnetic field line, the x and y components of the magnetic field will oscillate in time out of phase, such that when one is at its maximum value, the other is at the minimum, and vice-versa. The overall result is that the magnetic field lines will oscillate with the passage of the GW, following the deformations of the spacetime geometry, perfectly mimicking the (+) mode deformations. Figure 7.3 illustrate this phenomenon and was obtained using the expressions in Eqs. (7.151) and (7.152). The strength of the effect as a function of time is independent from the current I and it depends on the position (x, y) as well as on the GW parameters (see Fig. 7.4). It can be easily shown that the strength of the fluctuations are much stronger in specific regions of the (x, y) plane.

Charge density fluctuations induced by GWs

In the previous analysis we considered the behaviour of electric and magnetic fields in vacuum regions and did not take into account the effect of the propagating GWs on charge distributions. The effect of spacetime geometry can be understood from the charge conservation equation in curved spacetime ($\nabla_\mu j^\mu = 0$). As a result of this equation even in the absence of (intrinsic) currents, a non-static spacetime will induce a time variability in the charge density according to $\partial_t \rho = -\partial_t(\log(\sqrt{-g})\rho$, so we can write

$$\rho(t) = \rho_0 \sqrt{\frac{g_0}{g(t)}}, \tag{7.153}$$

where ρ_0 is the initial charge density before the passage of the wave and g_0 is the determinant of the initially unperturbed background metric. For the simpler case of GWs travelling along the z direction, seen as disturbances of Minkowski spacetime, we have the simple result

$$\rho(t) = \rho_0 \left(1 - f_\times^2 - f_+^2\right)^{-\frac{1}{2}} . \tag{7.154}$$

As an example, for the harmonic GW modes considered previously, we obtain

$$\rho(z,t) = \rho_0 \big[1 - b^2 \cos^2(kz - wt + \alpha) - a^2 \cos^2(kz - wt)\big]^{-\frac{1}{2}}. \tag{7.155}$$

Consequently, one naturally predicts charge density fluctuations and, therefore, currents due to the passage of GWs. Such density oscillations propagate along the z direction following the GW penetrating a conducting material medium. This is analogous to Alfvén waves in plasmas, which are density waves induced by magnetic disturbances which propagate along the magnetic field lines. In this case, astrophysical sources of GWs such as Gamma Ray Bursts or generic coalescing binaries that happen to be surrounded by plasmas in accretion disks or in stellar atmospheres, might generate similar mass density waves and charge density waves induced by the GW propagation. A more realistic treatment would require the equations of Magneto-Hydrodynamics in the background of a GW (see [154]). An interesting study would be to consider the backreaction of the relativistic plasma and electromagnetic fields on the GW properties such as the frequency, amplitude and polarizations, so that the travelling wave after detection could, in principle, contain information about the physical properties of the medium through which it propagated.

The above expression can also indicate another window for GW detection. Conductors in perfect electrostatic equilibrium or superconducting materials at very low temperatures might reveal very dim electric oscillations with well-defined characteristics, induced by GWs.

GW effects on electromagnetic radiation

The vacuum equations for the 4-potential in the presence of a background GW can be derived from Eq. (6.11). In terms of the electric and magnetic components of the 4-potential, we have

$$\partial_\mu \partial^\mu \phi + \frac{f_\times (\partial_t f_\times) + f_+ (\partial_t f_+)}{f_\times^2 + f_+^2 - 1} \partial_t \phi - \frac{f_\times (\partial_z f_\times) + f_+ (\partial_z f_+)}{f_\times^2 + f_+^2 - 1} \partial_z \phi = 0, \tag{7.156}$$

and

$$\partial_\mu \partial^\mu A^k + \frac{f_\times (\partial_t f_\times) + f_+ (\partial_t f_+)}{f_\times^2 + f_+^2 - 1} \partial_t A^k - \frac{f_\times (\partial_z f_\times) + f_+ (\partial_z f_+)}{f_\times^2 + f_+^2 - 1} \partial_z A^k = 0, \tag{7.157}$$

respectively. In the absence of GWs we recover the usual wave equations. The resulting expressions simplify significantly if one considers only one of the two possible GW modes. For example, for an electromagnetic wave travelling in the z direction and the harmonic GW in Eq. (7.122) with no $(\times)$ mode, we get the following wave equation for the electric potential

$$\Box \phi - \frac{w a^2 \sin(wt - kz) \cos(wt - kz)}{a^2 \cos^2(wt - kz) - 1} \partial_t \phi - \frac{k a^2 \sin(wt - kz) \cos(wt - kz)}{a^2 \cos^2(wt - kz) - 1} \partial_z \phi = 0, \tag{7.158}$$

with $\Box\phi \equiv \partial_{tt}^2\phi - \partial_{zz}^2\phi$. This equation in principle can be studied by applying Fourier transformation methods.

In order to study in depth the physical (measurable) effects of the passage of the GW on electromagnetic wave dynamics, one needs to solve these equations and then compute the gauge invariant electric and magnetic fields. We see in the above wave equations, the presence of terms proportional to the first derivatives which are completely absent in the electromagnetic wave equations in flat spacetime (in Cartesian coordinates). These terms are always induced by gravitational fields, but in this case the gravitational field is dynamical which represents a much richer electromagnetic wave signal with the signature of the GW (see also [177]). Such signals in the radio regime might possibly be detectable through methods of Long Baseline Interferometry, in order to amplify it. Nevertheless, we can see from the expressions above that the extra terms on the electromagnetic wave equations, induced by GWs are proportional to the frequency. Such gravitational effects might become important for sufficiently hight frequency GWs. Simulations are required to see the feasibility or not of such methods.

7.4.2 Discussion and summary

GW astronomy is an emerging field of science with the potential to revolutionize astrophysics and cosmology. The construction of GW observatories can also effectively boost major technological developments. Given the extremely low GW amplitudes reaching the Solar System, incredibly huge laser interferometers have been built and others are under development in order to reach the required sensitivities. These observatories represent an amazing technological effort and it is natural to investigate if there are alternative complementary routes towards GW detection and if the GW signal can be amplified.

One fundamental prediction of the coupling between gravity and electromagnetism is the generation of electromagnetic waves due to gravitational radiation. Therefore, in principle under the appropriate resonant conditions, the electromagnetic signal thus produced can be amplified allowing us to measure GWs, not through the motion of test masses but rather by transferring the GW signal directly into electromagnetic information. This fact might represent an important change in perspective for future ground and space GW detectors.

The fact that GWs can generate electromagnetic waves is of course not evident if one restricts the analysis to the propagation of light rays (in the geometrical optics limit) in curved spacetime. On the other hand, the full Einstein-Maxwell system of equations have to take into account the curved spacetime within Maxwell's equations and also the contribution of the electromagnetic stress-energy tensor to the gravitational field. The first aspect of this coupling was considered in this work, and it is sufficient to show that GWs can be sources of electromagnetic waves. The full gravity-electromagnetic coupling also shows the reverse phenomenon (for example in eliptical electromagnetic polarizations).

In this section, we obtained electric and magnetic field oscillations fully induced by a GW travelling along the z axis. For simplicity we assumed harmonic GWs. We considered the Gauss law for the cases of an electric field along the z axis, along the x axis and in the (x, y) plane. In the first case, the solutions in Eqs. (7.117) and (7.124) allowed to make an estimation of the energy flux of the resulting radiation. It is important to emphasize the fact that the electric fluctuation thus produced corresponds to a longitudinal wave. This means that a non-zero longitudinal mode in electromagnetic radiation can in general be induced by gravitational radiation. One should search for these GW signatures in the electromagnetic counterpart of GW sources. The solution we obtained shows the dependence on the amplitudes of the two GW modes, a and b, as well as on the frequency w and phase difference α. An important aspect of hypothetical electric-GW detectors is the fact that in general although the signal is very weak for any GW reaching the solar system, under appropriate resonant conditions it can be amplified. In fact, this can be used to improve the signal to noise ratio since a system analogous to optical resonators can act as a filter privileging the signal with a specific (resonant) frequency.

The changing electric field in Eq. (7.117) inside a capacitor, for instance, would also generate alternate currents in any conductor placed between the capacitor's charged plates. In particular, a diode placed in the appropriate orientation would allow a current signal in a single direction intermittently, following the rhythm of the GW fluctuations. In the (x, y) plane the electric field can be generated by two independent capacitors in perpendicular configuration. The approximate solutions obtained show electric field oscillations generated by the GW which propagate along the z axis with non-linear polarization. We can expect a spatial polarization pattern in our detector which changes with time. This contrasts with the other cases where the resulting wave was linearly polarized. This effect is shown in Figs. 7.1 and 7.2.

In all cases, the resulting electromagnetic signal has the signature of the GW that produces it, depending on a, b, w and α. In any of these examples, time varying electric fields are generated, which can contribute to the magnetic field via the Maxwell-Ampère law. In particular, they appear in the generalized Maxwell displacement current vector density, Eq. (6.9), induced by the GW. This in turn can generate a time varying magnetic field even in the absence of electric currents. Accordingly, GWs also induce magnetic field oscillations. We made an estimation of such an effect considering the case of a diagonal metric by setting the $(\times)$ GW mode to zero. We assumed a certain electric current I along the z axis and the electric field in Eq. (7.143) along the same direction. The magnetic field thus generated lies on the (x, y) plane and it is easy to see that for any point (x, y) fixed on the magnetic field lines, the x and y components of the magnetic field will oscillate in time out of phase, such that when one is at its maximum value, the other is at the minimum, and vice-versa. The overall result is that the magnetic field lines will oscillate with the passage of the GW, following the deformations of spacetime geometry. Figure 3 illustrate this phenomenon and was obtained using the expressions in Eqs. (7.151) and (7.152). In Fig. 4 we see the strength of the effect as a function of time. The signal to be measured is independent from the current I and it depends on the position (x, y) as well as on the GW parameters. It can be easily shown that the strength of the fluctuations are much stronger in specific regions of the (x, y) plane. Such small magnetic field changes could in principle be measured with SQUIDS (SuperConducting Quantum Interference Devices), which are sensitive to extremely small magnetic field changes [209]-[212]. SQUIDS have amazing applications in biophysics (in particular to biomagnetism) and medical sciences and also in theoretical physics: studies of majorana fermions [213], dark matter [214], gravity wave resonant bar detectors [215], cosmological fluctuations [217, 216].

The calculations in this work point to electromagnetic effects induced by GWs such that

$$\frac{\delta E}{E} \sim h, \qquad \frac{\delta B}{B} \sim h, \qquad h \sim 10^{-21}, \tag{7.159}$$

where h is the amplitude (strain) of the GW reaching the Solar System. SQUIDS have an incredible sensitivity [209]-[219] being able to measure magnetic fields of the order of $10^{-15}T$ or even $10^{-18}T$ for measurements performed over a sufficient period of time (the SQUIDS used in the GP-B experiment had this sensitivity). Using these values for the SQUIDS sensitivity, in order to be able to measure the tiny GW effects on magnetic fields we would require magnetic fields of the order of $B \sim 10^6T$ or in the best case $B \sim 10^3T$. Presently, the highest magnetic fields produced in the laboratory have values of $B \sim 45T$ (continuous) and $B \sim 100T - 10^3T$ (pulsed). therefore, although SQUIDS are extremely sensitive, there is a real limitation to perform these measurements coming from the huge magnetic fields required. Nevertheless, the science of SQUIDS and ultra-sensitive magnetometers is very active and evolving [218, 219] and it is natural to expect improvements in terms of sensitivities and noise reduction and modelling. For $B \sim 10T$ laboratory magnetic fields we would require extremely higher sensitivities ($\delta B \sim 10^{-20}$) which is not in the reach of present magnetometers. Besides these considerations, intrinsic and extrinsic noise should be extremely well modelled and if possible reduced by advanced cryogenics and filtering processes.

We may also consider the use of electromagnetic cavity resonators to amplify the electromagnetic waves induced by the GWs. For magnetometers with $\delta B \sim 10^{-18}T$ sensitivities and $10T$ reference magnetic fields, it means that the amplification of the signal would have to be about 2 orders of magnitude. Even if this cannot be achieved by present day electromagnetic resonators it might be in the near future. An important advantage of these cavities is that in practice they work as filters being able to amplify a signal centred around a specific frequency which corresponds to the fundamental frequency of the resonator. For cylindrical resonators with size L, the wavelength of the fundamental frequency is $\lambda \sim 2L$, meaning that different resonators of different sizes would be sensitive to the different parts of the GW spectrum. By effectively filtering and amplifying the signal around a certain frequency far from the noise peak, it is in principle possible to increase substantially the signal to noise ratio, which is essential for a good measurement/detection.

Let us consider the case where we use electric fields instead of magnetic fields in our electromagnetic detectors, for example the electric field inside a charged plane capacitor. By measuring the Voltage signal instead of electric field, we have the advantage of being able with the present technology to, on one hand, easily produce 10^3V or higher static fields and on the other hand, to reach sensitivities of $\delta V \sim 10^{-15}V$. This means that the signal should be amplified 2 to 3 orders of magnitude. The combination of electromagnetic cavity resonators and electronic amplifiers (for the Voltage signal) could make this a real possibility for GW detectors.

Moving now from human made laboratories on earth or in space to natural astrophysical observatories, we call the attention to the fact that the highest magnetic field values (indirectly) measured so far are those of neutron stars with intensities around $10^6T - 10^{11}T$. Radio and X-ray astronomy is able to indirectly measure these astrophysical magnetic fields by considering the properties of Cyclotron radiation. A stochastic GW background signal due to innumerable sources in the galaxy and beyond is expected to leave a measurable imprint on the magnetic field of normal pulsars and magnetars. In fact, this method could be used in a complementary way to that of PTA (Pulsar Timing Arrays) to measure a stochastic GW signal. The huge magnetic fields in the surroundings of pulsars makes them natural laboratories to study the effects of GWs on electromagnetic fields. The use of arrays of Pulsars could be advantageous in order to distinguish the GW signal from intrinsic fluctuations of the magnetic field and to better deal with extrinsic noise (via statistical methods). Pulsars are extremely precise clocks and if they behave as very stable dynamos, then it might be possible to generalize the methods and years of improvement in PTA by measuring the interaction of GWs with magnetic fields. Is this another window for GW astronomy through VLBL (Very Long Baseline Interferometry)? We leave this as an open question that deserves more research from both theorists and observation experts.

We also obtained charge density oscillations induced by GWs. These can propagate as density waves in the case of charged fluids, through which a GW is propagating. This effect deserves to be taken in consideration within more complete magnetohydrodynamical computations, in order to have simulations of the effects of GWs in plasmas near the cores of highly energetic GW sources. These plasma environments might occur in different astrophysical sources such as Gamma Ray Bursts and some specific coalescing binaries. Regarding electromagnetic waves in the presence of gravity, extra terms appear in the generalized wave equations which deserves further research to get a full analysis of the approximate solutions. Indeed, going beyond the geometrical optics limit, light deflection (null geodesics) and gravitational redshift are not the only effects arising from the coupling between light and gravity. More generally, all electromagnetic waves can experience gravitational effects on the amplitudes, frequencies and polarizations. The electric and magnetic wave dynamics can be coupled due to the non-stationary geometries, as is the case of GWs. Important studies have been made regarding the electromagnetic counterpart of GW sources (see for example [205] and [206]), but there is much to explore in the landscape of (multi-messenger) gravitational and electromagnetic astronomy

In general, one expects that GWs induce rich electromagnetic wave dynamics. These effects might become more significant for very high frequency GWs as one can see from Eq. (7.158). Moreover, the terms proportional to the first derivatives of the 4-potential have space and time varying coefficients. For the harmonic GWs considered in this work, these coefficients oscillate between positive and negative numbers, a fact that might imply a very distinctive wave modulation pattern of the resulting electromagnetic wave. This hypothesis and its implications requires further investigation as it might provide very rich GW information codified in the electromagnetic spectra of different astrophysical sources.

Part III

Towards a new spacetime paradigm: Open questions in unified field theories

Chapter 8

From geometrical methods and symmetry principles towards unified field theories and a new spacetime paradigm

Therefore, the endeavour of the theoreticians is directed toward finding natural generalisations of, or supplements to, Riemannian geometry in the hope of reaching a logical building in which all physical field concepts are unified by one single viewpoint.

(Einstein)

8.1 From geometrical methods and symmetries towards a new spacetime paradigm

Geometrical methods and symmetry principles, motivated by gauge theories of gravity, post-Riemannian geometries and Yang-Mills gauge fields (in the bundle formalism and exterior forms), are expected to play a vital role towards a unified field theory. This unified physical description might lead to a new spacetime paradigm extending the notion of the classical spacetime manifold where physical fields propagate on it and affect its geometry. Instead, it is plausible to expect a consistent unified physical manifold, where the properties of spacetime, matter fields and vacuum are manifestations of the same fundamental physical ontology.

8.1.1 Summary and open questions

In this thesis we explored the geometrical methods (post-Riemann spacetime geometries and Cartan's exterior calculus of forms) and symmetry principles in the gauge approach to gravity as well as the pre-metric formulation of classical electrodynamics, and how these topics might point to a new perspective over the spacetime paradigm. In this perspective, the conformal-causal structure is more fundamental than the metric structure (the primacy of the conformal geometry) and the absoluteness of the spacetime metric is abandoned at the fundamental level. We established the analogies between the pre-metric canonical formulation of gauge theories of gravity and the pre-metric equations and mathematical objects of general Yang-Mills fields. The theoretical formulation of Lagrangians in these theories (gravity and Yang-Mills) implicitly presuppose the assumption of the specific form for the constitutive relations between the field strengths (the generalized field velocities) and the excitations (conjugate field momenta). These relations can be interpreted as constitutive relations for the spacetime itself as suggested by Friedrich Hehl et.al. [39, 45, 46, 47], and we reinforce such interpretation here and moreover, we highlight the hypothesis that the physical constants, or coupling parameters, that enter in such relations, reflect physical properties of the spacetime manifold.

By endowing the classical spacetime with physical properties, the concept of classical vacuum with properties such as electric permittivity, magnetic permeability, etc, becomes somehow dispensable or simply dual to the very notion of a physical spacetime. The classical physical vacuum *is* the classical physical spacetime. Moreover, the properties of this physical spacetime might change from point to point and this scenario fits well within the scalar-tensor (Brans-Dickie, etc), vector-tensor or tensor-tensor extensions to GR. The idea that these properties of spacetime can be described by fields can have implications to spacetime symmetry considerations, i.e, the invariances of the physics under groups of spacetime coordinate transformations, and also a link to the Mach's ideas and the breaking of spacetime (metric) absoluteness. Therefore, this scenario of a physical spacetime with non-Riemann geometry and physical properties described by fields (not necessarily homogeneous and isotropic) fits naturally very well in the assumption of the primacy of the conformal-causal structure. This means that the so-called constants can change from place to place in space (non-homogeneity) and with spatial direction (anisotropy) and still preserve the local conformal-causal symmetry, and therefore, by extension, the causal structure.

The idea of an extended spacetime manifold with physical properties, puts forward the question of the objectivity of this construction. In fact what is assumed to be physically relevant (and endowed with physical objectivity) are the conformally-invariant properties. Therefore at this point the notion that is suggested at the classical level is highlighted below

An extended physical-spacetime manifold with non-Euclidean (post-Riemann) geometries, physical properties and a fundamental conformal symmetry.

The idea of a physical-spacetime manifold and certain motivations from several areas of physics suggests that the intrinsic physical properties of spacetime include energy-momentum, hypermomentum (including spin), electromagnetic and thermodynamical properties. The notion of thermodynamical properties associated to gravity and to spacetime suggests, by its turn, the existence of microscopic degrees of freedom, and if these are assumed to constitute a numerable set, then a consistent picture should go beyond the classical manifold and incorporate some degree of discreteness or gravity quantization. So we can consider at the next level of paradigm change:

A physical-spacetime manifold with locally invariant conformal-casual structure, with intrinsic physical properties (electromagnetic, thermodynamical, etc) and possibly a quantum nature.

This seems to be an appropriate hypothesis to address the unification of spacetime, matter fields and the quantum and classical vacuum.

In this chapter we will explore in more detail this hypothesis and its relation to various aspects of unification in physics and quantum gravity. The emphasis is more on the conceptual and philosophical side motivated also by mathematical considerations and in particular from geometrical methods and symmetry principles[1].

8.1.2 From symmetries and geometry towards unified field theory

The main relevant interpretations and hypothesis related to the above considerations and that are explored in this thesis are resumed below:

1. Geometrical methods in the pre-metric formalism of electromagnetism, Yang-Mills and gravity field equations, using the calculus of exterior forms, together with the corresponding (spacetime) constitutive relations, suggest the following:

    - The primacy of the conformal-causal structure (the conformally-invariant part of the metric) over the full metric structure. Corolary: The assumption of absoluteness of the spacetime metric ("absolute spacetime") is abandoned at the fundamental level.
    - Spacetime with physical properties. The fundamental coupling constants entering in the (vacuum) constitutive relations represent physical properties of spacetime. Corolary 1: The possibility that these properties are not *a priori* spatially homogeneous and isotropic (constants), but can change, following the isometries of spacetime, while respecting at the same time the local conformal symmetries (and the corresponding causal cone). Corolary 2: The identification of the physical classical vacuum with physical classical spacetime.

2. Gauge symmetries in gravitation and post-Riemann geometries

    - Classical spacetime with general metric-affine geometry: Curvature, torsion and non-metricity. The gauge approach to gravity clearly imply the existence of spacetime post-Riemann geometries associated to gravitational phenomena. This motivates the search for its signatures via astrophysical and cosmological effects, including GW probes and effects in particle physics.
    - Spacetime metric is not fundamental. Gauge theories of gravity have a robust mathematical consistency linking the symmetries of physics under spacetime coordinate transformations to the geometrical paradigm of spacetime. The theory identifies clearly, for each specific gauge group, the spacetime geometry, the Noether currents (which are the sources of the gravitational field) and the gauge potentials (geometrical degrees of freedom), and its field strengths. In this formalism the spacetime metric is not fundamental. This becomes more clear in the language of forms (Cartan calculus), where an explicit pre-metric approach to the field equations is completely general, coordinate-free and covariant. The metric structure is absent at the foundational level and can be assumed *a posteriori*, in particular via the constitutive relations. These relations relate the field strengths or field velocities, i.e curvature and torsion for example, to the canonically conjugate field momenta. Upon substitution

[1] For a recent exploration of historical, mathematical and philosophical consideration in gauge theories and gravity see [36]

in the inhomogeneous equations and through the Hodge star operation, only the conformally invariant part of the metric structure is involved in the coupling between the field strengths and the (Noether) sources, not the full metric structure. Therefore, symmetry principles and geometrical methods in gauge theories of gravity suggest again that the (full) metric structure is not fundamental and in models with wider symmetry groups beyond the Poincaré group, the paradigm of spacetime (metric) absoluteness is not valid. i.e, the metric changes under specific coordinate transformations between local observers (as in CGTG).

- Unified symmetry groups, symmetry breaking and phase transitions. General symmetry groups can be broken into smaller groups in processes of phase transitions. In principle, these are expected to have occurred in the early Universe in clear analogy to the standard model (and beyond standard model) of interactions, leading to first order phase transitions and the generation of GW emission in the form of a stochastic GW background. This GW signature of cosmological origin (which might include polarization features due to parity breaking effects of general quadratic Yang-Mills gauge theories of gravity) might be detectable by future missions such as LISA. Corolary 1: The breaking of scale invariance in the Early Universe. Of particular interest is the conformal group of Conformal Gauge Theories of Gravity (CGTG) that may be broken into the Poincaré group via symmetry breaking phase transitions in the early Universe. These necessarily include the breaking of scale invariance and the emergence of natural physical scales and the corresponding constants of nature. This presuppose the existence of a scale invariant cosmological epoch where the properties of the physical spacetime obey perfect conformal symmetry and its geometry, and also a transition into a broken symmetry phase where the spacetime metric appears to have a (local) absolute nature. Corolary 2: In specific "extreme" physical regimes (astrophysical compact objects and BHs) scale invariance, or the full conformal symmetry might be recovered.

3. Dualities and correspondences between metric-affine geometrical objects in gravitation (maps between curvature, torsion and non-metricity) together with the Bianchi identities suggest interesting possibilities

   - Curvature, torsion and non-metricity might be inter-convertible. This interpretation seems to be appropriate from the point of view of the generalized Bianchi identities of metric-affine geometries, see (3.31) and (3.24), relating the field strengths of gauge theories of gravity (curvature, torsion and also non-metricity). This is analogous, in some sense, to the magnetic-electric interconversion of electromagnetism. Recall that the equation $dF = 0$ is a Bianchi identity giving magnetic flux conservation and Faraday law. These post-Riemann Bianchi identities are implicit to the spacetime geometrical structures and express some sort of gravitational flux conservation and are compatible with the Noether currents conservations of the gauge approach to gravity. In terms of the relations between the field strengths, these relations seem to point towards the notion that curvature, torsion and non-metricity are interconvertible, which could open profound new avenues for the study of gravitational phenomena, extreme compact objects and blackholes, high-energy astrophysics, cosmology and GW astronomy. Corolary: An interesting analogy also to the Weyl-Ricci conversion (Weyl conjecture [220]) in cosmology, within GR. According to this conjecture, the Ricci curvature dominates completely the very early Universe and the Weyl curvature dominates the late-

Universe, asymptotically, around BHs before their final evaporation[2]. In this conjecture, the Ricci part of the Riemann tensor is converted or transformed into the Weyl part, as the Universe expands and forms gravitationally bound structures, assymptotically dominated by blackholes, and in accordance with the second law of thermodynamics[3]. Something similar can be postulated for the post-Riemann geometries at the cosmological level.

- Formal maps between "equivalent" descriptions of GR in the spacetimes with only torsion, or curvature or only non-metricity seem to reinforce this interpretation. Similar correspondences might be found also between generalizations to GR in the respective spacetime paradigms. Although under specific formal conditions, the phenomenology of these theories might be equivalent, the spacetime paradigms are clearly different and the minimal/non-minimal couplings of fermionic fields to these geometries can break the equivalences. Nevertheless, the mathematical structure of the geometrical approach to gravity seems to be compatible with the interpretation that these geometrical objects can be inter-convertible within more general Metric-Affine geometries. In fact, as we saw earlier, inside the full curvature there is a part corresponding to the Riemannian curvature, another part related to torsion and another part related to non-metricity. Dynamical transformations (conversions) between these three parts can be compatible to the (generalized) Bianchi identity involving the full curvature. Therefore, the existence of the previously mentioned formal maps might be pointing to a more fundamental physical link between curvature, torsion and non-metricity.

4. From geometrical methods and symmetry principles towards a unified physical spacetime-matter ontology.

   - A unified physical spacetime-matter manifold. Geometrical methods (bundle manifolds, connections, non-Riemann geometries, Cartan calculus of exterior forms) and symmetry principles (spacetime/external and internal symmetries) have a fundamental role in gauge theories of gravity and gauge field theories of the standard model (and beyond standard model) of particles. The importance of these geometrical and symmetry methods and its potential to establish analogies and connections between different branches of physics or different interactions, clearly suggests its vital contribution within the approaches to unification.

     Let us briefly consider the structure of the gauge approaches to gravity and to Yang-Mills fields, as represented in figures 8.1 and 8.2. It is clear that the fundamental difference resides in the fact that the gauge approach to gravity involves (external) spacetime symmetries, while the gauge approach to Yang-Mills fields of fundamental interactions is based on internal symmetries under phase transformations of spinor fields. Any attempt to a consistent unification between these two approaches therefore, between gravity and the other fundamental interactions, will plausibly require a unified symmetry group, on one hand, and some type of geometrical unification between internal and external spaces, on the other. This motivates geometrical methods for an extended spacetime paradigm together with symmetry principles that connect fermions and bosons. Regarding the first issue, it is also quite reasonable to assume

[2]This can also be generalized to cosmological cyclic models if one postulates the conversion Ricci → Weyl → Ricci, etc.

[3]The spacetime around Weyl type of singularities (inside BHs) is much more non-uniform or chaotic than the uniformity found around the Ricci type of (cosmological) past singularity. The entropy of gravitational fields is very high within BHs and very low in the very early Universe.

that the inclusion of complex numbers into the geometrical (extended) spacetime manifold is required in order to unify the (Bundle) connections related to internal and spacetime fibers. As for the second point, it is reasonable to consider the extensions of Lie algebras, as in the SUSY-SUGRA approaches so that bosonic (scalar, vector, tensor) and fermionic (spinor) fields are encompassed under the same mathematical formalism and linked by superymmetric transformations.

A fundamental geometrical unification could pressuppose a bundle connection with the bosonic sectors as in

$$\mathcal{A} = \mathcal{A}^{Gravity} + \mathcal{A}^{U(1)} + \mathcal{A}^{SU(2)} + \mathcal{A}^{SU(3)}, \tag{8.1}$$

or a single object, under a unified symmetry group, which under goes a differentiation into different pieces due to symmetry breaking phase transitions (of Higgs-like mechanisms)

$$\mathcal{A}^{Unified} \longrightarrow \mathcal{A}^{Gravity},\ \mathcal{A}^{GUT} \longrightarrow \mathcal{A}^{SU(3)},\ \mathcal{A}^{electroweack}... \tag{8.2}$$

Nevertheless, since fermionic fields constitute the fundamental nature of matter (quarks, leptons, hadronic structures), and provide the basic symmetries describing the electromagnetic and nuclear interactions, spinors are to be regarded as fundamental mathematical objects in any unified field theory. It might be the case that a consistent picture with a geometrical and group theory unification as sketched above is formulated within a bundle in which the base manifold is constructed with or from spinorial spaces.

A compact representation of the standard picture in field theories can be expressed in the following equations

$$\Psi = \int_{Feynman} \delta X e^{\frac{i}{\hbar}\left[S_g(g,\Gamma)+S_m(A,\psi,\varphi^H)\right]}, \tag{8.3}$$

$$\Psi = \int_{Feyn.} \delta X e^{\frac{i}{\hbar}\left[\frac{R}{16\pi G}+\Lambda g_{\mu\nu}+...+\frac{\gamma F^2}{4}+\frac{i\hbar}{2}(\bar{\psi}\gamma^\mu D_\mu\psi-(D_\mu\bar{\psi})\gamma^\mu\psi)-\lambda\varphi^H\bar{\psi}\psi+(D\varphi^H)^2-V(\varphi^H)\right]} \tag{8.4}$$

where the integral is a Feynman integral over all possible "histories", and we recognize scalar (Higgs), vector, tensor and spinor fields, spacetime geometry, gravity, cosmological constant, the coupling constants, and the wave function formalism, in the same mathematical expression. How to unify all these concepts? Is the Higgs field and also the coupling constants fundamental physical properties of the spacetime manifold? Can matter fields be geometrized? Is spacetime quantized? Do internal spaces of spinor symmetries and its underlying complex number structure, have an objective physical meaning? These questions remain open and motivate the searches for appropriate geometrical methods and symmetry principles that could provide a consistent picture, susceptible of being observationally tested.

As said before, one possibility and open question is that unification in physics must inevitably imply the merging between spacetime and physical fields into a single "physical spacetime-matter" manifold - the conceptual and mathematical representation of a unified physical ontology. It is quite consensual that the road to unification and quantum gravity will inevitably lead to a new spacetime paradigm. Here we reinforce this idea, starting from motivations already present at the classical level, using the geometrical and gauge symmetry considerations, and then include also various ideas from quantum theory and thermodynamical considerations, as relevant motivations for such new paradigm. A unified physical spacetime-matter manifold might have the following ingredients:

- Conformal symmetry at a fundamental level (possibly inside some unified symmetry group[4]);
- Internal and external symmetries;
- Have more than 4-dimensions;
- Include complex numbers (internal symmetries of spinors are embedded and intrinsic to the physical unified manifold);
- General metric-affine geometry and a gauge theory of gravity in the continuum (classical) limit;
- Have intrinsic internal and external physical properties such as stress-energy-momentum, hypermomentum (including spin density), U(1) and SU(N) internal charges;
- Have electromagnetic properties (such as electric permittivity and magnetic permeability), following the isometries and in accordance with local conformal symmetry;
- Thermodynamical properties (where the total local entropy possibly includes the contribution from internal degrees of freedom[5]);
- A fundamental quantum nature, possibly a discrete geometry incorporating the indeterminacy principle (this solves the quantum divergences);
- The possibility of phase transitions in cosmological evolution (pre-bigbang or during the GUT epoch), breaking the scale invariance and leading to well-deffined physical (natural) scales, the corresponding physical constants and the phenomenological laws.

The open questions regarding a consistent unified physical manifold are at the heart of great challenges at the frontiers of theoretical physics. In the topics above many ingredients and considerations were left out, and a more rigorous treatment to these issues is way beyond the scope of this thesis. It would be almost impossible to enumerate here an extensive list of valuable methods that are plausible to be important for unification and to address the quantum gravity challenge (causal sets, twistors, SUSY-SUGRA, strings and super strings, loop-gravity methods, etc.). Let us briefly look at some of the above considerations. The issue of the importance of the conformal structure has already been extensively discussed in this thesis. Let us consider then some ideas motivated from quantum physics. We start with a gentle note on the hypothesis of geometry discretization that is worth while to emphasize here: Both the thermodynamics of gravitational fields and the entropy of spacetime regions (horizons) and the fact that the metric-affine geometries describe the continuum-limit of the geometries of crystal lattices with defects, point to the idea of microscopic degrees of freedom in numerable sets. This links directly to the challenge of making compatible the spacetime causal and metric structures with the indeterminacy principle of physical fields. If physical fields are merged into spacetime in a single unified construct, then clearly such a manifold has to be quantized in a sense that it includes the indeterminacy principle in its intrinsic geometrical nature. If matter/energy is quantized (discretized) and if spacetime is also quantized, then it might be the case that matter is quantized because spacetime is quantized, or vice-versa. Instead of sustaining a reductionist approach and end up in the

[4]It is quite plausible to consider the fundamental role of SUSY and SUGRA in the unification of fermions, bosons and spacetime geometries.

[5]The internal degrees of freedom linked by SU(N) symmetries might be considered as physical degrees of freedom, analogous to the microscopic sates in statistical physics.

cyclic question of "what is more fundamental, spacetime or matter?", one can take the unified approach such that one cannot be considered without the other, i.e, both (spacetime and matter-energy) are fundamental and unified into a single physical ontology. In other words, in this perspective, spacetime and matter always come together and are inseparable. Under this hypothesis, at the deep fundamental level and way beyond the phenomenological realm, the differences between the two could fade away, and a common quantum-geometrical-physical nature might be revealed. Moreover, regarding the quantum challenge, it is worthwhile to recall here the note taken at the end of chapter 3 that emphasized the fact that a non-perturbative or semi-perturbative approach can be taken in both the 4-dimensional spacetime and momentum-energy manifolds, assuming local conformal symmetries, without the need for a background metric/vacuum.

The idea sketched above that matter-energy and spacetime are inseparable, in the context of the role of geometrical methods towards a new matter-spacetime paradigm, stimulates a geometrical unification between the spacetime and energy-momentum manifolds. In such construct, these 4-dimensional spaces can be seen as mutual internal spaces at each point (see figures 8.3 and 8.4). The unification in this approach can give new perspectives into the conciliation between the spacetime manifold (with its causal and metric structures) and the indeterminacy principle of matter fields. In quantum physics there is a well-known duality between a well defined spacetime representation and a well defined causal description, i.e, the indeterminacy principle, since in order to establish energy-momentum conservation to any physical process within a causal evolution, one needs to determine the energy and momentum of the system in the initial and final states, and this is limited by the Heisenberg uncertainty relations. When spacetime coordinates are completely determined, the energy-momentum is probabilistic and fully undetermined and vice-versa. Moreover, in the wave description, one says that the wave function collapses in spacetime into a point, while spreading in the energy-momentum space and vice-versa in a complementary way. This is, of course, related to the measurement problem and the Fourier transforms between both representations, in accordance to the Heisenberg relations. One can then picture a (unified) dynamical 8-dimensional energy-momentum-spacetime manifold (extended phase space) with post-Riemann (non-rigid) geometries that is compatible with the indeterminacy principle. To what extent the geometrical methods motivated by the gauge approaches to gravity and the hypothesis that gravity plays a fundamental role in quantum state reduction [221, 222], remains an interesting and open question, as well as the possible role of non-commutative and sympletic geometries. The duality between the spacetime and energy-momentum representations of a physical system, in connection to the measurement problem (wave collapse) is also represented in figures 8.3 and 8.4 .

Regarding the fundamental role played by spinors in gauge Yang-Mills theories, as already mentioned, it can motivate a geometrical construction that includes the complex numbers of spinors within the base (extended) spacetime manifold. Incorporating complex numbers into the physical manifold might lead us naturally to theories with extra dimensions, since the "hidden" imaginary numbers can be seen as spanning extra compactified dimensions. For example in the 5-dimensional case, the extra dimension can be thought as an imaginary time curved around the ordinary time dimension. This is reminiscent of Kaluza-Klein approaches and it calls for its extensions including complex numbers and non-Riemann geometries such as those with torsion and non-metricity. Since complex numbers are vital to quantum theory, it seems inevitable to consider a merging of spinorial matter fields and spacetime manifold using similar geometrical considerations such as these briefly sketched here. In the standard geometrical picture we have the following: The spinor fields of all matter are everywhere in spacetime, and at each point, there are four wavefunctions, which are complex numbers. At each spacetime point these complex numbers can be thought as vectors in an Argand plane and rotating. These phase transformations are in general different at different spacetime points, and the gradient of the complex phases in spacetime is basically the electromagnetic potential, while the curl of the potential is the electromagnetic field strength. Moreover the phase can change in space and in time but also at each spacetime point without any temporal or spatial changes, according to the gauge (phase) symmetries. This picture can be reformulated in a unified spacetime-matter approach, where the spinor fields are an intrinsic part of the base manifold. Indeed, if the manifold coordinates are allowed to be complex numbers $(t + i\tau, x + i\alpha, y + i\beta, z + i\gamma)$, then in practice each point of the manifold is represented by an object that is very similar to a spinor, since one has a set of four complex numbers. The extra imaginary parts can then be curved around the usual coordinates, encompassing four independent U(1) phase transformations, at each spacetime point. It should be possible to clarify the appropriate role of the torsion, non-metricity and curvature applied to the Bundle implicit in this extended complex spacetime.

In the first chapters of the thesis we elaborated the idea that the classical physical vacuum can be identified with the classical (physical) spacetime. Here we highlight the notion that if the quantum vacuum can also be identified with a certain extended spacetime manifold, then the matter particles/fields are seen as excitations of this physical-spacetime manifold.

The incorporation of complex numbers into the spacetime manifold has also the advantage of possible applications to the interesting topic of metric signature and signature changes. One can say that the signature, the conformal structure and the nature of time are fundamentally connected, and as such these topics also deserves our attention here.

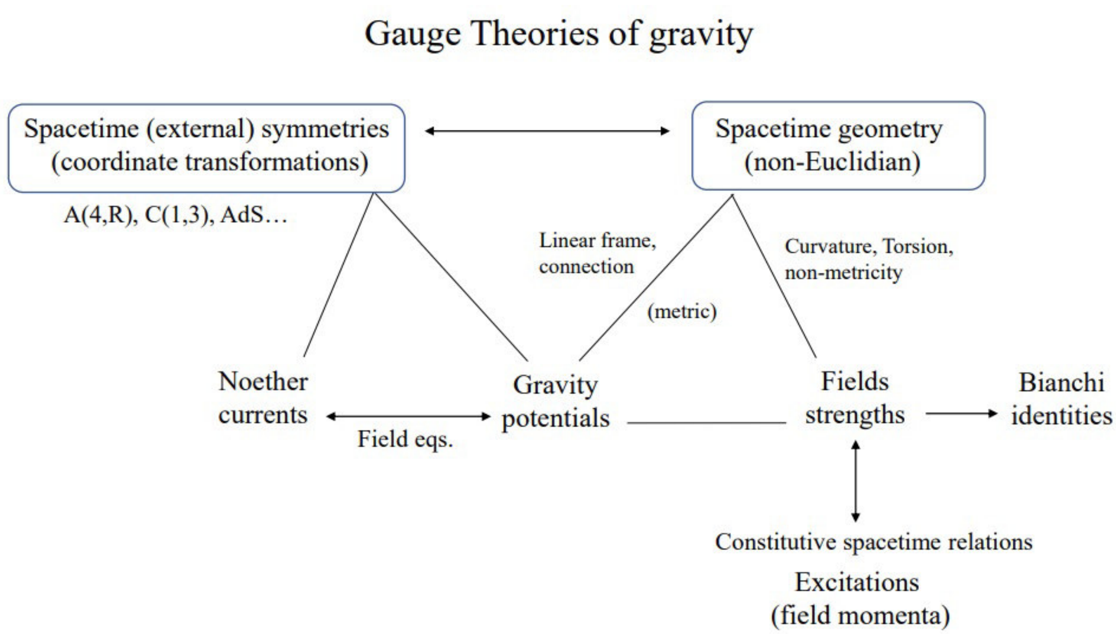


Figure 8.1: Gauge approach to gravity diagram.

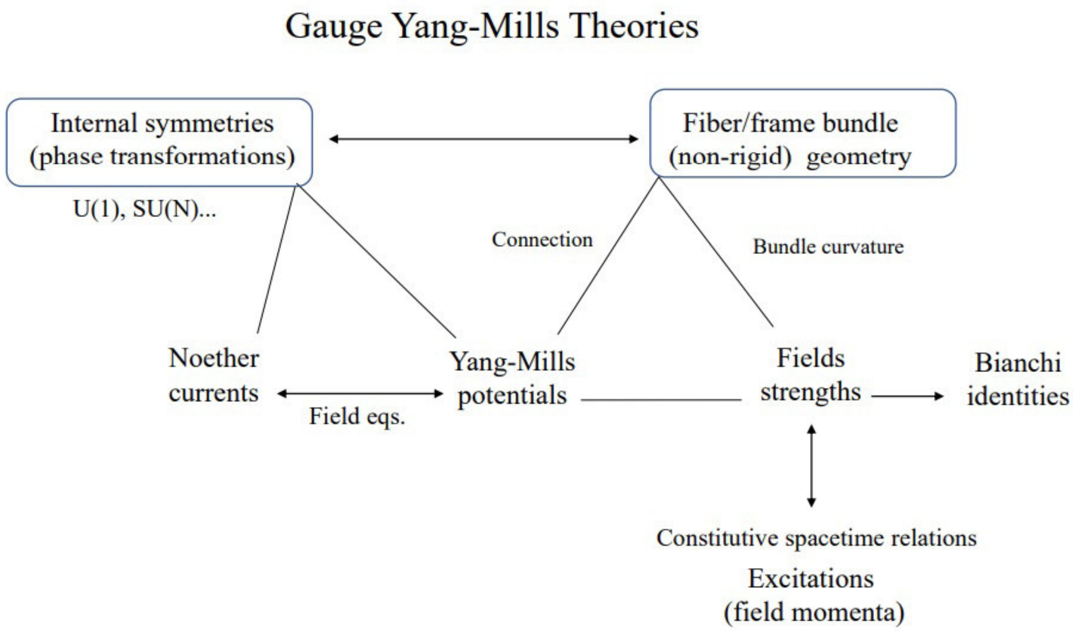


Figure 8.2: Gauge theories for Yang-Mills fields.

8.1.3 Causal structure, parity transformations, time-reversal and spacetime signature

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Let us consider the 4-dimensional spacetime with pseudo-euclidean geometry and metric η. The Lorentz transformations $L\eta L^T = \eta$ belong to a group which can be seen as the "union" of four sub-sets

$$L = L_+^\uparrow \cup L_-^\uparrow \cup L_+^\downarrow \cup L_-^\downarrow, \tag{8.5}$$

where $+$ stands for $det(L) > 0$ and $\uparrow$ for $L_{00} > 0$ and inversely for the $-$ and $\downarrow$ cases. The proper Lorentz sub-group $L_+^\uparrow$ therefore, preserves the "directionality" of time and the "axiality" of space. In fact, the parity transformations P (spatial reflexions) and the time inversions T obey to $P \in L_-$ and $T \in L^\downarrow$, respectively. Any Lorentz transformation belonging to any of the four subsets above can be transformed into another Lorentz transformation belonging to any of the other three subsets by application of P and T or products of these. These parity and time-reversal transformations are indeed very fundamental. Although usually one considers the proper subgroup $L_+^\uparrow$ (SO(1,3)) to represent the Lorentz group, in fact, in particle physics parity symmetry is sometimes broken as in the case of the weak nuclear interactions and possibly in some Great Unifying theories (GUT).

Now let us investigate how the parity and time-reversal are also surprisingly very fundamental to the spacetime paradigm, in relation to the metric structures, metric signature, causal structure and also to geometry. Consider first the relation to the causal/conformal structure of 4-dimensional pseudo-euclidean (flat) spacetime. Under a parity or a time-reversal transformation, the causal cone $ds^2 = 0$ remains invariant.

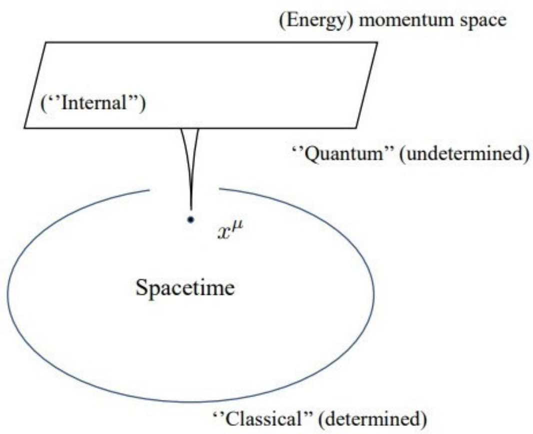

Figure 8.3: 4-momentum space as an internal space at each spacetime point. When the spacetime is determined according to the quantum state reduction, or wave function collapse in spacetime, the energy-momentum is completely undetermined.

One can think that with respect to two reference systems (or hypothetical observers) related by a P transformation the causal-cone is rotated by π, with respect to a time direction orthogonal to the 3d hypersurfaces, while for two reference systems related by time-reversal T, the causal-cone is reflected upwards/downwards with respect to the 3d hypersurfaces. Consider the world line of an idealized particle. Relative to one another, the observations of two observers related via a T transformation correspond to the particle moving forward or backward in time along the world-line. The positive time direction of one observer is symmetrical to the positive time-direction of the other and vice-versa. Both P and T preserve the metric structure ($L\eta L^T = \eta$), the line element and the causal structure, but change the directionality of time, that is, the arrow of time (conventionally taken as time moving in the positive direction) and the directionality/axiality of space (mirror reflexions).

Now, consider the issue of spacetime signature. In matrix form the parity and time-reversal transformations are $P = diag(+1-1-1-1)$ and $T = diag(-1+1+1+1)$, respectively. Therefore, curiously the two different choices for the Minkowski pseudo-euclidean metric correspond to the matrix representation of the parity P and time-reversal T transformations. Recall that in more general manifolds the Minkowski metric and its inverse establish a correspondence between the tangent and co-tangent spaces. If one chooses the $diag(+1-1-1-1)$ signature for the pseudo-euclidean tangent space, then the metric maps a Lorentz vector into a co-vector which corresponds exactly to the mirror-reflection of the original 4-vector. Similarly, the inverse metric coincides with the Minkowski metric and brings a co-vector into a vector. Therefore, one comes back into the original vector by first contracting with the metric (parity transformation) and then applying the inverse metric (again the parity mirror-reflection). Indeed the parity transformation is its own inverse. Analogously, by choosing the $diag(-1+1+1+1)$ signature for the pseudo-euclidean tangent space, then the metric maps a Lorentz vector into a co-vector which corresponds exactly to the time-reversed of the original 4-vector. And again the T transformation is of course its own inverse. So, given the tangent space with pseudo-euclidean metric, Lorentz co-vectors obtained by applying the $diag(+1-1-1-1)$ metric to the Lorentz vectors are equal to the mirror-reflections of these, while Lorentz co-vectors obtained by applying the $diag(-1+1+1+1)$ metric to the Lorentz vectors are equal to the time-reversed of the original 4-vectors. Also very interesting is

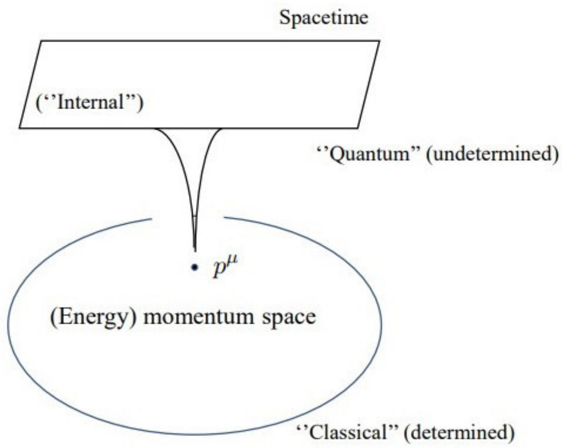

Figure 8.4: Spacetime as an internal space at each point of 4-momentum space. When the 4-momentum is determined according to the quantum state reduction, or wave function collapse in 4-momentum space, the spacetime is completely undetermined.

the fact that $P = -T$ and

$$-PT = -TP = diag(+1 + 1 + 1 + 1). \tag{8.6}$$

The Lorentzian signature $(+ - - -)$ or $(- + + +)$, is strongly related to the fact that the four dimensions don't have the same nature, some can be called spatial and others time-like. We can say that there is 1 time-like dimension and 3 spatial dimensions in strict relation to our experience and perception of space and to our experience and perception of time. So, in general the existence of time is encoded in the 4-dimensional manifold with a pseudo-Riemannian signature. A signature transition might have occured in the early Universe near the Big-Bang. In quantum cosmology it is discussed a scenario where one is lead from a set of metrics with so called "Euclidean" signature $(+ + + +)$ to a "Lorentzian" signature $(- + + +)$ or $(+ - - -)$. One speaks of signature transitions, also present within ECSK models [66, 92].

Possibly, one can also consider a regime in which the signature is not well defined. According to the basic change of spacetime paradigm that relaxes the accepted idea that the metric is fundamental and considers the primacy of the conformal structure, one is lead to abandon the idea of spacetime absoluteness and consider other invariant geometrical spacetime structures, such as the the causal cone. In the regime in which the conformal symmetry holds, then the 15-parametric C(1,3) conformal group preserves the causal cone while the metrics related via conformal transformations are different. One could in principle relax also the assumption on the metric signature to be $(- + + +)$ or $(+ - - -)$, and therefore consider all possible "Lorentzian" metrics conformally related to the same, absolute causal-cone, the incorporation of complex numbers into the spacetime paradigm might provide useful methods in order to address these signature changes.

As already mentioned, the change from an absolute metric structure to an absolute causal structure can also be useful for the quantization of gravity, due to its possible role in solving the background (metric) dependence problems of the perturbative approaches.

8.1.4 Conformal symmetries, cosmological phase transitions and the arrow of time

We briefly discuss here some ideas regarding symmetry breaking in cosmological phase transitions, also in the context of the gauge theories of gravity and the possibility of a new unified spacetime-matter paradigm.

At the classical level we suggest the conformal symmetries are fundamental and the spacetime has physical properties and metric-affine geometry. As for the conformal symmetries, if nature was purely obeying these, then scale invariance and self-similarity would be perfectly manifested in an almost fractal manner. There would not be natural units and specific scales nor hierarchies linked to the h, G, c, k_B constants. Let us look at this topic from another perspective. In Einstein's equations the constants c and G allows for the link between the energy-momentum of matter and spacetime (geometry). They are not equal, but conversely causally related. The physical dimensions of curvature and energy-momentum are not the same, informally one can say that these mathematical objects represent two different "things". There is no "unified physical ontology" in Einstein's paradigm and there is no explanation for the physical nature of this link at a fundamental level, since there is no explanation regarding the nature of the constants that allow for the relevant physical dimension conversion. According to the discussions in this chapter and in chapters 2 and 3, these constants can be thought of as representing physical properties of spacetime.

One might develop a model predicting a primordial stage of the Cosmos where conformal invariance occurs and a subsequent symmetry breaking phase transition takes place. Such a transition, from the conformal group to the Poincaré group, for example, is expected to generate the mass of particles/fields (in Higgs-like mechanisms), and to give rise to the emergence of well-defined physical scales. As mentioned in this thesis such transition would most likely produce GWs. This primordial stage could also be called pre-hot big-bang[6] phase and this leads us to the problem of the arrow of time. This arrow is absent in our time symmetric physical laws and in the spacetime concepts of special and general relativity. This is another motivation for a new spacetime paradigm already at the classical level. Some gauge gravity description with time symmetry, broken into a new symmetry group with explicit time asymmetry could provide the emergence of this arrow in a primordial Cosmological context. In other words, the arrow of time should be intimately linked (and consistent) to the spacetime manifold in this paradigm of unified spacetime-matter.

Related to these considerations is the assumption that one cannot think about time (and its arrow) without thinking about matter/energy (also in accordance with the entropy and second law descriptions), and one cannot think about space without thinking about matter as well. In that sense, also at the conceptual level we revisit the ideas related to Mach's principle (discussed in chapter 2), which abandons the absoluteness of spacetime and associates the definition of any inertial frame with respect to the matter distribution.

It might be the case that changes are required in both GR and quantum mechanics, in order to arrive at a consistent quantum gravity theory (as often defended by sir Roger Penrose). This might lead us to the idea strongly defended here of a new spacetime

[6]In ECSK the possibility of an expanding Universe emerging from a non-singular blackhole has also been explored (see [223, 224, 225, 226, 227]). There are many examples of cosmological models with pre-big bang phases and one of these also puts some emphasis in the conformal symmetries. This is the conformal cyclic cosmology of sir Roger Penrose. In his approach, via conformal mappings and in accordance to the second law of thermodynamics, a previous Universe can be related to an emerging big-bang, solving the big bang singularity.

paradigm in which space, time and matter/energy should come together, and possibly emerge from the same unified construct. Moreover, the arrow of time should also come from such formalism. In this perspective one could say that it is not possible to think about space or time without thinking about matter-energy. A "Physical spacetime-matter/energy" structure with microphysical degrees of freedom, possibly encompassing a discretization of the conformal structure, like in the causal sets approach.

The notion of microphysical degrees of freedom invokes the second law of thermodynamics, and the thermodynamics of spacetime regions and gravitational fields motivate the idea that the arrow of time might be intimately related to gravity, and therefore to the geometrical construction of gauge theories of gravitation. The fundamental time asymmetry should be incorporated in the model for the unified "physical matter-spacetime" manifold and the geometrical methods and symmetry principles of gauge theories of gravity might play a fundamental role in this important challenge. The arrow of time remains one of the most relevant open questions in physics and in scientific and philosophical thinking.

Chapter 9

The philosophy of science perspective

Einstein presented us with a remarkable insight: Gravity is spacetime geometrodynamics. Underlying this idea is the notion that spacetime has a physical nature that can be probed through phenomenological studies. Attributing a physical nature to spacetime is the same of saying that spacetime has a physical ontology. On the other hand, since Einstein's theory about gravity and spacetime rests on the principle of General Relativity, spacetime coordinates by itself have no absolute meaning, and therefore in his theory spacetime is "a physical object" and at the same time space and time are also mere elements of our language and thought that we assign to physical phenomena. In this respect, it touches two very important realms of the philosophy of science: ontology and epistemology.

The philosophy of science is vital to the movements of science itself as it not only investigates the history of ideas, the sociology of science or the place of science in society, but it also dives deeply into the heart of the paradigms, formal structures, conceptual thinking, the concepts and its applications and the underlying assumptions within any scientific theories and experimental results. Deeply routed in the history of both philosophical and scientific thinking, the philosophy of science perspective into any scientific theory or model nurtures the debate on the usually poorly discussed assumptions and on the deep meanings and open questions regarding the fundamental concepts used in such theories or models. To simplify a bit, and putting aside for the moment linguistic/semiotic considerations (syntax, semantics and pragmatics) implicit to any serious philosophical study, one could say that in the analysis of any scientific theory there are three basic levels: Ontology, epistemology and metaphysics. In any theory of physics there are fundamental ontological and epistemological issues not so usually discussed outside the circles of science philosophy (although many authors also address these topics to some extent). There are also metaphysical assumptions in any physical theory, in the sense that these theories involve concepts which are more or less explainable in terms of other more fundamental concepts and this ladder of reductionistic structure eventually reaches the level of the implicit assumptions which are not described or explained by the "physical theory". Indeed, the basic and most widely applied epistemological structure of physical sciences is that of reductionism. Of course there are various historical reasons that help to understand this fact. The history of ideas in physics is also to some extent the history of various philosophical perspectives, with its ontological and epistemological facets, accompanying the ideas, theories and empirical results.

Let us journey a long time ago, into the ancient Greeks of the Ionian school of pre-

Socratic philosophy, around the sixth century BC. Thinkers like Thales, Anaximander, Anaximenes, Anaxagoras and Heraclitus made profound philosophical inquires into the natural world [230] as Aristotle latter recognized. They were applying the philosophy of nature and deeply influenced the ideas and cosmologies (or cosmogonies) of many that followed them. The Eleatic school (from the Greek colony of Elea) was founded by Parmenides in the fith century and although this school also addressed fundamental epistemology, Parmenides had clear ideas on ontology, on *what is*. The ideas of Parmenides and Heraclitus were somehow in deep contrast and the impulse for a resolution of this tension by other thinkers deeply influenced the whole of history of physics and in fact of all natural sciences. For Parmenides there is only the "Being", immutable, undivided and indivisible, and any changes in the phenomenological world are mere illusions since true changes would require true emptiness and "non-being" cannot exist. For Heraclitus on the other hand, Nature is in constant change and transformation, it is fundamentally dynamical, nothing ever remains the same. In an attempt to reconcile these apparently antagonic views, the atomists (Leucippus and Democritus) assumed the "Being" of Parmenides manifested in immutable, indivisible elements called *atoms*, and the interactions, arrangements and re-arrangements of these give rise to all the changes in the world. What changes and what is invariant is therefore a very old and fundamental philosophical question. This question deeply influenced physics throughout the centuries and it still does in the most modern theories at the edge of the current paradigms and research. In fact, not only in western philosophy but also in the ancient eastern philosophy one can find both the questions of what is invariant and what is changing and the related question of what is truly fundamental at the deepest level of reality. The conceptual reconciliation of change and constancy can also be found in ancient eastern philosophy (particularly in India), with monistic views and (phenomenological) changes often coexisting under the same body of thought.

The legacy from the atomistic world view is astonishing, and although in its origin or at least in some of its formulations the atoms had the same nature of the "Being" of Parmenides since atoms were understood as manifestations of it, this world view gradually settled and crystallised within the emergent scientific culture throughout the centuries, as describing the materialistic physical reality. It is therefore interesting to note that the fundamental nature or ontology of matter in the ancient thinking was that of the "Being" of Parmenides, which was essentially spiritual, while the nature of matter in the western (so called "materialistic") world-view is that of something which is passive, devoid of *anima*. The idea of atoms turned out to be one of the most successful notions in science, with far-reaching consequences and applications, and therefore its routes deserved our attention here.

Although Newton developed a mechanics of point (material) particles, these were abstract notions. Nevertheless this created the foundations for all the amazing applications of mechanics with its extensions to the dynamics of systems of many particles and extended rigid objects, capable of explaining a multitude of phenomena, under the assumption of "real" material particles. In fact the nature of heat was finally understood as the result of the vibrations of the material constituents of macroscopic systems, therefore the study of heat and mechanics were merged into thermodynamics, and this link was revitalised through the tools of statistical physics, able to explain the macroscopic properties and phenomenological relations from the statistical mechanics of particle-like constituents. Latter, the existence of atoms was reinforced by the interpretations of the Brownian movement of suspended particles in a fluid that, as shown mathematically by Einstein in 1905, could be understood as the result of a stochastic process involving the random collisions of the atoms/molecules of the fluid. Before this, in the nineteenth century, advances in chemistry showed that the (hypothetical) atoms of different elements, with different masses, were responsible for different (chemical) properties of material substances, and Mendeleïv established the periodic table. The electron is discovered by J.J

Thomson in 1897 while the spectral analysis of atoms, the photoelectric and Compton effects and Planck's blackbody curve of thermal emission were putting into motion the conditions for the development and revolution of quantum mechanics. During the first decade of the twentieth century, the advances in the studies of radioactivity led to the probing of the atomic nucleus and to the Rutherford's model of the atom around 1912, although the neutron was only discovered in 1932 by James Chadwick.

While the physical existence of atoms gradually found its support from the solid interplay between the theories in classical physics and its empirical validation, the emergence of quantum mechanics and its establishment at the end of the 1920's was putting into question the ontological nature of electrons and atoms. The Copenhagen school led by Nields Bohr, presented at the time of the Solvay conference in 1927 the most robust interpretation which denies any ontological statement about quantum systems and physical reality. Although there were at that time also some "ontological interpretations", these could not find extensive support within the scientific community and the Copenhagen (orthodox) interpretation, which focused more on establishing the epistemological aspects of quantum theory, stood as the accepted interpretation. In any case, the atoms were at the very foundations of the classical picture of the physical world, and the question on whether one can consider its ontological nature and to what extent it makes any sense to inquire on it, was then put into debate and was an open question. The profound ontological, epistemological and metaphysical aspects of the philosophy of quantum physics remain open questions almost 100 years after the 1927 Solvay conference. Quantum mechanics brought a revolution into the concepts of causality, matter and light, one that was strongly motivated by the challenge of understanding the wave-particle duality. Not only the ontology of matter, its physical objectivity, was profoundly questioned, but also the study of the nature of light found the same basic challenges.

With the gradual discovery of more and more particles, the picture of the Greeks turned from that of indivisible atoms into the standard model of particles and interactions with a collection of twelve particles (leptons and quarks) and its anti-particles as the basic elementary structureless physical entities. These obey wave-particle dualities, have fields associated to them and are seen as excitations of the quantum vaccum. Related to the wave-particle dualities, the quantum measurement or quantum state reduction remains an open question. The same can be said for the fundamental nature of matter, of particles, of quantum fields and of quantum vaccum.

Regarding the wave-particle duality, let us recall that Schrödinger approached it from one direction (wave mechanics) while Heisenberg from another (matrix algebra), and Dirac showed the underlying consistency into a solid formal structure. Let us revisit the approach by Schrödinger. First he noticed that light had a wave description (wave optics) and a corresponding geometrical optics limit (eikonal equation), under the specific conditions, such as wavelengths much smaller than the typical sizes of the objects with which light interacts. Then he recognized that the Hamilton-Jacobi equation for particles obeying classical mechanics is formally analogous to the eikonal equation of the geometrical optics limit of wave optics. Therefore, by exploring this analogy, Schrödinger was led to his wave equation for matter, that leads to the Hamilton-Jacobi formulation under specific conditions formally analogous to those of the geometrical optics limit. In this respect it is relevant to mention the importance of metaphors and analogies in science. Indeed, a metaphor connects two or more semantic constellations, joins them together allowing for the development of important bridges between different perspectives. The same can be said for the exploration of analogies and this some times leads to major breakthrough and even unifications in science. In fact, the history of physics can also be seen as a history of unifications.

While Schrödinger explored formal analogies between light and matter to arrive at the wave equation with his name, Niels Bohr went further, also motivated by the wave particle duality and Heisenberg uncertainty relations, and explored analogies between conceptions within philosophy, psychology and physics. The philosophical influences in Niels Bohr from his father, Høfdding and Kierkegaard, for example, allow to put in perspective his thinking. Here we will only mention that the roots of Bohr's interpretation of Heisenberg's uncertainty relations as indeterminacy relations can be understood in the following way: Bohr recognized an analogy between two basic principles of psychology, the Kantian thinking and conceptions in physics. Accordingly, Kant's philosophy describes the *a priori* conditions of intelligibility and the *a priori* conditions of sensitivity. Among the first there is the notion of causality, fundamental to any intelligible construction, and among the second the are the notions of space and time, also presupposed in any experience. Now, Bohr recognized the correspondence between these two categories and the two aspects of human psychology, namely "seeing" and "understanding". These could be said to obey a certain principle of complementarity in the sense that there are limits to the possibility of seeing and knowing at the same time. The full attention on seeing and sensitivity is complementary to the full attention on knowing, understanding. Therefore, Bohr brought the principle of complementarity from psychology into physics, via philosophy, stating that in quantum physics there is a complementarity principle between a well defined description in spacetime and a well defined causal description. The last one requires the rigorous determination of the energy and momentum conservations in any physical process, and this faces the limits imposed by the Heisenberg relations. For Bohr, these relations were reflecting indeterminacy relations in the sense that were intrinsic to nature, although Bohr did not develop the ontology presupposed in such interpretation. One could say that to some extent, in Bohr's thinking there is an anticipation of the notion that the ideas about space and time had to change within quantum mechanics. Returning to Dirac, the advances of relativistic quantum mechanics and quantum field theory led to the intimate connection between the mass/energy spectrum of particles and 4-dimensional Minkowski spacetime.

Towards an ontology of matter and light? This remains a question under philosophical study. Some of the ideas discussed in this thesis are useful to understand that this question is fundamentally connected to another, namely: *towards an ontology of spacetime*? The philosophy of space and time is indeed a quite rich topic, with many discussions over the decades and centuries. While the thinkers of quantum mechanics gave an emphasis in the breaking of determinism and a well defined causality in physics, the breaking of a well defined description in spacetime via the deconstruction of the notion of the classical trajectory did not have profound implications about the spacetime paradigm. On the other hand, with Einstein, the door was open for physics to be also a science of spacetime. In this thesis we emphasized the notion that physics is also a science of the causal structure of spacetime and in the context of gauge theories of gravity and geometrical methods in field theories, the casual geometry and its conformal symmetries is fundamental and might play a relevant role in unified field theory and quantum gravity.

The science and philosophy of spacetime, together with the scientific and philosophical studies in quantum field theories and quantum gravity can jointly contribute to a vivid discussion on the open question regarding *the physical ontology of a matter-spacetime-vacuum unification*. In this endeavour, and as repeatedly emphasized in this thesis, the geometrical methods and symmetry principles motivated by the gauge approach to gravity and Yang-Mills fields should play an important role. But any ontological aspect in philosophy of physics, or philosophy of science in general, is always related to epistemological considerations. One may ask, what are the fundamental epistemological pillars of modern theoretical physics? This is a profound question which can be more or less developed depending on the depth of the analysis. One of the levels in this epistemological issue has to do with the more difficult questions related to the possibility of intelligibility of nature or the profound relation between mathematics, the physical realm and the bio-physical and neuro-psychological realm of mental formations. Indeed the mental sphere encompasses (besides the perceptions, memories, conscious physiological sensations, etc) the conceptual and philosophical structures that enable the formulation of scientific theories. This realm has a profound connection to both the physical realm via the neuro-physiological human condition in the body, and to mathematics which seems to be simultaneously a language and an "ontological level" of physical nature. Indeed, there is a remarkable, intimate connection between mathematics and physical phenomena, which continues to drive the theoretical research and be a source of wonder. Mathematics in physics presents basic epistemological pillars, related to symmetries, dynamics, geometrical patterns and structures, etc. In theoretical physics one clearly recognizes the fundamental role of the variational principle, from which the dynamical laws are derived. At this level the symmetry principles select the classes of functionals that can adequately describe certain phenomena, via the phenomenological dynamical laws that explain the changes in certain fundamental mathematical objects. The link between the symmetries and the dynamics is established through geometrical methods and structures (bundle formalism, connections), and spacetime geometry emerges also at this level as in gauge theories of gravity. Therefore, indeed the basic epistemological pillars of field theories in modern physics definitely include the functional, the symmetries and the variational principle, the geometrical methods and the fundamental objects (fields) which are assumed to be in direct connection to a certain level of physical reality or ontology, to the extent that correct phenomenological laws can be derived and empirically verified (the tests of GR, the confirmations of the standard model and quantum field theory in particle accelerators, etc).

The epistemological and ontological considerations in physics are indeed fundamentally related and the intimate connection between physics and mathematics can be approached from that perspective. One interesting issue is related to that old question that the Greeks explored, namely, *what can change*, *what is invariant* and *what is fundamental in Nature*. In the modern physics, as far as epistemology is concerned, there is still an attempt for a reductionist approach where some fundamental fields and particles and basic principles can adequately explain the physical nature. On the other hand, the fundamental ontological questions regarding the nature of matter, space and time, etc, remain open. The reductionist approach in natural science had a strong influence from the atomistic perspective, with many fields of knowledge also searching for the fundamental building blocks or units from which the more complicated systems could be understood. This was in some way accompanied by the perspective of linear science where the whole can be constructed by the composition of the parts in a linear way. In fact, physics and other sciences have only in the last century consistently moved into the realm of non-linear, complex systems, chaotic dynamics and criticality. The underlying paradigm changes in natural sciences seem to be gradually pointing towards systemic approaches and the perspectives of the sciences and epistemology of complexity, where a strong interdisciplinarity and transdisciplinarity is required. Non-integrable systems [231], non-linearities, complex interactions and retroactions from the whole to the parts, evolutionary paths with critical phenomena, emergent properties, self-organization and adaptable complex systems are all features of a more vast and rich paradigm that is emerging within natural and human sciences, that deeply contrasts with the reductionist world-view.

Coming back to the old questions posed by the Greeks, in modern language, the question on the invariances and changes in Nature, have a rigorous mathematical formulation. The symmetries of the mathematical theories in physics are in deep relation to the conservation laws of certain quantities (Noether currents). The principle of relativity, in its wider formulation, is a remarkable exposition of the essence of this question in the heart of physics. According to the principle of relativity, besides all relative perceptions and measurements, there are certain invariant quantities and mathematical laws that all observers can verify and agree. Although this implicitly presuppose that all observers are observing the same Nature, from different perspectives, and that they all share also that same fundamental Nature, physics has not turned into a science of the observer, in spite of some relevance given to it within quantum mechanics. The field of quantum cosmology has taken the debates on quantum physics up to a new level, that although fascinating we will not develop here, but nevertheless physics remains a science focused on the laws of Nature describing that which is observed by observers, not a science of the mind which observes Nature. A comprehensive dynamical interplay between natural sciences and human sciences, including the exploration of topics involving biology, physics (and biophysics), neurosciences and psychology (and psycho-sociology), might be able to radically transform some of the current paradigms, within the non-linear and systemic approaches mentioned before. Nevertheless, and taking into account the amazing complexity of physical nature, of the human condition and experience, the principle of relativity provides a valuable gift with a flavour from the positivist thinking of the ancient Greeks. It says that in spite of all the complexity, the Universe is knowable through the understanding of its fundamental laws that all observers can discover. In principle this idea of invariant laws in physics can be extended into all natural sciences, embracing biology, neurosciences, but also to psychology and sociology, and this can strengthen the humanistic view according to which the human diversity is embraced and it is recognized that beyond such beautiful diversity and complexity there are invariant and fundamental physical, biological, psychological and anthropological laws that prove that there is a true basic unity in all the human family.

One may think science should provide an objective description of physical reality, of Nature. It is quite more vast than that! In fact, science helps us develop an understanding on how to understand Nature. It gives us intellectual and experimental tools and paths that shed light on fundamental and deep epistemological and ontological questions. In search for an objective (description of) reality, we learn to recognize the relativity of perspectives and reference systems and the relativity, analogies, dualities and complementarity between different paradigms. Beyond all relative notions, can we unveil a fundamental invariant, absolute (physical) ontology? At any given time, science puts forward new perspectives about the fundamental nature of the Cosmos. Therefore, the relativistic approaches into Nature and our conceptions about what is objective, absolute, invariant, and of what is relative, changes over time, but it takes us into a more and more closer connection with Nature. A great lesson from this scientific spirit is that we learn to develop flexibility of mind and develop sensitivity, in order to continuously discover a deeper understanding. Rather than focus on the "rigid" construction of scientific paradigms, it is best to develop a meta-paradigmatic approach into the complementarities, dualities and correspondences between different paradigms. In fact these can be seen as different representations of epistemic constructions, intellectual maps, about Nature. In this way the scientific narrative about the world becomes more open, vast and rich, and flexible and dynamical, constantly inviting the interplay between different fields of science and philosophy.

A common assumption about science is that the scientific experience can be reproduced by any human being, under the same conditions, leading to the same conclusions. But, in the absence of a strong degree of inter-personal confidence within the communities, this scientific spirit fades away. The meaning of scientific proof for a certain person might therefore require that the experience is performed by himself or herself and the word from others might not have a meaningful validity. On the other hand, if the basis for inter-personal confidence exist, then it can grow and support the scientific spirit within the community. When such scientific spirit ripens there is a collective understanding of that which is "scientifically proven" as that set of ideas that potentially could be realized and empirically established by any individual (under the appropriate conditions), without the need for every single individual to test those ideas. This is the same as saying that a principle of relativity is understood and accepted by the community. It is at that level that one can talk about the ripening of scientific culture which depends on many factors (intellectual, socio-economical, educational, etc), including on the levels of inter-personal confidence. This brief commentary was to illustrate that the more technical epistemological aspect of field theories, such as the idea of invariances or symmetries under transformations between reference systems (the principle of relativity), can also be contextualized within the perspectives from the sociology of science, the construction of the scientific spirit, science education (and communication), and the psycho-sociology implicit in the dynamics of inter-personal confidence in human ecosystems.

Ring the bells that still can ring
Forget your perfect offering
There is a crack, a crack in everything
That's how the light gets in
Leonard Cohen

Chapter 10

Concluding remarks

The gauge theories of gravity are built upon a self-consistent approach that clarifies the fundamental connection between spacetime symmetries and spacetime geometries. In this context, the metric-affine formalism is clearly formulated within general spacetime geometries with curvature, torsion and non-metricity. In this thesis, the geometrical methods and symmetry principles in gravitation were explored motivating a new perspective into the spacetime paradigm. The effects of post-Riemann spacetime geometries with torsion were studied in applications to fundamental fermionic and bosonic fields, cosmology, astrophysics and gravitational waves. The physical implications and related phenomenological considerations of this study have been addressed, and the fundamental ideas related to spacetime physics, motivated by geometrical methods and symmetry principles, have been discussed in the context of the possible routes towards a new spacetime paradigm in gravitation and unified field theories.

We explored the analogies between the pre-metric approach to electrodynamics in the exterior calculus of forms plus electromagnetic-spacetime constitutive relations, and the gauge approach to gravity. These analogies were developed, reinforcing the hypothesis of the primacy of the conformal structure over the metric structure. The consideration of the fundamental link between electrodynamics and the conformal structure of spacetime at the foundational level, together with the basic principle of local symmetries in gravity, led to a stronger support of the idea that the metric structure is not as fundamental as the conformal structure. We also explored the analogies between the constitutive relations in electrodynamics and gravity to support the idea that the coupling constants entering in such relations reflect the electromagnetic and gravitational propagation properties of the spacetime, preserving local conformal invariances. Since the conformal symmetries seem to be broken symmetries in nature, the Poincaré gauge theories of gravity were taken into consideration. These theories live on a Riemann-Cartan spacetime geometry with curvature and torsion and motivate the search for effects in physical systems induced by torsion. We explored the effects of spacetime torsion in fundamental fermionic and bosonic fields, in cosmology and in gravitational waves.

Finally we discussed various philosophical considerations motivated by this research.

The endless river...
Pink Floyd

Appendix A

Geometrical methods

A.1 Exterior calculus of forms: bief outlook

In this appendix we provide a simple review on the basic operations within the exterior calculus of forms, that were used in the thesis, without providing a rigorous treatment of these.

Exterior derivative. The exterior derivative of a k-form $v = \frac{1}{k!} v_{a_1 \dots a_k} dx^{a_1} \wedge \dots dx^{a_k}$ is the $(k+1)$-form

$$dv = \partial_{[a_0} v_{a_1 \dots a_k]} dx^{a_0} \wedge \dots dx^{a_k}. \tag{A.1}$$

For the 1 form θ^a, we get the 2-form given by

$$d\theta^a = \partial_{[\mu} \theta^a{}_{\nu]} dx^\mu \wedge dx^\nu. \tag{A.2}$$

Wedge product. Similarly, if v is a p-form and w is a k-form, then $v \wedge w$ is a $(p+k)$-form with

$$v \wedge w = (-1)^{p \times k} w \wedge v. \tag{A.3}$$

Gauge covariant derivative. The (gauge) covariant exterior derivative of a generic tensor valued p-form, denoted by $V^a_{\ b}$, used in this paper is given by

$$DV^a_{\ b} = dV^a_{\ b} + \Gamma^a{}_c \wedge V^c_{\ b} + (-1)^p \Gamma^c{}_b \wedge V^a_{\ c}. \tag{A.4}$$

Interior product. In this thesis we use the symbol $\lrcorner$ for the interior product, also called contraction operator, which gives a contraction between a p-form and a vector, resulting in a (p-1)-form

Hodge star. In d dimensions the Hodge star operator maps p-forms to $(d-p)$-forms. For the case of $H = \star F$, where F is a 2-form, we get in components

$$H_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}g^{\alpha\lambda}g^{\beta\gamma}\epsilon_{\mu\nu\alpha\beta}F_{\lambda\gamma}. \tag{A.5}$$

η **-basis** The fundamental object $\eta = \star 1$ is the natural volume 4-form. From this we get the 3-form

$$\eta_a \equiv e_a \rfloor \eta = \star\theta_a, \tag{A.6}$$

the 2-form

$$\eta_{ab} \equiv e_b \rfloor \eta_a = \star(\theta_a \wedge \theta_b), \tag{A.7}$$

and the 1-form

$$\eta_{abc} \equiv e_c \rfloor \eta_{ab} = \star(\theta_a \wedge \theta_b \wedge \theta_c). \tag{A.8}$$

These fundamental objects can be used for e.g. in contractions with the field strengths of gauge theories of gravity, in order to obtain the required geometrical 4-form invariants suited for integration, within the gravitational action.

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