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Abstract

In this paper, we will treat a totally real non-cyclic cubic field k with discriminant $1396 = 2^2 \cdot 349$, which is unique up to isomorphism. Then the prime 5 splits completely in k. First we will introduce our previous results on Iwasawa invariants. And, by using these results, we will show that the Iwasawa λ_5 and μ_5-invariants of k vanish.

Key words : Iwasawa invariants (岩澤不変量), totally real cubic fields (総実 3 次代数体), $\mathbb{Z}_p$-extensions ($\mathbb{Z}_p$-拡大)

Mathematics Subject Classification. Primary 11R23; Secondary 11R16, 11R29.

1. Introduction

For a number field k and a prime number p, let k_∞ be the cyclotomic $\mathbb{Z}_p$-extension of k with n-th layer k_n. Let A_n be the p-Sylow subgroup of the ideal class group of k_n. Then there exist integers λ, μ and ν, depending only on k and p, such that $\# A_n = p^{\lambda n + \mu p^n + \nu}$ for sufficiently large n (cf. [Iw59], and also an excellent text book [Wa82]), where $\# G$ denotes the order of a finite group G. The integers $\lambda = \lambda_p(k)$, $\mu = \mu_p(k)$ and $\nu = \nu_p(k)$ are called the (cyclotomic) Iwasawa invariants of k for p. It is conjectured that both $\lambda_p(k)$ and $\mu_p(k)$ always vanish for any totally real number field k and any prime number p (cf. [Gr76], and also [Iw73]). This is called as Greenberg's conjecture. It is known by a theorem of Iwasawa [Iw56] that if p does not split in k and the class number of k is not divided by p, then Iwasawa $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$-invariants vanish. In particular, Greenberg's conjecture is valid for $k = \mathbb{Q}$, the field of rational numbers. Further, for any prime number p, it is shown by Ferrero and Washington [FW79] that the Iwasawa $\mu_p(k)$-invariant always vanishes if k is an abelian number field, but it is not known yet for the Iwasawa $\lambda_p(k)$-invariants of totally real number fields k, even if k has a low degree except when $k = \mathbb{Q}$.

Until now, several authors investigated Greenberg's conjecture in the case where k is a real abelian number field (cf. Greenberg [Gr76], Fukuda and Komatsu

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[FK86], Fukuda and the author [FT95], Ichimura and Sumida [IS96, IS97], Kraft and Schoof [KS97], Kurihara [Ku99], and the author [Ta96]). For instance, when $p = 3$, it is shown in [IS96] and [IS97] that the λ_3-invariants of real quadratic fields $\mathbb{Q}(\sqrt{m})$ vanish for all positive integers $m < 10{,}000$. Also, Ono [On99] and Byeon [By01, By03] proved that, for any prime number $p \geq 5$, there are infinitely many real quadratic fields k with $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ by estimating the number of such k.

Concerning cubic fields, we gave some affirmative computational date for totally real cubic fields (including cyclic cubic fields) and $p = 3$ (cf. [Ta99a]), for cyclic cubic fields and $p = 5, 7$ (cf. [Ta99b]), in the case where a given prime p splits completely. In this paper, we will treat a totally real non-cyclic cubic field k with discriminant $1396 = 2^2 \cdot 349$, which is unique up to isomorphism. Then the prime 5 splits completely in k. First, we will recall our previous results (cf. [Gr76], [Ta99a]). After that, we will calculate the order of some subgroups of the intermediate fields of the cyclotomic $\mathbb{Z}_5$-extension of k, and finally show by using the previous results that Iwasawa invariants λ_5 and μ_5 of k vanish.

2. Previous results

In this section, we will recall our previous results which we use in the next section. Let Γ be the Galois group of k_∞ over k, and let A_n^Γ be the subgroup of A_n consisting of ideal classes which are invariant under the action of Γ, namely, A_n^Γ is the Γ-invariant part of A_n. Let v_p be the p-adic valuation normalized by $v_p(p) = 1$. In the case where p splits completely in k, the following theorem, which is proved in [Ta99a], holds.

Theorem 2.1 Let k be a totally real number field and p an odd prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. Then, for every sufficiently large n,

$$\#A_n^\Gamma = \#A_0 p^{v_p(R_p(k)) - [k:\mathbb{Q}]+1},$$

where $R_p(k)$ is the p-adic regulator of k and $[k:\mathbb{Q}]$ the degree of k over $\mathbb{Q}$.

Let D_n is the subgroup of A_n consisting of ideal classes represented by products of prime ideals of k_n lying above p. It is clear that $D_n \subset A_n^\Gamma$. By using Theorem 2.1, we obtain the following alternative formulation of a theorem of Greenberg [Gr76, Theorem 2] on the vanishing of the Iwasawa invariants.

Theorem 2.2 Let k be a totally real number field and p an odd prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. Then the following conditions are equivalent:

(1) $\lambda_p(k) = \mu_p(k) = 0$,
(2) $D_n = \#A_0 p^{v_p(R_p(k)) - [k:\mathbb{Q}]+1}$ for some $n \geq 0$.

In particular, if $v_p(R_p) = [k:\mathbb{Q}] - 1$ and if $A_0 = D_0$, then $\lambda_p(k) = \mu_p(k) = 0$.

3. Example

In this section, we will study a totally real cubic field k defined by $f(x) = x^3 - x^2 - 7x + 5$, which is a non-Galois extension over $\mathbb{Q}$ (i.e., the Galois group of its Galois closure is the symmetric group of degree 3). This k is unique up to isomorphism, and also the prime 5 splits completely in k. Our purpose is to show that $\lambda_5(k) = \mu_5(k) = 0$ by applying Theorems 2.1 and 2.2.

Our computation has been carried out by means of excellent number theoretic calculator packages "KASH 3" [KASH3] and "GP/PARI Ver.2.7.0" [PARI2]. Also, we use the polynomials generating totally real cubic fields in a table made by M. Olivier, which is available at the site of "GP/PARI". Note that most of the previous effective methods to verify Greenberg's conjecture have been developed in the case where p is an odd prime number and k is a real abelian number field such that $[k:\mathbb{Q}]$ divides $p - 1$ (cf. [Gr76], [FK86], [FT95], [IS96], [IS97], [KS97]).

Now, we will give computational data of the total real cubic field k in which $p = 5$ splits completely, and show that $\lambda_5(k) = \mu_5(k) = 0$. Note that this k is the only one example such that k is a non-Galois cubic extension with $p = 5$ splitting completely and with discriminant less than 2000.

Example 3.1 Let k be a totally real cubic field defined by $f(x) = x^3 - x^2 - 7x + 5$ which is unique up to isomorphism. Then the discriminant of k is $1396 = 2^2 \cdot 349$ and $p = 5$ splits completely in k. Let θ be a root of $f(x) = 0$ and θ' one of its conjugates. By using KASH 3, we see that a system of fundamental units of k is

$$\{4 - 7\theta + 2\theta^2, 8 - \theta^2\}$$

and the class number of k is 1. Put $\varepsilon_1 = 4 - 7\theta + 2\theta^2$ and $\varepsilon_2 = 8 - \theta^2$. Further, put $\varepsilon'_1 = 4 - 7\theta' + 2\theta'^2$ and $\varepsilon'_2 = 8 - \theta'^2$, which are conjugates of ε_1 and ε_2 respectively. Since we may take the following values as θ and θ' (other pairs are possible and we obtain the same conclusion on the order of A_n^Γ and D_n for any other pairs):

$$\theta \equiv 177579 \pmod{5^{10}},$$
$$\theta' \equiv 734132 \pmod{5^{10}},$$

we obtain

$$\varepsilon_1 \equiv 953183 \pmod{5^{10}},$$
$$\varepsilon_2 \equiv 8667517 \pmod{5^{10}},$$
$$\varepsilon'_1 \equiv 3822928 \pmod{5^{10}},$$
$$\varepsilon'_2 \equiv 5284709 \pmod{5^{10}}.$$

Taking the 5-adic logarithms of these, we get

$$\log_5 \varepsilon_1 \equiv 8024605 \pmod{5^{10}},$$
$$\log_5 \varepsilon_2 \equiv 2861705 \pmod{5^{10}},$$
$$\log_5 \varepsilon'_1 \equiv 5566195 \pmod{5^{10}},$$
$$\log_5 \varepsilon'_2 \equiv 4923115 \pmod{5^{10}}.$$

Hence it follows that

$$R_5(k) \equiv 4 \cdot 5^2 \pmod{5^3}.$$

Thus, we have $v_5(R_5(k)) = 2$. In particular, Leopoldt's conjecture is valid in this case. Now, by Theorem 2.1, we obtain

$$\#A_n^\Gamma = \#A_0 \cdot 5^{v_5(R_5(k)) - [k:\mathbb{Q}]+1} = 1$$

for all integers $n \geq 0$, which implies that $\#D_n = 1$ for all integers $n \geq 0$. Hence it follows from Theorem 2.2 that $\lambda_5(k) = \mu_5(k) = 0$.

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