川崎医療福祉学会誌 Vol. 3 No. 2 1993

原 著

Maximum Principles for Finite Element Solutions on a Riemann Surface, II

Dedicated to Professor Fumi-Yuki Maeda on the occasion of his 60th birthday

Hisao MIZUMOTO[1] and Heihachiro HARA[2]

Department of Medical Informatics, Faculty of Medical Professions
Kawasaki University of Medical Welfare[1]
Kurashiki, 701-01, Japan
Department of Information Science, Faculty of Science
Shimane University[2]
Matsue, 690, Japan
(*Accepted Nov, 17, 1993*)

Key words : finite element approximation, Riemann surface

Abstract

In the previous paper [3] we established the maximum principles for the finite element solutions of the partial differential equation:
surface $\bar{\Omega}$. In the present paper we shall improve and extend the results in the paper [3]. First we construct a triangulation K of $\bar{\Omega}$ with width h and introduce a class $S=S(K)$ of element functions on K. For a partition to two parts C_1 and C_2 of the boundary $\partial\Omega$, we define the finite element approximation $\omega_h \in S$ of the boundary value problem: $\Delta u - qu = f$ on Ω, $u=\chi$ on C_1 and $*du=0$ along C_2, where by $*du$ we denote the conjugate differential of du. We assume that all angles of 2-simplices of K are $\leqq \pi/2$. Under the assumption weaker than one in the paper [3], we shall exhibit that the inequality

$$|\omega_h| \leqq \exp\left(\frac{4\pi M}{\sin\theta}\cdot\max_{\bar{\Omega}} q\right)\left(\max_{C_1}|\chi| + \frac{2}{\sin\theta}\iint_{\Omega}|f|\,dxdy\right)$$

holds for sufficiently small h, where θ is the smallest value of all angles of 2-simplices of K and M is a constant. The last inequality will be very useful to obtain error estimates of the finite element solutions.

Introduction

In the previous paper [3] we established the maximum principles for the finite element approximate solutions of the partial differential equation

$$\Delta u - qu \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - qu = f \quad (z = x + iy)$$

on a compact bordered Riemann surface $\bar{\Omega}$ under the assumption that there exists $\varepsilon > 0$ being independent of individual triangulation K such that all angles of 2-simplices of K are $\leqq \pi/2 - \varepsilon$. This assumption for angles of the 2-simplices is not convenient for the various applications of finite element method not only on a Riemann surface but also in the euclidean space. In the present paper we present the maximum principles under the assumption that all angles of 2-simplices of K are $\leqq \pi/2$, which is reasonable restriction in actual applications.

The present paper consists of the three sections:

§ 1 is devoted to construction of the triangulations K and K' of two kinds. K is a triangulation of $\bar{\Omega}$ and K' is a modification of K.

In § 2, first we define the original solution u as a solution of the boundary value problem:

$$\begin{aligned} \Delta u - qu &= f && \text{on } \Omega, \\ u &= \chi && \text{on } C_1, \\ *du &= 0 && \text{along } C_2, \end{aligned}$$

where $\{C_1, C_2\}$ is a partition to two parts of the boundary $\partial\Omega$, and by $*du$ we denote the conjugate differential of du. Next we state that u is the solution of the variational problem minimizing the functional

$$J[v] \equiv \iint_\Omega \left(\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + qv^2 + 2fv \right) dxdy$$

in a class $\mathfrak{F}_\chi$ of functions v satisfying $v = \chi$ on C_1. Further we introduce two classes of element functions on K and K': the *comparable class* $S = S(K)$ (with u) and the *computable class* $S' = S'(K')$. S' is a collection of modifications $v'_h = F(v_h)$ of $v_h \in S$, where F defines a one-to-one mapping of S onto S'.

In § 3, first we introduce the finite element approximations ω_h and u'_h of u in S and S' respectively which are defined as the functions $\omega_h \in S_\chi$ and $u'_h \in S'_\chi$ satisfying the minimalities

$$J[\omega_h] = \min_{v_h \in S_\chi} J[v_h]$$

and

$$J[u'_h] = \min_{v'_h \in S'_\chi} J[v'_h] \qquad \text{resp.,}$$

where S_χ is the subclass of S whose functions v_h satisfy the boundary condition $v_h = \hat{\chi}$ ($\hat{\chi}$ is the finite element interpolation of χ (see § 2.5 for the definition)) and $S'_\chi = \{v'_h \mid v'_h = F(v_h), v_h \in S_\chi\}$. The main results in the present paper are stated as follows: For sufficiently small $h > 0$, the inequalities

$$|\omega_h| \leqq \exp\left(\frac{4\pi M}{\sin\theta} \cdot \max_{\bar{\Omega}} q \right) \cdot \left(\max_{C_1} |\chi| + \frac{2}{\sin\theta} \iint_\Omega |f| \, dxdy \right) \quad \text{and}$$

$$|u_h| \leqq \exp\left(\frac{2\pi M}{\sin\theta} \cdot \max_{\bar{\Omega}} q \right) \cdot \left(\max_{C_1} |\chi| + \frac{1}{\sin\theta} \sum_{s \in K'} \iint_{|s|} |f| \, dxdy \right)$$

hold, where θ is the smallest value of all angles of the triangles $\varphi_j(s)$ ($\varphi_j(\natural s)$ in place of $\varphi_j(s)$ for a minor or major simplex s) ($s \in K_j$; $j = 1, \cdots, m$) (see § 1 for the definition), M is a constant defined in (i ′) of § 1.2.

The results in the present paper will be applied to error estimation for the finite element approximations in the forthcoming paper [4].

§ 1. Triangulation.

1. Collection Φ of local parameters. Let Ω be a subdomain of a Riemann surface W whose closure $\bar{\Omega}$ is a compact bordered subregion of W and whose boundary $\partial\Omega$ consists of a finite number of piecewise analytic closed curves. Let $\{p'_k\}_{k=1}^{\iota}$ be the collection of the vertices on $\partial\Omega$ at which the curves $\partial\Omega$ is not analytic. Then we assume that there exist parametric disks V_k ($k = 1, \cdots, \iota$) with the centers p'_k and the local parameters $z = \psi_k(p)$ by which $V_k \cap \bar{\Omega}$ are mapped onto sectors $\{|z| \leqq r_k\} \cap \{0 \leqq \arg z \leqq \beta_k\}$ $(0 < \beta_k \leqq 2\pi, \beta_k \neq \pi)$.

Let $\{C', C''\}$ be a partition to two parts of the boundary $\partial\Omega$ such that C' and C'' are sums of boundary components of $\partial\Omega$.

We assign 2σ points $p_1, \cdots, p_{2\sigma}$ ($\sigma \geqq 0$) on C'' in such a manner that an even number of $p_1, \cdots, p_{2\sigma}$ are assigned on each boundary component of C'' in the positive orientation with respect to Ω and in their order.

By $\Phi = \{ z = \varphi_j(p), U_j; j = 1, \cdots, m \}$ we denote a finite collection of local parameters $z = \varphi_j(p)$ ($j = 1, \cdots, m$) and parametric disks U_j ($j = 1, \cdots, m$) on W which satisfies the following conditions (i)~(v):

(i) Each U_j ($j = 1, \cdots, m$) is mapped onto a disk $|z| < \rho_j$ by the mapping $z = \varphi_j(p)$.

(ii) $\bar{\Omega}$ is covered by $\{U_j\}_{j=1}^m$.

(iii) If $U_j \cap U_k \neq \phi$, then there exists a constant L (>1) such that for the mapping $\zeta = \varphi(z) \equiv \varphi_k \circ \varphi_j^{-1}(z)$, $1/L < |\varphi'(z)| < L$ on $\varphi_j(U_j \cap U_k)$.

Let p_k ($k = 2\sigma + 1, \cdots, \nu$) be all the vertices of $\partial\Omega$ which are defined as the points of $\{p'_k\}_{k=1}^{\iota} - \{p_k\}_{k=1}^{2\sigma}$.

(iv) each U_j ($j = 1, \cdots, m$) contains at most one p_k ($k = 1, \cdots, \nu$) and if $p_k \in U_j$ then $\varphi_j(p_k) = 0$.

(v) If $U_j \cap \partial\Omega \neq \phi$ and U_j does not contain any p_k ($k = 1, \cdots, \nu$), then $\varphi_j(U_j \cap \Omega)$ is a half disk $\{|z| < \rho_j\} \cap \{\text{Im } z > 0\}$. If U_j contains some p_k ($k = 1, \cdots, \nu$), then $\varphi_j(U_j \cap \Omega)$ is a sector $\{|z| < \rho_j\} \cap \{0 < \arg z < \alpha_j\}$ ($0 < \alpha_j \leqq 2\pi$).

In the latter case of (v), if $p_k \neq p_1, \cdots, p_{2\sigma}$, then by the mapping $\zeta = \{\varphi_j(p)\}^{\pi/\alpha_j}$, $U_j \cap \Omega$ is mapped onto a half disk $\{|\zeta| < \rho_j^{\pi/\alpha_j}\} \cap \{\text{Im } \zeta > 0\}$. In this case we define anew $z = \varphi_j(p)$ and ρ_j by $\zeta = \{\varphi_j(p)\}^{\pi/\alpha_j}$ and ρ_j^{π/α_j} respectively. Further, in the latter case of (v), if U_j contains some p_k ($k = 1, \cdots, 2\sigma$), then by the mapping $\zeta = (\varphi_j(p))^{\pi/2\alpha_j}$, $U_j \cap \Omega$ is mapped onto a quadrant $\{|\zeta| < \rho_j^{\pi/2\alpha_j}\} \cap \{0 < \arg \zeta < \pi/2\}$. In this case we define anew $z = \varphi_j(p)$ and ρ_j by $\zeta = (\varphi_j(p))^{\pi/2\alpha_j}$ and $\rho^{\pi/2\alpha_j}$ respectively.

Then, in the case that U_j contains some p_k ($k = 1, \cdots, \nu$) the local parameter $z = \varphi_j(p)$ is no longer conformal at the center of U_j except for the case when U_j contains some p_k ($k = 1, \cdots, 2\sigma$) and $\alpha_j = \pi/2$.

2. Triangulation K associated to Φ. For the collection Φ of local parameters and parametric disks defined in § 1.1, and for a sufficiently small $h > 0$, we construct a triangulation $K = K^h$ of $\bar{\Omega}$ which satisfies the following conditions (i)~(v). This is called a *triangulation of $\bar{\Omega}$ with width h associated to Φ*.

(i) The points $p_1, \cdots, p_\nu$ are carriers of some 0-simplices of K.

(ii) K is the sum of subtriangulations $K_1, \cdots, K_m$ of K such that each 2-simplex of K belongs to one and only one K_j ($j = 1, \cdots, m$), and the carrier $|s|$ of each 2-simplex s of K_j is contained in U_j.

If a 1-simplex $e \in K_j$ does not belong to another K_k ($k \neq j$), or a 1-simplex e belongs to $K_j \cap K_k$ ($j \neq k$) and the mapping $\varphi_k \circ \varphi_j^{-1}$ is an affine transformation, then e is said to be *linear*. If each edge of a 2-simplex $s \in K_j$ is linear and $\varphi_j(s)$ is an ordinary triangle, then s is called a *natural simplex*.

(iii) Each 2-simplex $s \in K_j$ which has not a common edge with any 2-simplex of another K_k ($k \neq j$), is a natural simplex.

A 2-simplex of K_k which has a common edge with a 2-simplex $s \in K_j$ ($j \neq k$), is said to be an *adjoint* (simplex) of s and is denoted by s'.

(iv) For each pair of a 2-simplex $s \in K_j$ and its adjoint $s' \in K_k$ with a common edge e, either one of the following three cases (a), (b) and (c) occurs.

(a) Both s and s' are natural simplices.

(b) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly concave arc w.r.t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of s and s' except for e are linear (cf. Fig. 1). Then s is called a *minor simplex*. The case where s' is a minor simplex and s is its adjoint may also occur.

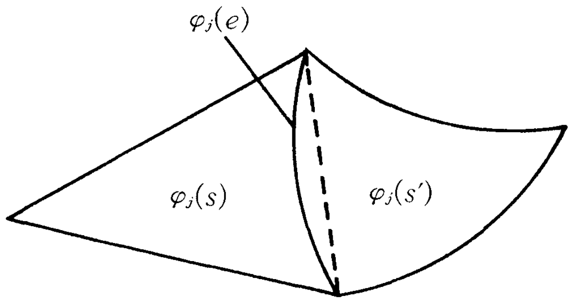

Fig. 1. Minor simplex s and its adjoint s′

(c) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly convex arc w.r.t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of s and s' except for e are linear (cf. Fig. 2). Then s is called a *major simplex*. The case where s' is a major simplex and s is its adjoint may also occur.

If s is a minor or major simplex of K_j, then it is assumed that $|s'|\subset U_j$ for its adjoint s'.

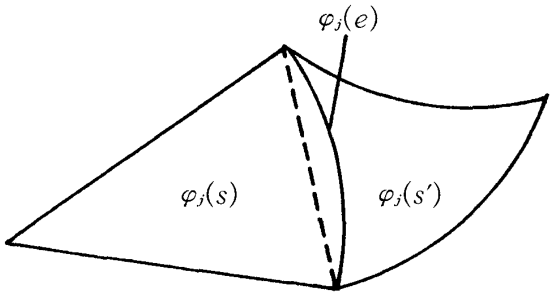

Fig. 2. Major simplex s and its adjoint s′

(v) For each 2-simplex $s\in K_j$ ($j=1, \cdots, m$), $d(\varphi_j(s))\leqq h$, where throughout the present paper we denote the diameter of a region G by $d(G)$.

Next, we assume that for the fixed Φ the class of the trianglations $K=K^h$ satisfies the following conditions (i′)~(iv′):

(i′) The number N of 2-simplices of K satisfies the inequality:

$$N\leqq M\cdot\frac{1}{h^2},$$

where M is a constant which is independent of the individual triangulation K.

For each minor (or major) simplex $s\in K_j$ we define the *naturalized simplex* $\natural s$ of s as the 2-simplex such that $|s|\subset|\natural s|$ ($|\natural s|\subset|s|$ resp.) and $\varphi_j(\natural s)$ is the ordinary triangle which has two common sides with $\varphi_j(s)$.

(ii′) All angles of the triangles $\varphi_j(s)$ ($\varphi_j(\natural s)$ in place of $\varphi_j(s)$ for a minor or major simplex s) ($s\in K_j$; $j=1, \cdots, m$) are $\leqq\pi/2$.

(iii′) For each $j=1, \cdots, m$ the union of carriers of all minor and major simplices of K_j, and all their adjoints is contained in a closed subset R_j of $U_j\cap\bar{\Omega}$ which is independent of the individual triangulation K.

(iv′) The number N' of minor and major simplices of K satisfies the inequality:

$$N'\leqq M'\cdot\frac{1}{h}, \tag{1.1}$$

where M' is a constant which is independent of the individual triangulation K.

3. Lune. For each minor (or major) simplex $s\in K_j$ we define a 2-simplex $\flat\ell=\flat\ell(s)$ ($\sharp\ell=\sharp\ell(s)$ resp.) with two edges whose carrier is the closed region $\overline{|\natural s|-|s|}$ ($\overline{|s|-|\natural s|}$ resp.). $\flat\ell(s)$ ($\sharp\ell(s)$ resp.) is called the *deficient* (*excessive* resp.) *lune* of s.

Each triple of a minor (or major) simplex $s\in K_j$, its adjoint $s'\in K_k$ and its deficient lune $\flat\ell$ (excessive lune $\sharp\ell$ resp.) is denoted by (s, s', $\flat\ell$) ((s, s', $\sharp\ell$) resp.), and is called a *triple for a minor* (*major* resp.) *simplex* s or *for a deficient* (*excessive* resp.) *lune* $\flat\ell$ ($\sharp\ell$ resp.) (cf. Figs. 3 and 4), where it is always assumed that $|\flat\ell|\subset|s'|$ for each (s, s', $\flat\ell$).

For simplicity of notation, we also denote $\flat\ell=\flat\ell(s)$ or $\sharp\ell=\sharp\ell(s)$ by $\ell=\ell(s)$. If a minor or major simplex s is in K_j, then we say that $\ell=\ell(s)$ is a *lune of* K_j and write $\ell\in K_j$.

Now we shall define the *naturalized triangulation* K' *associated to* K.

First, K'_j ($j=1, \cdots, m$) are defined as triangulations such that the collection of all 2-simplices of K'_j consists of all 2-simplices of K_j which are not minor or major, and of all

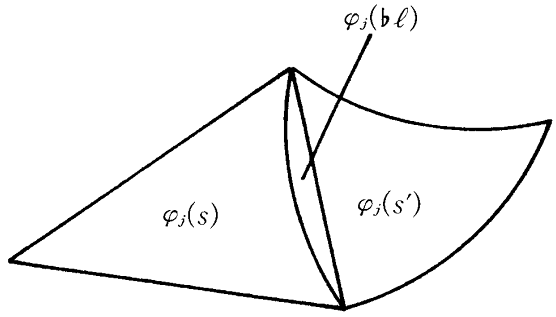

Fig. 3. Triple (s, s′, $\flat\ell$) for a minor simplex.

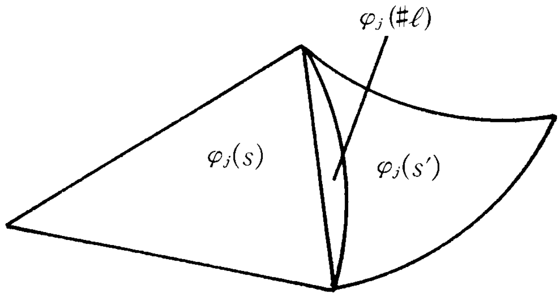

Fig. 4. Triple (s, s′, $\sharp\ell$) for a major simplex.

naturalized simplices of minor and major ones of K_j. Then the triangulation K' is defined as the sum of K'_j ($j=1, \cdots, m$). We should note that K' is no longer a triangulation of $\bar{\Omega}$, and also is not an ordinary triangulation.

4. Area of lune.

LEMMA 1.1. *Let* (s, s', ℓ) *be a triple for an arbitrary deficient or excessive lune* ℓ. *Then, the estimate*

$$A(\varphi_j(\ell)) \leqq \frac{h_1^3}{8}\left(\left|\frac{\psi''(\zeta_1)}{\psi'(\zeta_1)^2}\right| + \varkappa h_1\right) \tag{1.2}$$

holds, where throughout the present paper we denote the area of a region G by $A(G)$, $z=\psi(\zeta)\equiv\varphi_j\circ\varphi_k^{-1}(\zeta)$, $h_1=d(\varphi_j(\ell))$, ζ_1 *is one of the vertices of the lunar domain* $\varphi_k(\ell)$ *and* $\varkappa$ *is a constant depending only on* ψ.

See Lemma 1.1 of [1] for the proof.

§ 2. Classes of functions.

1. Class $\mathfrak{F}$. We consider the partial differential equation

$$L_z[u] \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - qu = f \tag{2.1}$$

on Ω, where $z=x+iy$ is a local parameter, u is an unknown function, q and f are the Hölder continuous (with exponent α ($0<\alpha\leqq 1$)) covariant tensors which are defined by satisfying

$$q(z)=\left|\frac{d\zeta}{dz}\right|^2 q(\varphi(z)) \text{ and } f(z)=\left|\frac{d\zeta}{dz}\right|^2 f(\varphi(z)) \tag{2.2}$$

by a change of local parameters $\zeta=\varphi(z)$, and we assume that $q\geqq 0$.

By $\mathfrak{F}$ we denote the class of all continuous functions v on $\bar{\Omega}=\Omega\cup\partial\Omega$, for which the partial derivatives $\partial v/\partial x$ and $\partial v/\partial y$ with respect to the local parameter $z=x+iy$ exist and are continuous on Ω at most except for a finite number of rectifiable curves on Ω, and for which the integral

$$\iint_\Omega \left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + qv^2 + 2\,|\,fv\,|\right) dxdy$$

is finite. Let v and w be functions in $\mathfrak{F}$, let $\mathfrak{f}$ be a covariant tensor on $\bar{\Omega}$, and let G be a subregion of $\bar{\Omega}$. We introduce the notations

$$(\mathfrak{f}, v)_{0,G} \equiv \iint_G \mathfrak{f} v\ dxdy,$$

$$(v, w)_G \equiv \iint_G \left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + qvw\right) dxdy$$

and

$$\|v\|_G \equiv (v, w)_G^{1/2}.$$

We define an *inner product* (v, w) of v and w, a *norm* $\|v\|$ of v and a functional $J[v]$ of v by

$$(v, w)=(v, w)_\Omega,$$
$$\|v\| = \|v\|_\Omega \quad \text{and}$$
$$J[v] = \|v\|^2 + 2\,(f, v)_{0,\Omega} = \iint_\Omega \left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + qv^2 + 2fv\right) dxdy$$

respectively. For simplicity, we set

$$(\mathfrak{f}, v)_0 = (\mathfrak{f}, v)_{0,\Omega}.$$

2. Original solution. Let γ_k ($k=1, \cdots, \sigma$) be the arcs (closed sets) on C'' from p_{2k-1} to p_{2k}, and let $C_1'=\sum_{k=1}^{\sigma}\gamma_k$, $C_1=C'+C_1'$ and $C_2=C''-C_1'$, where $\{p_k\}_{k=1}^{2\sigma}$ are the 2σ points on C'' defined in § 1.1. Let χ be a function in $\mathfrak{F}$ which satisfies the conditions:

(i) The 2nd ordered partial derivatives of χ are Hölder continuous (with exponent α ($0<\alpha\leqq 1$)) on a neighborhood of C'.

(ii) χ is constant on each γ_k ($k=1, \cdots, \sigma$).

By $\mathfrak{F}_\chi$ we denote the subclass of $\mathfrak{F}$ consisting of all functions v of $\mathfrak{F}$ satisfying the boundary condition:

$$v=\chi \qquad \text{on } C_1.$$

The solution u of the equation (2.1) on Ω satisfying the boundary conditions:

$$u=\chi \qquad \text{on } C_1, \quad \text{and} \tag{2.3}$$

$$*du=0 \qquad \text{along } C_2 \tag{2.4}$$

is called the *original solution* in $\mathfrak{F}_\chi$, where by $*du$ we denote the conjugate differential of du. It is shown that the original solution u is uniquely determined and the 2nd ordered partial derivatives of u are Hölder continuous (with exponent α ($0<\alpha\leqq 1$)) (cf. Gilbarg and Trudinger [5]).

The following lemma is easily verified by standard argument.

LEMMA 2.1 *The original solution u satisfies the minimality*

$$J[u]=\min_{v\in\mathfrak{F}_\chi} J[v]. \tag{2.5}$$

In the equality (2.5), *the minimum of the right hand side is attained if and only if $v=u$.*

3. Subclass S of $\mathfrak{F}$. We define a subclass $S=S(K)$ of $\mathfrak{F}$, called the *comparable class* (with u), as the class of functions v_h which satisfy the following conditions (i)~(iv):

(i) $v_h\in\mathfrak{F}$.

(ii) If $s\in K_j$ ($j=1, \cdots, m$) is a natural simplex, then v_h is a linear expression $ax+by+c$ (a, b, c: constants) of local variables x and y ($z=\varphi_j(p)=x+iy$) on $\varphi_j(s)$.

(iii) Let (s, s', $\flat\ell$) be a triple for a minor simplex s. Then v_h is a linear expression of local variables on $\varphi_j(s)$ and $\varphi_k(s')-\varphi_k(\flat\ell)$ respectively, and a harmonic function in $|\flat\ell|$.

(iv) Let (s, s', $\sharp\ell$) be a triple for a major simplex s. Then v_h is a linear expression of local variables on $\varphi_j(\natural s)$ and $\varphi_k(s')$ respectively, and a harmonic function in $|\sharp\ell|$.

4. Class S' of functions. Let K' be the naturalized triangulation associated to K. For each function $v_h\in S$, we define the *function v_h' on K' associated to v_h* as the function v_h' which satisfies the following conditions (i), (ii):

(i) For each 2-simplex $s\in K_j'$ ($j=1, \cdots, m$), v_h' is a linear expression of local variables on $\varphi_j(s)$.

(ii) $v_h'=v_h$ on the carrier of K minus all lunes.

We should note that the function v_h' is defined just twice on each deficient lune $\flat\ell$, while it is never defined on any exessive lune $\sharp\ell$. In the former case, for each triple (s, s', $\flat\ell$) we shall denote the function v_h' on $\natural s\in K_j'$ and $s'\in K_k'$ by v_{hs}' and $v_{hs'}'$ respectively.

The class of all functions v_h' associated to $v_h\in S$ is denoted by $S'=S'(K')$ and called the *computable class.* Let v_h' and w_h' be functions in S', and $\mathfrak{f}$ be a covariant tensor on $\bar{\Omega}$. We define an *inner product* (v_h', w_h') of v_h' and w_h', a *norm* $\|v_h'\|$ of v_h' and a functional $J[v_h']$ of v_h' by

$$(v_h', w_h')\equiv\sum_{s\in K'}(v_h', w_h')_s$$

$$=\sum_{s\in K'}\iint_{|s|}\left(\frac{\partial v_h'}{\partial x}\frac{\partial w_h'}{\partial x}+\frac{\partial v_h'}{\partial y}\frac{\partial w_h'}{\partial y}+q\, v_h'w_h'\right)dxdy,$$

$$\|v_h'\|\equiv(v_h', v_h')^{1/2} \qquad \text{and}$$

$$J[v'_h] \equiv \sum_{s\in K'} (\|v'_h\|_s^2 + 2(f, v'_h)_{0,s})$$

$$= \sum_{s\in K'} \iint_{|s|} \left(\left(\frac{\partial v'_h}{\partial x}\right)^2 + \left(\frac{\partial v'_h}{\partial y}\right)^2 + q v'^2_h + 2 f v'_h \right) dxdy$$

respectively. Further, we introduce the notation

$$(\mathfrak{f}, v'_h)_0 \equiv \sum_{s\in K'} (\mathfrak{f}, v'_h)_{0,s}$$

$$= \sum_{s\in K'} \iint_{|s|} \mathfrak{f} v'_h \, dxdy$$

We see that $v'_h = F(v_h)$ defines a one-to-one mapping of S onto S'.

5. Finite element interpolations. Let v be a function of $\mathfrak{F}$. We define the *finite element interpolation* $\hat{v}$ *of* v in the class S as the function uniquely determined by the following conditions (i) and (ii):

(i) $\hat{v} \in S$;

(ii) $\hat{v}(p) = v(p)$ at the carrier p of each 0-simplex of K.

§ 3. Finite element approximations.

1. Finite element approximation ω_h in S. Let $\hat{\chi}$ be the finite element interpolation of χ in the class S. By S_χ we denote the subclass of S consisting of all functions v_h satisfying the boundary condition:

$$v_h = \hat{\chi} \qquad \text{on } C_1.$$

By ω_h we denote the function satisfying the minimality

$$J[\omega_h] = \min_{v_h \in S_\chi} J[v_h]. \tag{3.1}$$

We call ω_h the *finite element approximation* of the original solution u in the class S.

2. Basis of S. Let $\{q_j\}_{j=1}^n$ be the collection of all 0-simplices of the triangulation K. We define the *basis* $\phi_1, \cdots, \phi_n$ *of the class* S as the functions of S which are uniquely determined by the condition

$$\phi_j(p) = \begin{cases} 1 & (p = |q_j|), \\ 0 & (p = |q_k|;\ k \neq j), \end{cases} \qquad (j, k = 1, \cdots, n). \tag{3.2}$$

Then an arbitrary function v_h of S can be represented as a linear combination

$$v_h = \sum_{j=1}^n v_j \phi_j$$

of ϕ_j $(j = 1, \cdots, n)$, where $v_j = v_h(|q_j|)$ $(j = 1, \cdots, n)$.

LEMMA 3.1. *The coefficients* u_j $(j = 1, \cdots, n)$ *of the function* $\omega_h = \sum_{j=1}^n u_j \phi_j$ *which minimizes the functional* $J[v_h]$ *in* S_χ, *are the solutions of the system of linear equations*:

$$\sum_{k=1}^n K_{jk} u_k = F_j \qquad (j = 1, \cdots, n'), \tag{3.3}$$

$$u_j = \chi(|q_j|) \qquad (j = n'+1, \cdots, n), \tag{3.4}$$

where

$$K_{jk} = (\phi_j, \phi_k) \qquad (j = 1, \cdots, n';\ k = 1, \cdots, n),$$

$$F_j = -(f, \phi_j)_0 \qquad (j = 1, \cdots, n')$$

and by $\{q_k\}_{k=n'+1}^n$ *we denote the collection of all* 0*-simplices on* C_1.

See lemma 3.1 of [3] and also Chapter 3 of [6] for the proof.

3. Maximum principle for ω_h.

THEOREM 3.1. *If* $h > 0$ *is sufficiently small, then for the finite element approximation* ω_h *of the original solution* u, *the inequality*

$$|\omega_h| \leqq \exp\left(\frac{4\pi M}{\sin\theta} \cdot \max_{\bar{\Omega}} q\right) \cdot \left(\max_{C_1} |\chi| + \frac{2}{\sin\theta} \iint_{\Omega} |f| \, dxdy\right) \tag{3.5}$$

holds, where θ *is the smallest value of all angles of the triangles* $\varphi_j(s)$ $(\varphi_j(\natural s)$ *in place of* $\varphi_j(s)$ *for a minor or major simplex* $s)$ $(s \in K_j;$ $j = 1, \cdots, m)$, M *is a constant defind in* (i′) *of* § 1.2 *and* $\max_{\bar{\Omega}} q$ *means*

$$\max_{\bar{\Omega}} q \equiv \max_{1 \leq j \leq m} \max_{\varphi_j(\bar{U}_j \cap \bar{\Omega})} q.$$

PROOF. By supp v we denote the support of a function v on $\bar{\Omega}$. Let q_0 be an arbitrarily fixed 0-simplex of K whose carrier belongs to $\bar{\Omega}-C_1$, let ϕ_0 be the base of S satisfying $\phi_0(|q_0|)=1$, and let ϕ_j $(j=1, \cdots, \upsilon)$ be the collection of basis of S which satisfy the condition

$$D_j \equiv \operatorname{supp} \phi_0 \cap \operatorname{supp} \phi_j \neq \phi.$$

We note that

$$\sum_{j=0}^{\upsilon} \phi_j = 1 \qquad \text{on supp } \phi_0. \tag{3.6}$$

The equations (3.3) can be written in the form

$$\sum_{j=0}^{\upsilon} (P_j+Q_j)u_j = F \qquad (|q_0| \in \bar{\Omega}-C_1), \tag{3.7}$$

where

$$P_j = \iint_{\Omega} \left(\frac{\partial \phi_0}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_0}{\partial y}\frac{\partial \phi_j}{\partial y}\right) dxdy,$$

$$Q_j = \iint_{\Omega} q\phi_0\phi_j \, dxdy \qquad (j=0, \cdots, \upsilon)$$

and

$$F = -\iint_{\Omega} f \, \phi_0 \, dxdy.$$

Let s_k $(k=1, \cdots, \upsilon'; \upsilon' \leqq \upsilon)$ be the collection of all 2-simplices of K, having q_0 as one of their vertices. Then, D_j $(j=\upsilon'+1, \cdots, \upsilon)$ are carriers of some lunes. By the condition (ii') of § 1.2 and Lemma 1.1 we can verify that for sufficiently small $h>0$ there hold the inequalities

$$-P_j \geqq 0, \qquad Q_j \geqq 0 \qquad (j=1, \cdots, \upsilon). \tag{3.8}$$

Furthermore, by (3.6) we have that

$$P_0 + \cdots + P_\upsilon = 0. \tag{3.9}$$

By (3.7)~(3.9) the inequalities

$$(P_0+Q_0)|u_0| \leqq (-(P_1+\cdots+P_\upsilon) - (Q_1+\cdots+Q_\upsilon))\cdot \max(|u_1|, \cdots, |u_\upsilon|) + |F|$$

$$\leqq \left(P_0 + \sum_{j=1}^{\upsilon} Q_j\right)\cdot \max(|u_1|, \cdots, |u_\upsilon|) + \iint_{\Omega} |f|\phi_0 \, dxdy$$

hold and thus the inequality

$$|u_0| \leqq \left(1 + \frac{1}{P_0}\sum_{j=1}^{\upsilon} Q_j\right)\cdot \max(|u_1|, \cdots, |u_\upsilon|) + \frac{1}{P_0}\iint_{\Omega} |f|\phi_0 \, dxdy \tag{3.10}$$

holds for sufficiently small $h>0$.

Let s be a 2-simplex of K, having q_0 as one of its vertices and let θ_1 be the angle of the triangle $\varphi_j(s)$ ($\varphi_j(|s)$ in place of $\varphi_j(s)$ for a minor or major simplex s) at the vertex $\varphi_j(|q_0|)$. Then we can easily verify that if s is a natural simplex, a major one or its adjoint, then from the property of ϕ_0 it follows that

$$\iint_{|s|} \left(\left(\frac{\partial \phi_0}{\partial x}\right)^2 + \left(\frac{\partial \phi_0}{\partial y}\right)^2\right) dxdy > \frac{1}{2}\sin\theta_1, \tag{3.11}$$

and if s is a minor simplex or its adjoint, then by Lemma 1.1 in addition to it, it follows that

$$\iint_{|s|} \left(\left(\frac{\partial \phi_0}{\partial x}\right)^2 + \left(\frac{\partial \phi_0}{\partial y}\right)^2\right) dxdy > \frac{1}{2}(1-\varkappa h)\sin\theta_1, \tag{3.12}$$

where $\varkappa$ is a constant depending only on $z=\boldsymbol{\phi}(\zeta) \equiv \varphi_j \circ \varphi_k^{-1}(\zeta)$. Since $\upsilon' \geqq 2$, by (3.11) and (3.12) the inequalities

$$P_0 \geqq \iint_{|s_1|+\cdots+|s_{\upsilon'}|} \left(\left(\frac{\partial \phi_0}{\partial x}\right)^2 + \left(\frac{\partial \phi_0}{\partial y}\right)^2\right) dxdy > \frac{\sin\theta}{2} \tag{3.13}$$

hold for sufficiently small $h>0$. Furthermore we can immediately see that

$$\sum_{j=1}^{\upsilon} Q_j \leqq 2\pi h^2 \cdot \max_{\bar{\Omega}} q. \tag{3.14}$$

(3.10), (3.13) and (3.14) imply that

$$|u_0| \leqq \left(1 + \frac{4\pi h^2}{\sin\theta}\cdot \max_{\bar{\Omega}} q\right)\cdot \max(|u_1|, \cdots, |u_\upsilon|) + \frac{2}{\sin\theta}\iint_{\Omega} |f|\phi_0 \, dxdy.$$

By (i') of § 1.2 and making use of the formula $(1+r/N)^N \leqq e^r$ $(r>0)$, the repeating of a finite number of the above argument implies the inequality (3.5).

4. Finite element approximation u_h' in S'. Let $S_\chi' = \{v_h' \mid v_h' = F(v_h), v_h \in S_\chi\}$. By u_h' we denote the function of S'_χ such that

(3.15) $$J[u'_h]=\min_{v_h'\in S_\chi'} J[v'_h].$$

We call u'_h the *finite element approximation of u in the class S'*.

5. Basis of S'. Let $\{q_j\}_{j=1}^n$ be the collection of all 0-simplices of the triangulation K'. We define the *basis* $\phi'_1, \cdots, \phi'_n$ *of the class* S' as the functions of S' which are uniquely determined by the condition

(3.16) $$\phi'_j(p)=\begin{cases}1 & (p=|q_j|),\\ 0 & (p=|q_k|;\ k\neq j).\end{cases}$$
$(j, k=1, \cdots, n)$

Then an arbitrary function v'_h of S' can be represented as a linear combination

$$v'_h=\sum_{j=1}^n v'_j\phi'_j$$

of ϕ'_j $(j=1, \cdots, n)$, where $v'_j=v'_h(|q_j|)$ $(j=1, \cdots, n)$.

Lemma 3.2. *The coefficients* u'_j $(j=1, \cdots, n)$ *of the function* $u'_h=\sum_{j=1}^n u'_j\phi'_j$ *which minimizes the functional* $J[v'_h]$ *in* S'_χ, *are the solutions of the system of linear equations*:

(3.17) $$\sum_{k=1}^n K'_{jk}u'_k=F'_j \qquad (j=1, \cdots, n'),$$

(3.18) $$u'_j=\chi(|q_j|) \qquad (j=n'+1, \cdots, n),$$

where

$$K'_{jk}=(\phi'_j, \phi'_k)$$
$$(j=1, \cdots, n';\ k=1, \cdots, n),$$
$$F'_j=-(f, \phi'_j)_0 \qquad (j=1, \cdots, n')$$

and by $\{q_k\}_{k=n'+1}^n$ *we denote the collection of all 0-simplices on* C_1.

The proof is similar to that of Lemma 3.1.

Theorem 3.2. *If* $h>0$ *is sufficiently small, then for the finite element approximation* u'_h *of the original solution u, the inequality*

(3.19) $$|u_h| \leqq \exp\left(\frac{2\pi M}{\sin\theta}\cdot\max_{\bar{\Omega}} q\right)\cdot \left(\max_{C_1}|\chi| + \frac{1}{\sin\theta}\sum_{s\in K'}\iint_{|s|}|f|\,dxdy\right)$$

holds, where $u_h=F^{-1}(u'_h)$ *and other notations are the same as in Theorem* 3.1.

Proof. Let q_0 be an arbitrarily fixed 0-simplex of K' whose carrier belongs to Ω. Let s_j $(j=1, \cdots, \upsilon)$ be the collection of 2-simplices of K' satisfying the conditions $|q_0|\in|s_j|$ $(j=1, \cdots, \upsilon)$ and $|s_j|\cap|s_{j+1}|\neq\phi$ $(j=1, \cdots, \upsilon;\ s_{\upsilon+1}=s_1)$, and let q_j $(j=1, \cdots, \upsilon)$ be the collection of 0-simplices such that q_0, q_j, q_{j+1} $(j=1, \cdots, \upsilon;\ q_{\upsilon+1}=q_1)$ are the vertices of s_j. Let ϕ'_j $(j=0, \cdots, \upsilon)$ be the basis of S' satisfying $\phi'_j(|q_j|)=1$. We note that

(3.20) $$\phi'_0+\phi'_j+\phi'_{j+1}\equiv 1 \qquad \text{on } |s_j|$$
$$(j=1, \cdots, \upsilon;\ \phi'_{\upsilon+1}\equiv\phi'_1).$$

The equations (3.17) can be written in the form

(3.21) $$\sum_{j=0}^{\upsilon}(P'_j+Q'_j)u'_j=F' \qquad (|q_0|\in\Omega),$$

where

$$P'_0=\sum_{j=1}^{\upsilon}\iint_{|s_j|}\left(\left(\frac{\partial\phi'_0}{\partial x}\right)^2+\left(\frac{\partial\phi'_0}{\partial y}\right)^2\right)dxdy,$$

$$Q'_0=\sum_{j=1}^{\upsilon}\iint_{|s_j|}q\phi_0'^2\,dxdy,$$

$$P'_j=\iint_{|s_{j-1}|+|s_j|}\left(\frac{\partial\phi'_0}{\partial x}\frac{\partial\phi'_j}{\partial x}+\frac{\partial\phi'_0}{\partial y}\frac{\partial\phi'_j}{\partial y}\right)dxdy$$

and

$$Q'_j=\iint_{|s_{j-1}|+|s_j|}q\phi'_0\phi'_j\,dxdy$$
$$(j=1, \cdots, \upsilon;\ s_0\equiv s_\upsilon).$$

By the condition (ii′) of § 1.2 we can verify that there hold the inequalities

(3.22) $$-P'_j\geqq 0, \quad Q'_j\geqq 0 \qquad (j=1, \cdots, \upsilon).$$

Furthermore, by (3.20) it follows that

(3.23) $$P'_0+\cdots+P'_\upsilon=0.$$

By (3.21)~(3.23) the inequalities

$$(P'_0+Q'_0)\,|u'_0|$$
$$\leqq(-(P'_1+\cdots+P'_\upsilon)-(Q'_1+\cdots+Q'_\upsilon))\cdot \max(|u'_1|, \cdots, |u'_\upsilon|)+|F'|$$
$$\leqq\left(P'_0+\sum_{j=1}^{\upsilon}Q'_j\right)\cdot\max(|u'_1|, \cdots, |u'_\upsilon|)$$

$$+\sum_{s\in K'}\iint_{|s|}|f|\phi_0'\,dxdy$$

hold and thus the inequality

$$(3.24)\quad |u_0'| \leqq \left(1+\frac{1}{P_0'}\sum_{j=1}^{\upsilon}Q_j'\right)\cdot$$

$$\cdot\max(|u_1'|,\cdots,|u_\upsilon'|)+\frac{1}{P_0'}\sum_{s\in K'}\iint_{|s|}|f|\phi_0'\,dxdy$$

holds.

Let s be a 2-simplex of K', having q_0 as one of its vertices and let θ_1 be the angle of the triangle $\varphi_j(s)$ at the vertex $\varphi_j(|q_0|)$. Then we can easily verify that from the property of ϕ_0' it follows that

$$(3.25)\quad \iint_{|s|}\left(\left(\frac{\partial\phi_0'}{\partial x}\right)^2+\left(\frac{\partial\phi_0'}{\partial y}\right)^2\right)dxdy>\frac{1}{2}\sin\theta_1.$$

Since $\upsilon\geqq 4$, by (3.25) the inequalities

$$P_0'=\iint_{|s_1|+\cdots+|s_\upsilon|}\left(\left(\frac{\partial\phi_0'}{\partial x}\right)^2+\left(\frac{\partial\phi_0'}{\partial y}\right)^2\right)dxdy$$

$$>\sin\theta$$

hold. Hence we have that

$$(3.26)\quad |u_0'| \leqq \left(1+\frac{1}{\sin\theta}\sum_{j=1}^{\upsilon}Q_j'\right)\cdot$$

$$\cdot\max(|u_1'|,\cdots,|u_\upsilon'|)+\frac{1}{\sin\theta}\sum_{s\in K'}\iint_{|s|}|f|\phi_0'\,dxdy.$$

In the case where q_0 is a 0-simplex of K' whose carrier belongs to C_2, by the similar argument except some minor change we also obtain the inequality (3.26).

The remained parts of the proof are similar to ones of Theorem 3.1.

References

1) Mizumoto H, Hara H (1988) Finite element approximations of harmonic differentials on a Riemann surface. *Hiroshima Mathematical Journal*, **18**, 617—654.

2) Hara H, Mizumoto H (1990) Determination of the modulus of quadrilaterals by finite element methods. *Journal of Mathematical Society of Japan*, **42**, 295—326.

3) Mizumoto H, Hara H (1992) Maximum principles for finite element solutions on a Riemann surface. *Kawasaki Medical Welfare Journal*, **2**, 267—275.

4) Hara H, Mizumoto H, Kikuchi S (to appear) Finite element approximations for $\Delta u-qu=f$ on a Riemann surface.

5) Gilbarg D, Trudinger NS (1977) Elliptic Partial Differential Equations of Second Order. Springer, Berlin, 1—401.

6) Mizumoto H, Hara H (1983) Finite Element Method in Engineering Science (in Japanese). Morikita, Tokyo, pp 1—200.

リーマン面上の有限要素解に対する最大値の原理，II

水本久夫[1]　原平八郎[2]

川崎医療福祉大学　医療技術学部　医療情報学科[1]
島根大学理学部　情報科学科[2]

（平成５年11月17日受理）

要　　約

前の論文［３］では，縁をもつコンパクトなリーマン面 $\bar{\Omega}$ 上で定義された偏微分方程式：$\Delta u-qu=f$ の有限要素解に対する最大最小値の原理を確立したが，本論文では，論文［３］の結果を改良し，拡張する．まず，$\bar{\Omega}$ の幅 h の三角形分割 K を作成し，K 上の要素関数のクラス $S=S(K)$ を導入する．境界 $\partial\Omega$ の二つの部分 C_1，C_2 への分割に対して，境界値問題：Ω 上で $\Delta u-qu=f$，C_1 上で $u=\chi$，C_2 に沿って $*du=0$ の有限要素近似 $\omega_h\in S$ を定義する，ここで，$*du$ は，du の共役微分を表す．K の２-単体のすべての内角は $\leqq\pi/2$ であると仮定する．論文［３］の仮定より弱い，この仮定のもとで，十分小さい $h>0$ に対して，不等式

$$|\omega_h|\leqq\exp\left(\frac{4\pi M}{\sin\theta}\cdot\max_{\bar{\Omega}}q\right)\left(\max_{C_1}|\chi|+\frac{2}{\sin\theta}\iint_{\Omega}|f|\,dxdy\right)$$

が成り立つことが示される，ここで，θ は K のすべての２-単体の内角の最小値，M は定数である．この不等式は，有限要素解の理論解に対する誤差評価をするときに，非常に有用となるものである．