川崎医療福祉学会誌 Vol. 2 No. 1 1992

原 著

Maximum Principles for Finite Element Solutions on a Riemann Surface

Hisao MIZUMOTO[1)] **and Heihachiro HARA**[2)]

Department of Medical Informatics, Faculty of Medical Professions
Kawasaki University of Medical Welfare[1)]
Kurashiki, Okayama 701-01, Japan
Department of Information Science, Faculty of Science
Shimane University[2)]
Matsue, Shimane 690, Japan

(*Accepted April 7, 1992*)

Key words: finite element approximation, Riemann surface

Abstract

In the present paper we establish the maximum and minimum principles for the finite element approximate solutions of the partial differential equation: $\Delta u - qu = f$ on a compact bordered Riemann Surface $\bar{\Omega}$. First we construct a triangulation K of $\bar{\Omega}$ with width h and introduce a class $S = S(K)$ of element functions on K. For a partition to two parts C_1 and C_2 of the boundary $\partial\Omega$, we define the finite element approximation $\omega_h \in S$ of the boundary value problem: $\Delta u - qu = f$ on Ω, $u = \chi$ on C_1 and $\partial u/\partial n = 0$ on C_2, where by $\partial u/\partial n$ we denote the inner normal derivative of u on C_2. The main result in the present paper is stated as follows: For sufficiently small $h > 0$, the inequality

$$|\omega_h| \leq \max_{C_1} \chi + \frac{1}{\sin\theta} \iint_{\Omega} |f|\, dxdy$$

holds, where θ is the smallest value of all angles of the 2-simplices of K. The last inequality will be very useful to obtain error estimates of the finite element solutions for the theoretic ones.

Introduction

The aim of the present paper is to establish the maximum and minimum principles for the finite element approximate solutions of the partial differential equation

$$\Delta u - qu \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - qu = f \qquad (z = x + iy)$$

on a compact bordered Riemann surface $\bar{\Omega}$. Classical results of the maximum and minimum principles for the solutions of the boundary value problems of $\Delta u - qu = f$ are well known (e. g. cf. Courant-Hilbert [1] and Protter-Weinberger [2]). Also the ones for the finite element approximations in the euclidean space have been studied by many authors, especially by Ciarlet-Raviart [3], Lorenz [4], Hohn-Mittelmann [5], Ruas

Santos [6] and Ciarlet [7]. In the present paper we shall exhibit that the maximum and minimum principles analogous to these results hold for the finite element approximations on a compact bordered Riemann surface $\bar{\Omega}$. The function-theoretic treatments of the finite element approximations on a Riemann surface are found in papers by the present authors [8] and [9].

The present paper consists of the three sections:

§ 1 is devoted to construction of the triangulation K of $\bar{\Omega}$ with width h associated to local parameters and parameter disks.

In § 2, first we define the original solution u as a solution of the boundary value problem:

$$\Delta u - qu = f \qquad \text{on } \Omega,$$
$$u = \chi \qquad \text{on } C_1,$$
$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } C_2,$$

where C_1 and C_2 are a partition to two parts of the boundary $\partial\Omega$, and by $\partial u/\partial n$ we denote the inner normal derivative of u on C_2. Next we state that u is the solution of the variational problem minimizing the functional

$$J[v] \equiv \iint_\Omega \left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + qv^2 + 2fv\right) dxdy$$

in a class $\mathfrak{F}_\chi$ of functions v satisfying $v=\chi$ on C_1. Further we introduce a class $S=\dot{S}(K)$ of element functions on K.

In § 3, first we introduce the finite element approximation of the original solution u which is defined as the function $\omega_h \in S_\chi$ satisfying the minimality

$$J[\omega_h] = \max_{v_h \in S_\chi} J[v_h],$$

where S_χ is the subclass of S whose functions v_h satisfy the boundary condition $v_h = \hat{\chi}$ ($\hat{\chi}$ is the finite element interpolation of χ (see § 2.4 for the definition)). The main result in the present paper is stated as follows: For sufficiently small $h>0$, the inequality

$$|\omega_h| \leqq \max_{C_1} |\chi| + \frac{1}{\sin\theta} \iint_\Omega |f|\, dxdy$$

holds, where θ is the smallest value of all angles of the triangles $\varphi_j(s)$ ($\varphi_j(\natural s)$ in place of $\varphi_j(s)$ for a minor or major simplex s) ($s \in K_j$; $j=1, \cdots, m$) (see § 1 for the definition).

The results in the present paper will be applied to error estimates for the finite element approximations in the forthcoming paper [10].

§ 1. Triangulation.

1. Collection Φ of local parameters. Let Ω be a subdomain of a Riemann surface W whose closure $\bar{\Omega}$ is a compact bordered subregion of W. We assume that the boundary $\partial\Omega$ consists of a finite number of analytic arcs meeting at vertices p'_k ($k=1, \cdots, \iota$), and there exist parametric disks V_k ($k=1, \cdots, \iota$) with the centers p'_k and the local parameters $z=\psi_k(p)$ by which $V_k \cap \bar{\Omega}$ are mapped onto sectors $\{|z| \leqq r_k\} \cap \{0 \leqq \arg z \leqq \beta_k\}$ ($0<\beta_k \leqq 2\pi$, $\beta_k \neq \pi$).

Let $\{C', C''\}$ be a partition to two parts of the boundary $\partial\Omega$ such that C' and C'' are sums of boundary components of $\partial\Omega$.

We assign 2σ points $p_1, \cdots, p_{2\sigma}$ ($\sigma \geqq 0$) on C'' in such a manner that an even number of $p_1, \cdots, p_{2\sigma}$ are assigned on each boundary component of C'' in the positive orientation with respect to Ω and in their order.

By $\Phi = \{z=\varphi_j(p),\ U_j;\ j=1, \cdots, m\}$ we denote a finite collection of local parameters $z=\varphi_j(p)$ ($j=1, \cdots, m$) and parametric disks U_j ($j=1, \cdots, m$) on W which satisfies the following conditions (i) ~ (v):

(i) Each U_j ($j=1, \cdots, m$) is mapped onto a disk $|z|<\rho_j$ by the mapping $z=\varphi_j(p)$.

(ii) $\bar{\Omega}$ is covered by $\{U_j\}_{j=1}^m$.

(iii) If $U_j \cap U_k \neq \phi$, then there exists a constant L (>1) such that for the mapping $\zeta = \varphi(z) \equiv \varphi_k \circ \varphi_j^{-1}(z)$, $1/L < |\varphi'(z)| < L$ on $\varphi_j(U_j \cap U_k)$.

Let p_k $(k=2\sigma+1, \cdots, v)$ be all the vertices of $\partial\Omega$ which are defined as the points of $\{p'_k\}_{k=1}^{\iota}-\{p_k\}_{k=1}^{2\sigma}$.

(iv) each U_j $(j=1, \cdots, m)$ contains at most one p_k $(k=1, \cdots, v)$ and if $p_k \in U_j$ then $\varphi_j(p_k)=0$.

(v) If $U_j \cap \partial\Omega \neq \phi$ and U_j does not contain any p_k $(k=1, \cdots, v)$, then $\varphi_j(U_j \cap \Omega)$ is a half disk $\{|z|<\rho_j\} \cap \{\mathrm{Im}\ z>0\}$. If U_j contains some p_k $(k=1, \cdots, v)$, then $\varphi_j(U_j \cap \Omega)$ is a sector $\{|z|<\rho_j\} \cap \{0<\arg z<\alpha_j\}$ $(0<\alpha_j \leqq 2\pi)$.

In the latter case of (v), if $p_k \neq p_1, \cdots, p_{2\sigma}$, then by the mapping $\zeta=(\varphi_j(p))^{\pi/\alpha_j}$, $U_j \cap \Omega$ is mapped onto a half disk $\{|\zeta|<\rho_j^{\pi/\alpha_j}\} \cap \{\mathrm{Im}\ \zeta>0\}$. In this case we define anew $z=\varphi_j(p)$ and ρ_j by $\zeta=(\varphi_j(p))^{\pi/\alpha_j}$ and ρ_j^{π/α_j} respectively.

Further, in the latter case of (v), if U_j contains some p_k $(k=1, \cdots, 2\sigma)$, then by the mapping $\zeta=(\varphi_j(p))^{\pi/2\alpha_j}$, $U_j \cap \Omega$ is mapped onto a quadrant $\{|\zeta|<\rho_j^{\pi/2\alpha_j}\} \cap \{0<\arg \zeta<\pi/2\}$. In this case we define anew $z=\varphi_j(p)$ and ρ_j by $\zeta=(\varphi_j(p))^{\pi/2\alpha_j}$ and $\rho_j^{\pi/2\alpha_j}$ respectively.

Then, in the case that U_j contains some p_k $(k=1, \cdots, v)$ the local parameter $z=\varphi_j(p)$ is no longer conformal at the center of U_j except for the case when U_j contains some p_k $(k=1, \cdots, 2\sigma)$ and $\alpha_j=\pi/2$.

2. Triangulation K associated to Φ. For the collection Φ of local parameters and parametric disks defined in § 1.1, and for a sufficiently small positive number h, we construct a triangulation $K=K^h$ of $\bar{\Omega}$ which satisfies the following conditions (i)～(v). This is called a *triangulation of $\bar{\Omega}$ with width h associated to Φ.*

(i) The points $p_1, \cdots, p_v$ are carriers of some 0-simplices of K.

(ii) K is the sum of subtriangulations $K_1, \cdots, K_m$ of K such that each 2-simplex of K belongs to one and only one K_j $(j=1, \cdots, m)$, and the carrier $|s|$ of each 2-simplex s of K_j is contained in U_j.

If a 1-simplex $e \in K_j$ does not belong to another K_k $(k \neq j)$, or a 1-simplex e belongs to $K_j \cap K_k$ $(j \neq k)$ and the mapping $\varphi_k \circ \varphi_j^{-1}$ is an affine transformation, then e is said to be *linear*. If each edge of a 2-simplex $s \in K_j$ is linear and $\varphi_j(s)$ is an ordinary triangle, then s is called a *natural simplex*.

(iii) Each 2-simplex $s \in K_j$ which has not a common edge with any 2-simplex of another $K_k (k \neq j)$, is a natural simplex.

A 2-simplex of K_k which has a common edge with a 2-simplex $s \in K_j$ $(j \neq k)$, is said to be an *adjoint* (simplex) of s and is denoted by s'.

(iv) For each pair of a 2-simplex $s \in K_j$ and its adjoint $s' \in K_k$ with a common edge e, either one of the following three cases (a), (b), (c) occurs.

(a) Both s and s' are natural simplices.

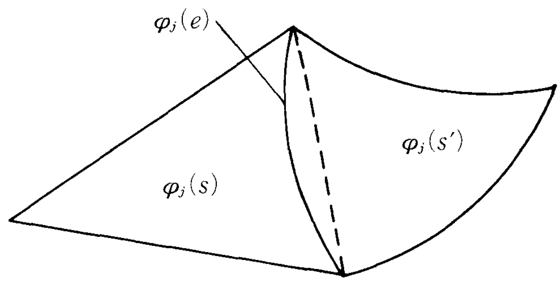

Figure 1. Minor simplex s and its adjoint s'.

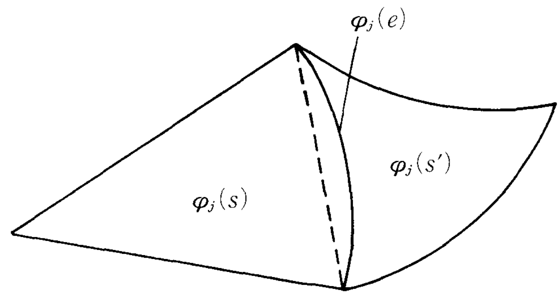

Figure 2. Major simplex s and its adjoint s'.

(b) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strict concave arc w. r. t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of s and s' except for e are linear (cf. Figure 1). Then s is called a *minor simplex*. The case where s' is a minor simplex and s is its adjoint may also occur.

(c) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly convex arc w. r. t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of s and s' except for e are linear (cf. Figure 2). Then s is called a *major simplex*. The case where s' is a major simplex and s is its adjoint may also occur.

If s is a minor or major simplex of K_j, then it is assumed that $|s'| \subset U_j$ for its adjoint s'.

(v) For each 2-simplex $s \in K_j$ ($j=1, \cdots, m$), $d(\varphi_j(s)) \leqq h$, where throughout the present paper we denote the diameter of a region G by $d(G)$.

Next, we assume that for the fixed Φ the class of the triangulations $K=K^h$ satisfies the following conditions (i′), (ii′) and (iii′):

(i′) For each $j=1, \cdots, m$ the union of carriers of all minor and major simpleces of K_j, and all their adjoints is contained in a closed subset R_j of $U_j \cap \bar{\Omega}$ which is independent of the individual triangulation K.

(ii′) The number N of minor and major simplices of K satisfies the inequality:

$$N \leqq M \cdot \frac{1}{h},$$

where M is a constant which is independent of the individual triangulation K.

For each minor (or major) simplex $s \in K_j$ we define the *naturalized simplex* $\natural s$ of s as the 2-simplex such that $|s| \subset |\natural s|$ ($|\natural s| \subset |s|$ resp.) and $\varphi_j(\natural s)$ is the ordinary triangle which has two common sides with $\varphi_j(s)$.

(iii′) There exists $\varepsilon > 0$ being independent of individual triangulation K such that all angles of the triangles $\varphi_j(s)$ ($\varphi_j(\natural s)$ in place of $\varphi_j(s)$ for a minor or major simplex s) ($s \in K_j$; $j=1, \cdots, m$) are $\leqq \pi/2-\varepsilon$.

3. Lune. We define a 2-simplex $\flat\ell=\flat\ell(s)$ ($\sharp\ell=\sharp\ell(s)$ resp.) with two edges whose carrier is the closed region $\overline{|\natural s|-|s|}$ ($\overline{|s|-|\natural s|}$ resp.). $\flat\ell(s)$ $\sharp\ell(s)$ resp.) is called the *deficient* (*excessive* resp.) *lune* of s.

Each triple of a minor (or major) simplex $s \in K_j$, its adjoint $s' \in K_k$ and its deficient lune $\flat\ell$ (excessive lune $\sharp\ell$ resp.) is denoted by (s, s', $\flat\ell$) ((s, s', $\sharp\ell$) resp.), and is called a *triple for a minor* (*major* resp.) *simplex* s or *for a deficient* (*excessive* resp.) *lune* $\flat\ell$ ($\sharp\ell$ resp.) (cf. Figures 3 and 4), where it is always assumed that $|\flat\ell| \subset |s|$ for each (s, s', $\flat\ell$).

For simplicity of notation, we also denote $\flat\ell=\flat\ell(s)$ or $\sharp\ell=\sharp\ell(s)$ by $\ell=\ell(s)$. If a minor or major simplex s is in K_j, then we say that $\ell=\ell(s)$ is a lune of K_j and write $\ell \in K_j$.

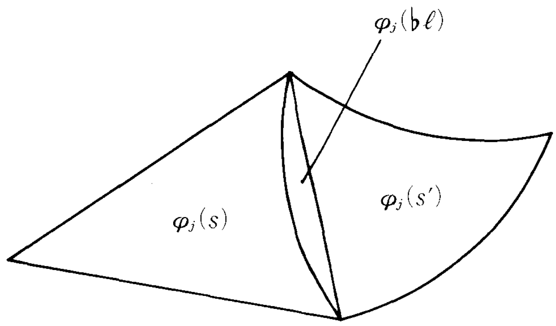

Figure 3. Triple for a minor simplex (s, s', $\flat\ell$).

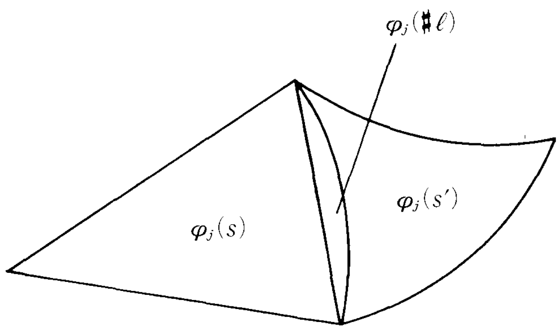

Figure 4. Triple for a major simplex (s, s', $\sharp\ell$).

4. Area of lune.

LEMMA 1.1. *Let* (s, s', ℓ) *be a triple for an arbitrary deficient or excessive lune* ℓ. *Then, the estimate*

$$A(\varphi_j(\ell)) \leqq \frac{h_1^3}{8}\left(\left|\frac{\psi''(\zeta_1)}{\psi'(\zeta_1)^2}\right| + \varkappa h_1\right)$$

holds, where throughout the present paper we denote the area of a region G by $A(G)$, $z=\psi(\zeta)\equiv\varphi_j\circ\varphi_k^{-1}(\zeta)$, $h_1=d(\varphi_j(\ell))$, ζ_1 *is one of the vertices of the lunar domain* $\varphi_k(\ell)$ *and* $\varkappa$ *is a constant depending only on* ψ.

See Lemma 1.1 of [8] for the proof.

§ 2. Classes of functions.

1. Class $\mathfrak{F}$. We consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-qu=f \tag{2.1}$$

on Ω, where $z=x+iy$ is a local parameter, u is an unknown function, q and f are the covariant tensors of the class $C^1(\bar{\Omega})$ which are defined by saisfying

$$q(z)=\left|\frac{d\zeta}{dz}\right|^2 q(\varphi(z)) \quad \text{and} \tag{2.2}$$

$$f(z)=\left|\frac{d\zeta}{dz}\right|^2 f(\varphi(z)) \tag{2.3}$$

by a change of local parameters $\zeta=\varphi(z)$, and we assume that $q\geqq 0$.

By $\mathfrak{F}$ we denote the class of all continuous functions v on $\bar{\Omega}=\Omega\cup\partial\Omega$, for which the partial derivatives $\partial v/\partial x$ and $\partial v/\partial y$ with respect to the local parameter $z=x+iy$ exist and are continuous on Ω at most except for a finite number of rectifiable curves on Ω, and for which the integral

$$\iint_\Omega\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+qv^2+2|fv|\right)dxdy$$

is finite. Let v and w be functions in $\mathfrak{F}$, and let G be a subregion of $\bar{\Omega}$. We introduce the notations

$$(f, v)_{0,G}\equiv\iint_G fv\, dxdy,$$

$$(v, w)_G\equiv\iint_G\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}+qvw\right)dxdy$$

and

$$\|v\|_G\equiv(v, v)_G^{1/2}.$$

We define an *inner product* (v, w) of v and w, a *norm* $\|v\|$ of v and a functional $J[v]$ of v by

$$(v, w)=(v, w)_\Omega,$$

$$\|v\|=\|v\|_\Omega, \qquad \text{and}$$

$$J[v]=\|v\|^2+2(f, v)_{0,\Omega}$$

$$=\iint_\Omega\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+qv^2+2fv\right)dxdy$$

respectively. For simplicity, we set

$$(f, v)_0=(f, v)_{0,\Omega}.$$

2. Original solution. Let γ_k $(k=1, \cdots, \sigma)$ be the arcs (closed sets) on C'' from p_{2k-1} to p_{2k} $(k=1, \cdots, \sigma)$, and let $C_1'=\Sigma_{k=1}^{\sigma}\gamma_k$, $C_1=C'+C_1'$ and $C_2=C''-C_1'$, where $\{p_k\}_{k=1}^{2\sigma}$ are the 2σ points on C'' defined in § 1.1. Let χ be a function in $\mathfrak{F}$ which satisfies the conditions:

(i) If $U_j\cap C'\neq\phi$, then $\chi\circ\varphi_j^{-1}$ satisfies the equeation (2.1) on a neighborhood of $\varphi_j(U_j\cap C')$.

(ii) χ is constant on each γ_k $(k=1, \cdots, \sigma)$.

By $\mathfrak{F}_\chi$ we denote the subclass of $\mathfrak{F}$ consisting of all functions v of $\mathfrak{F}$ satisfying the boundary conditions:

$$v=\chi \quad \text{on } C_1.$$

By u we denote the solution of the equation (2.1) on Ω uniquely determined by the boundary conitions:

$$u=\chi \qquad \text{on } C_1, \qquad \text{and} \tag{2.4}$$

$$\frac{\partial u}{\partial n}=0 \qquad \text{on } C_2, \tag{2.5}$$

where by $\partial u/\partial n$ we denote the inner normal derivative of u w. r. t. a local parameter on C_2. The unique function u in $\mathfrak{F}_\chi$ is called the *original solution* in $\mathfrak{F}_\chi$.

The following lemma is easily verified by standard arguments.

LEMMA 2.1. *The original solution u sat-*

isfies the minimality

(2.6) $$J[u]=\min_{v\in\mathfrak{F}_\chi} J[v].$$

In the equality (2.6), *the minimum of the right hand side is attained if and only if* $v=u$.

3. Subclass S of $\mathfrak{F}$. We define a subclass $S=S(K)$ of $\mathfrak{F}$ as the class of functions v_h which satisfy the following conditions (i) ~ (iv):

(i) $v_h\in\mathfrak{F}$.

(ii) If $s\in K_j$ $(j=1, \cdots, m)$ is a natural simplex, then v_h is a linear expression $ax+by+c$ (a, b, c: constants) of local variables x and y $(z=\varphi_j(p)=x+iy)$ on $\varphi_j(s)$.

(iii) Let $(s, s', \flat\ell)$ be a triple for a minor simplex s. Then v_h is a linear expression of local variables on $\varphi_j(s)$ and $\varphi_k(s')-\varphi_k(\flat\ell)$ respectively, and a harmonic function in $|\flat\ell|$.

(iv) Let $(s, s', \sharp\ell)$ be a triple for a major simplex s. Then v_h is a linear expression of local variables on $\varphi_j(\natural s)$ and $\varphi_k(s')$ respectively, and a harmonic function in $|\sharp\ell|$.

4. Finite element interpolations. Let v be a function of $\mathfrak{F}$. We define the *finite element interpolation* $\hat{v}$ of v in the class S as the function uniquely determined by the following conditions (i) and (ii):

(i) $\hat{v}\in S$;

(ii) $\hat{v}(p)=v(p)$ at carrier p of each 0-simplex of K.

§ 3. Finite element approximations.

1. Finite element approximation ω_h in S. Let $\hat{\chi}$ be the finite element interpolation of χ in the class S. By S_χ we denote the subclass of S consisting of all functions v_h satisfying the boundary condition:

$$v_h=\hat{\chi} \qquad \text{on } C_1.$$

By ω_h we denote the function satisfying the minimality

(3.1) $$J[\omega_h]=\min_{v_h\in S_\chi} J[v_h].$$

We call ω_h the *finite element approximation* of the original solution u in the class S.

2. Basis of S. Let $\{q_j\}_{j=1}^n$ be the collection of all 0-simplices of the triangulation K. We define the *basis* $\phi_1, \cdots, \phi_n$ of the class S as the functions of S which are uniquely determined by the condition

(3.2) $$\phi_j(p)=\begin{cases}1 & (p=|q_j|),\\ 0 & (p=|q_k|;\ k\neq j),\end{cases} \qquad (j, k=1, \cdots, n).$$

Then an arbitrary function v_h of S can be represented as a linear combination

$$v_h=\sum_{j=1}^n v_j\phi_j$$

of ϕ_j $(j=1, \cdots, n)$, where $v_j=v_h(|q_j|)$ $(j=1, \cdots, n)$.

Lemma 3.1. *The coefficients u_j $(j=1,\cdots, n)$ of the function $\omega_h=\Sigma_{j=1}^n u_j\phi_j$ which minimizes the functional $J[v_h]$ in S_χ, are the solutions of the system of linear equations*:

(3.3) $$\sum_{k=1}^n K_{jk}u_k=F_j \qquad (j=1, \cdots, n'),$$

(3.4) $$u_j=\chi(|q_j|) \qquad (j=n'+1, \cdots, n),$$

where

$$K_{jk}=(\phi_j, \phi_k) \qquad (j=1, \cdots, n';\ k=1, \cdots, n),$$

$$F_j=-(f, \phi_j)_0 \qquad (j=1, \cdots, n')$$

and by $\{q_k\}_{n'+1}^n$ we denote the collection of all 0-simplices on C_1.

Proof. (Cf. Chapter 3 of [11].) If $\omega_h=\Sigma_{j=1}^n u_j\phi_j$ minimizes the functional

$$J[v_h]=\iint_\Omega\left(\left(\sum_{j=1}^n v_j\frac{\partial\phi_j}{\partial x}\right)^2+\left(\sum_{j=1}^n v_j\frac{\partial\phi_j}{\partial y}\right)^2 +q\left(\sum_{j=1}^n v_j\phi_j\right)^2+2f\sum_{j=1}^n v_j\phi_j\right)dxdy \qquad \left(v_h=\sum_{j=1}^n v_j\phi_j\right)$$

in S_χ, then it needs to satisfy the condition

$$\frac{\partial J}{\partial u_j}=0 \qquad (j=1, \cdots, n'),$$

which implies the equations (3.3).

3. Main theorem.

THEOREM 3.1. *If $h>0$ is sufficiently small, then for the finite element approximation ω_h of the original solution u, the inequality*

(3.5)

$$|\omega_h| \leqq \max_{C_1} |\chi| + \frac{1}{\sin\theta} \iint_{\Omega} |f|\, dxdy,$$

holds, where θ is the smallest value of all angles of the triangles $\varphi_j(s)$ ($\varphi_j(\natural s)$ in place of $\varphi_j(s)$ for a minor or major simplex s) ($s \in K_j$; $j=1, \cdots, m$).

PROOF. By supp v we denote the support of a function v on $\bar{\Omega}$. Let q_0 be an arbitrarily fixed 0-simplex of K whose carrier belongs to $\bar{\Omega}-C_1$. Let ϕ_0 be the base of S satisfying $\phi_0(|q_0|)=1$, and let ϕ_j ($j=1,\cdots, v$) be the collection of basis of S which satisfy the condition

$$D_j \equiv \mathrm{supp}\ \phi_0 \cap \mathrm{supp}\ \phi_j \neq \phi.$$

We note that

$$\sum_{j=0}^{v} \phi_j = 1 \qquad \text{on supp } \phi_0. \tag{3.6}$$

The equations (3.3) can be written in the form

(3.7)

$$\sum_{j=0}^{v} (P_j+Q_j)\, u_j = F \qquad (|q_0| \in \bar{\Omega}-C_1),$$

where

$$P_j = \iint_{\Omega} \left(\frac{\partial\phi_0}{\partial x} \frac{\partial\phi_j}{\partial x} + \frac{\partial\phi_0}{\partial y} \frac{\partial\phi_j}{\partial y} \right) dxdy,$$

$$Q_j = \iint_{\Omega} q\, \phi_0\phi_j\, dxdy \qquad (j=0, \cdots, v),$$

and

$$F = -\iint_{\Omega} f\phi_0\, dxdy.$$

Let s_k ($k=1,\cdots, v'$; $v' \leqq v$) be all the 2-simplices of K, having q_0 as one of their vertices. Then, D_j ($j=v'+1, \cdots, v$) are carriers of some lunes. By the condition (iii′) of § 1.2 and Lemma 1.1 we can verify that for sufficiently small $h>0$ there hold the inequalities

$$-P_j > Q_j \geqq 0 \qquad (j=1, \cdots, v). \tag{3.8}$$

Furthermore, by (3.6) we have that

$$P_0+\cdots+P_v=0. \tag{3.9}$$

By (3.7) ~ (3.9) the inequalites

$$\begin{aligned}(P_0+Q_0)\,|u_0| &\leqq (-(P_1+\cdots+P_v)-(Q_1+\cdots+Q_v)) \cdot \\ &\quad \cdot \max(|u_1|, \cdots, |u_v|) + |F| \\ &\leqq P_0 \max(|u_1|, \cdots, |u_v|) + \iint_{\Omega} |f|\,\phi_0\, dxdy\end{aligned}$$

hold and thus the inequality

$$|u_0| \leqq \max(|u_1|, \cdots, |u_v|) + \frac{1}{P_0} \iint_{\Omega} |f|\,\phi_0\, dxdy \tag{3.10}$$

holds for sufficiently small $h>0$.

Let s be a 2-simplex of K, having q_0 as one of its vertices and let θ_1 be the angle of the triangle $\varphi_j(s)$ ($\varphi_j(\natural s)$ in place of $\varphi_j(s)$ for a minor or major simplex s) at the vertex $\varphi_j(|q_0|)$. Then we can easily verify that if s is a natural simplex, a major one or its adjoint, then from the property of ϕ_0 it follows that

(3.11)

$$\iint_{|s|} \left(\left(\frac{\partial\phi_0}{\partial x}\right)^2 + \left(\frac{\partial\phi_0}{\partial y}\right)^2 \right) dxdy > \frac{1}{2}\sin\theta_1,$$

and if s is a minor simplex or its adjoint, then by Lemma 1.1 in addition to it, it follows that

(3.12)

$$\iint_{|s|} \left(\left(\frac{\partial\phi_0}{\partial x}\right)^2 + \left(\frac{\partial\phi_0}{\partial y}\right)^2 \right) dxdy > \frac{1}{2}(1-\varkappa h)\sin\theta_1,$$

where $\varkappa$ is a constant depending only on $z=\psi(\zeta)\equiv\varphi_j\circ\varphi_k^{-1}(\zeta)$. Since $v' \geqq 3$, by (3.11) and (3.12) the inequalities

$$P_0 \geqq \iint_{|s_1|+\cdots+|s_{v'}|} \left(\left(\frac{\partial\phi_0}{\partial x}\right)^2 + \left(\frac{\partial\phi_0}{\partial y}\right)^2 \right) dxdy > \sin\theta$$

hold for sufficiently small $h>0$. Hence we have that

$$|u_0| \leqq \max(|u_1|, \cdots, |u_v|)$$

$$+\frac{1}{\sin\theta}\iint_{\Omega}|f|\phi_0\,dxdy.$$

The last inequality implies (3.5).

4. Maximum and minimum principles. Now we consider the partial differential equation

(3.13) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-qu=0$$

on Ω which is the case of $f\equiv 0$ in the equation (2.1). Let u be the original solution of the equation (3.13) satisfying the boundary conditions (2.4) and (2.5), and let ω_h be the finite element approximation of u. Then we obtain the following theorem.

THEOREM 3.2. *For the finite element approximation ω_h of the original solution u, the inequalities*

(3.14) $$\omega_h\leqq\max\,(\max_{C_1}\chi,\,0),\qquad and$$

(3.15) $$\omega_h\geqq\min\,(\min_{C_1}\chi,\,0)$$

hold.

PROOF. We note that the equation (3.7) with $F=0$ holds. Then by (3.8) the inequality

(3.16) $$(P_0+Q_0)\,u_0 \leqq(-(P_1+\cdots+P_v)-(Q_1+\cdots+Q_v))\cdot \cdot\max\,(u_1,\,\cdots,\,u_v)$$

holds for sufficiently small $h>0$. Further if $u_0>0$, then by (3.8) and (3.9) it implies the inequality

(3.17) $$u_0\leqq\max(u_1,\,\cdots,\,u_v).$$

Hence we have the inequality (3.14).

By a similar method the inequality (3.15) is verified.

References

1) Courant R, Hilbert D (1962) Methods of Mathematical Physics, II. Interscience, New York, pp 1—680.

2) Protter M, Weinberger H (1984) Maximum Principles in Differential Equations. Springer, Berlin, pp 1—261.

3) Ciarlet P G, Raviart P -A (1973) Maximum principle and uniform convergence for the finite element method. Comput. Methods Appl. Mech. Engrg., **2**, 17—31.

4) Lorenz J (1977) Zur Inversmonotonie diskreter Probleme. Numer. Math, **27**, 227—238.

5) Hohn W, Mittelmann H D (1981) Some remarks on the discrete maximum principle for finite elements of higher order. Computing, **27**, 145—154.

6) Ruas Santos V (1982) On the strong maximum principle for some piecewise linear finite element approximate problems of non-positive type. J. Fac. Sci. Univ. Tokyo, Sec. IA, **29**, 473—491.

7) Ciarlet P G (1991) Basic error estimates for elliptic problems. In: Ciarlet P G, Lions J L (eds) Handbook of Numerical Analysis Vol. II, Elsevier Science Publishers B. V., North-Holland, pp 17—351.

8) Mizumoto H, Hara H (1988) Finite element approximations of harmonic differentials on a Riemann surface. Hiroshima Math. J., **18**, 617—654.

9) Hara H, Mizumoto H (1990) Determination of the modulus of quadrilaterals by finite element methods. J. Math. Soc. Japan, **42**, 295—326.

10) Hara H, Mizumoto H, Kikuchi S (to appear) Finite element approximations for $\Delta u-qu=f$ on a Riemann surface.

11) Mizumoto H, Hara H (1983) Finite element methods in engineering science (in Japanese). Morikita, Tokyo, pp 1—200.

要　　約

この論文では，縁をもつコンパクトなリーマン面 $\bar{\Omega}$ 上で定義された偏微分方程式：$\Delta u-qu=f$ の有限要素解に対する最大最小値の原理を確立する．まず，$\bar{\Omega}$ の幅 h の三角形分割 K を作成し，K 上の要素関数のクラス $S=S(K)$ を導入する．境界 $\partial\Omega$ の二つの部分 C_1，C_2 への分割に対して，境界値問題：Ω 上で $\Delta u-qu=f$，C_1 上で $u=\chi$，C_2 上で $\partial u/\partial n=0$ の有限要素近似 $\omega_h\in S$ を定義する，ここで，$\partial u/\partial n$ は，u の C_2 上での内法線方向微分を表す．この論文の主要結果は，つぎのように述べられる：十分に小さい $h>0$ に対して，不等式

$$|\omega_h| \leqq \max_{C_1} \chi+\frac{1}{\sin\theta}\iint_{\Omega}|f|\,dxdy$$

が成り立つ，ここで，θ は K のすべての 2-単体の内角の最小値である．この不等式は，有限要素解の理論解に対する誤差評価をするときに，非常に有用となるものである．