5-2022

Homomorphic Images and Related Topics

Alejandro Martinez
California State University - San Bernardino

Homomorphic Images and Related Topics

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Alejandro Martinez

May, 2022

Homomorphic Images and Related Topics

_____

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

_____

by

Alejandro Martinez

May, 2022

Approved by:

Zahid Hasan, Committee Chair

Cory Johnson, Committee Member

Corey Dunn, Committee Member

Madeleine Jetter , Chair, Department of Mathematics

Corey Dunn, Graduate Coordinator

Abstract

In this thesis, we have demonstrated our method of writing symmetric presentations of permutation progenitors, finding monomial representations and symmetric presentations of monomial progenitors. We have also explained how various types of additional relations are found. We have discovered original symmetric presentations and original constructions of numerous groups.

We have constructed $PSL(2,11) \times 2$, A_7, $2^3 : (2^2 : S_3)$, $U_3(3) : 2$, S_6, $PSL(2,11) \times 2$, $PSL(2,11)$, G, and $PSL(2,7) \times 2^6$, as homomorphic images of $2^{*6} : D_6$, $2^{*5} : (D_5 : 2)$, $2^{*6} : A_4$, $2^{*28} : PSL(2,7)$, $2^{*5} : (5x4)$, $2^{*6} : A_5$, $2^{*10} : A_5$, $2^{*28} : PSL(2,7)$, and $2^{*28} : PSL(2,7)$, respectively. We have found monomial representation of degree 3 for S_4 over Z_3 and degree 5 over Z_7 for $3 \times A_5$. We have given symmetric presentations of the corresponding progenitors $3^{*3} : S_4$ and $7^{*5} : (3 \times A_5)$. In addition, we have given isomorphism types of almost every image that we have discovered.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

One of the great desires of the finite group theorist was to classify all finite simple groups. The classification theorem states that any finite simple group is cyclic, alternating, classical, it belongs to a broad infinite class called the groups of Lie type, or else it is one of twenty-six sporadic groups. Simple groups can be seen as the basic building blocks of all finite groups. Now every finite nonabelian simple group is a homomorphic image of a progenitor of the form $2^{*n} : N$, where $N \leq S_n$. We are interested in finding simple groups.

This thesis is devoted to homomorphic images of such progenitors and others. In Chapter 1 we give important definitions and theorems which are used for our research. In the Chapter 2 we construct nine important groups by using the technique of double coset enumeration. In Chapter 3, we give permutation progenitors, first order relations, famous lemma relations. We demonstrate the process of finding monomial representations and have given presentations for several progenitors of the form $p^{*n} \underset{m}{:} N$ ($p = 3, 5, 7, \dots$). We have also introduced Wreath Products in this chapter. In Chapter 4 we demonstrate our technique of finding isomorphism types of groups and give isomorphism types of several groups. In Chapter 5, we have listed several progenitors (mostly unsuccessful in terms of images) and their images.

1.1 Definitions

Definition 1.2 Group [Rot 95]: *A **group** $G(G, *)$ is a nonempty collection of elements with an associative operation $*$, such that:*

- *there exists an identity element, $e \in G$ such that $e * a = a * e$ for all $a \in G$;*

- *for every $a \in G$, there exists an element $b \in G$ such that $a * b = e = b * a$.*

Definition 1.3 Operation [Rot 95]: *Let G be a set. A (binary) **operation** of G is a function that assigns each ordered pair of elements of G an element on G.*

Definition 1.4 Subgroup [Rot 95]: *For group G, a **subgroup** S of G is a nonempty subset where $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$. We denote subgroup S of G as $S \leq G$.*

Definition 1.5 Proper Subgroup [Rot 95]: *Let H be a subgroup of group G. H is a **proper** subgroup of G if $H \neq G$. We denote this as $H < G$.*

Definition 1.6 Symmetric group [Rot 95]: *A **symmetric group,** S_X, is the group of all permutations of X, where $X \in \mathbb{N}$. S_X is a group under compositions.*

Definition 1.7 Permutation [Rot 95]: *If X is a nonempty set, a **permutation** of X is a bijection $\phi : X \longrightarrow X$.*

Definition 1.8 Semigroup [Rot 95]: *A **semigroup** $(G, *)$ is a nonempty set G equipped with an associative operation.*

Definition 1.9 ϕ fixes x [Rot 95]: *If $x \in X$ and $\phi \in S_X$, then ϕ **fixes** x if $\phi(x) = x$ and ϕ **moves** x if $\phi(x) \neq x$.*

Definition 1.10 Disjoint [Rot 95]: *For permutations $\alpha, \beta \in S_X$, α and β are **disjoint** if every element moved by one permutation is fixed by the other. Precisely,*

$$if \ \alpha(x) \neq x, \ then \ \beta(a) = a \ and \ if \ \alpha(y) = y, \ then \ \beta(y) \neq y.$$

Definition 1.11 Transposition [Rot 95]: *A permutation which interchanges a pair of elements is a **transposition.***

Definition 1.12 Commute [Rot 95]: *In group G, if $a, b \in G$, a and b **commute** if $a * b = b * a$.*

Definition 1.13 Abelian [Rot 95]: *A group G is **abelian** if every pair of elements in G commutes with one another.*

Definition 1.14 Presentation [Rot 95]: *Let X be a set and Δ by a family of words on X. A group G has* **generators** *X and* **relations** *Δ if $G \cong F/R$, where F is a free group with basis X and R is the normal subgroup of F generated by Δ. We say $< X|\Delta >$ is a* **presentation** of G.

Definition 1.15 Progenitor [Cur07]: *Let G be a group and $T = t_1, t_2, ..., t_n$ be a symmetric generating set for G with $|t_i| = m$. Then if $N = N_G(\bar{T})$, then we define the* **progenitor** *to be the semi direct product $m^{*n} : N$, where m^{*n} is the free product of n copies of the cyclic group C_n.*

Definition 1.16 Normalizer [Rot 95]: *Let G be a group. If $H \leq G$, the* **normalizer** *of H in G is defined by*

$$N_G(H) = \{a \in G | aHa^{-1} = H.\}$$

Definition 1.17 Centralizer [Rot 95]: *Let G be a group. If $H \leq G$, the* **centralizer** *of H in G is:*

$$C_G(H) = x \in G : [x, h] = 1 \ for \ all \ h \in H.$$

Definition 1.18 Elementary Abelian [Rot 95]: *Let p be prime. If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times ... \times \mathbb{Z}_p$, then we say G is* **elementary abelian**.

Definition 1.19 Isomorphism [Rot 95]: *Let $(G, *)$ and $(H, \circ)$ be groups. The function $\phi : G \longrightarrow H$ is a* **homomorphism** *if $\phi(a * b) = \phi(a) \circ \phi(b)$, for all $a, b \in G$. An* **isomorphism** *is a bijective homomorphism. We say G is isomorphic to $H, G \cong H$, if there is exists an isomorphism $f : G \longrightarrow H$.*

Definition 1.20 Kernel of a Homomorphism [Rot 95]: *Let $f : G \longrightarrow H$ be a homomorphism. The* **kernel of a homomorphism** *is the set $\{x \in G | f(x) = 1\}$, where 1 is the identity in H. We denote the kernel of f as* **ker f** .

Definition 1.21 Word [Rot 95]: *Let X be a nonempty subset of a group G. Let $w \in G$ where $w = x_1^{e_1} x_1^{e_2} ... \ x_n^{e_n}$, with $x_i \in X$ and $e_i = \pm 1$. We say that w is a* **word** *on X.*

Definition 1.22 Conjugacy Class [Rot 95]: *Let $a \in G$, where G is a group. The* **conjugacy class** *of a is given by $a^G = \{a^g | g \in G\} = \{g^{-1}ag | g \in G\}$.*

Definition 1.23 Dihedral Group [Rot 95]: *The* **Dihedral Group** *D_n, n even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $< a, b | a^{\frac{n}{2}}, b^2, (ab)^2 >$.*

Definition 1.24 General Linear Group [Rot 95]: *A general linear group,* $GL(n, \mathbb{F})$ *is the set of all* $n \times n$ *matrices with nonzero determinant over field* $\mathbb{F}$.

Definition 1.25 Special Linear Group [Rot 95]: *A special linear group,* $SL(n, \mathbb{F})$ *is the set of all* $n \times n$ *matrices with determinant* 1 *over field* $\mathbb{F}$.

Definition 1.26 Projective Special Linear Group [Rot 95]: *A projective special* **linear group**, $PSL(n, \mathbb{F})$ *is the set of all* $n \times n$ *matrices with determinant* 1 *over field* $\mathbb{F}$ *factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F}))}.$$

Definition 1.27 Projective General Linear Group [Rot 95]: *A projective general* **linear group**, $PGL(n, \mathbb{F})$ *is the set of all* $n \times n$ *matrices with nonzero determinant over field* $\mathbb{F}$ *factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F}))}.$$

Definition 1.28 (Monomial Character) [Led87]: *Let* G *be a finite group and* $H \leq G$. *The character* X *of* G *is monomial if* $X = \lambda^G$, *where* λ *is a linear character of H.*

Definition 1.29 (Character) [Led87]: *Let* $A(x) = (A_{ij}(x))$ *be a matrix representation of* G *of degree m. We consider the character polynomial of A(x), namely*

$$det(\lambda I - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & -a_{12}(x) & ... & -a_{1m}(x) \\ \lambda - a_{11}(x) & -a_{12}(x) & ... & -a_{1m}(x) \\ ... & ... & ... & ... \\ \lambda - a_{m1}(x) & -a_{m2}(x) & ... & -a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree m in λ, *and inspection shows that the coefficient of* $-\lambda^{m-1}$ *is equal to*

$$\phi = a_{11}(x) + a_{22}(x) + ... + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of A(x), abbreviated to trA(x), so that

$$\phi(x) = trA(x)$$

We regard $\phi(x)$ *as a function on* G *with values in* K, *and we call it the* **character** *of A(x).*

Definition 1.30 Degree of a Character [Led87]: *The sun of squares of the degrees of the s=distinct irreducible characters of* G *is equal to* $|G|$. *The* **degree of a character**

χ *is* $\chi(1)$*. Note that a character whose degree is 1 is called a linear character*

Definition 1.31 (Lifting Process) [Led87]: *Let N be a normal subgroup of G and suppose that $A_0(N_x)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(N(x)$ defines a representation of G/N lifted from G/N. If $\phi_0(N_x)$ is a character of $A_0(N_x)$, then $\phi(x) = \phi_0(N_x)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m, \phi(u) = m = \phi(1)$. Then lifting process preserves irreducibility.*

Definition 1.32 (Induced Character) [Led87]: Let $H \leq G$ and $\phi(u)$ be a character of H and defined $\phi(x) = 0$ if $x \in H$, then

$$\phi^G(x) = \begin{cases} \phi(x) & x \in H \\ 0 & x \notin H \end{cases}$$

is an induced character of G.

Definition 1.33 [Led87]: *Let G be a finite group and H be a subgroup such that $[G : H] = n$. Let $C_\alpha, \alpha = 1, 2, ..., m$ be the conjugacy classes of G with $|C_\alpha| = h_\alpha, \alpha = 1, 2, 3, ..., m$. Let ϕ be a character of H and ϕ G be the character of G induced from the character ϕ of H up to G. The values of ϕ^G on the m classes of G are given by:*

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w), \alpha = 1, 2, 3, ..., m.$$

Definition 1.34 Order [Rot95]: *Let G be a group. The* **order** *of G is the number of elements contained in G. We denote the order of G by $|G|$.*

Definition 1.35 Normal [Rot95]: *Let G be a group such that $K \leq G$. K is* **normal** *in G if $gKg^{-1} = K$, for every $g \in G$. We will use $K \triangleleft G$ to denote K as being normal in G.*

Definition 1.36 Representative [Rot95]: *Let G be a group and $S \subseteq G$. For $t \in G$, a* **right coset** *of S in G is the subset of G such that $St = st : s \in G$. We say t is a* **representative** *of the coset St.*

Definition 1.37 Index [Rot95]: *Let G be a group. The* **index** *of $H \leq G$, denoted $[G : H]$, is the number of right cosets of H in G.*

Definition 1.38 Double Coset [Rot95]: *Let G be a group and H and K be subgroups of G. A* **double coset** *of H and K of the form $HgK = \{HgK | k \in K\}$ is determined by $g \in G$.*

Definition 1.39 Point Stabilizer [Rot95]: *Let N be a group. The* **point stabilizer** *of w in N is given by:*

$$N^w = n \in N | w^n = w, where\ w\ is\ a\ word\ in\ the\ ti's.$$

Definition 1.40 Coset Stabilizer [Rot95]: *Let N be a group. The* **coset stabilizer** *of Nw in N is given by:*

$$N^{(w)} = n \in N | Nw^n = N_w, where\ w\ is\ a\ word\ of\ the\ ti's.$$

Definition 1.41 Center [Rot95]: *Let G be a group. The* **center** *of G, Z(G), is the set of all elements in G that commute with all elements of G.*

Definition 1.42 [Rot95]: *A symmetric presentation of a group G is a definition of G of the form:*

$$G \cong \frac{2^{*n} : N}{\pi_1 \omega_1, \pi_2 \omega_2, ...}$$

where 2^{*n} denotes a free product of n copies of the cyclic group of order 2, N is transitive permutation group of degree n which permutes the n generators of the cyclic group by conjugation, thus defining semi-direct product, and the relators $\pi_1 \omega_1, \pi_2 \omega_2, ...$ have been factored out.

Definition 1.43 [Rot95]: *We defined*

$$\mathcal{N}^i = C_\mathcal{N}(t_i); \mathcal{N}^{ij} = C_\mathcal{N}(\langle t_i, t_j \rangle)\ etc,$$

single point and two point stabilizer in $\mathcal{N}$ respectively. The coset stabilizing subgroup, $\mathcal{N}^{(w)}$, of $\mathcal{N}$ is given by

$$\mathcal{N}^w = \pi \in \mathcal{N} : \mathcal{N}w\pi = \mathcal{N}w$$

for w a word in the symmetric generators. Clearly $\mathcal{N}^w \le \mathcal{N}^{(w)}$, and the number of cosets in the double coset$[w] = \mathcal{N}w\mathcal{N}$ is given by $|\mathcal{N}|/|\mathcal{N}^{(w)}|$, since $\mathcal{N}w\pi_1 \ne \mathcal{N}w\pi_2$

$\Longleftrightarrow \mathcal{N}w\pi_1\pi_2^{-1} \ne \mathcal{N}w$

$\Longleftrightarrow \pi_1\pi_2 \notin \mathcal{N}^w$

$\Longleftrightarrow \mathcal{N}^{(w)}\pi_1\pi_2^{-1} \ne \mathcal{N}^w$

$\Longleftrightarrow \mathcal{N}^{(w)}\pi_1 \ne \mathcal{N}^{(w)}\pi_2.$

Double Coset Enumeration Arithmetic [Cur07]

In order to obtain the index of $\mathcal{N}$ in $\mathcal{G}$ we shall perform a manual double coset enumeration of $\mathcal{G}$ over $\mathcal{N}$; thus we must find all double cosets [w] and work out how many single cosets each of them contains. We shall know that we have completed the double coset enumeration when the set of right cosets obtained is closed under right multiplication. Moreover, the completion test above is best performed by obtaining the orbits of $\mathcal{N}^{(w)}$ on the symmetric generators. We need only identify, for each [w], the double coset to which $\mathcal{N}wt_i$ belongs for one symmetric generator t_i from each orbit.

Definition 1.44 First Isomorphism Theorem(F.I.T) [Rot95]: *Let $\phi : G \longrightarrow H$ is a homomorphism with $Ker\phi$. Then,*

$$\bullet\ Ker\phi \trianglelefteq G$$

$$\bullet\ G/Ker\phi \cong img\phi$$

Definition 1.43.1 monomial representation [Cur07]: *A monomial representation of a group G is a homomorphism from G into GL(n, F), the group of nonsingular n x n matrices over the field F, in which the image of every element of G is a monomial matrix over F*

Definition 1.43.2 monomial matrices [Led87]: *A matrix in which there is precisely one non-zero term in each row and in each column is said to be monomial.*

Theorem 1.45 [Cur07]: *The number of irreducible character of G is equal to the number of conjugacy classes of G.*

Theorem 1.46 [Rot95]: *Let $\phi : G \longrightarrow H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong im\phi$.*

Theorem 1.47 [Rot95]: *Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$*

Theorem 1.48 [Rot95]: *Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles.*

Theorem 1.49 [Rot95]: *Let $f : (G, *) \longrightarrow (G', \circ)$ be a homomorphism. The following hold true:*

- *$f(e) = e'$, where e' is the identity in G',*

- *If $a \in G$, then $f(a^{-1}) = f(a)^{-1}$,*

- If $a \in G$ and $n \in \mathbb{Z}$, then $f(a^n) = f(a)^n$.

Theorem 1.50 [Rot95]: *The intersection of any family of subgroups of a group G is again a subgroup of G.*

Theorem 1.51 [Rot95]: *If $S \leq G$, then any two right (or left) cosets of S in G are either identical or disjoint.*

Theorem 1.52 [Rot95]: *If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$.*

Theorem 1.53 [Rot95]: *If S and T are subgroups of a finite group G, then*

$$|ST||S \cap T| = |S||T|.$$

Theorem 1.54 [Rot95]: *If $N \triangleleft G$, then the cosets of N in G form a group, denoted by G/N, of order $[G : N]$.*

Theorem 1.55 [Rot95]: *The commutator subgroup G' is a normal subgroup of G. Moreover, if $H \triangleleft G$, then G/H is abelian if and only if $G' \leq H$.*

Theorem 1.56 [Rot95]: *Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$, then $G \cong H \times K$.*

Theorem 1.57 [Rot95]: *If $a \in G$, the number of conjugates of a is equal to the index of its centeralizer:*

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of $|G|$ when G is finite.

Theorem 1.58 [Rot95]: *If $H \leq G$, then the number c of conjugates of H in G is equal to the index of its normalizer: $c = [G : N_G(H)]$, and c divides $|G|$ when G is finite. Moreover, $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N_G(H)$.*

Theorem 1.59 [Rot95]: *Every group G can be imbedded as a subgroup of S_G. In particular, if $|G| = n$, then G can be imbedded in S_n.*

Theorem 1.60 [Rot95]: *If $H \leq G$ and $[G : H] = n$, then there is a homomorphism $\rho : G \to S_n$ with $\ker \rho \leq H$. The homomorphism ρ is called the representation of G on the cosets of H.*

Theorem 1.61 [Rot95]: *If X is a G-set with action α, then there is a homomorphism $\tilde{\alpha} : S_X$ given by $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$. Conversely, every homomorphism $\varphi : G \to S_X$ defines an action, namely, $gx = \varphi(g)x$, which makes X into a G-set.*

Theorem 1.62 Jordan-Hölder Theorem [Rot95]: *Every two composition series of a group G are equivalent. We will refer to this Theorem as the* **Jordan-Hölder Theorem**.

Theorem 1.63 [Rot95]: *Let X be a faithful primitive G-set of degree $n \geq 2$. If $H \triangleleft G$ and if $H \neq 1$, then X is a transitive H-set. Also, n divides $|H|$.*

1.2 Lemmas

Lemma 1.64 [Rot95]: *Let X be a G-set, and let $xy \in X$.*

- *If $H \leq G$, then $Hx \cap Hy \neq \emptyset$ implies $Hx = Hy$.*

- *If $H \triangleleft G$, then the subsets Hx are blocks of X.*

Lemma 1.65 [Famous Lemma]: $N \cap <t_i, t_j> \leq C_N(N_{ij})$ *where N_{ij} denotes the stabilzer in N of the two points i and j.*

Proof: Let $w \in N \cap <t_i, t_j>$. We need to show that $w \in C = Centralizer(N, N^{t_i t_j})$. We use the defintion of a centralizer to perform the following:

$w \in C$ if w commutes with every element of N^{ij}. Let $\pi \in N^{ij}$ then

- $w^\pi = w$

- $\pi^{-1} w \pi = w$

- $w\pi = \pi w$

Thus π commutes with every element of N_{ij} completing the proof.

Chapter 2

Manual Double Coset Enumeration

2.1 Construction of $PSL(2, 11) \times 2$

We perform double coset enumeration of PSL (2,11) x 2 over the maximal subgroup $D_6 : 2$ and D_6

Let $G = \frac{2^{*6}:D_6}{x^5 t_4 t_3 t_2 t_1 t_6 = e,\, x^3 t_2 t_3 t_1 t_2 t_6 t_1 = e}$. We will show $G \cong PSL(2, 11)$.

Our relations are

$$x^5 t_4 t_3 t_2 t_1 t_6 = e, \tag{2.1}$$

$$x^3 t_2 t_3 t_1 t_2 t_6 t_1 = e. \tag{2.2}$$

Let
$x = (1, 2, 3, 4, 5, 6)$,
$y = (1, 5)(2, 4)$ and let $N = < x, y > \cong D_6$.
and t $\sim t_1$.
We consider the maximal subgroup $H = < x, y, x^3 t_6 t_3 t_6 > \cong D_6 : 2$ containing N. We perform double coset enumeration of G over H and N.

We first prove the following lemmas.

Lemma 1: $Ht_1t_4 = Nt_1$.

We have $t_1t_4 = xyt_1$.

Then $Ht_1t_4 = Ht_1 \in [1]$.

Lemma 2: $Ht_1t_2t_3 = Ht_5t_4$.

Now we have $t_1t_2t_3 = x^{-1}t_5t_4$.

Then $Ht_1t_2t_3 = Ht_5t_4 \in [12]$.

Lemma 3: $Ht_1t_3t_1 = Ht_1t_3$.

$(t_1t_3)^{(1,3)(4,6)} = t_3t_1$.

Now we have $t_1t_3t_1 = x^2yt_3t_1$

$\implies Ht_1t_3t_1 = Ht_1t_3 \in [13]$.

Lemma 4: $Ht_1t_3t_5 = Ht_1t_3$.

$(t_1t_3)^{(1,3)(4,6)} = t_3t_1$.

Now we have $t_1t_3t_5 = x^{-2}t_3t_1$

$\implies Ht_1t_3t_5 = Ht_3t_1 \in [13]$.

Lemma 5: $Ht_1t_3t_6 = Ht_1t_2$.

$(t_1t_2)^{(1,2,3,4,5,6)} = t_2t_3$.

Now we have $t_1t_3t_6 = x^3yt_2t_3$

$\implies Ht_1t_3t_6 = Ht_2t_3 \in [12]$.

Lemma 6: $Ht_1t_3t_4t_2 = Ht_1t_3t_4$.

$(t_1t_3t_4)^{(1,4)(2,3)(5,6)} = t_4t_2t_1$.

Now we have $t_1t_3t_4t_2 = x^2yt_4t_2t_1$

$\implies Ht_1t_3t_4t_2 = Ht_1t_3t_4 \in [134]$.

Lemma 7: $Ht_1t_3t_4t_5 = Ht_6t_1t_6$.

$(t_1t_2t_1)^{(1,6,5,4,3,2)} = t_6t_1t_6$.

Now we have $t_1t_3t_4t_5 = x^{-1}yt_6t_1t_6$

$\implies Ht_1t_3t_4t_5 = Ht_6t_1t_6 \in [121]$.

Lemma 8: $Ht_1t_2t_6t_2 = Ht_3t_2t_4$.

$(t_1t_2t_6)^{(1,3)(4,6)} = t_3t_2t_4$.

Now we have $t_1t_2t_6t_2 = x^{-1}t_3t_2t_4$

$\implies Ht_1t_2t_6t_2 = Ht_3t_2t_4 \in [126]$.

Lemma 9: $Ht_1t_2t_4t_3 = Ht_5t_3t_4$.

$(t_1t_3t_2)^{(1,5)(2,4)} = t_5t_3t_4$.

Now we have $t_1t_2t_4t_3 = x^{-1}t_5t_3t_4$

$\implies Ht_1t_2t_4t_3 = Ht_5t_3t_4 \in [132]$.

Lemma 10: $Ht_1t_2t_4t_6 = Ht_5t_4t_2$.

$(t_1t_2t_4)^{(1,5)(2,4)} = t_5t_4t_2$.

Now we have $t_1t_2t_4t_6 = x^{-2}t_5t_4t_2$

$\implies Ht_1t_2t_4t_6 = Ht_5t_4t_2 \in [124]$.

Lemma 11: $Ht_1t_2t_4t_1 = Ht_3t_4t_3$.

$(t_1t_2t_1)^{(1,3,5)(2,4,6)} = t_3t_4t_3$.

Now we have $t_1t_2t_4t_1 = xyt_3t_4t_3$

$\implies Ht_1t_2t_4t_1 = Ht_3t_4t_3 \in [121]$.

Lemma 12: $Ht_1t_3t_2t_3 = Ht_5t_4t_2$.

$(t_1t_2t_4)^{(1,5)(2,4)} = t_5t_4t_2$.

Now we have $t_1t_3t_2t_3 = x^{-1}t_5t_4t_2$

$\implies Ht_1t_3t_2t_3 = Ht_5t_4t_2 \in [124]$.

Lemma 13: $Ht_1t_2t_1t_3 = Ht_1t_5t_4$.

$(t_1t_3t_4)^{(2,6)(3,5)} = t_1t_5t_4$.

Now we have $t_1t_2t_1t_3 = x^{-1}t_1t_5t_4$

$\implies Ht_1t_2t_1t_3 = Ht_1t_5t_4 \in [134]$.

Lemma 14: $Ht_1t_2t_1t_4 = Ht_4t_3t_1$.

$(t_1t_2t_4)^{(1,4)(2,3)(5,6)} = t_4t_3t_1$.

Now we have $t_1t_2t_1t_4 = x * yt_4t_3t_1$

$\implies Ht_1t_2t_1t_4 = Ht_4t_3t_1 \in [124]$.

We now perform the double coset enumeration.

The First Double Coset HeN = [*]

H

The Second Double Coset $Ht_1N = [1]$

To determine the number of single right cosets in [1] we do $\frac{|12|}{|2|}$, where $N^{(1)}$is a coset stabilizer that is greater than or equal to the point-stabilizer of 1. We note that $N^{(1)} = N^1 =< (2,6)(3,5) >$.

The number of right single cosets in the double cosets [1] is $\frac{N}{N^1} = \frac{|12|}{|2|} = 6$. The orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{4\}, \{2, 6\}, \{3, 5\}$.

We choose representatives 1, 4, 2 and 3 from the four orbits and form the cosets $Ht_1t_1, Ht_1t_4,$ Ht_1t_2, Ht_1t_3. We need to determine the double cosets of these cosets.

We have,

$Ht_1t_1 = N \in [*]$,

$Ht_1t_2 \in Ht_1t_2H = [12]$,

$Ht_1t_3 \in Ht_1t_3H = [13]$,

$Ht_1t_4 \in Ht_1t_4H = [1]$.

The Third Double Coset $Ht_1t_2N = [12]$

To determine the number of single right cosets in [12] we do $\frac{|N|}{|N^{(12)}|}$, where $N^{(12)}$is the coset stabilizer of the coset Ht_1t_2 that is greater than or equal to the point-stabilizer of 1. We note that $N^{(12)} = N^{12} =< e >$.

The number of right cosets in the double cosets [12] is $\frac{N}{N^{12}} = \frac{|12|}{|1|} = 12$. The

orbits of $N^{(12)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

We choose representatives 1, 2, 3, 4, 5, 6 from the six orbits and form the cosets $Ht_1t_2t_1, Ht_1t_2t_2, Ht_1t_2t_3, Ht_1t_2t_4, Ht_1t_2t_5, Ht_1t_2t_6$. We need to determine the double cosets of these cosets.

We have,

$Ht_1t_2t_1 \in Ht_1t_2t_1H = [121]$,

$Ht_1t_2t_2 \in Ht_1H = [1]$,

$Ht_1t_2t_3 = Ht_5t_4H$

$\quad\quad = H(t_1t_2)^{(1,5)(2,4)} \implies (1,5)(2,4) \in [12]$,

$Ht_1t_2t_4 \in Ht_1t_2t_4H = [124]$,

$Ht_1t_2t_5 \in Ht_1t_3H = [13]$,

$Ht_1t_2t_6 \in Ht_1t_2t_6H = [126]$.

The Fourth Double Coset $Ht_1t_3N = [13]$

To determine the number of single right cosets in [13] we do $\frac{|N|}{|N^{(13)}|}$, where $N^{(13)}$ is the coset stabilizer of the coset Ht_1t_3 that is greater than or equal to the point-stabilizer of 1. Note that $N^{(13)} = N^{13} = <e>$.

The number of right cosets in the double cosets [13] is $\frac{N}{N^{13}} = \frac{|12|}{|1|} = 12$. The orbits of $N^{(13)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

We choose representatives 1, 2, 3, 4, 5, 6 from the 6 orbit and form the cosets $Ht_1t_3t_1, Ht_1t_3t_2,$ $Ht_1t_3t_3, Ht_1t_3t_4, Ht_1t_3t_5, Ht_1t_3t_6$. We need to determine the double cosets of these cosets.

We have,

$Ht_1t_3t_1 \in Ht_1t_3H = [13]$,

$Ht_1t_3t_2 \in Ht_1t_3t_2H = [132]$,

$Ht_1t_3t_3 \in Ht_1H = [1]$,

$Ht_1t_3t_4 \in Ht_1t_3t_4H = [134]$,

$Ht_1t_3t_5 \in Ht_1t_3H = [13]$,

$Ht_1t_3t_6 \in Ht_1t_2H = [12]$.

The Fifth Double Coset $Ht_1t_3t_4N = [134]$

To determine the number of single right cosets in [134] we do $\frac{|N|}{|N^{(134)}|}$, where $N^{(134)}$ is the coset stabilizer of the coset $Ht_1t_3t_4$ that is greater than or equal to the point-stabilizer of 1. Note that Point Stabilizer $N^{134} = \{e\}$.

$Nt_1t_3t_4 = Nt_4t_2t_1 \implies (1,4)(2,3)(5,6)$
belongs to $N^{(134)}$ and $Nt_1t_3t_4^{(1,4)(2,3)(5,6)} = Nt_4t_2t_1$
Coset stabilizer is $N^{(134)} \geq < N^{134}, (1,4)(2,3)(5,6) >$.

The number of right cosets in the double cosets [134] is $\frac{N}{N^{134}} = \frac{|12|}{|2|} = 6$. The orbits of $N^{(134)}$ on $\{1,2,3,4,5,6\}$ are $\{1,4\}, \{2,3\}, \{5,6\}$.

We choose representatives 4, 2, 5 from the three orbits and form the cosets.

$Ht_1t_3t_4t_4, Ht_1t_3t_4t_2, Ht_1t_3t_4t_5$. We need to determine the double cosets of these cosets.

We have,

$Ht_1t_3t_4t_4 \in Ht_1t_3H = [13]$,

$Ht_1t_3t_4t_2 \in Ht_1t_3t_4t_2 = [134]$,

$Ht_1t_3t_4t_5 \in Ht_1t_3t_4t_5 = [121]$.

The Sixth Double Coset $Ht_1t_2t_6N = [126]$

To determine the number of single right cosets in [126] we do $\frac{|N|}{|N^{(126)}|}$, where $N^{(126)}$ is the coset stabilizer of the coset $Ht_1t_2t_6$ that is greater than or equal to the point-stabilizer of 1. Note that the point stabilizer $N^{126} = \{e\}$.

$Nt_1t_2t_6 = Nt_6t_5t_1 \implies (1,6)(2,5)(3,4)$
belongs to $N^{(126)}$ and $Nt_1t_2t_6^{(1,6)(2,5)(3,4)} = Nt_6t_5t_1$.

$Nt_1t_2t_6 = Nt_4t_5t_3 \implies (1,4)(2,5)(3,6)$
belongs to $N^{(126)}$ and $Nt_1t_2t_6^{(1,4)(2,5)(3,6)} = Nt_4t_5t_3$.
Coset stabilizer is $N^{(126)} \geq < N^{126}, (1,4)(2,5)(3,6), (1,6)(2,5)(3,4) >$.

The number of right cosets in the double cosets [126] is $\frac{N}{N^{126}} = \frac{|12|}{|4|} = 3$. The

orbits of $N^{(126)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{1, 6\}, \{2, 5\}, \{3, 4\}$.

We choose representatives 6, 2, 3 from three orbits and form the cosets.

$Ht_1t_2t_6t_6, Ht_1t_2t_6t_2, Ht_1t_2t_6t_3$. We need to determine the double cosets of these cosets. We have,

$Ht_1t_2t_6t_2 \in Ht_1t_2t_6 = [126]$,

$Ht_1t_2t_6t_6 \in Ht_1t_2 = [12]$.

The Seventh Double Coset $Ht_1t_2t_4N = [124]$

To determine the number of single right cosets in [124] we do $\frac{|N|}{|N^{(124)}|}$, where $N^{(124)}$ is the coset stabilizer of the coset $Ht_1t_2t_4$ that is greater than or equal to the point-stabilizer of 1. Point Stabilizer $N^{124} = \{e\}$.

$Nt_1t_2t_4 = Nt_5t_4t_2 \implies (1, 5)(2, 4)$

belongs to $N^{(124)}$ and $Nt_1t_2t_4^{(1,5)(2,4)} = Nt_5t_4t_2$

Coset stabilizer is $N^{(124)} \geq < N^{124}, (1, 5)(2, 4) >$.

The number of right cosets in the double cosets [124] is $\frac{N}{N^{124}} = \frac{|12|}{|2|} = 6$. The orbits of $N^{(124)}$ on $\{1, 2, 3, 4, 5, 6\}$ are $\{3\}, \{6\}, \{1, 5\}, \{2, 4\}$.

We choose representatives 3, 6, 1, 4 from the four orbits and form the cosets $Ht_1t_2t_4t_3$, $Ht_1t_2t_4t_6, Ht_1t_2t_4t_1, Ht_1t_2t_4t_4$. We need to determine the double cosets of these cosets. We have,

$Ht_1t_2t_4t_3 \in Ht_1t_3t_2 = [132]$ Lemma 9

$Ht_1t_2t_4t_6 \in Ht_1t_2t_4 = [124]$ Lemma 10

$Ht_1t_2t_4t_1 \in Ht_1t_2t_1 = [121]$ Lemma 11

$Ht_1t_2t_4t_4 \in Ht_1t_2 = [12]$.

The Eighth Double Coset $Ht_1t_3t_2N = [132]$

To determine the number of single right cosets in [132] we do $\frac{|N|}{|N^{(132)}|}$, where $N^{(132)}$ is the coset stabilizer of the coset $Ht_1t_3t_2$ that is greater than or equal to the point-stabilizer of 1. Point Stabilizer $N^{132} = \{e\}$.

$Nt_1t_3t_2 = Nt_5t_3t_4 \implies (1,5)(2,4)$

belongs to $N^{(132)}$ and $Nt_1t_3t_2^{(1,5)(2,4)} = Nt_5t_3t_4$

$Nt_1t_3t_2 = Nt_4t_6t_5 \implies (1,4)(2,5)(3,6)$

belongs to $N^{(132)}$ and $Nt_1t_3t_2^{(1,4)(2,5)(3,6)} = Nt_4t_6t_5$

Coset stabilizer is $N^{(132)} \geq < N^{132}, (1,5)(2,4), (1,4)(2,5)(3,6) >$.

The number of right cosets in the double cosets $[132]$ is $\frac{N}{N^{132}} = \frac{|12|}{|4|} = 3$. The orbits of $N^{(132)}$ on $\{1,2,3,4,5,6\}$ are $\{3\}, \{6\}, \{1,5\}, \{2,4\}$.

We choose representatives 3, 6, 1, 4 from the four orbits and form the cosets $Ht_1t_3t_2t_3$, $Ht_1t_3t_2t_6, Ht_1t_3t_2t_1, Ht_1t_3t_2t_2$. We need to determine the double cosets of these cosets. We have,

$Ht_1t_3t_2t_3 \in Ht_1t_2t_4 = [124]$,

$Ht_1t_3t_2t_2 \in Ht_1t_3 = [13]$.

The Ninth Double Coset $Ht_1t_2t_1N = [121]$

To determine the number of single right cosets in $[121]$ we do $\frac{|N|}{|N^{(121)}|}$, where $N^{(121)}$ is the coset stabilizer of the coset $Ht_1t_2t_1$ that is greater than or equal to the point-stabilizer of 1. Point Stabilizer $N^{121} = \{e\}$.

$Nt_1t_2t_1 = Nt_2t_1t_2 \implies (1,2)(3,6)(4,5)$

belongs to $N^{(121)}$ and $Nt_1t_2t_1^{(1,2)(3,6)(4,5)} = Nt_2t_1t_2$

Coset stabilizer is $N^{(121)} \geq < N^{121}, (1,2)(3,6)(4,5) >$.

The number of right cosets in the double cosets $[121]$ is $\frac{N}{N^{121}} = \frac{|12|}{|2|} = 6$. The orbits of $N^{(121)}$ on $\{1,2,3,4,5,6\}$ are $\{1,2\}, \{3,6\}, \{4,5\}$.

We choose representatives 1, 3, 4 from the three orbits and form the cosets.

$Ht_1t_2t_1t_1, Ht_1t_2t_1t_3, Ht_1t_2t_1t_4$. We need to determine the double cosets of these cosets. We have,

$Ht_1t_2t_1t_1 \in Ht_1t_2 = [12]$,

$Ht_1t_2t_1t_3 \in Ht_1t_3t_4 = [134]$,

$Ht_1t_2t_1t_4 \in Ht_1t_2t_4 = [124]$.
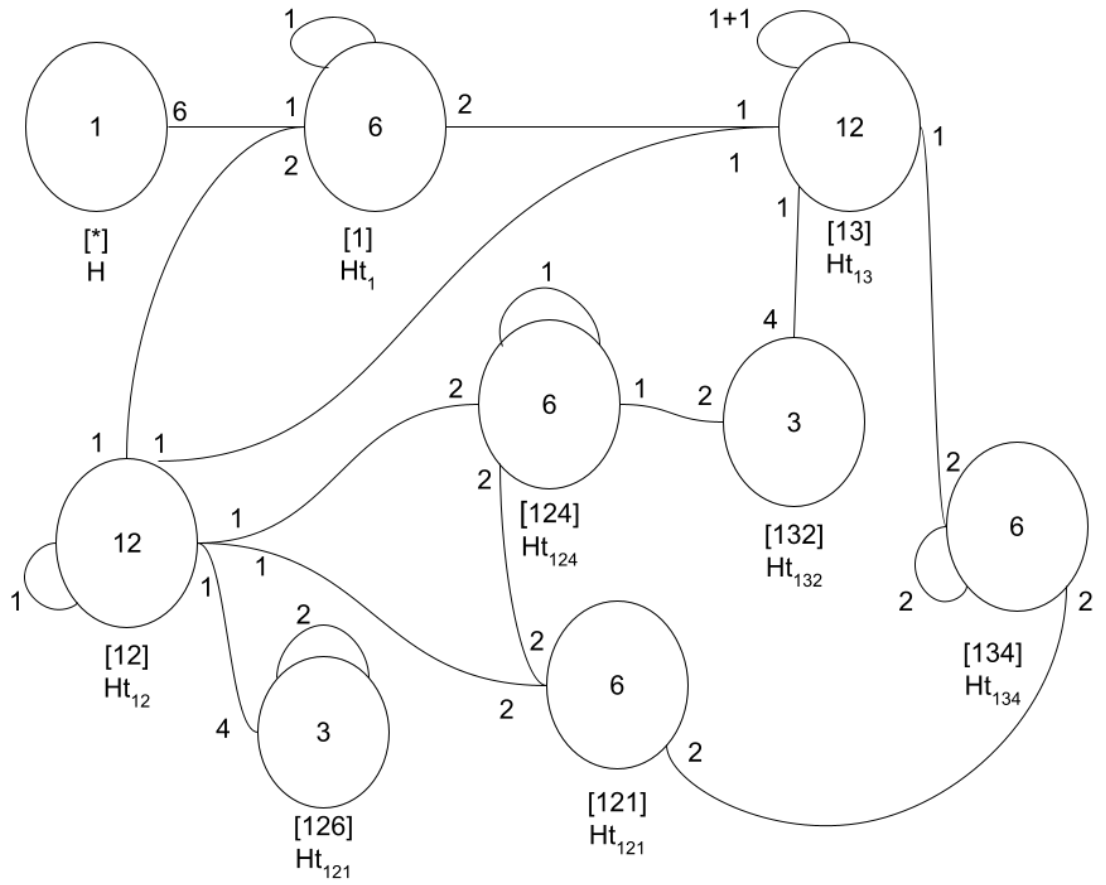
Figure 2.1: Cayley Diagram of PSL $(2, 11) \times 2$ over The Maximal Subgroup $D_6 : 2$ and D_6

2.2   Constructions of A_7

We perform double coset enumeration of A_7 over the maximal subgroup S_5 and $D_5 : 2$.

Let G= $\frac{2^*6:(D_5:2)}{yx^{-1}t_2t_4t_5t_3t2=e}$. We will show that $G \cong A_7$.

Our relations are

$$yx^{-1}t_2t_4t_5t_3t2 = e \tag{2.3}$$

Let

$x = (1, 2, 3, 4, 5)$,

$y = (1, 2, 4, 3)$ and let $N =< x, y > \cong D_5 : 2$.

and t $\sim t_1$.

Let H be a subgroup generated by N, where $N = A_7$ and $t_{15}t_{24}t_{11}$.

Thus H $= < N, t_{15}t_{24}t_{11} >$, where H is a maximal subgroup of G and $H \cong S_5$. We perform double coset enumeration of G over H and N.

We first prove the following lemmas.

Lemma 1: $Ht_1t_2 = Nt_1$.

$t_1t_4 = xyt_1$

Then $Ht_1t_4 = Ht_1 \in [1]$.

We now perform the double coset enumeration.

The First Double Coset HeN = [*]

H is the only coset that is contained in [*]. This implies that the number of single cosets of H in [*] is 1.

N is transitive on $\{1, 2, 3, 4, 5\}$. So we can choose a representative from the orbits say, 1 to form a new double coset Ht_1N. Thus all the 6 generators will move onto the new double coset $Ht_1N = [1]$.

The Second Double Coset $Ht_1N = [1]$

To determine the number of single right cosets in $[1]$ we do $\frac{|N|}{|N^{(1)}|}$, where $N^{(1)}$is a coset stabilizer that is greater than or equal to the point-stabilizer of 1. Note that $N^{(1)} = N^1 =< (2,5)(3,4), (2,3,5,4) > .$

The number of right cosets in the double cosets $[1]$ is $\frac{N}{N^1} = \frac{|N|}{|N^{(1)}|} = \frac{20}{4} = 5$. The orbits of $N^{(1)}$ on $\{1, 2, 3, 4, 5\}$ are $\{1\}, \{2, 4, 5, 3\}$.

We choose representatives 1 and 2 from the two orbits and form the cosets Ht_1t_1, Ht_1t_2. We need to determine the double cosets of these cosets.

We have,

$Ht_1t_1 = N \in [*],$

$Ht_1t_2 \in Ht_1t_2H = [12].$
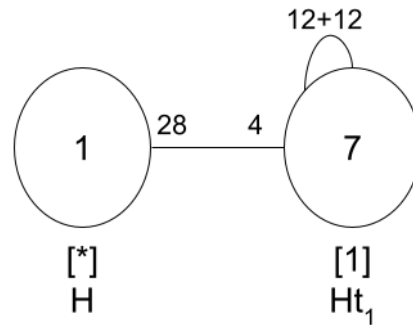
Figure 2.2: Cayley Diagram of A_7 over The Maximal Subgroup S_5 and $D_5 : 2$

2.3   Construction of $2^3 : (2^2 : S_3)$

We perform double coset enumeration of $2^3 : (2^2 : S_3)$ over A_4.

G= $\frac{2^{*6}:A_4}{(x^{-1}t)^4,(x^{-1}t^y)^4}$. We will show that $G \cong 2^3 : (2^2 : S_3)$.

Let

$x = (1,3,5)(2,4,6)$,

$y = (1,2,6)(3,4,5)$ and let $N =< x, y > \cong A_4$.

and t $\sim t_1$.

Expanding my given relation: $(x^{-1}t^y)^4 \implies (x^{-1}t_2)^4$

$=x^{-4}t_2^{x^{-3}}t_2^{x^{-2}}t_2^{x^{-1}}t_2$

$=x^{-1}t_2t_4t_6t_2$

$=x^2t_2t_4t_6t_2$

$\implies x^2t_2t_4 = t_2t_6$

$$x^2t_2t_4 = t_2t_6 \tag{2.4}$$

Expanding my second relation in a similar way leads to:

$$x^2t_1t_3 = t_1t_5 \tag{2.5}$$

Our Group N has 11 elements; namely,

$N =$

$(1,3,5)(2,4,6) \sim x = x^{-2}$,

$(1,2,6)(3,4,5) \sim y$,

$(1,5,3)(2,6,4) \sim x^{-1}$,

$(1,6,2)(3,5,4) \sim y^{-}1$,

$(1,4)(2,5) \sim x * y$,

$(1,5,6)(2,3,4) \sim x * y - 1$,

$(1,4)(3,6) \sim y * x$,

$(1,6,5)(2,4,3) \sim y * x^{-1}$,

$(1,3,2)(4,6,5) \sim x^{-1} * y = (x^{-1} * y)^{-1}$,

$(1,2,3)(4,5,6) \sim y^{-1} * x$,

$(2,5)(3,6) \sim x * y^{-1} * x.$

We first prove the following lemmas.

Lemma 1: $Nt_1t_2 = Nt_1t_3$.

$x^2t_2t_4 = t_2t_6$ (2.4)

Rewrite: $(153)(264)t_2t_4 = t_2t_6$

$(153)(264)t_2t_4^{(162)(354)} = t_2t_6^{(162)(354)}$

$=(123)(645)t_1t_3 = t_1t_2$.

Now we have $Nt_1t_3 = Nt_1t_2$

Note by R2 we know that $Nt_1t_3 = t_1t_5$

So, $Nt_1t_2 = Nt_1t_3 = Nt_1t_5$.

Lemma 2: $Nt_1t_6 = Nt_1t_5$.

$x^2t_1t_3 = t_1t_5$ (2.5)

Rewrite: $(135)(246)t_1t_3 = t_1t_5$

$(153)(264)t_2t_4^{(132)(465)} = t_2t_6^{(132)(465)}$

$= (342)(156)t_1t_6 = t_1t_5$.

Now we have $Nt_1t_6 = Nt_1t_5$.

Lemma 2.1:

By Lemma 1: $Nt_1t_2 = Nt_1t_3 = Nt_1t_5$ and $Nt_1t_5 = Nt_1t_6$ (By Lemma 2).

So that implies that $Nt_1t_2 = Nt_1t_3 = Nt_1t_5 = Nt_1t_6$.

Lemma 3: $Nt_1t_2t_3 = Nt_1$.

Note from above: $Nt_1t_2 = Nt_1t_3$

Multiply right sides by $t_3 \implies Nt_1t_2t_3t_3 = Nt_1t_3t_3$

$Nt_1t_2t_3t_3 = Nt_1$.

Lemma 3.1: $Nt_4t_3 = Nt_4t_6$.

$Nt_1t_3 = Nt_1t_5$ (2.5)

$(153)(264)t_1t_3^{(14)(25)} = t_1t_5^{(14)(25)}$

$(423)(561)t_4t_3 = t_4t_2$.

Now we look at:

$Nt_2t_4 = t_2t_6$ (2.4)

$(153)(264)t_2t_4^{(135)(246)} = t_2t_6^{(135)(246)}$

$(153)(264)t_4t_6 = t_4t_2$

Substituting the above results in the following:$(156)(234)t_4t_3 = (153)(246)Nt_4t_6$

$(156)(234)(153)(246)t_4t_3 = t_4t_6$

$(14)(36)t_4t_3 = t_4t_6$.

Lemma 4: $Nt_1t_4 = Nt_3t_6$.

Consider: 1463(look at 46)

From Lemma 3.1 we know that $(14)(36)t_4t_3 = t_4t_6$

$= 1(14)(36)433$

$= (14)(36)44$

$= (14)(36)$.

So $1463 = (14)(36) \in N$

$\implies Nt_1t_4t_6t_3 = N(14)(36)$

$\implies Nt_1t_4 = Nt_3t_6$.

Lemma 5: $Nt_1t_2t_1 = Nt_5t_6t_5$.

Note: $Nt_1t_2 = Nt_1t_6$ from Lemma 2

Multiply on right side by $t_1 \implies Nt_1t_2t_1 = Nt_1t_6t_1$

From our given relations $x^2t_2t_4 = x^2t_2t_6$ (conjugate by $(126)(345)$)

$=(156)(234)t_6t_5 = t_6t_1$.

Now substitute the new relation above into $Nt_1t_2t_1 = Nt_1t_6t_1$

$Nt_1t_2t_1 = Nt_1(156)(234)t_6t_5$

$=Nt_5t_6t_5$

So, $Nt_1t_2t_1^{(153)(264)} = Nt_5t_6t_5t$.

Lemma 6: $Nt_1t_2t_1 = Nt_1t_5t_1$.

Note: $Nt_1t_2 = Nt_1t_5$ from Lemma 1

Multiply on right side by t_1 so, $Nt_1t_2t_1 = Nt_1t_5t_1$.

Lemma 7: $Nt_1t_2t_4 = Nt_6t_4t_3$.

Note: $Nt_1t_2t_4t_3t_4t_6 = N$ (By Lemma 4)

Rewrite:

$Nt_1t_2t_4 = t_6t_4t_3$.

Lemma 8: $Nt_1t_2t_4 = Nt_1t_5t_4$.

Note: $Nt_1t_2 = Nt_1t_5$ from Lemma 1

Multiply on right side by t_4 so, $Nt_1t_2t_4 = Nt_1t_5t_4$.

We now perform the double coset enumeration.

1st Double Coset $[*] = NeN$

$$[*] = NeN = \{N(e)^n | n \in N\}$$

Now we plug in each element that is in N and this will return N. The coset stabilizer of the coset Ne is N.

So the number of single right cosets in NeN is $= \frac{|N|}{|N|} = 1$.

Now the orbits of N are on $X = \{1, 2, 3, 4, 5, 6\}$ are $\{1, 2, 3, 4, 5, 6\}$.

Now we select a representative from the single orbit say 1. So $Nt_1 \in [1] \implies$ all of the six ti's extend to the same double coset which we will call [1].
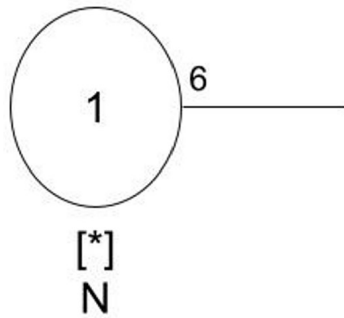
2nd Double Coset $[1] = Nt_1N$

$$[1] = Nt_1N = \{N(t_1)^n | n \in N\}$$

Now we plug in each element that is in N

$= \{N(t_1)^e, N(t_1)^{(1,3,5)(2,4,6)}, N(t_1)^{(1,2,6)(3,4,5)}, N(t_1)^{(1,5,3)(2,6,4)}...N(t_1)^{(2,5)(3,6)}\}$

$= \{Nt_1, Nt_3, Nt_2, Nt_5, ...Nt_6\}$.

Point Stabilizer on $N^1 = \{e, (2,5)(3,6)\}$

Coset Stabilizer of $N^{(1)} =< N^{(1)} >= 2$

$N^{(1)} = N^1 = \{e, (2,5)(3,6)\}$.

So the number of single right cosets in Nt_1N is $= \frac{|N|}{|N^{(1)}|} = \frac{12}{2} = 2$.

Now the orbits of N on $X = \{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{4\}, \{2, 5\}, \{3, 6\}$.

Now we select a representative from each orbit say 1,4,2,3 and determine the double coset of $Nt_1t_1, Nt_1t_4, Nt_1t_2, Nt_1t_3$.

We have,

$Nt_1t_1 = N \in [*]$,

$Nt_1t_4 \in Nt_1t_4N = [14]$,

$Nt_1t_2 \in Nt_1t_2N = [12]$,

$Nt_1t_3 \in Nt_1t_3 = [12]$.
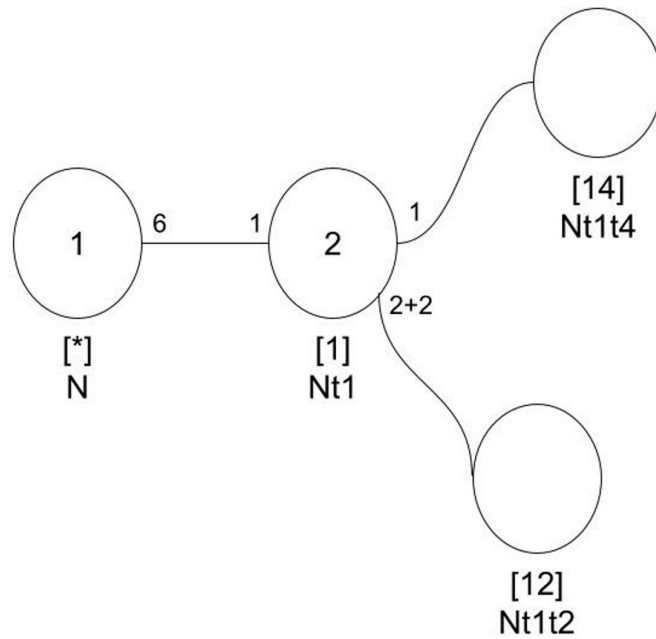
3rd Double Coset $[12] = Nt_1t_2N$

$[12] = Nt_1t_2N = \{N(t_1t_2)^n | n \in N\}$

Now we plug in each element that is in N

$= \{N(t_1t_2)^e, N(t_1t_2)^{(1,3,5)(2,4,6)}, N(t_1t_2)^{(1,2,6)(3,4,5)}, N(t_1t_2)^{(1,5,3)(2,6,4)}...N(t_1t_2)^{(2,5)(3,6)}\}$

$= Nt_1t_2, Nt_3t_4, Nt_2t_6, Nt_5t_6, Nt_6t_1, Nt_4t_5, Nt_5t_3, Nt_4t_2, Nt_6t_4, Nt_3t_1, Nt_2t_3, Nt_1t_5$.

Point Stabilizer on $N^{12} = \{e\}$

Note: $Nt_1t_2 = Nt_1t_5$ (By Lemma 1).

We know from Lemma 1: $Nt_1t_2^{(25)(36)} = Nt_1t_5$.

So, $(25)(36) \in N^{(12)}$ and $|N^{(12)}| = 2$.

Therefore $N^{(12)} \geq \{e, (2,5)(3,6)\}$.

So the number of single right cosets in Nt_1t_2N is $= \frac{|N|}{|N^{(12)}|} = \frac{12}{2} = 6$.

Now the orbits of the coset stabilizer $N^{(12)}$ on $X = \{1,2,3,4,5,6\}$ are $\{1\}, \{4\}, \{2,5\},$ and $\{3,6\}$.

Now we select a representative from each orbit say 1,4,2,3 and determine the double cosets

of $Nt_1t_2t_1, Nt_1t_2t_4, Nt_1t_2t_2, Nt_1t_2t_3$.

We have,

$Nt_1t_2t_1 \in Nt_1t_2t_1N = [121]$,

$Nt_1t_2t_4 \in Nt_1t_2t_4N = [124]$,

$Nt_1t_2t_2 \in Nt_1N = [1]$,

$Nt_1t_2t_3 \in Nt_1N = [1]$.
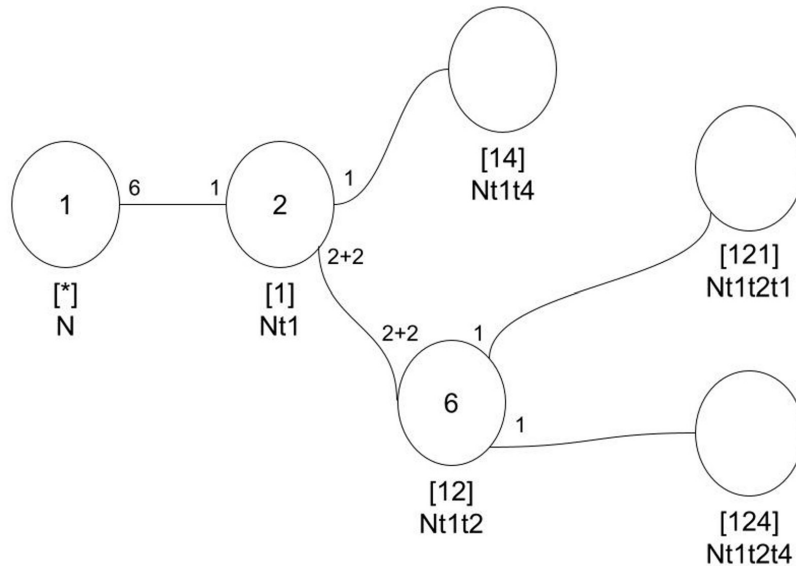
4th Double Coset $[14] = Nt_1t_4N$

$[14] = Nt_1t_4N = \{N(t_1t_4)^n | n \in N\}$

Now we plug in each element that is in N

$= \{N(t_1t_4)^e, N(t_1t_4)^{(1,3,5)(2,4,6)}, N(t_1t_4)^{(1,2,6)(3,4,5)}, N(t_1t_4)^{(1,5,3)(2,6,4)}...N(t_1t_4)^{(2,5)(3,6)}\}$

$= Nt_1t_4, Nt_3t_6, .., Nt_4t_1.$

Point Stabilizer on $N^{14} = \{e, (2,5)(3,6)\}$

Note: $Nt_1t_4 = Nt_3t_6$ (by Lemma 4).

We know from Lemma 4: $Nt_1t_4^{(135)(246)} = t_3t_6.$

So, $(135)(246) \in N^{(14)}.$

Therefore $N^{(14)} \geq < N^{14}, (135)(246) >= N.$

So the number of single right cosets in $N^{(14)}$ has the order of 12 and Nt_1t_4N is

$= \frac{|N|}{|N^{(14)}|} = \frac{12}{12} = 1.$

Now the orbits of the coset stabilizer $N^{(14)}$ on $X = \{1, 2, 3, 4, 5, 6\}$ are $\{1, 2, 3, 4, 5, 6\}$

Now we select a representative from each orbit, say 4 and determine the double cosets of $Nt_1t_4t_4.$

We have,

$Nt_1t_4t_4 \in Nt_1t_4t_4N = [1].$
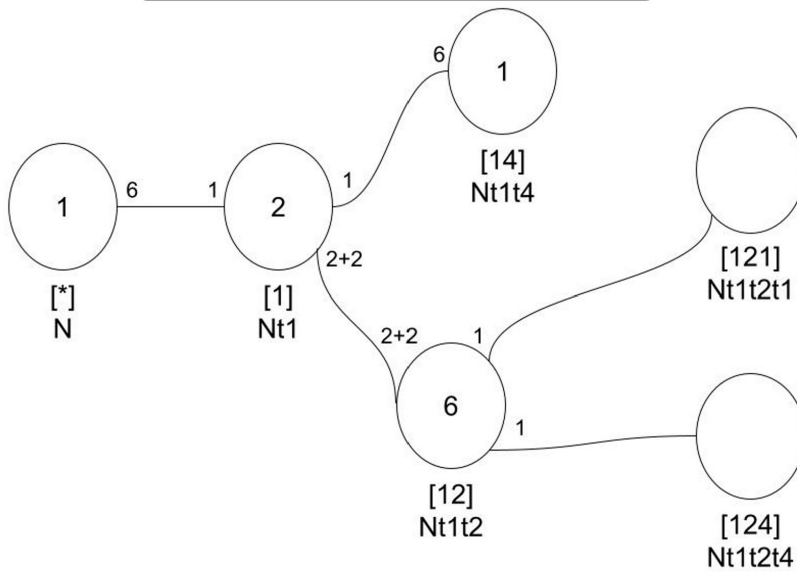
5th Double Coset $[121] = Nt_1t_2t_1N$

$$[121] = Nt_1t_2t_1N = \{N(t_1t_2t_1)^n | n \in N\}$$

Now we plug in each element that is in N

$= \{N(t_1t_2t_1)^e, N(t_1t_2t_1)^{(1,3,5)(2,4,6)}, N(t_1t_2t_1)^{(1,2,6)(3,4,5)}, N(t_1t_2t_1)^{(1,5,3)(2,6,4)}...N(t_1t_2t_1)^{(2,5)(3,6)}\}$

$= Nt_1t_2t_1, Nt_3t_4t_3, Nt_2t_6t_2, Nt_5t_6t_5, Nt_6t_1t_6, Nt_4t_5t_4, Nt_5t_3t_5, Nt_4t_2t_4, Nt_6t_4t_6,$

$Nt_3t_1t_3, Nt_2t_3t_2, Nt_1t_5t_1.$

Point Stabilizer on $N^{121} = \{e\}$

Note: $Nt_1t_2t_1 = Nt_5t_6t_5$ (By Lemma 5).

We know from Lemma 5: $Nt_1t_2t_1^{(153)(264)} = t_5t_6t_5$.

So, $(153)(264) \in N^{(121)}$.

Note: $Nt_1t_2t_1 = Nt_1t_5t_1$.

From lemma 6 we get that $Nt_1t_2t_1^{(25)(36)} = t_1t_5t_1$.

So, $(25)(36) \in N^{(121)}$

Therefore $N^{(121)} \geq < (25)(36), (153)(264) > = N$.

So the order $N^{(121)}$ is 12 and the number of single right cosets in $Nt_1t_2t_1N$

is $= \frac{|N|}{|N^{(121)}|} = \frac{12}{12} = 1$.

Now the orbits of the coset stabilizer $N^{(121)}$ on $X = \{1, 2, 3, 4, 5, 6\}$ are $\{1, 3, 5\}, \{2, 4, 6\}$

Now we select a representative from each orbit, say 1 and 2 and determine the double

coset of $Nt_1t_2t_1t_1, Nt_1t_2t_1t_2$.

We have,

$Nt_1t_2t_1t_1 \in Nt_1t_2N = [12],$

$Nt_1t_2t_1t_2 \in Nt_1t_2t_1t_2 = [12].$
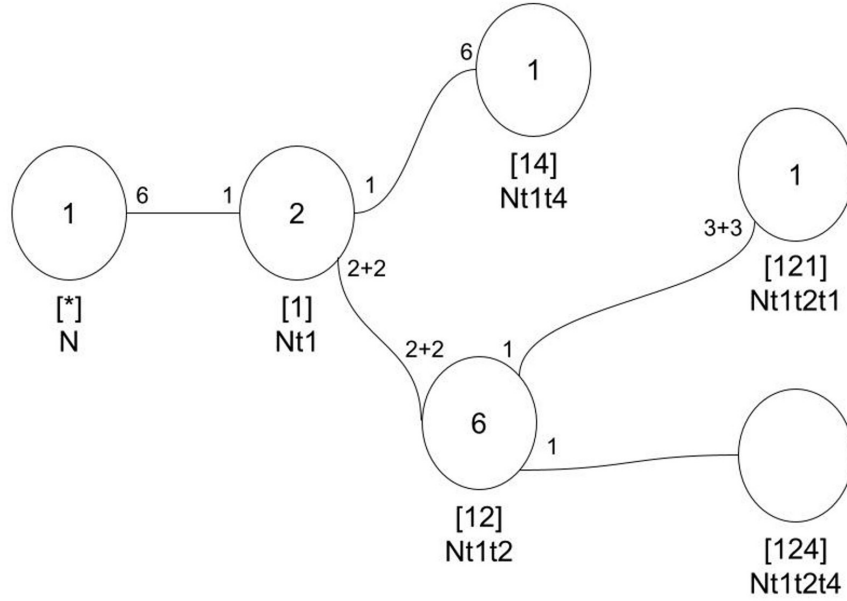
6th Double Coset $[124] = Nt_1t_2t_4N$

$$[124] = Nt_1t_2t_4N = \{N(t_1t_2t_4)^n | n \in N\}$$

Now we plug in each element that is in N

$= \{N(t_1t_2t_4)^e, N(t_1t_2t_4)^{(1,3,5)(2,4,6)}, N(t_1t_2t_4)^{(1,2,6)(3,4,5)}, N(t_1t_2t_4)^{(1,5,3)(2,6,4)} ... N(t_1t_2t_4)^{(2,5)(3,6)}\}$

$= Nt_1t_2t_4, Nt_3t_4t_6, Nt_2t_6t_5, Nt_5t_6t_2, Nt_6t_1t_3, Nt_4t_5t_1, Nt_5t_3t_2, Nt_4t_2t_1, Nt_6t_4t_3,$

$Nt_3t_1t_6, Nt_2t_3t_5, Nt_1t_5t_4.$

Point Stabilizer on $N^{124} = \{e\}$

Note: $Nt_1t_2t_4 = Nt_6t_4t_3$ (By Lemma 7).

We know from Lemma 7: $Nt_1t_2t_4^{(165)(243)} = t_6t_4t_3$.

So, $(165)(243) \in N^{(124)}$

Note: $Nt_1t_2t_4 = Nt_1t_5t_4$ (By Lemma 8).

We know from Lemma 8 $Nt_1t_2t_4^{(25)(36)} = t_1t_5t_4$.

So, $(25)(36) \in N^{(124)}$.

Therefore $N^{(124)} \geq < (25)(36), (165)(243) >= N$.

So the number of single right cosets in $Nt_1t_2t_4N$ is $= \frac{|N|}{|N^{(124)}|} = \frac{12}{12} = 1$.

Now the orbits of the coset stabilizer $N^{(124)}$ on $X = \{1, 2, 3, 4, 5, 6\}$ are $\{1, 2, 3, 4, 5, 6\}$.

Now we select a representative from each orbit, say 1 and determine the double cosets of $Nt_1t_2t_4t_4$.

We have,

$Nt_1t_2t_4t_4 \in Nt_1t_2 = [12]$.
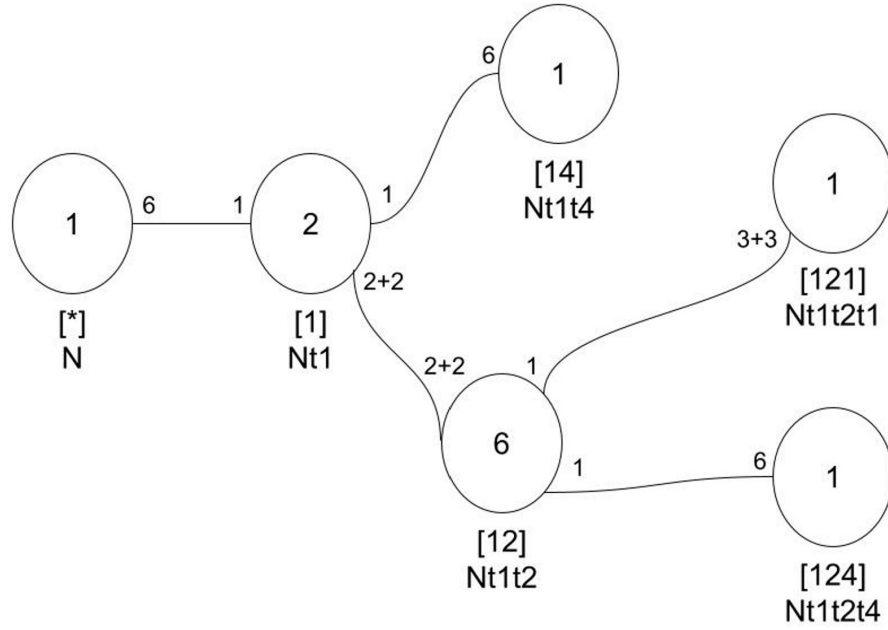
Figure 2.3: Cayley Diagram of $2^3 : (2^2 : S_3)$ over A_4

2.4   Construction $U_3(3) : 2$

We perform double coset enumeration of $U_3(3) : 2$ over $PSL(2,7)$.

G= $\frac{2^*28:PSL(2,7)}{t_1t_2t_1(12)(37)(46)(59),t_1t_3t_7=(1,2,3,8,7)(4,10,6,5,9)}$. We will show that $G \cong U_3(3) : 2$.

Let

$x = (1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)$,

$y = (1,3)(2,4,7,10)(5,8,12,17)(6,9,14,20)(11,16,13,18)(15,21,19,23)(22,24,25,26)(27,28)$,

and let $N =< x,y >\cong PSL(2,7)$.

and t $\sim t_1$.

Expanding given relation: $(yxyt^{xy^2})^3 = e$

Let $q = yxy$ and $s = xy^2$

Note: $t^s = t_7$

Rewrite: $(qt^s)^3 \implies q^3 t_7^{q^3} t_7^{q^2} \ t_7^q$

$q^3 t_7 t_{20} t_{21} = e \implies t_7 t_{20} t_{21} = e.$

$$q^3 t_7 t_{20} t_{21} = e \tag{2.6}$$

Expanding given relation: $((yxy^-1xyxy^-1)t)^4 = e$

Let c = $yxy^{-1}xyxy^{-1}$

Rewrite: $(ct)^4 \implies c^4 t^{c^3} t^{c^2} t^c t = e$

$\implies c^4 t_2 t_1 t_2 t_1 = e.$

$$t_2 t_1 t_2 t_1 = e \tag{2.7}$$

We will prove that $\|G\| = 15624$.

Thus we must find all double cosets of G over N and determine the number of right cosets in each double coset. We will know that the double enumeration is complete if the set of right cosets is closed under the right multiplication by the t_i. We note that for each double coset NwN it suffices to determine the double coset of Nwti for one t_i from each orbit of the coset stabilising group $N^{(w)}$ of the right coset representative Nw of the

double coset NwN. In order to do this we must first perform the manual DCE of G over N.

We find it convenient to list the permutations, and the corresponding words, of N.

$N =$

$(1, 3, 6)(2, 4, 7)(5, 8, 9) \implies y,$

$(1, 2)(3, 5)(7, 9)(8, 10) \implies x,$

$(1, 4, 7, 5, 6)(2, 3, 8, 10, 9) \implies xy,$

$(1, 3, 6)(2, 4, 7)(5, 8, 9) \implies x^2 * y,$

$(1, 2)(3, 5)(7, 9)(8, 10) \implies x^3,$

$(1, 4, 7, 5, 6)(2, 3, 8, 10, 9) \implies x^3 y,$

$(1, 3, 6)(2, 4, 7)(5, 8, 9) \implies x^4 y,$

$(1, 2)(3, 5)(7, 9)(8, 10) \implies x^5,$

$(1, 4, 7, 5, 6)(2, 3, 8, 10, 9) \implies x^5 y.$

Lemma 1: $Nt_1 t_2 = Nt_2 t_1.$

$t_1 t_2 = t_2 t_1 \implies Nt_1 t_2 = Nt_2 t_1.$

1st Double Coset $[*] = NeN$

$$[*] = NeN = \{N(e)^n | n \in N\}$$

Now we plug in each element that is in N.

$= \{N(e)^e, N(e)^{(1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)},$

$N(e)^{(1,3)(2,4,7,10)(5,8,12,17)(6,9,14,20)(11,16,13,18)(15,21,19,23)(22,24,25,26)(27,28)},$

$...N(e)^{(1,27)(2,4)(3,28)(6,26)(7,10)(8,17)(9,25)(11,16)(13,18)(14,24)(20,22)(21,23)}\}$

This will just give us $\{N\}$ and the coset stabilizer of $N = N.$

So the number of single right cosets in NeN is $= \frac{|N|}{|N|} = 1.$

Now the orbits of N on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,$

$17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$ is x.

Now we select a representative from the single orbit say 1. So $Nt_1 \in [1] \implies$ all of the

28 ti's extend to the same double coset which we will call [1].
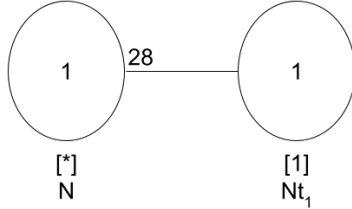
2nd Double Coset $[1] = Nt_1N$

$$[1] = Nt_1N = \{N(t_1)^n | n \in N\} = \{Nt_1, ..., Nt_{28}\}$$

Point Stabilizer of 1 on $N^1 = \{e, (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)$
$(11,13)(15,19)(16,18)(21,23)(22,25)(24,26),$
$(2,28)(3,20)(4,23)(5,26)(6,15)(10,25)(11,12)(13,24)(14,18)(16,19)(17,27)(21,22)\}.$

So $N^{(1)} = N^1 = < (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)(16,18)(21,23)(22,25)(24,26),$
$(2,28)(3,20)(4,23)(5,26)(6,15)(10,25)(11,12)(13,24)(14,18)(16,19)(17,27)(21,22) > .$

The number of single right cosets in Nt_1N is $= \frac{|N|}{|N^{(1)}|} = \frac{168}{6} = 28.$

Orbits of N on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,$
$22, 23, 24, 25, 26, 27, 28\}$ are $\{1\}, \{2, 7, 28\}, \{3, 20, 9\}, \{8, 17, 27\}, \{4, 10, 23, 25, 21, 22\},$
$\{5, 12, 26, 11, 24, 13\}, \{6, 14, 15, 18, 19, 16\}.$

Now we select a representative from each orbit say 1,2,3,8,4,5,6 and determine the double
coset of $Nt_1t_1, Nt_1t_4, Nt_1t_2, Nt_1t_3, Nt_1t_8, Nt_1t_4, Nt_1t_5, Nt_1t_6.$

We have,

$Nt_1t_1 = N \in [*]$

$Nt_1t_2 \in N = [*],$

$Nt_1t_3 \in Nt_1t_3 = [13],$

$Nt_1t_8 \in Nt_1t_8 = [13],$

$Nt_1t_4 \in Nt_1t_4 = [13],$

$Nt_1t_5 \in Nt_1t_5 = [13],$

$Nt_1t_6 \in Nt_1t_6 = [13].$
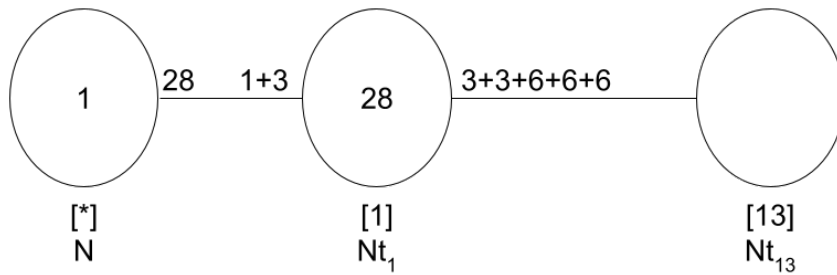
3rd Double Coset $[13] = Nt_1t_3N$

$$[13] = Nt_1t_3N = \{N(t_1t_3)^n | n \in N\} = \{Nt_1, ..., Nt_{28}\}$$

Point Stabilizer $N^{13} = \{e, (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)$
$(16,18)(21,23)(22,25)(24,26)\}.$

$Nt_1t_3 = Nt_3t_1 \implies (1,3)(2,10,7,4)(5,17,12,8)(6,20,14,9)(11,18,13,16)$
$(15,23,19,21)(22,26,25,24)(27,28)$
belongs to $N^{(13)}$ and $Nt_1t_3^{(1,3)(2,10,7,4)(5,17,12,8)(6,20,14,9)(11,18,13,16)(15,23,19,21)(22,26,25,24)(27,28)}$
$= Nt_3t_1.$

$Nt_1t_3 = Nt_{27}t_{28} \implies (1,27)(2,4)(3,28)(6,26)(7,10)(8,17)(9,25)(11,16)(13,18)$

$(14, 24)(20, 22)(21, 23)$

belongs to $N^{(13)}$ and $Nt_1t_3^{(1,27)(2,4)(3,28)(6,26)(7,10)(8,17)(9,25)(11,16)(13,18)(14,24)(20,22)(21,23)}$

$= Nt_{27}t_{28}$.

Coset stabilizer is $N^{(13)} \geq < N^{13}, (1, 3)(2, 10, 7, 4)(5, 17, 12, 8)(6, 20, 14, 9)$

$(11, 18, 13, 16)(15, 23, 19, 21)(22, 26, 25, 24)(27, 28),$

$(1, 3)(2, 4, 7, 10)(5, 8, 12, 17)(6, 9, 14, 20)(11, 16, 13, 18)(15, 21, 19, 23)$

$(22, 24, 25, 26)(27, 28), (1, 27)(2, 4)(3, 28)(6, 26)(7, 10)(8, 17)(9, 25)(11, 16)(13, 18)$

$(14, 24)(20, 22)(21, 23), (1, 27)(2, 10)(3, 28)(4, 7)(5, 12)(6, 24)(9, 22)(11, 18)$

$(13, 16)(14, 26)(15, 19)(20, 25) > .$

The number of single right cosets in $N^{(13)}$ has the order is $\frac{|N|}{|N^{(13)}|} = \frac{168}{8} = 21$.
Orbits of N on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,$

$21, 22, 23, 24, 25, 26, 27, 28\}$ are $\{1, 3, 27, 28\}, \{2, 7, 4, 10\}, \{5, 12, 8, 17\}, \{11, 13, 16, 18\},$

$\{15, 19, 21, 23\}, \{6, 14, 9, 20, 26, 24, 25, 22\}.$

Now we select a representative from each orbit say 3,2,5,11,15,6 and determine the double

cosets of $Nt_1t_3t_3, Nt_1t_3t_2, Nt_1t_3t_5, Nt_1t_3t_{11}, Nt_1t_3t_{15}, Nt_1t_3t_6$.

We have,

$Nt_1t_3t_3 \in Nt_1 \in [1],$

$Nt_1t_3t_2 \in Nt_1 = [1],$

$Nt_1t_3t_5 \in Nt_1t_3t_5 = [135],$

$Nt_1t_3t_{11} \in Nt_1t_3t_{11} = [1311],$

$Nt_1t_3t_{15} \in Nt_1t_3t_5 = [135],$

$Nt_1t_3t_6 \in Nt_1t_3t_5 = [135].$
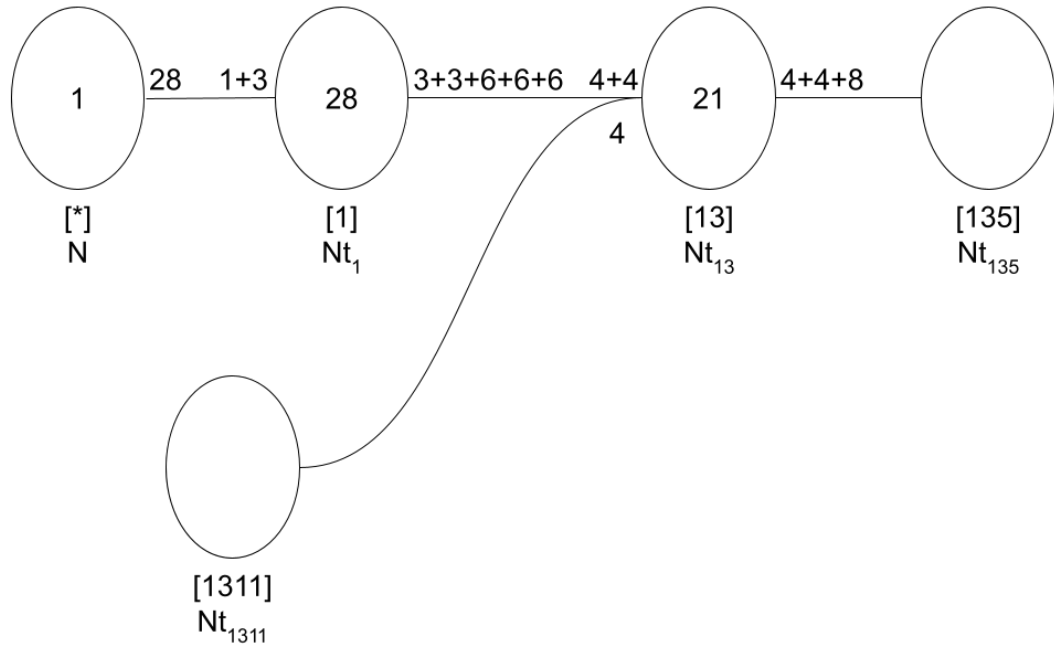
4th Double Coset $[135] = Nt_1t_3t_5N$

$$[135] = Nt_1t_3t_5N = \{N(t_1t_3t_5)^n | n \in N\}$$

Point Stabilizer on $N^{135} = \{e\}$.

$Nt_1t_3t_5 = Nt_1t_3t_{12} \implies (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)$
$(11,13)(15,19)(16,18)(21,23)(22,25)(24,26)$
belongs to $N^{(135)}$ and $Nt_1t_3t_5^{(2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)(16,18)(21,23)(22,25)(24,26)}$
$= Nt_1t_3t_{12}$.

$Nt_1t_3t_5 = Nt_{19}t_5t_{28} \implies (1,19,27,15)(2,24,10,6)(3,5,28,12)(4,26,7,14)(8,23)$
$(9,25,22,20)(11,13,18,16)(17,21)$
belongs to $N^{(135)}$ and $Nt_1t_3t_5^{(1,19,27,15)(2,24,10,6)(3,5,28,12)(4,26,7,14)(8,23)(9,25,22,20)(11,13,18,16)(17,21)}$
$= Nt_{19}t_5t_{28}$.
Coset stabilizer is $N^{(135)} \geq < N^{135}, (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)$
$(11,13)(15,19)(16,18)(21,23)(22,25)(24,26), (1,19,27,15)(2,24,10,6)(3,5,28,12)(4,26,7,14)(8,23)$
$(9,25,22,20)(11,13,18,16)(17,21) > .$

The number of single right cosets in $N^{(135)}$ is $\frac{|N|}{|N^{(135)}|} = \frac{168}{8} = 21$.

Orbits of N on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,$
$21, 22, 23, 24, 25, 26, 27, 28\}$ are $\{1, 19, 15, 27\}, \{3, 5, 12, 28\},$
$\{8, 17, 23, 21\}, \{9, 20, 25, 22\}, \{11, 13, 18, 16\}, \{2, 7, 24, 14, 26, 10, 6, 4\}$.

Now we select a representative from each orbit say 1,5,8,9,11,2 and determine the double
cosets of $Nt_1t_3t_5t_1, Nt_1t_3t_5t_5, Nt_1t_3t_5t_8, Nt_1t_3t_5t_9, Nt_1t_3t_5t_{11}, Nt_1t_3t_5t_2$.

We have,

$Nt_1t_3t_5t_1 \in Nt_1t_3 = [13]$,

$Nt_1t_3t_5t_5 \in Nt_1t_3 = [13]$,

$Nt_1t_3t_5t_8 \in Nt_1t_3t_5t_8 = [1358]$,

$Nt_1t_3t_5t_9 \in Nt_1t_3t_5t_8 = [1358]$,

$Nt_1t_3t_5t_{11} \in Nt_1t_3t_5t_{11} = [13511]$,

$Nt_1t_3t_5t_2 \in Nt_1t_3 = [13]$.
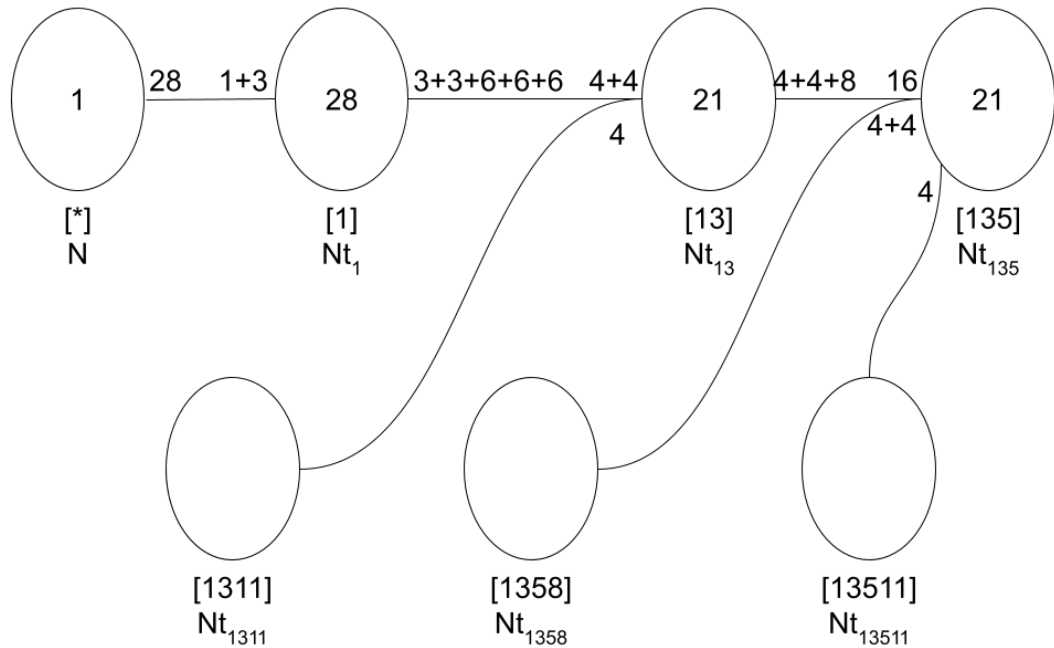
5th Double Coset $[1311] = Nt_1t_3t_{11}N$

$$[1311] = Nt_1t_3t_{11}N = \{N(t_1t_3t_{11})^n | n \in N\}$$

Point Stabilizer on $N^{1311} = \{e\}$.

$Nt_1t_3t_{11} = Nt_3t_1t_{16} \implies (1,3)(2,4,7,10)(5,8,12,17)(6,9,14,20)(11,16,13,18)$
$(15,21,19,23)(22,24,25,26)(27,28)$
belongs to $N^{(1311)}$ and $Nt_1t_3t_{11}^{(1,3)(2,4,7,10)(5,8,12,17)(6,9,14,20)(11,16,13,18)(15,21,19,23)(22,24,25,26)(27,28)}$
$= Nt_3t_1t_{16}$.

$Nt_1t_3t_{11} = Nt_{11}t_7t_4 \implies (1,11,4)(2,16,27)(3,7,18)(5,9,25)(6,8,20)(10,28,13)(12,24,14)$
$(15,21,23)(17,26,22)$
belongs to $N^{(1311)}$ and $Nt_1t_3t_{11}^{(1,11,4)(2,16,27)(3,7,18)(5,9,25)(6,8,20)(10,28,13)(12,24,14)(15,21,23)(17,26,22)}$
$= Nt_{11}t_7t_4$.

Coset stabilizer is $N^{(1311)} \geq < N^{1311}, (1,3)(2,4,7,10)(5,8,12,17)(6,9,14,20)(11,16,13,18)$
$(15,21,19,23)(22,24,25,26)(27,28),$
$(1,11,4)(2,16,27)(3,7,18)(5,9,25)(6,8,20)(10,28,13)(12,24,14)$
$(15,21,23)(17,26,22) > .$

The number of single right cosets in $N^{(1311)}$ is $\frac{|N|}{|N^{(1311)}|} = \frac{168}{24} = 7$.

Orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,$
$21,22,23,24,25,26,27,28\}$ are $\{15,21,19,23\}, \{1,3,11,7,16,4,10,18,13,27,2,28\},$
$\{5,8,9,12,20,14,25,17,24,6,26,22\}$.

Now we select a representative from each orbit say 15,11,5 and determine the double cosets of $Nt_1t_3t_{11}t_{15}, Nt_1t_3t_{11}t_{11}, Nt_1t_3t_{11}t_5$.

We have,

$Nt_1t_3t_{11}t_{15} \in Nt_1t_3t_5t_{11} = [13511]$,

$Nt_1t_3t_{11}t_{11} \in Nt_1t_3 = [13]$,

$Nt_1t_3t_{11}t_5 \in Nt_1t_3t_5t_{11} = [13511]$.
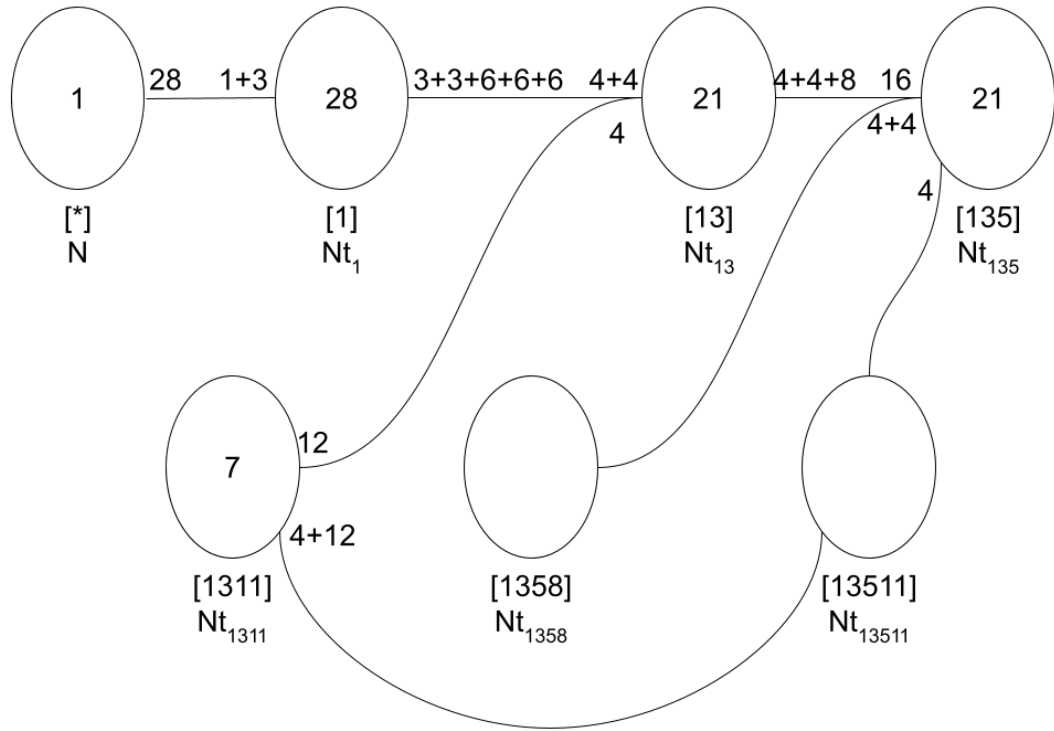
6th Double Coset $[1358] = Nt_1t_3t_5t_8N$

$$[1358] = Nt_1t_3t_5t_8N = \{N(t_1t_3t_5t_8)^n | n \in N\}$$

Point Stabilizer on $N^{1358} = \{e\}$.

$Nt_1t_3t_5t_8 = Nt_2t_5t_3t_{11} \implies (1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)$
$(17,20)(18,21)(23,25)(26,27)$
belongs to $N^{(1358)}$ and $Nt_1t_3t_5t_8^{(1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)}$
$= Nt_2t_5t_3t_{11}$.

$Nt_1t_3t_5t_8 = Nt_8t_4t_6t_1 \implies (1,8)(2,21)(3,4)(5,6)(7,9)(11,19)(12,16)(13,24)$
$(14,18)(15,26)(20,23)(22,28)$
belongs to $N^{(1358)}$ and $Nt_1t_3t_5t_8^{(1,8)(2,21)(3,4)(5,6)(7,9)(11,19)(12,16)(13,24)(14,18)(15,26)(20,23)(22,28)}$
$= Nt_8t_4t_6t_1$.

$Nt_1t_3t_5t_8 = Nt_{10}t_{16}t_{22}t_{14} \implies (1,10,11)(2,27,18)(3,16,7)(4,13,28)(5,22,24)(6,9,12)$
$(8,14,26)(15,21,19)(17,25,20)$
belongs to $N^{(1358)}$ and $Nt_1t_3t_5t_8^{(1,10,11)(2,27,18)(3,16,7)(4,13,28)(5,22,24)(6,9,12)(8,14,26)(15,21,19)(17,25,20)}$

$= Nt_{10}t_{16}t_{22}t_{14}.$

Coset stabilizer is $N^{(1358)} \geq < N^{1358}, (1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)$
$(17,20)(18,21)(23,25)(26,27), (1,8)(2,21)(3,4)(5,6)(7,9)(11,19)(12,16)(13,24)$
$(14,18)(15,26)(20,23)(22,28), (1,10,11)(2,27,18)(3,16,7)(4,13,28)(5,22,24)(6,9,12)$
$(8,14,26)(15,21,19)(17,25,20) > .$

The number of single right cosets in $N^{(1358)}$ is $\frac{|N|}{|N^{(1358)}|} = \frac{168}{24} = 7.$

Orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,$
$21,22,23,24,25,26,27,28\}$ are $\{17,20,25,23\}, \{1,2,8,10,21,27,11,14,15,18,19,26\},$
$\{3,5,4,16,6,22,13,12,7,9,28,24\}.$

Now we select a representative from each orbit say 17,8,3 and determine the double cosets
of $Nt_1t_3t_5t_8t_{17}, Nt_1t_3t_5t_8t_8, Nt_1t_3t_5t_8t_3.$

We have,

$Nt_1t_3t_5t_8t_{17} \in Nt_1t_3t_5t_8t_{17} = [135817],$

$Nt_1t_3t_5t_8t_8 \in Nt_1t_3t_5 = [135],$

$Nt_1t_3t_5t_8t_3 \in Nt_1t_3t_5t_8t_3 = [135].$
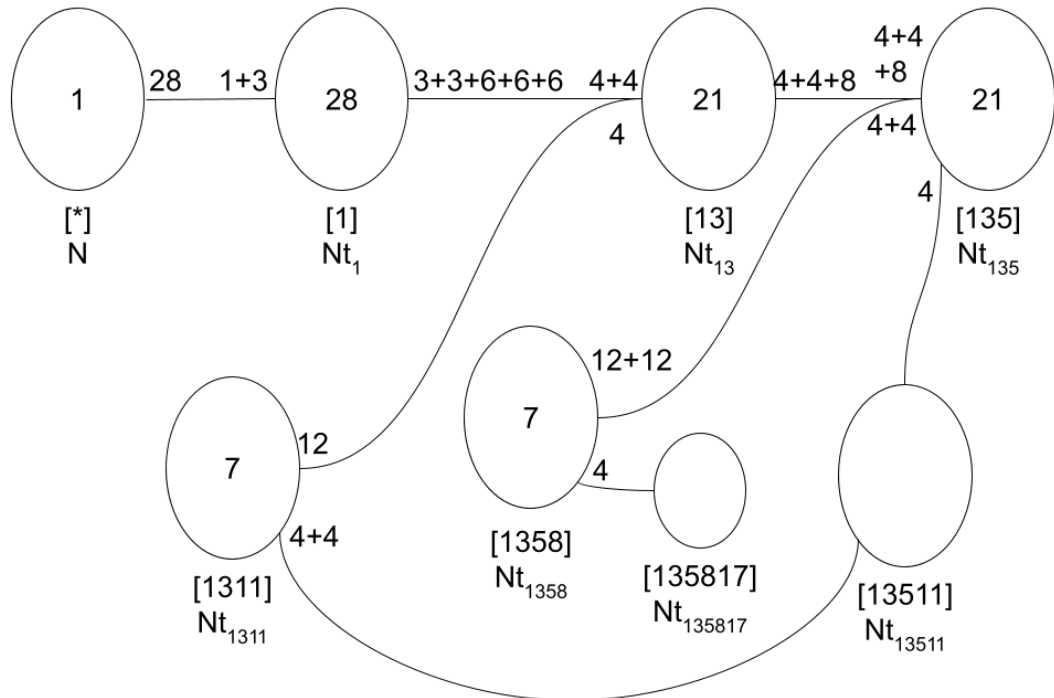
7th Double Coset $[13511] = Nt_1t_3t_5t_{11}N$

$$[13511] = \mathrm{N}t_1t_3t_5t_{11}N = \{N(t_1t_3t_5t_{11})^n | n \in N\}$$

Point Stabilizer on $N^{13511} = \{e\}$.

$Nt_1t_3t_5t_{11} = Nt_2t_5t_3t_8 \implies (1,2)(3,5)(4,6)(8,11)(9,13)$

$(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)$

belongs to $N^{(13511)}$ and $Nt_1t_3t_5t_{11}^{(1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)}$

$= Nt_2t_5t_3t_8$.

$Nt_1t_3t_5t_{11} = Nt_1t_3t_{12}t_{13} \implies (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)$

$(16,18)(21,23)(22,25)(24,26)$

belongs to $N^{(13511)}$ and $Nt_1t_3t_5t_{11}^{(2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)(16,18)(21,23)(22,25)(24,26)}$

$= Nt_1t_3t_{12}t_{13}$.

$Nt_1t_3t_5t_{11} = Nt_{27}t_{28}t_5t_{16} \implies (1,27)(2,4)(3,28)(6,26)(7,10)$

$(8,17)(9,25)(11,16)(13,18)(14,24)(20,22)(21,23)$

belongs to $N^{(13511)}$ and $Nt_1t_3t_5t_{11}^{(1,27)(2,4)(3,28)(6,26)(7,10)(8,17)(9,25)(11,16)(13,18)(14,24)(20,22)(21,23)}$

$= Nt_{27}t_{28}t_5t_{16}$.

Coset stabilizer is $N^{(13511)} \geq\ < N^{13511}, (1,2)(3,5)(4,6)(8,11)(9,13)$

$(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27), (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)$

$(11,13)(15,19)(16,18)(21,23)(22,25)(24,26), (1,27)(2,4)(3,28)(6,26)(7,10)$

$(8,17)(9,25)(11,16)(13,18)(14,24)(20,22)(21,23) > $.

The number of single right cosets in $N^{(13511)}$ is $\frac{|N|}{|N^{(13511)}|} = \frac{168}{24} = 7$.

Orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,$

$21,22,23,24,25,26,27,28\}$ are $\{3,5,28,12\}, \{1,2,27,7,4,26,10,6,24,15,14,19\},$

$\{8,11,17,13,16,20,9,18,22,25,21,23\}$.

Now we select a representative from each orbit say 3,1,11 and determine the double cosets

of $Nt_1t_3t_5t_{11}t_3, Nt_1t_3t_5t_{11}t_1, Nt_1t_3t_5t_{11}t_{11}$.

We have,

$Nt_1t_3t_5t_{11}t_3 \in Nt_1t_3t_5t_{11}t_3 = [135113]$,

$Nt_1t_3t_5t_{11}t_1 \in Nt_1t_3t_5t_{11}t_1 = [1311]$,

$Nt_1t_3t_5t_{11}t_{11} \in Nt_1t_3t_5t_{11}t_{11} = [135]$.

1 — 28 | 1+3 — 28 — 3+3+6+6+6 | 4+4 — 21 — 4+4+8 | 4+4+8 — 21

[*]   [1]   [13]   [135]
N Nt_1 Nt_{13} Nt_{135}

7   12+12   7

[1311]   [1358]   [135817]   [13511]
Nt_{1311} Nt_{1358} Nt_{135817} Nt_{13511}
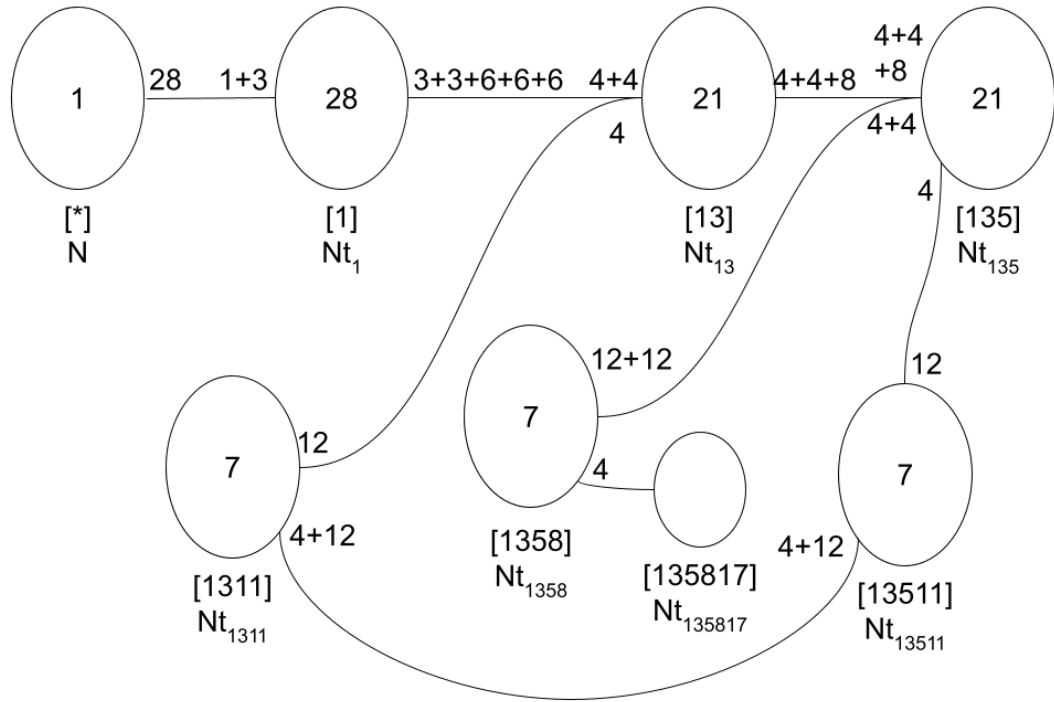
8th Double Coset $[135817] = Nt_1t_3t_5t_8t_{17}N$

$$[135817] = Nt_1t_3t_5t_8t_{17}N = \{N(t_1t_3t_5t_8t_{17})^n | n \in N\}$$

Point Stabilizer on $N^{135817} = \{e\}$.

$Nt_1t_3t_5t_8t_{17} = Nt_{27}t_{28}t_5t_{17}t_8 \implies (1,2)(3,5)(4,6)(8,11)(9,13)$
$(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)$
belongs to $N^{(13511)}$ and $Nt_1t_3t_5t_{11}^{(1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)}$
$= Nt_2t_5t_3t_8$.

$Nt_1t_3t_5t_8t_{17} = Nt_4t_5t_{28}t_{16}t_{22} \implies (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)$
$(16,18)(21,23)(22,25)(24,26)$
belongs to $N^{(13511)}$ and $Nt_1t_3t_5t_{11}^{(2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)(16,18)(21,23)(22,25)(24,26)}$
$= Nt_1t_3t_{12}t_{13}$.

$Nt_1t_3t_5t_8t_{17} = Nt_3t_6t_{22}t_{12}t_{28} \implies (1,27)(2,4)(3,28)(6,26)(7,10)$
$(8,17)(9,25)(11,16)(13,18)(14,24)(20,22)(21,23)$
belongs to $N^{(13511)}$ and $Nt_1t_3t_5t_{11}^{(1,27)(2,4)(3,28)(6,26)(7,10)(8,17)(9,25)(11,16)(13,18)(14,24)(20,22)(21,23)}$

$= N t_{27} t_{28} t_5 t_{16}$.

Coset stabilizer is $N^{(135817)} \geq < N^{135817}, (1,27)(2,4)(3,28)(6,26)(7,10)(8,17)(9,25)(11,16)$
$(13,18)(14,24)(20,22)(21,23), (1,4,26)(2,27,6)(3,5,28)$
$(7,10,15)(8,16,20)(9,18,23)(11,17,22)(13,25,21)(14,19,24),$
$(1,3,6,21,24,18,20)(2,27,5,22,19,13,25)$
$(4,15,9,14,11,17,28)(7,10,26,8,12,16,23) > .$

The number of single right cosets in $N^{(135817)}$ is $\frac{|N|}{|N^{(135817)}|} = \frac{168}{168} = 1$.
Orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,$
$21,22,23,24,25,26,27,28\}$ are $\{1,27,4,3,6,5,2,26,15,28,21,22,8,7,9,23,13,24,20,$
$11,19,17,16,12,10,25,18,14\}$.

Now we select a representative from each orbit say 17 and determine the double cosets
of $N t_1 t_3 t_5 t_8 t_{17} t_{17}$.

We have,

$N t_1 t_3 t_5 t_8 t_{17} t_{17} \in N t_1 t_3 t_5 t_8 t_{17} t_{17} = [1358]$.
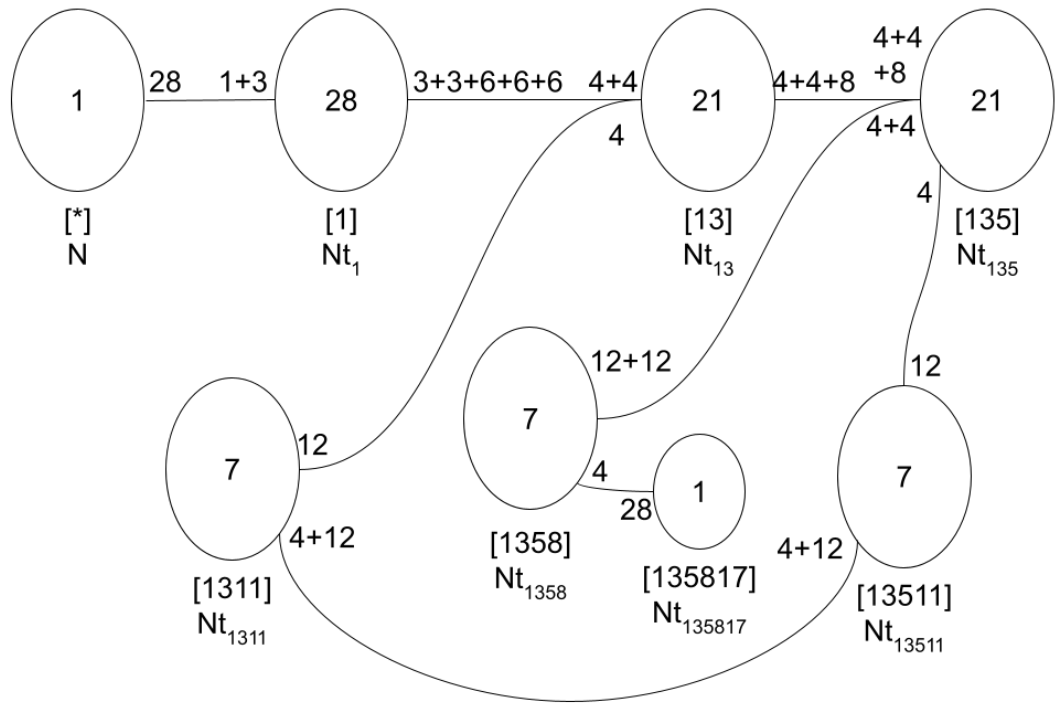
Figure 2.4: Cayley Diagram of $U_3(3) : 2$ over $PSL(2,7)$

So $|G| \leq \frac{|N|}{|N|} + \frac{|N|}{|N^{(1)}|} = 1 + 28 + 21 + 21 + 7 + 7 + 7 + 1$

Then we multiply the result by the order of N to get the following: $93 * 168 = 15624$.

2.5 Construction of S_6

We perform double coset enumeration of S_6 over 5:4.

G$\cong \frac{2^{*5}:(5:4)}{(2,3,5,4)t_2t_4=t_2t_3t_5}$. We will show that $G \cong S_6$.

Our relations are

$$x^2yt_2t_4 = t_2t_3t_5 = e \tag{2.8}$$

Let

$x = (1, 2, 3, 4, 5)$

$y = (1, 2, 4, 3)$ and let $N =< x, y >\cong (5:4)$.

and t $\sim t_1$.

Our Group N has 20 elements; namely,

N =

(1, 3)(4, 5),

(1, 4, 5, 2),

(1, 3, 2, 5),

(1, 4, 3, 5),

(1, 2, 4, 3),

(1, 3, 4, 2),

(1, 3, 5, 2, 4),

(1, 2, 5, 4),

(1, 5, 2, 3),

(1, 5, 4, 3, 2),

(1, 5)(2, 4),

(1, 4, 2, 5, 3),

(1, 4)(2, 3),

(1, 5, 3, 4),

(1, 2, 3, 4, 5),

(2, 4, 5, 3),

Id(N),

(1, 2)(3, 5),

(2, 5)(3, 4),

(2, 3, 5, 4).

First we are going to obtain additional relations by conjugating all the previous relations of N and the following is the resulting list:

$(2, 3, 5, 4)[2, 4, 5, 3, 2], \sim (2354)24 = 235$

$(1, 5, 3, 4)[3, 5, 1, 4, 3], \sim (1534)35 = 341$

$(1, 2, 4, 3)[1, 3, 4, 2, 1], \sim (1243)13 = 124$

$(1, 4, 5, 2)[4, 1, 2, 5, 4], \sim (1452)41 = 452$

$(2, 3, 5, 4)[4, 5, 3, 2, 4], \sim (2354)45 = 423$

$(1, 5, 3, 4)[4, 3, 5, 1, 4], \sim (1534)43 = 415$

$(1, 4, 5, 2)[5, 4, 1, 2, 5], \sim (1452)54 = 521$

$(2, 3, 5, 4)[3, 2, 4, 5, 3], \sim (2354)32 = 354$

$(1, 3, 2, 5)[1, 5, 2, 3, 1], \sim (1325)15 = 132$

$(1, 3, 2, 5)[5, 2, 3, 1, 5], \sim (1325)52 = 513$

$(1, 4, 5, 2)[1, 2, 5, 4, 1], \sim (1452)12 = 145$

$(1, 3, 2, 5)[2, 3, 1, 5, 2], \sim (1325)23 = 251$

$(1, 5, 3, 4)[5, 1, 4, 3, 5], \sim (1534)51 = 534$

$(1, 2, 4, 3)[3, 4, 2, 1, 3], \sim (1243)34 = 312$

$(1, 5, 3, 4)[1, 4, 3, 5, 1], \sim (1534)14 = 153$

$(1, 3, 2, 5)[3, 1, 5, 2, 3], \sim (1325)31 = 325$

$(1, 2, 4, 3)[4, 2, 1, 3, 4], \sim (1243)42 = 431$

$(1, 4, 5, 2)[2, 5, 4, 1, 2], \sim (1452)25 = 214$

$(2, 3, 5, 4)[5, 3, 2, 4, 5], \sim (2354)53 = 542$

$(1, 2, 4, 3)[2, 1, 3, 4, 2], \sim (1243)21 = 243$

We now perform the double coset enumeration.

1st Double Coset $[*] = NeN$

Point Stabilizer on N^* conjugate by the elements in N. Also note that that Coset Stabilizer of $N^{(*)} = N^*$ since $|N^{(*)}| \geq |N^*| = 20$

So the number of single right cosets in NeN is $= \frac{|N|}{|N|} = 1$.

This tells us that there is a single Coset in NeN

Orbits are found by conjugating each of the elements of N by the following: (1^N). So we obtain the orbit of N on (1,2,3,4,5) are (1,2,3,4,5). Now we select a representative from the single orbit say 1. So $Nt_1 \in [1] \implies$ all of the five ti's extend to the same double coset which we will call [1].
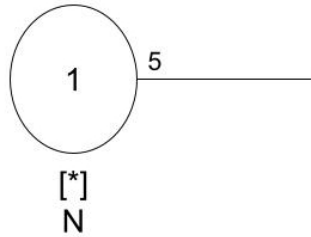
2nd Double Coset $[1] = Nt_1N$

Point Stabilizer on $N^1 = \{(2,3,5,4), e, (2,5)(3,4), (2,4,5,3)\}$

Coset Stabilizer on $N^{(1)} = N^1$ since $|N^{(1)}| \geq |N^1| = 4$

So the number of single right cosets in Nt_1N is $= \frac{|N|}{|N^{(1)}|} = \frac{20}{4} = 5$.

This tells us that there are 5 single Cosets in Nt_1N.

Now the Orbits on $N^{(1)}$ are $\{1\}$ and $\{2,3,4,5\}$. Then we select a representative of the orbit say 1 and $1 \in \{1\} \implies Nt_1t_1 \in [*] \implies NeN \in [*]$

Then we select 2 and $2 \in \{2,3,4,5\} \implies Nt_1t_2 \in [12]$.

So we have a new double coset [12].
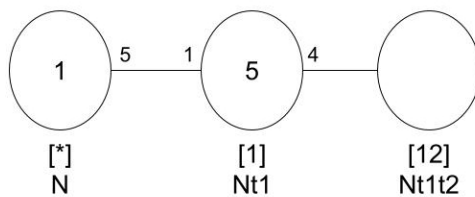
3rd Double Coset $[12] = Nt_1t_2N$

Point Stabilizer on $N^{12} = \{e\}$

Coset Stabilizer on $N^{(12)} = N^{12}$ since $|N^{(12)}| \geq |N^{12}| = 1$

So the number of single right cosets in Nt_1t_2N is $= \frac{|N|}{|N^{(12)}|} = \frac{20}{1} = 20$.

Now the orbits of the coset stabilizer $N^{(12)}$ are $\{1\}$ and $\{2\}\{3\}\{4\}\{5\}$. To be able to find each distinct coset we have to find all the right cosets of N^{12} in N Right cosets of N^{12} are

$Nt_1t_2(1,3)(4,5), Nt_1t_2(1,4,5,2), Nt_1t_2(1,3,2,5), Nt_1t_2(1,4,3,5), Nt_1t_2(1,2,4,3), Nt_1t_2(1,3,4,2),$

$Nt_1t_2(1,3,5,2,4), Nt_1t_2(1,2,5,4), Nt_1t_2(1,5,2,3), Nt_1t_2(1,5,4,3,2), Nt_1t_2(1,5)(2,4),$

$Nt_1t_2(1,4,2,5,3), Nt_1t_2(1,4)(2,3), Nt_1t_2(1,5,3,4), Nt_1t_2(1,2,3,4,5), Nt_1t_2(2,4,5,3),$

$Nt_1t_2(1,2)(3,5), Nt_1t_2(2,5)(3,4), Nt_1t_2(2,3,5,4)$.

Then we can get the transversal by taking an element representative from each coset.

So we get $T_{12} = (1,3)(4,5), (1,4,5,2), (1,3,2,5), (1,4,3,5), (1,2,4,3),$

$(1,3,4,2), (1,3,5,2,4), (1,5,2,3), (1,5,4,3,2), (1,4,2,5,3),$

$(1,5,3,4), (1,2,3,4,5), (2,4,5,3), (2,3,5,4)$.

Now we can find the double cosets by conjugating Nt_1t_2 by the transversals. So we get the following:

$Nt_1t_2N = \{Nt_2t_3, Nt_2t_4, Nt_5t_1, Nt_3t_1, Nt_1t_2, Nt_3t_2, Nt_3t_5, Nt_4t_3, Nt_5t_2, Nt_1t_3, Nt_3t_4,$

$Nt_4t_1, Nt_1t_5, Nt_2t_5, Nt_4t_5, Nt_5t_3, Nt_5t_4, Nt_4t_2, Nt_1t_4, \}$ Note: that the above will help in identifying which cosets are equal by the relation.

Then we select a representative of the orbit:

First we select 1 and $1 \in \{1\} \implies Nt_1t_2t_1 \in [121]$ and this is a new double coset $[121]$.

Second we select 2 and $2 \in \{2\} \implies Nt_1t_2t_2 \in [1]$ and this goes back to $[1]$.

Third we select 3 and $3 \in \{3\} \implies Nt_1t_2t_3 \in [123]$ and creates a new double coset $[123]$.

Fourth we select 4 and $4 \in \{4\} \implies Nt_1t_2t_4$. For this one lets look the relations we found in the beginning:

$Nt_2t_4 = Nt_2t_3t_5 \implies (Nt_2t_4)^{-1} = (Nt_2t_3t_5)^{-1} \implies Nt_1t_3 = Nt_1t_2t_4$ where $Nt_1t_3 \in [12] \implies Nt_1t_2t_4 \in [12]$.

Fifth we select 5 and $5 \in \{5\} \implies Nt_1t_2t_5$. For this one lets look the relations we found in the beginning:

$Nt_2t_4 = Nt_2t_3t_5 \implies (Nt_2t_4)^{(214)} = (Nt_2t_3t_5)^{(214)} \implies Nt_1t_2 = Nt_1t_3t_5$ Now multiply by $t_5 \implies Nt_1t_2t_5 = Nt_1t_3$ where $Nt_1t_3 \in [12] \implies Nt_1t_2t_5 \in [12]$.
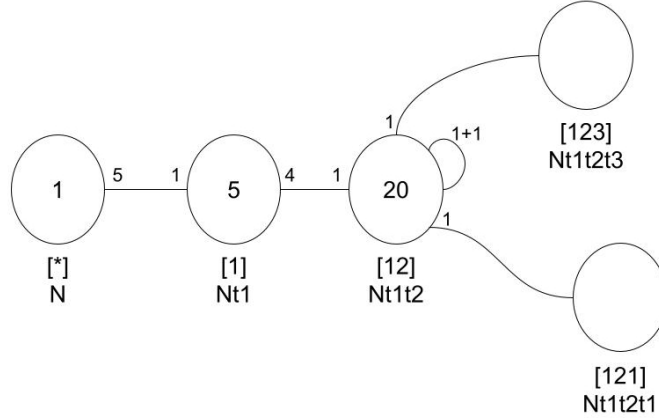
4th Double Coset $[121] = Nt_1t_2t_1N$

Point Stabilizer on $N^{121} = \{e\}$

Coset Stabilizer on $N^{(121)} = N^{121}$ since $|N^{(121)}| \geq |N^{121}| = < e >$.

Now we should look at our relations from the beginning specifically we are going to look at $(1452)t_1t_2 = t_1t_4t_5$. We can multiply by a t_5 on the left sides $\implies t_5(1452)t_1t_2 = t_5t_1t_4t_5$ note: $(1534)t_5t_1 = t_5t_3t_4$ so we can rewrite $t_5(1452)t_1t_2 = t_5t_1t_4t_5$ into

$(1452)t_2t_1t_2 = (1534)t_5t_3t_4t_4t_5$

$(1452)t_2t_1t_2 = (1534)t_5t_3t_5$

$(1452)t_2t_1t_2 = (1534)t_5t_3t_5$ (Now we are going to conjugate) $\implies (1452)(1452)^{-1}t_2t_1t_2 = (1534)(1534)^{-1}t_5t_3t_5 \implies t_2t_1t_2^{(1452)} = t_5t_3t_5^{(1534)}$

$Nt_1t_2t_1 = Nt_3t_5t_3$.

Now we can say that $Nt_1t_2t_1 = Nt_3t_5t_3^{(1,3,2,5)}$ and this implies that $N^{(121)} \geq N^{121} = < (1,3,2,5) >$.

So $N^{121} = \{e, (1325), (12)(35), (1523)\}$.

So the number of single right cosets in $Nt_1t_2t_1N$ is $= \frac{|N|}{|N^{(121)}|} = \frac{20}{5} = 5$.

So Orbits on N^{121} are $\{4\}$ and $\{1,2,3,5\}$. This gives us five distinct single cosets in $[121]$. Then we can get the transversal by taking an element representative from each coset.

So we get $T_{121} = e, (1, 2, 3, 4, 5), (1, 4, 3, 5), (1, 4, 2, 5, 3), (1, 3, 5, 2, 4)$

So we can get the distinct double cosets:

$Nt_1t_2t_1 = Nt_3t_5t_3 = Nt_2t_1t_2 = Nt_5t_3t_5,$

$Nt_2t_3t_2 = Nt_4t_1t_4 = Nt_3t_2t_3 = Nt_1t_4t_1,$

$Nt_4t_2t_4 = Nt_5t_1t_5 = Nt_2t_4t_2 = Nt_1t_5t_1,$

$Nt_4t_5t_4 = Nt_1t_3t_1 = Nt_5t_4t_5 = Nt_3t_1t_3,$

$Nt_3t_4t_3 = Nt_5t_2t_5 = Nt_4t_3t_4 = Nt_2t_5t_2.$

Then we select a representative of the orbit:

First we select 1 and $1 \in \{1, 2, 3, 4, 5\} \implies Nt_1t_2t_1t_1 \implies Nt_1t_2 \in [12]$.

Second we select 4 and $4 \in \{4\} \implies Nt_1t_2t_1t_4$. For this one lets look the relations we found in the beginning:

$(1452)t_2t_5 = t_2t_1t_4$ then we can multiply by t_1 on left side $\implies t_1(1452)t_2t_5 = t_1t_2t_1t_4 \implies t_4t_2t_5 = t_1t_2t_1t_4$.

$Nt_4t_2t_5 = Nt_1t_2t_1t_4$ From the next double coset of [123] we will show that $Nt_1t_2t_1t_4 \in [123]$.
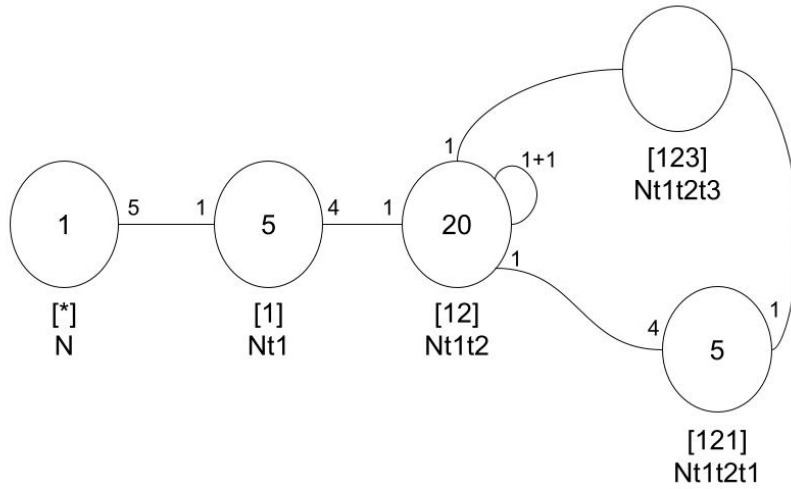
5th Double Coset $[123] = Nt_1t_2t_3N$

Point Stabilizer on $N^{123} = \{e\}$

Coset Stabilizer on $N^{(123)} = N^{123}$ since $|N^{(123)}| \geq |N^{123}| = <e>$.

Now we should look at our relations from the beginning specifically we are going to look at:

$(1243)t_1t_3 = t_1t_2t_4 \implies (1243)t_1t_3 = t_1(2354)t_2t_3t_5$ because $(2354)t_2t_4 = t_2t_3t_5$

Multiply by t_5 on right side and rewrite $(1243)t_1t_3t_5 = (2354)t_1t_2t_3t_5t_5$

$(1243)t_1t_3t_5 = (2354)t_1t_2t_3$

$Nt_1t_3t_5 = Nt_1t_2t_3.$

Now we can say that $Nt_1t_2t_3 = Nt_1t_3t_5^{(2,3,5,4)}$ and this implies that $N^{(123)} \geq N^{123} =< (2,3,5,4) > .$

So, N^{123} has 4 elements $= \{e, (2354), (25)(34), (2453)\}.$

So, the number of single right cosets in $Nt_1t_2t_3N$ is $= \frac{|N|}{|N^{(123)}|} = \frac{20}{4} = 5.$

Orbits on N^{123} are $\{1\}$ and $\{2,5,3,4\}$. This gives us five distinct single cosets in $[123]$.

Then we can get the transversal by taking an element representative from each coset.

So we get $T_{123} = \{e, (1,2,3,4,5), (1,5,4,3,2), (1,3,4,2), (1,4,2,5,3)\}.$

So we can get the distinct double cosets:

$Nt_1t_2t_3 = Nt_1t_3t_5 = Nt_1t_5t_4 = Nt_1t_4t_2,$

$Nt_2t_3t_4 = Nt_2t_4t_1 = Nt_2t_1t_5 = Nt_2t_5t_3,$

$Nt_5t_1t_2 = Nt_5t_2t_4 = Nt_5t_4t_3 = Nt_5t_3t_1,$

$Nt_3t_1t_4 = Nt_3t_4t_5 = Nt_3t_5t_2 = Nt_3t_2t_1,$

$Nt_4t_5t_1 = Nt_4t_1t_3 = Nt_4t_3t_2 = Nt_4t_2t_5.$

Now to prove what we left off in the previous double coset $[121]$ we are going to look at the relation: $Nt_4t_5t_1$

$(Nt_1t_2t_3)^{(1,2,3,4,5)^{-2}} = Nt_4t_5t_1 \implies Nt_4t_5t_1 = Nt_4t_2t_5$(from the above)

$\implies Nt_4t_2t_5 = Nt_1t_2t_1t_4 \in [123].$

Then we select a representative of the orbit:

First we select 1 and $1 \in \{1\} \implies Nt_1t_2t_3t_1.$

So we have to look at the relation $(1325)t_2t_3 = t_2t_5t_1.$

We can multiply by t_5 on the left side: $t_5(1325)t_2t_3 = t_5t_2t_5t_1$

$(1325)t_1t_2t_3 = t_5t_2t_5t_1.$

We can multiply by t_1 on the right side: $(1325)t_1t_2t_3t_1 = t_5t_2t_5t_1t_1$

$(1325)t_1t_2t_3t_1 = t_5t_2t_5$

$Nt_1t_2t_3t_1 = t_5t_2t_5$.

We can look at the relations from the double coset $[121] \implies Nt_3t_4t_3 = Nt_5t_2t_5 = Nt_4t_3t_4 = Nt_2t_5t_2$.

Looking at the fourth coset $[121]$ we get $(Nt_1t_2t_1)^{(1,2,3,4,5)2} \implies Nt_3t_4t_3$ and we know that $Nt_3t_4t_3 = Nt_5t_2t_5$ so we have that $Nt_5t_2t_5 = Nt_1t_2t_3t_1 \in [121]$.
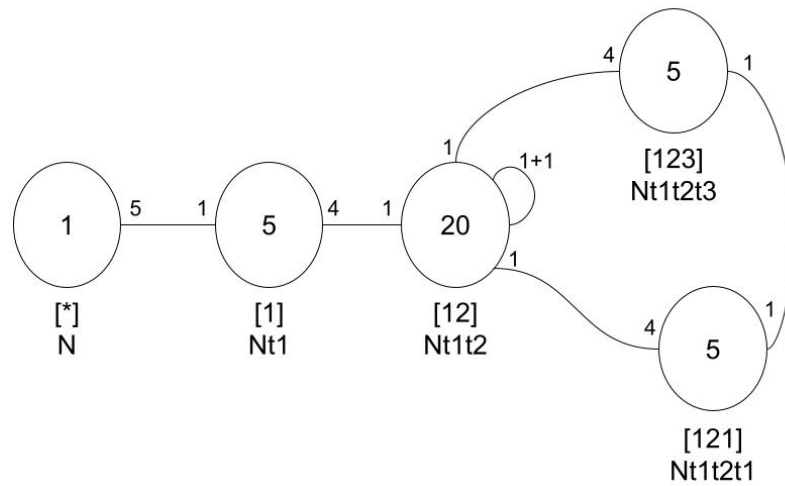
Figure 2.5: Cayley Diagram of S_6 over 5:4

2.6 Construction of $PSL(2, 11) \times 2$

We perform double coset enumeration of PSL (2,11) x 2 over A_5

Let $G = \frac{2^{*6}:A_5}{((x^{-1}*y)^2*t)^4}$. We will show $G \cong A_5$.

Expanding my relation: $((x^{-1} * y)^2 * t)^4$

Using a shortcut we can expand the relation:

$= ((x^{-1} * y)^2)^4 * t^{((x^{-1}*y)^2)^3} * t^{((x^{-1}*y)^2)^2} * t^{(x^{-1}*y)^2} * t$

Let $z = (x^{-1} * y)^2 = (16325)$

Rewrite: $z^4 * t_1^{z^3} * t_1^{z^2} * t_1^z * t_1$

$= (15236)t_2 t_3 t_6 t_1 \implies (15236)t_2 t_3 = t_1 t_6$

Rewrite in terms of x: $(yx)^2 t_2 t_3 = t_1 t_6$.

$$(yx)^2 t_2 t_3 = t_1 t_6 \tag{2.9}$$

Let
$x = (1, 4, 3)(2, 5, 6),$
$y = (1, 4)(2, 3)$ and let $N = <x, y> \cong A_5$.
and t $\sim t_1$.
Our Group N has the 60 elements; namely,
$N =$
$(1, 4, 3)(2, 5, 6), (1, 4)(2, 3), (1, 3, 4)(2, 6, 5), (2, 5, 6, 3, 4), (1, 3, 5, 6, 2),$
$(2, 4, 3, 6, 5),$
$(1, 2, 6, 5, 3),$
$(1, 4, 5, 2, 6),$
$(1, 3)(4, 6),$
$(1, 2, 4)(3, 5, 6),$
$(1, 4, 2)(3, 6, 5),$
$(1, 5)(3, 4),$

$(1, 6, 2, 5, 4),$

$(2, 6, 4, 5, 3),$

$(1, 2, 3, 4, 6),$

$(1, 5, 2, 3, 6),$

$(1, 6, 4, 3, 2),$

$(1, 3, 2, 4, 5),$

$(2, 3, 5, 4, 6),$

$(1, 5, 4, 2, 3),$

$(1, 6, 3, 2, 5),$

$(1, 4, 6, 3, 5),$

$(1, 3, 6)(2, 5, 4),$

$(1, 5, 6, 4, 2),$

$(1, 6, 3)(2, 4, 5),$

$(1, 5, 3, 6, 4),$

$(1, 6)(2, 4),$

$(1, 2)(4, 5),$

$(1, 6, 2)(3, 4, 5),$

$(1, 2, 4, 6, 5),$

$(1, 2, 6)(3, 5, 4),$

$(1, 2, 5)(3, 6, 4),$

$(1, 5, 6)(2, 4, 3),$

$(1, 2, 3)(4, 5, 6),$

$(1, 5, 2)(3, 4, 6),$

$(1, 5, 3)(2, 4, 6),$

$(1, 6, 5)(2, 3, 4),$

$(1, 3, 2)(4, 6, 5),$

$(1, 3, 5)(2, 6, 4),$

$(1, 5, 4)(2, 6, 3),$

$(1, 6)(3, 5),$

$(1, 6, 4)(2, 3, 5),$

$(1, 2)(3, 6),$

$(1, 3, 4, 5, 6),$

$(1, 6, 5, 4, 3),$

$(2, 4)(3, 5),$

$(1, 4, 5)(2, 3, 6),$

$(3, 6)(4, 5),$

$(1, 4, 6)(2, 5, 3),$

$(1, 5)(2, 6),$

$(1, 3, 6, 2, 4),$

$(2, 5)(4, 6),$

$(1, 4, 2, 6, 3),$

$(1, 2, 5, 3, 4),$

$(1, 4, 3, 5, 2),$

$(2, 3)(5, 6),$

$(2, 6)(3, 4),$

$(1, 3)(2, 5),$

$(1, 4)(5, 6).$

We first prove the following lemmas.

Lemma 1: $Nt_1t_2 = Nt_3t_5$.

$(yx)^2 t_2 t_3 = t_1 t_6$ (1)

$(15236)t_2 t_3^{(132)(465)} = t_1 t_6^{(132)(465)}$

$(34125)t_1 t_2 = t_3 t_5.$

Therefore $Nt_1t_2 = Nt_3t_5$.

Lemma 1.1: $Nt_2t_1 = Nt_3t_4$.

$(yx)^2 t_2 t_3 = t_1 t_6$ (1)

$(15236)t_2 t_3^{(13)(46)} = t_1 t_6^{(13)(46)}$

$(35214)t_2 t_1 = t_3 t_4$

$t_2 t_1 = (41253)t_3 t_4.$

Therefore $Nt_2t_1 = Nt_3t_4$.

Lemma 1.2: $Nt_2t_3 = Nt_4t_5$.

$(yx)^2 t_2 t_3 = t_1 t_6$ (1)

$(15236)t_2 t_3^{(14)(56)} = t_1 t_6^{(14)(56)}$

$(46235)t_2 t_3 = t_4 t_5$

$t_2 t_3 = (53264)t_4 t_5$.

Therefore $Nt_2 t_3 = Nt_4 t_5$.

Lemma 2: $Nt_1 t_2 t_1 = Nt_4 t_5 t_4$.

Consider 121:

$\mathbf{121} = 1(41253)34$ (By Lemma 1.1)

$= (41253)234$

$= (41253)\mathbf{23}4 \implies (41253)(53264)454$ (By Lemma 1.2)

$= (164)(235)454$.

Therefore $Nt_1 t_2 t_1 = Nt_4 t_5 t_4$.

We now perform the double coset enumeration.

1st Double Coset $[*] = NeN$

$$[*] = NeN = \{N(e)^n | n \in N\}$$

Now we plug in each element that is in N.

The coset stabilizer of the coset Ne is N.

The number of single right cosets in NeN is $= \frac{|N|}{|N|} = \frac{60}{60} = 1$.

Now the orbits of N on $X = \{1, 2, 3, 4, 5, 6\}$ are $\{1, 2, 3, 4, 5, 6\}$.

Now we select a representative from the single orbit say 1. So $Nt_1 \in [1] \implies$ all of the

six ti's extend to the same double coset which we will call [1].
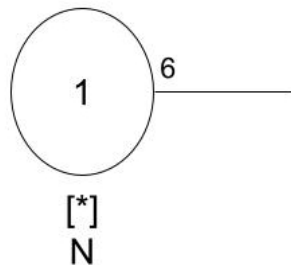
2nd Double Coset $[1] = Nt_1N$

$$[1] = Nt_1N = \{N(t_1)^n | n \in N\}$$

Now we plug in each element that is in N

$$= \{N(t_1)^e, N(t_1)^{(1,4,3)(2,5,6)}, N(t_1)^{(1,4)(2,3)}, N(t_1)^{(1,3,4)(2,6,5)}...N(t_1)^{(1,4)(5,6)}\}$$

$$= \{Nt_1, Nt_2, Nt_3, Nt_4, ...Nt_6\}.$$

Point Stabilizer on $N^1 = \{e, (3,6)(4,5), (2,4,3,6,5), (2,6)(3,4), (2,6,4,5,3),$

$(2,5,6,3,4), (2,4)(3,5), (2,3)(5,6), (2,3,5,4,6), (2,5)(4,6)\} = <(3,6)(4,5), (2,4,3,6,5)> \cong$

D_5.

Coset Stabilizer $N^{(1)} \geq N^1$ and $N^{(1)} = N^1$

So the order of $N^{(1)}$ is 10, because D_5 is of order 10.

The number of single right cosets in Nt_1N is $= \frac{|N|}{|N^{(1)}|} = \frac{60}{10} = 6$.

Now the orbits of N on X are $\{1\}, \{2, 4, 6, 5, 3\}$.

Now we select a representative from each orbit, say 1 and 2 and determine the double coset of Nt_1t_1 and Nt_1t_2.

We have,

$Nt_1t_1 = N \in [*]$,

$Nt_1t_2 \in Nt_1t_2N = [12]$.
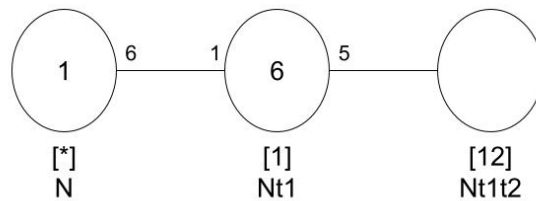
3rd Double Coset $[12] = Nt_1t_2N$

$$[12] = Nt_1t_2N = \{N(t_1t_2)^n | n \in N\}$$

$$= \{N(t_1t_2)^e, N(t_1t_2)^{(1,4,3)(2,5,6)}, N(t_1t_2)^{(1,4)(2,3)}, N(t_1t_2)^{(1,3,4)(2,6,5)}...N(t_1t_2)^{(1,4)(5,6)}\}$$

$= \{Nt_1t_2, Nt_4t_5, Nt_4t_3, Nt_3t_6...Nt_4t_2\}.$

Point Stabilizer $N^{12} = \{e, (36)(45)\}$

Coset Stabilizer $N^{(12)} \geq \{N^{12}, (16)(24), (163)(245), (13)(25), (136)(254)\}.$

We have $Nt_3t_5 = Nt_1t_2$ (By Lemma 1) and $Nt_1t_2^{(1,3)(2,5)} = Nt_3t_5.$

So $Nt_1t_2^{(1,3)(2,5)} = Nt_3t_5 = Nt_1t_2$

$\implies (1,3)(2,5) \in N^{12}$

So the order of $N^{(12)}$ is 6.

The number of single right cosets in Nt_1t_2N is $= \frac{|N|}{|N^{(12)}|} = \frac{60}{6} = 10.$

Now the orbits of the coset stabilizer $N^{(12)}$ on X are $\{1, 6, 3\}$, and $\{2, 4, 5\}.$

Now we select a representative from each orbit, say 1 and 2 and determine the double cosets of $Nt_1t_2t_1$ and $Nt_1t_2t_2.$

We have,

$Nt_1t_2t_1 \in Nt_1t_2t_1 = [121],$

$Nt_1t_2t_2 \in [1].$
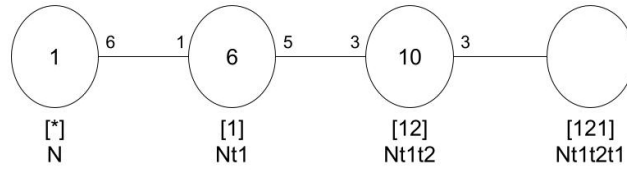
4th Double Coset $[121] = Nt_1t_2t_1N$

$[121] = Nt_1t_2t_1N = \{N(t_1t_2t_1)^n | n \in N\}$

Now we plug in each element that is in N

$= \{N(t_1t_2t_1)^e, N(t_1t_2t_1)^{(1,4,3)(2,5,6)}, N(t_1t_2t_1)^{(1,4)(2,3}, N(t_1t_2t_1)^{(1,3,4)(2,6,5)}...N(t_1t_2t_1)^{(1,4)(5,6)}\}$

$= \{Nt_1t_2t_1Nt_4t_5t_4, Nt_4t_3t_4, Nt_3t_6t_3...Nt_4t_2t_4\}.$

Point Stabilizer on $N^{121} = \{e, (36)(45)\}$

Coset Stabilizer on $N^{(121)} \geq < N^{121}, (1,5,6)(2,4,3), (1,2)(3,6), (1,6,5)(2,3,4), (1,3,5)(2,6,4),$

$(1,4,3)(2,5,6), (1,2)(4,5), (1,5,3)(2,4,6), (1,3,4)(2,6,5),$

$(1,6,4)(2,3,5), (1,4,6)(2,5,3) > .$

We have $Nt_1t_2t_1 = Nt_4t_5t_4$ (By Lemma 2).

Now $Nt_1t_2t_1^{(1,4,3)(2,5,6)} = Nt_4t_5t_4$

So $Nt_1t_2t_1^{(1,4,3)(2,5,6)} = Nt_4t_5t_4$

$\implies (1,4,3)(2,5,6) \in N^{121}$.

The order of $N^{(121)}$ is equal to 12.

The number of single right cosets in $Nt_1t_2t_1N$ is $= \frac{|N|}{|N^{(121)}|} = \frac{60}{12} = 5$.

Now the orbits of N on X are $\{1,4,5,3,6,2\}$.

Now we select a representative from each orbit, say 1 and determine the double cosets of $Nt_1t_2t_1t_1$,

We have,

$Nt_1t_2t_1t_1 \in Nt_1t_2 = [12]$.
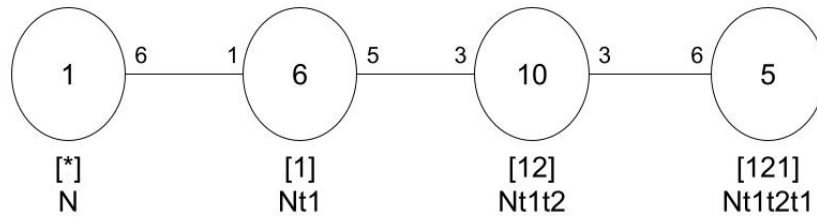
Figure 2.6: Cayley Diagram of $PSL(2,11) \times 2$ over A_5

2.7   Construction of PSL (2,11)

We perform double coset enumeration of PSL(2,11) over A_5.

G= $\frac{2^{*}10:A_5}{t_1t_2t_1(12)(37)(46)(59),t_1t_3t_7=(1,2,3,8,7)(4,10,6,5,9)}$. We will show that $G \cong PSL(2,11)$.

Expanding my given relation: $(x*y*x*y^-1*x*y*x*y^-1*x*y*t)^3$

Let $x*y*x*y^-1*x*y*x*y^-1*x*y = q$

$=q^3*t^{q^2}*t^q*t$

$=(1,2)(3,7)(4,6)(5,9)*t^e*t^{(1,2)(3,7)(4,6)(5,9)}*t.$

$\implies Nt_1t_2t_1q=e.$

$$Nt_1t_2t_1*q=e \tag{2.10}$$

Expanding my given relation: $(y*x*y*t^(x*y^-1))^3$

$=(y*x*y)^3*t_7^{(y*x*y)^2}*t_7^{(y*x*y)^3}$

Let $(y*x*y)^3 = r$

$\implies rt_1t_3t_7=e$

$$rt_1t_3t_7=e. \tag{2.11}$$

Let

$x=(1,2)(3,5)(7,9)(8,10),$

$y=(1,3,6)(2,4,7)(5,8,9)$ and let $N=<x,y>\cong A_5.$

and t $\sim t_1$.

We perform double coset enumeration of G over N.

Our Group N has the 10 elements; namely,

$N=$

$(1,3,6)(2,4,7)(5,8,9) \sim y,$

$(1,2)(3,5)(7,9)(8,10) \sim x,$

$(1,4,7,5,6)(2,3,8,10,9) \sim xy,$

$(1,3,6)(2,4,7)(5,8,9) \sim x^2*y,$

$(1,2)(3,5)(7,9)(8,10) \sim x^3,$

$(1,4,7,5,6)(2,3,8,10,9) \sim x^3y,$

$(1, 3, 6)(2, 4, 7)(5, 8, 9) \sim x^4 y,$

$(1, 2)(3, 5)(7, 9)(8, 10) \sim x^5,$

$(1, 4, 7, 5, 6)(2, 3, 8, 10, 9) \sim x^5 y.$

We first prove the following lemmas.

Lemma 1: $Nt_1 t_3 = Nt_4 t_2$.

$x^8 t_3 t_1 t_5 t_3 = e$ (1)

$x^8 t_3 t_1 t_5 t_3^{(1,3)(4,6)} = x^4 t_1 t_3 t_5 t_1$

$= x^4 t_1 t_3 = t_1^{-1} t_5^{-1}$

$\implies x^4 t_1 t_3 = t_4 t_2$

So $Nt_1 t_3 = Nt_4 t_2$.

Lemma 2 : $Nt_1 t_2 t_2 = Nt_4 t_6$.

$x^8 t_3 t_1 t_5 t_3 = e$ (1)

$x^8 t_3 t_1 t_5 t_3^{(1,6)(2,5)(3,4)} = x^4 t_4 t_6 t_2 t_4$

$\implies x^4 t_4 t_6 t_2 t_4 = e$ Rewrite: $x^4 t_4 t_6 = t_4^{-1} t_2^{-1}$ (Note: $t_4^{-1} = t_1, t_2^{-1} = t_5$)

$\implies x^4 t_4 t_6 = t_1 t_5$ (Note: $t_5 = t_2 t_2$)

So $Nt_1 t_2 t_2 = Nt_4 t_6$.

1st Double Coset $[*] = NeN$

$$[*] = NeN = \{N(e)^n | n \in N\}$$

Now we plug in each element that is in N.

$= \{N(e)^e, N(e)^{(1,2)(3,5)(7,9)(8,10)}, N(e)^{(1,3,6)(2,4,7)(5,8,9)}, N(e)^{(1,6,3)(2,7,4)(5,9,8)} ... N(e)^{(1,2)(3,7)(4,6)(5,9)}\}$

This will just give us N and the coset stabilizer of $N = N$.

The number of single right cosets in NeN is $= \frac{|N|}{|N|} = 1$.

Now the orbits of N on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Now we select a representative from the single orbit say 1. So $Nt_1 \in [1] \implies$ all of the ten t_i's extend to the same double coset which we will call [1].
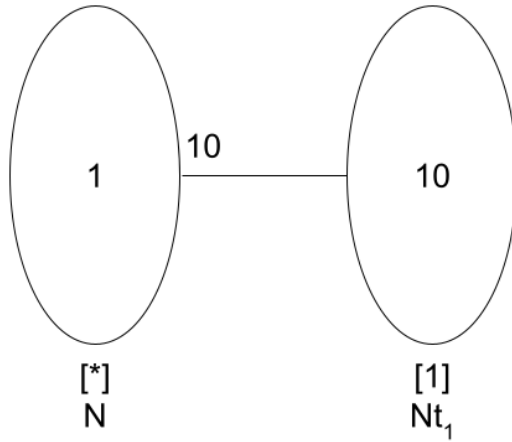
2nd Double Coset $[1] = Nt_1N$

$$[1] = Nt_1N = \{N(t_1)^n | n \in N\}$$

Finding the point stabilizer of Nt_1

Point Stabilizer on $N^1 = \{e, (2,5)(3,4)(6,8)(9,10), (3,9)(4,6)(5,7)(8,10)\}$

So $N^{(1)} = N^1 =< (2,5)(3,4)(6,8)(9,10), (3,9)(4,6)(5,7)(8,10) >$.

The number of single right cosets in Nt_1N is $= \frac{|N|}{|N^{(1)}|} = \frac{60}{6} = 10$.

Now the orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10\}$ are $\{1\}, \{2,7,5\}, \{3,6,8,10,9,4\}$.

Now we select a representative from each orbit, say 1, 2, 3 and determine the double cosets of $Nt_1t_1, Nt_1t_2, Nt_1t_3$.

We have,

$Nt_1t_1 \in N = [1]$,

$Nt_1t_2 \in Nt_1t_2 = [1]$,

$Nt_1t_3 \in Nt_1t_3 = [1]$.
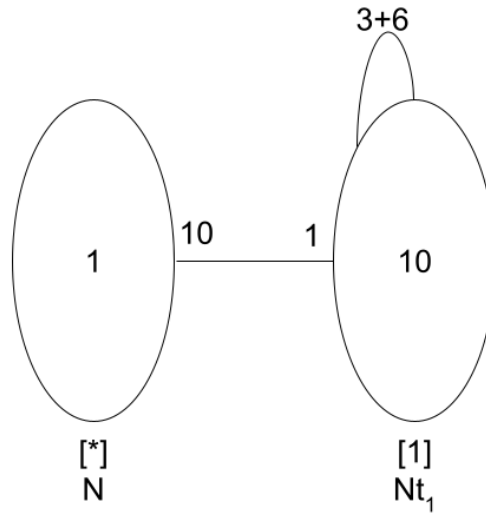
Figure 2.7: Cayley Diagram of PSL(2,11) over $2^*10 : A_5$

So $|G| \leq \frac{|N|}{|N|} + \frac{|N|}{|N^{(1)}|} = 1 + 10$

Then we multiply the result by the order of N to get the following: $11 * 60 = 660$.

2.8 Construction of G

We perform double coset enumeration of G over $PSL(2,7)$.

Let $G = \frac{2^*28:PSL(2,7)}{t_{11}t_{15}t_{24}=e,t_2t_1=e}$.

Expanding my given relation: $(yxyt^{(yx)^2})^3$

Let $q = yxy$ and $s = (yx)^2$

Note: $t^s = t_{11}$

Rewrite: $(qt^s)^3 \implies q^3 t_{11}^{q^3} t_{11}^{q^2} t_{11}^q$

$q^3 t_{11} t_{25} t_{24} = e \implies t_{11} t_{15} t_{24} = e.$

$$q^3 t_{11} t_{15} t_{24} = e \tag{2.12}$$

Expanding my given relation: $((yxy^{-1}xyxy^{-1})t)^2$

Let c $= yxy^{-1}xyxy^{-1}$

Rewrite: $c^2 t_2 t_1$

$\implies c^2 t_2 t_1 = e.$

$$t_2 t_1 = e \tag{2.13}$$

Let

$x = (1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)$,

$y = (1,3)(2,4,7,10)(5,8,12,17)(6,9,14,20)(11,16,13,18)(15,21,19,23)(22,...28)$ and let

$N = <x,y> \cong PSL(2,7)$.

and $t \sim t_1$.

In order to do this we must first perform the manual DCE of G over N.

We first prove the following lemmas.

Lemma 1: $Nt_1t_2 = Nt_2t_1$

Lemma 2 : $Nt_2 = Nt_1$

$t_2t_1 = e$ (2.12)

Inverse on both sides:

$t_2t_1t_1 = t_1$

So $t_2 = t_1$

1st Double Coset $[*] = NeN$

$$[*] = NeN = \{N(e)^n | n \in N\}$$

Now we plug in each element that is in N.

$= \{N(e)^e, N(e)^{(1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27)}$,

$N(e)^{(1,3)(2,4,7,10)(5,8,12,17)(6,9,14,20)(11,16,13,18)(15,21,19,23)(22,24,25,26)(27,28)}$,

$...N(e)^{(1,27)(2,4)(3,28)(6,26)(7,10)(8,17)(9,25)(11,16)(13,18)(14,24)(20,22)(21,23)}\}$. This will just give us

N and the coset stabilizer of $N = N$.

The number of single right cosets in NeN is $= \frac{|N|}{|N|} = 1$.

Now the orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,$

$22,23,24,25,26,27,28\}$ are all 28 ti's.

Now we select a representative from the single orbit say 1. So $Nt_1 \in [1] \implies$ all of the

28 ti's extend to the same double coset which we will call [1].
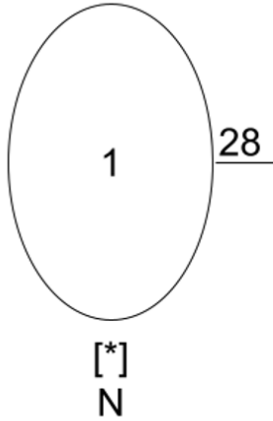
2nd Double Coset $[1] = Nt_1N$

$$[1] = Nt_1N = \{N(t_1)^n | n \in N\}$$

Finding the point stabilizer of Nt_1

Point Stabilizer on $N^1 = \{e, (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)(16,18)$
$(21,23)(22,25)(24,26)$ $(2,28)(3,20)(4,23)(5,26)(6,15)(10,25)(11,12)(13,24)$
$(14,18)(16,19)(17,27)(21,22)\}$.

So $N^{(1)} = N^1 = < (2,7)(4,10)(5,12)(6,14)(8,17)(9,20)(11,13)(15,19)(16,18)(21,23)$
$(22,25)(24,26)$ $(2,28)(3,20)(4,23)(5,26)(6,15)(10,25)(11,12)(13,24)(14,18)$
$(16,19)(17,27)(21,22) > .$

The number of single right cosets in Nt_1N is $= \frac{|N|}{|N^{(1)}|} = \frac{168}{6} = 28$.

Now the orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,$
$25,26,27,28\}$ are $\{1\}, \{2,7,28\}, \{3,20,9\}, \{8,17,27\}, \{4,10,23,25,21,22\}, \{5,12,26,11,24,13\},$
$\{6,14,15,18,19,16\}$.

Now we select a representative from each orbit, say 1, 2, 3,4,5,6,8 and determine the
double cosets of $Nt_1t_1, Nt_1t_2, Nt_1t_3, Nt_1t_8, Nt_1t_4, Nt_1t_5, Nt_1t_6$.

We have,

$Nt_1t_1 = N \in [*]$,

$Nt_1t_2 \in Nt_1t_2 = [*]$,

$Nt_1t_3 \in Nt_1t_3 = [1]$,

$Nt_1t_8 \in Nt_1t_8 = [1],$

$Nt_1t_4 \in Nt_1t_4 = [1],$

$Nt_1t_5 \in Nt_1t_5 = [1],$

$Nt_1t_6 \in Nt_1t_6 = [1].$
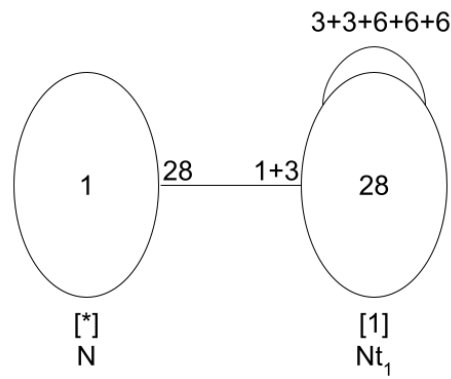
Figure 2.8: Cayley Diagram of $2^7 : PSL(2,7)$ over $PSL(2,7)$

So $|G| \leq \frac{|N|}{|N|} + \frac{|N|}{|N^{(1)}|} = 1 + 28.$

Then we multiply the result by the order of N to get the following: $29 * 168 = 4872.$

2.9 Construction of PSL (2,7) $\times 2^6$

We perform double coset enumeration of PSL (2,7) x 2^6 over A_7.

Let $G = \frac{2^*28:A_7}{t_1t_2t_1(12)(37)(46)(59),t_1t_3t_7=(1,2,3,8,7)(4,10,6,5,9)}$. We will show $G \cong A_7$.

Expanding my given relation: $(y * x * y * t^{x*y^2})^3$

Let $q = y * x * y$ and $s = x * y^2$

Note: $t^s = t_7$

Rewrite: $(qt^s)^3 \implies q^3 * t_7^{q^3} * t_7^{q^2} * t_7^q$

$q^3 t_7 t_{20} t_{21} = e \implies t_7 * t_{20} * t_{21} = e.$

$$q^3 t_7 * t_{20} * t_{21} = e \tag{2.14}$$

Expanding my given relation: $((y * x * y^-1 * x * y * x * y^-1) * t)^4$

Let c $= y * x * y^{-1} * x * y * x * y^{-1}$

Rewrite: $(ct)^4 \implies c^4 * t^{c^3} * t^{c^2} * t^c * t = e$

$\implies c^4 * t_2 t_1 t_2 t_1 = e.$

$$t_2 t_1 t_2 t_1 = e \tag{2.15}$$

Let

$x = (1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)(18,21)(23,25)(26,27),$

$y = (1,3)(2,4,7,10)(5,8,12,17)(6,9,14,20)(11,16,13,18)(15,21,19,23)(22,...)(27,28)$ and

let $N = < x, y > \cong A_7$.

and t $\sim t_1$.

In order to do this we must first perform the manual DCE of G over N.

We first prove the following lemmas.

Lemma 1: $Nt_1t_2 = Nt_2t_1.$

1st Double Coset $[*] = NeN$

$\qquad [*] = NeN = \{N(e)^n | n \in N\}$

Now we plug in each element that is in N.

Coset Stabilizer of the coset Ne is N

The number of single right cosets in NeN is $= \frac{|N|}{|N|} = 1$.

Now the orbits of N are on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,$
$19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$ and all 28 are going to the next coset.

Now we select a representative from the single orbit say 1. So $Nt_1 \in [1] \implies$ all of the
28 ti's extend to the same double coset which we will call [1].
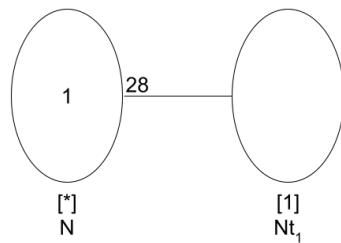
2nd Double Coset $[1] = Nt_1N$

$$[1] = Nt_1N = \{N(t_1)^n | n \in N\}$$

Point Stabilizer on $Nt_1 = N^1 = \{e, (3,9)(4,22)(5,13)(6,16)(7,28)(8,27)(10,21)(11,26)$

$(12,24)(14,19)(15,18)(23,25)\ (2,28)(3,20)(4,23)(5,26)(6,15)(10,25)$

$(11,12)(13,24)(14,18)(16,19)(17,27)(21,22)\}$.

Coset Stabilizer $N^{(1)} \geq N^1$

$N^{(1)} = N^1$.

The number of single right cosets in Nt_1N is $= \frac{|N|}{|N^{(1)}|} = \frac{168}{6} = 28$.

Orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,$

$26,27,28\}$ are $\{1\}, \{2,7,28\}, \{3,20,9\}, \{8,17,27\}, \{4,10,23,25,21,22\},$

$\{5,12,26,11,24,13\}, \{6,14,15,18,19,16\}$.

Now we select a representative from each orbit, say 1,2,3,8,4,5 and 6 and determine the
double coset of Nt_1t_1, Nt_1t_2, Nt_1t_3, Nt_1t_8, Nt_1t_4, Nt_1t_5, Nt_1t_6.

We have,

$Nt_1t_1 = N \in [*]$,

$Nt_1t_2 \in Nt_1t_2N = [12]$,

$Nt_1t_3 \in Nt_1t_3N = [127]$,

$Nt_1t_8 \in Nt_1t_8N = [124]$,

$Nt_1t_4 \in Nt_1t_4N = [1]$,

$Nt_1t_5 \in Nt_1t_5N = [1]$,

$Nt_1t_6 \in Nt_1t_6N = [12]$.
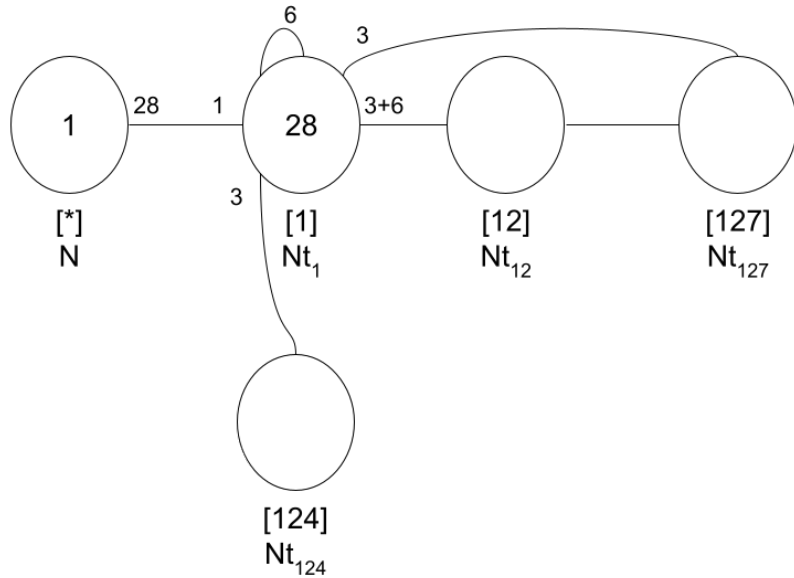
3rd Double Coset $[12] = Nt_1t_2N$

$$[12] = Nt_1t_2N = \{N(t_1t_2)^n | n \in N\}$$

Point Stabilizer on $N^{12} = \{e, (3,9)(4,22)(5,13)(6,16)(7,28)(8,27)(10,21)(11,26)(12,24)$
$(14,19)(15,18)(23,25)\}$.

Note: $Nt_1t_2 = Nt_2t_1$

$Nt_1t_2 = Nt_{17}t_{20}$

Coset stabilizer is $N^{(12)} \geq \{N^{12}, (1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)$
$(18,21)(23,25)(26,27),$

$(1,17)(2,20)(3,21)(4,22)(5,15)(7,25)(8,27)(9,10)(12,19)(13,18)(14,24)(23,28),$

$(1,2)(3,13)(4,16)(5,9)(6,22)(7,28)(8,26)(10,18)(11,27)(12,24)(15,21)(17,20),$

$(1,17)(2,20)(3,10)(5,18)(6,16)(7,23)(9,21)(11,26)(12,14)(13,15)(19,24)(25,28),$

$(3,9)(4,22)(5,13)(6,16)(7,28)(8,27)(10,21)(11,26)(12,24)(14,19)(15,18)(23,25)\}$.

The number of single right cosets in Nt_1t_2N has the order is $\frac{|N|}{|N^{(12)}|} = \frac{168}{2} = 84$.

Orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,$

$24, 25, 26, 27, 28\}$ are

$\{1, 2, 17, 20\}, \{4, 22, 6, 16\}, \{7, 28, 23, 25\}, \{8, 27, 11, 26\}, \{12, 24, 14, 19\}, \{3, 9, 5, 13, 10, 21, 18, 15\}$.

Now we select a representative from each orbit, say 1,4,7,8,12 and 3 and determine the

double cosets of $Nt_1t_2t_1, Nt_1t_2t_4, Nt_1t_2t_7, Nt_1t_2t_8, Nt_1t_2t_12, Nt_1t_2t_3$.

We have,

$Nt_1t_2t_2 = N \in [1]$,

$Nt_1t_2t_4 \in Nt_1t_8N = [124]$,

$Nt_1t_2t_7 \in Nt_1t_3N = [127]$,

$Nt_1t_2t_8 \in Nt_1t_2N = [12]$,

$Nt_1t_2t_{12} \in Nt_1t_2N = [12]$,

$Nt_1t_2t_3 \in Nt_1N = [1]$.
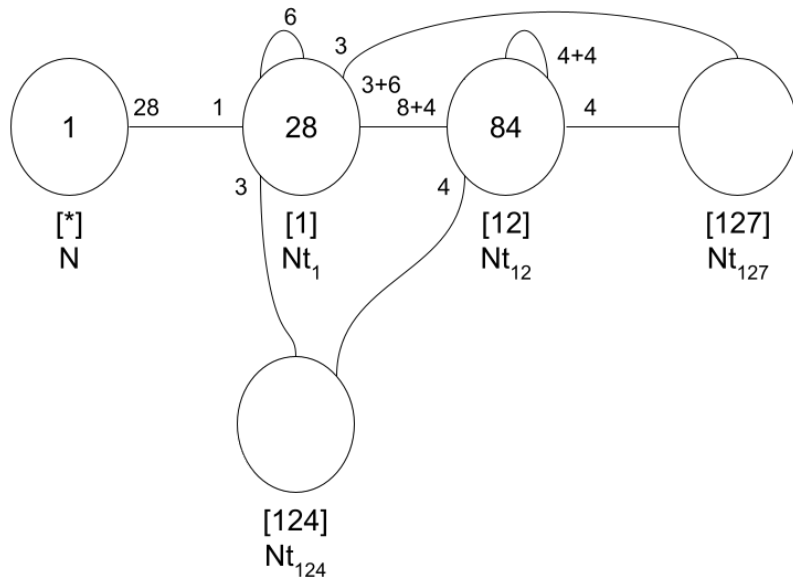
4th Double Coset $[127] = Nt_1t_2t_7N$

$$[127] = Nt_1t_2t_7N = \{N(t_1t_2t_7)^n | n \in N\}$$

Point Stabilizer on $N^{127} = \{e\}$

Note: $Nt_1t_2t_7 = Nt_2t_1t_7$

$Nt_1t_2t_7 = Nt_{19}t_{24}t_{14}$.

Coset stabilizer is $N^{(127)} \geq \{N^{127}, (1,2)(3,5)(4,6)(8,11)(9,13)(10,15)(14,19)(16,22)(17,20)$
$(18,21)(23,25)(26,27),$
$(1,19,27,15)(2,24,10,6)(3,5,28,12)(4,26,7,14)(8,23)(9,25,22,20)(11,13,18,16)(17,21)\}$.

The number of single right cosets in $Nt_1t_2t_7N$ is $= \frac{|N|}{|N^{(127)}|} = \frac{168}{24} = 7$.

Orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,$
$25,26,27,28\}$ are $\{3,5,28,12\}, \{1,2,19,24,14,27,10,4,26,15,6,7\},$
$\{8,11,23,13,25,9,18,22,21,16,20,17\}$.

Now we select a representative from each orbit, say 3,7 and 8 and determine the double
coset of $Nt_1t_2t_7t_3$, $Nt_1t_2t_7t_7$, and $Nt_1t_2t_7t_8$.

We have,

$Nt_1t_2t_7t_3 \in Nt_1t_2t_4N = [124]$,

$Nt_1t_2t_7t_7 \in Nt_1t_2N = [12]$,

$Nt_1t_2t_7t_8 \in Nt_1N = [1]$.
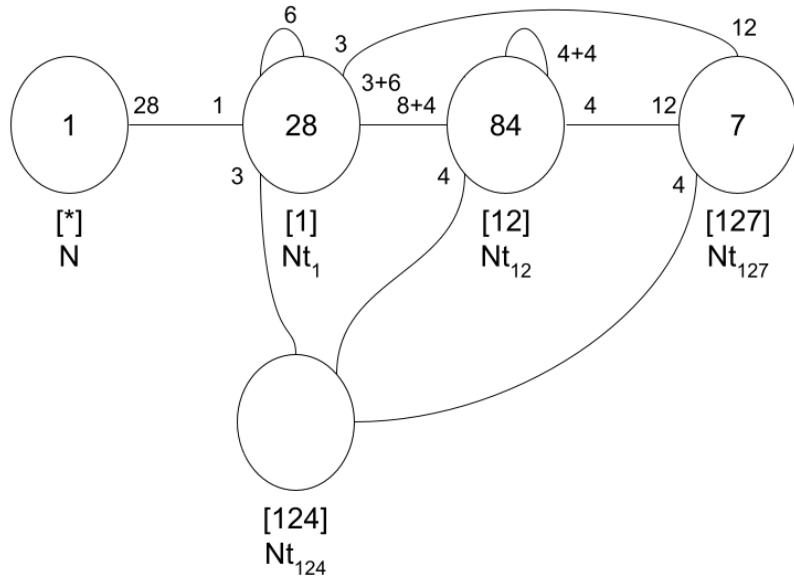
5th Double Coset $[124] = Nt_1t_2t_4N$

$$[124] = Nt_1t_2t_4N = \{N(t_1t_2t_4)^n | n \in N\}$$

Point Stabilizer on $N^{124} = \{e\}$

Note: $Nt_1t_2t_4 = Nt_6t_5t_{13}$

$Nt_1t_2t_4 = Nt_{17}t_{20}t_4$.

Coset stabilizer is $N^{(124)} \geq \{N^{124}, (1,17)(2,20)(3,10)(5,18)(6,16)(7,23)(9,21)(11,26)$

$(12,14)(13,15)(19,24)(25,28), (1,6,7,15)(2,5)(3,11,12,20)(4,13,19,17)(8,22,9,21)$

$(10,18,14,25)(16,24,23,27)(26,28)\}$.

The number of single right cosets in $Nt_1t_2t_4$ is $= \frac{|N|}{|N^{(124)}|} = \frac{168}{24} = 7$.

Orbits of N on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,$
$26,27,28\}$ are $\{8,22,9,21\}, \{1,6,17,7,16,4,15,23,24,13,27,19\},$
$\{2,5,20,18,3,14,11,10,25,12,26,28\}$.

Now we select a representative from each orbit, say 8,4 and 2 and determine the double

cosets of $Nt_1t_2t_4t_8$, $Nt_1t_2t_4t_4$ and $Nt_1t_2t_4t_2$.

We have,

$Nt_1t_2t_4t_8 \in Nt_1t_2t_7N = [127]$,

$Nt_1t_2t_4t_4 \in Nt_1t_2N = [12]$,

$Nt_1t_2t_4t_2 \in Nt_1N = [1]$.
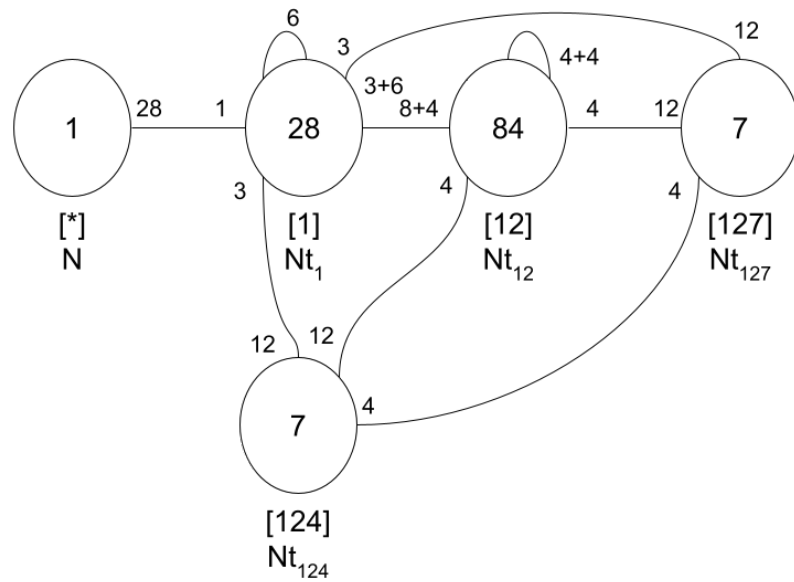
Figure 2.9: Cayley Diagram of PSL (2,7) x 2^6 over A_7

So $|G| \leq \frac{|N|}{|N|} + \frac{|N|}{|N^{(1)}|} = 1 + 28$

Then we multiply the result by the order of N to get the following: $29 * 168 = 4872$.

Chapter 3

Symmetric Presentations of Progenitors and Additional Relations

Let N $= S_4 =< xx, yy >$, where xx $=$ (1,2,3,4) and yy $=$ (1,2). We give a symmetric presentations of the progenitors $2^*4 : N$ below:

We first note that N has presentations

$< x, y | x^4, y^2, (x*y)^3 >$. Let $t \sim t_1$. The stabilizer of 1 in N $< xy, y^x > \cong S_3 < xy, y^x >$

Symmetric presentation for the progenitor is obtained by listing the above presentation of N. Followed by t^2 and then the relations that insure that t commutes with the stabilizer of 1. Thus,

$G =< x, y, t | x^4, y^2, (x*y)^3, t^2, (t, xy), (t, y^x) >$. G is an infinite group and we want to add relations called additional relations to see finite homomorphic images. Here, I will demonstrate the method of finding what we call the first order relations. Note that a first order relation is of the form $(\pi t_i)^a$ where $\pi \in N$ and a is a parameter.

S_4 has 24 elements; namely,

$(1, 2, 3, 4) \sim x,$

$(1, 2) \sim y,$

$(1, 4, 3, 2) \sim x^{-1},$

$(1, 3)(2, 4) \sim x^2,$

$(2, 3, 4) \sim x * y,$

$(1, 3, 4) \sim y * x,$

$(2, 4, 3) \sim y * x^{-1},$

$(1, 4, 3) \sim x^{-1} * y,$

$(1, 3, 2, 4) \sim x^2 * y,$

$(1, 2, 4, 3) \sim x * y * x,$

$(1, 4) \sim x * y * x^{-1},$

$(1, 4, 2, 3) \sim y * x^2,$

$(1, 3, 4, 2) \sim y * x * y,$

$(2, 3) \sim y^x,$

$(1, 4, 2) \sim x^2 * y * x,$

$(1, 2, 3) \sim x^2 * y * x^{-1},$

$(1, 3, 2) \sim x * y * x^2,$

$(1, 4)(2, 3) \sim y * x^2 * y,$

$(1, 2, 4) \sim y * x * y * x^{-1},$

$(3, 4) \sim x^2 * y * x^2,$

$(1, 3) \sim x * y * x^2 * y,$

$(2, 4) \sim y * x^2 * y * x,$

$(1, 2)(3, 4) \sim (x^2 * y)^2.$

There are a total of 92 relations if we do not include the identity. We acquire these relations by multiplying the element in N by every word from t_1 to t_4 where $t \sim t_1, t^x \sim t_2, t^{x^2} \sim t_3, t^{x^3} \sim t_4$

However we can shorten the process by using exactly 1 representative from the orbits of the centraliser of each class representative on $X = \{1, 2, 3, 4\}$. Thus we need to find the orbits of the centralizer of each of the elements $(12)(34), (12), (123), (1234)$.

Now the Orbits of $(12)(34)$:

$\{1\}, \{2\}, \{3,4\}$

Now we take a representative from each orbit and obtain the following:

$(12)(34)t_1 \sim (x^2 * y)^2 t_1$

$(12)(34)t_2 \sim (x^2 * y)^2 t_2$

$(12)(34)t_3 \sim (x^2 * y)^2 t_3$

So far our first order relations are $((x^2 * y)^2 t)^a$

Orbits of (12):

{1}, {2}, {3,4}

Now we take a representative from each orbit and obtain the following:

$(12)t_1 \sim yt_1$

$(12)t_2 \sim yt_2$

$(12)t_3 \sim yt_3$

This relation is added to the list $(yt)^b$

Orbits of (123):

{1}, {2}, {3},{4}

Now we take a representative from each orbit and obtain the following:

$(123)t_1 \sim (x^2 * y * x^{-1})t_1$

$(123)t_2 \sim (x^2 * y * x^{-1})t_2$

$(123)t_3 \sim (x^2 * y * x^{-1})t_3$

$(123)t_4 \sim (x^2 * y * x^{-1})t_4$

This relation is added to the list $(x^2 * y * x^{-1}t)^c$

Orbits of (1234):

{1}, {2}, {3},{4}

Now we take a representative from each orbit and obtain the following:

$(1234)t_1 \sim xt_1$

$(1234)t_2 \sim xt_2$

$(1234)t_3 \sim xt_3$

$(1234)t_4 \sim xt_4$

This relation is added to the list $(xt)^d$

Then $G = < x, y, t | x^4, y^2, (x*y)^3, t^2, (t, xy), (t, y^x), ((x^2 * y)^2 t)^a, (yt)^b, (x^2 * y * x^{-1}t)^c, (xt)^d >$
will provide finite homomorphic images for $a, b, c, d \in [0..10]$

We handle the progenitor $2^{*3} : S_3 = < x, y, t | x^3, y^2, (xy)^2, t^2, (t, xy) >$ in a similar manner.

3.1 Using The Famous Lemma

In order to find a progenitor of the form $m^{*n} : N$ factored by a single relation will produce a simple group. Finding these relations are not simple so in order to find groups of interests we need to write elements of our group N in terms of symmetric generators. This method was founded by Robert Curtis and is used to find factor the progenitor in order to find the images easier then before. Robert Curtis method is called the Famous Lemma and is shown below.

Lemma 4.64 [Famous Lemma]: $N \cap < t_i, t_j > \leq C_N(N_{ij})$ *where* N_{ij} *denotes the stabilzer in N of the two points i and j.*

Proof: Let $w \in N \cap < t_i, t_j >$. *We need to show that* $w \in C = Centralizer(N, N^{t_i t_j})$. *We use the defintion of a centralizer to perform the following:*

$w \in C$ if w commutes with every element of N^{ij}. Let $\pi \in N^{ij}$ then

- $w^{\pi} = w$

- $\pi^{-1} w \pi = w$

- $w\pi = \pi w$

Thus π commutes with every element of N_{ij} completing the proof.

Example 3.1

Take $N = S_{10} = < (1,2)(3,5)(7,9)(8,10), (1,3,6)(2,4,7)(5,8,9)]$. Next we find the Centralizer(S_4, Stabilizer(S_4, [1, 2]). The stabilizer of Stabilizer(S_4, [1, 2]) stabilizes the points 1 and 2 to get us a result of Stabiliser$(N, [1, 2]) = < (3,9)(4,6)(5,7)(8,10) >$. The centralizer is computed by finding what elements of N commute with $\{e, (3,9)(4,6)(5,7)(8,10)\}$. We now have the following relations using Lemma 4.64 [Famous Lemma]:

$$(t_1 t_2)^m = (3,9)(4,6)(5,7)(8,10), m >= 2$$

$((1,\ 2)(3,\ 5)(7,\ 9)(8,\ 10)\text{t}_1)^k$, k is odd,

$((1,2)(3,7)(4,6)(5,9)t_1)^q$, q is odd

3.2   Finding The First Order Relation

In this section we want find a method that will exhaust all possible relations of a particular progenitor. The purpose of this would be to find all possible finite homomorphic images. J.N. Bray, A.N.A Hammus, and R.T. Curtis developed a way to exhaust all relations of the form $((\pi)t_i^a)^b = 1$, where $\pi \in N$ and w is a word in the t_is, which they called the first order relations. In order to find these relations, we begin by finding the Centralizer(elements in conjugacy class). Next we find the orbits of the centralizer and take a single t_i from each orbit (respectively) in order to write out relation. An example of this process is shown below.

Example 3.2

In order to find the first order relations we need to compute the classes of N where N = $< (1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11, 20, 17)(12, 19, 18)(13, 21, 16)(14, 22, 15),$
$(1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7, 11, 9, 13)(8, 12, 10, 14)(15, 16)(17, 18) >$. We show the classes of N below:

Class	Order	Class Representative	Length	
1	1	e	1	
2	2	(1, 23)(2, 24)(7, 9)(8, 10)(11, 13)(12, 14)(19, 22)(20, 21)	3	
3	2	(1, 18)(2, 17)(3, 14)(4, 13)(5, 12)(6, 11)(7, 8)(9, 10)(15, 23)(16, 24)...	12	
4	3	(1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11, 20, 17)(12, 19, 18)(13, 21,16)...	32	
5	4	(3, 16, 5, 17)(4, 15, 6, 18)(7, 19, 9, 22)(8, 20, 10, 21)	3	
6	4	(3, 17, 5, 16)(4, 18, 6, 15)(7, 22, 9, 19)(8, 21, 10, 20)	3	
7	4	(1, 13, 23, 11)(2, 14, 24, 12)(3, 17, 5, 16)(4, 18, 6, 15)(7, 9)(8, 10)(19,22)...	6	
8	4	(1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7, 11, 9, 13)(8, 12, 10, 14)(15,16)..	12	
9	8	(1, 14)(2, 13)(3, 22, 16, 7, 5, 19, 17, 9)(4, 21, 15, 8, 6, 20, 18, 10)(11,24)...	12	
10	8	(1, 14)(2, 13)(3, 7, 17, 22, 5, 9, 16, 19)(4, 8, 18, 21, 6, 10, 15, 20)(11,24)...	12	

Table 3.1: Classes of example 3.2

The next step is that we compute the centralizer of each class representative in N and then find the orbits of the corresponding centralizer. We can then build the table below which shows how each relation is written.

Let $x = (1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11, 20, 17)(12, 19, 18)(13, 21, 16)(14, 22, 15)$ and $y = (1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7, 11, 9, 13)(8, 12, 10, 14)(15, 16)(17, 18)$

Class Number	Class Rep	Cent(N,Class Rep)	Orbit	Relations
2	y^2	$(1, 23)...$	$\{1,...\}$	$(y^2 * t)$
3	$x * y^{-1} * x^{-1} * y^2$	$(1, 18)...$	$\{2,...\}$	$(x * y^{-1} * x^{-1} * y^2) * t^{(y*x*y*x^{-1}*y^{-1})}$
4	x	$(1, 10, 5)...$	$\{3,...\}$	$(x) * t^{(y^2*x^{-1})}$
5	$(x * y)^2$	$(3, 16, 5, 17)...$	$\{4,...\}$	$(x * y)^2 * t^{(x^{-1}*y)}$
6	$(x * y^{-1})^2$	$(3, 17, 5, 16)...$	$\{1,...\}$	$(x * y^{-1})^2) * t$
7	$x^{-1} * y * x^{-1} * y^{-1}$	$(1, 13, 23, 11)...$	$\{7,...\}$	$(x^{-1} * y * x^{-1} * y^{-1}) * t^{(x^{-1}*y*x^{-1})}$
8	y	$(1, 19, 23, 22)...$	$\{15,...\}$	$(y) * t^{(y^{-1}*x)}$
9	$x * y$	$(1, 14)...$	$\{11,...\}$	$(x * y) * t^{(y*x)^2}$
10	$y * x^{-1} * y^2$	$(1, 14)...$	$\{12,...\}$	$(y * x^{-1} * y^2) * t^{(x*y^{-1})}$

Table 3.2: First order relations

By using this technique we are able to build progenitors.

3.3   Monomial Progenitors

In this section we will discuss building a monomial progenitor. In order to start this process we need to have a monomial representation that is formed from n x n matrices over a certain field (Definition 1.43.1). These matrices have monomial columns and rows (Definition 1.43.2).

Example 3.4

Consider G $= xx$ and yy, where $xx = (3, 6)$, and $yy = (1, 3, 5)(2, 4, 6)$. We want to find a faithful, irreducible, monomial representation of degree 3. Thus we need to find a subgroup of index 3 in G and induce a linear character of H up to the character ϕ^G of G.

The conjugacy classes of group G are
$C1 = Id(G),$
$C2 = (1, 4)(2, 5)(3, 6),$
$C3 = (1, 4),$
$C4 = (1, 4), (3, 6).$
$C5 = (1, 3, 5)(2, 4, 6).$

$C6 = (1, 5, 3)(2, 6, 4).$

$C7 = (1, 6, 2, 4, 3, 5).$

$C8 = (1, 5, 3, 4, 2, 6).$

Consider the subgroup H of G given below.

$H = Id(G), (2, 5), (2, 5)(3, 6), (1, 4)(2, 5).$

The conjugacy classes of H are

$D1 = Id(G),$

$D2 = (1, 4)(2, 5),$

$D3 = (2, 5)(3, 6),$

$D4 = (1, 4)(3, 6).$

$D5 = (2, 5).$

$D6 = (1, 4).$

$D7 = (3, 6).$

$D8 = (1, 4)(2, 5)(3, 6).$

Consider the irreducible characters ϕ (of H) and ϕ^G (of G) given below.

Class	D1	D2	D3	D4	D5	D6	D7	D8
Size	1	1	1	1	1	1	1	1
Representative	$Id(G)$	$(1, 4)(2, 5)$	$(2, 5)(3, 6)$	$(1, 4)(3, 6)$	$(2, 5)$	$(1, 4)$	$(3, 6)$	$(1, 4)(2, 5)(3, 6)$
ϕ	1	1	-1	-1	-1	-1	1	1

Class	C1	C2	C3	C4	C5	C6	..	C8
Size	1	1	3	3	4	4	4	4
Representative	$Id(G)$	$(1, 4)(2, 5)(3, 6)$	$(1, 4)$	$(1, 4)(3, 6)$	$(1, 3, 5)(2, 4, 6)$	$(1, 5, 3)(2, 6, 4)$	..	$(1, 5, 3, 4, 2, 6)$
ϕ^G	3	3	-1	-1	0	0	..	0

Do the following.

(a) Induce the character ϕ of H up to G to obtain the character ϕ^G of G.

$\phi \uparrow_H^G$

$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{24}{8} = 3$.

$\phi_1^G = \frac{3}{1} \sum_{w \in H \cap C_1} \phi(w)$

So, $\phi_1^G = 3(\phi(1)) = 3(1) = 3$.

$\phi_2^G = \frac{3}{1} \sum_{w \in H \cap C_2} \phi(w)$

So, $\phi_2^G = 3(\phi((1,4)(2,5)(3,6))) = 3(1) = 3$.

$\phi_3^G = \frac{3}{3} \sum_{w \in H \cap C_3} \phi(w)$

So, $\phi_3^G = \frac{3}{3}(\phi(1,4) + \phi(2,5) + \phi(3,6)) = 1(-1 + -1 + 1) = -1$.

$\phi_4^G = \frac{3}{4} \sum_{w \in H \cap C_4} \phi(w)$

So, $\phi_4^G = \frac{3}{-1}(\phi(0)) = \frac{3}{-1}(\phi(1,4)(3,6) + \phi(2,5)(3,6) + \phi(1,4)(2,5)) = -3(-1 + -1 + 1) = 3$.

$\phi_5^G = \frac{3}{4} \sum_{w \in H \cap C_\alpha} \phi(w)$

So, $\phi_5^G = \frac{3}{4}(\phi(0)) = \frac{3}{4}(0) = 0$.

$\phi_6^G = \frac{3}{4} \sum_{w \in H \cap C_\alpha} \phi(w)$

So, $\phi_6^G = \frac{3}{4}(\phi(0)) = \frac{3}{4}(0) = 0.$

$\phi_7^G = \frac{3}{4}\sum_{w \in H \cap C_\alpha} \phi(w)$

So, $\phi_7^G = \frac{3}{4}(\phi(0)) = \frac{3}{4}(0) = 0.$

$\phi_8^G = \frac{3}{4}\sum_{w \in H \cap C_\alpha} \phi(w)$

So, $\phi_8^G = \frac{3}{4}(\phi(0)) = \frac{3}{4}(0) = 0.$

$\phi \uparrow_H^G = 3 \quad 3 \quad -1 \quad 3 \quad 0 \quad 0 \quad 0 \quad 0.$

(b) Show the monomial representation has the generators

Let $t_1 = e$, $t_2 = (1,3,5)(2,4,6)$, and $t_3 = (1,5,3)(2,6,4)$.

Then

$$A(xx) = \begin{bmatrix} \phi(t_1 x t_1^{-1}) & \phi(t_1 x t_2^{-1}) & \phi(t_1 x t_3^{-1}) \\ \phi(t_2 x t_1^{-1}) & \phi(t_2 x t_2^{-1}) & \phi(t_2 x t_3^{-1}) \\ \phi(t_3 x t_1^{-1}) & \phi(t_3 x t_2^{-1}) & \phi(t_3 x t_3^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \phi(exe) & \phi(ex(1,5,3)(2,6,4)) & \phi(ex((1,3,5)(2,4,6)) \\ \phi((1,3,5)(2,4,6)xe) & \phi((1,3,5)(2,4,6)x(1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)x(1,5,3)(2,6,4)) \\ \phi((1,5,3)(2,6,4)xe) & \phi((1,5,3)(2,6,4)x(1,5,3)(2,6,4)) & \phi((1,5,3)(2,6,4)x(1,3,5)(2,4,6)) \end{bmatrix}$$

$$= \begin{bmatrix} \phi((3,6)) & \phi((3,6)(1,5,3)(2,6,4)) & \phi((3,6)(1,3,5)(2,4,6)) \\ \phi((1,3,5)(2,4,6)(3,6)(1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)(3,6)(1,5,3)(2,6,4)) & \phi(((1,3,5)(2,4,6)(3,6)(1,5,3)(2,6,4)) \\ \phi((1,5,3)(2,6,4)(3,6)) & \phi((1,5,3)(2,6,4)(3,6)(1,3,5)(2,4,6)) & \phi((1,5,3)(2,6,4)(3,6)(1,5,3)(2,6,4)) \end{bmatrix}$$

$$= \begin{bmatrix} \phi((3,6)) & \phi((1,5,3,4,2,6)) & \phi((1,3,2,4,6,5)) \\ \phi((1,6,2,4,3,5)) & \phi((1,4)) & \phi((1,2,6,4,5,3)) \\ \phi((1,5,6,4,2,3)) & \phi((1,3,5,4,6,2)) & \phi((2,5)) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A(yy) = \begin{bmatrix} \phi(t_1 y t_1^{-1}) & \phi(t_1 y t_2^{-1}) & \phi(t_1 y t_3^{-1}) \\ \phi(t_2 y t_1^{-1}) & \phi(t_2 y t_2^{-1}) & \phi(t_2 y t_2^{-1}) \\ \phi(t_3 y t_1^{-1}) & \phi(t_3 y t_2^{-1}) & \phi(t_3 y t_3^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \phi(eye) & \phi(ey(1,5,3)(2,6,4)) & \phi(ey((1,3,5)(2,4,6)) \\ \phi((1,3,5)(2,4,6)ye) & \phi((1,3,5)(2,4,6)y(1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)y(1,5,3)(2,6,4)) \\ \phi((1,5,3)(2,6,4)ye) & \phi((1,5,3)(2,6,4)y(1,5,3)(2,6,4)) & \phi((1,5,3)(2,6,4)y(1,3,5)(2,4,6)) \end{bmatrix}$$

$$= \begin{bmatrix} \phi((1,3,5)(2,4,6)) & \phi((1,3,5)(2,4,6)(1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)(1,3,5)(2,4,6)) \\ \phi((1,3,5)(2,4,6)(1,3,5)(2,4,6)(1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)(1,3,5)(2,4,6)(1,5,3)(2,6,4)) & \phi(((1,3,5)(2,4,6)(1,3,5)(2,4,6)\ldots) \\ \phi((1,5,3)(2,6,4)(1,3,5)(2,4,6)) & \phi((1,5,3)(2,6,4)(1,3,5)(2,4,6)(1,5,3)(2,6,4)) & \phi((1,5,3)(2,6,4)(1,3,5)(2,4,6)\ldots) \end{bmatrix}$$

$$= \begin{bmatrix} \phi((1,3,5)(2,4,6)) & \phi(e) & \phi((1,5,3)(2,6,4)) \\ \phi((1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)) & \phi(e) \\ \phi(e) & \phi((1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) Give a permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation of part (b).

$$A(xx) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \text{ where } a_{11} = 1,\ a_{22} = 2,\ a_{33} = 2.$$

Note:

$$a_{ij} = a \iff t_i \rightarrow t_j^a$$

Therefore,

$t_1 \rightarrow t_1^1,$

$t_2 \rightarrow t_2^2,$

$t_3 \rightarrow t_3^2.$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^2	t_2^2	t_3^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1^1	t_2^2	t_2^2	t_1^2	t_2	t_3
1	5	6	4	2	3

Therefore, $A(xx) = (2,5)(3,6).$

$$A(yy) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ where } a_{12} = 1,\ a_{23} = 1,\ a_{31} = 1.$$

Therefore,

$t_1 \rightarrow t_2,$

$$t_2 \rightarrow t_3,$$
$$t_3 \rightarrow t_1.$$

1	2	3	4	5	6
t_1	t_1^2	t_2	t_2^2	t_3	t_3^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_2^2	t_3	t_3^2	t_1	t_1^2
3	4	5	6	1	2

Therefore, $A(yy) = (1,3,5)(2,4,6)$.

Example 3.5

Consider $G = S_4$, xx and yy, where $xx = (1, 2, 3, 4)$, and $yy = (1, 2)$. We want to find a faithful, irreducible, monomial representation of degree 3. Thus we need to find a subgroup of index 3 in G and induce a linear character of H up to the character ϕ^G of G.

The conjugacy classes of group G are

$$C1 = Id(G),$$

$$C2 = (1, 2)(3, 4),$$

$$C3 = (1, 2),$$

$$C4 = (1, 2, 3).$$

$$C5 = (1, 2, 3, 4).$$

Consider the subgroup H of G given below.

$$H = Id(G), (1, 2), (1, 3, 2, 4), (3, 4), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4, 2, 3), (1, 4)(2, 3).$$

The conjugacy classes of H are

$$D1 = Id(G),$$

$$D2 = (1, 2)(3, 4),$$

$$D3 = (1, 3)(2, 4),$$

$$D4 = (1, 2).$$

$D5 = (1, 4, 2, 3)$.

Consider the irreducible characters ϕ (of H) and ϕ^G (of G) given below.

Class	D1	D2	D3	D4	D5
Size	1	1	2	2	2
Representative	$Id(G)$	$(1,2)(3,4)$	$(1,3)(2,4)$	$(1,2)$	$(1,4,2,3)$
ϕ	1	1	-1	1	-1

Table 3.3: Character table ϕ (of H) of $3^{*3} :_m S_4$

Class	C1	C2	C3	C4	C5
Size	1	3	6	8	6
Representative	$Id(G)$	$(1,2)(3,4)$	$(1,2)$	$(1,2,3)$	$(1,2,3,4)$
ϕ^G	3	-1	1	0	-1

Table 3.4: Character table ϕ^G (of G) of $3^{*3} :_m S_4$

Do the following.

(a) Induce the character ϕ of H up to G to obtain the character ϕ^G of G.

$\phi \uparrow_H^G$

$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{24}{8} = 3$.

$\phi_1^G = \frac{3}{1} \sum_{w \in H \cap C_1} \phi(w)$

So, $\phi_1^G = 3(\phi(1)) = 3(1) = 3$.

$\phi_2^G = \frac{3}{3} \sum_{w \in H \cap C_2} \phi(w)$

So, $\phi_2^G = 1(\phi((1,3)(2,4)) + \phi((1,3)(2,4)) + \phi((1,4)(2,3))) = 1(1 - 1 - 1) = -1.$

$\phi_3^G = \frac{3}{6} \sum_{w \in H \cap C_3} \phi(w)$

So, $\phi_3^G = \frac{1}{2}(\phi((1,2)) + \phi((3,4))) = \frac{1}{2}(1 + 1) = 1.$

$\phi_4^G = \frac{3}{8} \sum_{w \in H \cap C_\alpha} \phi(w)$

So, $\phi_4^G = \frac{3}{8}(\phi(e)) = \frac{3}{8}(0) = 0.$

$\phi_5^G = \frac{3}{6} \sum_{w \in H \cap C_\alpha} \phi(w)$

So, $\phi_5^G = \frac{3}{6}(\phi((1,4,2,3)) + \phi((1,3,2,4))) = \frac{1}{2}(-1 - 1) = -1.$

$\phi \uparrow_H^G = 3 \quad -1 \quad 1 \quad 0 \quad -1.$

(b) Show the monomial representation has the generators

$G = He \cup H(1,2,3,4) \cup H(2,3,4)$

Let $t_1 = e$, $t_2 = (1,2,3,4)$, and $t_3 = (2,3,4)$.

Then

$$A(xx) = \begin{bmatrix} \phi(t_1 x t_1^{-1}) & \phi(t_1 x t_2^{-1}) & \phi(t_1 x t_3^{-1}) \\ \phi(t_2 x t_1^{-1}) & \phi(t_2 x t_2^{-1}) & \phi(t_2 x t_3^{-1}) \\ \phi(t_3 x t_1^{-1}) & \phi(t_3 x t_2^{-1}) & \phi(t_3 x t_3^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \phi(exe) & \phi(ex(1,4,3,2)) & \phi(ex(2,4,3)) \\ \phi((1,2,3,4)x) & \phi((1,2,3,4)x(1,4,3,2)) & \phi((1,2,3,4)x(1,4,3,2)) \\ \phi((2,3,4)xe) & \phi((2,3,4)x(1,4,3,2)) & \phi((2,3,4)x(2,4,3)) \end{bmatrix}$$

$$
= \begin{bmatrix}
\phi((1,2)(3,4)) & \phi((1,2,3,4)(1,4,3,2)) & \phi((1,2,3,4)(2,4,3)) \\
\phi((1,2,3,4)(1,2,3,4)) & \phi((1,2,3,4)(1,2,3,4)(1,4,3,2)) & \phi((1,2,3,4)(1,2,3,4)(2,4,3)) \\
\phi((2,3,4)(1,2,3,4)) & \phi((2,3,4)(1,2,3,4)(1,4,3,2)) & \phi((2,3,4)(1,2,3,4)(2,4,3))
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\phi((1,2,3,4)) & \phi(e) & \phi((1,4)) \\
\phi((1,3)(2,4)) & \phi((1,2,3,4)) & \phi((1,2,3)) \\
\phi((1,2,4,3)) & \phi((2,3,4)) & \phi((1,4,2,3))
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
$$

$$
A(yy) = \begin{bmatrix}
\phi(t_1 y t_1^{-1}) & \phi(t_1 y t_2^{-1}) & \phi(t_1 y t_3^{-1}) \\
\phi(t_2 y t_1^{-1}) & \phi(t_2 y t_2^{-1}) & \phi(t_2 y t_3^{-1}) \\
\phi(t_3 y t_1^{-1}) & \phi(t_3 y t_2^{-1}) & \phi(t_3 y t_3^{-1})
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\phi(eye) & \phi(ey(1,4,3,2)) & \phi(ey(2,4,3)) \\
\phi((1,2,3,4)y) & \phi((1,2,3,4)y(1,4,3,2)) & \phi((1,2,3,4)y(2,4,3)) \\
\phi((2,4,3)ye) & \phi((2,3,4)y(1,4,3,2)) & \phi((2,3,4)y(2,4,3))
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\phi((1,2)) & \phi((1,2)(1,4,3,2)) & \phi((1,2)(2,4,3)) \\
\phi((1,2,3,4)(1,2)) & \phi((1,2,3,4)(1,2)(1,4,3,2)) & \phi((1,2,3,4)(1,2)(2,4,3)) \\
\phi((2,3,4)(1,2)) & \phi((2,3,4)(1,2)(1,4,3,2)) & \phi((2,3,4)(1,2)(2,4,3))
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\phi((1,2)) & \phi(2,4,3) & \phi((1,4,3,2)) \\
\phi((2,3,4)) & \phi((1,4)) & \phi(e) \\
\phi(1,2,3,4) & \phi(e) & \phi((1,4))
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
$$

(c) Give a permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation of part (b).

$$
A(xx) = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
2 & 0 & 0 \\
0 & 0 & 2
\end{bmatrix} \text{(Mod3), where } a_{12} = 1, \; a_{21} = 2,
$$

$a_{33} = 2.$

Therefore,

$$t_1 \rightarrow t_2^1,$$

$$t_2 \rightarrow t_1^2,$$

$$t_3 \rightarrow t_3^2.$$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^2	t_2^2	t_3^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_1^2	t_3^2	t_2^2	t_1	t_3
2	4	6	5	1	3

Table 3.5: Computation of permutation representation of matrices A(xx)

Therefore, $A(xx) = (1, 2, 4, 5)(3, 6)$.

$$A(yy) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ where } a_{12} = 1, a_{23} = 1, a_{32} = 1.$$

Therefore,

$$t_1 \rightarrow t_1,$$

$$t_2 \rightarrow t_3,$$

$$t_3 \rightarrow t_2.$$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^2	t_2^2	t_3^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1	t_3	t_2	t_2^2	t_3^2	t_2^2
1	3	2	5	6	5

Table 3.6: Computation of permutation representation of matrices A(yy)

Therefore, $A(yy) = (2, 3)(5, 6)$.

(d) Give a presentation of the monomial progenitor $3^{*3} :_m S_4$.

We can find a presentation for S_4 by using the followin magma code:

```
S:=Sym(6);
xx:=S!(1,2,4,5)(3,6);
yy:=S!(2,3)(5,6);
N:=sub<S|xx,yy>;
#N;
/* 24 */
#G;
/* 24*/
IsIsomorphic(N,G);
/*true Mapping from: GrpPerm: N to GrpPerm: G
Composition of Mapping from: GrpPerm: N to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G
FPGroup(N);

Finitely presented group on 2 generators
Relations
    $.1^4 = Id($)
    $.2^2 = Id($)
    ($.2 * $.1^{-1})^3 = Id($)
```

A presentation for S_4 is $< x, y | x^4, y^2, (yx^{-1})^3 >$.

Stabiliser$(N, \{1, 4\}) = <(2, 3)(5, 6), (2, 6)(3, 5), (1, 4)(2, 5)>$.

Therefore, t commutes with $<(2, 3)(5, 6), (2, 6)(3, 5), (1, 4)(2, 5)>$.

We can use the Schreier system to get the rest of our permutations:

```
word:=function(A);
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
for i in [2..#G] do
 P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=(xx); end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=(xx)^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=(yy); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=(yy^-1); end if;
end for;
PP:=Id(N);
for k in [1..#P] do
 PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;
A:=N!(2, 3)(5, 6); (t,A)
B:=N!(2, 6)(3, 5); (t,B)
C:=N!(1, 4)(2, 5);
word(A);
/* b */
word(B);
/* a^2 * b * a^2 */
word(C);
/* a^2 */
```

Thus, our presentation is $<x, y, t | x^4, y^2, (yx^{-1})^3, t^3, t^y = t, t^{x^2 * y * x^2} = t, t^{x^2} = t^2>$.

```
G<x,y,t>:=Group<x,y,t| x^4,y^2,(y*x^(-1))^3,t^3, t^y=t,
t^(x^2*y*x^2) = t, t^(x^2) = t^2>;
#G;
648
```

```
> 3^3*24;
648
```

Example 3.6

Consider $G = S_{15}$, xx and yy, where $xx = (1, 10, 9, 13, 3, 6, 15, 2, 14, 7, 8, 5, 4, 12, 11)$, and $yy = (1, 5, 3, 2, 12, 6, 10, 7, 4, 13, 8, 15, 11, 9, 14)$. We want to find a faithful, irreducible, monomial representation of degree 5. Thus we need to find a subgroup of index 5 in G and induce a linear character of H up to the character ϕ^G of G.

The conjugacy classes of group G are

$C1 = Id(G)$,

$C2 = (1, 2)(3, 4)$,

$C3 = (1, 3)(2, 4)$,

$C4 = (1, 4)(2, 3)$.

Consider the subgroup H of G given below.

$H = Id(G), (1, 2), (1, 3, 2, 4), (3, 4), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4, 2, 3), (1, 4)(2, 3)$.

The conjugacy classes of H are

$D1 = Id(G)$,

$D2 = (1, 2)(3, 4)$,

$D3 = (1, 3)(2, 4)$,

$D4 = (1, 4)(2, 3).$

Consider the irreducible characters ϕ (of H) and ϕ^G (of G) given below.

Class	D1	D2	D3	D4
Size	1	3	1	1
Representative	Id(H)	(1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14)	(1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)	(1, 6, 8)(2, 4, 9)(3, 7, 11)(5, 10, 15)(12, 13, 14)
ϕ	1	1	9	-9-1

Class	D5	D6	D7	D8
Size	4	4	4	4
Representative	(1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12, 11)	(1, 3, 13)(2, 4, 9)(5, 15, 10)(6, 7, 14)(8, 11, 12)	(1, 12, 7)(2, 4, 9)(3, 8, 14)(6, 13, 11)	(1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13)
ϕ	9	-9-1	-9-1	9

Class	D9	D10	D11	D12
Size	4	4	3	3
Representative	(2, 7, 13)(3, 12, 9)(4, 11, 14)(5, 15, 10)	(2, 13, 7)(3, 9, 12)(4, 14, 11)(5, 10, 15)	(1, 3, 8, 11, 6, 7)(2, 14, 9, 13, 4, 12)(5, 10, 15)	(1, 7, 6, 11, 8, 3)(2, 12, 4, 13, 9, 14)(5, 15, 10)
ϕ	1	1	-9-1	9

Class	C1	C2	C3	C4
Size	1	15	1	20
Representative	Id(H)	(1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14)	(1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)	(1, 6, 8)(2, 4, 9)(3, 7, 11)(5, 10, 15)(12, 13, 14)
ϕ	1	1	9	-9-1

Class	C5	C6	C7	C8
Size	20	20	20	12
Representative	(1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12, 11)	(1, 3, 13)(2, 4, 9)(5, 15, 10)(6, 7, 14)(8, 11, 12)	(1, 12, 7)(2, 4, 9)(3, 8, 14)(6, 13, 11)	(1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13)
ϕ	9	-9-1	-9-1	9

Class	C9	C10	C11	C12
Size	12	15	15	12
Representative	(2, 7, 13)(3, 12, 9)(4, 11, 14)(5, 15, 10)	(2, 13, 7)(3, 9, 12)(4, 14, 11)(5, 10, 15)	(1, 3, 8, 11, 6, 7)(2, 14, 9, 13, 4, 12)(5, 10, 15)	(1, 7, 6, 11, 8, 3)(2, 12, 4, 13, 9, 14)(5, 15, 10)
ϕ	1	1	-9-1	9

Do the following.

(a) Induce the character ϕ of H up to G to obtain the character ϕ^G of G.

$\phi \uparrow_H^G$

$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{180}{36} = 5$.

$\phi_1^G = \frac{5}{1} \sum_{w \in H \cap C_1} \phi(w)$

So, $\phi_1^G = 5(\phi(1)) = 5(1) = 5$.

$\phi_2^G = \frac{5}{15} \sum_{w \in H \cap C_2} \phi(w)$

So, $\phi_2^G = \frac{1}{3}(\phi((1,4)(2,8)(3,12)(6,9)(7,13)(11,14) + \phi((1,14)(2,7)(3,9)(4,11)(6,12)(8,13)) + \phi((1,11)(2,13)(3,6)(4,14)(7,8)(9,12))) = \frac{1}{3}(1+1+1) = 1$.

$\phi_3^G = \frac{5}{1} \sum_{w \in H \cap C_3} \phi(w)$

So, $\phi_3^G = 5(\phi((1,6,8)(2,4,9)(3,7,11)(5,10,15)(12,13,14)) = 5(zeta(3)) = 5 * zeta(3)$.

If we continue this process we will arrive at the following:

$\phi \uparrow_H^G = 3 \; 1 \; 5*zeta(3)_3 \; -5*zeta(3)_3 -5 \; -1 \; zeta(3)_3+1 \; -zeta(3)_3 \; 0 \; 0 \; zeta(3)_3 - zeta(3)_3 - 1 \; 0 \; 0 \; 0 \; 0$.

(b) Show the monomial representation has the generators

$$A(xx) = \begin{bmatrix} \phi(t_1 x x t_1^{-1}) & \phi(t_1 x x t_2^{-1}) & \phi(t_1 x x t_3^{-1}) & \phi(t_1 x x t_4^{-1}) & \phi(t_1 x x t_5^{-1}) \\ \phi(t_2 x x t_1^{-1}) & \phi(t_2 x x t_2^{-1}) & \phi(t_2 x x t_3^{-1}) & \phi(t_2 x x t_4^{-1}) & \phi(t_1 x x t_2^{-1}) \\ \phi(t_3 x x t_1^{-1}) & \phi(t_3 x x t_2^{-1}) & \phi(t_3 x x t_3^{-1}) & \phi(t_3 x x t_4^{-1}) & \phi(t_3 x x t_5^{-1}) \\ \phi(t_4 x x t_1^{-1}) & \phi(t_4 x x t_2^{-1}) & \phi(t_4 x x t_3^{-1}) & \phi(t_4 x x t_4^{-1}) & \phi(t_4 x x t_5^{-1}) \\ \phi(t_5 x x t_1^{-1}) & \phi(t_5 x x t_2^{-1}) & \phi(t_5 x x t_3^{-1}) & \phi(t_5 x x t_4^{-1}) & \phi(t_5 x x t_5^{-1}) \end{bmatrix}$$

$$A(yy) = \begin{bmatrix} \phi(t_1 y y t_1^{-1}) & \phi(t_1 y y t_2^{-1}) & \phi(t_1 y y t_3^{-1}) & \phi(t_1 y y t_4^{-1}) & \phi(t_1 y y t_5^{-1}) \\ \phi(t_2 y y t_1^{-1}) & \phi(t_2 y y t_2^{-1}) & \phi(t_2 y y t_3^{-1}) & \phi(t_2 y y t_4^{-1}) & \phi(t_1 y y t_2^{-1}) \\ \phi(t_3 y y t_1^{-1}) & \phi(t_3 y y t_2^{-1}) & \phi(t_3 y y t_3^{-1}) & \phi(t_3 y y t_4^{-1}) & \phi(t_3 y y t_5^{-1}) \\ \phi(t_4 y y t_1^{-1}) & \phi(t_4 y y t_2^{-1}) & \phi(t_4 y y t_3^{-1}) & \phi(t_4 y y t_4^{-1}) & \phi(t_4 y y t_5^{-1}) \\ \phi(t_5 y y t_1^{-1}) & \phi(t_5 y y t_2^{-1}) & \phi(t_5 y y t_3^{-1}) & \phi(t_5 y y t_4^{-1}) & \phi(t_5 y y t_5^{-1}) \end{bmatrix}$$

$$A(xx) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$A(yy) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $t_1 = e$, $t_2 = (1, 2, 3, 4)$, and $t_3 = (2, 3, 4)$.

Then

$$A(yy) = \begin{bmatrix} \phi(t_1 y t_1^{-1}) & \phi(t_1 y t_2^{-1}) & \phi(t_1 y t_3^{-1}) \\ \phi(t_2 y t_1^{-1}) & \phi(t_2 y t_2^{-1}) & \phi(t_2 y t_3^{-1}) \\ \phi(t_3 y t_1^{-1}) & \phi(t_3 y t_2^{-1}) & \phi(t_3 y t_3^{-1}) \end{bmatrix}$$

(c) Give a permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation of part (b).

Formula: $a_{ij} = q \implies t_i = t_j^q$

$$A(xx) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $a_{12} = 1$, $a_{24} = 1$, $a_{35} = 4$, $a_{43} = 1$, $a_{51} = 1$.

Therefore,

$$t_1 \rightarrow t_2^1,$$

$$t_2 \rightarrow t_4^1,$$

$t_3 \rightarrow t_5^4.$

$t_4 \rightarrow t_3^1.$

$t_5 \rightarrow t_1^1.$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_1	t_2	t_3	t_4	t_5	t_1^2	t_1^3	t_1^4	t_1^5	t_1^6	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2	t_4	t_5^4	t_3	t_1	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6	t_4^2	t_4^3	t_4^4	t_4^5	t_4^6
2	4	28	3	1	11	12	13	14	15	21	22	23	24	25

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
t_3^2	t_3^3	t_3^4	t_3^5	t_3^6	t_4^2	t_4^3	t_4^4	t_4^5	t_4^6	t_5^2	t_5^3	t_5^4	t_5^5	t_5^6
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_5	t_5^5	t_5^2	t_5^6	t_5^3	t_3^2	t_3^3	t_3^4	t_3^5	t_3^6	t_1^2	t_1^3	t_1^4	t_1^5	t_1^6
5	29	26	30	27	16	17	18	19	20	6	7	8	9	10

Therefore, $A(xx) = (1, 2, 4, 3, 28, 8, 13, 23, 18, 26, 6, 11, 21, 16, 5)$
$(7, 12, 22, 17, 29, 9, 14, 24, 19, 30, 10, 15, 25, 20, 27)$

$$B(yy) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $a_{13} = 1$, $a_{24} = 2$, $a_{32} = 2$, $a_{45} = 2$, $a_{51} = 4$.

Therefore,

$t_1 \rightarrow t_3^1,$

$$t_2 \rightarrow t_4^2,$$

$$t_3 \rightarrow t_2^2,$$

$$t_4 \rightarrow t_5^2,$$

$$t_5 \rightarrow t_1^4,$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_1	t_2	t_3	t_4	t_5	t_1^2	t_1^3	t_1^4	t_1^5	t_1^6	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_3	t_4^2	t_2^2	t_5^2	t_1^4	t_3^2	t_3^3	t_3^4	t_3^5	t_3^6	t_4^4	t_4^6	t_4^1	t_4^3	t_4^5
3	21	11	26	8	16	17	18	19	20	23	25	4	22	24

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
t_3^2	t_3^3	t_3^4	t_3^5	t_3^6	t_4^2	t_4^3	t_4^4	t_4^5	t_4^6	t_5^2	t_5^3	t_5^4	t_5^5	t_5^6
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2^4	t_2^6	t_2	t_2^3	t_2^5	t_5^4	t_5^6	t_5	t_5^3	t_5^5	t_1	t_1^5	t_1^2	t_1^6	t_1^3
13	15	2	12	14	28	30	5	27	29	1	9	6	10	7

Therefore, $B(yy) = (1, 3, 11, 23, 5, 8, 18, 2, 21, 28, 6, 16, 13, 4, 26)$
$(7, 17, 15, 24, 27, 9, 19, 12, 25, 29, 10, 20, 14, 22, 30)$

(d) Give a presentation of the monomial progenitor $3_m^{*3} S_4$.

We can find a presentation for S_4 by using the followin magma code:

```
S:=Sym(30);
xx:=S!(1,2,4,3,28,8,13,23,18,26,6,11,21,16,5)(7,12,22,17,29
,9,14,24,19,30,10,15,25,20,27);
yy:=S!(1,3,11,23,5,8,18,2,21,28,6,16,13,4,26)(7,17,15,24,27
,9,19,12,25,29,10,20,14,22,30);
N:=sub<S|xx,yy>;
```

```
#N;
/* 180 */
IsIsomorphic(N,G);
/*true Mapping from: GrpPerm: N to GrpPerm: G
Composition of Mapping from: GrpPerm: N to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G
FPGroup(N);

Finitely presented group on 2 generators
Relations
    ($.2^-1 * $.1)^3 = Id($)  --> (y^-1 *x)^3
    $.2^-1 * $.1^-3 * $.2^-1 * $.1^2 = Id($) --> y^-1*x^-3*y^-1*x^2
    $.2^-3 * $.1^-1 * $.2^2 * $.1^-1 = Id($) --> y^-3*x^-1*y^2*x^-1
```

A presentation for S_{15} is $< x, y | (y^{-1} * x)^3, y^{-1} * x^{-3} * y^{-1} * x^2, y^{-3} * x^{-1} * y^2 * x^{-1} >$.

Stabiliser$(N, \{1, 4\}) = (2, 3)(5, 6)(2, 6)(3, 5)(1, 4)(2, 5)$.

Therefore, t commutes with $(2, 3)(5, 6)$ $(2, 6)(3, 5)$ $(1, 4)(2, 5)$.

We can use the Schreier system to get the rest of our permutations:

```
word:=function(A);
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
for i in [2..#G] do
 P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=(xx); end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=(xx)^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=(yy); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=(yy^-1); end if;
end for;
PP:=Id(N);
for k in [1..#P] do
 PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;
A:=N!(2, 3)(5, 6);
```

```
B:=N!(2, 6)(3, 5);
C:=N!(1, 4)(2, 5);
word(A);
/* b */
word(B);
/* a^2 * b * a^2 */
word(C);
/* a^2 */
```

Thus, our presentation is $< x, y, t | x^4, y^2, (yx^{-1})^3, t^3, t^y, t^{x^2*y*x^2} = t, t^{x^2} = t^2 >$.

```
G<x,y,t>:=Group<x,y,t| x^4,y^2,(y*x^(-1))^3,t^3, t^y,
t^(x^2*y*x^2) = t, t^(x^2) = t^2>;
#G;
/* 24 */
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
s:=IsIsomorphic(N,G1);
s;
/* True */
```

3.4 Monomial Representation

In this section we are going to focus on finding a monomial representation. This monomial representation can then be used to write our monomial progenitor. The process of finding a monomial presentation is to find a subgroup H of a group G whose index is equal to the degree of a faithful and irreducible character of G. The next step is to induce a linear character of H up to G. The final step will be to use the formula for finite monomial representation to represent our G. This process will be shown below for $G = 3 \times A_5$.

We are given that $3 \times A_5$ has a monomial irreducible representation of degree 5. Knowing that the order of 3 x A_5 is 180, the order of our subgroup H must be 36, since $\frac{|3xA_5|}{|H|} = \frac{180}{36} = 5, |H| = 36$. Let $H =< x, y > \leq G$, where $x \sim (1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13)$, and $y \sim (1, 5)(2, 14)(4, 12)(6, 10)(8, 15)(9, 13)$. The character table of H and G are given below:

We have $G = < xx, yy >$ where, x = (1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13),

y = (1, 5)(2, 14)(4, 12)(6, 10)(8, 15)(9, 13) and $H = < x, y >$.

Character Table of Group H

Class	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$	$C7$	$C8$	$C9$	$C10$	$C11$	$C12$
Order	1	2	3	3	3	3	3	3	3	3	6	6
ϕ_1	1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	1	1	J	-1-J-1	-J	J-1	-J	J	1	1-1	-J	J
ϕ_3	1	1	1	1	J-1	-J-1	-J	J	J-1	-J	1	1
ϕ_4	1	1	-1-J	J-1	-J	J	1	1	J-1	-J	J-1	-J
ϕ_5	1	1	-1-J	J	J-1	-J	J-1	-J	1	1	J-1	-J
ϕ_6	1	1	1	1-1	-J	J	J-1	-J	-1-J	J	1	1
ϕ_7	1	1	-1-J	J	1	1-1	-J	J-1	-J	J	J-1	-J
ϕ_8	1	1	J	-1-J	1	1	J-1	-J	J-1	-J-1	-J	J
ϕ_9	1	1	J	-1-J	J-1	-J	1	1-1	-J	J-1	-J	J
ϕ_{10}	3	-1	3	3	0	0	0	0	0	0	-1	-1
ϕ_{11}	3	-1	3*J	-3-3*J	0	0	0	0	0	0	1+J	-J
ϕ_{12}	3	-1	-3-3*J	3*J	0	0	0	0	0	0	-J	1+J

Character Table of Group G

Class	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14	C15
Order	1	2	3	3	3	3	3	5	5	6	6	15	15	15	15
ϕ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	1	1	J	-1-J-1	-J	J	1	1	1-1	-J	J	-1-J	J	-1-J	J
ϕ_3	1	1	-1-J	J	J-1	-J	1	1	1	J-1	-J	J	-1-J	J	-1-J
ϕ_4	3	-1	3	3	0	0	0	Z1	Z12	-1	-1	Z12	Z1	Z1	Z12
ϕ_5	3	-1	3	3	0	0	0	Z12	Z1	-1	-1	Z1	Z12	Z12	Z1
ϕ_6	3	-1	3*J	-3-3*J	0	0	0	Z12	Z1	1+J	-J	Z2	Z22	Z27	Z211
ϕ_7	3	-1	3*J	-3-3*J	0	0	0	Z1	Z12	1+J	-J	Z27	Z211	Z2	Z22
ϕ_8	3	-1	-3-3*J	3*J	0	0	0	Z1	Z12	-J	1+J	Z22	Z2	Z211	Z27
ϕ_9	3	-1	-3-3*J	3*J	0	0	0	Z12	Z1	-J	1+J	Z211	Z27	Z22	Z2
ϕ_{10}	4	0	4	4	1	1	1	-1	-1	0	0	-1	-1	-1	-1
ϕ_{11}	4	0	4*J	-4-4*J-1	-J	J	1	-1	-1	0	0	1+J	-J	1+J	-J
ϕ_{12}	4	0	-4-4*J	4*J	J-1	-J	1	-1	-1	0	0	-J	1+J	-J	1+J
ϕ_{13}	5	1	5	5	-1	-1	-1	0	0	1	1	0	0	0	0
ϕ_{14}	5	1	5*J	-5-5*J	1+J	-J	-1	0	0	-1-J	J	0	0	0	0
ϕ_{15}	5	1	-5-5*J	5*J	-J	1+J	-1	0	0	J-1	-J	0	0	0	0

We show below that G is isomorphic to $3 \times A_5$. We are given G, a transitive group on 15 letters, where G is generated by

xx=(1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13)

and

yy=(1, 5)(2, 14)(4, 12)(6, 10)(8, 15)(9, 13)

The Composition Factors and Normal Lattice of G are below:

```
CompositionFactors(G);
    G
    |  Alternating(5)
    *
    |  Cyclic(3)
    1
```

From the composition factors above we see that G is the direct product of (3 x A_5). We can further verify this using the following code:

```
IsIsomorphic(G, DirectProduct(CyclicGroup(3), Alt(5)));
True
```

Therefore, G is a direct product (3 x A_5) and we can find a representation of G.

Faithful Irreducible Monomial Representation of (3 x A_5)

Using the code below we can show that G has an irreducible monomial representation of degree 5.

```
H:=sub<G| (1, 6, 8)(2, 15, 13)(3, 11, 7)(4, 5, 14)(9, 10, 12),
(2, 12)(3, 10)(4, 13)(5, 11)(7, 15)(9, 14),
(2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15),
(1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)>;
#H;
/* 36 */
CH:=CharacterTable(H);
CG:=CharacterTable(G);
for l in [1..#CH] do if
Induction(CH[4],G) eq CG[15];
/* true */
#G/#H --> 180/36 = 5 (Note this is also the number of transversals)
```

Note: The above tells us we are going to have a 5 x 5 matrices because we have 5 t's. Therefore, our progenitor is $7^{*5} : (3xA5)$.

$$A(xx) = \begin{bmatrix} \phi(t_1xxt_1^{-1}) & \phi(t_1xxt_2^{-1}) & \phi(t_1xxt_3^{-1}) & \phi(t_1xxt_4^{-1}) & \phi(t_1xxt_5^{-1}) \\ \phi(t_2xxt_1^{-1}) & \phi(t_2xxt_2^{-1}) & \phi(t_2xxt_3^{-1}) & \phi(t_2xxt_4^{-1}) & \phi(t_1xxt_2^{-1}) \\ \phi(t_3xxt_1^{-1}) & \phi(t_3xxt_2^{-1}) & \phi(t_3xxt_3^{-1}) & \phi(t_3xxt_4^{-1}) & \phi(t_3xxt_5^{-1}) \\ \phi(t_4xxt_1^{-1}) & \phi(t_4xxt_2^{-1}) & \phi(t_4xxt_3^{-1}) & \phi(t_4xxt_4^{-1}) & \phi(t_4xxt_5^{-1}) \\ \phi(t_5xxt_1^{-1}) & \phi(t_5xxt_2^{-1}) & \phi(t_5xxt_3^{-1}) & \phi(t_5xxt_4^{-1}) & \phi(t_5xxt_5^{-1}) \end{bmatrix}$$

We have, $A(yy) = \begin{bmatrix} \phi(t_1yyt_1^{-1}) & \phi(t_1yyt_2^{-1}) & \phi(t_1yyt_3^{-1}) & \phi(t_1yyt_4^{-1}) & \phi(t_1yyt_5^{-1}) \\ \phi(t_2yyt_1^{-1}) & \phi(t_2yyt_2^{-1}) & \phi(t_2yyt_3^{-1}) & \phi(t_2yyt_4^{-1}) & \phi(t_1yyt_2^{-1}) \\ \phi(t_3yyt_1^{-1}) & \phi(t_3yyt_2^{-1}) & \phi(t_3yyt_3^{-1}) & \phi(t_3yyt_4^{-1}) & \phi(t_3yyt_5^{-1}) \\ \phi(t_4yyt_1^{-1}) & \phi(t_4yyt_2^{-1}) & \phi(t_4yyt_3^{-1}) & \phi(t_4yyt_4^{-1}) & \phi(t_4yyt_5^{-1}) \\ \phi(t_5yyt_1^{-1}) & \phi(t_5yyt_2^{-1}) & \phi(t_5yyt_3^{-1}) & \phi(t_5yyt_4^{-1}) & \phi(t_5yyt_5^{-1}) \end{bmatrix}$

$$A(xx) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \text{ and}$$

$$A(yy) = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now we want to see if it is a faithful representation and we use Cyclotomic Field 3 since we have a cube root of unity.

```
C:=CyclotomicField(3);
A:=[[C.1,0,0,0,0] : i in [1..5]];
B:=[[C.1,0,0,0,0] : i in [1..5]];
GG:=GL(5,C);
IsIsomorphic(G,sub<GG|GG!A,GG!B>);
/* True */
```

Hence $< A(xx), A(yy) >$ is a faithful monomial representation of G.

Permutation Representation of the Monomial Representation of $(3 \text{ x } A_5)$

Formula: $a_{ij} = q \implies t_i = t_j^q$

$$A(xx) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

where $a_{12} = 1$, $a_{24} = 1$, $a_{33} = 1$, $a_{41} = 1$, $a_{55} = 4$.

Therefore,

$$t_1 \rightarrow t_2^1,$$

$$t_2 \rightarrow t_4^1,$$

$$t_3 \rightarrow t_3^1.$$

$$t_4 \rightarrow t_1^1.$$

$$t_5 \rightarrow t_5^4.$$

We label $t_1, t_2, t_3, t_4, t_5, t_1^2, .., t_5^2, ..., t_1^6, ...,$ and t_5^6 by 1,2,3,4,5,6,..,10, 26, respectively, and compute the permutation representations of the matrices A(xx), and A(yy) through the following tables.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_1	t_2	t_3	t_4	t_5	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_1^3	t_2^3	t_3^3	t_4^3	t_5^3
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2	t_4	t_3	t_1	t_5^4	t_2^2	t_4^2	t_3^2	t_1^2	t_5	t_2^3	t_4^3	t_3^3	t_1^3	t_5^5
2	4	3	1	20	7	9	8	6	5	12	14	13	11	25

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
t_4^1	t_2^4	t_3^4	t_4^4	t_5^4	t_1^5	t_2^5	t_3^5	t_4^5	t_5^5	t_1^6	t_2^6	t_3^6	t_4^6	t_5^6
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2^4	t_4^4	t_3^4	t_1^4	t_5^2	t_2^5	t_4^5	t_3^5	t_1^5	t_5^6	t_2^6	t_4^6	t_3^6	t_1^6	t_5^3
17	19	18	16	10	22	24	23	21	30	27	29	28	26	15

Therefore, $A(xx) = (1, 2, 4)(5, 20, 10)(6, 7, 9)(11, 12, 14)(15, 25, 30)(16, 17, 19)$
$(21, 22, 24)(26, 27, 29)$

Now

$$A(yy) = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore,

$$t_1 \to t_3^2,$$

$$t_2 \to t_2^1,$$

$$t_3 \to t_1^4,$$

$$t_4 \to t_5^1,$$

$$t_5 \to t_4^1.$$

We have

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_1	t_2	t_3	t_4	t_5	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_1^3	t_2^3	t_3^3	t_4^3	t_5^3
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_3^2	t_2^1	t_1^4	t_5^1	t_4^1	t_3^4	t_2^2	t_1	t_5^2	t_4^2	t_3^6	t_2^3	t_1^5	t_5^3	t_4^3
8	2	16	5	4	18	7	1	10	9	28	12	21	15	14

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_1^5	t_2^5	t_3^5	t_4^5	t_5^5	t_1^6	t_2^6	t_3^6	t_4^6	t_5^6
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_3	t_2^4	t_1^2	t_5^4	t_4^4	t_3^3	t_2^5	t_1^6	t_5^5	t_4^5	t_3^5	t_2^6	t_1^3	t_5^6	t_4^6
3	17	6	20	19	13	22	26	25	24	23	27	11	30	29

Therefore, $A(yy) = (1,8)(3,16)(4,5)(6,18)(9,10)(11,28)(13,21)(14,15)$
$(19,20)(23,26)(24,25)(29,30)$

Symmetric Presentation of the Monomial Progenitor of $7^{*5} : m(3 \text{ x } A_5)$

First we need to check that the permutation representation $< A(xx), A(yy)$ is isomorphic to G and we can do that by using the following magma code.

```
S:=Sym(30);
xx:=S!(1,2,4)(5,20,10)(6,7,9)(11,12,14)(15,25,30)(16,17,19)(
21,22,24)(26,27,29);
yy:=S!(1,8)(3,16)(4,5)(6,18)(9,10)(11,28)(13,21)(14,15)(19,20
)(23,26)(24,25)(29,30);
N:=sub<S|xx,yy>;
#N;
/*180*/
IsIsomorphic(sub<GG|GG!A,GG!B>,N);
/* true */
```

We now give symmetric presentations of the progenitor $7^{*5} : (3xA5)$. We first find a presentation of N.

Thus, a presentation of N is

$N < x, y > = < x, y, t | x^3, y^2, (y * x^-1 * y * x * y * x^{-1})^3, x^-1 * y * x^{-1} * y * x^{-1} * y *$
$x^-1 * y * x^-1 * y * x * y * x * y * x * y * x * y * x * y >.$

Now we need to find the normalizer of $< t > = < t_1 > = \{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6\}$. We can then translate $\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6\}$ to $t_1 = 1$, $t_1^2 = 6$, $t_1^3 = 11$, $t_1^4 = 16$, $t_1^5 = 21$, $t_1^6 = 26$. Now we can find the stabilizer of $\{1, 6, 11, 16, 21, 26\}$ by using the function below:

```
Nt:=Stabiliser(N,{1,6,11,16,21,26});
```

Next step is to find where 1 is sent in all the stabilizer elements. We have

$1^{x^{-1}*y*x*y^{-1}*x*y*x^-1*y*x^2*y^{-1}*x^2*y*x*y*x*y} = 1,$

$\implies 1^{x^{-1}*y*x*y^{-1}*x*y*x^{-1}*y*x^2*y^{-1}*x^2*y*x*y*x*y} = t,$

$1^{x*y*x^{-1}*y*x*y} = 1 \implies 1^{x*y*x^{-1}*y*x*y} = t,$

$1^{y*x^{-1}*y*x*y^{-1}*x*y*x*y*x*y*x*y} = 1,$

$\implies 1^{y*x^{-1}*y*x*y^{-1}*x*y*x*y*x*y*x*y} = t^4.$

We will ensure that Q is a monomial progenitor by testing it with the stabiliser of Stabiliser(N,1)

Now we need to find the permutations in N that take 1 to 6,11,16,21,26,2,12 (these are the orbits of our stabiliser).

Therefore our presentation will include the following:

$(t, t^{x^{-1}*y*x^{-1}*y*x}),$

$(t, t^e),$

$(t, t^{x^{-1}*y*x*y*x}),$

$(t, t^{x^{-1}*y*x*y*x}),$

$(t, t^{x^{-1}*y*x^{-1}*y*x}),$

$(t, t^x),$

$(t^3, t^x),$

$Q < x, y, t >=< x, y, t | x^3, y^2, (y*x^-1*y*x*y*x^-1)^3, x^-1*y*x^-1*y*x^-1*y*x^-1*$
$y*x^-1*y*x*y*x*y*x*y*x*y*x*y, t^7, t^{(}x^-1*y*x*y^-1*x*y*x^-1*y*x^2*y^-1*x^2*y*x*$
$y*x*y) = t, t^{(}x*y*x^-1*y*x*y) = t, t^{(}y*x^-1*y*x*y^-1*x*y*x*y*x*y*x*y) = t^4,$
$(t, t^{x^-1*y*x^-1*y*x}), (t, t^e), (t, t^{x^-1*y*x*y*x}), (t, t^{x^-1*y*x*y*x}), (t, t^{x^-1*y*x^-1*y*x}), (t, t^x), (t^3, t^x) >$

.

The index of Q should be 7^5 and the order of Q will be $7^5 \times 180 = 3025260$. We then use the Grindstaff's Lemma to confirm this.Therefore the monomial progenitor of $7^{*5} : m(3 \text{ x } A_5)$ is

$Q < x, y, t >=< x, y, t | (y^-1*x)^3, y^-1*x^-3*y^-1*x^2, y^-3*x^-1*y^2*x^-1, t^7, (t, x^2*$
$y^2), (t, x^2*y^2*x^3*y^-1), t^{(}x^3*y^-1*x^2*y^2*x^2*y^2*x^-1*y) = t^4$

Appendix Magma Code Monomial Representation of $7^{*5} : m(3 \text{ x } A_5)$

```
T:=TransitiveGroups(15);
G:=T[15]; CG:=CharacterTable(G); S:=Subgroups(G); H:=S[19]'subgroup;
CH:=CharacterTable(H); Induction(CH[2],G) eq CG[15];
/* True */
/*
for g,h in G do if sub<G|g,h> eq G then A:=g; B:=h; end if; end for;
*/
/* interrupt it */
A,B; /* These values change */
/*
(1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13)
(1, 5)(2, 14)(4, 12)(6, 10)(8, 15)(9, 13) */
S:=Sym(15);
xx:=S!(1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13);
yy:=S!(1, 5)(2, 14)(4, 12)(6, 10)(8, 15)(9, 13);
G:=sub<S|xx,yy>;

/* Fixing value from H
Permutation group H acting on a set of cardinality 15
Order = 36 = 2^2 * 3^2
    (1, 6, 8)(2, 15, 13)(3, 11, 7)(4, 5, 14)(9, 10, 12)
    (2, 12)(3, 10)(4, 13)(5, 11)(7, 15)(9, 14)
    (2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15)
    (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)

*/
H:=sub<G| (1, 6, 8)(2, 15, 13)(3, 11, 7)(4, 5, 14)(9, 10, 12),
(2, 12)(3, 10)(4, 13)(5, 11)(7, 15)(9, 14),
(2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15),
(1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)>;
CH:=CharacterTable(H);
CG:=CharacterTable(G);
for l in [1..#CH] do  if
Induction(CH[l],G) eq CG[15] then l; end if; end for;
/*
4
5
*/
Induction(CH[4],G) eq CG[15];
/* true */
```

```
Index(G,H);
/* 5 */
#H;
/* 36 */
5*36;
Factorization(180);
/* [ <2, 2>, <3, 2>, <5, 1> ] 2^2*3^2*5^1  */


CH:=CharacterTable(H);
/* How many linear does it have?
How many linear chars does it have?
*/
for i in [1..#CH] do if CH[i](Id(H)) eq 1 then i; end if; end for;
/*
1
2
3
4
5
6
7
8
9
*/
CH;
/*
Character Table of Group H
--------------------------
```

```
-------------------------------------------------------------
Class |  1  2     3      4    5    6    7    8    9   10   11   12
Size  |  1  3     1      1    4    4    4    4    4    4    3    3
Order |  1  2     3      3    3    3    3    3    3    3    6    6
-------------------------------------------------------------
p  =  2  1  1     4      3    6    5    8    7   10    9    3    4
p  =  3  1  2     1      1    1    1    1    1    1    1    2    2
-------------------------------------------------------------
X.1   +  1  1     1      1    1    1    1    1    1    1    1    1
X.2   0  1  1     J   -1-J-1-J   J-1-J   J    1  1-1-J   J
X.3   0  1  1     1      1  J-1-J-1-J   J  J-1-J   1    1
X.4   0  1  1  -1-J    J-1-J   J    1    1  J-1-J  J-1-J
X.5   0  1  1  -1-J     J  J-1-J  J-1-J   1    1  J-1-J
X.6   0  1  1     1    1-1-J   J  J-1-J-1-J   J    1    1
```

```
X.7   0   1  1     -1-J       J   1   1-1-J   J-1-J    J   J-1-J
X.8   0   1  1      J    -1-J  1   1   J-1-J   J-1-J-1-J   J
X.9   0   1  1      J    -1-J  J-1-J  1  1-1-J  J-1-J   J
X.10  +   3 -1      3       3  0   0   0   0   0   0  -1  -1
X.11  0   3 -1    3*J  -3-3*J  0   0   0   0   0   0 1+J  -J
X.12  0   3 -1 -3-3*J    3*J   0   0   0   0   0   0  -J 1+J


Explanation of Character Value Symbols
--------------------------------------


J = RootOfUnity(3)
*/
T:=Transversal(G,H);
#T;
/* 5 */
for i in[1..#T] do i,T[i]; end for;
/*
1 Id(G)
2 (1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13)
3 (1, 5, 2, 13, 11, 8, 15, 9, 12, 7, 6, 10, 4, 14, 3)
4 (1, 12, 7)(2, 4, 9)(3, 8, 14)(6, 13, 11)
5 (1, 4, 13, 11, 10, 6, 9, 14, 3, 15, 8, 2, 12, 7, 5)
*/
 CH[5];
/*
( 1, 1, -zeta(3)_3 - 1, zeta(3)_3, zeta(3)_3, -zeta(3)_3 - 1, zeta(3)_3,
-zeta(3)_3- 1, 1, 1, zeta(3)_3, -zeta(3)_3 - 1 )
*/


C:=CyclotomicField(3);
A:=[[C.1,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do A[i,j]:=0; end for;
/* A consists of 0's */
for i,j in [1..5] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[5](T[i]*xx*T[j]^-1);end if; end for;
GG:=GL(5,C);
GG!A;
/*
[        0          1          0          0          0]
[        0          0          0          1          0]
[        0          0          1          0          0]
[        1          0          0          0          0]
[        0          0          0          0 -zeta_3 - 1]
```

```
*/
Order(xx);
/* 3 */
Order(GG!A);
/* 3 */

B:=[[C.1,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do B[i,j]:=0; end for;
/* A consists of 0's */
for i,j in [1..5] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[5](T[i]*yy*T[j]^-1);end if; end for;
GG:=GL(5,C);
GG!B;
/* ASK about Matrix changing
[         0         0    zeta_3         0         0]
[         0         1         0         0         0]
[-zeta_3 - 1         0         0         0         0]
[         0         0         0         0         1]
[         0         0         0         1         0]

*/
Order(yy);
/* 2 */
Order(GG!B);
/* 2 */
IsIsomorphic(G,sub<GG|GG!A,GG!B>);
/* true */
/* Note that zeta_3 = 9 Ask Hasan how this happens one more time*/
CH[5];
/*Replace zeta_3 with 9

( 1, 1, -zeta(3)_3 - 1, zeta(3)_3,
zeta(3)_3, -zeta(3)_3 - 1,
zeta(3)_3,-zeta(3)_3 - 1, 1, 1,
zeta(3)_3, -zeta(3)_3 - 1 )
 */
PrimitiveRoot(7);
/* 3 */
CH5seq:= [ 1, 1, -9 - 1, 9, 9, -9 - 1, 9,-9 - 1, 1, 1, 9, -9 - 1  ];


A:=[[0,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do if T[i]*xx*T[j]^-1
in H then for k in [1..#CH[5]] do if
```

```
CH[5] (T[i]*xx*T[j]^-1) eq CH[5][k] then
A[i,j]:=CH5seq[k] mod 7 ;end if; end for; end if; end for;
GG:=GL(5,7);
GG!A;
/*
[0 1 0 0 0]
[0 0 0 1 0]
[0 0 1 0 0]
[1 0 0 0 0]
[0 0 0 0 4]
*/

B:=[[0,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do if T[i]*yy*T[j]^-1
in H then for k in [1..#CH[5]] do if
CH[5] (T[i]*yy*T[j]^-1) eq CH[5][k] then
B[i,j]:=CH5seq[k] mod 7 ;end if; end for; end if; end for;
GG:=GL(5,7);
GG!B;
/*
[0 0 2 0 0]
[0 1 0 0 0]
[4 0 0 0 0]
[0 0 0 0 1]
[0 0 0 1 0]
*/
IsIsomorphic(G,sub<GG|GG!A,GG!B>);
/* true */
S:=Sym(30);

xx:=S!(1,2,4)(5,20,10)(6,7,9)(11,12,14)
(15,25,30)(16,17,19)(21,22,24)(26,27,29);
yy:=S!(1,8)(3,16)(4,5)(6,18)(9,10)(11,28)
(13,21)(14,15)(19,20)(23,26)(24,25)(29,30);
N:=sub<S|xx,yy>;
#N;
/*180*/
IsIsomorphic(sub<GG|GG!A,GG!B>,N);
/* true */

FPGroup(N);
/*
 $.1^3 = Id($) --> x^3
    $.2^2 = Id($)
```

```
    ($.2 * $.1^-1 * $.2 * $.1 * $.2 * $.1^-1)^3 = Id($)
    $.1^-1 * $.2 * $.1^-1 * $.2 * $.1^-1
    * $.2 * $.1^-1 * $.2 * $.1^-1 * $.2
    * $.1 * $.2 * $.1 * $.2 * $.1 * $.2
    * $.1 * $.2 * $.1 * $.2 = Id($)

x^3 = Id($)
y^2 = Id($)
(y *x^-1 *y *x *y *x^-1)^3 = Id($)
x^-1 *y *x^-1 *y *x^-1 *y *x^-1 *y
*x^-1 *y * x *y *x *y *x *y *x *y *x *y = Id($)

G<x,y,t>:=Group<x,y,t|x^3, y^2,(y *x^-1
*y *x *y *x^-1)^3,x^-1 *y *x^-1 *y *x^-1
*y *x^-1 *y *x^-1 *y * x *y *x *y *x *y
*x *y *x *y,t^7,
*/


Nt:=Stabiliser(N,{1,6,11,16,21,26}); /* Get the #'s
from the perm building chart */
 /* These perms are in the stabiliser */
for n in N do if {1,6,7,8,9,10}^n eq {1,6,7,8,9,10}
 then n; end if; end for;
Nt;
/*
Permutation group Nt acting on a set of cardinality 30
Order = 36 = 2^2 * 3^2
    (2, 3)(4, 20)(5, 9)(7, 8)(10, 19)(12, 13)(14, 25)(15,
    29)(17, 18)(22,
        23)(24, 30)(27, 28)
    (2, 18, 4)(3, 9, 7)(5, 20, 10)(8, 19, 17)(12, 23,
    14)(13, 29, 27)(15, 25,
        30)(22, 28, 24)
    (1, 16, 6)(2, 3, 10)(5, 17, 18)(7, 8, 20)(11, 21,
    26)(12, 13, 30)(15, 22,
        23)(25, 27, 28)
*/
Set(Nt);
w:=WordGroup(N);
rho:=InverseWordMap(N);
A:=N!(2, 3)(4, 20)(5, 9)(7, 8)(10,
19)(12, 13)(14, 25)(15, 29)(17, 18)(22,
23)(24, 30)(27, 28);/* t^()= t (t,_) */
B:=N!(2, 18, 4)(3, 9, 7)(5, 20, 10)(8,
```

```
19, 17)(12, 23, 14)(13, 29, 27)(15, 25,
30)(22, 28, 24); /* t^()= t (t,_) */
C:=N!(1, 16, 6)(2, 3, 10)(5, 17, 18)(7,
8, 20)(11, 21, 26)(12, 13, 30)(15, 22,
23)(25, 27, 28);
A@rho;
AA:=function(w)
    w1 := w.1^-1; w3 := w1 * w.2; w2 := w.2^-1; w4 := w2 * w.1; w5 := w.1 * w4;
    w6 := w3 * w5; w8 := w6 * w.2; w9 := w8 * w6; w10 := w9 * w6; w11 := w10 *
    w.1; w12 := w11 * w.2; w13 := w12 * w.1; w14 := w13 * w.2; w15 := w14 * w.1;
    w16 := w15 * w.2; return w16;
end function;
AA(FPGroup(N));
/*
$.1^-1 * $.2 * $.1 * $.2^-1 * $.1 * $.2 * $.1^-1 * $.2 * $.1^2 * $.2^-1 * $.1^2
* $.2 * $.1 * $.2 * $.1 * $.2

x^-1 *y *x *y^-1 *x *y *x^-1 *y *x^2 *y^-1 *x^2
*y *x *y *x *y
*/
B@rho;
BB:= function(w)
    w17 := w.1 * w.2; w1 := w.1^-1; w18 := w17 * w1; w19 := w18 * w.2; w20 :=
    w19 * w.1; w21 := w20 * w.2; return w21;
end function;
BB(FPGroup(N));
/*
$.1 * $.2 * $.1^-1 * $.2 * $.1 * $.2

x * y * x^-1 * y * x * y
*/
C@rho;
CC:=function(w)
    w1 := w.1^-1; w3 := w1 * w.2; w2 := w.2^-1; w4 := w2 * w.1; w5 := w.1 * w4;
    w6 := w3 * w5; w8 := w.2 * w6; w9 := w8 * w.2; w10 := w9 * w.1; w11 := w10 *
    w.2; w12 := w11 * w.1; w13 := w12 * w.2; w14 := w13 * w.1; w15 := w14 * w.2;
    return w15;
end function;
CC(FPGroup(N));
/*
$.2 * $.1^-1 * $.2 * $.1 * $.2^-1 * $.1 * $.2 * $.1 * $.2 * $.1 * $.2 * $.1 *
$.2

y * x^-1 * y * x * y^-1 * x * y * x * y * x * y * x * y
```

```
*/
Orbits(Stabiliser(N,1));
/*
 GSet{@ 1 @},
    GSet{@ 6 @},
    GSet{@ 11 @},
    GSet{@ 16 @},
    GSet{@ 21 @},
    GSet{@ 26 @},
    GSet{@ 2, 10, 4, 20, 8, 18, 7, 5, 17, 9, 19, 3 @},
    GSet{@ 12, 30, 14, 25, 28, 23, 27, 15, 22, 29, 24, 13 @}
*/

/* Work with this one */
G<x,y,t>:=Group<x,y,t|x^3, y^2,(y *x^-1
*y *x *y *x^-1)^3,x^-1 *y *x^-1 *y *x^-1
*y *x^-1 *y *x^-1 *y * x *y *x *y *x *y
*x *y *x *y,t^7,t^(x^-1 *y *x *y^-1 *x
*y *x^-1 *y *x^2 *y^-1 *x^2 *y *x *y *x
*y)=t,t^(x * y * x^-1 * y * x *
y)=t,t^(y * x^-1 * y * x * y^-1 * x * y
* x * y * x * y * x * y)=t^4,(t,t^x)/*
Need to remove after test*/>;

/* #G Should get 7^5*180 */
Index(G,sub<G|x,y>); /* should be 7^5 */
/* 16807 */
#sub<G|x,y>;
```

Example 3.9

Monomial Representation of $7^5 :_m (3xA_5)$

Isomorphism Type of G

We are given G, a transitive group on 15 letters, where G is generated by

xx=(1, 10, 9, 13, 3, 6, 15, 2, 14, 7, 8, 5, 4, 12, 11)

and

yy=(1, 5, 3, 2, 12, 6, 10, 7, 4, 13, 8, 15, 11, 9, 14)

We will now evaluate the Composition Factors and Normal Lattice of G using the Magma codes below:

```
S:=Sym(15);
xx:=S!(1, 10, 9, 13, 3, 6, 15, 2, 14, 7, 8, 5, 4, 12, 11);
yy:=S!(1, 5, 3, 2, 12, 6, 10, 7, 4, 13, 8, 15, 11, 9, 14);
G:=sub<S|xx,yy>;
#G;
/* 180 */
CompositionFactors(G);
/*
    G
    |  Alternating(5)
    *
    |  Cyclic(3)
    1
```

From the composition factor results we believe that we will get a direct product of $(3 \times A_5)$. We can further verify this using the following code:

```
IsIsomorphic(G, DirectProduct(CyclicGroup(3), Alt(5)));
True
```

Therefore, G is a direct product $(3 \times A_5)$ and we can find a representation of G.

Faithful Irreducible Monomial Representation of $(3 \times A_5)$

Using the code below we can show that G has an irreducible monomial representation of degree 5.

```
H:=sub<G| (1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12, 11),
    (1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14),
    (1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8, 13),
    (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)>;
#H;
/* 36 */
CH:=CharacterTable(H);
CG:=CharacterTable(G);
for l in [1..#CH] do if
Induction(CH[5],G) eq CG[14];
/* true */
#G/#H --> 180/36 = 5 (Note this is also the number of transversals)
```

Note: The above tells us we are going to have a 5 x 5 matrices because we have 5 t's.

$$A(xx) = \begin{bmatrix} \phi(t_1xxt_1^{-1}) & \phi(t_1xxt_2^{-1}) & \phi(t_1xxt_3^{-1}) & \phi(t_1xxt_4^{-1}) & \phi(t_1xxt_5^{-1}) \\ \phi(t_2xxt_1^{-1}) & \phi(t_2xxt_2^{-1}) & \phi(t_2xxt_3^{-1}) & \phi(t_2xxt_4^{-1}) & \phi(t_1xxt_2^{-1}) \\ \phi(t_3xxt_1^{-1}) & \phi(t_3xxt_2^{-1}) & \phi(t_3xxt_3^{-1}) & \phi(t_3xxt_4^{-1}) & \phi(t_3xxt_5^{-1}) \\ \phi(t_4xxt_1^{-1}) & \phi(t_4xxt_2^{-1}) & \phi(t_4xxt_3^{-1}) & \phi(t_4xxt_4^{-1}) & \phi(t_4xxt_5^{-1}) \\ \phi(t_5xxt_1^{-1}) & \phi(t_5xxt_2^{-1}) & \phi(t_5xxt_3^{-1}) & \phi(t_5xxt_4^{-1}) & \phi(t_5xxt_5^{-1}) \end{bmatrix}$$

$$A(yy) = \begin{bmatrix} \phi(t_1yyt_1^{-1}) & \phi(t_1yyt_2^{-1}) & \phi(t_1yyt_3^{-1}) & \phi(t_1yyt_4^{-1}) & \phi(t_1yyt_5^{-1}) \\ \phi(t_2yyt_1^{-1}) & \phi(t_2yyt_2^{-1}) & \phi(t_2yyt_3^{-1}) & \phi(t_2yyt_4^{-1}) & \phi(t_1yyt_2^{-1}) \\ \phi(t_3yyt_1^{-1}) & \phi(t_3yyt_2^{-1}) & \phi(t_3yyt_3^{-1}) & \phi(t_3yyt_4^{-1}) & \phi(t_3yyt_5^{-1}) \\ \phi(t_4yyt_1^{-1}) & \phi(t_4yyt_2^{-1}) & \phi(t_4yyt_3^{-1}) & \phi(t_4yyt_4^{-1}) & \phi(t_4yyt_5^{-1}) \\ \phi(t_5yyt_1^{-1}) & \phi(t_5yyt_2^{-1}) & \phi(t_5yyt_3^{-1}) & \phi(t_5yyt_4^{-1}) & \phi(t_5yyt_5^{-1}) \end{bmatrix}$$

$$A(xx) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$A(yy) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we want to see if it is a faithful representation and we use Cyclotomic Field 3 since we have a cube root of unity.

```
C:=CyclotomicField(3);
A:=[[C.1,0,0,0,0] : i in [1..5]];
B:=[[C.1,0,0,0,0] : i in [1..5]];
```

```
GG:=GL(5,C);
IsIsomorphic(G,sub<GG|GG!A,GG!B>);
/* True */
```

Hence $< A(xx), A(yy) >$ is a faithful monomial representation of G.

Permutation Representation of the Monomial Representation of $(3 \text{ x } A_5)$

Formula: $a_{ij} = q \implies t_i = t_j^q$

$$A(xx) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $a_{12} = 1$, $a_{24} = 1$, $a_{35} = 4$, $a_{43} = 1$, $a_{51} = 1$.

Therefore,

$$t_1 \to t_2^1,$$

$$t_2 \to t_4^1,$$

$$t_3 \to t_5^4.$$

$$t_4 \to t_3^1.$$

$$t_5 \to t_1^1.$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_1	t_2	t_3	t_4	t_5	t_1^2	t_1^3	t_1^4	t_1^5	t_1^6	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2	t_4	t_5^4	t_3	t_1	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6	t_4^2	t_4^3	t_4^4	t_4^5	t_4^6
2	4	28	3	1	11	12	13	14	15	21	22	23	24	25

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
t_3^2	t_3^3	t_3^4	t_3^5	t_3^6	t_4^2	t_4^3	t_4^4	t_4^5	t_4^6	t_5^2	t_5^3	t_5^4	t_5^5	t_5^6
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_5	t_5^5	t_5^2	t_5^6	t_5^3	t_3^2	t_3^3	t_3^4	t_3^5	t_3^6	t_1^2	t_1^3	t_1^4	t_1^5	t_1^6
5	29	26	30	27	16	17	18	19	20	6	7	8	9	10

Therefore, $A(xx) = (1, 2, 4, 3, 28, 8, 13, 23, 18, 26, 6, 11, 21, 16, 5)$

$(7, 12, 22, 17, 29, 9, 14, 24, 19, 30, 10$

$, 15, 25, 20, 27)$

$$B(yy) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $a_{13} = 1$, $a_{24} = 2$, $a_{32} = 2$, $a_{45} = 2$, $a_{51} = 4$.

Therefore,

$t_1 \rightarrow t_3^1,$

$t_2 \rightarrow t_4^2,$

$t_3 \rightarrow t_2^2,$

$t_4 \rightarrow t_5^2,$

$$t_5 \to t_1^4,$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_1	t_2	t_3	t_4	t_5	t_1^2	t_1^3	t_1^4	t_1^5	t_1^6	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_3	t_4^2	t_2^2	t_5^2	t_1^4	t_3^2	t_3^3	t_3^4	t_3^5	t_3^6	t_4^4	t_4^6	t_4^1	t_4^3	t_4^5
3	21	11	26	8	16	17	18	19	20	23	25	4	22	24

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
t_3^2	t_3^3	t_3^4	t_3^5	t_3^6	t_4^2	t_4^3	t_4^4	t_4^5	t_4^6	t_5^2	t_5^3	t_5^4	t_5^5	t_5^6
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2^4	t_2^6	t_2	t_2^3	t_2^5	t_5^4	t_5^6	t_5	t_5^3	t_5^5	t_1	t_1^5	t_1^2	t_1^6	t_1^3
13	15	2	12	14	28	30	5	27	29	1	9	6	10	7

Therefore, $B(yy) = (1, 3, 11, 23, 5, 8, 18, 2, 21, 28, 6, 16, 13, 4, 26)$
$$(7, 17, 15, 24, 27, 9, 19, 12, 25, 29, 10, 20, 14, 22, 30)$$

Symmetric Presentation of the Monomial Progenitor of $7^{*5} : m(3 \text{ x } A_5)$

First we need to check that the permutation representation is isomorphic to G and we can do that will the following magma code.

```
S:=Sym(30);
xx:=S!(1,2,4,3,28,8,13,23,18,26,6,11,21,16,5)(7,12,22,17,29,9,14,24,19,
30,10,15,25,20,27);
yy:=S!(1,3,11,23,5,8,18,2,21,28,6,16,13,4,26)(7,17,15,24,27,9,19,12,25,
29,10,20,14,22,30);
N:=sub<S|xx,yy>;
#N;
/*180*/
IsIsomorphic(N,G);
/* true */
FPGroup(N);
/*
```

```
Finitely presented group on 2 generators
Relations
    ($.2^-1 * $.1)^3 = Id($)   --> (y^-1 *x)^3
    $.2^-1 * $.1^-3 * $.2^-1 * $.1^2 = Id($) --> y^-1*x^-3*y^-1*x^2
    $.2^-3 * $.1^-1 * $.2^2 * $.1^-1 = Id($) --> y^-3*x^-1*y2*x^-1
```

We can now build the presentation in terms of x and y:

$Q < x, y, t >=< x, y | (y^-1 * x)^3, y^-1 * x^{-3} * y^-1 * x^2, y^{-3} * x^-1 * y^2 * x^-1 >$. Now we need to find the normalizer of $< t >=< t_1 >= \{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6\}$. We can then translate $\{t_1, t_1^2, t_1^3, t_1^4, t_1^5, t_1^6\}$ to $t_1 = 1$ $t_1^2 = 6$ $t_1^3 = 7$ $t_1^4 = 8$ $t_1^5 = 9$ $t_1^6 = 10$ and we get this from our prevous labeling on page 3. Now we can find the stabilizer of $\{1, 6, 7, 8, 9, 10\}$ by using the function below:

```
Nt:=Stabiliser(N,{1,6,7,8,9,10});
```

Next step is to find where 1 is sent in all the stabilizer elements. We can see that
$1^{x^2*y^2} = 1 \implies 1^{x^2*y^2} = t$
$1^{x^2*y^2*x^3*y^{-1}} = 1 \implies 1^{x^2*y^2*x^3*y^{-1}} = t$
$1^{x^3*y^{-1}*x^2*y^2*x^2*y^2*x^{-1}*y} = 1 \implies 1^{x^3*y^{-1}*x^2*y^2*x^2*y^2*x^{-1}*y} = t^4$

We will ensure that Q is a monomial progenitor by testing it with the stabiliser of Stabiliser(N,1)

Now we need to find the permutations in N that take 1 to 2,6,7,8,9,10,12.

Therefore our presentation will include the following:

(t, t^x),

(t, t^{y^-1*x}),

$(t, t^{x^{-1}})$,

$(t, t^{x^{-1}*y})$,

(t, t^{y^-2}),

(t, t^{x*y*x}),

(t, t^e),

$Q < x, y, t >=< (y^-1 * x)^3, y^-1 * x^{-3} * y^-1 * x^2, y^{-3} * x^-1 * y^2 * x^-1, t^7,$

$(t, x^2 * y^2), (t, x^2 * y^2 * x^3 * y^-1), t^{(}x^3 * y^-1 * x^2 * y^2 * x^2 * y^2 * x^-1 * y) = t^4,$

$(t, t^x), (t, t^{y^-1*x}), (t, t^{x^{-1}}), (t, t^{x^{-1}*y}), (t, t^{y^-2}), (t, t^{x*y*x}), (t, t^e) >$

The index of Q should be 7^5 and the order of Q will be $7^5 \times 180 = 3025260$. We then use the Grindstaff's Lemma to confirm this.Therefore the monomial progenitor of $7^{*5} : m(3 \text{ x } A_5)$ is

$Q < x, y, t >=< (y^-1 * x)^3, y^-1 * x^-3 * y^-1 * x^2, y^-3 * x^-1 * y^2 * x^-1, t^7,$

$(t, x^2 * y^2), (t, x^2 * y^2 * x^3 * y^-1), t^{(}x^3 * y^-1 * x^2 * y^2 * x^2 * y^2 * x^-1 * y) = t^4 >$

Appendix Magma Code Monomial Representation of $7^{*5} : m(3 \text{ x } A_5)$

```
T:=TransitiveGroups(15);
G:=T[15];
T:=TransitiveGroups(15);
G:=T[15]; CG:=CharacterTable(G); S:=Subgroups(G);
H:=S[19]'subgroup; CH:=CharacterTable(H);
for g,h in G do if sub<G|g,h> eq G then A:=g;
B:=h; end if; end for;
*/
/* interrupt it */
A,B; /* These values change and give us our x and y */
S:=Sym(15);
xx:=S!(1, 10, 9, 13, 3, 6, 15, 2, 14, 7, 8, 5, 4, 12, 11);
yy:=S!(1, 5, 3, 2, 12, 6, 10, 7, 4, 13, 8, 15, 11, 9, 14);
G:=sub<S|xx,yy>;
CompositionFactors(G);
/*
    G
    |  Alternating(5)
    *
    |  Cyclic(3)
    1
*/
G eq T[15];
/* true */
#G;
/* 180 */
/*
Permutation group H acting on a set of cardinality 15
Order = 36 = 2^2 * 3^2
    (1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12, 11)
    (1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14)
    (1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8, 13)
    (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)
*/
H:=sub<G| (1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12, 11),
    (1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14),
    (1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8, 13),
    (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)>;
```

```
#H;
/* 36 */
CH:=CharacterTable(H);
CG:=CharacterTable(G);
for l in [1..#CH] do if
Induction(CH[l],G) eq CG[14] then l; end if; end for;
/*
5
6
*/
Induction(CH[5],G) eq CG[14];
/* true */
Index(G,H);
/* 5 */
#H;
/* 36 */
36*5;
Factorization(180);
/* [ <2, 2>, <3, 2>, <5, 1> ] 2^2*3^2*5^1  */
H;
/*
Permutation group H acting on a set of cardinality 15
Order = 36 = 2^2 * 3^2
    (1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12, 11)
    (1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14)
    (1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8, 13)
    (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)
*/
H eq sub<G|(1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14,
7)(8, 12, 11),(1, 4)(2, 8)(3, 12)(6, 9)(7,
13)(11, 14),(1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8,
13),(1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12,
14, 13)>;

H:=sub<G|(1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14,
7)(8, 12, 11),(1, 4)(2, 8)(3, 12)(6, 9)(7,
13)(11, 14),(1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8,
13),(1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12,
14, 13)>;
CH:=CharacterTable(H);
/* How many linear does it have?
How many linear chars does it have?
*/
```

```
for i in [1..#CH] do if CH[i](Id(H)) eq 1 then i; end if; end for;
/*
1
2
3
4
5
6
7
8
9
*/
CH;
/*
Character Table of Group H
--------------------------
```

```
----------------------------------------------------------------
Class |  1  2      3      4   5   6   7   8   9  10  11  12
Size  |  1  3      1      1   4   4   4   4   4   4   3   3
Order |  1  2      3      3   3   3   3   3   3   3   6   6
----------------------------------------------------------------
p  =  2  1  1      4      3   6   5   8   7  10   9   3   4
p  =  3  1  2      1      1   1   1   1   1   1   1   2   2
----------------------------------------------------------------
X.1   +  1  1      1      1   1   1   1   1   1   1   1   1
X.2   0  1  1   -1-J    J-1-J   J  J-1-J   1   1    J-1-J
X.3   0  1  1   -1-J      J J-1-J  1  1-1-J   J    J-1-J
X.4   0  1  1      1   1-1-J J-1-J J-1-J   J   1   1
X.5   0  1  1      J    -1-J J-1-J-1-J  J   1  1-1-J   J
X.6   0  1  1      J  -1-J-1-J   J   1   1 J-1-J-1-J   J
X.7   0  1  1      1      1 J-1-J J-1-J J-1-J   1   1
X.8   0  1  1      J    -1-J   1   1 J-1-J-1-J J-1-J   J
X.9   0  1  1   -1-J      J   1 1-1-J   J J-1-J J-1-J
X.10  +  3 -1      3      3   0   0   0   0   0   0  -1  -1
X.11  0  3 -1    3*J -3-3*J   0   0   0   0   0   0 1+J  -J
X.12  0  3 -1 -3-3*J    3*J   0   0   0   0   0   0  -J 1+J
```

```
Explanation of Character Value Symbols
--------------------------------------
```

```
J = RootOfUnity(3)
```

```
*/
T:=Transversal(G,H);
#T;
/* 5 */
for i in[1..#T] do i,T[i]; end for;
/*
1 Id(G)
2 (1, 10, 9, 13, 3, 6, 15, 2, 14, 7, 8, 5, 4, 12, 11)
3 (1, 5, 3, 2, 12, 6, 10, 7, 4, 13, 8, 15, 11, 9, 14)
4 (1, 9, 3, 15, 14, 8, 4, 11, 10, 13, 6, 2, 7, 5, 12)
5 (1, 11, 12, 4, 5, 8, 7, 14, 2, 15, 6, 3, 13, 9, 10)
*/
T[1]*xx*T[1]^-1 in H;
/* False */
T[1]*xx*T[2]^-1 in H;
/* True */
 CH[5];
/*
( 1, 1, zeta(3)_3, -zeta(3)_3 - 1, zeta(3)_3, -zeta(3)_3 - 1,
-zeta(3)_3 - 1, zeta(3)_3, 1, 1, -zeta(3)_3 - 1, zeta(3)_3 )
*/
T[11]*xx*T[11]^-1 in H;
/* true */
T[11]*xx*T[11]^-1;


C:=CyclotomicField(3);
A:=[[C.1,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do A[i,j]:=0; end for;
/* A consists of 0's */
for i,j in [1..5] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[5](T[i]*xx*T[j]^-1);end if; end for;
GG:=GL(5,C);
GG!A;
/*
[         0         1         0         0         0]
[         0         0         0         1         0]
[         0         0         0         0 -zeta_3 - 1]
[         0         0         1         0         0]
[         1         0         0         0         0]
*/
Order(xx);
/* 15 */
Order(GG!A);
```

```
/*15 */

B:=[[C.1,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do B[i,j]:=0; end for;
/* A consists of 0's */
for i,j in [1..5] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[5](T[i]*yy*T[j]^-1);end if; end for;
GG:=GL(5,C);
GG!B;
/*
[          0          0          1          0          0]
[          0          0          0      zeta_3          0]
[          0      zeta_3          0          0          0]
[          0          0          0          0     zeta_3]
[-zeta_3 - 1          0          0          0          0]
*/
Order(yy);
/*15*/
Order(GG!B);
/* 15 */
IsIsomorphic(G,sub<GG|GG!A,GG!B>);
/* true */
/* Note that zeta_3 = 9 Ask Hasan how this happens one more time*/
CH[5];
/*Replace zeta_3 with 9

( 1, 1, zeta(3)_3, -zeta(3)_3 - 1, zeta(3)_3, -zeta(3)_3 - 1, -zeta(3)_3 - 1,
    zeta(3)_3, 1, 1, -zeta(3)_3 - 1, zeta(3)_3 )
 */
PrimitiveRoot(7);
/* 3 */
CH5seq:= [1, 1, (9), -(9) - 1, (9), -(9) - 1, -(9) - 1,
    (9), 1, 1, -(9) - 1, (9) ];

T:=Transversal(G,H);
A:=[[0,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do if T[i]*xx*T[j]^-1 in H then
for k in [1..#CH[5]] do if CH[5]
(T[i]*xx*T[j]^-1) eq CH[5][k] then
A[i,j]:=CH5seq[k] mod 7 ;end if; end for; end if; end for;
GG:=GL(5,7);
GG!A;
/*
[0 1 0 0 0]
```

```
[0 0 0 1 0]
[0 0 0 0 4]
[0 0 1 0 0]
[1 0 0 0 0]
*/
T:=Transversal(G,H);
B:=[[0,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do if T[i]*yy*T[j]^-1 in H then
for k in [1..#CH[5]] do if CH[5]
(T[i]*yy*T[j]^-1) eq CH[5][k] then
B[i,j]:=CH5seq[k] mod 7 ;end if; end for; end if; end for;
GG:=GL(5,7);
GG!B;
/*
[0 0 1 0 0]
[0 0 0 2 0]
[0 2 0 0 0]
[0 0 0 0 2]
[4 0 0 0 0]
*/

S:=Sym(30);
xx:=S!(1,2,4,3,28,8,13,23,18,26,6,11,21,16,5)(7,1
2,22,17,29,9,14,24,19,30,10,15,25,20,27);
yy:=S!(1,3,11,23,5,8,18,2,21,28,6,16,13,4,26)(7,1
7,15,24,27,9,19,12,25,29,10,20,14,22,30);
N:=sub<S|xx,yy>;
#N;
/*180*/
IsIsomorphic(sub<GG|GG!A,GG!B>,N);
/* true */
FPGroup(N);
/*
Finitely presented group on 2 generators
Relations
    ($.2^-1 * $.1)^3 = Id($)  --> (y^-1 *x)^3
    $.2^-1 * $.1^-3 * $.2^-1 * $.1^2 = Id($) --> y^-1*x^-3*y^-1*x^2
    $.2^-3 * $.1^-1 * $.2^2 * $.1^-1 = Id($) --> y^-3*x^-1*y2*x^-1

G<x,y,t>:=Group<x,y,t| (y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y^-3*x^-1*y2*x^-1,
t^7,
*/
Nt:=Stabiliser(N,{1,6,7,8,9,10});
 /* These perms are in the stabiliser */
```

```
for n in N do if {1,6,7,8,9,10}^n eq {1,6,7,8,9,10}
 then n; end if; end for;
Set(Nt);
w:=WordGroup(N);
rho:=InverseWordMap(N);
A:=N!(2, 23, 3)(4, 16, 11)(5, 28, 26)(12, 24,
17)(13, 21, 18)(14, 25, 19)(15, 22,20)(27, 29,
30);
B:=N!(2, 28)(3, 21)(4, 18)(5, 11)(12, 29)(13,
26)(14, 30)(15, 27)(16, 23)(17,25)(19, 22)(20,
24);
C:=N!(1, 8, 6)(2, 18, 4)(3, 21, 11)(7, 9, 10)(12,
19, 22)(13, 16, 23)(14, 20,24)(15, 17, 25);
A@rho;
AA:=function(w)
    w6 := w.1^2; w7 := w.2^2; w8 := w6 * w7; return w8;
end function;
AA(FPGroup(N));

B@rho;
BB:= function(w)
    w6 := w.1^2; w7 := w.2^2; w8 := w6 * w7; w3 := w.1^3; w2 := w.2^-1; w4 := w3
    * w2; w10 := w8 * w4; return w10;
end function;
BB(FPGroup(N));

C@rho;
CC:=function(w)
    w3 := w.1^3; w2 := w.2^-1; w4 := w3 * w2; w6 := w.1^2; w7 := w.2^2; w8 := w6
    * w7; w11 := w4 * w8; w12 := w11 * w8; w1 := w.1^-1; w13 := w12 * w1; w14 :=
    w13 * w.2; return w14;
end function;
CC(FPGroup(N));

/* Work with this one */
G<x,y,t>:=Group<x,y,t| (y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y^-3*x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*x^2*y^2*x^-1*y) = t^4>;
Orbits(Stabiliser(N,1));
/*
[
    GSet{@ 1 @},
    GSet{@ 6 @},
    GSet{@ 7 @},
    GSet{@ 8 @},
```

```
    GSet{@ 9 @},
    GSet{@ 10 @},
    GSet{@ 2, 21, 23, 26, 18, 28, 3, 5, 13, 11, 16, 4 @},
    GSet{@ 12, 25, 24, 30, 19, 29, 17, 27, 14, 15, 20, 22 @}
]

*/

/* Do not use this one it was only to test it
(t^x), Has an extra relation at the end */
G<x,y,t>:=Group<x,y,t| (y^-1 *x)^3,
y^-1*x^-3*y^-1*x^2,y^-3*x^-1*y^2*x^-1,
t^7, (t,x^2*y^2),
(t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*x^2*y^2*
x^-1*y) = t^4, (t,t^x)>;
#G;
/* Should get 7^5*180 */
Index(G,sub<G|x,y>);/* should be 7^5 */
/* 16807 */
```

3.5 Wreath Product of $\mathbb{Z}_3 \wr \mathbb{S}_3$

Let $H = < (1, 2, 3, 4) > \cong \mathbb{Z}_4$ and $K = < (5, 6, 7, 8) > \cong \mathbb{Z}_4$ be permutation groups on $X = \{1,2,3,4\}$ and $Y = \{5,6,7,8\}$, respectively.

Let $Z = X \times Y$
Z = {(1,5)(1,6)(1,7)(1,8)(2,5)(2,6)(2,7)(2,8)(3,5)(3,6)(3,7)(3,8)(4,5)(4,6)(4,7)(4,8)}

We can now label each 2 cycle permutation to help us find our generators of H and K.

9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
(1,5)	(1,6)	(1,7)	(1,8)	(2,5)	(2,6)	(2,7)	(2,8)	(3,5)	(3,6)	(3,7)	(3,8)	(4,5)	(4,6)	(4,7)	(4,8)

Table 3.7: Permutation cycle of $\mathbb{Z}_3 \wr \mathbb{S}_3$

Let $\gamma = (1, 2, 3, 4)$
$\gamma(5):$
$\mathbf{9}(1, 5) \rightarrow (2, 5)\mathbf{13}$
$\mathbf{13}(2, 5) \rightarrow (3, 5)\mathbf{17}$
$\mathbf{17}(3, 5) \rightarrow (4, 5)\mathbf{21}$
$\mathbf{21}(4, 5) \rightarrow (1, 5)\mathbf{9}$

$\gamma(5) = (9, 13, 17, 21)$

$\gamma(6):$
$\mathbf{10}(1, 6) \rightarrow (2, 6)\mathbf{14}$
$\mathbf{14}(2, 6) \rightarrow (3, 6)\mathbf{18}$
$\mathbf{18}(3, 6) \rightarrow (4, 6)\mathbf{22}$
$\mathbf{22}(4, 6) \rightarrow (1, 6)\mathbf{14}$

$\gamma(6) = (10, 14, 18, 22)$

$\gamma(7) :$
$\mathbf{11}(1, 7) \rightarrow (2, 7)\mathbf{15}$
$\mathbf{15}(2, 7) \rightarrow (3, 7)\mathbf{19}$
$\mathbf{19}(3, 7) \rightarrow (4, 7)\mathbf{23}$
$\mathbf{23}(4, 7) \rightarrow (1, 7)\mathbf{11}$

$\gamma(7) = (11, 15, 19, 23)$

$\gamma(8) :$
$\mathbf{12}(1, 8) \rightarrow (2, 8)\mathbf{16}$
$\mathbf{16}(2, 8) \rightarrow (3, 8)\mathbf{20}$
$\mathbf{20}(3, 8) \rightarrow (4, 8)\mathbf{24}$
$\mathbf{24}(4, 8) \rightarrow (1, 8)\mathbf{12}$

$\gamma(8) = (12, 16, 20, 24)$

Let $B = \gamma(5) \times \gamma(6) \times \gamma(7) \times \gamma(8)$
$B = < (9, 13, 17, 21), (10, 14, 18, 22), (11, 15, 19, 23), (12, 16, 20, 24) >$

Let $k = < (5, 6, 7, 8) >$
$k^* :$
$\mathbf{9}(1, 5) \rightarrow (1, 6)\mathbf{10}$
$\mathbf{10}(1, 6) \rightarrow (1, 7)\mathbf{11}$
$\mathbf{11}(1, 7) \rightarrow (1, 8)\mathbf{12}$
$\mathbf{12}(1, 8) \rightarrow (1, 5)\mathbf{9}$
$\mathbf{13}(2, 5) \rightarrow (2, 6)\mathbf{14}$
$\mathbf{14}(2, 6) \rightarrow (2, 7)\mathbf{15}$
$\mathbf{15}(2, 7) \rightarrow (2, 8)\mathbf{16}$
$\mathbf{16}(2, 8) \rightarrow (2, 5)\mathbf{13}$

17$(3,5) \to (3,6)$**18**
18$(3,6) \to (3,7)$**19**
19$(3,7) \to (3,8)$**20**
20$(3,8) \to (3,5)$**17**
21$(4,5) \to (4,6)$**22**
22$(4,6) \to (4,7)$**23**
23$(4,7) \to (4,8)$**24**
24$(4,8) \to (4,5)$**21**

$e = k^* = < (9,10,11,12)(13,14,15,16)(17,18,19,20)(21,22,23,24) >$

Label the B permutations a,b,c and d

B =

a	b	c	d
(9,13,17,21)	(10,14,18,22)	(11,15,19,23)	(12,16,20,24)

Table 3.8: Labeling B permutations

Presentation of $H \wr K$

$< a,b,c,d,e | a^4, b^4, c^4, d^4, (a,b), (a,c), (a,d), (b,c), (b,d), (c,d), e^4, a^e = b, b^e = c, c^e = d, d^e = a > .$

Note from using B and k^*:
$a^e = (10,14,18,22) = b,$
$b^e = (11,15,19,23) = c,$
$c^e = (12,16,20,24) = d,$
$d^e = (9,13,17,21) = a.$

Verifying answer permutation generator and presentation by using magma code below:

```
Z_$
S:= Sym(24);
B:= sub<S| (9,13,17,21), (10,14,18,22),(11,15,19,23),(12,16,20,24)>;

Reath := sub$<S|B,(9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24)$>;
#Reath;
/*1024 */


G <a,b,c,d,e> := Group <a,b,c,d,e | a^4, b^4, c^4, d^4, (a,b),(a,c),(a,d),(b,c),
(b,d), (c,d), e^4, a^e=b,b^e=c,c^e=d,d^e=a>;
#G;
/*1024 */
```

Using Magma we can compute the minimal faithful permutation representation of
G.

```
/* Below is Z_3 wr S_3 */
G <a,b,c,d,e> := Group <a,b,c,d,e| a^3,b^3,c^3,d^3,e^2,(d*e)^2,
(a,b),(a,c),(b,c), a^d=b,b^d=c,c^d=a,a^e=b,b^e=a,c^e=c>;
#G;
/* 162 */
W:= WreathProduct(Alt(3),Sym(3));
#W;
/* 162 */
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
IsIsomorphic(G1,W);
/* true */
G1;
/* 162 */
/* Gives you that the group is on carditionality 9 (meaning 9 letters) */
W;
/*

Permutation group W acting on a set of cardinality 9
Order = 162 = 2 * 3^4
    (1, 4, 7)(2, 5, 8)(3, 6, 9)
    (1, 4)(2, 5)(3, 6)
    (1, 2, 3)

*/


/*
Loop below finds the minimal faithful permutation representation ( i.e 648 --> 9
```

```
Looking at kernels that have one and storing those
(Core means kernel)
*/
SL:=Subgroups(G1);

T := {X`subgroup: X in SL};

#T;

TrivCore := {H:H in T| #Core(G1,H) eq 1};

mdeg := Min({Index(G1,H):H in TrivCore});

Good := {H: H in TrivCore| Index(G1,H) eq mdeg};

#Good;

H := Rep(Good);


#H;

f,G1,K := CosetAction(G1,H);
G1;
```

Chapter 4

Isomorphism Types

In this chapter we demonstrate how to determine the isomorphism type of a group.

4.1 Semi-Direct Product

We determine the isomorphism type of the group $N = < xx, yy, zz >$, where

xx = (1, 20, 37, 9, 28, 5, 18, 35, 13, 26, 4, 21, 34, 12, 30)(2, 19, 38, 10, 27, 6, 17, 36, 14, 25, 3, 22, 33, 11, 29)(7, 23, 40, 16, 31)(8, 24, 39, 15, 32),

yy = (1, 32, 13, 34, 24, 5, 26, 15, 37, 18, 8, 30, 9, 39, 21)(2, 31, 14, 33, 23, 6, 25, 16, 38, 17, 7, 29, 10, 40, 22)(3, 27, 11, 36, 19)(4, 28, 12, 35, 20),

zz = (1, 19, 38, 10, 28, 5, 18, 36, 14, 25, 4, 21, 34, 11, 29, 2, 20, 37, 9, 27, 6, 17, 35, 13, 26, 3, 22, 33, 12, 30)(7, 24, 40, 15, 31, 8, 23, 39, 16, 32).

We note that $|N| = 480$ and we use the magma code below to find the normal lattice of N:

```
S:=Sym(40);
xx:=S!(1, 20, 37, 9, 28, 5, 18, 35, 13, 26, 4,
21, 34, 12, 30)(2, 19, 38, 10, 27, 6, 17,
```

```
36, 14, 25, 3, 22, 33, 11, 29)(7, 23, 40, 16,
31)(8, 24, 39, 15, 32);

yy:=S!(1, 32, 13, 34, 24, 5, 26, 15, 37, 18, 8,
30, 9, 39, 21)(2, 31, 14, 33, 23, 6, 25,
16, 38, 17, 7, 29, 10, 40, 22)(3, 27, 11, 36,
19)(4, 28, 12, 35, 20);

zz:=S!(1, 19, 38, 10, 28, 5, 18, 36, 14, 25, 4,
21, 34, 11, 29, 2, 20, 37, 9, 27, 6, 17,
35, 13, 26, 3, 22, 33, 12, 30)(7, 24, 40, 15, 31,
8, 23, 39, 16, 32);

N:=sub<S|xx,yy,zz>;
#N;
/* 480 */
NL:= NormalLattice(N);
NL;
```


Figure 4.1: Normal Lattice of Semi Direct Product

The largest normal abelian subgroup of N can be found using the following magma

code:

```
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;

The largest number it outputs will be the largest normal Abelian
subgroup of N. In this
case it is 13
1
2
3
4
5
6
8
11
12
13
```

From the info above we can see that NL[13] is the largest normal abelian subgroup and the order of NL[13] to be 40.

N has 3 Normal subgroups of order 2 and one of order 5 so they are all contained in NL[13]. This means that N is not a direct product and can be a mixed extension or a semi-direct product. We now determine the isomorphism type of NL[13].

Now NL[13] $= <A, B, C, D>$, where
$A \sim (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)(25,26)$
$(27,28)(29,30)(31,32)(33,34)(35,36)(37,38)(39,40)$
$B \sim (1,9,18,26,34)(2,10,17,25,33)(3,11,19,27,36)(4,12,20,28,35)(5,13,21,30,37)(6,14,$
$22,29,38)(7,16,23,31,40)(8,15,24,32,39)$
$C \sim (1,7)(2,8)(3,5)(4,6)(9,16)(10,15)(11,13)(12,14)(17,24)(18,23)(19,21)(20,22)(25,32)$
$(26,31)(27,30)(28,29)(33,39)(34,40)(35,38)(36,37)$
$D \sim (1,4)(2,3)(5,8)(6,7)(9,12)(10,11)(13,15)(14,16)(17,19)(18,20)(21,24)(22,23)(25,27)$
$(26,28)(29,31)(30,32)(33,36)(34,35)(37,39)(38,40)$.

At this point we see that N is isomorphic to NL[13] by N/NL[13]. We can see that q = N/NL[13] is generated by C and D. Using magma we can find the presentation of q.

q,ff:=quo< $N|NL13$ >;

T:=Transversal(N,NL13);

T;

12

q;

ff(T[3]) eq q.2;// ff(T[2]) eq q.1;// ff(T[4]) eq q.3; // Note ff(T[1]) is Id(q)

FPGroup(q);

Finitely presented group on 3 generators Relations

$.1^3 = \text{Id}() \longrightarrow e^3$

$.2^3 = \text{Id}() \longrightarrow f^3$

$.3^3 = \text{Id}() \longrightarrow g^3$

$.2 * .1^{-1} * .3^{-1} = \text{Id}() \longrightarrow f * e^{-1} * g^{-1}$

$.1^{-1} * .2^{-1} * .1 * .3^{-1} = \text{Id}() \longrightarrow e^{-1} * f^{-1} * e * g^{-1}$

The presentation of q is$< e, f, g | e^3, f^3, g^3, f * e^{-1} * g^{-1}, e^{-1} * f^{-1} * e * g^{-1} >$.

Semi-Direct Product of NL[13] by N/NL[13]

We determine the action of $q =< E, F, G >$ on $NL[13] =< A, B, C, D >$. Thus we must conjugate A, B, C, and D by $NL[13] =< A, B, C, D >$. We find that $a^c = a * b^2, a^d = a * b^2, b^c = a^2 * b, b^d = a^2 * b^3$. We will include these in our presentation of G and verify that it is isomorphic to N to find the presentation of N.

G< a, b, c, d >:=Group< $a, b, c, d | a^2, b^4, (a, b), c^2, d^2, (c * d)^2, a^c = a * b^2, a^d = a * b^2, b^c = a^2 * b, b^d = a^2 * b^3$ >;

f,G1,k:=CosetAction(G,sub< $G|Id(G)$ >);

s:=IsIsomorphic(N,G1);

s;

TRUE

Thus N is a semi-direct product of $(4x2) : 2^2$ and the presentation of N is:

$< a, b, c, d | a^2, b^4, (a, b), c^2, d^2, (c*d)^2, a^c = a*b^2, a^d = a*b^2, b^c = a^2*b, b^d = a^2*b^3 >$

The following normal lattice shows the path we followed for this proof.
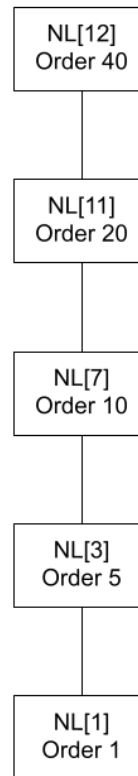
Figure 4.2: Normal Lattice of a Semi-Direct Product of $(4x2) : 2^2$

4.2 Mixed Extension Example

We determine the isomorphism type of the group $N =< xx, yy, zz >$, where

xx = (1, 2, 3, 4, 5, 6, 7, 8),

yy = (1, 5)(3, 7),

zz = (1, 6)(2, 5)(3, 4)(7, 8).

We note that $|N| = 32$ and use the magma code below to find the normal lattice of N:

S:=Sym(8);

xx:=S!(1, 2, 3, 4, 5, 6, 7, 8);

yy:=S!(1, 5)(3, 7);

zz:=S!(1, 6)(2, 5)(3, 4)(7, 8);

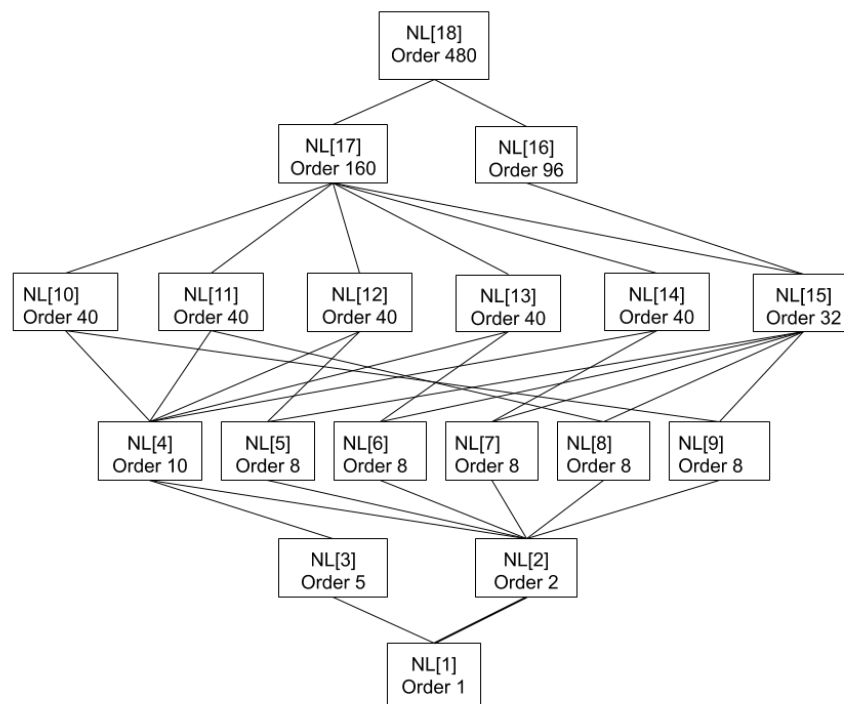N:=sub$< S|xx, yy, zz >$;

N;

32

NL:= NormalLattice(N);

NL;
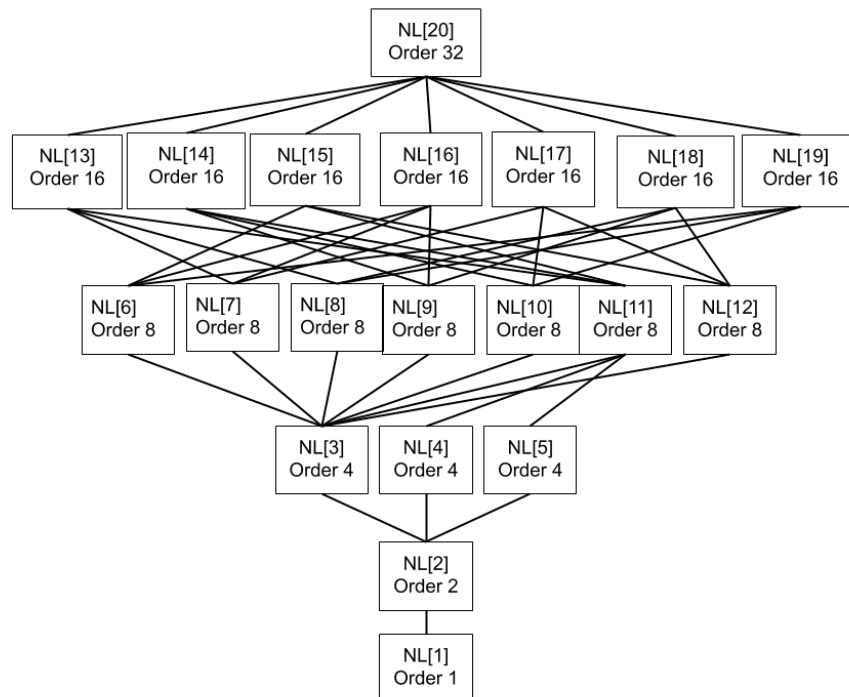
Figure 4.3: Normal Lattice of a Mixed Extension

The largest normal abelian subgroup of N can be found using the following magma code:

```
 for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;\\
The largest number it outputs will be the largest normal Abelian subgroup of N.
1
2
3
4
5
9
10
11
(In this case it is 11)
```

From the info above we can see that NL[11] is the largest normal abelian subgroup and the order of NL[11] is 8 or 2^3.

N has 3 Normal subgroups of order 4 and order 2 so they are all contained in NL[11]. This means that N is not a direct product and can be a mixed extension or a semi-direct product. Now $NL[11] =< A, B >$, where
A= ((1, 5)(3, 7)),
B=((1, 7, 5, 3)(2, 8, 6, 4)).
A presentation of NL[11] is $< a, b | a^2, b^4, (a, b) >$.

Now q = N/NL[11] is generated by C and D. Using magma we can find the presentation of q.
q,ff:=quo$< N|NL11 >$.
T:=Transversal(N,NL11);
Let C = T[2] and D = T[3]. A presentation of q is $< c, d | c^2, d^2, (c \times d)^2 >$.
We determine the action of $q =< C, D >$ on $NL[11] =< A, B >$. Thus we must conjugate A and B by C and D . We find that $a^c = a * b^2, a^d = a * b^2, b^c = a^2 * b, b^d = a^2 * b^3$. We will include these in our presentation of G and verify that it is isomorphic to N.

G$< a, b, c, d >$:=Group$< a, b, c, d | a^2, b^4, (a, b), c^2, d^2, (c * d)^2, a^c = a * b^2, a^d = a * b^2, b^c = a^2 * b, b^d = a^2 * b^3 >$;

f,G1,k:=CosetAction(G,sub$< G|Id(G) >$);
s:=IsIsomorphic(N,G1);
s;
TRUE

Thus N is a semi direct product of $(4 \times 2)n : 2^2$ and the presentation of N is:
$< a, b, c, d | a^2, b^4, (a, b), c^2, d^2, (c * d)^2, a^c = a * b^2, a^d = a * b^2, b^c = a^2 * b, b^d = a^2 * b^3 >$
The following normal lattice shows the path we followed for this proof.
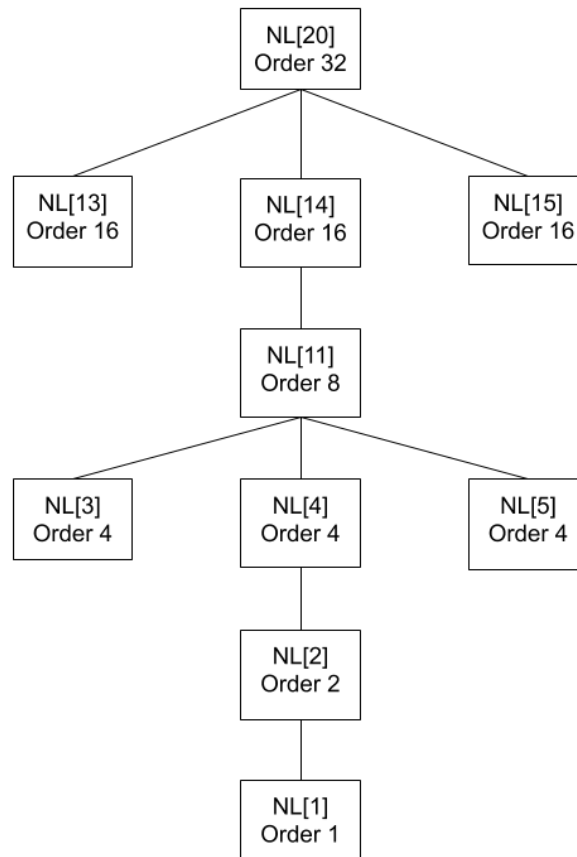
Figure 4.4: Shortened Normal Lattice of Mixed Extension

4.3   Semi-Direct Product $N \sim 2^5 : S_5$

We determine the isomorphism type of the group $N = <xx, yy>$, where
xx = (1, 4, 2, 10, 8, 6, 9, 7, 5, 3)

yy = (1, 6)(2, 5, 9, 3)(4, 8, 7, 10)

We note that $|N| = 3840$ and we will use the magma code below to find the normal lattice of N:

```
S:=Sym(10);
xx:=S!(1, 4, 2, 10, 8, 6, 9, 7, 5, 3);
yy:=S!(1, 6)(2, 5, 9, 3)(4, 8, 7, 10);
N:=sub<S|xx,yy>;
#N;
/* 3840 */
NL:= NormalLattice(N);
NL;
```

Figure 4.5: Semi-Direct Product $N \sim 2^5 : S_5$

The largest normal abelian subgroup of N can be found using the following magma code:

```
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
```

```
The largest number it outputs will be the largest normal abelian
subgroup of N. In this case it is 4
1
2
3
4
```

From the info above we can see that NL[4] is the largest normal abelian subgroup and the order of NL[4] is 32.

N has 1 normal subgroup of order 2 and it is contained in NL[4]. This means that N is not a direct product and can be a mixed extension or a semi-direct product. We now determine the isomorphism type of NL[4].

Now NL[4] $= < A, B, C, D, E >$, where

$A \sim (1,6)(2,7)(3,8)(4,9)(5,10)$,

$B \sim (4,9)(5,10)$,

$C \sim (2,7)(3,8)$,

$D \sim (3,8)(5,10)$,

$E \sim (3,8)$.

At this point we see that N is isomorphic to NL[4] by N/NL[4]. We can see that q = N/NL[4]. Using magma we can find the presentation of q.

q,ff:=quo$< N|NL4 >$;

T:=Transversal(N,NL4);

q;

FPGroup(q);

Finitely presented group on 3 generators Relations

$.1^5 = \text{Id}() \longrightarrow f^5$,

$.2^4 = \text{Id}() \longrightarrow g^4$,

$(.1 *.2 * .1)^2 = \text{Id}() \longrightarrow (f * g * f)^2$,

$.2^{-1} *.1^{-1} *.2^{-2} *.1^{-1} *.2^2 *.1^{-1} *.2^{-1} = \text{Id}() \longrightarrow g^-1 * f^{-1} * g^{-2} * f^{-1} * g^2 * f^{-1} * g^{-1}$,

The presentation of q:$< f, g | f^5, g^4, (f * g * f)^2, g^-1 * f^{-1} * g^{-2} * f^{-1} * g^2 * f^{-1} * g^{-1} > .$

Semi-Direct Product of NL[4] by N/NL[4]

We determine the action of $q = < F, G >$ on $NL[4] = < A, B, C, D, E >$. Thus we must conjugate A, B, C, and D by $NL[4] = < A, B, C, D, E >$. We find that $a^f = a, a^g = a, b^f = c, b^g = b * d, c^f = a * b * c * d * e, c^g = c * d, d^f = a * b * c * e, d^g = b * c * d, e^f = a * b * c, e^g = c * e$. We will include these in our presentation of G and verify that it is isomorphic to N to find the presentation of N.

H$< a, b, c, d, e, f, g >$:=Group$< a, b, c, d, e, f, g | a^2, b^2, c^2, d^2, e^2, (a * b)^2, (a * c)^2, (b * c)^2, (a * d)^2, (b * d)^2, (c * d)^2, (a * e)^2, (b * e)^2, (c * e)^2, (d * e)^2,$

$\qquad$ f$^5, g^4, (f * g * f)^2, g^-1 * f^{-1} * g^{-2} * f^{-1} * g^2 * f^{-1} * g^{-1}$

$\qquad$ a$^f = a, a^g = a, b^f = c, b^g = b * d, c^f = a * b * c * d * e, c^g = c * d, d^f = a * b * c * e, d^g = b * c * d, e^f = a * b * c, e^g = c * e >$;

f,H1,k:=CosetAction(H,sub$< G | Id(H) >$);

s:=IsIsomorphic(N,H1);

s;

TRUE

$\qquad$ Thus N is a semi-direct product $2^5 : S_5$ and the presentation of N is:

$< a, b, c, d, e, f, g | a^2, b^2, c^2, d^2, e^2, (a * b)^2, (a * c)^2, (b * c)^2, (a * d)^2, (b * d)^2, (c * d)^2, (a * e)^2, (b * e)^2, (c * e)^2, (d * e)^2,$

$\qquad$ f$^5, g^4, (f * g * f)^2, g^-1 * f^{-1} * g^{-2} * f^{-1} * g^2 * f^{-1} * g^{-1}$

$\qquad$ a$^f = a, a^g = a, b^f = c, b^g = b * d, c^f = a * b * c * d * e, c^g = c * d, d^f = a * b * c * e, d^g = b * c * d, e^f = a * b * c, e^g = c * e >$

4.4 Semi-Direct Product $N \sim 2^4 : S_3$

We determine the isomorphism type of the group $N = < xx, yy >$, where
xx = (1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11, 20, 17)(12, 19, 18)(13, 21, 16)(14, 22, 15)

yy = (1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7, 11, 9, 13)(8, 12, 10, 14)(15, 16)(17, 18)

We note that $|N| = 96$ and will use the magma code below to find the normal lattice of N:

```
S:=Sym(24);
xx:=S!(1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11, 20,
17)(12, 19, 18)(13, 21, 16)(14, 22, 15);
yy:=S!(1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7, 11,
9, 13)(8, 12, 10, 14)(15, 16)(17, 18);
N:=sub<S|xx,yy>;
#N;
/* 96 */
NL:= NormalLattice(N);
NL;
```
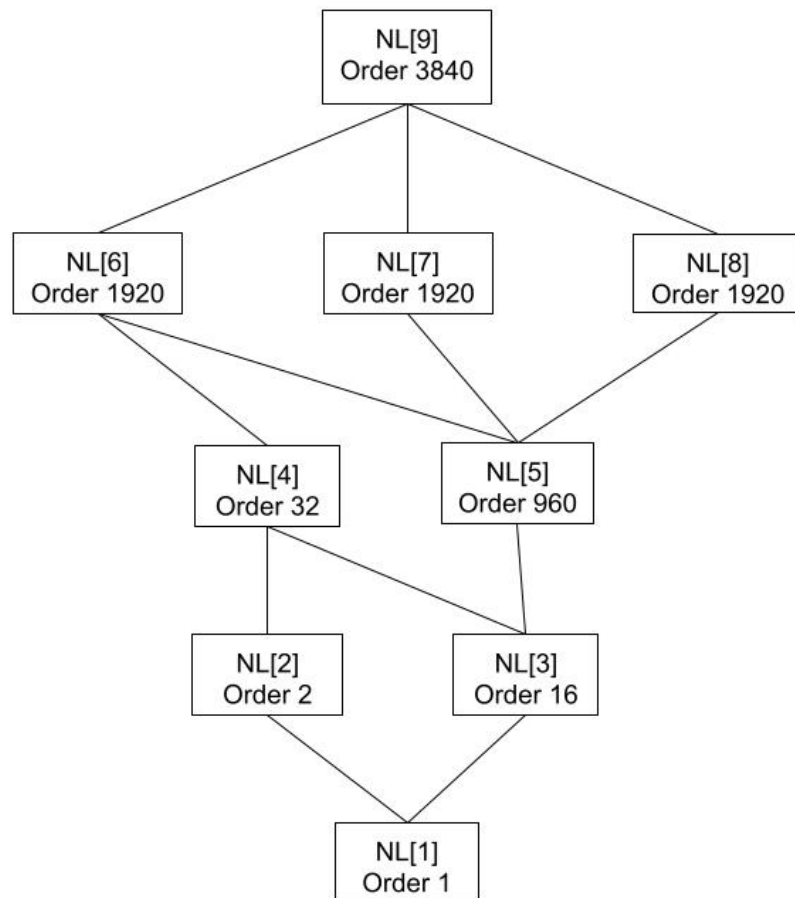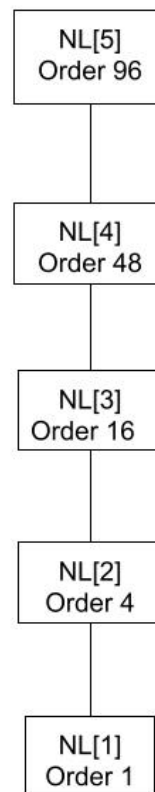
Figure 4.6: Semi-Direct Product $N \sim 2^4 : S_3$

The largest normal abelian subgroup of N can be found using the following magma code:

```
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
```

```
The largest number it outputs will be the largest normal abelian
subgroup of N. In this case it is 3
1
2
3
```

From the info above we can see that NL[3] is the largest normal abelian subgroup and the order of NL[3] is 16.

N has 1 normal subgroup of order 2 and it is contained in NL[3]. This means that N is not a direct product and can be a mixed extension or a semi-direct product. We now determine the isomorphism type of NL[3].

Now $NL[3] =< A, B, C, D >$, where

$A \sim (3, 16, 5, 17)(4, 15, 6, 18)(7, 19, 9, 22)(8, 20, 10, 21)$

$B \sim (1, 11, 23, 13)(2, 12, 24, 14)(3, 16, 5, 17)(4, 15, 6, 18)(7, 9)(8, 10)(19, 22)(20, 21)$

$C \sim (1, 23)(2, 24)(3, 5)(4, 6)(11, 13)(12, 14)(15, 18)(16, 17)$

$D \sim (3, 5)(4, 6)(7, 9)(8, 10)(15, 18)(16, 17)(19, 22)(20, 21)$.

At this point we see that N is isomorphic to NL[3] by N/NL[3]. We can see that q = N/NL[3]. Using magma we can find the presentation of q.

q,ff:=quo$< N|NL3 >$;

T:=Transversal(N,NL3);

q;

FPGroup(q);

Finitely presented group on 3 generators Relations

$.2^2 = \text{Id}() \longrightarrow f^2$,

$.1^{-3} = \text{Id}() \longrightarrow e^{-3}$,

$(.1^{-1} * .2)^2 = \text{Id}() \longrightarrow (e^{-1} * f)^2$,

The presentation of q:$< e, f | f^2, e^{-3}, (e^{-1} * f)^2 >$.

Semi-Direct Product of NL[3] by N/NL[3]

We determine the action of $q =< E, F >$ on $NL[3] =< A, B, C, D >$. Thus we must conjugate A, B, C, and D by $NL[3] =< A, B, C, D >$. We find that $a^e = b * d, a^f = b * d, b^e = a * b, b^f = a * c, c^e = c * d, c^f = d, d^e = c, d^f = c$. We will include these in our presentation of G and verify that it is isomorphic to N to find

the presentation of N.

H$< a, b, c, d, e, f >$:=Group$< a, b, c, d, e, f | a^4, b^4, c^2, d^2, a^{-2} * d, (a, b), b^{-2} * c,$
$\qquad$ $f^2, e^{-3}, (e^{-1} * f)^2,$
$\qquad$ a$^e = b * d, a^f = b * d, b^e = a * b, b^f = a * c, c^e = c * d, c^f = d, d^e = c, d^f = c >;$

f,H1,k:=CosetAction(H,sub$< G | Id(H) >$);

s:=IsIsomorphic(N,H1);

s;

TRUE

$\qquad$ Thus N is a semi-direct product $2^4 : S_3$ and the presentation of N is:

$< a, b, c, d, e, f | a^4, b^4, c^2, d^2, a^{-2} * d, (a, b), b^{-2} * c,$
$\qquad$ $f^2, e^{-3}, (e^{-1} * f)^2,$
$\qquad$ a$^e = b * d, a^f = b * d, b^e = a * b, b^f = a * c, c^e = c * d, c^f = d, d^e = c, d^f = c >$

4.5   Central Extension

We determine the isomorphism type of the group $N = <xx, yy>$, where
xx = (1, 6, 2)(3, 12, 7)(4, 9, 11)(5, 8, 10),

yy = (1, 12)(4, 8, 10, 6)(5, 9, 11, 7).

Next we will use the magma code below to find the size and normal lattice of N:

```
S:=Sym(12);
xx:=S!(1, 6, 2)(3, 12, 7)(4, 9, 11)(5, 8, 10);
yy:=S!(1, 12)(4, 8, 10, 6)(5, 9, 11, 7);
N:=sub<S|xx,yy>;
#N;
NL:=NormalLattice(N);
NL;
```
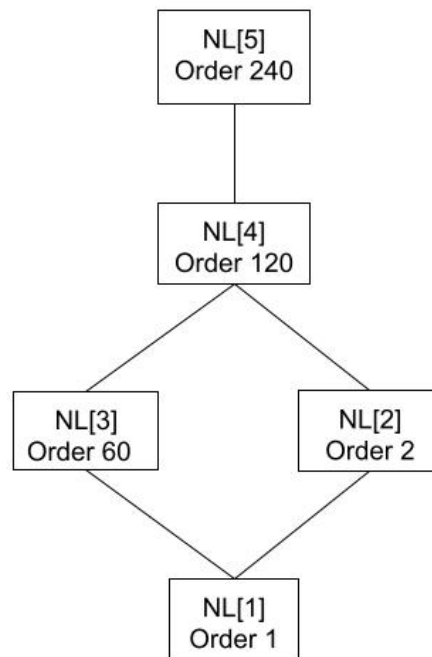
Figure 4.7: Normal Lattice of a Central Extension

The largest normal abelian subgroup of N can be found using the following magma code:

```
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
```

```
The largest number it outputs will be the largest normal abelian
subgroup of N. In this case it is 2.
1
2
```

From the info above we can see that NL[2] is the largest normal abelian subgroup and the order of NL[2] is 2. However, NL[2] is the center of N. Thus we have a central extension of NL[2] by N/NL[2].

Now NL[2] $= < A >$, where

$A \sim (1, 12)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)$.

Let q = N/NL[2]. Using magma we can find the presentation of q.

q,ff:=quo$< N|NL2 >$;

T:=Transversal(N,NL2);

q;

FPGroup(q);

Finitely presented group on 4 generators Relations

$.1^3 = \text{Id}() \longrightarrow b^3,$

$.2^4 = \text{Id}() \longrightarrow c^4,$

$(.1^{-1}, .2^{-1})^2 = \text{Id}() \longrightarrow (b^{-1}, c^{-1})^2,$

$.1^{-1} * .2^{-2} * .1^{-1} * .2^{-1} * .1^{-1} * .2^2 * .1^{-1} * .2^{-1} = \text{Id}() \longrightarrow b^{-1} * c^{-2} * b^{-1} * c^{-1} * b^{-1} * c^2 * b^{-1} * c^{-1}.$

The presentation of q is

$< b, c|b^3, c^4, (b^{-1}, c^{-1})^2, b^{-1} * c^{-2} * b^{-1} * c^{-1} * b^{-1} * c^2 * b^{-1} * c^{-1} >.$

Next we must conjugate A by B, C. We find that $a^b = a, a^c = a$ and $b^{-1} * c^{-2} * b^{-1} * c^{-1} * b^{-1} * c^2 * b^{-1} * c^{-1} = a$. We will include these in our presentation of H. In our presentation of H given below.

H$< a, b, c >$:=Group$< a, b, c|a, b, c|a^2,$
$\qquad$ b$^3, c^4, (b^{-1}, c^{-1})^2, b^{-1} * c^{-2} * b^{-1} * c^{-1} * b^{-1} * c^2 * b^{-1} * c^{-1} = a,$
$\qquad$ a$^b = a, a^c = a >.$

We verify that H is isomorphic to N in Magma.

f,H1,k:=CosetAction(H,sub< $G|Id(H)$ >);

s:=IsIsomorphic(N,H1);

s;

TRUE.

Thus N is a central extension of $2 : S_5$ and the presentation of N is:

$< a, b, c >$:=Group$< a, b, c|a^2$,

b$^3, c^4, (b^{-1}, c^{-1})^2, b^{-1} * c^{-2} * b^{-1} * c^{-1} * b^{-1} * c^2 * b^{-1} * c^{-1} = a$,

a$^b = a, a^c = a >$;

Chapter 5

Homomorphic Images of Progenitors

5.1 Unsuccessful Homorphic Images and Progenitors

5.01

m	n	o	p	q	[G:N]
2	0	3	0	0	3

```
S:=Sym(24);
xx:=S!(1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11,
20, 17)(12, 19, 18)(13, 21, 16)(14, 22, 15);
yy:=S!(1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7,
11, 9, 13)(8, 12, 10, 14)(15,16)(17, 18);
N:=sub<S|xx,yy>;
#N;

G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1
*
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2)/* Stab */,
((y^2*t))^a, ((y^2*t^(y * x * y * x^-1 *
y^-1)))^b,((y^2*t^(y^2 * x^-1)))^c,(((x * y^-1 *
x^-1
* y^2)*t^(x^-1 * y * x^-1)))^d,(((x * y^-1 * x^-1 *
y^2)*t^(y * x * y * x * y)))^e,(((x * y^-1 * x^-1 *
```

```
y^2)*t^(y)))^f,(((x * y^-1 * x^-1 * y^2)*t^(y * x *
y
* x^-1)))^g,(((x * y^-1 * x^-1 * y^2)*t))^h,(((x *
y^-1 * x^-1 * y^2)*t^(y * x * y * x^-1 *
y^-1)))^i,(((x)*t),((x)*t^(y * x * y * x^-1 *
y^-1)))^j,(((x)*t^(y^2 * x^-1)),((x)*t^(x^-1 *
y)))^k,(((x)*t^((y * x)^2)),((x)*t^(x *
y^-1)))^l,(((x)*t^(x^-1 * y * x^-1 *
y^-1)))^m,(((x)*t^(x * y)),(((x * y)^2)*t))^n,
(y^2 * x^-1 * y^-1 * x*t)^o,
(y^2 * x^-1 *y^-1 * x*t)^p,
(t*t^(y * x * y * x^-1 * y^-1))^q = (x^-1 * y *
x)^2>;



G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1
*
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2), (((x)*t^(x^-1 * y * x^-1 *
y^-1)))^2,
(y^2 * x^-1 * y^-1 * x*t)^3>;

#G;
/* 288 */
f,G1,k:=CosetAction(G,sub<G|x,y>);

NL:=NormalLattice(G1);

NL;
/*
Normal subgroup lattice
-----------------------

[3]  Order 6  Length 1  Maximal Subgroups: 2
---
[2]  Order 3  Length 1  Maximal Subgroups: 1
---
[1]  Order 1  Length 1  Maximal Subgroups:

*/
CompositionFactors(G1);
/*
 G
    |  Cyclic(2)
```

```
    *
    |  Cyclic(3)
    1
*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
/*
1

2
*/
X:=[3];

IsIsomorphic(NL[2],AbelianGroup(GrpPerm,X));
/*
true Isomorphism of GrpPerm: $, Degree 3, Order 3
into GrpPerm: $, Degree 3, Order 3 induced
by
    (1, 3, 2) |--> (1, 3, 2)
```

5.02

m	n	o	p	q	[G:N]
2	0	5	5	0	5

```
S:=Sym(24);
xx:=S!(1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11,
20, 17)(12, 19, 18)(13, 21, 16)(14, 22, 15);
yy:=S!(1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7,
11, 9, 13)(8, 12, 10, 14)(15,16)(17, 18);
N:=sub<S|xx,yy>;
#N;

G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1
*
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2)/* Stab */,
((y^2*t))^a, ((y^2*t^(y * x * y * x^-1 *
y^-1)))^b,((y^2*t^(y^2 * x^-1)))^c,(((x * y^-1 *
x^-1
* y^2)*t^(x^-1 * y * x^-1)))^d,(((x * y^-1 * x^-1 *
y^2)*t^(y * x * y * x * y)))^e,(((x * y^-1 * x^-1 *
```

```
y^2)*t^(y)))^f,(((x * y^-1 * x^-1 * y^2)*t^(y * x *
y
* x^-1)))^g,(((x * y^-1 * x^-1 * y^2)*t))^h,(((x *
y^-1 * x^-1 * y^2)*t^(y * x * y * x^-1 *
y^-1)))^i,(((x)*t),((x)*t^(y * x * y * x^-1 *
y^-1)))^j,(((x)*t^(y^2 * x^-1)),((x)*t^(x^-1 *
y)))^k,(((x)*t^((y * x)^2)),((x)*t^(x *
y^-1)))^l,(((x)*t^(x^-1 * y * x^-1 *
y^-1)))^m,(((x)*t^(x * y)),(((x * y)^2)*t))^n,
(y^2 * x^-1 * y^-1 * x*t)^o,
(y^2 * x^-1 *y^-1 * x*t)^p,
(t*t^(y * x * y * x^-1 * y^-1))^q = (x^-1 * y *
x)^2>;



G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1 *
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2), (((x)*t^(x^-1 * y * x^-1 *
y^-1)))^2,
(y^2 * x^-1 * y^-1 * x*t)^5,
(y^2 * x^-1 *y^-1 * x*t)^5>;

#G;
/* 480 */
f,G1,k:=CosetAction(G,sub<G|x,y>);

NL:=NormalLattice(G1);

NL;
/*
Normal subgroup lattice
-----------------------

[3]  Order 10  Length 1  Maximal Subgroups: 2
---
[2]  Order 5   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:
*/
CompositionFactors(G1);
/*
  G
    |  Cyclic(2)
    *
```

```
    |  Cyclic(5)
    1
*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
/*
1

2
*/
X:=[5];

IsIsomorphic(NL[2],AbelianGroup(GrpPerm,X));
/*
true Mapping from: GrpPerm: $, Degree 5, Order 5 to GrpPerm: $, Degree 5, Order 5
Composition of Mapping from: GrpPerm: $, Degree 5, Order 5 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 5, Order 5
*/
```

5.03

m	n	o	p	q	[G:N]
2	0	7	7	0	7

```
S:=Sym(24);
xx:=S!(1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11,
20, 17)(12, 19, 18)(13, 21, 16)(14, 22, 15);
yy:=S!(1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7,
11, 9, 13)(8, 12, 10, 14)(15,16)(17, 18);
N:=sub<S|xx,yy>;
#N;

G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1
*
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2)/* Stab */,
((y^2*t))^a, ((y^2*t^(y * x * y * x^-1 *
y^-1)))^b,((y^2*t^(y^2 * x^-1)))^c,(((x * y^-1 *
x^-1
* y^2)*t^(x^-1 * y * x^-1)))^d,(((x * y^-1 * x^-1 *
y^2)*t^(y * x * y * x * y)))^e,(((x * y^-1 * x^-1 *
y^2)*t^(y)))^f,(((x * y^-1 * x^-1 * y^2)*t^(y * x *
y
* x^-1)))^g,(((x * y^-1 * x^-1 * y^2)*t))^h,(((x *
y^-1 * x^-1 * y^2)*t^(y * x * y * x^-1 *
y^-1)))^i,(((x)*t),((x)*t^(y * x * y * x^-1 *
y^-1)))^j,(((x)*t^(y^2 * x^-1)),((x)*t^(x^-1 *
y)))^k,(((x)*t^((y * x)^2)),((x)*t^(x *
y^-1)))^l,(((x)*t^(x^-1 * y * x^-1 *
y^-1)))^m,(((x)*t^(x * y)),(((x * y)^2)*t))^n,
(y^2 * x^-1 * y^-1 * x*t)^o,
(y^2 * x^-1 *y^-1 * x*t)^p,
(t*t^(y * x * y * x^-1 * y^-1))^q = (x^-1 * y *
x)^2>;


G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1 *
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2),
(((x)*t^(x^-1 * y * x^-1 *
y^-1)))^2,
(y^2 * x^-1 * y^-1 * x*t)^7,
```

```
(y^2 * x^-1 *y^-1 * x*t)^7>;

#G;
/* 672 */
f,G1,k:=CosetAction(G,sub<G|x,y>);

NL:=NormalLattice(G1);

NL;
/*
Normal subgroup lattice
-----------------------

[3]  Order 14  Length 1  Maximal Subgroups: 2
---
[2]  Order 7   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:

*/
CompositionFactors(G1);
/*
    G
     |  Cyclic(2)
     *
     |  Cyclic(7)
     1
*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
/*
1

2
*/
X:=[7];

IsIsomorphic(NL[2],AbelianGroup(GrpPerm,X));
/*
true Mapping from: GrpPerm: $, Degree 7, Order 7 to GrpPerm: $, Degree 7, Order 7
Composition of Mapping from: GrpPerm: $, Degree 7, Order 7 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 7, Order 7
*/
```

5.04

m	n	o	p	q	[G:N]
2	0	9	0	0	9

```
S:=Sym(24);
xx:=S!(1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11,
20, 17)(12, 19, 18)(13, 21, 16)(14, 22, 15);
yy:=S!(1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7,
11, 9, 13)(8, 12, 10, 14)(15,16)(17, 18);
N:=sub<S|xx,yy>;
#N;

G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1
*
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2)/* Stab */,
((y^2*t))^a, ((y^2*t^(y * x * y * x^-1 *
y^-1)))^b,((y^2*t^(y^2 * x^-1)))^c,(((x * y^-1 *
x^-1
* y^2)*t^(x^-1 * y * x^-1)))^d,(((x * y^-1 * x^-1 *
y^2)*t^(y * x * y * x * y)))^e,(((x * y^-1 * x^-1 *
y^2)*t^(y)))^f,(((x * y^-1 * x^-1 * y^2)*t^(y * x *
y
* x^-1)))^g,(((x * y^-1 * x^-1 * y^2)*t))^h,(((x *
y^-1 * x^-1 * y^2)*t^(y * x * y * x^-1 *
y^-1)))^i,(((x)*t),((x)*t^(y * x * y * x^-1 *
y^-1)))^j,(((x)*t^(y^2 * x^-1)),((x)*t^(x^-1 *
y)))^k,(((x)*t^((y * x)^2)),((x)*t^(x *
y^-1)))^l,(((x)*t^(x^-1 * y * x^-1 *
y^-1)))^m,(((x)*t^(x * y)),(((x * y)^2)*t))^n,
(y^2 * x^-1 * y^-1 * x*t)^o,
(y^2 * x^-1 *y^-1 * x*t)^p,
(t*t^(y * x * y * x^-1 * y^-1))^q = (x^-1 * y *
x)^2>;


G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1 *
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2),
(((x)*t^(x^-1 * y * x^-1 *
y^-1)))^2,
(y^2 * x^-1 * y^-1 * x*t)^9>;
```

```
#G;
/* 864 */
f,G1,k:=CosetAction(G,sub<G|x,y>);

NL:=NormalLattice(G1);

NL;
/*
Normal subgroup lattice
-----------------------

[4]  Order 18  Length 1  Maximal Subgroups: 3
---
[3]  Order 9   Length 1  Maximal Subgroups: 2
---
[2]  Order 3   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:

*/
CompositionFactors(G1);
/*
   G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
/*
1
2
3
*/
X:=[9];

IsIsomorphic(NL[3],AbelianGroup(GrpPerm,X));
/*
true Mapping from: GrpPerm: $, Degree 9, Order 3^2
to GrpPerm: $, Degree 9, Order 3^2
Composition of Mapping from: GrpPerm: $, Degree 9,
```

```
Order 3^2 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 9, Order
3^2
*/
```

5.05

m	n	o	p	q	[G:N]
2	0	10	10	0	10

```
S:=Sym(24);
xx:=S!(1, 10, 5)(2, 9, 6)(3, 23, 8)(4, 24, 7)(11,
20, 17)(12, 19, 18)(13, 21, 16)(14, 22, 15);
yy:=S!(1, 19, 23, 22)(2, 20, 24, 21)(3, 6)(4, 5)(7,
11, 9, 13)(8, 12, 10, 14)(15,16)(17, 18);
N:=sub<S|xx,yy>;
#N;

G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1
*
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2)/* Stab */,
((y^2*t))^a, ((y^2*t^(y * x * y * x^-1 *
y^-1)))^b,((y^2*t^(y^2 * x^-1)))^c,(((x * y^-1 *
x^-1
* y^2)*t^(x^-1 * y * x^-1)))^d,(((x * y^-1 * x^-1 *
y^2)*t^(y * x * y * x * y)))^e,(((x * y^-1 * x^-1 *
y^2)*t^(y)))^f,(((x * y^-1 * x^-1 * y^2)*t^(y * x *
y
* x^-1)))^g,(((x * y^-1 * x^-1 * y^2)*t))^h,(((x *
y^-1 * x^-1 * y^2)*t^(y * x * y * x^-1 *
y^-1)))^i,(((x)*t),((x)*t^(y * x * y * x^-1 *
y^-1)))^j,(((x)*t^(y^2 * x^-1)),((x)*t^(x^-1 *
y)))^k,(((x)*t^((y * x)^2)),((x)*t^(x *
y^-1)))^l,(((x)*t^(x^-1 * y * x^-1 *
y^-1)))^m,(((x)*t^(x * y)),(((x * y)^2)*t))^n,
(y^2 * x^-1 * y^-1 * x*t)^o,
(y^2 * x^-1 *y^-1 * x*t)^p,
(t*t^(y * x * y * x^-1 * y^-1))^q = (x^-1 * y *
x)^2>;


G<x,y,t>:=Group<x,y,t|x^3,y^4,y^-1 * x^-1 * y^-2 *
x^-1 * y^2 * x^-1 * y^-1,x^-1 * y^-1 * x^-1 * y^-1 *
x^-1 * y * x^-1 * y,t^2, (t,(x * y)^2),
(((x)*t^(x^-1 * y * x^-1 *
y^-1)))^2,
(y^2 * x^-1 * y^-1 * x*t)^10,
```

```
(y^2 * x^-1 *y^-1 * x*t)^10>;

#G;
/* 960 */
f,G1,k:=CosetAction(G,sub<G|x,y>);

NL:=NormalLattice(G1);

NL;
/*
Normal subgroup lattice
-----------------------

[7]  Order 20  Length 1  Maximal Subgroups: 4 5 6
---
[6]  Order 10  Length 1  Maximal Subgroups: 2 3
[5]  Order 10  Length 1  Maximal Subgroups: 3
[4]  Order 10  Length 1  Maximal Subgroups: 3
---
[3]  Order 5   Length 1  Maximal Subgroups: 1
[2]  Order 2   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:

*/
CompositionFactors(G1);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(2)
    1
*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
/*
1
2
3
6
*/
X:=[2,5];
```

```
IsIsomorphic(NL[6],AbelianGroup(GrpPerm,X));
/*
true Mapping from: GrpPerm: $, Degree 10, Order 2 *
5 to GrpPerm: $, Degree 7, Order 2 * 5
Composition of Mapping from: GrpPerm: $, Degree 10,
Order 2 * 5 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 7, Order
2 * 5
*/
```

5.2 2 by S_{10}

g	h	i	j	[G:N]
0	0	0	2	2

```
S:=Sym(10);
xx:=S!(1, 4, 2, 10, 8, 6, 9, 7, 5, 3);
yy:=S!(1, 6)(2, 5, 9, 3)(4, 8, 7, 10);
N:=sub<S|xx,yy>;
#N;


G<x,y,t>:=Group<x,y,t|x^10,y^4,y^-1 * x^-1 * y * x^3 * y * x * y *
x,x * y^-2 * x^-1 * y * x^2 * y^2 * x^-1 * y * x ,x * y * x^-1 *
y^-2 * x^2 * y * x^-1 * y^2 * x,

t^2,(t,x^-1 * y * x^2 * y * x^-1 * y * x),(t,x * y^-1 * x^-2 * y *
x^-2 * y^-1 * x ),

(((x^5)*t))^a,(((((x * y^-1)^4)*t^(x^2)))^b,(((((x *
y^-1)^4)*t))^c,(((((x * y^2)^3)*t^(x)))^d,(((((x *
y^2)^3)*t))^e,(((((x * y * x)^2)*t^(x^-1)))^f,(((((x * y *
x)^2)*t))^g,(((x^3 * y^-1 * x^-2 * y^-1)*t))^h,

x^3 * y^-1 * x^-2 * y^-1 =(t*t^(x^2))^2,(y * x^-1 * y * x^2 * y
*t)^i,(x^2 * y * x^2 * y^2 * x^2*t)^j>;

G<x,y,t>:=Group<x,y,t|x^10,y^4,y^-1 * x^-1 * y * x^3 * y * x * y *
x,x * y^-2 * x^-1 * y * x^2 * y^2 * x^-1 * y * x ,x * y * x^-1 *
y^-2 * x^2 * y * x^-1 * y^2 * x,

t^2,(t,x^-1 * y * x^2 * y * x^-1 * y * x),(t,x * y^-1 * x^-2 * y *
x^-2 * y^-1 * x ),

(x^2 * y * x^2 * y^2 * x^2*t)^2>;

#G;
/* 7680 */
f,G1,k:=CosetAction(G,sub<G|x,y>);

NL:=NormalLattice(G1);
```

```
NL;
/*
Normal subgroup lattice
-----------------------

[2]  Order 2  Length 1  Maximal Subgroups: 1
---
[1]  Order 1  Length 1  Maximal Subgroups:
*/
CompositionFactors(G1);
/*
  G
    |  Cyclic(2)
    1
*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
/*
1

2
*/
X:=[2];
IsIsomorphic(NL[2],AbelianGroup(GrpPerm,X));
/*
true Isomorphism of GrpPerm: $, Degree 2, Order 2
into GrpPerm: $, Degree 2, Order 2 induced
by
    (1, 2) |--> (1, 2)

*/
```

5.3 $2^4 \cdot (2^3 : PSL(2,7))$

n	o	p	q	[G:N]
0	0	3	0	128

```
S:=Sym(28);
xx:=S!(1, 2)(3, 5)(4, 6)(8, 11)(9, 13)(10, 15)(14,
19)(16, 22)(17, 20)(18, 21)(23,25)(26, 27);
yy:=S!(1, 3)(2, 4, 7, 10)(5, 8, 12, 17)(6, 9, 14,
20)(11, 16, 13, 18)(15, 21, 19,23)(22, 24, 25,
26)(27, 28);

N:=sub<S|xx,yy>;
#N;



G<x,y,t>:=Group<x,y,t|x^2,y^4,y^-2 * x * y^2 * x * y^2 * x,(x * y^-1)^7,

t^2, (t,x * y * x * y^-1 * x * y * x * y), (t,x * y^-1 * x *
y * x * y * x * y^-1 * x * y^-1),

((y * x * y^-1 * x * y * x * y^-1)*t)^r,

(y^2 *t)^a,(y^2 *t^(x))^b,(y^2 *t^(y * x))^c,(y^2 *t^((y *
x)^2))^d,(y^2 *t^(x * y^-1 * x))^e,(y^2 *t^(x * y * x))^f,(y
* x * y *t^(x * y^-1 * x * y^-1 * x * y * x))^g,(y * x * y
*t)^h,(y * x * y *t^(x))^l,(y * x * y *t^(y))^m,(y * x * y
*t^( y * x))^n,(y * x * y *t^(x * y * x))^o,(y * x * y *t^(x
* y^2))^p,(y * x * y *t^((y * x)^2))^q>;



G<x,y,t>:=Group<x,y,t|x^2,y^4,y^-2 * x * y^2 * x * y^2 *

x,(x * y^-1)^7,

t^2, (t,x * y * x * y^-1 * x * y * x * y), (t,x *
y^-1 * x * y * x * y * x * y^-1 * x * y^-1), (y * x
* y *t^(x * y^2))^3>;
#G;
/* 21504 */
f,G1,k:=CosetAction(G,sub<G|x,y>);
```

```
NL:=NormalLattice(G1);

NL;
/*
Normal subgroup lattice
----------------------

[5]  Order 21504  Length 1  Maximal Subgroups: 4
---
[4]  Order 128    Length 1  Maximal Subgroups: 3
---
[3]  Order 16     Length 1  Maximal Subgroups: 2
---
[2]  Order 2      Length 1  Maximal Subgroups: 1
---
[1]  Order 1      Length 1  Maximal Subgroups:

*/
CompositionFactors(G1);

/*
    G
    |  A(1, 7)                    = L(2, 7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
/*
1
2
3
```

```
*/
X:=[2,2,2,2];

IsIsomorphic(NL[3],AbelianGroup(GrpPerm,X));
/*
true Mapping from: GrpPerm: $, Degree 128, Order 2^4 to GrpPerm: $, Degree 8,

Order 2^4

Composition of Mapping from: GrpPerm: $, Degree 128, Order 2^4 to GrpPC and

Mapping from: GrpPC to GrpPC and

Mapping from: GrpPC to GrpPerm: $, Degree 8, Order 2^4
*/
q,ff:=quo<G1|NL[3]>;

q;
/*
Permutation group q acting on a set of cardinality 8

Order = 1344 = 2^6 * 3 * 7

    (2, 3)(6, 8)

    (2, 4)(3, 5, 7, 8)

    (1, 2)(3, 5)(4, 6)(7, 8)
*/
CompositionFactors(q);
/*
    G

    |  A(1, 7)                    = L(2, 7)

    *

    |  Cyclic(2)

    *

    |  Cyclic(2)

    *
```

```
    |  Cyclic(2)

    1
*/
nl:=NormalLattice(q);

nl;
/*
Normal subgroup lattice

----------------------



[3]  Order 1344  Length 1  Maximal Subgroups: 2

---

[2]  Order 8     Length 1  Maximal Subgroups: 1

---

[1]  Order 1     Length 1  Maximal Subgroups:
*/

X:=[2,2,2];

IsIsomorphic(nl[2],AbelianGroup(GrpPerm,X));
/*
true Mapping from: GrpPerm: $, Degree 8, Order 2^3 to GrpPerm: $, Degree 6,

Order 2^3

Composition of Mapping from: GrpPerm: $, Degree 8, Order 2^3 to GrpPC and

Mapping from: GrpPC to GrpPC and

Mapping from: GrpPC to GrpPerm: $,
*/
```

Chapter 6

Appendix

Double Coset Enumeration Codes

6.1 Double Coset Enumeration of PSL (2,11) x 2 over The Maximal Subgroup $D_6 : 2$ and D_6

```
S:=Sym(6);
xx:=S!(1,2,3,4,5,6);
yy:=S!(1,5)(2,4);
N:=sub<S|xx,yy>;
#N;
/* 12 */
sub<N|yy> eq Stabiliser(N,6);
Sch:=SchreierSystem(G,sub<G|Id(G)>);
NN<x,y>:=Group<x,y|x^6, y^2,(x*y)^2>;
Sch1:=SchreierSystem(NN,sub<NN|Id(NN)>);
Set(Sch1);
G<x,y,t>:=Group<x,y,t|x^6, y^2,(x*y)^2, t^2,(t,y),
(x*t*t^x)^3, (t*t*x*t)^5>;
#G;
/* 1320 */
```

```
A:={ Id(G), y * x, x^3, x^2 * y, x^-2, x^-1, y * x^2, x, y,
     x^2, x^3 * y, x * y };
Sch:=SchreierSystem(G,sub<G|Id(G)>);
H:=sub<G|x,y,x^3 * y * t * x^3 * t * x^3 * t>;
f,G1,k:=CosetAction(G,H);
#DoubleCosets(G,H,sub<G|x,y>);
/* 9 */
IN:=sub<G1|f(x),f(y)>;
IH:=sub<G1|f(x),f(y), f(x^3 * y * t * x^3 * t * x^3 * t)>;
CompositionFactors(IH);
/*
 G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
*/
Index(G,H);
/* 55 */
DC:=[ Id(G1), f(t), f( t * x * t), f( t * x^2 * t), f(t * x
    ^2 * t * x * t), f(t * x * t * x^-2 * t ), f(t * x * t *
    x^2 * t), f(t * x^2 * t * x^-1 * t),f(t * x * t * x^-1 *
    t)];

ts := [Id(G1): i in [1 .. 6] ];
ts[6]:=f(t); ts[2]:=f(t^(x^2)); ts[3]:=f(t^(x ^3));
ts[4]:=f(t^(x^4)); ts[5]:=f(t^(x^5)); ts[1]:=f(t^(x));
```

```
cst:=[null : i in [1 .. 110]] where null is [Integers() |
    ];
prodim:=function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;



for i := 1 to 6 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
for i in [1..110] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/* 6 */

N1:=Stabiliser(N,1);
N1s:=N1;
Orbits(N1s);
/*
[
    GSet{@ 1 @},
    GSet{@ 4 @},
    GSet{@ 2, 6 @},
    GSet{@ 3, 5 @}
]
*/
```

```
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*1*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[4] eq g*(DC[i])^h then i;
 break i; break g; break h; end if; end for;end for;end for
    ;
/*2*/
for m in IH do for n in A do
if ts[1]*ts[4] eq m*(ts[1])^f(n) then
for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i]; break
   i; end if; end for;
n;break m; break n;
end if; end for; end for;
/* x * y
Id(G)
This is telling me the following:
t_1t_4 = xyt_1^e
The bottom code is checking if this is true
*/

ts[1]*ts[4] eq f(x*y)*ts[1];
/* true */

/*
So we can say the following:
Ht_1t_4 \in [1]: We have t_1t_4 = xyt_1(need a proof)
```

```
So  Ht_1t_4 = Ht_1 \in [1]
*/

for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/
for m in IH do for n in A do
if ts[1]*ts[3] eq m*(ts[1]*ts[2])^f(n) then
for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i]; break
    i; end if; end for;
n;break m; break n;
end if; end for; end for;
/* 3rd DCE */
S:={[1,2]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
end if; end for; end for;
/*
```

```
[1,2]

*/

N12:=Stabiliser(N,[1,2]);
#N12;
/* 2 */
N12s:=N12;
Orbits(N12s);
/*
 GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*9*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
```

```
/*3*/
for m in IH do for n in A do
if ts[1]*ts[2]*ts[3] eq m*(ts[1]*ts[2])^f(n) then
for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i]; break
   i; end if; end for;
n;break m; break n;
end if; end for; end for;
/*
x^-1
y
> [1,2]^yy;
[ 5, 4 ]
> ts[1]*ts[2]*ts[3] eq f(x^-1)*(ts[5]*ts[4]);
true
*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*7*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[5] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/
for m in IH do for n in A do
if ts[1]*ts[2]*ts[5] eq m*(ts[1]*ts[3])^f(n) then
for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i]; break
   i; end if; end for;
n;break m; break n;
end if; end for; end for;
```

```
/*
y * x
x^-1
*/
[1,3]^(xx^-1);
/* [ 6, 2 ] */
> ts[1]*ts[2]*ts[5] eq f(y*x)*ts[6]*ts[2];
true
for n in N do if [1,3]^n eq [6,2] then n; break n; end if;
    end for;
(1, 6, 5, 4, 3, 2)
> x;
x
> xx;
(1, 2, 3, 4, 5, 6)
> [1,3]^(xx^-1);
[ 6, 2 ]

*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[6] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*6*/

/*
for m in IH do
for> if ts[1]*ts[2]*ts[6] eq m*(ts[4]*ts[5]*ts[3]) then
for|if> for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i
    ]; break i; end if; end \
for;
```

```
for | if > break m;
for | if > end if; end for;
x^2 * y
> ts[1]*ts[2]*ts[6]  eq  f(x^2*y)*ts[4]*ts[5]*ts[3];
true
*/



Generators(N12s);
/*
{

}

*/
[1,2]^N12s;
/*



*/

tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 18 */
#sub<G|x,y>/#N12s+6;
```

```
/* 4th DCE */
S:={[1,3]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
end if; end for; end for;
/*

[1,3]

*/

N13:=Stabiliser(N,[1,3]);
#N13;
/* 2 */
N13s:=N13;
Orbits(N13s);
/*
 GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
*/
for i in [1..9] do
```

```
for g in IH do for h in IN do
if ts[1]*ts[3]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/

for m in IH do for n in A do
if ts[1]*ts[3]*ts[1] eq m*(ts[1]*ts[3])^f(n) then
for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i]; break
   i; end if; end for;
n;break m; break n;
end if; end for; end for;
/*
x^2 * y
Id(G)
*/

ts[1]*ts[3]*ts[1] eq f(x^2*y)*ts[1]*ts[3];
/* true */

for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[3]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*8*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[3]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/
for i in [1..9] do
for g in IH do for h in IN do
```

```
if  ts [1]* ts [3]* ts [4]  eq  g*(DC[ i ])^h  then  i ;
break  i ;  break  g;  break  h;  end  if ;  end  for ; end  for ; end  for ;
/*5*/
for  i  in  [1 . . 9]  do
for  g  in  IH  do  for  h  in  IN  do
if  ts [1]* ts [3]* ts [5]  eq  g*(DC[ i ])^h  then  i ;
break  i ;  break  g;  break  h;  end  if ;  end  for ; end  for ; end  for ;
/*4*/

for  m  in  IH  do  for  n  in  A  do
if  ts [1]* ts [3]* ts [5]  eq  m*( ts [1]* ts [3])^ f (n)  then
for  i  in  [1 . .# Sch ]  do  if  m  eq  f (Sch[ i ])  then  Sch[ i ];  break
    i ;  end  if ;  end  for ;
n ; break  m;  break  n;
end  if ;  end  for ;  end  for ;
/*
x^−2
x^2  *  y
*/
[1 ,3]^( xx^2*yy ) ;

ts [1]* ts [3]* ts [5]  eq  f (x^(−2))* ts [3]* ts [1];
/* true */

for  i  in  [1 . . 9]  do
for  g  in  IH  do  for  h  in  IN  do
if  ts [1]* ts [3]* ts [6]  eq  g*(DC[ i ])^h  then  i ;
break  i ;  break  g;  break  h;  end  if ;  end  for ; end  for ; end  for ;
/*3*/
for  m  in  IH  do  for  n  in  A  do
if  ts [1]* ts [3]* ts [6]  eq  m*( ts [1]* ts [2])^ f (n)  then
```

```
for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i]; break
    i; end if; end for;
n;break m; break n;
end if; end for; end for;
/*
x^3 * y
x
*/
[1,2]^(xx);
/* [2,3] */
ts[1]*ts[3]*ts[6] eq f(x^(3)*y)*ts[2]*ts[3];
/* true */

Generators(N13s);
/*
{

}

*/
[1,3]^N13s;
/*



*/

tr1:=Transversal(N,N13s);
for i:=1 to #tr1 do
ss:=[1,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
```

```
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 30 */
#sub<G|x,y>/#N13s+18;



/* 5th DCE */
S:={[1,3,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IH do if ts[1]*ts[3]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])
   [3]] then print SSS[i];
end if; end for; end for;
/*

{
    [ 1, 3, 4 ]
}
{
    [ 4, 2, 1 ]
}
*/

/*
Proof of Ht_1t_3t_4 = Ht_4t_2t_1
for m in IH do
if ts[1]*ts[3]*ts[4] eq m*(ts[4]*ts[2]*ts[1]) then
```

```
for i in [1..#Sch] do if f(Sch[i]) eq m then Sch[i]; break
    i; end if; end for;
end if; end for;
/*
x
> ts[1]*ts[3]*ts[4] eq f(x)*(ts[4]*ts[2]*ts[1]);
true

Proof of Ht_1t_3t_4 = Ht_4t_2t_1
We have t_1t_3t_4 = xt_4t_2t_1 ( need proof)
Then Ht_1t_3t_4 = NEED SOME CODE HERE

*/
N134:=Stabiliser(N,[1,3,4]);
#N134;
/* 1 */
N134s:=N134;
for n in N do if [1,3,4]^n eq [4,2,1] then N134s:=sub<N|
    N134s,n>; end if; end for;
#N134s;
/* 2 */
[1,3,4]^N134s;

Orbits(N134s);
/*
 GSet{@ 1, 4 @},
    GSet{@ 2, 3 @},
    GSet{@ 5, 6 @}
*/
for i in [1..9] do
for g in IH do for h in IN do
```

```
if ts[1]*ts[3]*ts[4]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[3]*ts[4]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*5*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[3]*ts[4]*ts[5] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*9*/

for m in IH do for n in A do
if ts[1]*ts[3]*ts[4]*ts[5] eq m*(ts[1]*ts[2]*ts[1])^f(n)
    then
for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i]; break
    i; end if; end for;
n;break m; break n;
end if; end for; end for;
/*
x^-1
x^-1
This is telling me the following:
t_1t_3t_4t_5 = x^-1(t_1t_2t_1)^(x^-1)
[1,2,1]^(xx^-1);
/*
[ 6, 1, 6 ]
*/
The bottom code is checking if this is true
```

t_1t_3t_4t_5 = x^−1t_6t_1t_6

ts[1]*ts[3]*ts[4]*ts[5] eq f(x^−1)*ts[6]*ts[1]*ts[6];
/* true */
We now proof that Ht_1t_3t_4t_5 belongs to [1,2,1] =
    Ht_1t_2t_1N

We have t_1t_3t_4t_5 = x^−1(t_6t_1t_6) (need to proof)

Then Ht_1t_3t_4t_5 = H x^−1(t6_t_1t_6) = Ht_6t_1t_6 (x^−1
    in H)

Now Ht_1t_2t_1N = { H(t_1t_2t_1)^n | n in N}
*/
for n in N do if [6,1,6]^n eq [1,2,1] then n; break n; end
    if; end for;
/* (1, 2, 3, 4, 5, 6)

Now Ht_1t_2t_1N = { H(t_1t_2t_1)^n | n in N}
Then Ht_1t_3t_4t_5 = H x^−1(t6_t_1t_6) = Ht_6t_1t_6 (x^−1
    in H) = H(t_1t_2t_1)^{(1, 2, 3, 4, 5, 6)} in Ht_1t_2t_1N
    = [1,2,1]

 */

Generators(N134s);
/*
(1, 4)(2, 3)(5, 6)

```
*/

tr1:=Transversal(N,N134s);
for i:=1 to #tr1 do
ss:=[1,3,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 36 */
#sub<G|x,y>/#N134s+30;


/* 6th DCE */
S:={[1,2,6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IH do if ts[1]*ts[2]*ts[6]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])
   [3]] then print SSS[i];
end if; end for; end for;
/*

{
    [ 1, 2, 6 ]
}
{
    [ 6, 5, 1 ]
}
```

```
{
    [ 4, 5, 3 ]
}
{
    [ 3, 2, 4 ]
}

*/

N126:= Stabiliser (N,[1,2,6]);
#N126;
/* 1 */
N126s:=N126;
for n in N do if [1,2,6]^n eq [6,5,1] then N126s:=sub<N|
   N126s,n>; end if; end for;
#N126s;
/* 2 */
for n in N do if [1,2,6]^n eq [4,5,3] then N126s:=sub<N|
   N126s,n>; end if; end for;
#N126s;
[1,2,6]^N126s;

/*
for m in IH do
if ts[1]*ts[2]*ts[6] eq m*(ts[4]*ts[5]*ts[3]) then
for i in [1..#Sch] do if m eq f(Sch[i]) then Sch[i]; break
   i; end if; end for;
break m;
end if; end for;

ts[1]*ts[2]*ts[6] eq f(x^2*y)*ts[4]*ts[5]*ts[3];
```

```
*/



Orbits(N126s);
/*
        GSet{@ 2, 5 @},
    GSet{@ 1, 6, 4, 3 @}


*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[6]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*6*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[6]*ts[6] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/



Generators(N126s);
/*
(1, 4)(2, 5)(3, 6),
    (1, 6)(2, 5)(3, 4)
*/

tr1:=Transversal(N,N126s);
for i:=1 to #tr1 do
ss:=[1,2,6]^tr1[i];
```

```
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 39 */
#sub<G|x,y>/#N126s+36;




/* 7th DCE */
S:={[1,2,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IH do if ts[1]*ts[2]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])
   [3]] then print SSS[i];
end if; end for; end for;
/*

    [ 1, 2, 4 ]
}
{
    [ 5, 4, 2 ]

*/

N124:=Stabiliser(N,[1,2,4]);
#N124;
/* 1 */
N124s:=N124;
```

```
for n in N do if [1,2,4]^n eq [5,4,2] then N124s:=sub<N|
    N124s,n>; end if; end for;
#N124s;
/* 2 */
[1,2,4]^N124s;

Orbits(N124s);
/*

 GSet{@ 3 @},
    GSet{@ 6 @},
    GSet{@ 1, 5 @},
    GSet{@ 2, 4 @}
*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*8*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4]*ts[6] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*7*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*9*/
for i in [1..9] do
for g in IH do for h in IN do
```

```
if ts[1]*ts[2]*ts[4]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

Generators(N124s);
/*
 (1, 5)(2, 4)
*/

tr1:=Transversal(N,N124s);
for i:=1 to #tr1 do
ss:=[1,2,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 45 */
#sub<G|x,y>/#N124s+42;


/* 8th DCE */
S:={[1,3,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IH do if ts[1]*ts[3]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])
   [3]] then print SSS[i];
end if; end for; end for;
/*
```

```
{
    [ 1, 3, 2 ]
}
{
    [ 5, 3, 4 ]
}
{
    [ 4, 6, 5 ]
}
{
    [ 2, 6, 1 ]
}

*/

N132:=Stabiliser(N,[1,3,2]);
#N132;
/* 1 */
N132s:=N132;
for n in N do if [1,3,2]^n eq [5,3,4] then N132s:=sub<N|
    N132s,n>; end if; end for;
#N132s;
/* 2 */
for n in N do if [1,3,2]^n eq [4,6,5] then N132s:=sub<N|
    N132s,n>; end if; end for;
#N132s;
[1,3,2]^N132s;

Orbits(N132s);
/*
```

```
[
    GSet{@ 3, 6 @},
    GSet{@ 1, 5, 4, 2 @}
]

*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[3]*ts[2]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*7*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[3]*ts[2]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/


Generators(N132s);
/*
  (1, 4)(2, 5)(3, 6),
    (1, 5)(2, 4)
*/

tr1:=Transversal(N,N132s);
for i:=1 to #tr1 do
ss:=[1,3,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
```

```
then m:=m+1;
end if; end for;m;
/* 48 */
#sub<G|x,y>/#N132s+48;



/* 9th DCE */
S:={[1,2,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IH do if ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])
   [3]] then print SSS[i];
end if; end for; end for;
/*
  [ 1, 2, 1 ]
}
{
    [ 2, 1, 2 ]
*/


N121:=Stabiliser(N,[1,2,1]);
#N121;
/* 1 */
N121s:=N121;
for n in N do if [1,2,1]^n eq [2,1,2] then N121s:=sub<N|
   N121s,n>; end if; end for;
#N121s;
/* 2 */
[1,2,1]^N121s;
```

```
Orbits(N121s);
/*

  GSet{@ 1, 2 @},
    GSet{@ 3, 6 @},
    GSet{@ 4, 5 @}

*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[1]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[1]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*5*/
for i in [1..9] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[1]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*7*/

Generators(N121s);
/*
 (1, 2)(3, 6)(4, 5)
*/

tr1:=Transversal(N,N121s);
```

```
for i:=1 to #tr1 do
ss:=[1,2,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..110] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 54 */
#sub<G|x,y>/#N121s+54;


N ~ D_6


> NL:=NormalLattice(IH);
IsIsomorphic(NL[20],DihedralGroup(6));


IH ~ D_6 : 2



Relations:
There are two relations in this group:
> ts := [Id(G1): i in [1 .. 6] ];
> ts[1]:=f(t); ts[2]:=f(t^x); ts[3]:=f(t^(x ^2));
> ts[4]:=f(t^(x^3)); ts[5]:=f(t^(x^4)); ts[6]:=f(t^(x^5));


> /* (x*t*t^x)^3 where t^x = t_2, t=t_6*/
/* x^3 (t*t^x)^(x^2) (t*t^x)^x (t*t^x) */
> /*x^3 (t6 t1)^(x^2) (t6 t1)^x (t6t1)*/
> /*x^3 (t2t3) (t1t2)(t6t1)*/
> f(x^3)*ts[2]*ts[3]*ts[1]*ts[2]*ts[6]*ts[1];
Id(G1)
```

```
> /* x^3 * y * t * x^3 * t * x^3 * t in H*/
> /*  x^3 * y * t * x^3 * t * x^3 * t= x^3 * y * t * x^6x
    ^-3 * t * x^3     *t
>  x^3 * y  t*e*t^(x^3)     t= x^3*y*t*t^(x^3)     *t=x^3t6t3t6
    */
> f(x^3*y)*ts[6]*ts[3]*ts[6]  in IH;
true
So  Ht6t3=Ht6
```

```
/* NEW GROUP BELOW */
S:=Sym(15);
xx:=S!(1,  15,  12,  8,  3,  9,  14,  13,  7,  4)(2,  11,  5,  6,  10);
yy:=S!(1,  11,  14,  6,  12,  2,  7,  5,  3,  10)(4,  8,  13,  15,  9);
N:=sub<S|xx,yy>;
#N;
/*150*/
G<x,y,t>:=Group<x,y,t |  (y^-1 * x^-1)^3 ,  (y^-1 * x)^3, x
    ^-1* y^-1 * x^3 * y^-1 * x^-1 * y   ,x^2 * y * x^2 *y^3,t
    ^2,
(t,y^-1 * x^2 * y^-1),  (t,x^3 * y^-1 * x) ,(x^(-1)*y*t)^5,(x
    ^(-2)*t^(y*x^(-1)))^3>;
#G;
/*124800*/
/* N~  5^2:S3 */


/* G~U(3,4):2*/
```

```
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*12*/
H:=sub<G|x,y,x * y * t * x^-2 * t * x^2 * t * y^-1>;
#H;
/*300*/
Index(G,H);
/*416*/
f,G1,k:=CosetAction(G,H);


IN:=sub<G1|f(x),f(y)>;


IH:=sub<G1|f(x),f(y),f(x * y * t * x^-2 * t * x^2 * t * y
    ^-1)>;
#DoubleCosets(G,H,sub<G|x,y>);
/*7*/


/* Do DCE of G over H and N*/


ts := [ Id(G1): i in [1 .. 15] ];


ts[1]:=f(t);


ts[2]:=f(t^(y * x^-1)); ts[3]:=f(t^(y^-2)); ts[4]:=f(t^(x
    ^-1)); ts[5]:=f(t^(y * x));


ts[6]:=f(t^(y^-1 * x^-1)); ts[7]:=f(t^(x^-2));


ts[8]:=f(t^(x^-1 * y)); ts[9]:=f(t^(x*y));ts[10]:=f(t^(y^-1
    )); ts[11]:=f(t^y); ts[12]:=f(t^(x^2 )); ts[13]:=f(t^(x*
    y^-1));
```

```
ts [14]:= f ( t ^( y ^2 ) ) ;  ts [15]:= f ( t ^x ) ;


DC:=[ Id (G1) , f ( t ) , f ( t  *  x  *  t ) , f ( t  *  x ^−1  *  t  *  y  *  t ) , f ( t  *
      x  *  t  *  x  *  t ) , f ( t  *  x  *  t  *  y ^−1  *  t ) ,  f ( t  *  y  *  t  *
   y  *  t )   ] ;
L<u , v>:=Group<u , v | ( v ^−1  *  u ^−1)^3  ,  ( v ^−1  *  u )^3  ,
u ^−1  *  v ^−1  *  u ^3  *  v ^−1  *  u ^−1  *  v   , u ^2  *  v  *  u ^2  *  v ^3
   >;
Sch:= SchreierSystem (L , sub<L | Id (L)>);
h:=hom<L−>N | u−>xx , v−>yy >;
g:=hom<IN−>N | f ( x)−>xx , f ( y)−>yy >;
Sch1:= SchreierSystem (G, sub<G | Id (G)>);
cst  :=  [ null  :  i  in  [1  ..  416]]  where  null  is  [ Integers ()  |
     ] ;
prodim  :=  function ( pt ,  Q,  I )
v  :=  pt ;
for  i  in  I  do
v  :=  v ^(Q[ i ]) ;
end  for ;
return  v ;
end  function ;


N1:= Stabiliser (N,1) ;


for  g  in  IH  do  for  i  in  [1..15]  do  if  ts [1]  eq  g*ts [ i ]  then
     i ;  end  if ;  end  for ;  end  for ;


S:={[1]};


SS:=S ^N;
```

```
SSS:=Setseq(SS);

for i in [1..#SSS] do

for g in IH do if ts[1]

eq g*ts[Rep(SSS[i])[1]]

then print SSS[i];

end if; end for; end for;

/*

{

    [ 1 ]

}



*/

N1:=Stabiliser(N,1);

N1s:=N1;

#N1s;

/*10*/
```

```
tr1:=Transversal(N,N1s);

for  i:=1  to  #tr1  do

ss:=[1]^tr1[i];

cst[prodim(1,ts,ss)]:=ss;

end  for;

/*15*/

m:=0; for i in [1..416] do if cst[i] ne []

then m:=m+1;

end if; end for;m;

Orbits(N1s);

/*

[

    GSet{@ 1 @},

    GSet{@ 3 @},
```

GSet{@ 7 @},

GSet{@ 12 @},

GSet{@ 14 @},

GSet{@ 2, 5, 13, 10, 8, 15, 11, 4, 9, 6 @}

]

*/

```
for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[1] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*1*/

for i in [1..7] do
```

```
for  g  in  IH  do  for  h  in  IN  do

if  ts [1]* ts [3]  eq  g*(DC[ i ])^h  then  i ;

break  i ;  break  g;  break  h;  end  if ;  end  for ;end  for ;end  for ;

/*2*/
for  m  in  IH  do  for  n  in  IN  do
if  ts [1]* ts [3]  eq  m*( ts [1])^n
then
for  i  in  [1..#Sch]  do  if  g(n)  eq  h(Sch[ i ])  then  Sch[ i ];
   break  i ;end  if ;  end  for ;
for  j  in  [1..#Sch1]  do  if  m  eq  f(Sch1[ j ])  then  Sch1[ j ];
   break  j ;end  if ;  end  for ;
break  m;  break  n;  end  if ;  end  for ;  end  for ;
/*
u^-2
(x  *  y^-1  *  x^-1)^2
1^( xx^-2);
7
>  ts [1]* ts [3]  eq  f (( x  *  y^-1  *  x^-1)^2)* ts [7];
true
>
*/
for  i  in  [1..7]  do

for  g  in  IH  do  for  h  in  IN  do

if  ts [1]* ts [7]  eq  g*(DC[ i ])^h  then  i ;
```

```
break i; break g; break h; end if; end for;end for;end for;

/*2*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[12] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*2*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[14] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*2*/


for i in [1..7] do
```

```
for g in IH do for h in IN do

if ts[1]*ts[2] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*3*/




S:={[1,2]};

SS:=S^N;

SSS:=Setseq(SS);

for i in [1..#SSS] do

for g in IH do if ts[1]*ts[2]

eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]

then print SSS[i];

end if; end for; end for;

/*

{
```

```
    [ 1, 2 ]

}


*/

N12:=Stabiliser(N,[1,2]);

N12s:=N12;

#N12s;

/*1*/

tr1:=Transversal(N,N12s);

for i:=1 to #tr1 do

ss:=[1,2]^tr1[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..416] do if cst[i] ne []

then m:=m+1;

end if; end for;m;
```

/*165*/

Orbits(N12s);

/*

[

    GSet{@ 1 @},

    GSet{@ 2 @},

    GSet{@ 3 @},

    GSet{@ 4 @},

    GSet{@ 5 @},

    GSet{@ 6 @},

    GSet{@ 7 @},

    GSet{@ 8 @},

    GSet{@ 9 @},

    GSet{@ 10 @},

    GSet{@ 11 @},

```
    GSet{@ 12 @},

    GSet{@ 13 @},

    GSet{@ 14 @},

    GSet{@ 15 @}

]



*/

for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[1] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*3*/
for m in IH do for n in IN do
if ts[1]*ts[2]*ts[1] eq m*(ts[1]*ts[2])^n
then
for i in [1..#Sch] do if g(n) eq h(Sch[i]) then Sch[i];
    break i;end if; end for;
for j in [1..#Sch1] do if m eq f(Sch1[j]) then Sch1[j];
    break j;end if; end for;
break m; break n; end if; end for; end for;
```

```
/*
Id(L)
x * t * x^2 * t * x^-2 * t * x^-1
> ts[1]*ts[2]*ts[1] eq f(x * t * x^2 * t * x^-2 * t * x^-1)
  *ts[1]*ts[2];
true
>

*/

for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[2] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*2*/

for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[3] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*7*/
```

```
for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[4] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*6*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[5] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*3*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[6] eq g*(DC[i])^h then i;
```

```
break i; break g; break h; end if; end for;end for;end for;

/*3*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[7] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*5

*/

for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[8] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*3*/


for i in [1..7] do
```

```
for g in IH do for h in IN do

if ts[1]*ts[2]*ts[9] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*4*/



for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[10] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*3*/



for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[11] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;
```

```
/*3*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[12] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*5*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[13] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*6*/


for i in [1..7] do

for g in IH do for h in IN do
```

```
if ts[1]*ts[2]*ts[14] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*7*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[15] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*4*/




/*

{

    [ 1, 2, 9 ]

}
```

```
{

    [ 13, 2, 7 ]

}

for m in IH do if ts[1]*ts[2]*ts[9] eq m*ts[13]*ts[2]*ts[7]
    then
for j in [1..#Sch1] do if m eq f(Sch1[j]) then Sch1[j];
    break j;end if; end for;
break m; end if; end for;
x * t * x^2 * t * x * y^-1 * x * t
> ts[1]*ts[2]*ts[9] eq f(x * t * x^2 * t * x * y^-1 * x * t
    )*ts[13]*ts[2]*ts[7\
];
true
*/




N129:=Stabiliser(N,[1,2,9]);

N129s:=N129;

for g in N do if [1,2,9]^g eq [13,2,7] then N129s:=sub<N|
    N129s,g>; end if ; end for;

[1,2,9]^N129s;

/*
```

GSet{@

    [1, 2, 9] ,

    [13, 2, 7]

@}

*/

#N/#N129s;

/*75*/

```
tr1:=Transversal(N,N129s);

for i:=1 to #tr1 do

ss:=[1,2,9]^tr1[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..416] do if cst[i] ne []

then m:=m+1;

end if; end for;m;
```

```
/*240*/


S:={[1,2,7]};

SS:=S^N;

SSS:=Setseq(SS);

for i in [1..#SSS] do

for g in IH do if ts[1]*ts[2]*ts[7]

eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])
    [3]]

then print SSS[i];

end if; end for; end for;

/*

{

    [ 1, 2, 7 ]

}

{
```

```
    [  2,  15,  11  ]

}

{

    [  15,  1,  4  ]

}

*/

N127:= Stabiliser (N,[1,2,7]);

N127s:=N127;

for  g  in  N  do  if  [1,2,7]^g  eq  [2,15,11]  then  N127s:=sub<N|
    N127s,g>; end if ; end for;

[1,2,7]^N127s;

#N/#N127s;

/*50*/




tr1:=Transversal(N,N127s);

for  i:=1  to  #tr1  do
```

```
ss :=[1 ,2 ,7]^ tr1 [ i ];

cst [ prodim (1 , ts , ss )]:= ss ;

end for ;

m:=0; for i in [1..416] do if cst [i] ne []

then m:=m+1;

end if ; end for ;m;

/*290*/



S:={[1 ,2 ,4]};

SS:=S^N;

SSS:= Setseq (SS) ;

for i in [1..#SSS] do

for g in IH do if ts [1]* ts [2]* ts [4]

eq g* ts [Rep(SSS[ i ]) [1]]* ts [Rep(SSS[ i ]) [2]]* ts [Rep(SSS[ i ])
    [3]]

then print SSS[ i ];
```

```
end if; end for; end for;

/*

{

    [ 1, 2, 4 ]

}

{

    [ 6, 13, 7 ]

}

{

    [ 15, 14, 10 ]

}

*/


N124:=Stabiliser(N,[1,2,4]);
```

```
N124s:=N124;

for g in N do if [1,2,4]^g eq [6,13,7] then N124s:=sub<N|
    N124s,g>; end if ; end for;

[1,2,4]^N124s;

/*

GSet{@

    [ 1, 2, 4 ],

    [ 6, 13, 7 ],

    [ 15, 14, 10 ]

@}

*/

#N/#N124s;

/*50*/



tr1:=Transversal(N,N124s);

for i:=1 to #tr1 do
```

```
ss:=[1,2,4]^tr1[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in [1..416] do if cst[i] ne []

then m:=m+1;

end if; end for;m;

/*340*/



S:={[1,2,3]};

SS:=S^N;

SSS:=Setseq(SS);

for i in [1..#SSS] do

for g in IH do if ts[1]*ts[2]*ts[3]

eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])
    [3]]

then print SSS[i];
```

```
end if; end for; end for;

/*

{

    [ 1, 2, 3 ]

}

{

    [ 5, 3, 2 ]

}

*/



N123:=Stabiliser(N,[1,2,3]);

N123s:=N123;

for g in N do if [1,2,3]^g eq [5,3,2] then N123s:=sub<N|
    N123s,g>; end if ; end for;

[1,2,3]^N123s;

/*
```

```
GSet{@

    [ 1, 2, 3 ],

    [ 5, 3, 2 ]

@}
```

```
*/
```

```
#N/#N123s;
```

```
/*75*/
```

```
Orbits(N123s);
```

```
/*[

    GSet{@ 4 @},

    GSet{@ 8 @},

    GSet{@ 9 @},

    GSet{@ 13 @},

    GSet{@ 15 @},

    GSet{@ 1, 5 @},
```

```
        GSet{@ 2,  3 @},

        GSet{@ 6,  7 @},

        GSet{@ 10,  14 @},

        GSet{@ 11,  12 @}

]



*/



tr1:=Transversal(N,N123s);

for  i:=1  to  #tr1  do

ss:=[1,2,3]^tr1[i];

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0;  for  i  in  [1..416]  do  if  cst[i]  ne  []

then  m:=m+1;
```

```
end if; end for;m;
```

```
/*415*/
```

```
for i in [1..7] do
```

```
for g in IH do for h in IN do
```

```
if ts[1]*ts[2]*ts[3]*ts[4] eq g*(DC[i])^h then i;
```

```
break i; break g; break h; end if; end for;end for;end for;
```

```
/*7*/
```

```
for i in [1..7] do
```

```
for g in IH do for h in IN do
```

```
if ts[1]*ts[2]*ts[3]*ts[8] eq g*(DC[i])^h then i;
```

```
break i; break g; break h; end if; end for;end for;end for;
```

```
/*4*/
```

```
for i in [1..7] do
```

```
for g in IH do for h in IN do

if ts[1]*ts[2]*ts[3]*ts[9] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*4*/
```

```
for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[3]*ts[13] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*4*/
```

```
for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[3]*ts[15] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;
```

```
/*7*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[3]*ts[1] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*5*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[3]*ts[2] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*3*/


for i in [1..7] do

for g in IH do for h in IN do
```

```
if ts[1]*ts[2]*ts[3]*ts[6] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*3*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[3]*ts[10] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*7*/


for i in [1..7] do

for g in IH do for h in IN do

if ts[1]*ts[2]*ts[3]*ts[11] eq g*(DC[i])^h then i;

break i; break g; break h; end if; end for;end for;end for;

/*5*/
```

6.2 Double Coset Enumeration of 2^3:PSL (2,7) over PSL (2,7)

```
S:=Sym(10);
xx:=S!(1, 2)(3, 5)(7, 9)(8, 10);
yy:=S!(1, 3, 6)(2, 4, 7)(5, 8, 9);
N:=sub<S|xx,yy>;

#N;
/* 60 */


G<x,y,t>:=Group<x,y,t|x^2,y^3,(x * y^-1)^5,

t^2, (t,y * x * y^-1 * x * y * x), (t,x^y),

(x * y * x * y^-1 * x * y * x * y^-1 * x * y*t)^3>;
#G;
for i in [0..9] do for j in [0..1] do printf "%o" , i;j;xx^
   i*yy^j; end for;end for;
/*
Id(S)
01
(1, 3, 6)(2, 4, 7)(5, 8, 9)
10
(1, 2)(3, 5)(7, 9)(8, 10)
11
(1, 4, 7, 5, 6)(2, 3, 8, 10, 9)
20
Id(S)
21
(1, 3, 6)(2, 4, 7)(5, 8, 9)
```

```
30
(1, 2)(3, 5)(7, 9)(8, 10)
31
(1, 4, 7, 5, 6)(2, 3, 8, 10, 9)
40
Id(S)
41
(1, 3, 6)(2, 4, 7)(5, 8, 9)
50
(1, 2)(3, 5)(7, 9)(8, 10)
51
(1, 4, 7, 5, 6)(2, 3, 8, 10, 9)
60
Id(S)
61
(1, 3, 6)(2, 4, 7)(5, 8, 9)
70
(1, 2)(3, 5)(7, 9)(8, 10)
71
(1, 4, 7, 5, 6)(2, 3, 8, 10, 9)
80
Id(S)
81
(1, 3, 6)(2, 4, 7)(5, 8, 9)
90
(1, 2)(3, 5)(7, 9)(8, 10)
91
(1, 4, 7, 5, 6)(2, 3, 8, 10, 9)

*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
```

```
#k ;
/* 1 */
#sub<G|x,y>;
/* 60 */
CompositionFactors(G1);
/*
    G
    |   A(1, 11)                        = L(2, 11)
    1
*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/* 2 */

NN<a,b>:=Group<a,b|a^2,b^3,(a * b^-1)^5>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
Sch;

a:=xx;
b:=yy;

/*
(1, 2)(3, 5)(7, 9)(8, 10)
(1, 3, 6)(2, 4, 7)(5, 8, 9)
(1, 6, 3)(2, 7, 4)(5, 9, 8)
(1, 4, 7, 5, 6)(2, 3, 8, 10, 9)
(1, 7, 8, 10, 5)(2, 6, 3, 9, 4)
(1, 5, 10, 8, 7)(2, 4, 9, 3, 6)
(1, 6, 5, 7, 4)(2, 9, 10, 8, 3)
(1, 4, 9)(2, 5, 6)(3, 10, 7)
(1, 9, 4)(2, 6, 5)(3, 7, 10)
(1, 8, 2, 7, 3)(4, 5, 10, 9, 6)
```

(1, 9)(4, 8)(5, 10)(6, 7)
(2, 5)(3, 4)(6, 8)(9, 10)
(1, 3, 7, 2, 8)(4, 6, 9, 10, 5)
(1, 7, 6, 4, 5)(2, 8, 9, 3, 10)
(1, 2, 9, 6, 7)(3, 10, 4, 8, 5)
(1, 5, 4, 3, 2)(6, 8, 9, 7, 10)
(1, 8, 5, 7, 10)(2, 3, 4, 6, 9)
(1, 10, 7, 5, 8)(2, 9, 6, 4, 3)
(1, 7, 6, 9, 2)(3, 5, 8, 4, 10)
(1, 2, 3, 4, 5)(6, 10, 7, 9, 8)
(1, 5, 4, 6, 7)(2, 10, 3, 9, 8)
(1, 9, 5, 2, 10)(3, 8, 7, 6, 4)
(2, 7)(3, 8)(4, 10)(6, 9)
(1, 3)(4, 5)(6, 10)(7, 8)
(1, 10, 2, 5, 9)(3, 4, 6, 7, 8)
(1, 10, 4)(2, 8, 6)(3, 7, 9)
(1, 2, 3, 8, 7)(4, 10, 6, 5, 9)
(1, 4, 10)(2, 6, 8)(3, 9, 7)
(1, 4, 8)(2, 6, 10)(3, 7, 5)
(1, 7, 8, 3, 2)(4, 9, 5, 6, 10)
(1, 8, 4)(2, 10, 6)(3, 5, 7)
(1, 3, 9)(4, 10, 7)(5, 8, 6)
(1, 6, 8)(2, 4, 10)(3, 5, 9)
(1, 6, 10)(2, 4, 8)(5, 7, 9)
(2, 7, 5)(3, 6, 10)(4, 9, 8)
(1, 8, 6)(2, 10, 4)(3, 9, 5)
(2, 5, 7)(3, 10, 6)(4, 8, 9)
(1, 9, 3)(4, 7, 10)(5, 6, 8)
(1, 10, 6)(2, 8, 4)(5, 9, 7)
(1, 5, 10, 9, 2)(3, 7, 4, 8, 6)
(1, 2, 9, 10, 5)(3, 6, 8, 4, 7)

```
(1, 9, 8)(2, 10, 7)(3, 5, 6)
(1, 5)(2, 10)(3, 8)(4, 7)
(1, 3, 10)(2, 8, 5)(4, 9, 7)
(1, 6)(2, 9)(3, 10)(4, 5)
(1, 5)(2, 4)(6, 9)(7, 10)
(1, 8, 9)(2, 7, 10)(3, 6, 5)
(1, 10)(2, 9)(3, 6)(7, 8)
(1, 10, 3)(2, 5, 8)(4, 7, 9)
(1, 8)(2, 3)(4, 9)(5, 10)
(1, 9, 7, 2, 6)(3, 4, 5, 10, 8)
(1, 4, 2, 5, 3)(6, 9, 10, 8, 7)
(1, 7)(2, 8)(5, 6)(9, 10)
(1, 6, 2, 7, 9)(3, 8, 10, 5, 4)
(1, 3, 5, 2, 4)(6, 7, 8, 10, 9)
(1, 7)(2, 6)(3, 4)(5, 8)
(1, 4)(2, 3)(6, 7)(8, 9)
(3, 9)(4, 6)(5, 7)(8, 10)
(1, 2)(3, 7)(4, 6)(5, 9)

*/

G<x, y, t>:=Group<x, y, t | a^2, b^3, (a * b^-1)^5>;
G1:=N; G:=NN;
Sch:=SchreierSystem(G, sub<G| Id(G)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
P:=[Id(G1): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
```

```
if Eltseq(Sch[i])[j] eq −2 then P[j]:=yy^−1; end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
N1:=Stabiliser(N,1);
N1;

/*
Do the DCE of G over N
These are the t's
*/
for i in [1..#N] do for j in [2..10] do if 1^ArrayP[i] eq j
    then j, Sch[i]; break; end if; end for; end for;
/*
2 x
3 y
6 y^−1
4 x ∗ y
2 x ∗ t
7 x ∗ y^−1
5 y ∗ x
3 y ∗ t
2 t ∗ x
3 t ∗ y
6 t ∗ y^−1
6 y^−1 ∗ x
6 y^−1 ∗ t
4 x ∗ y ∗ x
```

4 x * y * t

4 x * t * y

7 x * t * y^−1

9 x * y^−1 * x

7 x * y^−1 * t

8 y * x * y

5 y * x * t

9 y * x * y^−1

5 y * t * x

6 y * t * y

4 t * x * y

2 t * x * t

7 t * x * y^−1

5 t * y * x

3 t * y * t

6 t * y^−1 * x

6 t * y^−1 * t

6 y^−1 * x * t

3 y^−1 * x * y^−1

6 y^−1 * t * x

3 y^−1 * t * y^−1

7 (x * y)^2

4 x * y * x * t

2 x * y * x * y^−1

4 x * y * t * x

7 x * y * t * y

2 x * y * t * y^−1

3 x * t * x * y

6 x * t * x * y^−1

4 x * t * y * x

4 x * t * y * t

```
9 x * t * y^-1 * x
7 x * t * y^-1 * t
5 x * y^-1 * x * y
9 x * y^-1 * x * t
8 (x * y^-1)^2
9 x * y^-1 * t * x
2 x * y^-1 * t * y
4 x * y^-1 * t * y^-1
*/
N;
/*
Permutation group N acting on a set of cardinality 10
Order = 60 = 2^2 * 3 * 5
    (1, 2)(3, 5)(7, 9)(8, 10)
    (1, 3, 6)(2, 4, 7)(5, 8, 9)
*/
S:=Sym(10);
xx:=S!(1, 2)(3, 5)(7, 9)(8, 10);
yy:=S!(1, 3, 6)(2, 4, 7)(5, 8, 9);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^2,y^3,(x * y^-1)^5,
t^2, (t,y * x * y^-1 * x * y * x), (t,x^y),
(x * y * x * y^-1 * x * y * x * y^-1 * x * y*t)^3>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1;
IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 10] ];
ts[1]:=f(t); ts[2]:=f(t^(x)); ts[3]:=f(t^(y)); ts[4]:=f(t^(
    x*y)); ts[5]:=f(t^(y*x));
ts[6]:=f(t^(y^2));ts[7]:=f(t^(x*y^-1));ts[8]:=f(t^(y*x*y));
```

```
    ts[9]:=f(t^(x * y^-1 * x));ts[10]:=f(t^(x*(x*y)^3));
#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
/* 2 */
DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
/* { <GrpFP, Id(G), GrpFP>, <GrpFP, t, GrpFP> } */
DC:=[ f( Id(G)), f( t)];
Index(G1,IN);
/* 11 */
cst := [null : i in [1 .. Index(G1,IN)]] where null is [
    Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 10 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..11] do if cst[i] ne [] then m:=m+1; end
    if; end for;m;
/* 10 */


/*




ts := [ Id(G1): i in [1 .. 10] ];
> ts[1]:=f(t); ts[2]:=f(t^(x)); ts[3]:=f(t^(y)); ts[4]:=f(t
```

```
            ^(x*y));  ts[5]:<  t^(x*y));  ts[5]:=f(t
                                                    > ts
    [1]:=f(t);  ts[2]:=f(t^(x));  ts[3]:=f(t^(y));  ts[4]:=f(t^(
    x*y));  ts[5]:=f(t\
^(y*x));
> ts[6]:=f(t^(y^2));ts[7]:=f(t^(x*y^-1));ts[8]:=f(t^(y*x*y)
    );ts[9]:=f(t^(x< ));ts[9]:=f(t^(x * y
                                                    > ts
    [6]:=f(t^(y^2));ts[7]:=f(t^(x*y^-1));ts[8]:=f(t^(y*x*y));
    ts[9]:=f(t^(x * y\
^-1 * x));ts[10]:=f(t^(x*(x*y)^3));
> cst := [null : i in [1 .. Index(G1,IN)]] where null is [
    Integers() | ];
> prodim := function(pt, Q, I)
function> v := pt;
function> for i in I do
function|for> v := v^(Q[i]);
function|for> end for;
function> return v;
function> end function;
> for i := 1 to 10 do
for> cst[prodim(1, ts, [i])] := [i];
for> end for;
> m:=0; for i in [1..11] do if cst[i] ne [] then m:=m+1;
    end if; end for;m> m:=0; for i in [1..11] do if cst[i] ne
    [] then m:=m+1; end if; end for;m;
10
> xx;
(1, 2)(3, 5)(7, 9)(8, 10)
> yy;
(1, 3, 6)(2, 4, 7)(5, 8, 9)
```

```
>
> for i in [1..10] do ts[3]*ts[1] eq n*ts[i] end for;

>> for i in [1..10] do ts[3]*ts[1] eq n*ts[i] end for;
                                                    ^
User error: bad syntax
> for i in [1..10] do ts[3]*ts[1] eq n*ts[i]; end for;

>> for i in [1..10] do ts[3]*ts[1] eq n*ts[i]; end for;
                                              ^
User error: Identifier 'n' has not been declared or
    assigned
> for i in [1..10] do ts[3]*ts[1] eq n*ts[i] end for;

>> for i in [1..10] do ts[3]*ts[1] eq n*ts[i] end for;
                                                  ^
User error: bad syntax
> for n in IN for i in [1..10] do if ts[3]*ts[1] eq n*ts[i]
    end if;

>> for n in IN for i in [1..10] do if ts[3]*ts[1] eq n*ts[i
   ] end if;
                    ^
User error: bad syntax
> for n in IN do for i in [1..10] do if ts[3]*ts[1] eq n*ts
   [i] end if;

>> for n in IN do for i in [1..10] do if ts[3]*ts[1] eq n*
   ts[i] end if;
                                                          ^
```

```
User error: bad syntax
> for n in IN do for i in [1..10] do if ts[3]*ts[1] eq n*ts
    [i] then i;
for|for|if> end if; end for; end for;
8
> for n in IN do for i in [1..10] do if ts[2]*ts[1] eq n*ts
    [i] then i;
for|for|if> end if; end for; end for;
2
> for n in IN do for i in [1..10] do if ts[4]*ts[1] eq n*ts
    [i] then i;
for|for|if> end if; end for; end for;
6
> for n in IN do for i in [1..10] do if ts[5]*ts[1] eq n*ts
    [i] then i;
for|for|if> end if; end for; end for;
5
> for n in IN do for i in [1..10] do if ts[6]*ts[1] eq n*ts
    [i] then i;
for|for|if> end if; end for; end for;
4
> for n in IN do for i in [1..10] do if ts[7]*ts[1] eq n*ts
    [i] then i;
for|for|if> end if; end for; end for;
7
> for n in IN do for i in [1..10] do if ts[8]*ts[1] eq n*ts
    [i] then i;
for|for|if> end if; end for; end for;
3
> for n in IN do for i in [1..10] do if ts[9]*ts[1] eq n*ts
    [i] then i;
```

```
for|for|if> end if; end for; end for;
10
> for n in IN do for i in [1..10] do if ts[10]*ts[1] eq n*
    ts[i] then i;
for|for|if> end if; end for; end for;
9
> S:Sym(11);

>> S:Sym(11);
          ^
User error: bad syntax
>
> fx:=S!(2,3)(4,6)(8,10)(9,11);

>> fx:=S!(2,3)(4,6)(8,10)(9,11);
                         ^
Runtime error in elt< ... >: Bad arguments to cycle or
    commutator construction

> fy:=S!(2,4,7)(3,5,8)(6,9,10);
> ft:=S!(1,2)(4,9)(5,7)(10,11);

>> ft:=S!(1,2)(4,9)(5,7)(10,11);
                        ^
Runtime error in elt< ... >: Bad arguments to cycle or
    commutator construction

> S:=Sym(11);
>
> fx:=S!(2,3)(4,6)(8,10)(9,11);
> fy:=S!(2,4,7)(3,5,8)(6,9,10);
```

```
>  ft:=S!(1,2)(4,9)(5,7)(10,11);
> #sub<S|xx,yy>:
> #sub<S|xx,yy>;

>> #sub<S|xx,yy>;
     ^

User error: bad syntax
> #sub<S|xx,yy>;

>> #sub<S|xx,yy>;
        ^

Runtime error in sub< ... >: Can not build a generator from
     the arguments given

> #sub<S|xx,yy>;

>> #sub<S|xx,yy>;
       ^

Runtime error in sub< ... >: Can not build a generator from
     the arguments given

> #sub<S|fx,fy>;
60
> GG1:=sub<S|fx,fy,ft>;
> #GG1;
660
> s,t:=IsIsomorphic(GG1,PSL(2,11));
> t(fx);
(1, 3)(2, 8)(4, 9)(5, 6)(7, 12)(10, 11)
> S:=Sym(12);
> F7:=GF(7);
```

```
> {a^2a in F7};

>> {a^2a in F7};
          ^
User error: bad syntax
> {a^2: in F7};

>> {a^2: in F7};
            ^
User error: bad syntax
> {a^2:a in F7};
{ 0, 1, 2, 4 }
> /* All non zero squares are 1,2,4 and powers of 2 give
   1,2,4 */
> /* x->2x */
> S:=Sym(11);
> S:=Sym(12);
> a:=S!(11,1,2,3,4,5,6,7,8,9,10);
> b:=S!(1,2,4,8,5,10,9,7);
> b:=S!(1,2,4,8,5,10,9,7,3,6);
> g:=S!(11,12)(1,10)(2,5)(3,7)(4,8)(6,9);
> #sub<S|a,b,g>;
1320
> b;
(1, 2, 4, 8, 5, 10, 9, 7, 3, 6)
> F11:=GF(11);
> {a^2:a in F11};
{ 0, 1, 3, 4, 5, 9 }
> /* x -> 3x */
> b:=S!(1,3,9,5,4)(2,6,7,10,8);
> #sub<S|a,b,g>;
```

660

```
> psl211:=sub<S|a,b,g>;
> IsIsomorphic(psl211,PSL(2,11));
```

true Homomorphism of GrpPerm: psl211, Degree 12, Order 2^2
    * 3 * 5 * 11 into

GrpPerm: $, Degree 12, Order 2^2 * 3 * 5 * 11 induced by
    (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11) |--> (2, 9, 10, 7,
        4, 5, 12, 8, 11, 3,
        6)
    (1, 3, 9, 5, 4)(2, 6, 7, 10, 8) |--> (2, 6, 8, 3, 4)(5,
        7, 11, 9, 10)
    (1, 10)(2, 5)(3, 7)(4, 8)(6, 9)(11, 12) |--> (1, 12)(2,
        11)(3, 10)(4, 9)(5,
        8)(6, 7)

```
> IsIsomorphic(GG1,psl211);
```

true Homomorphism of GrpPerm: GG1, Degree 11, Order 2^2 * 3
    * 5 * 11 into

GrpPerm: psl211, Degree 12, Order 2^2 * 3 * 5 * 11 induced
    by
    (2, 3)(4, 6)(8, 10)(9, 11) |--> (1, 12)(2, 3)(4, 9)(5,
        7)(6, 8)(10, 11)
    (2, 4, 7)(3, 5, 8)(6, 9, 10) |--> (1, 6, 12)(2, 10, 3)
        (4, 8, 5)(7, 9, 11)
    (1, 2)(4, 9)(5, 7)(10, 11) |--> (1, 7)(2, 9)(3, 6)(4,
        10)(5, 8)(11, 12)
```

/*
```

Monomial Progenitors and Finding Relations Codes

6.3 Monomial Progenitors Examples 3.4, 3.5 and First Order Relation

```
T:=TransitiveGroups (15);
G:=T[15]; CG:=CharacterTable (G); S:=Subgroups (G); H:=S[19]'
    subgroup;
CH:=CharacterTable (H); Induction (CH[3],G) eq CG[14];



T:=TransitiveGroups (15);
G:=T[15]; CG:=CharacterTable (G); S:=Subgroups (G); H:=S[19]'
    subgroup;
CH:=CharacterTable (H); Induction (CH[2],G) eq CG[15];

*/


/* Problem #1 */

T:=TransitiveGroups (15);
G:=T[15];
T:=TransitiveGroups (15);
G:=T[15]; CG:=CharacterTable (G); S:=Subgroups (G); H:=S[19]'
    subgroup; CH:=CharacterTable (H);
for g,h in G do if sub<G|g,h> eq G then A:=g; B:=h; end if;
    end for;
*/
/* interrupt it */
```

```
A,B; /* These values change and give us our x and y */
S:=Sym(15);
xx:=S!(1, 10, 9, 13, 3, 6, 15, 2, 14, 7, 8, 5, 4, 12, 11);
yy:=S!(1, 5, 3, 2, 12, 6, 10, 7, 4, 13, 8, 15, 11, 9, 14);
G:=sub<S|xx,yy>;
CompositionFactors(G);
/*
    G
    |   Alternating(5)
    *
    |   Cyclic(3)
    1
*/
G eq T[15];
/* true */
#G;
/* 180 */
/*
Permutation group H acting on a set of cardinality 15
Order = 36 = 2^2 * 3^2
    (1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12, 11)
    (1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14)
    (1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8, 13)
    (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)
*/
H:=sub<G| (1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12,
    11),
    (1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14),
    (1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8, 13),
    (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)>;
```

```
#H;
/* 36 */
CH:=CharacterTable(H);
CG:=CharacterTable(G);
for l in [1..#CH] do if
Induction(CH[l],G) eq CG[14] then l; end if; end for;
/*
5
6
*/
Induction(CH[5],G) eq CG[14];
/* true */
Index(G,H);
/* 5 */
#H;
/* 36 */
36*5;
Factorization(180);
/* [ <2, 2>, <3, 2>, <5, 1> ] 2^2*3^2*5^1 */
H;
/*
Permutation group H acting on a set of cardinality 15
Order = 36 = 2^2 * 3^2
    (1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12, 11)
   (1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14)
   (1, 14)(2, 7)(3, 9)(4, 11)(6, 12)(8, 13)
   (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)
*/
H eq sub<G|(1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12,
```

```
11) ,(1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14) ,(1, 14)(2,
    7)(3, 9)(4, 11)(6, 12)(8, 13) ,(1, 8, 6)(2, 9, 4)(3, 11,
    7)(5, 15, 10)(12, 14, 13)>;
```

```
H:=sub<G|(1, 13, 3)(2, 9, 4)(5, 10, 15)(6, 14, 7)(8, 12,
    11) ,(1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14) ,(1, 14)(2,
    7)(3, 9)(4, 11)(6, 12)(8, 13) ,(1, 8, 6)(2, 9, 4)(3, 11,
    7)(5, 15, 10)(12, 14, 13)>;
CH:=CharacterTable(H);
/* How many linear does it have?
How many linear chars does it have?
*/
for i in [1..#CH] do if CH[i](Id(H)) eq 1 then i; end if;
    end for;
/*
1
2
3
4
5
6
7
8
9
*/
CH;
/*
Character Table of Group H
_____
```

Class	1	2	3	4	5	6	7	8	9	10	11	12
Size	1	3	1	1	4	4	4	4	4	4	3	3
Order	1	2	3	3	3	3	3	3	3	3	6	6
p = 4	2	1	1	4	3	6	5	8	7	10	9	3
p = 2	3	1	2	1	1	1	1	1	1	1	2	2
X.1 +	1	1	1	1	1	1	1	1	1	1	1	1
X.2 0	1	1	−1−J	J	−1−J	J	J	−1−J	1	1	J	−1−J
X.3 0	1	1	−1−J	J	J	−1−J	1	1	−1−J	J	J	−1−J
X.4 0	1	1	1	1	−1−J	J	−1−J	J	−1−J	J	1	1
X.5 0	1	1	J	−1−J	J	−1−J	−1−J	J	1	1	−1−J	J
X.6 0	1	1	J	−1−J	−1−J	J	1	1	J	−1−J	−1−J	J
X.7 0	1	1	1	1	J	−1−J	J	−1−J	J	−1−J	1	1
X.8 0	1	1	J	−1−J	1	1	J	−1−J	−1−J	J	−1−J	J

J

X.9	0	1	1	−1–J	J	1	1–1–J	J	J–1–J	J–1–J	
X.10	+	3	−1	3	3	0	0	0	0	0	0	−1 −1
X.11	0	3	−1	3*J	−3–3*J	0	0	0	0	0	0	1+J −J
X.12	0	3	−1	−3–3*J	3*J	0	0	0	0	0	0	−J 1+J

Explanation of Character Value Symbols
_____

```
J = RootOfUnity(3)
*/
T:=Transversal(G,H);
#T;
/* 5 */
for i in[1..#T] do i,T[i]; end for;
/*
1 Id(G)
2 (1, 10, 9, 13, 3, 6, 15, 2, 14, 7, 8, 5, 4, 12, 11)
3 (1, 5, 3, 2, 12, 6, 10, 7, 4, 13, 8, 15, 11, 9, 14)
4 (1, 9, 3, 15, 14, 8, 4, 11, 10, 13, 6, 2, 7, 5, 12)
5 (1, 11, 12, 4, 5, 8, 7, 14, 2, 15, 6, 3, 13, 9, 10)
*/
T[1]*xx*T[1]^-1 in H;
/* False */
T[1]*xx*T[2]^-1 in H;
/* True */
```

```
 CH[5];
/*
( 1, 1, zeta(3)_3, −zeta(3)_3 − 1, zeta(3)_3, −zeta(3)_3 −
    1, −zeta(3)_3 − 1, zeta(3)_3, 1, 1, −zeta(3)_3 − 1, zeta
    (3)_3 )
*/
T[11]*xx*T[11]^−1 in H;
/* true */
T[11]*xx*T[11]^−1;

C:=Classes(G);
C;
D:=Classes(H);
D;
Set(H) meet Class(G,C[2][3]);

C:=CyclotomicField(3);
A:=[[C.1,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do A[i,j]:=0; end for;
/* A consists of 0's */
for i,j in [1..5] do if T[i]*xx*T[j]^−1 in H then
A[i,j]:=CH[5](T[i]*xx*T[j]^−1);end if; end for;
GG:=GL(5,C);
GG!A;
/*
[            0            1            0            0
    0]
[            0            0            0            1
    0]
[            0            0            0            0 −zeta_3 −
    1]
```

```
[              0             0             1             0
   0]
[              1             0             0             0
   0]
*/
Order(xx);
/* 15 */
Order(GG!A);
/*15 */

B:=[[C.1,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do B[i,j]:=0; end for;
/* A consists of 0's */
for i,j in [1..5] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[5](T[i]*yy*T[j]^-1);end if; end for;
GG:=GL(5,C);
GG!B;
/*
[              0             0             1             0
   0]
[              0             0             0       zeta_3
   0]
[              0       zeta_3             0             0
   0]
[              0             0             0             0
   zeta_3]
[-zeta_3 - 1             0             0             0
   0]
*/
Order(yy);
/*15*/
```

```
Order (GG!B);
/* 15 */
IsIsomorphic (G, sub<GG|GG!A,GG!B>);
/* true */
/* Note that zeta_3 = 9 Ask Hasan how this happens one more
    time*/
CH[5];
/*Replace zeta_3 with 9


( 1, 1, zeta(3)_3, -zeta(3)_3 - 1, zeta(3)_3, -zeta(3)_3 -
    1, -zeta(3)_3 - 1,
    zeta(3)_3, 1, 1, -zeta(3)_3 - 1, zeta(3)_3 )
 */
PrimitiveRoot(7);
/* 3 */
CH5seq:= [1, 1, (9), -(9) - 1, (9), -(9) - 1, -(9) - 1,
    (9), 1, 1, -(9) - 1, (9) ];

T:=Transversal(G,H);
A:=[[0,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do if T[i]*xx*T[j]^-1 in H then for k in
    [1..#CH[5]] do if CH[5] (T[i]*xx*T[j]^-1) eq CH[5][k]
    then
A[i,j]:=CH5seq[k] mod 7 ;end if; end for; end if; end for;
GG:=GL(5,7);
GG!A;
/*
[0 1 0 0 0]
[0 0 0 1 0]
[0 0 0 0 4]
[0 0 1 0 0]
```

```
[1 0 0 0 0]
*/
T:=Transversal(G,H);
B:=[[0,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do if T[i]*yy*T[j]^-1 in H then for k in
    [1..#CH[5]] do if CH[5] (T[i]*yy*T[j]^-1) eq CH[5][k]
    then
B[i,j]:=CH5seq[k] mod 7 ;end if; end for; end if; end for;
GG:=GL(5,7);
GG!B;
/*
[0 0 1 0 0]
[0 0 0 2 0]
[0 2 0 0 0]
[0 0 0 0 2]
[4 0 0 0 0]
*/

S:=Sym(30);
xx:=S!(1,2,4,3,28,8,13,23,18,26,6,11,21,16,5)
    (7,12,22,17,29,9,14,24,19,30,10,15,25,20,27);
yy:=S!(1,3,11,23,5,8,18,2,21,28,6,16,13,4,26)
    (7,17,15,24,27,9,19,12,25,29,10,20,14,22,30);
N:=sub<S|xx,yy>;
#N;
/*180*/
IsIsomorphic(sub<GG|GG!A,GG!B>,N);
/* true */
FPGroup(N);
/*
Finitely presented group on 2 generators
```

```
Relations
    ($.2^-1 * $.1)^3 = Id($)   --> (y^-1 *x)^3
    $.2^-1 * $.1^-3 * $.2^-1 * $.1^2 = Id($) --> y^-1*x^-3*
        y^-1*x^2
    $.2^-3 * $.1^-1 * $.2^2 * $.1^-1 = Id($) --> y^-3*x^-1*
        y2*x^-1


G<x,y,t>:=Group<x,y,t| (y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y
    ^-3*x^-1*y2*x^-1,
t^7,
*/
Nt:=Stabiliser(N,{1,6,7,8,9,10});
 /* These perms are in the stabiliser */
for n in N do if {1,6,7,8,9,10}^n eq {1,6,7,8,9,10}
 then n; end if; end for;
Set(Nt);
w:=WordGroup(N);
rho:=InverseWordMap(N);
A:=N!(2,  23,  3)(4,  16,  11)(5,  28,  26)(12,  24,  17)(13,  21,
    18)(14,  25,  19)(15,  22,20)(27,  29,  30);
B:=N!(2,  28)(3,  21)(4,  18)(5,  11)(12,  29)(13,  26)(14,  30)
    (15,  27)(16,  23)(17,
        25)(19,  22)(20,  24);
C:=N!(1,  8,  6)(2,  18,  4)(3,  21,  11)(7,  9,  10)(12,  19,  22)
    (13,  16,  23)(14,  20,
        24)(15,  17,  25);
A@rho;
AA:=function(w)
    w6 := w.1^2; w7 := w.2^2; w8 := w6 * w7; return w8;
end function;
AA(FPGroup(N));
```

```
B@rho;
BB:= function(w)
    w6 := w.1^2; w7 := w.2^2; w8 := w6 * w7; w3 := w.1^3;
        w2 := w.2^-1; w4 := w3
    * w2; w10 := w8 * w4; return w10;
end function;
BB(FPGroup(N));


C@rho;
CC:=function(w)
    w3 := w.1^3; w2 := w.2^-1; w4 := w3 * w2; w6 := w.1^2;
        w7 := w.2^2; w8 := w6
    * w7; w11 := w4 * w8; w12 := w11 * w8; w1 := w.1^-1;
        w13 := w12 * w1; w14 :=
    w13 * w.2; return w14;
end function;
CC(FPGroup(N));


/* Work with this one */
G<x,y,t>:=Group<x,y,t| (y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y
    ^-3*x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*
    x^2*y^2*x^-1*y) = t^4>;
Orbits(Stabiliser(N,1));
/*
[
    GSet{@ 1 @},
    GSet{@ 6 @},
    GSet{@ 7 @},
    GSet{@ 8 @},
```

```
    GSet{@ 9 @},
    GSet{@ 10 @},
    GSet{@ 2, 21, 23, 26, 18, 28, 3, 5, 13, 11, 16, 4 @},
    GSet{@ 12, 25, 24, 30, 19, 29, 17, 27, 14, 15, 20, 22 @
        }
  }
]

*/


/* Do not use this one it was only to test it (t^x), Has an
    extra relation at the end */
G<x,y,t>:=Group<x,y,t | (y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y
    ^-3*x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*
    x^2*y^2*x^-1*y) = t^4, (t,t^x)>;
#G;
/* Should get 7^5*180 */
Index(G,sub<G|x,y>);/* should be 7^5 */
/* 16807 */


/* FOR */



S:=Sym(30);
xx:=S!(1,2,4,3,28,8,13,23,18,26,6,11,21,16,5)
    (7,12,22,17,29,9,14,24,19,30,10,15,25,20,27);
yy:=S!(1,3,11,23,5,8,18,2,21,28,6,16,13,4,26)
    (7,17,15,24,27,9,19,12,25,29,10,20,14,22,30);
N:=sub<S|xx,yy>;
#N;
/*180*/
```

```
C:=Classes(N);
#C;
/*15*/
FPGroup(N);
/*
Finitely presented group on 2 generators
Relations
    ($.2^-1 * $.1)^3 = Id($)  —> (y^-1 *x)^3
    $.2^-1 * $.1^-3 * $.2^-1 * $.1^2 = Id($) —> y^-1*x^-3*
        y^-1*x^2
    $.2^-3 * $.1^-1 * $.2^2 * $.1^-1 = Id($) —> y^-3*x^-1*
        y^2*x^-1
*/
NN<x,y>:=Group<x,y| (y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y^-3*x
    ^-1*y^2*x^-1>;
/* 180 */
Sch:=SchreierSystem(NN,sub<NN| Id(NN)>);
 ArrayP:=[Id(N): i in [1.. #N]];
 for i in [2.. #N] do P:=[Id(N): l in [1.. #Sch[i]]];
 for j in [1..# Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
 PP:=Id(N);
 for k in [1.. #P] do
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
 end for;
for i in [1..#N] do if 1^ArrayP[i] eq 2 then Sch[i]; break;
```

```
    end if; end for;
/* x */
 for i in [1..#N] do if 1^ArrayP[i] eq 3 then Sch[i]; break
    ;end if; end for;
/* y */
 for i in [1..#N] do if 1^ArrayP[i] eq 4 then Sch[i]; break
    ;end if; end for;
/* x^2 */
for i in [1..#N] do if 1^ArrayP[i] eq 5 then Sch[i]; break;
    end if; end for;
/* x^−1 */
 for i in [1..#N] do if 1^ArrayP[i] eq 6 then Sch[i]; break
    ;end if; end for;
/* y^−1 * x*/
for i in [1..#N] do if 12^ArrayP[i] eq 7 then Sch[i]; break
    ;end if; end for;
/* x^−1 */
Orbits(N);
/*
[
    GSet{@ 1, 2, 3, 4, 21, 28, 11, 26, 16, 8, 6, 23, 5, 13,
        18 @},
    GSet{@ 7, 12, 17, 22, 25, 29, 15, 30, 20, 9, 10, 24,
        27, 14, 19 @}
*/
for i in [2..#C] do i, C[i][3]; for j in [1..#N] do
 if ArrayP[j] eq C[i][3]
 then Sch[j]; end if; end for;
 Orbits(Centraliser(N,C[i][3])); end for;
/*
2 (1, 11)(2, 8)(3, 28)(5, 16)(6, 13)(7, 15)(9, 12)(10, 14)
```

(17, 29)(18, 26)(19, 30)(20, 27)

x * y * x * y^−1 * x

[

    GSet{@ 4, 21, 23 @},

    GSet{@ 22, 25, 24 @},

    GSet{@ 1, 11, 26, 6, 18, 13, 28, 8, 3, 2, 5, 16 @},

    GSet{@ 7, 15, 30, 10, 19, 14, 29, 9, 17, 12, 27, 20 @}

((x * y * x * y^−1 * x)*t^(x^2)), ((x * y * x * y^−1 * x)*(

   t^x^2)^3),((x * y * x * y^−1 * x)*t),((x * y * x * y^−1 *

   x)*t^3),

]

3 (1, 8, 6)(2, 13, 11)(3, 18, 16)(4, 23, 21)(5, 28, 26)(7,

   9, 10)(12, 14, 15)(17, 19,

    20)(22, 24, 25)(27, 29, 30)

x^5

[

    GSet{@ 1, 2, 3, 4, 21, 28, 11, 26, 16, 8, 6, 23, 5, 13,

       18 @},

    GSet{@ 7, 12, 17, 22, 25, 29, 15, 30, 20, 9, 10, 24,

       27, 14, 19 @}

((x^5)*t),((x^5)*t^3),

]

4 (1, 6, 8)(2, 11, 13)(3, 16, 18)(4, 21, 23)(5, 26, 28)(7,

   10, 9)(12, 15, 14)(17, 20,

    19)(22, 25, 24)(27, 30, 29)

x^−5

[

    GSet{@ 1, 2, 3, 4, 21, 28, 11, 26, 16, 8, 6, 23, 5, 13,

       18 @},

```
        GSet{@ 7, 12, 17, 22, 25, 29, 15, 30, 20, 9, 10, 24,
            27, 14, 19 @}
((x^−5)∗t) ,((x^−5)∗t^3) ,


]
5 (1, 18, 4)(2, 13, 11)(3, 21, 6)(5, 26, 28)(7, 19, 22)(8,
    16, 23)(9, 20, 24)(10, 17,
        25)(12, 14, 15)(27, 30, 29)
x ∗ y^−1
[
        GSet{@ 2, 13, 11 @},
        GSet{@ 5, 26, 28 @},
        GSet{@ 12, 14, 15 @},
        GSet{@ 27, 30, 29 @},
        GSet{@ 1, 18, 23, 4, 8, 3, 16, 21, 6 @},
        GSet{@ 7, 19, 24, 22, 9, 17, 20, 25, 10 @}
((x ∗ y^−1)∗t^x) ,((x ∗ y^−1)∗t^(x^−1)) ,((x ∗ y^−1)∗(t^x)^3)
    ,((x ∗ y^−1)∗(t^x^−1)^3) ,((x ∗ y^−1)∗t) ,((x ∗ y^−1)∗t^3) ,


]
6 (2, 21, 26)(3, 16, 18)(4, 5, 13)(11, 23, 28)(12, 25, 30)
    (14, 22, 27)(15, 24, 29)(17, 20,
        19)
x^3 ∗ y^−1
[
        GSet{@ 1, 8, 6 @},
        GSet{@ 3, 16, 18 @},
        GSet{@ 7, 9, 10 @},
        GSet{@ 17, 20, 19 @},
        GSet{@ 2, 21, 4, 26, 5, 28, 13, 11, 23 @},
```

GSet{@ 12, 25, 22, 30, 27, 29, 14, 15, 24 @}

((x^3 * y^−1)*t), ((x^3 * y^−1)*t^y), ((x^3 * y^−1)*t^3), ((x^3 * y^−1)*(t^y)^3), ((x^3 * y^−1)*t^x), ((x^3 * y^−1)*(t^x)^3),

]

7 (2, 26, 21)(3, 18, 16)(4, 13, 5)(11, 28, 23)(12, 30, 25)(14, 27, 22)(15, 29, 24)(17, 19, 20)

y * x^−3

[

   GSet{@ 1, 6, 8 @},
   GSet{@ 3, 18, 16 @},
   GSet{@ 7, 10, 9 @},
   GSet{@ 17, 19, 20 @},
   GSet{@ 2, 26, 23, 21, 11, 5, 28, 4, 13 @},
   GSet{@ 12, 30, 24, 25, 15, 27, 29, 22, 14 @}

((y * x^−3)*t), ((y * x^−3)*t^y), ((y * x^−3)*t^3), ((y * x^−3)*(t^y)^3), ((y * x^−3)*t^x), ((y * x^−3)*(t^x)^3),

]

8 (1, 3, 13, 26, 21)(2, 28, 23, 6, 16)(4, 8, 18, 11, 5)(7, 17, 14, 30, 25)(9, 19, 15, 27, 22)(10, 20, 12, 29, 24)

x^3

[

   GSet{@ 1, 3, 28, 13, 23, 18, 26, 6, 11, 21, 16, 5, 2, 4, 8 @},
   GSet{@ 7, 17, 29, 14, 24, 19, 30, 10, 15, 25, 20, 27, 12, 22, 9 @}

((x^3)*t), ((x^3)*t^3),

]

9 (1, 13, 21, 3, 26)(2, 23, 16, 28, 6)(4, 18, 5, 8, 11)(7,
    14, 25, 17, 30)(9, 15, 22, 19,
     27)(10, 12, 24, 20, 29)

y * x * y^-1 * x * y

[

    GSet{@ 1, 13, 28, 21, 6, 18, 3, 2, 5, 26, 23, 8, 16,
        11, 4 @},
    GSet{@ 7, 14, 29, 25, 10, 19, 17, 12, 27, 30, 24, 9,
        20, 15, 22 @}

((y * x * y^-1 * x * y)*t),((y * x * y^-1 * x * y)*t^3),

]

10 (1, 13, 8, 11, 6, 2)(3, 5, 18, 28, 16, 26)(4, 21, 23)(7,
    14, 9, 15, 10, 12)(17, 27, 19,
     29, 20, 30)(22, 25, 24)

x^2 * y^-1

[

    GSet{@ 4, 21, 23 @},
    GSet{@ 22, 25, 24 @},
    GSet{@ 1, 11, 26, 13, 18, 6, 3, 8, 28, 2, 5, 16 @},
    GSet{@ 7, 15, 30, 14, 19, 10, 17, 9, 29, 12, 27, 20 @}

((x^2 * y^-1)*t^(x^2)),((x^2 * y^-1)*(t^x^2)^3),((x^2 * y
    ^-1)*t),((x^2 * y^-1)*t^3),

]

11 (1, 2, 6, 11, 8, 13)(3, 26, 16, 28, 18, 5)(4, 23, 21)(7,
    12, 10, 15, 9, 14)(17, 30, 20,
     29, 19, 27)(22, 24, 25)

y * x^-2

[
    GSet{@ 4, 23, 21 @},
    GSet{@ 22, 24, 25 @},
    GSet{@ 1, 11, 26, 2, 18, 8, 16, 6, 5, 13, 28, 3 @},
    GSet{@ 7, 15, 30, 12, 19, 9, 20, 10, 27, 14, 29, 17 @}
((y * x^−2)*t^(x^2)),((y * x^−2)*(t^x^2)^3),((y * x^−2)*t)
    ,((y * x^−2)*t^3),

]
12 (1, 2, 4, 3, 28, 8, 13, 23, 18, 26, 6, 11, 21, 16, 5)(7,
    12, 22, 17, 29, 9, 14, 24, 19,
    30, 10, 15, 25, 20, 27)
x
[
    GSet{@ 1, 2, 4, 3, 28, 8, 13, 23, 18, 26, 6, 11, 21,
      16, 5 @},
    GSet{@ 7, 12, 22, 17, 29, 9, 14, 24, 19, 30, 10, 15,
      25, 20, 27 @}
((x)*t),((x)*t^3),

]
13 (1, 4, 28, 13, 18, 6, 21, 5, 2, 3, 8, 23, 26, 11, 16)(7,
    22, 29, 14, 19, 10, 25, 27, 12,
    17, 9, 24, 30, 15, 20)
x^2
[
    GSet{@ 1, 4, 28, 13, 18, 6, 21, 5, 2, 3, 8, 23, 26, 11,
      16 @},
    GSet{@ 7, 22, 29, 14, 19, 10, 25, 27, 12, 17, 9, 24,
      30, 15, 20 @}
((x^2)*t),((x^2)*t^3),

```
]
14 (1, 23, 5, 13, 16, 8, 21, 28, 11, 3, 6, 4, 26, 2, 18)(7,
    24, 27, 14, 20, 9, 25, 29, 15,
    17, 10, 22, 30, 12, 19)
y^−1 * x * y * x * y^−1
[
    GSet{@ 1, 23, 5, 13, 16, 8, 21, 28, 11, 3, 6, 4, 26, 2,
        18 @},
    GSet{@ 7, 24, 27, 14, 20, 9, 25, 29, 15, 17, 10, 22,
        30, 12, 19 @}
((y^−1 * x * y * x * y^−1)*t),((y^−1 * x * y * x * y^−1)*t
    ^3),

]
15 (1, 11, 23, 3, 5, 6, 13, 4, 16, 26, 8, 2, 21, 18, 28)(7,
    15, 24, 17, 27, 10, 14, 22, 20,
    30, 9, 12, 25, 19, 29)
x^−4
[
    GSet{@ 1, 11, 23, 3, 5, 6, 13, 4, 16, 26, 8, 2, 21, 18,
        28 @},
    GSet{@ 7, 15, 24, 17, 27, 10, 14, 22, 20, 30, 9, 12,
        25, 19, 29 @}
((x^−4)*t),((x^−4)*t^3),
]
*/

/*FOR: Started 4/23   (7star5_180.out)*/
for a,b,c,d,e,f,g,h in [0..10] do
G<x,y,t>:=Group<x,y,t|(y^−1 *x)^3, y^−1*x^−3*y^−1*x^2,y^−3*
```

```
        x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*
    x^2*y^2*x^-1*y) = t^4, ((x * y * x * y^-1 * x)*t^(x^2))^a
    , ((x * y * x * y^-1 * x)*(t^x^2)^3)^b,((x * y * x * y^-1
    * x)*t)^c,((x * y * x * y^-1 * x)*t^3)^d,((x^5)*t)^e,((x
    ^5)*t^3)^f,((x^-5)*t)^g,((x^-5)*t^3)^h>;
a,b,c,d,e,f,g,h, Index(G,sub<G|x,y>); end for;



/*FOR: Started 4/28 (7star5_180_1.out)*/
for a,b,c,d,e,f,g,h in [0..10] do
G<x,y,t>:=Group<x,y,t|(y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y^-3*
    x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*
    x^2*y^2*x^-1*y) = t^4, (((x * y^-1)*t^x))^a, (((x * y^-1)
    *t^(x^-1)))^b,(((x * y^-1)*(t^x)^3))^c,(((x * y^-1)*(t^x
    ^-1)^3))^d,(((x * y^-1)*t))^e,(((x * y^-1)*t^3))^f,(((x^3
    * y^-1)*t))^g,(((x^3 * y^-1)*t^y))^h>;
a,b,c,d,e,f,g,h, Index(G,sub<G|x,y>); end for;



/*FOR: Started 4/28 (7star5_180_2.out)*/
for a,b,c,d,e,f,g,h in [0..10] do
G<x,y,t>:=Group<x,y,t|(y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y^-3*
    x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*
    x^2*y^2*x^-1*y) = t^4, (((x^3 * y^-1)*t^3))^a, (((x^3 * y
    ^-1)*(t^y)^3))^b,(((x^3 * y^-1)*t^x))^c,(((x^3 * y^-1)*t^
    x))^d,(((x^3 * y^-1)*(t^x)^3))^e,(((x^3)*t),((x^3)*t^3))^
    f,(((y * x * y^-1 * x * y)*t))^g,(((y * x * y^-1 * x * y)
    *t^3))^h>;
```

```
a,b,c,d,e,f,g,h, Index(G,sub<G|x,y>); end for;




/*FOR: Started 4/28 (7star5_180_3.out)*/
for a,b,c,d,e,f,g,h in [0..10] do
G<x,y,t>:=Group<x,y,t|(y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y^-3*
    x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*
    x^2*y^2*x^-1*y) = t^4, (((x^2 * y^-1)*t^(x^2)))^a, (((x^2
    * y^-1)*(t^x^2)^3))^b,(((x^2 * y^-1)*t))^c,(((x^2 * y
    ^-1)*t^3))^d,(((y * x^-2)*t^(x^2)))^e,(((y * x^-2)*(t^x
    ^2)^3))^f,(((y * x^-2)*t))^g,(((y * x^-2)*t^3),((x)*t))^h
    >;
a,b,c,d,e,f,g,h, Index(G,sub<G|x,y>); end for;


/*FOR: Started 4/28 (7star5_180_4.out)*/
for a,b,c,d,e,f,g in [0..10] do
G<x,y,t>:=Group<x,y,t|(y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y^-3*
    x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*
    x^2*y^2*x^-1*y) = t^4, (((x)*t^3))^a, (((x^2)*t))^b,(((x
    ^2)*t^3))^c,(((y^-1 * x * y * x * y^-1)*t))^d,(((y^-1 * x
    * y * x * y^-1)*t^3))^e,(((x^-4)*t))^f,(((x^-4)*t^3))^g
    >;
a,b,c,d,e,f,g, Index(G,sub<G|x,y>); end for;



/* Saved for Later Split this into two parts.



*/
```

/∗ Running it in the background ∗/

1. nano 7star5:180

2. Insert data

3. Ctrl + x

4. Choose yes

5. Enter to save file

If we want to check the input file 7star5:180 type nano 7
  star5:180

nohup magma "7star5:180"&>7star5:180.out&

After the above command has been run if we want to check
  output(or whether the program is running). We use the
  command nano 7star5:180.out

/∗ Checking Status of Progenitor ∗/

nano 7star5_180.out

/∗ Problem #2 ∗/

T:=TransitiveGroups(15);

G:=T[15]; CG:=CharacterTable(G); S:=Subgroups(G); H:=S[19]'
  subgroup;

CH:=CharacterTable(H); Induction(CH[2],G) eq CG[15];

/∗ True ∗/

/∗

```
for g,h in G do if sub<G|g,h> eq G then A:=g; B:=h; end if;
    end for;
*/
/* interrupt it */
A,B; /* These values change */
/*
(1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13)
(1, 5)(2, 14)(4, 12)(6, 10)(8, 15)(9, 13) */
S:=Sym(15);
xx:=S!(1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13);
yy:=S!(1, 5)(2, 14)(4, 12)(6, 10)(8, 15)(9, 13);
G:=sub<S|xx,yy>;


/* Fixing value from H
Permutation group H acting on a set of cardinality 15
Order = 36 = 2^2 * 3^2
    (1, 6, 8)(2, 15, 13)(3, 11, 7)(4, 5, 14)(9, 10, 12)
    (2, 12)(3, 10)(4, 13)(5, 11)(7, 15)(9, 14)
    (2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15)
    (1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)

*/
H:=sub<G| (1, 6, 8)(2, 15, 13)(3, 11, 7)(4, 5, 14)(9, 10,
    12),
(2, 12)(3, 10)(4, 13)(5, 11)(7, 15)(9, 14),
(2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15),
(1, 8, 6)(2, 9, 4)(3, 11, 7)(5, 15, 10)(12, 14, 13)>;
CH:=CharacterTable(H);
CG:=CharacterTable(G);
for l in [1..#CH] do  if
Induction(CH[l],G) eq CG[15] then l; end if; end for;
```

```
/*
4
5
*/
Induction(CH[4],G) eq CG[15];
/* true */
Index(G,H);
/* 5 */
#H;
/* 36 */
5*36;
Factorization(180);
/* [ <2, 2>, <3, 2>, <5, 1> ] 2^2*3^2*5^1 */

CH:=CharacterTable(H);
/* How many linear does it have?
How many linear chars does it have?
*/
for i in [1..#CH] do if CH[i](Id(H)) eq 1 then i; end if;
   end for;
/*
1
2
3
4
5
6
7
8
9
*/
```

CH;
/*
Character Table of Group H

———————————————————

		1	2	3	4	5	6	7	8	9	10	11	12
Class	|	1	2	3	4	5	6	7	8	9	10	11	12
Size	|	1	3	1	1	4	4	4	4	4	4	3	3
Order	|	1	2	3	3	3	3	3	3	3	3	6	6
p = 2		1	1	4	3	6	5	8	7	10	9	3	4
p = 3		1	2	1	1	1	1	1	1	1	1	2	2
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1
X.2	0	1	1	J	−1−J	−1−J	J	−1−J	J	1	1	−1−J	J
X.3	0	1	1	1	1	J	−1−J	−1−J	J	J	−1−J	1	1
X.4	0	1	1	−1−J	J	−1−J	J	1	1	J	−1−J	J	−1−J
X.5	0	1	1	−1−J	J	J	−1−J	J	−1−J	1	1	J	−1−

J

X.6	0	1	1	1	1–1–J	J	J–1–J–1–J	J	1	1		
X.7	0	1	1	–1–J	J	1	1–1–J	J–1–J	J	J–1–J		
X.8	0	1	1	J	–1–J	1	1	J–1–J	J–1–J–1–J	J		
X.9	0	1	1	J	–1–J	J–1–J	1	1–1–J	J–1–J	J		
X.10	+	3	–1	3	3	0	0	0	0	0	0	–1	–1
X.11	0	3	–1	3∗J	–3–3∗J	0	0	0	0	0	0	1+J	–J
X.12	0	3	–1	–3–3∗J	3∗J	0	0	0	0	0	0	–J	1+J

Explanation of Character Value Symbols

——————————————————————————————

J = RootOfUnity(3)

```
*/
T:=Transversal(G,H);
#T;
/* 5 */
for i in[1..#T] do i,T[i]; end for;
/*
1 Id(G)
2 (1, 7, 12)(2, 9, 4)(3, 14, 8)(6, 11, 13)
3 (1, 5, 2, 13, 11, 8, 15, 9, 12, 7, 6, 10, 4, 14, 3)
4 (1, 12, 7)(2, 4, 9)(3, 8, 14)(6, 13, 11)
```

```
5 (1, 4, 13, 11, 10, 6, 9, 14, 3, 15, 8, 2, 12, 7, 5)
*/
 CH[5];
/*
( 1, 1, −zeta(3)_3 − 1, zeta(3)_3, zeta(3)_3, −zeta(3)_3 −
   1, zeta(3)_3, −zeta(3)_3− 1, 1, 1, zeta(3)_3, −zeta(3)_3
   − 1 )
*/

C:=CyclotomicField(3);
A:=[[C.1,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do A[i,j]:=0; end for;
/* A consists of 0's */
for i,j in [1..5] do if T[i]*xx*T[j]^−1 in H then
A[i,j]:=CH[5](T[i]*xx*T[j]^−1);end if; end for;
GG:=GL(5,C);
GG!A;
/*
[               0               1               0               0
   0]
[               0               0               0               1
   0]
[               0               0               1               0
   0]
[               1               0               0               0
   0]
[               0               0               0               0 −zeta_3 −
   1]
*/
Order(xx);
/* 3 */
```

```
Order(GG!A);
/* 3 */


B:=[[C.1,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do B[i,j]:=0; end for;
/* A consists of 0's */
for i,j in [1..5] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[5](T[i]*yy*T[j]^-1);end if; end for;
GG:=GL(5,C);
GG!B;
/* ASK about Matrix changing
[              0              0        zeta_3             0
    0]
[              0              1             0             0
    0]
[-zeta_3 - 1              0             0             0
    0]
[              0              0             0             0
    1]
[              0              0             0             1
    0]

*/
Order(yy);
/* 2 */
Order(GG!B);
/* 2 */
IsIsomorphic(G,sub<GG|GG!A,GG!B>);
/* true */
/* Note that zeta_3 = 9 Ask Hasan how this happens one more
    time*/
```

```
CH[5];
/*Replace zeta_3 with 9

( 1, 1, −zeta(3)_3 − 1, zeta(3)_3, zeta(3)_3, −zeta(3)_3 −
    1, zeta(3)_3,−zeta(3)_3 − 1, 1, 1, zeta(3)_3, −zeta(3)_3
    − 1 )
 */
PrimitiveRoot(7);
/* 3 */
CH5seq:= [ 1, 1, −9 − 1, 9, 9, −9 − 1, 9,−9 − 1, 1, 1, 9,
    −9 − 1   ];



A:=[[0,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do if T[i]*xx*T[j]^−1 in H then for k in
    [1..#CH[5]] do if CH[5] (T[i]*xx*T[j]^−1) eq CH[5][k]
    then
A[i,j]:=CH5seq[k] mod 7 ;end if; end for; end if; end for;
GG:=GL(5,7);
GG!A;
/*
[0 1 0 0 0]
[0 0 0 1 0]
[0 0 1 0 0]
[1 0 0 0 0]
[0 0 0 0 4]
*/



B:=[[0,0,0,0,0] : i in [1..5]];
for i,j in [1..5] do if T[i]*yy*T[j]^−1 in H then for k in
    [1..#CH[5]] do if CH[5] (T[i]*yy*T[j]^−1) eq CH[5][k]
```

```
    then
B[i,j]:=CH5seq[k] mod 7 ;end if; end for; end if; end for;
GG:=GL(5,7);
GG!B;
/*
[0  0  2  0  0]
[0  1  0  0  0]
[4  0  0  0  0]
[0  0  0  0  1]
[0  0  0  1  0]
*/
IsIsomorphic(G,sub<GG|GG!A,GG!B>);
/* true */
S:=Sym(30);
xx:=S!(1,2,4)(5,20,10)(6,7,9)(11,12,14)
(15,25,30)(16,17,
19)(21,22,24)(26,27,29);
yy:=S!(1,8)(3,16)(4,5)(6,18)(9,10)(11,28)(13,21)
(14,15)(19,20)(23,26)(24,25)(29,30);
N:=sub<S|xx,yy>;
#N;
/*180*/
IsIsomorphic(sub<GG|GG!A,GG!B>,N);
/* true */

FPGroup(N);
/*
 $.1^3 = Id($) --> x^3
    $.2^2 = Id($)
    ($.2 * $.1^-1 * $.2 * $.1 * $.2 * $.1^-1)^3 = Id($)
    $.1^-1 * $.2 * $.1^-1 * $.2 * $.1^-1 * $.2 * $.1^-1 * $
```

```
         .2 * $.1^−1 * $.2 * $.1 * $.2 * $.1 * $.2 * $.1 * $.2
           * $.1 * $.2 * $.1 * $.2 = Id($)


x^3 = Id($)
y^2 = Id($)
(y *x^−1 *y *x *y *x^−1)^3 = Id($)
x^−1 *y *x^−1 *y *x^−1 *y *x^−1 *y *x^−1 *y * x *y *x *y *x
    *y *x *y *x *y = Id($)


G<x,y,t>:=Group<x,y,t|x^3, y^2,(y *x^−1 *y *x *y *x^−1)^3,x
   ^−1 *y *x^−1 *y *x^−1 *y *x^−1 *y *x^−1 *y * x *y *x *y *
   x *y *x *y *x *y,t^7,
*/


Nt:=Stabiliser(N,{1,6,11,16,21,26}); /* Get the #'s from
    the perm building chart */
 /* These perms are in the stabiliser */
for n in N do if {1,6,7,8,9,10}^n eq {1,6,7,8,9,10}
 then n; end if; end for;
Nt;
/*
Permutation group Nt acting on a set of cardinality 30
Order = 36 = 2^2 * 3^2
    (2, 3)(4, 20)(5, 9)(7, 8)(10, 19)(12, 13)(14, 25)(15,
        29)(17, 18)(22,
         23)(24, 30)(27, 28)
    (2, 18, 4)(3, 9, 7)(5, 20, 10)(8, 19, 17)(12, 23, 14)
        (13, 29, 27)(15, 25,
         30)(22, 28, 24)
    (1, 16, 6)(2, 3, 10)(5, 17, 18)(7, 8, 20)(11, 21, 26)
        (12, 13, 30)(15, 22,
```

```
        23)(25, 27, 28)
*/
Set(Nt);
w:=WordGroup(N);
rho:=InverseWordMap(N);
A:=N!(2, 3)(4, 20)(5, 9)(7, 8)(10, 19)(12, 13)(14, 25)(15,
    29)(17, 18)(22, 23)(24, 30)(27, 28);/* t^()= t (t,_) */
B:=N!(2, 18, 4)(3, 9, 7)(5, 20, 10)(8, 19, 17)(12, 23, 14)
    (13, 29, 27)(15, 25, 30)(22, 28, 24); /* t^()= t (t,_) */
C:=N!(1, 16, 6)(2, 3, 10)(5, 17, 18)(7, 8, 20)(11, 21, 26)
    (12, 13, 30)(15, 22, 23)(25, 27, 28);
A@rho;
AA:=function(w)
    w1 := w.1^-1; w3 := w1 * w.2; w2 := w.2^-1; w4 := w2 *
        w.1; w5 := w.1 * w4;
    w6 := w3 * w5; w8 := w6 * w.2; w9 := w8 * w6; w10 := w9
        * w6; w11 := w10 *
    w.1; w12 := w11 * w.2; w13 := w12 * w.1; w14 := w13 * w
        .2; w15 := w14 * w.1;
    w16 := w15 * w.2; return w16;
end function;
AA(FPGroup(N));
/*
$.1^-1 * $.2 * $.1 * $.2^-1 * $.1 * $.2 * $.1^-1 * $.2 * $
    .1^2 * $.2^-1 * $.1^2
* $.2 * $.1 * $.2 * $.1 * $.2

x^-1 *y *x *y^-1 *x *y *x^-1 *y *x^2 *y^-1 *x^2
*y *x *y *x *y
*/
B@rho;
```

```
BB:= function(w)
    w17 := w.1 * w.2; w1 := w.1^-1; w18 := w17 * w1; w19 :=
        w18 * w.2; w20 :=
    w19 * w.1; w21 := w20 * w.2; return w21;
end function;
BB(FPGroup(N));
/*
$.1 * $.2 * $.1^-1 * $.2 * $.1 * $.2


x * y * x^-1 * y * x * y
*/
C@rho;
CC:=function(w)
    w1 := w.1^-1; w3 := w1 * w.2; w2 := w.2^-1; w4 := w2 *
        w.1; w5 := w.1 * w4;
    w6 := w3 * w5; w8 := w.2 * w6; w9 := w8 * w.2; w10 :=
        w9 * w.1; w11 := w10 *
    w.2; w12 := w11 * w.1; w13 := w12 * w.2; w14 := w13 * w
        .1; w15 := w14 * w.2;
    return w15;
end function;
CC(FPGroup(N));
/*
$.2 * $.1^-1 * $.2 * $.1 * $.2^-1 * $.1 * $.2 * $.1 * $.2 *
    $.1 * $.2 * $.1 *
$.2


y * x^-1 * y * x * y^-1 * x * y * x * y * x * y * x * y
*/
Orbits(Stabiliser(N,1));
/*
```

```
  GSet{@ 1 @},
      GSet{@ 6 @},
      GSet{@ 11 @},
      GSet{@ 16 @},
      GSet{@ 21 @},
      GSet{@ 26 @},
      GSet{@ 2, 10, 4, 20, 8, 18, 7, 5, 17, 9, 19, 3 @},
      GSet{@ 12, 30, 14, 25, 28, 23, 27, 15, 22, 29, 24, 13 @
          }
*/


/* Work with this one */
G<x,y,t>:=Group<x,y,t|x^3, y^2,(y *x^-1 *y *x *y *x^-1)^3,x
    ^-1 *y *x^-1 *y *x^-1 *y *x^-1 *y *x^-1 *y * x *y *x *y *
    x *y *x *y *x *y,t^7,t^(x^-1 *y *x *y^-1 *x *y *x^-1 *y *
    x^2 *y^-1 *x^2 *y *x *y *x *y)=t,t^(x * y * x^-1 * y * x
    * y)=t,t^(y * x^-1 * y * x * y^-1 * x * y * x * y * x * y
    * x * y)=t^4,(t,t^x)/* Need to remove after test*/>;


/* #G Should get 7^5*180 */
Index(G,sub<G|x,y>); /* should be 7^5 */
/* 16807 */
#sub<G|x,y>;
```

```
/* Find the FOR */

S:=Sym(30);
xx:=S!(1,2,4)(5,20,10)(6,7,9)(11,12,14)(15,25,30)(16,17,19)
    (21,22,24)(26,27,29);
yy:=S!(1,8)(3,16)(4,5)(6,18)(9,10)(11,28)(13,21)(14,15)
    (19,20)(23,26)(24,25)(29,30);
N:=sub<S|xx,yy>;
#N;
/*180*/
C:=Classes(N);
#C;
/*15*/
FPGroup(N);
/*
Finitely presented group on 2 generators
Relations
    $.1^3 = Id($)
    $.2^2 = Id($)
    ($.2 * $.1^-1 * $.2 * $.1 * $.2 * $.1^-1)^3 = Id($)
    $.1^-1 * $.2 * $.1^-1 * $.2 * $.1^-1 * $.2 * $.1^-1 * $
        .2 * $.1^-1 * $.2 * $.1 *
    $.2 * $.1 * $.2 * $.1 * $.2 * $.1 * $.2 * $.1 * $.2 =
        Id($)

*/
NN<x,y>:=Group<x,y|x^3, y^2,(y *x^-1 *y *x *y *x^-1)^3,x^-1
    *y *x^-1 *y *x^-1 *y *x^-1 *y *x^-1 *y * x *y *x *y *x *
    y *x *y *x *y>;
#NN;
/* 180 */
```

```
Sch:=SchreierSystem(NN,sub<NN| Id(NN)>);
 ArrayP:=[Id(N): i in [1.. #N]];
 for i in [2.. #N] do P:=[Id(N): l in [1.. #Sch[i]]];
 for j in [1..# Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
end for;
 PP:=Id(N);
 for k in [1.. #P] do
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
 end for;
for i in [1..#N] do if 1^ArrayP[i] eq 2 then Sch[i]; break;
   end if; end for;
/* x */
 for i in [1..#N] do if 1^ArrayP[i] eq 3 then Sch[i]; break
    ;end if; end for;
/* x^-1 * y * x * y * x * y */
 for i in [1..#N] do if 1^ArrayP[i] eq 4 then Sch[i]; break
    ;end if; end for;
/* x^-1 */
for i in [1..#N] do if 1^ArrayP[i] eq 5 then Sch[i]; break;
   end if; end for;
/* x^-1*y */
 for i in [1..#N] do if 1^ArrayP[i] eq 6 then Sch[i]; break
    ;end if; end for;
/* x^-1 * y * x^-1 * y * x */
 for i in [1..#N] do if 11^ArrayP[i] eq 12 then Sch[i];
    break;end if; end for;
/* x */
```

```
for i in [2..#C] do i, C[i][3]; for j in [1..#N] do
 if ArrayP[j] eq C[i][3]
 then Sch[j]; end if; end for;
 Orbits(Centraliser(N,C[i][3])); end for;
Orbits(N);
/*
2 (2, 10)(3, 19)(4, 8)(5, 17)(7, 20)(9, 18)(12, 30)(13, 24)
   (14, 28)(15, 22)(23,
     29)(25, 27)
x * y * x^-1
[
    GSet{@ 1, 6, 16 @},
    GSet{@ 11, 26, 21 @},
    GSet{@ 2, 10, 19, 7, 3, 20, 4, 17, 8, 5, 9, 18 @},
    GSet{@ 12, 30, 24, 27, 13, 25, 14, 22, 28, 15, 29, 23 @
        }
]
//((x * y * x^-1)*t),((x * y * x^-1)*(t^x)^2), ((x * y * x
   ^-1)*t^x), ((x * y * x^-1)*((t^x)^3)),

3 (1, 16, 6)(2, 17, 7)(3, 18, 8)(4, 19, 9)(5, 20, 10)(11,
   21, 26)(12, 22, 27)(13,
     23, 28)(14, 24, 29)(15, 25, 30)
(x * y)^5
[
    GSet{@ 1, 2, 8, 4, 5, 20, 10, 19, 9, 16, 6, 17, 3, 7,
        18 @},
    GSet{@ 11, 12, 28, 14, 15, 25, 30, 24, 29, 21, 26, 22,
        13, 27, 23 @}
]
```

//((x * y)^5*t),((x * y)^5*(t^x)^2),

4 (1, 6, 16)(2, 7, 17)(3, 8, 18)(4, 9, 19)(5, 10, 20)(11,
   26, 21)(12, 27, 22)(13,
    28, 23)(14, 29, 24)(15, 30, 25)
(y * x^−1)^5
[
    GSet{@ 1, 2, 8, 4, 5, 20, 10, 19, 9, 16, 6, 17, 3, 7,
       18 @},
    GSet{@ 11, 12, 28, 14, 15, 25, 30, 24, 29, 21, 26, 22,
       13, 27, 23 @}
]
//((y * x^−1)^5*t),((y * x^−1)^5*(t^x)^2),

5 (1, 2, 4)(5, 20, 10)(6, 7, 9)(11, 12, 14)(15, 25, 30)(16,
   17, 19)(21, 22, 24)(26,
    27, 29)
x
[
    GSet{@ 3, 18, 8 @},
    GSet{@ 5, 20, 10 @},
    GSet{@ 13, 23, 28 @},
    GSet{@ 15, 25, 30 @},
    GSet{@ 1, 2, 17, 4, 19, 9, 16, 6, 7 @},
    GSet{@ 11, 12, 22, 14, 24, 29, 21, 26, 27 @}
]
// (x*t^(x^−1 * y * x * y * x * y)), (x*t^(x^−1 * y)),(x*(t
   ^x)^4),(x*(t^x)^6),(x*t),(x*(t^x)^2),

6 (1, 4, 2)(5, 10, 20)(6, 9, 7)(11, 14, 12)(15, 30, 25)(16,
   19, 17)(21, 24, 22)(26,

```
    29, 27)
xˆ−1
[
    GSet{@ 3, 18, 8 @},
    GSet{@ 5, 10, 20 @},
    GSet{@ 13, 23, 28 @},
    GSet{@ 15, 30, 25 @},
    GSet{@ 1, 4, 17, 2, 16, 9, 19, 7, 6 @},
    GSet{@ 11, 14, 22, 12, 21, 29, 24, 27, 26 @}
]
((xˆ−1)∗tˆ(xˆ−1 ∗ y ∗ x ∗ y ∗ x ∗ y)),((xˆ−1)∗tˆ(xˆ−1 ∗ y))
    ,((xˆ−1)∗(tˆx)ˆ4),((xˆ−1)∗(tˆx)ˆ6),((xˆ−1)∗t),((xˆ−1)∗(tˆ
    x)ˆ2),


7 (1, 3, 19)(2, 7, 17)(4, 6, 8)(5, 20, 10)(9, 16, 18)(11,
    13, 24)(12, 27, 22)(14,
        26, 28)(15, 25, 30)(21, 23, 29)
xˆ−1 ∗ y ∗ x ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1
[
    GSet{@ 2, 7, 17 @},
    GSet{@ 5, 20, 10 @},
    GSet{@ 12, 27, 22 @},
    GSet{@ 15, 25, 30 @},
    GSet{@ 1, 4, 3, 18, 6, 19, 9, 8, 16 @},
    GSet{@ 11, 14, 13, 23, 26, 24, 29, 28, 21 @}
]
// ((xˆ−1 ∗ y ∗ x ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1)∗tˆx),((
    xˆ−1 ∗ y ∗ x ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1)∗tˆ(xˆ−1 ∗
    y)),((xˆ−1 ∗ y ∗ x ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1)∗(tˆx
    )ˆ3),((xˆ−1 ∗ y ∗ x ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1)∗(tˆ
    x)ˆ6),((xˆ−1 ∗ y ∗ x ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1 ∗ y ∗ xˆ−1)∗t)
```

,((x^−1 ∗ y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x^−1 ∗ y ∗ x^−1)∗(t^x)
^2),
8 (1, 19, 17, 18, 10)(2, 3, 20, 6, 4)(5, 16, 9, 7, 8)(11,
24, 22, 23, 30)(12, 13,
25, 26, 14)(15, 21, 29, 27, 28)
(x ∗ y)^3
[
    GSet{@ 1, 19, 5, 17, 16, 3, 18, 9, 20, 10, 7, 6, 8, 4,
        2 @},
    GSet{@ 11, 24, 15, 22, 21, 13, 23, 29, 25, 30, 27, 26,
        28, 14, 12 @}
(((x ∗ y)^3)∗t),(((x ∗ y)^3)∗(t^x)^2),

]
9 (1, 17, 10, 19, 18)(2, 20, 4, 3, 6)(5, 9, 8, 16, 7)(11,
22, 30, 24, 23)(12, 25,
14, 13, 26)(15, 29, 28, 21, 27)
x^−1 ∗ y ∗ x ∗ y ∗ x ∗ y ∗ x ∗ y ∗ x
[
    GSet{@ 1, 17, 5, 10, 9, 3, 19, 8, 6, 18, 16, 2, 7, 20,
        4 @},
    GSet{@ 11, 22, 15, 30, 29, 13, 24, 28, 26, 23, 21, 12,
        27, 25, 14 @}
((x^−1 ∗ y ∗ x ∗ y ∗ x ∗ y ∗ x ∗ y ∗ x)∗t),((x^−1 ∗ y ∗ x ∗
y ∗ x ∗ y ∗ x ∗ y ∗ x)∗(t^x)^2),

]
10 (1, 3, 16, 18, 6, 8)(2, 9, 17, 4, 7, 19)(5, 10, 20)(11,
13, 21, 23, 26, 28)(12,
29, 22, 14, 27, 24)(15, 30, 25)
x^−1 ∗ y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x^−1 ∗ y

[
    GSet{@ 5, 10, 20 @},
    GSet{@ 15, 30, 25 @},
    GSet{@ 1, 18, 2, 3, 4, 6, 9, 16, 7, 8, 17, 19 @},
    GSet{@ 11, 23, 12, 13, 14, 26, 29, 21, 27, 28, 22, 24 @
       }
((x^−1 ∗ y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x^−1 ∗ y)∗t^(x^−1 ∗ y)),((x
   ^−1 ∗ y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x^−1 ∗ y)∗(t^x)^6),((x^−1 ∗
   y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x^−1 ∗ y)∗t),((x^−1 ∗ y ∗ x ∗ y ∗
   x^−1 ∗ y ∗ x^−1 ∗ y)∗(t^x)^2),

]
11 (1, 8, 6, 18, 16, 3)(2, 19, 7, 4, 17, 9)(5, 20, 10)(11,
   28, 26, 23, 21, 13)(12,
    24, 27, 14, 22, 29)(15, 25, 30)
y ∗ x ∗ y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x
[
    GSet{@ 5, 20, 10 @},
    GSet{@ 15, 25, 30 @},
    GSet{@ 1, 18, 4, 8, 2, 16, 17, 6, 19, 3, 9, 7 @},
    GSet{@ 11, 23, 14, 28, 12, 21, 22, 26, 24, 13, 29, 27 @
       }
((y ∗ x ∗ y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x)∗t^(x^−1 ∗ y)),((y ∗ x ∗
   y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x)∗(t^x)^6),((y ∗ x ∗ y ∗ x ∗ y ∗
   x^−1 ∗ y ∗ x)∗t),((y ∗ x ∗ y ∗ x ∗ y ∗ x^−1 ∗ y ∗ x)∗(t^
  x)^2),

]
12 (1, 2, 5, 19, 3, 16, 17, 20, 9, 18, 6, 7, 10, 4, 8)(11,
   12, 15, 24, 13, 21, 22,
    25, 29, 23, 26, 27, 30, 14, 28)

```
x * y
[
    GSet{@ 1, 2, 5, 19, 3, 16, 17, 20, 9, 18, 6, 7, 10, 4,
        8 @},
    GSet{@ 11, 12, 15, 24, 13, 21, 22, 25, 29, 23, 26, 27,
        30, 14, 28 @}
((x * y)*t),((x * y)*(t^x)^2),

]
13 (1, 5, 3, 17, 9, 6, 10, 8, 2, 19, 16, 20, 18, 7, 4)(11,
    15, 13, 22, 29, 26, 30,
    28, 12, 24, 21, 25, 23, 27, 14)
(x * y)^2
[
    GSet{@ 1, 5, 3, 17, 9, 6, 10, 8, 2, 19, 16, 20, 18, 7,
        4 @},
    GSet{@ 11, 15, 13, 22, 29, 26, 30, 28, 12, 24, 21, 25,
        23, 27, 14 @}
((x * y)^2*t),((x * y)^2*(t^x)^2),

]
14 (1, 20, 8, 17, 4, 16, 10, 3, 7, 19, 6, 5, 18, 2, 9)(11,
    25, 28, 22, 14, 21, 30,
    13, 27, 24, 26, 15, 23, 12, 29)
x^-1 * y * x * y * x * y * x * y * x^-1 * y
[
    GSet{@ 1, 20, 8, 17, 4, 16, 10, 3, 7, 19, 6, 5, 18, 2,
        9 @},
    GSet{@ 11, 25, 28, 22, 14, 21, 30, 13, 27, 24, 26, 15,
        23, 12, 29 @}
((x^-1 * y * x * y * x * y * x * y * x^-1 * y)*t),((x^-1 *
```

```
        y * x * y * x * y * x * y * x^-1 * y)*(t^x)^2),


]
15 (1, 7, 20, 19, 8, 6, 17, 5, 4, 18, 16, 2, 10, 9, 3)(11,
    27, 25, 24, 28, 26, 22,
     15, 14, 23, 21, 12, 30, 29, 13)
(y * x^-1)^4
[
    GSet{@ 1, 7, 20, 19, 8, 6, 17, 5, 4, 18, 16, 2, 10, 9,
        3 @},
        GSet{@ 11, 27, 25, 24, 28, 26, 22, 15, 14, 23, 21, 12,
            30, 29, 13 @}
(((y * x^-1)^4)*t) ,(((y * x^-1)^4)*(t^x)^2),


]

*/


/*FOR ALL Together:*/
for a,b,c,d,e,f,g,h in [0..10] do
G<x,y,t>:=Group<x,y,t|(y^-1 *x)^3, y^-1*x^-3*y^-1*x^2,y^-3*
    x^-1*y^2*x^-1,
t^7, (t,x^2*y^2), (t,x^2*y^2*x^3*y^-1),t^(x^3*y^-1*x^2*y^2*
    x^2*y^2*x^-1*y) = t^4, ((x * y * x * y^-1 * x)*t^(x^2))^a
    , ((x * y * x * y^-1 * x)*(t^x^2)^3)^b,((x * y * x * y^-1
    * x)*t)^c,((x * y * x * y^-1 * x)*t^3)^d,((x^5)*t)^e,((x
    ^5)*t^3)^f,((x^-5)*t)^g,((x^-5)*t^3)^h>;
a,b,c,d,e,f,g,h, Index(G,sub<G|x,y>); end for;


/* Saved for Later Split this into two parts.
```

```
*/
```

```
/* Running it in the background
1. nano 7star5:180
2. Insert data
3. Ctrl + x
4. Choose yes
5. Enter to save file
If we want to check the input file 7star5:180 type nano 7
    star5:180

nohup magma "7star5:180"&>7star5:180.out&
After the above command has been run if we want to check
    output(or whether the program is running). We use the
    command  nano 7star5:180.out



Checking Status of Progenitor
nano 7star5:180.out
*/
```

Wreath Products Codes

6.4 Wreath Product of $\mathbb{Z}_3 \wr \mathbb{S}_3$

```
Z_3
S:= Sym(24);
B:= sub<S| (9,13,17,21), (10,14,18,22),(11,15,19,23)
    ,(12,16,20,24)>;
/* This is where my K^* goes ( Should be 4^4*4)
```

```
Verify 's part a *\

Reath := sub<S|B,(9,10,11,12)(13,14,15,16)(17,18,19,20)
    (21,22,23,24)>;
#Reath;
/*1024 */
/* Note that comma in paranthesis means commute
Give the presentation of K wreath K and verify if (Should
    be 1024 */

G <a,b,c,d,e> := Group <a,b,c,d,e | a^4, b^4, c^4, d^4, (a,
    b),(a,c),(a,d),(b,c),
(b,d), (c,d), e^4, a^e=b,b^e=c,c^e=d,d^e=a>;
#G;
/*1024 */


/* #2 Part */
/* Below is Z_3 wr S_3 */
G <a,b,c,d,e> := Group <a,b,c,d,e| a^3,b^3,c^3,d^3,e^2,(d*e
    )^2,
(a,b),(a,c),(b,c), a^d=b,b^d=c,c^d=a,a^e=b,b^e=a,c^e=c>;
#G;
/* 162 */
W:= WreathProduct(Alt(3),Sym(3));
#W;
/* 162 */
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
IsIsomorphic(G1,W);
/* true */
G1;
```

```
/* 162 */
/* Gives you that the group is on carditionality 9 (meaning
    9 letters) */
W;
/*

Permutation group W acting on a set of cardinality 9
Order = 162 = 2 * 3^4
    (1, 4, 7)(2, 5, 8)(3, 6, 9)
    (1, 4)(2, 5)(3, 6)
    (1, 2, 3)

*/

/*
Loop below finds the minimial faithful permutation
    representation ( i.e 648 —> 9
Looking at keranls that have one and storing those
(Core means kernal)
*/
SL:=Subgroups(G1);

T := {X'subgroup: X in SL};

#T;

TrivCore := {H:H in T| #Core(G1,H) eq 1};

mdeg := Min({Index(G1,H):H in TrivCore});

Good := {H: H in TrivCore| Index(G1,H) eq mdeg};
```

```
#Good;

H := Rep(Good);


#H;

f,G1,K := CosetAction(G1,H);
G1;
```

Isomorphism Types Codes

6.5 Semi-Direct Product of $(7\mathrm{x}5 : 6), N = S_{40}, T[401]$

```
T:=TransitiveGroups(40);
T[401];

/*
Permutation group acting on a set of cardinality 40
Order = 480 = 2^5 * 3 * 5
    (1, 20, 37, 9, 28, 5, 18, 35, 13, 26, 4, 21, 34, 12,
        30)(2, 19, 38, 10, 27, 6, 17, 36, 14, 25, 3, 22, 33,
        11, 29)(7, 23, 40, 16, 31)(8, 24, 39, 15, 32)
    (1, 32, 13, 34, 24, 5, 26, 15, 37, 18, 8, 30, 9, 39,
        21)(2, 31, 14, 33, 23, 6, 25, 16, 38, 17, 7, 29, 10,
        40, 22)(3, 27, 11, 36, 19)(4, 28, 12, 35, 20)
    (1, 19, 38, 10, 28, 5, 18, 36, 14, 25, 4, 21, 34, 11,
        29, 2, 20, 37, 9, 27, 6, 17, 35, 13, 26, 3, 22, 33,
        12, 30)(7, 24, 40, 15, 31, 8, 23, 39, 16, 32)
 */

S:=Sym(40);
xx:=S!(1, 20, 37, 9, 28, 5, 18, 35, 13, 26, 4, 21, 34, 12,
    30)(2, 19, 38, 10, 27, 6, 17, 36, 14, 25, 3, 22, 33, 11,
    29)(7, 23, 40, 16, 31)(8, 24, 39, 15, 32);

yy:=S!(1, 32, 13, 34, 24, 5, 26, 15, 37, 18, 8, 30, 9, 39,
    21)(2, 31, 14, 33, 23, 6, 25, 16, 38, 17, 7, 29, 10, 40,
    22)(3, 27, 11, 36, 19)(4, 28, 12, 35, 20);

zz:=S!(1, 19, 38, 10, 28, 5, 18, 36, 14, 25, 4, 21, 34, 11,
```

```
      29, 2, 20, 37, 9, 27, 6, 17, 35, 13, 26, 3, 22, 33, 12,
      30)(7, 24, 40, 15, 31, 8, 23, 39, 16, 32);

N:=sub<S|xx,yy,zz>;
#N;
/* 480
For finding the isomorphism type look for a minimial
    faithful Perm Rep */

SL:=Subgroups(N);
T:={X' subgroup: X in SL};
#T;
/* 92 */

TrivCore := {H:H in T| #Core(N,H) eq 1};
mdeg := Min ({Index(N,H):H in TrivCore});
Good   := {H:H in TrivCore | Index(N,H) eq mdeg};
#Good;
/* 3 */

H:= Rep(Good);
#H;
/* 12 */

f2,N1,K2:= CosetAction(N,H);
N1;
/*
Permutation group N1 acting on a set of cardinality 8
Order = 32 = 2^5
    (1, 2, 5, 4, 3, 6, 8, 7)
    (1, 3)(5, 8)
```

```
        (1 , 4) (2 , 5) (3 , 7) (6 , 8)
*/


N;
/*
Permutation group N acting on a set of cardinality 40
Order = 480 = 2^5 * 3 * 5
        (1 , 20, 37, 9, 28, 5, 18, 35, 13, 26, 4, 21, 34, 12,
            30) (2 , 19, 38, 10, 27, 6, 17, 36, 14, 25, 3, 22, 33,
            11, 29) (7 , 23, 40, 16, 31) (8 , 24, 39, 15, 32)
        (1 , 32, 13, 34, 24, 5, 26, 15, 37, 18, 8, 30, 9, 39,
            21) (2 , 31, 14, 33, 23, 6, 25, 16, 38, 17, 7, 29, 10,
            40, 22) (3 , 27, 11, 36, 19) (4 , 28, 12, 35, 20)
        (1 , 19, 38, 10, 28, 5, 18, 36, 14, 25, 4, 21, 34, 11,
            29, 2, 20, 37, 9, 27, 6, 17, 35, 13, 26, 3, 22, 33,
            12, 30) (7 , 24, 40, 15, 31, 8, 23, 39, 16, 32)
*/


Order (xx) ;
/* 15
Algorithm for Isom Type  */
CompositionFactors (N) ;


/*

 G
    |   Cyclic (3)
    *
    |   Cyclic (2)
    *
    |   Cyclic (2)
```

```
        *
        |   Cyclic(5)
        *
        |   Cyclic(2)
        *
        |   Cyclic(2)
        *
        |   Cyclic(2)
        1


*/


NL:=  NormalLattice(N);
NL;
/*
Normal  subgroup  lattice
_____


[18]   Order  480   Length  1   Maximal  Subgroups:  16  17
____
[17]   Order  160   Length  1   Maximal  Subgroups:  10  11  12  13
     14  15
[16]   Order  96    Length  1   Maximal  Subgroups:  15
____
[15]   Order  32    Length  1   Maximal  Subgroups:  5  6  7  8  9
____
[14]   Order  40    Length  1   Maximal  Subgroups:  4  7
[13]   Order  40    Length  1   Maximal  Subgroups:  4  6
[12]   Order  40    Length  1   Maximal  Subgroups:  4  5


[11]   Order  40    Length  1   Maximal  Subgroups:  4  8
```

```
[10]    Order  40    Length  1   Maximal Subgroups: 4 9
————

[ 9]    Order  8     Length  1   Maximal Subgroups: 2
[ 8]    Order  8     Length  1   Maximal Subgroups: 2
[ 7]    Order  8     Length  1   Maximal Subgroups: 2
[ 6]    Order  8     Length  1   Maximal Subgroups: 2
[ 5]    Order  8     Length  1   Maximal Subgroups: 2
————

[ 4]    Order  10    Length  1   Maximal Subgroups: 2 3
————

[ 3]    Order  5     Length  1   Maximal Subgroups: 1
[ 2]    Order  2     Length  1   Maximal Subgroups: 1
————

[ 1]    Order  1     Length  1   Maximal Subgroups:

*/

for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if;
   end for;
/*
1
2
3
4
5
6
8
11
12
13
```

```
*/

/* Choose the largest i in this case 13 */

Generators (NL[13]);

/*
    (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)
        (17, 18)(19, 20)(21,
          22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)
              (35, 36)(37, 38)(39, 40),
    (1, 9, 18, 26, 34)(2, 10, 17, 25, 33)(3, 11, 19, 27,
        36)(4, 12, 20, 28, 35)(5,
          13, 21, 30, 37)(6, 14, 22, 29, 38)(7, 16, 23, 31,
              40)(8, 15, 24, 32, 39),
    (1, 7)(2, 8)(3, 6)(4, 5)(9, 16)(10, 15)(11, 14)(12, 13)
        (17, 24)(18, 23)(19,
          22)(20, 21)(25, 32)(26, 31)(27, 29)(28, 30)(33, 39)
              (34, 40)(35, 37)(36, 38),
    (1, 4)(2, 3)(5, 7)(6, 8)(9, 12)(10, 11)(13, 16)(14, 15)
        (17, 19)(18, 20)(21,
          23)(22, 24)(25, 27)(26, 28)(29, 32)(30, 31)(33, 36)
              (34, 35)(37, 40)(38, 39)
 */

A:=N!((1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15,
    16)(17, 18)(19, 20)(21,22)(23, 24)(25, 26)(27, 28)(29,
    30)(31, 32)(33, 34)(35, 36)(37, 38)(39, 40));

B:=N!((1, 9, 18, 26, 34)(2, 10, 17, 25, 33)(3, 11, 19, 27,
    36)(4, 12, 20, 28, 35)(5,13, 21, 30, 37)(6, 14, 22, 29,
```

38)(7, 16, 23, 31, 40)(8, 15, 24, 32, 39));

C:=N!((1, 7)(2, 8)(3, 6)(4, 5)(9, 16)(10, 15)(11, 14)(12, 13)(17, 24)(18, 23)(19,22)(20, 21)(25, 32)(26, 31)(27, 29)(28, 30)(33, 39)(34, 40)(35, 37)(36, 38));

D:=N!((1, 4)(2, 3)(5, 7)(6, 8)(9, 12)(10, 11)(13, 16)(14, 15)(17, 19)(18, 20)(21,23)(22, 24)(25, 27)(26, 28)(29, 32)(30, 31)(33, 36)(34, 35)(37, 40)(38, 39));

/*
This means that we have a (2x5x2x2): ? (later we find that
    it is A_4)
A,B,C,D is the generator os the largest abelian */

NL13:=sub<N|A,B,C,D>;
/* Check that NL[13] = NL13*/
NL[13] eq NL13;
/*
True
*/
Order (NL13);

/* 40 = 5 *2^3 */
Order (N);
/* 480 */

Order (N)/Order(NL13);
/*
12

```
IsIsomorphic(NL[13], AbelianGroup(GrpPerm,[5,2,2,2]));
/*
true Mapping from: GrpPerm: $, Degree 40, Order 2^3 * 5 to
    GrpPerm: $, Degree 11,
Order 2^3 * 5
Composition of Mapping from: GrpPerm: $, Degree 40, Order
    2^3 * 5 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 11, Order 2^3 * 5
*/


Does NL have a subgroup (normal) of order 12
It does so it is a direct product (check normal lattice)


Now N = <xx, yy, zz> and NL13=<A,B,C,D>. Then N/NL14 = <NL6xx
    , NL6yy, NL6zz>
*/


q, ff:=quo<N|NL13>;
/* q is the isom type of N/NL13; that is, q N/NL13 */
T:=Transversal(N,NL13);
/* T gives right cosets of NL6 in N
Thus, N/NL13=<NL13T[2]> */
T[2] eq xx; T[3] eq yy;T[4] eq zz;
/* true, true, true */

#T;
/* 12 */

q;
/*
```

```
Permutation group q acting on a set of cardinality 4
Order = 12 = 2^2 * 3
    (2, 4, 3)
    (1, 2, 4)
    (1, 3, 4)


IsIsomorphic(q,Alt(4));
/*
true Isomorphism of GrpPerm: q, Degree 4, Order 2^2 * 3
   into GrpPerm: $, Degree 4,
Order 2^2 * 3 induced by
    (2, 4, 3) |--> (2, 4, 3)
    (1, 2, 4) |--> (1, 2, 4)
    (1, 3, 4) |--> (1, 3, 4)
*/


FPGroup(q);
/*
Finitely presented group on 3 generators
Relations
    $.1^3 = Id($)  --> e^3
    $.2^3 = Id($)  --> f^3
    $.3^3 = Id($)  --> g^3
    $.2 * $.1^-1 * $.3^-1 = Id($) -->  f * e^-1 * g^-1
    $.1^-1 * $.2^-1 * $.1 * $.3^-1 = Id($) --> e^-1 * f^-1
       * e * g^-1
*/


ff(T[3]) eq q.2;
/* true */
ff(T[2]) eq q.1;
```

```
/* true */
ff(T[4]) eq q.3;


/* Now I can write the presenation using the above
   information
we get the powers of a,b,c,d from the top A:=N!,B:=N!, etc.


<a,b,c,d,e,f,g | a^2,b^5,c^2,d^2, (a,b),(a,c),(a,d),(b,c),(
   b,d),(c,d), e^3,f^3,g^3,f *e^(-1) * g^(-1),e^(-1) * f
   ^(-1) * e * g^(-1),a^e,a^f,a^g,b^e,b^f,b^g,c^e,c^f,c^g,
d^e,d^f,d^g)


Next piece of code will help us find what a^e =, a^f= and
   etc..
*/



for i,j,k in [0..1] do for l in [0..4] do
if A^T[2] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
     end for;
/* 1 0 0 0 —> a^e = a*/


for i,j,k in [0..1] do for l in [0..4] do
if A^T[3] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
     end for;
/* 1 0 0 0 —> a^f = a */


for i,j,k in [0..1] do for l in [0..4] do
if A^T[4] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
     end for;
/* 1 0 0 0 —> a^g = a */
```

```
for i,j,k in [0..1] do for l in [0..4] do
if B^T[2] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
    end for;
/* 0 0 0 1 ---> b^e = b*/


for i,j,k in [0..1] do for l in [0..4] do
if B^T[3] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
    end for;
/* 0 0 0 1 ---> b^f = b */


for i,j,k in [0..1] do for l in [0..4] do
if B^T[4] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
    end for;
/* 0 0 0 1 ---> b^g = b */


for i,j,k in [0..1] do for l in [0..4] do
if C^T[2] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
    end for;
/* 0 1 1 0 ---> c^e = c*d */


for i,j,k in [0..1] do for l in [0..4] do
if C^T[3] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
    end for;
/* 0 0 1 0 --->c^f = d */


for i,j,k in [0..1] do for l in [0..4] do
if C^T[4] eq A^i*C^j*D^k*B^l then i,j,k,l; end if; end for;
    end for;
/* 0 1 1 0 ---> c^g = c*d */
```

```
for  i , j , k  in  [ 0 . . 1 ]  do  for  l  in  [ 0 . . 4 ]  do
if  D^T[2]  eq  A^i*C^j*D^k*B^l  then  i , j , k , l ;  end  if ;  end  for ;
    end  for ;
/*  0  1  0  0  —>  d^e  =  c  */


for  i , j , k  in  [ 0 . . 1 ]  do  for  l  in  [ 0 . . 4 ]  do
if  D^T[3]  eq  A^i*C^j*D^k*B^l  then  i , j , k , l ;  end  if ;  end  for ;
    end  for ;
/*  1  1  1  0  —>  d^f  =  a*c*d  */


for  i , j , k  in  [ 0 . . 1 ]  do  for  l  in  [ 0 . . 4 ]  do
if  D^T[4]  eq  A^i*C^j*D^k*B^l  then  i , j , k , l ;  end  if ;  end  for ;
    end  for ;
/*  1  1  0  0  —>  d^g  =  a*c


Thus  a  pres  of :
<a , b , c , d , e , f , g  |  a^2 , b^5 , c^2 , d^2 ,  ( a , b ) , ( a , c ) , ( a , d ) , ( b , c ) , (
   b , d ) , ( c , d ) ,  e^3 , f^3 , g^3 , f  *e^(−1)  *  g^(−1) , e^(−1)  *  f
   ^(−1)  *  e  *  g^(−1) , a^e  =  a ,  a^f  =  a , a^g  =  a , b^e  =  b , b^f  =
   b , b^g  =  b , c^e  =  c*d ,  c^f  =  d ,  c^g  =  c*d , d^e  =  c , d^f  =  a*
   c*d , d^g  =  a*c>
*/


G<a , b , c , d , e , f , g>:=Group<a , b , c , d , e , f , g  |  a^2 , b^5 , c^2 , d^2 ,  ( a
   , b ) , ( a , c ) , ( a , d ) , ( b , c ) , ( b , d ) , ( c , d ) ,  e^3 , f^3 , g^3 , f  *e^(−1)
   *  g^(−1) , e^(−1)  *  f^(−1)  *  e  *  g^(−1) , a^e  =  a ,  a^f  =  a , a^
   g  =  a , b^e  =  b , b^f  =  b , b^g  =  b , c^e  =  c*d ,  c^f  =  d ,  c^g  =  c
   *d , d^e  =  c , d^f  =  a*c*d , d^g  =  a*c>;
#G;
/*  480
Check  if  G~N.  We  need  a  perm  rep  of  G
```

```
*/
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
#G1;
s:=IsIsomorphic(N,G1);
s;
/True*/
```

WE PROVED THAT   N~ (5x2^3):Alt(4)

6.6 Mixed Extension Product of $(4x2 : 2^2), N = S_8, T[15]$

```
T:=TransitiveGroups(8);
N:=T[15];


N;
/*
Permutation group N acting on a set of cardinality 8
Order = 32 = 2^5
    (1, 2, 3, 4, 5, 6, 7, 8)
    (1, 5)(3, 7)
    (1, 6)(2, 5)(3, 4)(7, 8)
*/


S:=Sym(8);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8);


yy:=S!(1, 5)(3, 7);


zz:=S!(1, 6)(2, 5)(3, 4)(7, 8);
N:=sub<S|xx,yy,zz>;
/*
How to shorten down your generators (optional if you have 5
    or more generators on top)
for g, h in N do
if sub<N|g,h> eq N then a:= g; b:= h; end if; end for;
- when you try to output a you get ident error then you
    need all generators


for g, h in N do
if sub<N|g,h> eq N then g,h; end if; end for;
```

```
–  If  there  is  no  output  you  need  all  generators  given  by
   magma


*/
#N;
/*32 */
SL:=Subgroups(N);
T:={X' subgroup: X in SL};
#T;
/* 34 */


TrivCore  :=  {H:H in T| #Core(N,H) eq 1};
mdeg  :=  Min ({Index(N,H):H in TrivCore});
Good   :=  {H:H in TrivCore | Index(N,H) eq mdeg};
#Good;
/* 2 */


H:= Rep(Good);
#H;
/* 4 */


f2,N1,K2:= CosetAction(N,H);
N1;
/*
Permutation group N1 acting on a set of cardinality 8
Order = 32 = 2^5
    (1, 2, 5, 4, 3, 6, 8, 7)
    (1, 3)(5, 8)
    (1, 4)(2, 5)(3, 7)(6, 8)
*/
CompositionFactors(N);
```

```
/*
  G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1


*/
NL:= NormalLattice(N);
NL;
/*
Normal subgroup lattice
————————————————————————————

[20]   Order 32   Length 1   Maximal Subgroups: 13 14 15 16 17
    18 19

———

[19]   Order 16   Length 1   Maximal Subgroups: 6 8 10
[18]   Order 16   Length 1   Maximal Subgroups: 8 9 12
[17]   Order 16   Length 1   Maximal Subgroups: 7 10 12
[16]   Order 16   Length 1   Maximal Subgroups: 6 7 9
[15]   Order 16   Length 1   Maximal Subgroups: 6 11 12
[14]   Order 16   Length 1   Maximal Subgroups: 9 10 11
[13]   Order 16   Length 1   Maximal Subgroups: 7 8 11

———
```

```
[12]    Order 8    Length 1   Maximal Subgroups: 3
[11]    Order 8    Length 1   Maximal Subgroups: 3 4 5
[10]    Order 8    Length 1   Maximal Subgroups: 3
[ 9]    Order 8    Length 1   Maximal Subgroups: 3
[ 8]    Order 8    Length 1   Maximal Subgroups: 3
[ 7]    Order 8    Length 1   Maximal Subgroups: 3
[ 6]    Order 8    Length 1   Maximal Subgroups: 3
____

[ 5]    Order 4    Length 1   Maximal Subgroups: 2
[ 4]    Order 4    Length 1   Maximal Subgroups: 2
[ 3]    Order 4    Length 1   Maximal Subgroups: 2
____

[ 2]    Order 2    Length 1   Maximal Subgroups: 1
____

[ 1]    Order 1    Length 1   Maximal Subgroups:

*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if;
   end for;
/*
1
2
3
4
5
9
10
11
Choose the largest i in this case 11

Checking these because order of NL[11] is 8:
```

```
8
4*2
2*2*2
*/

IsIsomorphic(NL[11], AbelianGroup(GrpPerm,[8]));

IsIsomorphic(NL[11], AbelianGroup(GrpPerm,[4,2]));


Generators(NL[11]);
/*
 (1, 5)(3, 7),
    (1, 7, 5, 3)(2, 8, 6, 4),
    (1, 5)(2, 6)(3, 7)(4, 8)
*/
A:=N!((1, 5)(3, 7));

B:=N!((1, 7, 5, 3)(2, 8, 6, 4));

/*
Notice that B^2 = (1, 5)(2, 6)(3, 7)(4, 8)

A,B is the generator of the largest abelian */
NL11:=sub<N|A,B>;
/* Check that NL[11] = NL11*/
NL[11] eq NL11;
/*
True
*/
Order(NL11);
```

```
/* 8 = 2^3 */
Order (N);
/* 32 */


Order (N)/Order(NL11);
/* 4 this is the order of q */
q,ff:=quo<N|NL11>;
T:=Transversal(N,NL11);
#T;
/* 4 */
q;
/*
Permutation group q acting on a set of cardinality 4
Order = 4 = 2^2
    (1, 2)(3, 4)
    Id(q)
    (1, 3)(2, 4)
*/
ff(T[2]) eq q.1;
/* q.2 is just the identity */
ff(T[3]) eq q.3;


FPGroup(q);
/*
Finitely presented group on 3 generators
Relations
    $.1^2 = Id($) --> c^2
    $.3^2 = Id($) --> d^2
    ($.1 * $.3)^2 = Id($) --> (c * d)^2
```

```
    $.2 = Id($) --> c ( Ident so dont use it )
Writing the presenation:
G<a,b,c,d>:=Group<a,b,c,d| a^2,b^4,(a,b),c^2, d^2, (c * d)
   ^2,
a^c =, a^d=,b^c=,b^d=>;
*/


for i in [1..2] do for j in [1..4] do
if A^T[2] eq A^i*B^j then i,j; end if; end for; end for;
/* 1 2 */



for i in [1..2] do for j in [1..4] do
if A^T[3] eq A^i*B^j then i,j; end if; end for; end for;
/* 1 2 */


for i in [1..2] do for j in [1..4] do
if B^T[2] eq A^i*B^j then i,j; end if; end for; end for;
/* 2 1 */


for i in [1..2] do for j in [1..4] do
if B^T[3] eq A^i*B^j then i,j; end if; end for; end for;
/* 2 3 */


G<a,b,c,d>:=Group<a,b,c,d| a^2,b^4,(a,b),c^2, d^2, (c * d)
   ^2,
a^c = a*b^2, a^d=a*b^2, b^c=a^2*b, b^d=a^2*b^3>;
#G;


f,G1,k:=CosetAction(G,sub<G|Id(G)>);
s:=IsIsomorphic(N,G1);
```

```
s;
/* We got false so we need to need to check the
Order (T[2],T[3],(T[2]*T[3])) */

for i in [1..2] do for j in [1..4] do
if T[2]^2 eq A^i*B^j then i,j; end if; end for; end for;
/* 2 3*/

G<a,b,c,d>:=Group<a,b,c,d| a^2,b^4,(a,b),c^2=a^2*b^3, d^2,
    (c * d)^2,
a^c = a*b^2, a^d=a*b^2,b^c=a^2*b,b^d=a^2*b^3>;
#G;
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
s:=IsIsomorphic(N,G1);
s;

/* This is a Mixed extension of (4x2):2^2 */
```

6.7 Mixed Extension Product of $(4\mathrm{x}2 : 2^2), N = S_8, T[15]$

```
/* T:=TransitiveGroup(12,124); */
T:=TransitiveGroup(12,124);
#T;
/* 240 */
N:=TransitiveGroup(12,124);
#N;
/* 240 */
N;/* Has 3 generators so use the for loop below to see if
   we can shorten it */
/*
for g,h in N do if sub<N|g,h> eq N then A:=g;B:=h; end if;
   end for;
> A;
(1, 6, 2)(3, 12, 7)(4, 9, 11)(5, 8, 10)
> B;
(1, 12)(4, 8, 10, 6)(5, 9, 11, 7)

*/
S:=Sym(12);
xx:=S!(1, 6, 2)(3, 12, 7)(4, 9, 11)(5, 8, 10);
yy:=S!(1, 12)(4, 8, 10, 6)(5, 9, 11, 7);
N:=sub<S|xx,yy>;
#N;
/* 240 */
N12:=Stabiliser(N,[1,2]);
/*
Permutation group N12 acting on a set of cardinality 12
Order = 2
    (4, 10)(5, 11)(6, 8)(7, 9)
```

```
*/
C:=Centraliser(N,N12);
Set(C);
/*
    (1, 3, 12, 2)(4, 8, 5, 9)(6, 11, 7, 10),
    (1, 3, 12, 2)(4, 6, 5, 7)(8, 11, 9, 10),
    (2, 3)(4, 7, 10, 9)(5, 6, 11, 8),
    (2, 3)(4, 9, 10, 7)(5, 8, 11, 6),
    (1, 12)(4, 6, 10, 8)(5, 7, 11, 9),
    (1, 12)(2, 3)(4, 11)(5, 10)(6, 9)(7, 8),
    (1, 12)(4, 8, 10, 6)(5, 9, 11, 7),
    (1, 2)(3, 12)(4, 5)(6, 8)(7, 9)(10, 11),
    Id(C),
    (1, 3)(2, 12)(6, 9)(7, 8),
    (1, 12)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11),
    (1, 2, 12, 3)(4, 7, 5, 6)(8, 10, 9, 11),
    (1, 2, 12, 3)(4, 9, 5, 8)(6, 10, 7, 11),
    (1, 3)(2, 12)(4, 10)(5, 11)(6, 7)(8, 9),
    (1, 2)(3, 12)(4, 11)(5, 10),
    (4, 10)(5, 11)(6, 8)(7, 9)

*/

/*
This code is only needed when the set output is too large,
    it gives you what you need only
for n in C do
if ((1^n eq 1 and 2^n eq 2) or (1^n eq 2 and 2^n eq 1)) and
    Order(n) eq 2 then n;
end if; end for;
```

```
*/

FPGroup(N);
/*

Finitely presented group on 2 generators
Relations
    $.1^3 = Id($)
    $.2^4 = Id($)
    ($.1^-1, $.2^-1)^2 = Id($)
    ($.1 * $.2^-1)^4 = Id($)

 x^3 = Id($)
    y^4 = Id($)
    (x^-1, y^-1)^2 = Id($)
    (x * y^-1)^4 = Id($)

*/

NN<x,y>:=Group<x,y | x^3,y^4,(x^-1, y^-1)^2,(x * y^-1)^4 >;
#NN;

*/
/* #N/ #N1 */
S:=Sym(12);
xx:=S!(1, 6, 2)(3, 12, 7)(4, 9, 11)(5, 8, 10);
yy:=S!(1, 12)(4, 8, 10, 6)(5, 9, 11, 7);
N:=sub<S|xx,yy>;
#N;

G:=NN; G1:=N;
```

```
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(G1): i in [1..#G1]];
for i in [2..#G1] do
P:=[Id(G1): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
N1:=Stabiliser(N,1);
N1;
/*
Permutation group N1 acting on a set of cardinality 12
Order = 20 = 2^2 * 5
    (4, 10)(5, 11)(6, 8)(7, 9)
    (2, 7, 4, 11)(3, 6, 5, 10)(8, 9)

*/
for i in [1..#N] do if ArrayP[i] eq N!(4, 10)(5, 11)(6, 8)
   (7, 9) then Sch[i]; end if; end for;
/* y^2 */
for i in [1..#N] do if ArrayP[i] eq N!(2, 7, 4, 11)(3, 6,
   5, 10)(8, 9) then Sch[i]; end if; end for;
/*y^-1 * x^-1 * y^-1 * x * y^-1 */
```

```
/* FPGroup and Stab included

G<x,y,t>:=Group<x,y,t|x^3,y^4,(x^-1, y^-1)^2,(x * y^-1)^4,
    t^2, (t,y^2),(t,y^-1 * x^-1 * y^-1 * x * y^-1)>;

*/
Orbits(N1);
/*
[
    GSet{@ 1 @},
    GSet{@ 12 @},
    GSet{@ 2, 7, 9, 4, 8, 10, 11, 6, 3, 5 @}

*/

for n in C do
if ((1^n eq 1 and 2^n eq 2) or (1^n eq 2 and 2^n eq 1)) and
    Order(n) eq 2 then n;
end if; end for;
/*
(4, 10)(5, 11)(6, 8)(7, 9) --> y^2 = (t*t^(x^2))^k
(1, 2)(3, 12)(4, 11)(5, 10) --> (x * y^-1 * x^-1 * y^2 * x
    * y*t)^i
(1, 2)(3, 12)(4, 5)(6, 8)(7, 9)(10, 11) --> (x * y^-1 * x
    ^-1 * y^2 * x * y^-1*t)^j
*/
for i in [1..#N] do if ArrayP[i] eq N!(4, 10)(5, 11)(6, 8)
    (7, 9) then Sch[i]; end if; end for;
for i in [1..#N] do if ArrayP[i] eq N!(1, 2)(3, 12)(4, 11)
    (5, 10) then Sch[i]; end if; end for;
```

```
for i in [1..#N] do if ArrayP[i] eq N!(1, 2)(3, 12)(4, 5)
    (6, 8)(7, 9)(10, 11) then Sch[i]; end if; end for;
```

```
for k in [2..10 by 2] do
```

```
G<x,y,t>:=Group<x,y,t|x^3,y^4,(x^-1, y^-1)^2,(x * y^-1)^4,
    t^2, (t,y^2),(t,y^-1 * x^-1 * y^-1 * x * y^-1),y^2 = (t*t
    ^(x^2))^k>; k, #G; end for;
```

```
/* Results
> for k in [2..10 by 2] do
for>
for> G<x,y,t>:=Group<x,y,t|x^3,y^4,(x^-1, y^-1)^2,(x * y
    ^-1)^4, t^2, (t,y^2),(t,y^\
-1 * x^-1 * y^-1 * x * y^-1),y^2 = (t*t^(x^2))^k>; k, #G;
    end for;
2 4320
4 0
6 0
8 0
10 0

*/
```

```
G<x,y,t>:=Group<x,y,t|x^3,y^4,(x^-1, y^-1)^2,(x * y^-1)^4,
    t^2, (t,y^2),(t,y^-1 * x^-1 * y^-1 * x * y^-1),y^2 = (t*t
    ^(x^2))^2>; #G;
```

```
/* FOR */
S:=Sym(12);
xx:=S!(1, 6, 2)(3, 12, 7)(4, 9, 11)(5, 8, 10);
yy:=S!(1, 12)(4, 8, 10, 6)(5, 9, 11, 7);
N:=sub<S|xx,yy>;
#N;
C:=Classes(N);
#C;
FPGroup(N);
NN<x,y>:=Group<x,y |x^3,y^4,(x^-1, y^-1)^2,(x * y^-1)^4 >;
#NN;

Sch:=SchreierSystem(NN,sub<NN| Id(NN)>);
 ArrayP:=[Id(N): i in [1.. #N]];
 for i in [2.. #N] do P:=[Id(N): l in [1.. #Sch[i]]];
 for j in [1..# Sch[i]] do
 if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
 PP:=Id(N);
 for k in [1.. #P] do
 PP:=PP*P[k]; end for;
 ArrayP[i]:=PP;
 end for; end for;

Orbits(N);
/*
```

```
[
    GSet{@ 1, 6, 12, 2, 4, 7, 9, 8, 3, 5, 11, 10 @}
]

for i in [1..#N] do if 1^ArrayP[i] eq 2 then Sch[i]; break;
  end if; end for;
/* x^-1 */
 for i in [1..#N] do if 1^ArrayP[i] eq 3 then Sch[i]; break
    ;end if; end for;
/* y* x^-1 */
 for i in [1..#N] do if 1^ArrayP[i] eq 4 then Sch[i]; break
    ;end if; end for;
/* x * y */
for i in [1..#N] do if 1^ArrayP[i] eq 5 then Sch[i]; break;
  end if; end for;
/* x * y^-1 * x */
 for i in [1..#N] do if 1^ArrayP[i] eq 6 then Sch[i]; break
    ;end if; end for;
/* x */
for i in [1..#N] do if 1^ArrayP[i] eq 7 then Sch[i]; break;
  end if; end for;
/* y * x */
for i in [1..#N] do if 1^ArrayP[i] eq 8 then Sch[i]; break;
  end if; end for;


*/
for i in [2..#C] do i, C[i][3]; for j in [1..#N] do
 if ArrayP[j] eq C[i][3]
 then Sch[j]; end if; end for;
 Orbits(Centraliser(N,C[i][3])); end for;
/*
```

2 (1, 12)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)
(x * y * x * y^2)^2
[
    GSet{@ 1, 6, 12, 2, 4, 7, 9, 8, 3, 5, 11, 10 @}
]
(((x * y * x * y^2)^2)*t),

3 (1, 4)(2, 7)(3, 6)(5, 12)
(x^-1 * y)^2
[
    GSet{@ 8, 11, 10, 9 @},
    GSet{@ 1, 5, 4, 3, 12, 7, 6, 2 @}
]
(((x^-1 * y)^2)*t^(x * y^2)),(((x^-1 * y)^2)*t),

4 (1, 12)(2, 5)(3, 4)(6, 11)(7, 10)(8, 9)
(y^-1, x)
[
    GSet{@ 1, 8, 12, 9 @},
    GSet{@ 2, 5, 4, 10, 3, 7, 6, 11 @}
]
(((y^-1, x))*t),(((y^-1, x))*t^(x^-1)),

5 (1, 6, 2)(3, 12, 7)(4, 9, 11)(5, 8, 10)
x
[
    GSet{@ 1, 6, 7, 4, 2, 3, 9, 8, 12, 11, 10, 5 @}
]
((x)*t),

6 (1, 8, 12, 9)(2, 10, 3, 11)(4, 6, 5, 7)

(x ∗ y)^3
[
  GSet{@ 1, 6, 8, 3, 5, 12, 11, 7, 9, 2, 4, 10 @}
]
(((x ∗ y)^3)∗t),

7 (1, 9, 12, 8)(2, 11, 3, 10)(4, 7, 5, 6)
x ∗ y^−1 ∗ x^−1 ∗ y^−1 ∗ x^−1 ∗ y^−1
[
  GSet{@ 1, 6, 9, 3, 4, 5, 12, 10, 7, 8, 2, 11 @}
]
((x ∗ y^−1 ∗ x^−1 ∗ y^−1 ∗ x^−1 ∗ y^−1)∗t)

8 (1, 2, 4, 7)(3, 5, 6, 12)(8, 9)
x^−1 ∗ y
[
  GSet{@ 8, 9 @},
  GSet{@ 10, 11 @},
  GSet{@ 1, 2, 6, 4, 12, 7, 3, 5 @}
]
((x^−1 ∗ y)∗t^(x ∗ y^2)),((x^−1 ∗ y)∗t^(x ∗ y^−1)),((x^−1 ∗
  y)∗t),

9 (1, 7, 4, 2)(3, 12, 6, 5)(8, 9)
y^−1 ∗ x
[
  GSet{@ 8, 9 @},
  GSet{@ 10, 11 @},
  GSet{@ 1, 7, 6, 4, 5, 2, 3, 12 @}
]
((y^−1 ∗ x)∗t^(x ∗ y^2)),((y^−1 ∗ x)∗t^(x ∗ y^−1)),((y^−1 ∗

x)∗t) ,

10 (1 , 8 , 7 , 3 , 10)(2 , 11 , 12 , 9 , 6)
(xˆ−1 ∗ yˆ2)ˆ2
[
    GSet{@ 4 , 5 @} ,
    GSet{@ 1 , 8 , 9 , 7 , 6 , 3 , 2 , 10 , 11 , 12 @}
]
(((xˆ−1 ∗ yˆ2)ˆ2)∗t ˆ(x ∗ y)) ,(((xˆ−1 ∗ yˆ2)ˆ2)∗t) ,

11 (1 , 11 , 6 , 12 , 10 , 7)(2 , 5 , 9 , 3 , 4 , 8)
(x ∗ y)ˆ2
[
    GSet{@ 1 , 11 , 4 , 6 , 8 , 12 , 2 , 10 , 5 , 7 , 9 , 3 @}
]
12 (1 , 2 , 8 , 11 , 7 , 12 , 3 , 9 , 10 , 6)(4 , 5)
xˆ−1 ∗ yˆ2
[
    GSet{@ 4 , 5 @} ,
    GSet{@ 1 , 2 , 8 , 11 , 7 , 12 , 3 , 9 , 10 , 6 @}
]
13 (1 , 4 , 11 , 8 , 6 , 2 , 12 , 5 , 10 , 9 , 7 , 3)
x ∗ y
[
    GSet{@ 1 , 4 , 11 , 8 , 6 , 2 , 12 , 5 , 10 , 9 , 7 , 3 @}
]
14 (1 , 5 , 11 , 9 , 6 , 3 , 12 , 4 , 10 , 8 , 7 , 2)
y ∗ x ∗ y ∗ xˆ−1 ∗ yˆ2 ∗ x ∗ yˆ−1
[
    GSet{@ 1 , 5 , 11 , 9 , 6 , 3 , 12 , 4 , 10 , 8 , 7 , 2 @}
]

```
*/

/* FOR until 10

(((x * y * x * y^2)^2)*t),
(((x^-1 * y)^2)*t^(x * y^2)),
(((x^-1 * y)^2)*t),
(((y^-1, x))*t),
(((y^-1, x))*t^(x^-1)),
((x)*t),
(((x * y)^3)*t),
((x * y^-1 * x^-1 * y^-1 * x^-1 * y^-1)*t),
((x^-1 * y)*t^(x * y^2)),((x^-1 * y)*t^(x * y^-1)),((x^-1 *
    y)*t),((y^-1 * x)*t^(x * y^2)),((y^-1 * x)*t^(x * y^-1))
    ,((y^-1 * x)*t),(((x^-1 * y^2)^2)*t^(x * y)),(((x^-1 * y
    ^2)^2)*t),



*/
for a,b,c,d,e,f,g,h,i,j in [0..10] do
G<x,y,t>:=Group<x,y,t|x^3,y^4,(x^-1, y^-1)^2,(x * y^-1)^4,

t^2, (t,y^2),(t,y^-1 * x^-1 * y^-1 * x * y^-1),

(((((x * y * x * y^2)^2)*t))^a,(((((x^-1 * y)^2)*t^(x * y^2))
    ))^b,(((((x^-1 * y)^2)*t))^c,(((((y^-1, x))*t))^d,
((y^-1, x)*t^(x^-1))^e,(((x)*t))^f,(((((x * y)^3)*t))^g,(((x
    * y^-1 * x^-1 * y^-1 * x^-1 * y^-1)*t))^h,y^2 = (t*t^(x
    ^2))^2,(x * y^-1 * x^-1 * y^2 * x * y*t)^i,(x * y^-1 * x
```

```
^-1 * y^2 * x * y^-1*t )^j >;
a,b,c,d,e,f,g,h,i,j, Index(G,sub<G|x,y>);  end for;
```

```
/* Running it in the background */
```
1. nano Proj_240
2. Insert data
3. Ctrl + x
4. Choose yes
5. Enter to save file

If we want to check the input file 7star5:180 type nano 7
star5:180

```
nohup magma "Proj_240"&>Proj_240.out&
```

After the above command has been run if we want to check
output(or whether the program is running). We use the
command nano Proj_192.out

```
/* Checking Status of Progenitor */
nano Proj_192.out
```

Bibliography

[BC94] W. Bosma and J. Cannon. Handbook of MAGMA Functions. University
of Sydney, 1994.

[Cur07] Robert Curtis. Symmetric Generation of Groups. volume 111 of
Encyclopedia of Mathematics and its Applications.
Cambridge University Press, Cambridge,2007.

[Gri15] Dustin Grindsta. Symmetric Presentations and Generation. CSUSB, 2015.

[Hum96] John F. Humphreys. A course in Group Theory. Oxford University Press,
1996.

[Isa76] I. Martin Isaacs. Character Theory of Finite Groups. Academic Press,
New York, 1976.

[Lam04] AlNazzal Lamies. Homomorphic images of semi-direct products. CSUSB,
2004.

[Rot95] Joseph J. Rotman. An Introduction to the Theory of Groups, volume
148 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[WB99] Robert Wilson and John Bray. Atlas of finite group representations.
http://brauer.maths.qmul.ac.uk/Atlas/v3/, 1999.