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SYMMETRIC PRESENTATIONS OF FINITE GROUPS AND RELATED TOPICS

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Symmetric Presentations of Finite Groups and Related Topics

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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ABSTRACT

A progenitor is an infinite semi-direct product of the form $m^{*n} : N$, where $N \leq S_n$ and $m^{*n} : N$ is a free product of n copies of a cyclic group of order m. A progenitor of this type, in particular $2^{*n} : N$, gives finite non-abelian simple groups and groups involving these, including alternating groups, classical groups, and the sporadic group. We have conducted a systematic search of finite homomorphic images of numerous progenitors. In this thesis we have presented original symmetric presentations of the sporadic simple groups, M_{12}, J_1 as homomorphic images of the progenitor $2^{*12} : (2 \times A_5)$, M_{22} and $M_{22} : 2$ as homomorphic images of $2^{*14} : (2^3 : 7)$ and J_2 as a homomorphic image of $2^{*160} : PSL(2, 11)$. We have also given original symmetric presentations of a number of alternating and classical groups and symmetric groups such as $PSL(2, 7)$, $PSL(2, 19)$, $PSL(2, 41)$, $PSL(2, 8)$, A_8, S_7, and S_8. We have also searched for finite homomorphic images of the monomial progenitor: $2^3 : 3 :_m 2^3 : A_6$ and found the original symmetric presentations of the image $2^6 : Sym(5)$. We construct the following images by using our technique of double coset enumeration: $3^4 : (2^2 : S_3)$ over $(3^3 : (3 : 2))$, $J1$ over $(2 \times A_5)$, $(2^4) : (S_5 : 2)$ over S_5, $7^3 : S_3 : 2$ over $(21 \times (S_6)$, $2^7 : PSL(2, 7)$ over $PSL(2, 7)$, and $2^7 : (7 : 3)$ over $(2^3 : (7 : 3)$. In addition, we give isomorphism class of each image that we have discovered.

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Chapter 1

Introduction

Some of the most beautiful mathematical objects are the sporadic simple groups, but gaining familiarity with these groups presents problems for two reasons. Firstly, they are discovered in many different ways, so to understand their constructions in depth, one needs to study many different techniques. Secondly, since each of them is, in a sense, recording some exceptional symmetry in space of certain dimensions, they are by their nature highly complicated objects with a rich underlying combinatorial structure [Cur07].

The main purpose of this thesis is to demonstrate a concise but informative alternative method for representing group elements of finite simple groups. A finite simple group is any group G, having a finite number of elements that does not have any normal subgroups except for the trivial group and G itself [Rot94]. This method will be useful for the sporadic groups, the 26 exceptional groups found in the classification of finite simple groups. Five of the sporadic groups were discovered by Mathieu in the 1860s and the other 21 were found between 1965 and 1975. Several of these groups were predicted to exist before they were constructed. Most of the groups are named after the mathematician(s) who first predicted their existence [Cur07].

In order to determine the original symmetric presentations of some of these groups, we will search various progenitors, including permutation, wreath product, and monomial progenitors, and to do this we first need to know some important definitions. To find finite images of any group, we add additional relations to the semi-direct product. In an introduction to the theory of groups, Alexandroff uses the example of three people: David, John, and Peter who are sitting in an order from left to right. They can regroup

in six different ways, and the change from one seating-arrangement to another is called a permutation [Ale54]. There are two methods that are commonly used to represent elements of finite groups as permutations: left and right regular representation. In left regular representation of G, the degree of the permutation representation is $|G|$. In this representation, if $|G| = 1000$ then G is a permutation of 1000 letters. According to the left regular representation, every group of order n is isomorphic to a subgroup of S_n. On the other hand, to represent S_n on the right coset H, every element of G can be written as nw, where n is a permutation on n letters and w is a word in at most n letters.

We will use manual double coset enumeration to construct these images and show by hand that the images are finite, then, if G is an image of a progenitor with control group N, then $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$, where g_i's are words in t_i's. Now N acts on the right cosets of N in G by conjugation. If a progenitor is generated by $\langle x, y, t \rangle$ then we can say $G \cong \langle f_x, f_y, f_t \rangle$ [Has06]. We will decompose the group G as right cosets of N in G, then we need to find all double cosets $[w]$ and find out how many single cosets each of them contain, where $[w] = \{Nw^n | n \in N\}$.

This thesis presents our discovery of some original symmetric presentations and constructions of important finite groups. In particular, in the second chapter, we provide some main definitions that will help the reader to grasp our work. The construction by hand of wreath product of Z_3 by S_3 and the magma verification is in chapter three. We include some symmetric presentations in chapter four such as, 2^{10} on $(5^2 : 2^2)$, 2^{10} on $Sym(6)$, 2^{12} on $(2^5 : S_3)$, and 2^{24} on $(2^5 : 3)$. In chapter five, we write a linear map that generates $PSL(2, 23)$ using α, β, γ. Then we compute the linear fractional maps and see what they give us using the induced permutations that magma gives us. In chapter six, we evaluate each image of the progenitor G that is noted to be faithful. There are four types of extension. Direct product such as $(A5 \times A5 \times A5 \times A5 \times A5) : ((2 \times A5) : 2)$ and $(A_5 x A_5) : 4$. Semi-direct product such as $5^2 : 4$ and $5^2 : 2^5$. Also, mixed extension such as $3 \dot{:} PGL(2 \times 9)$ and $5^5 : (2^7 : 5)$.

Chapter seven has many examples about constructing Double Coset Enumeration such as $3^4 : (2^2 : S_3)$ over $(3^3 : (3 : 2))$, $3^4 : (2^3 : S_3)$ over $(3^3 : (3 : 2))$, $5^3 : (2 \times A_5)$ over $A_5 \times 2$ $A_5 \times A_5$ over $A_5 \times 2$, $(2 : A_5 \times A_5)$ over $(A_5 \times 2)$, $J1$ over $(2 \times A_5)$, $(3 : 2^4) : (S_5 : 2)$ over S_5, $(2^4) : (S_5 : 2)$ over S_5, $(2^2 : 3) : (2^4 : (7 : 3))$ over $(21 \times (S_6)$, $(2^6 : S_6)(7 : 3)$ over $(21 \times (S_6))$, $7^3 : S_3 : 2$ over $(21 \times (S_6)$, $2^6 : PSL(2, 7)$

over $PSL(2,7)$, $2^7 : PSL(2,7)$ over $PSL(2,7)$. In the same chapter, we show how to use Lemma 3.3 in the process of constructing $2^{*7} : (7 : 3))$ over $(2^3 : (7 : 3)$. Chapter eight includes one example of double coset enumeration involving maximal subgroup, S_5 over $2^4 : A_4$ and $2^4 : 3$. In chapter nine, we discuss a different type of progenitors called the monomial progenitors. We use magma to view the character table of each group with the goal of writing the monomial representation of each group such as, $37^{*2} :_m (2^3 : S_3)$ and $37^{*2} :_m 2^3 : S_3$. We also include linear lifting characters of $5^{*6} :_m (G)$ and $5^{*10} :_m (G)$ in the same chapter. We list composition charts in chapter ten, and show the isomorphic images of some of our progenitors that mostly do not produce a good image.

In our last chapter, we include some unsuccessful progenitors. Although we wish that all of our progenitors produce images, we have some fail such as $Sym(84)$ and $Sym(160)$.

Chapter 2

Preliminary Information

Preliminary Definitions

Definition 2.1. *A **binary operation** on a nonempty set G is a function $\mu : G \times G \to G$* *[Rot94].*

Definition 2.2. *A **semi-group** $(G, *)$ is a nonempty set G equipped with an associative operation $(*)$ [Rot94].*

Definition 2.3. *A **group** is a semi-group G containing an element e such that*
(i) $e * a = a = a * e$, $\forall a \in G$
(ii) For every $a \in G$, there is an element $b \in G$, with $a * b = e = b * a$ [Rot94].

Definition 2.4. *A nonempty subset S of a group G, is a **subgroup** of G if $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$ [Rot94].*

Definition 2.5. *A subgroup $K \leq G$ is a **normal subgroup** denoted by $K \triangle G$. If $gKg^{-1} = K$ for every $g \in G$ [Rot94].*

Definition 2.6. *A subgroup $H \leq G$ is a **maximal normal subgroup** of G if there is no normal subgroup N of G with $H < N < G$ [Rot94].*

Definition 2.7. *If X is a subset of a group F, then F is a **free group** with basis X if, for every group G and every function $f : X \to G$, there exists a unique homomorphism $\phi : F \to G$ extending f [Rot94].*

Definition 2.8. *A group $G \neq 1$ is **Simple** if it has no normal subgroups other than G and 1 [Rot94].*

Definition 2.9. *Let $(G, *$ and $(H, \circ)$ be groups. A function $f : G \to H$ is a **homomorphism** if, for all $a, b \in G$, $f(a * b) = f(a) \circ f(b)$ [Rot94].*

Definition 2.10. *An **isomorphism** is a homomorphism that is also a bijection. We say that G is isomorphic to H, denoted by $G \cong H$, if there exists an isomorphism $f : G \to H$ [Rot94].*

Definition 2.11. *S_x is a group with composition as operation; it is called the **symmetric group** on X. When $X = \{1, 2, ..., n\}$, then S_x is denoted by S_n and it is called the symmetric group on n letters [Rot94].*

Definition 2.12. *If $a, b \in G$, the **commutator** of a and b, denoted by $[a, b]$, is $[a, b] = aba^{-1}b^{-1}$ [Rot94].*

Definition 2.13. *A pair of elements a and b in a semi-group commutes if $a * b = b * a$. A group (or a semi-group) is **abelian** if every pair of its elements commutes [Rot94].*

Definition 2.14. *If X is a set and G is a group, then X is a **G-set** if there is a function $\alpha : G \times X \to X$ called an **action** denoted by $\alpha : (g, x) \to gx$, such that:*
(i) $1x = x$ for all $x \in X$
(ii) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.
*One also says that G **acts** on X. If $|X| = n$, then n is called the **degree** of the $G - set$ X [Rot94].*

Definition 2.15. *If X is a $G - set$ and $x \in X$, then the **G-orbit** of x is $O(x) = \{gx : g \in G \subset X\}$. One often denotes the orbit $O(x)$ by Gx [Rot94].*

Definition 2.16. *A $G - set$ X is **transitive** if it has only one orbit; that is, for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$ [Rot94].*

Definition 2.17. *If S is a subgroup of G and if $t \in G$, then a **right coset** of S in G is $St = \{st | s \in S\}$. A **left coset** of S is $tS = \{ts | s \in S\}$. One calls t a **representative** of St and also tS [Rot94].*

Definition 2.18. A ***normal series*** of a group G is a sequence of subgroups $G = G_0 \geq G_1 \geq ... \geq G_n = 1$ in which $G_{i+1} \triangle G_i$ for all i. The ***factor groups*** of this normal series are the groups $\frac{G_i}{G_{i+1}}$ for $i = 0, 1, ..., n-1$ [Rot94].

Definition 2.19. A ***composition series*** is a normal series $G = G_0 \geq G_1 \geq ... \geq G_n = 1$, in which, for all i, either G_{i+1} is a maximal normal subgroup of G_i or $G_{i+1} = G_i$. The factor groups of a composition series are called the ***composition factors*** of that group [Rot94].

Definition 2.20. If $S \leq G$, then ***index*** of S in G, denoted by $[G : S]$, is the number of right cosets of S in G [Rot94].

Definition 2.21. If $H \leq G$ and $g \in G$, then the ***Conjugate*** gHg^{-1} is $\{ghg^{-1} : h \in H\}$. The conjugate gHg^{-1} is often denoted by H^g [Rot94].

Definition 2.22. If X is a $G-set$ and $x \in X$, then the ***Stabilizer*** of x, denoted by G_x is the subgroup $G_x = \{g \in G | gx = x\}$ [Rot94].

Definition 2.23. If X is a nonempty subset of a group G, then a ***Word*** on X is an element $\omega \in G$ of the form $\omega = x_1^{e_1} x_2^{e_2} ... x_n^{e_n}$ where $x_i \in X$, $e_i = \mp 1$, and $n \geq 1$ [Rot94].

Definition 2.24. Let ω be a word in the t_i's, then the (***Point-Stabilizer***): of $N^\omega = \{n \in N | \omega^n = \omega\}$ [Rot94].

Definition 2.25. Let ω be a word in the t_i's then $N^{(\omega)} = \{n \in N | N\omega^n = N\omega\}$ is called the ***Coset-Stabilizing group*** of the coset $N\omega$ [Rot94].

Definition 2.26. The infinite semi-direct product of the form $m^{*n} : N$, where m^{*n} is a free product of n cyclic groups generated by t_i, where $1 \leq i \leq n$, of order m is called ***Progenitor***. The elements of a progenitor can be written as $n\omega$, where $n \in N$ and ω is a word in the t_i's [Rot94].

Definition 2.27. If X is a nonempty set, a ***permutation*** of X is a bijection $\alpha : X \to X$. We denote the set of all permutations of X by S_x [Rot94]

Definition 2.28. Let G acts on X, then there exists a homomorphism $\alpha^n : G \to S_x$
$g \leftrightarrow \alpha \in S_x$ such that $\alpha x = gx$
α^n is called ***The Permutation Representation*** of G on X. $|x|$ is called ***the degree***

of the permutation representation.

Note: α^n is faithful if $\ker \alpha^n = 1$, also every group G has a permutation of degree $|G|$ [Rot94].

Definition 2.29. *__Magma__ is a mathematical computer program that is designed for computations in algebra, number theory and many other algebraic fields.*

Preliminary Theorems and Lemmas

Theorem 2.30. *__First Isomorphism Theorem - FIT__: Let $f : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong \mathrm{im}(f)$ [Rot94].*

Theorem 2.31. *__Lagrange's Theorem__: If G is a finite group and $S \leq G$, then $|S|$ divides $|G|$ and $[G : S] = \frac{|G|}{|S|}$ [Rot94].*

Theorem 2.32. *$\frac{2^{*n}:N}{t_i,t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$ [Rot94].*

Lemma 2.33. *__GrindStaff/ Factorng Lemma__: Factoring the progenitor $m^{*n} : N$ by (t_i, t_j) for $1 \leq i \leq j \leq n$ gives the group $m^n : N$ [Led77]*

Prove an Example of a Theorem $\frac{2^{*n}:S_n}{t_1t_2t_1=(1,2)} \cong S_{n+1}$

We have S_7, where $n = 7$ and is generated by $\langle (1,2,3,4,5,6,7),(1,2) \rangle$ where $x = (1,2,3,4,5,6,7)$ and $y = (1,2)$. By using magma we can find the following

```
S:=Sym(7);
xx:=S!(1,2,3,4,5,6,7);
yy:=S!(1,2);
N:=sub<S|xx,yy>;
N1:=Stabiliser(N,1);
N1;
/*
Permutation group N1 acting on a set of cardinality 7
Order = 720 = 2^4 * 3^2 * 5
    (2, 3)
    (3, 4)
    (4, 5)
    (5, 6)
    (6, 7)
*/
```

Then we try to find which relations can give us these permutations. We check,

$yy^{xx} = (2,3)$

$yy^{(xx^2)} = (3,4)$

$yy^{(xx^3)} = (4,5)$

$yy^{(xx^4)} = (5,6)$

$yy^{(xx^5)} = (6,7)$

Then we check if these permutations are equal to $N1$ by using the command:

```
N1 eq sub<N|yy^(xx),yy^(xx^2),yy^(xx^3),yy^(xx^4),yy^(xx^5)>;
true
```

We start writing the group by using the command $FPGroup(N)$ which gives:

```
NN<a,b>:=Group<a,b|a^7,b^2,
(a * b * a^-2 * b * a)^2,
(a^-1 * b)^6, (a^-1 * b * a * b)^3,
#NN;
5040
```

Then to test the progenitor by adding some relations after replacing a by x and b by y and put t to the order of the group which is 2 in this case.

```
 G<x,y,t>:=Group<x,y,t|x^7,y^2,(x * y * x^-2 * y * x)^2,
(x^-1 * y)^6, (x^-1 * y * x * y)^3,
t^2,
(t,(y^(x))),(t,y^(x^2)),(t,y^(x^3)),(t,y^(x^4)),(t,y^(x^5)),
(t,t^y)>;
#G;
645120
Index(G,sub<G|x,y>);
128
```

The order of G must be $|2^7 : S_7| = 5040 \times 128 = 645120$. Thus our progenitor is good. Now we need to find the DCE of G over N where $G = \frac{2^{(*7):S_8}}{t_1 t_2 t_1 = (12}$ and $N = S_8$

First Double Coset [*]

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of the right coset in N is $\frac{|N|}{|N|} = \frac{5040}{5040} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7\}$, we need only to determine the double coset of the right coset Nt_1.

Thus seven t_i's extend the double coset [*] to the double coset $Nt_1N = [1]$.
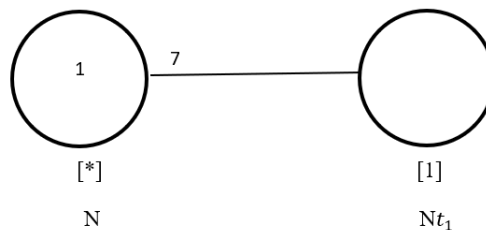
Cayley Diagram

Figure 2.1: Cayley Diagram for G over S_7

Second Double Coset [1]

$[1] = Nt_1N = \{N(t_1)^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7\}$.

Note that the point stabiliser N^1 are the elements that fix 1, and the coset stabiliser $N^{(1)}$

are the elements that fix Nt_1. In this case $N^{(1)} = \langle e, (2,3,4,5,6,7) \rangle$.

This shows that there are 7 single cosets in the double coset Nt_1.

The number of distinct right cosets is calculated by the formula:

$\frac{|N|}{|N^{(1)}|} = \frac{5040}{720} = 7$

Nt_1t_2 belongs to [1] since our relation is $Nt_1t_2t_1 = Nt(12) \implies Nt_1t_2 = N(12)t1 \in [1]$.

Cayley Diagram
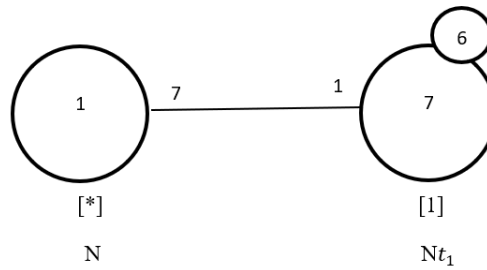
Figure 2.2: Cayley Diagram for G over S_7

We have $|G| \geq |1 \times |N| + 7 \times |N| = 1 \times 5040 + 7 \times 5040$.

G acts on $X = \{N, Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7\}$,

and lets us find permutation representation of $< x, y, t >$.

Conjugate with x

$x = (1, 2, 3, 4, 5, 6, 7)$

Conjugate by X				
Labeling	Element	Conjugate x	Element	Labeling
(1)	N	$N^{(1,2,3,4,5,6,7)}$	N	(1)
(2)	Nt_1	$Nt_1^{(1,2,3,4,5,6,7)}$	Nt_2	(3)
(3)	Nt_2	$Nt_2^{(1,2,3,4,5,6,7)}$	Nt_3	(4)
(4)	Nt_3	$Nt_3^{(1,2,3,4,5,6,7)}$	Nt_4	(5)
(5)	Nt_4	$Nt_4^{(1,2,3,4,5,6,7)}$	Nt_5	(6)
(6)	Nt_5	$Nt_5^{(1,2,3,4,5,6,7)}$	Nt_6	(7)
(7)	Nt_6	$Nt_6^{(1,2,3,4,5,6,7)}$	Nt_7	(8)
(8)	Nt_7	$Nt_7^{(1,2,3,4,5,6,7)}$	Nt_1	(2)

Table 2.1: Conjugation of Nt by x

which leads us to $f(x) \sim (2, 3, 4, 5, 6, 7, 8)$.

Conjugate with y

$y = (1, 2)$

Conjugate by y				
Labeling	Element	Conjugate y	Element	Labeling
(1)	N	$N^{(1,2)}$	N	(1)
(2)	Nt_1	$Nt_1^{(1,2)}$	Nt_2	(3)
(3)	Nt_2	$Nt_2^{(1,2)}$	Nt_1	(2)
(4)	Nt_3	$Nt_3^{(1,2)}$	Nt_3	(4)
(5)	Nt_4	$Nt_4^{(1,2)}$	Nt_4	(5)
(6)	Nt_5	$Nt_5^{(1,2)}$	Nt_5	(6)
(7)	Nt_6	$Nt_6^{(1,2)}$	Nt_6	(7)
(8)	Nt_7	$Nt_7^{(1,2)}$	Nt_7	(8)

Table 2.2: Conjugation of Nt by y

thus $f(y) \sim (2, 3)$.

Before the conjugating table of t is computed, we will conjugate our relation $t_1 t_2 t_1 = (1, 2)$

by the following permutations $(1,2),(2,3),(2,4),(2,5),(2,6),(2,7)$.

Which gives us:

$\alpha(1,2):(t_1t_2t_1=(1,2))^{(1,2)}=t_2t_1t_2=(2,1)$

$\beta(2,3):(t_1t_2t_1=(1,2))^{(2,3)}=t_1t_3t_1=(1,3)$

$\gamma(2,4):(t_1t_2t_1=(1,2))^{(2,4)}=t_1t_4t_1=(1,4)$

$\delta(2,5):(t_1t_2t_1=(1,2))^{(2,5)}=t_1t_5t_1=(1,5)$

$\zeta(2,6):(t_1t_2t_1=(1,2))^{(2,6)}=t_1t_6t_1=(1,6)$

$\eta(2,7):(t_1t_2t_1=(1,2))^{(2,7)}=t_1t_7t_1=(1,7)$

<u>Conjugate with t</u>

$t=t_1$

Conjugate by t_1				
Labeling	Element	Conjugate t_1	Element	Labeling
(1)	N	$N*t_1$	Nt_1	(2)
(2)	Nt_1	Nt_1*t_1	$Nt_1^2=N$	(1)
(3)	Nt_2	Nt_2*t_1	Nt_2t_1 by α	(3)
(4)	Nt_3	Nt_3*t_1	$Nt_3t_1$by β	(4)
(5)	Nt_4	Nt_4*t_1	Nt_4t_1 by γ	(5)
(6)	Nt_5	Nt_5*t_1	Nt_5t_1 by δ	(6)
(7)	Nt_6	Nt_6*t_1	Nt_6t_1 by ζ	(7)
(8)	Nt_7	Nt_7*t_1	Nt_7t_1 by η	(8)

Table 2.3: Conjugation of Nt by t_1

Thus $f(t)\sim(1,2)$

$G<x,y,t>$ is homomorphic to permutation representation $f(x),f(y),f(t)$. $\frac{G}{ker\phi}$ is homomorphic to $\langle f(x),f(y),f(t)\rangle=\langle(2,3,4,5,6,7,8),(2,3),(1,2)\rangle$ order 40320, where $\frac{|G|}{|ker\phi|}=40320$, which means $|G|=5040|ker\phi|$, which means $|G|\geq40320$ but Cayley shows that $|G|\leq40320$. Therefore, $|G|=40320$. Now G is homomorphic to $\langle f(x),f(y),f(t)\rangle$ and

$f(t)=(1,2)$,

$f(t^x)=((f(t)^{f(x)})=(12)^{(2,3,4,5,6,7,8)}=(13)$,

$f(t^{(x^2)})=(f(t)^{f(x^2)})=(12)^{(2,4,6,8,3,5,7)}=(14)$,

$$f(t^{(x^3)}) = (f(t)^{f(x^3)}) = (12)^{(2,5,8,4,7,3,6)} = (15),$$

$$f(t^{(x^4)}) = (f(t)^{f(x^4)}) = (12)^{(2,6,3,7,4,8,5)} = (16),$$

$$f(t^{(x^5)}) = (f(t)^{f(x^5)}) = (12)^{(2,7,5,3,8,6,4)} = (17),$$

$$f(t^{(x^6)}) = (f(t)^{f(x^6)}) = (12)^{(2,8,7,6,5,4,3)} = (18).$$

$$f(t^{(x^7)}) = (f(t)^{f(x^7)}) = (12)^{(Id)} = (12).$$

$|\langle f(x), f(y), f(t) \rangle| = 40320$. But $f(t), f(t^x), f(t^{x^2}) \in \langle f(x), f(y), f(t) \rangle$

$\implies (12), (13), (14), (15), (16), (17), (18) \in \langle f(x), f(y), f(t) \rangle$

$\implies S_8$ belongs to $\langle f(x), f(y), f(t) \rangle$.

Using magma we confirm the following:

```
S:=Sym(8);
f(x):=S!(2,3,4,5,6,7,8);
f(y):=S!(2,3);
f(t):=S!(1,2);
G1:=sub<S|fx,fy,ft>;
#G1 eq Factorial(8);
true

sub<G1|(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8)> eq G1;
true

IsIsomorphic(G1,Sym(8));
true
```

We conclude that $\frac{2^{*n}:S_n}{t_1 t_2 t_1 = (1,2)} \cong S_{n+1}$

Chapter 3

Wreath Product

3.1 Constructing Wreath Product of Z_3 by S_3 denote by $Z_3 \wr S_3$ by hand

The wreath product of the given groups H by K, denoted $H \wr K$ is a semi-direct product composed of as many copies of H as the number of letters on which the permutation group K acts on. We define the wreath product below.

Definition 3.1. *Let X and Y be non-empty sets. Let H be a permutation group on X and K on Y. Let $Z = X \times Y$, then **The wreath product** is a semi-direct product of X and Y. We define a permutation group on Z such that we let $\gamma \in H$ and define a permutation of $\gamma(y)$ of Z by*

$$\gamma(y) = \begin{Bmatrix} (x, y_1) \to (\gamma(x), y_1) \ if \ y_1 = y \\ (x, y_1) \to (x, y_1) \ if \ y_1 \neq y \end{Bmatrix}$$

Also, for $k \in K$, define $K^* : (x; y) = (x; (y)K)$ such that $B = X_{y \in Y} H(y)$ is a direct product of the group generated by the y's. Thus, $G = B : K^*$ is called a wreath product of H and K, where H is normal subgroup, denoted by $H \wr K$. [Der96].

Consider $H = \langle (1, 2, 3) \rangle \cong 3$ and $K = \langle (4, 5, 6), (4, 5) \rangle \cong S_3$ which are permutation groups on $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$, respectively.

We will construct permutation generators using wreath product $H \wr K$ of H and K, as well as give its presentation.

Let H be on 3 letters and K be on 3 letters, then

$|H \wr K| = |H|^m \times |K|$

$S_3 = \langle x, y | x^3 = y^2 = (x * y)^2 = 1 \rangle$

We have $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$. Then,

$Z = X \times Y = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

Now, $3 \wr S_3$ is a permutation group on Z.

$|x \times y| = |X||Y| = 3 \times 3 = 9$.

Let $\gamma \in H$ define the permutation $\gamma(y_1)$ where $y_1 \in Y$ as follows.

We will assign γ to all the elements in K. Using wreath product definition we let $\gamma = (123) \in H$ and $y \in Y$ we will compute $\gamma(4), \gamma(5), \gamma(6)$

$$\gamma(y) = \begin{cases} (x, y_1) \to (\gamma(x), y_1) \text{ if } y_1 = y \\ (x, y_1) \to (x, y_1) \text{ if } y_1 \neq y \end{cases}$$

Notice by definition this application of $\gamma(4)$ will only change elements which contain 1, 2, and 3 as the x-coordinate and 4 as the y-coordinate.

		Compute $\gamma(4)$		
Labeling	Element	Compute γ	Element	Labeling
(7)	$(1,4)$	$(\gamma(1), 4)$	$(2,4)$	(8)
(8)	$(2,4)$	$(\gamma(2), 4)$	$(3,4)$	(9)
(9)	$(3,4)$	$(\gamma(3), 4)$	$(1,4)$	(7)
(10)	$(1,5)$	$(\gamma(1), 5)$	$(1,5)$	(10)
(11)	$(2,5)$	$(\gamma(2), 5)$	$(2,5)$	(11)
(12)	$(3,5)$	$(\gamma(3), 5)$	$(3,5)$	(12)
(13)	$(1,6)$	$(\gamma(1), 6)$	$(1,6)$	(13)
(14)	$(2,6)$	$(\gamma(2), 6)$	$(2,6)$	(14)
(15)	$(3,6)$	$(\gamma(3), 6)$	$(3,6)$	(15)

Table 3.1: Compute $\gamma(4)$

Thus, we get

$\gamma(4) = (7, 8, 9)$.

By definition this computation of $\gamma(5)$ will only change elements which contain 1, 2, and

3 as the x-coordinate and 5 as the y-coordinate.

Compute $\gamma(5)$				
Labeling	Element	Compute γ	Element	Labeling
(7)	$(1,4)$	$(\gamma(1),4)$	$(1,4)$	(7)
(8)	$(2,4)$	$(\gamma(2),4)$	$(2,4)$	(8)
(9)	$(3,4)$	$(\gamma(3),4)$	$(3,4)$	(9)
(10)	$(1,5)$	$(\gamma(1),5)$	$(2,5)$	(11)
(11)	$(2,5)$	$(\gamma(2),5)$	$(3,5)$	(12)
(12)	$(3,5)$	$(\gamma(3),5)$	$(1,5)$	(10)
(13)	$(1,6)$	$(\gamma(1),6)$	$(1,6)$	(13)
(14)	$(2,6)$	$(\gamma(2),6)$	$(2,6)$	(14)
(15)	$(3,6)$	$(\gamma(3),6)$	$(3,6)$	(15)

Table 3.2: Compute $\gamma(5)$

Thus, we get

$\gamma(5) = (10, 11, 12)$.

By definition this computation of $\gamma(6)$ will only change elements which contain 1, 2, and 3 as the x-coordinate and 6 as the y-coordinate.

Compute $\gamma(6)$				
Labeling	Element	Compute γ	Element	Labeling
(7)	$(1,4)$	$(\gamma(1),4)$	$(1,4)$	(7)
(8)	$(2,4)$	$(\gamma(2),4)$	$(2,4)$	(8)
(9)	$(3,4)$	$(\gamma(3),4)$	$(3,4)$	(9)
(10)	$(1,5)$	$(\gamma(1),5)$	$(1,5)$	(10)
(11)	$(2,5)$	$(\gamma(2),5)$	$(2,5)$	(11)
(12)	$(3,5)$	$(\gamma(3),5)$	$(3,5)$	(12)
(13)	$(1,6)$	$(\gamma(1),6)$	$(2,6)$	(14)
(14)	$(2,6)$	$(\gamma(2),6)$	$(3,6)$	(15)
(15)	$(3,6)$	$(\gamma(3),6)$	$(1,6)$	(13)

Table 3.3: Compute $\gamma(6)$

Thus, we get

$\gamma(6) = (13, 14, 15)$.

Let $k \in K$ then define the permutation k_1^* and k_2^* of z as follows:

Now we have $K = \langle (4,5,6), (4,5) \rangle \cong S_3$. Let $k_1 = (4,5,6)$ and $k_2 = (4,5)$. Then as the definition shows, k_1^*, k_2^* will change all Y elements. Let us compute k_1^* then we will get:

		Action of k_1^*		
Labeling	Element	Action k_1^*	Element	Labeling
(7)	$(1,4)$	$(1, k_1^*(4))$	$(1,5)$	(10)
(8)	$(2,4)$	$(2, k_1^*(4))$	$(2,5)$	(11)
(9)	$(3,4)$	$(3, k_1^*(4))$	$(3,5)$	(12)
(10)	$(1,5)$	$(1, k_1^*(5))$	$(1,6)$	(13)
(11)	$(2,5)$	$(2, k_1^*(5))$	$(2,6)$	(14)
(12)	$(3,5)$	$(3, k_1^*(5))$	$(3,6)$	(15)
(13)	$(1,6)$	$(1, k_1^*(6))$	$(1,4)$	(7)
(14)	$(2,6)$	$(2, k_1^*(6))$	$(2,4)$	(8)
(15)	$(3,6)$	$(3, k_1^*(6))$	$(3,4)$	(9)

Table 3.4: Action of k_1^*

So, $k_1^* = (7, 10, 13), (8, 11, 14), (9, 12, 15)$

Then computing k_2^* we will get

Action of k_2^*				
Labeling	Element	Action of k_2^*	Element	Labeling
(7)	$(1,4)$	$(1,k_2^*(4))$	$(1,5)$	(10)
(8)	$(2,4)$	$(2,k_2^*(4))$	$(2,5)$	(11)
(9)	$(3,4)$	$(3,k_2^*(4))$	$(3,5)$	(12)
(10)	$(1,5)$	$(1,k_2^*(5))$	$(1,4)$	(7)
(11)	$(2,5)$	$(2,k_2^*(5))$	$(2,4)$	(8)
(12)	$(3,5)$	$(3,k_2^*(5))$	$(3,4)$	(9)
(13)	$(1,6)$	$(1,k_2^*(6))$	$(1,6)$	(13)
(14)	$(2,6)$	$(2,k_2^*(6))$	$(2,6)$	(14)
(15)	$(3,6)$	$(3,k_2^*(6))$	$(3,6)$	(15)

Table 3.5: Action of k_2^*

So, $k_2^* = (7,10),(8,11),(9,12)$

$H \wr K = (H_1 \times H_2 \times H_3) : K^*$

We have formed $B = \{< r(4) > \times < r(5) > \times < r(6) >\}$

where $H_1 =< r(4) >, H_2 =< r(5) >, H_3 =< r(6) >$

Now we will write the presentation of this group. We will label them as follows,

$a = (7,8,9),$

$b = (10,11,12),$

$c = (13,14,15),$

$d = (7,10,13),(8,11,14),(9,12,15),$

$e = (7,10),(8,11),(9,12)$

We will conjugate the elements from H with the elements from K such as,

$a^d = (7,8,9)^{(7,10,13),(8,11,14),(9,12,15)} = (10,11,12) = b$

$a^e = (7,8,9)^{(7,10),(8,11),(9,12)} = (10,11,12) = b$

$b^d = (10,11,12)^{(7,10,13),(8,11,14),(9,12,15)} = (13,14,15) = c$

$b^e = (10,11,12)^{(7,10),(8,11),(9,12)} = (7,8,9) = a$

$c^d = (13,14,15)^{(7,10,13),(8,11,14),(9,12,15)} = (7,8,9) = a$

$$c^e = (13, 14, 15)^{(7,10),(8,11),(9,12)} = (13, 14, 15) = c$$

Thus, we will use the result in our presentation.

```
wrp<a,b,c,d,e>:=Group<a,b,c,d,e|
a^3,b^3,c^3,
(a,b),(a,c),(b,c),d^3,e^2,(d*e)^2,
a^d=b,a^e=b,
b^d=c, b^e=a,
c^d=a, c^e=c>.
```

Then in magma we will compute the order of wreath product by the following command,

```
 # wrp
162
```

We know that the order of 162 is correct, since X, Y, and Z are all of order three we can write them as 3^3 and our K is of order 6.

Thus to find the order of our group we will have the order of $3^3 \times 6 = 162$.

Next we will find the isomorphism type of the wreath product and N by putting the following command into magma,

```
N:= WreathProduct(CyclicGroup(3) , Sym(3))
True
```

That will conclude our construction by hand. We give a symmetric presentation for the progenitor $2^{*9} : 3 \wr S_3$

3.2 Wreath Product : $Z_3 \wr S_3$ using Magma

In this section, we will build our presentation using magma. We will use the following code to find the permutaion of group N:

```
FPGroup(N);
  x^3,
  y^6,
   y * x * y^-1 * x^-1 * y^-1 * x * y
```

Next we want to find our stabiliser of N with respects to 1, and input them into our presentation. Therefore in magma we input the following:

```
N1:=Stabiliser(N,[1]);
N1;
(y^2 * x^-1 * y^-1)
(x^-1 * y^2 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-2)
```

Next we add the stabilisers into our presentation which gives us:

```
G<x,y,t>:=Group<x,y,t| x^3, y^6,
y * x * y^-1 * x^-1 * y^-1 * x * y,
t^2,
(t,(y^2 * x^-1 * y^-1)),
(t,(x^-1 * y^2 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-2))>;
```

Lastly, to know if our presentation is correct we will apply GrindStaff's lemma. To do so, we will look at the orbits of N with respects to 1, and pick an orbit representative and raise 1 by powers of x and y to get the respected orbit representative.

```
Orbits(N1s);
{1}, {2}, {3}, {4, 8, 6, 9, 7, 5}
```

Selecting the following orbit representatives we have the following

$1^{y^2} = 2$

$1^{y^5} = 3$

$1^y = 6$

Therefore, we will add $(t, t^{(y^2)}), (t, t^{(y^5)}), (t, t^{(y)})$ to our presentation.

```
G<x,y,t>:=Group<x,y,t|x^3, y^6,
y * x * y^-1 * x^-1 * y^-1 * x * y,
t^2,
(t,(y^2 * x^-1 * y^-1)),
(t,(x^-1 * y^2 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-2)),
(t,t^{(y^2)}),(t,t^{(y^5)}),(t,t^{(y)})>
```

The order of G must be 82944 which can be derived from the order of N being 162×2^9, where 2^9 comes from $t^2 = e$ and $Sym(9)$. Thus in magma we input:

```
G<x,y,t>:=Group<x,y,t| x^3, y^6,
y * x * y^-1 * x^-1 * y^-1 * x * y,
t^2,
(t,(y^2 * x^-1 * y^-1)),
(t,(x^-1 * y^2 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-2)),
(t,t^(y^2)),(t,t^(y^5)),(t,t^(y))>;
#G;
82944
2^9*162;
82944
```

Thus, our presentation is correct.

Next we begin the process to make a progenitor. We begin by checking the number of classes of N, which we find to be 22 classes by using the following code in magma:

```
For i in [2..#C] do i, C[i][3]; word (C[i][3]);
Orbits(Centraliser(N,C[i][3]));end for;
2 (1, 4)(2, 5)(3, 6)
y^3
[
    GSet{@ 7, 9, 8 @},
    GSet{@ 1, 4, 6, 3, 2, 5 @}
]
3 (1, 2, 3)(4, 5, 6)(7, 8, 9)
x * y * x^-1 * y * x * y^-2
[
    GSet{@ 1, 9, 6, 5, 7, 2, 8, 4, 3 @}
]
4 (1, 3, 2)(4, 6, 5)(7, 9, 8)
x * y^-1 * x^-1 * y^2 * x^-1 * y^-1
[
    GSet{@ 1, 9, 6, 5, 7, 2, 8, 4, 3 @}
]
5 (1, 2, 3)(4, 5, 6)(7, 9, 8)
y^2
[
    GSet{@ 7, 9, 8 @},
    GSet{@ 1, 2, 4, 3, 5, 6 @}
]
6 (1, 3, 2)(4, 6, 5)(7, 8, 9)
```

```
y^-2
[
    GSet{@ 7, 8, 9 @},
    GSet{@ 1, 3, 4, 2, 6, 5 @}
]
7 (1, 2, 3)(7, 8, 9)
(x * y^-1)^2
[
    GSet{@ 4, 6, 5 @},
    GSet{@ 1, 2, 9, 3, 7, 8 @}
]
8 (1, 3, 2)(7, 9, 8)
(y * x^-1)^2
[
    GSet{@ 4, 6, 5 @},
    GSet{@ 1, 3, 9, 2, 8, 7 @}
]
9 (7, 8, 9)
x^-1 * y^3 * x^-1 * y^-1
[
    GSet{@ 7, 8, 9 @},
    GSet{@ 1, 2, 4, 3, 5, 6 @}
]
10 (7, 9, 8)
x * y^3 * x * y
[
    GSet{@ 7, 9, 8 @},
    GSet{@ 1, 2, 4, 3, 5, 6 @}
]
11 (1, 3, 2)(7, 8, 9)
x^-1 * y^-1 * x^-1 * y
[
    GSet{@ 1, 3, 2 @},
    GSet{@ 4, 6, 5 @},
    GSet{@ 7, 8, 9 @}
]
12 (1, 9, 5)(2, 7, 6)(3, 8, 4)
x
[
    GSet{@ 1, 9, 8, 5, 4, 6, 3, 2, 7 @}
]
13 (1, 6, 2, 4, 3, 5)(7, 8, 9)
y
[
```

```
        GSet{@ 7, 8, 9 @},
        GSet{@ 1, 4, 6, 3, 2, 5 @}
]
14 (1, 5, 3, 4, 2, 6)(7, 9, 8)
y^-1
[
        GSet{@ 7, 9, 8 @},
        GSet{@ 1, 4, 5, 2, 3, 6 @}
]
15 (1, 8, 2, 9, 3, 7)
x * y^-1
[
        GSet{@ 4, 5, 6 @},
        GSet{@ 1, 8, 2, 9, 3, 7 @}
]
16 (1, 7, 3, 9, 2, 8)
y * x^-1
[
        GSet{@ 4, 6, 5 @},
        GSet{@ 1, 7, 3, 9, 2, 8 @}
]
17 (1, 5)(2, 6)(3, 4)(7, 9, 8)
y^2 * x^-1 * y^-1 * x^-1
[
        GSet{@ 7, 9, 8 @},
        GSet{@ 1, 5, 2, 6, 3, 4 @}
]
18 (1, 5)(2, 6)(3, 4)(7, 8, 9)
x * y * x * y^-2
[
        GSet{@ 7, 8, 9 @},
        GSet{@ 1, 5, 2, 6, 3, 4 @}
]
19 (1, 2, 3)(4, 7, 6, 9, 5, 8)
y^2 * x^-1 * y
[
        GSet{@ 1, 2, 3 @},
        GSet{@ 4, 9, 7, 5, 6, 8 @}
]
20 (1, 3, 2)(4, 8, 5, 9, 6, 7)
y^-1 * x * y^-2
[
        GSet{@ 1, 3, 2 @},
        GSet{@ 4, 9, 8, 6, 5, 7 @}
```

```
]
21 (1, 6, 8, 2, 4, 9, 3, 5, 7)
y^2 * x^-1
[
    GSet{@ 1, 6, 8, 2, 4, 9, 3, 5, 7 @}
]
22 (1, 8, 4, 3, 7, 6, 2, 9, 5)
y * x * y^-2 * x * y^-1
[
    GSet{@ 1, 8, 4, 3, 7, 6, 2, 9, 5 @}
]
```

After using the following code to find the name of the respective $ts[i]$:

```
for j in [2..#N] do for i in [1..#Setseq(Set(N))] do if 1^Setseq(Set(N))[i]
eq j then j, word(Setseq(Set(N))[i]); break;end if; end for;end for;
2 y^2
3 x * y * x^-1 * y * x
4 x^-1 * y^-1 * x^-1
5 y * x * y^-2 * x^-1
6 x * y^-2 * x^-1 * y^-1
7 x * y^-2
8 y^-1 * x^-1 * y^-1
9 x
```

We will build a first order relation which is composed of a representative of a class of N and a $ts[i]$ representative. For example, we will use the following information from the class 21 which has the representative $(y^2 * x^{-1})$. Next we will multiply the class representative by a $ts[i]$ representative which we will choose to be $ts[2]$ that has the representative of $t^{(y^2)}$.

Thus with this information we can build the following first order relation $((y^2 * x^{-1}) * t^{(y^2)})$. By constructing different first order relations we can build a progenitor as follows:

```
for a,b,c,d,e,f,g,h,i,j in [0..10] do
G<x,y,t>:=Group<x,y,t| x^3, y^6,
y * x * y^-1 * x^-1 * y^-1 * x * y,
t^2,
((y^3)*t^( x * y^-2 * x^-1 * y^-1))^a,
((x * y * x^-1 * y * x * y^-2)*t^( x))^b,
((y^-2)*t^( x * y * x^-1 * y * x))^c,
(((y * x^-1)^2)*t^( y^2))^e,
((x^-1 * y^-1 * x^-1 * y)*t^( x * y^-2))^f,
((x)*t^( x * y * x^-1 * y * x))^g,
```

```
((x * y^-1)*t^( y * x * y^-2 * x^-1))^h,
((y * x^-1)*t^( x * y^-2 * x^-1 * y^-1))^i,
((y^2 * x^-1 * y^-1 * x^-1)*t^( x * y * x^-1 * y * x))^j>;
a,b,c,d,e,f,g,h,i,j, Index(G,sub<G|x,y>); end for; end for;
```

Now our progenitor is ready to put in magma to produce homomorphic images that we can use to construct the double coset enumeration which we will introduce later.

Chapter 4

Symmetric Presentations

Transitive progenitors of the form $m^{*n} : N$ require N to be transitive. First, we find the number of transitive groups on n letters. Second, we choose a group i from the sequence to investigate further. The group is stored as $N := TransitiveGroup(n, i)$. Then we find the generators of N to begin writing the progenitor.

4.1 Symmetric Presentation of $2^{10} : (5^2 : 2^2)$

To write a symmetric presentation for the progenitor determined by $T(10, 9)$ we implement GrindStaff's lemma. Then we add additional relations to this progenitor and run it interactively to determine finite images.

Let $N = (5^2 : 2^2)$ where N is of order 100 and generated by $x \sim (1, 5)(4, 10)(6, 8)(7, 9)$ and $y \sim (1, 2, 7, 6, 3, 10, 9, 4, 5, 8)$. We are able to build a progenitor for our N. We write symmetric presentation for the progenitor $2^{10} : (5^2 : 2^2)$. A presentation for N is,

```
G<x,y,t>:=Group<x,y,t|x^2,
(y * x * y)^2, y^10,
t^2 >;
```

We need to find the stabilisers of $(N, 1)$ meaning we need to find permutations contained in N, that fix 1. We find the following permutations stabilise 1,

$(2, 8, 4, 10, 6) = y * x * y^{-1} * x * y^2$, and

$(3, 9)(4, 10)(5, 7)(6, 8) = x * y^{-1} * x * y * x * y^{-2}$

We will add these additional words into our progenitor to complete it.

```
G<x,y,t>:=Group<x,y,t|x^2, (y * x * y)^2, y^10,
t^2,
(t, (y * x * y^-1 * x * y^2)),
(t, (x * y^-1 * x * y * x * y^-2))>;
```

To check if we build the correct progenitor, we apply Grindstaff's Lemma, where we look at the orbits of $N1$ and find what word (permutation) takes 1 to an orbit representative and we continue this process until all orbits are exhausted.

```
Orbits(N1);
GSet{@ 1 @},
GSet{@ 3, 9 @},
GSet{@ 5, 7 @},
GSet{@ 2, 8, 4, 6, 10 @}
```

We will choose the following orbit representatives: 2,3, and 5

$1^{(y)}$;

2

$1^{(y^4)}$;

3

$1^{(x)}$;

5

Next we will put the following permutations into a t-cycle and then add them into our progenitor.

$(t, t^{(x)})$,

$(t, t^{(y)})$,

$(t, t^{(y^4)})$

```
G<x,y,t>:=Group<x,y,t|x^2, (y * x * y)^2, y^10,
t^2,
(t, (y * x * y^-1 * x * y^2)),
(t, (x * y^-1 * x * y * x * y^-2)),
y^-1 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-1 *
x * y * x * y * x * y * x * y * x * y^-1 * x,
(t,t^(x)),
(t,t^(y)),
(t,t^(y^4))>;

Index(G,sub<G|x,y>);
1024
```

```
#G;
102400

2^10*100;
102400
```

This proves that our progenitor is correct. Next we will use Lemma 3.3 to find additional relations to add to our progenitor. Lemma 3.3 is only applicable to $t_1 t_2$ where permutations need to be of order two, and need to stabilise $t_1 t_2$. We will look at the set of permutations of the centraliser in N of the point-stabiliser in N of t_1 and t_2. We find the following permutations:

```
(1, 2)(3, 10)(4, 9)(5, 8)(6, 7)= y * x * y^-1 * x * y^3,
(3, 9)(4, 10)(5, 7)(6, 8)= x * y^-1 * x * y * x * y^-2,
(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)= (x * y)^5
```

For the relations found using Lemma 3.3, they are of even or odd power. If they are of even power, it means that the permutation(s) found do not have $(1, 2)$ contained in their permutation hence why it is necessary to input $t_1 t_2$ to be equal to that said permutation(s). When it comes to adding the permutation to first order relations the even relation found from Lemma 3.3 is the permutation(s) that contains $(1, 2)$ to be added into the progenitor. It is the word of that progenitor multiplied by t. After finding the classes of N, we use all the information we collect to build this progenitor and put it in magma to find isomorphic images of N:

```
for a,b,c,d,e,f,g,h,i,j in [0..10] do
G<x,y,t>:=Group<x,y,t|x^2, (y * x * y)^2, y^10,
t^2,
((y^5)*t^(x * y^-1 * x * y * x * y^-1 * x ))^a,
((x * y * x * y * x * y * x * y * x * y^-1)*t^(x * y)^2)^b,
((y * x * y)*t^(x * y * x * y^-1 * x * y ))^c,
((y^2)*t^(x * y * x * y * x * y^-1 * x * y * x ))^d,
(((y * x)^2)*t^(x * y * x * y^-1 * x * y * x * y^-2 ))^e,
((y^-4)*t^(y^2 ))^f,
(((x * y)^4)*t^(x * y * x * y^-4 ))^g,
((x * y * x * y^-1)*t^(y * x * y^-1 * x)^2)^h,
(((x * y * x * y^-1)^2)*t^(x * y^-1 * x ))^i,
((y * x * y * x * y * x * y^-1 * x)*t)^j>;
a,b,c,d,e,f,g,h,i,j, #G; end for;
```

4.2 Symmetric Presentation of $2^{10} : Sym(6)$

Let $N = Sym(6)$ where N is of order 720 and generated by $x \sim (1, 2, 10)(3, 4, 5)(6, 7, 8)$, $y \sim (1, 3, 2, 6)(4, 5, 8, 7)$, $z \sim (1, 2)(4, 7)(5, 8)(9, 10)$ and $w \sim (3, 6)(4, 7)(5, 8)$. We are able to build a progenitor for our N. We write a symmetric presentation for the progenitor $2^{10} : Sym(6)$. A presentation for N is:

```
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3, y^4,
z^2,w^2,(y^-1 * z)^2,
x^-1 * w * x * w,
(y^-1 * w)^2,(z * w)^2,
y^-2 * x^-1 * y^2 * x^-1,
(z * x^-1)^3,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y * x * y,
(x^-1 * y^-1 * x * z)^3,
t^2>;
```

We need to find the stabilisers of $(N, 1)$ meaning we need to find permutations contained in N, that fix 1. We find the following permutations stabilise 1,

$(3, 6)(4, 7)(5, 8) = w$

$(2, 7, 8)(3, 5, 10, 9, 4, 6) = z * x * w * y * x$

$(2, 7)(4, 10)(5, 6) = w * x^{-1} * y^{-1} * x$

We will add these additional words into our progenitor to complete it,

```
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3, y^4,
z^2,w^2,(y^-1 * z)^2,
x^-1 * w * x * w,
(y^-1 * w)^2,(z * w)^2,
y^-2 * x^-1 * y^2 * x^-1,
(z * x^-1)^3,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y * x * y,
(x^-1 * y^-1 * x * z)^3,
t^2,
(t,w),(t,(z * x * w * y * x)),
(t,(w * x^-1 * y^-1 * x))>;
```

To check if we build the correct progenitor, we apply Grindstaff's Lemma, where we look at the orbits of $N1$, and find what word (permutation) takes one to an orbit representative and we continue this process until all orbits are exhausted.

```
Orbits(N1);
GSet{@ 1 @},
GSet{@ 2, 4, 6, 7, 10, 3, 8, 9, 5 @}
```

We will choose the following orbit representative: 2

$1^{(x)}$;

2

Next we will put the following permutations into a t-cycle and then add them into our progenitor.

$(t, t^{(x)})$

```
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3, y^4,
z^2,w^2,
(y^-1 * z)^2,
x^-1 * w * x * w,
(y^-1 * w)^2,
(z * w)^2,
y^-2 * x^-1 * y^2 * x^-1,
(z * x^-1)^3,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y * x * y,
(x^-1 * y^-1 * x * z)^3,
t^2,
(t,w),(t,(z * x * w * y * x)),
(t,(w * x^-1 * y^-1 * x)),
(t,t^x)>;

Index(G,sub<G|x,y,z,w>);
1024

#G;
737280

2^10*720;
737280
```

This proves that our progenitor is correct. Next we will use Lemma 3.3 to find additional relations to add to our progenitor. Lemma 3.3 is only applicable to $t_1 t_2$ where permutations need to be of order two, and need to stabilise $t_1 t_2$. We will look at the set of permutations of the centraliser in N of the point-stabiliser in N of t_1 and t_2. We find the following permutations:

```
(1, 2)(4, 5)(7, 8)= b^2 * d,
Id(C),
(3, 6)(4, 5)(7, 8)(9, 10)= b^2 * c,
(1, 2)(3, 6)(9, 10)=c * d
```

For the relations found using Lemma 3.3, they are of even or odd power. If they are of even power, it means that the permutation(s) found do not have $(1,2)$ contained in their permutation hence why it is necessary to input $t_1 t_2$ to be equal to that said permutation(s). When it comes to adding the permutation to first order relations the even relation found from Lemma 3.3 the permutation(s) that contains $(1,2)$ to be added into the progenitor. It is the word of that progenitor multiplied by t. After finding the classes of N. We use all the information we collect to build this progenitor and put it in magma to find isomorphic images of N:

```
for a,b,c,d,e,f,g,h,i,j in [0..10] do
G<x,y,z,w,t>:=Group<x,y,z,w,t|x^3, y^4,
z^2,w^2,
(y^-1 * z)^2,
x^-1 * w * x * w,
(y^-1 * w)^2,
(z * w)^2,
y^-2 * x^-1 * y^2 * x^-1,
(z * x^-1)^3,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y * x * y,
(x^-1 * y^-1 * x * z)^3,
t^2,
(w*t^(x * y * z * x * y * z * x^-1 * y^-1))^a,
(z * w*t^(x * y * w))^b,
(z*t^(x * y * z * x * y^-1 * x^-1 * z))^c,
(z * x*t^(y * x^-1))^d,
(z * x * y * x^-1*t^(x * y * z * x * y * w * x^-1))^e,
(z * x * y * x*t^(z * y * x))^f,
(x * z * y * w * x^-1*t^(x * y * x * z * y * w))^g,
(z * x * y*t)^h,
(z * x * w*t^(w * x^-1 * z))^i,
(z * x * w * y * x*t^(x^-1 * y * x * z * x * y))^j>;
a,b,c,d,e,f,g,h,i,j, #G; end for;
```

4.3   Symmetric Presentation of $2^{12} : (2^5 : S_3)$

Let $N = (2^5 : S_3)$ where N is of order 192 and generated by $x \sim (1,7,12,6)(2,5,9,11)(3,4,8,10)$ and $y \sim (1,2,7,8)(3,6,9,12)(4,10)(5,11)$. We are able to build a progenitor for our N. We write symmetric presentation for the progenitor $2^{12} : (2^5 : S_3)$. A presentation for N is:

```
G<x,y,t>:=Group<x,y,t|x^4, y^4 ,
x^-1 * y^-1 * x^2 * y^-1 * x^-1 ,
y^-2 * x^-1 * y^-2 * x^-1 * y^2 * x * y^2 * x ,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y^-1 * x * y * x *
y^-1 * x^-1 * y^-1 ,
x^-1 * y * x^-1 * y^-1 * x^-1 * y^-1 * x * y^-1 * x * y * x * y,
t^2 >;
```

We need to find the stabilisers of $(N, 1)$ meaning we need to find permutations contained in N, that fix 1. We find the following permutations stabilise 1,

$(2, 5, 3, 4)(6, 7)(8, 10, 9, 11) = x * y^2$,

$(2, 10)(3, 11)(4, 8)(5, 9) = y * x * y * x * y^-1$,

$(4, 5)(6, 7)(8, 9) = (x * y^-1)^3$

We will add these additional words into our progenitor to complete it.

```
G<x,y,t>:=Group<x,y,t|x^4, y^4 ,
x^-1 * y^-1 * x^2 * y^-1 * x^-1 ,
y^-2 * x^-1 * y^-2 * x^-1 * y^2 * x * y^2 * x ,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y^-1 * x * y * x *
y^-1 * x^-1 * y^-1 ,
x^-1 * y * x^-1 * y^-1 * x^-1 * y^-1 * x * y^-1 * x * y * x * y,
t^2,
(t,x * y^2),(t,y * x * y * x * y^-1),
(t,(x * y^-1)^3)>;
```

To check if we build the correct progenitor, we apply Grindstaff's Lemma, where we look at the orbits of $N1$, and find what word (permutation) takes one to an orbit representative and we continue this process until all orbits are exhausted.

```
Orbits(N1);
GSet{@ 1 @},
GSet{@ 12 @},
GSet{@ 6, 7 @},
GSet{@ 2, 5, 10, 3, 9, 4, 11, 8 @}
```

We will choose the following orbit representative: 2,6, and 12

$1^{(y)}$;

2

$1^{(x^3)}$;

6

$1^{(x^2)}$;

12

Next we will put the following permutations into a t-cycle and then add them into our progenitor.

$(t, t^{(y)})$

$(t, t^{(x^3)})$

$(t, t^{(x^2)})$

```
G<x,y,t>:=Group<x,y,t|x^4, y^4 ,
x^-1 * y^-1 * x^2 * y^-1 * x^-1 ,
y^-2 * x^-1 * y^-2 * x^-1 * y^2 * x * y^2 * x ,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y^-1 * x * y * x *
y^-1 * x^-1 * y^-1 ,
x^-1 * y * x^-1 * y^-1 * x^-1 * y^-1 * x * y^-1 * x * y * x * y,
t^2,
(t,x * y^2),(t,y * x * y * x * y^-1),(t,(x * y^-1)^3),
(t,t^y),(t,t^(x^2)),(t,t^(x^3))>;

Index(G,sub<G|x,y>);
4096

#G;
786432

2^12*192;
786432
```

This proves that our progenitor is correct. Next we will use Lemma 3.3 to find additional relations to add to our progenitor. Lemma 3.3 is only applicable to $t_1 t_2$ where permutations need to be of order two, and need to stabilise $t_1 t_2$. We will look at the set of permutations of the centraliser in N of the point-stabiliser in N of t_1 and t_2. We find the following permutations:

$(4, 5)(6, 7)(8, 9)$.

For the relations found using Lemma 3.3, they are of even or odd power. If they are of even power, it means that the permutation(s) found do not have $(1, 2)$ contained in their permutation hence why it is necessary to input $t_1 t_2$ to be equal to that said permutation(s). When it comes to adding the permutation to first order relations the even relation found from Lemma 3.3 the permutation(s) that contains $(1, 2)$ to be added into the progenitor it is the word of that progenitor multiplied by t. After finding the classes of N. We use all the information we collect to build this progenitor and put it in magma

to find isomorphic images of N:

```
for a,b,c,d,e,f,g,h,i,j in [0..10] do
G<x,y,t>:=Group<x,y,t|x^4, y^4 ,
x^-1 * y^-1 * x^2 * y^-1 * x^-1 ,
y^-2 * x^-1 * y^-2 * x^-1 * y^2 * x * y^2 * x ,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y^-1 * x * y * x *
y^-1 * x^-1 * y^-1 ,
x^-1 * y * x^-1 * y^-1 * x^-1 * y^-1 * x * y^-1 * x * y * x * y,
t^2,
((x * y^2)^2*t^(x * y * x * y^-1 * x^-1 * y^-1))^a,
(x * y * x * y^-1 * x^-1 * y^-1*t^(y * x^-1))^b,
(x^2*t^(y^2 * x * y^-1)^2)^c,
(y^2*t^(y^-1 * x * y^-1 * x^-1 * y^2))^d,
(x * y^-1 * x*t^(x * y^2 * x * y * x^-1))^e,
(x * y^2 * x * y * x * y^-1*t^(x * y^2 * x * y * x^-1))^f,
((y * x)^2*t^(y * x * y^2))^g,
(x * y^2*t^(x * y * x * y^-1 * x^-1 * y))^h,
(x * y^2 * x^-1 * y*t^(y * x * y * x * y))^i,
(x * y^-1 * x * y * x * y*t^(y * x * y^-1 * x * y^2))^j>;
a,b,c,d,e,f,g,h,i,j, #G; end for;
```

4.4 Symmetric Presentation of $2^{24} : (2^5 : 3)$

Let $N = (2^5 : 3)$ where N is of order 96 and generated by

$x \sim (1,4,2,3)(5,23,6,24)(7,10)(8,9)(15,21,17,19)(16,22,18,20)$ and

$y \sim (1,8,21)(2,7,22)(3,12,16)(4,11,15)(5,14,18)(6,13,17)(9,20,24)(10,19,23)$.

We are able to build a progenitor for our N. We write symmetric presentation for the progenitor $2^{24} : (2^5 : 3)$. A presentation for N is:

```
G<x,y,t>:=Group<x,y,t|x^4, y^3 ,
(y^-1 * x^-1)^3 ,
x^-1 * y^-1 * x * y^-1 * x^-1 * y * x^2 * y ,
t^2 >;
```

We need to find the stabilisers of $(N, 1)$ meaning we need to find permutations contained in N that fix 1. We find the following permutations stabilise 1,

$(7,9)(8,10)(11,13)(12,14)(15,16)(17,18)(19,20)(21,22)$

$(3,6)(4,5)(7,13,9,11)(8,14,10,12)(15,20,16,19)(17,22,18,21)$

We will add these additional words into our progenitor to complete it.

```
G<x,y,t>:=Group<x,y,t|x^4, y^3 ,
(y^-1 * x^-1)^3 ,
x^-1 * y^-1 * x * y^-1 * x^-1 * y * x^2 * y ,
t^3,
(t,(y * x * y^-1)^2),
(t,(x * y^-1 * x * y))>;
```

To check if we build the correct progenitor, we apply Grindstaff's Lemma, where we look at the orbits of $N1$, and find what word (permutation) takes one to an orbit representative and we continue this process until all orbits are exhausted.

```
Orbits(N1);
GSet{@ 1 @},
GSet{@ 2 @},
GSet{@ 23 @},
GSet{@ 24 @},
GSet{@ 3, 6 @},
GSet{@ 4, 5 @},
GSet{@ 7, 9, 13, 11 @},
GSet{@ 8, 10, 14, 12 @},
GSet{@ 15, 16, 20, 19 @},
GSet{@ 17, 18, 22, 21 @}
```

We will choose the following orbit representative: 2,3,4,7,8,15,17,23, and 24

$1^{(x^2)}$;

2

$1^{(x^3)}$;

3

$1^{(x)}$;

4

$1^{(y^3*x^2*y)}$;

7

$1^{(y)}$;

8

$1^{(y^{-1}*x^{-1})}$;

15

$1^{(x*y^2*x^2)}$;

17

$1^{(y^2*x^2*y)}$;

23

$1^{(y*x^3*y^2)}$;

24

Next we will put the following permutations into a t-cycle and then add them into our progenitor.

$(t, t^{(x^2)})$,

$(t, t^{(x^3)})$,

$(t, t^{(x)})$,

$(t, t^{(y^3*x^2*y)})$,

$(t, t^{(y)})$,

$(t, t^{(y^{-1}*x^{-1})})$,

$(t, t^{(x*y^2*x^2)})$,

$(t, t^{(y^2*x^2*y)})$,

$(t, t^{(y*x^3*y^2)})$

```
G<x,y,t>:=Group<x,y,t|x^4, y^3 ,
(y^-1 * x^-1)^3 ,
x^-1 * y^-1 * x * y^-1 * x^-1 * y * x^2 * y ,
```

```
t^3,
(t,(y * x * y^-1)^2),
(t,(x * y^-1 * x * y)),
(t,t^(x^2)),
(t,t^(x^3)),(t,t^(x)),(t,t^(y^3*x^2*y)),
(t,t^(y)),(t,t^(y^-1*x^-1)),
(t,t^(x*y^2*x^2)),(t,t^(y^2*x^2*y)),(t,t^(y*x^3*y^2))>;

#G;
1610612736

2^24*96;
1610612736
```

This proves that our progenitor is correct. Next we will use Lemma 3.3 to find additional relations to add to our progenitor. Lemma 3.3 is only applicable to $t_1 t_2$ where permutations need to be of order two and need to stabilise $t_1 t_2$. We will look at the set of permutations of the centraliser in N of the point-stabiliser in N of t_1 and t_2. We find the following permutations:

$(7,9)(8,10)(11,13)(12,14)(15,16)(17,18)(19,20)(21,22)$,

$(3,6)(4,5)(7,11,9,13)(8,12,10,14)(15,19,16,20)(17,21,18,22)$.

For the relations found using Lemma 3.3, they are of even or odd power. If they are of even power, it means that the permutation(s) found do not have $(1,2)$ contained in their permutation hence why it is necessary to input $t_1 t_2$ to be equal to that said permutation(s). When it comes to adding the permutation to first order relations the even relation found from lemma 3.3 the permutation(s) that contains $(1,2)$ to be added into the progenitor it is the word of that progenitor multiplied by t. After finding the classes of N. We use all the information we collect to build this progenitor and put it in magma to find isomorphic images of N,

```
for a,b,c,d,e,f,g,h,i,j in [0..10] do
G<x,y,t>:=Group<x,y,t|x^4, y^3 ,
(y^-1 * x^-1)^3 ,
x^-1 * y^-1 * x * y^-1 * x^-1 * y * x^2 * y ,
t^2,
((x * y^-1)^3*t^(y^x))^a,
((y * x * y^-1)^2*t^(x * y^-1)^2)^b,
(x * y * x^-1 * y * x^-1 * y*t^(y^-1 * x * y^-1 * x * y^-1))^c,
(y*t^(x * y * x^2 * y))^d,
(y^-1*t^(y^-1 * x))^e,
```

```
(y * x * y^-1*t^(x * y^-1 * x^-1))^f,
(y * x^-1 * y^-1*t^(y * x^-1 * y))^g,
((x^-1, y^-1)*t^(y^-1))^h,
((y^-1, x^-1)*t^(x * y * x^-1 * y * x^-1 * y))^i,
(y * x^-1*t^(y * x^-1 * y^-1))^j>;
a,b,c,d,e,f,g,h,i,j, #G; end for;
```

Chapter 5

Linear Map of $PSL(2, 23)$

Writing a linear map that generate $PSL(2, 23)$ using α, β, γ where

$\alpha : x \to x + 1$

$\beta : x \to Kx$

K is a nonzero square in F_{23} whose powers give all of the squares of F_{23}

$\gamma : x \to \frac{-1}{x} = -x^{-1}$.

Starting with α our permutations are:

$(\infty), (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22)$.

In order to find β permutations, we need to find all nonzero squares for F_{23}

Compute β		
square power	$\cong 23$	Result
0^2	modulo 23	0
1^2	modulo 23	1
2^2	modulo 23	4
3^2	modulo 23	9
4^2	modulo 23	16
$5^2 = 25$	modulo 23	2
$6^2 = 36$	modulo 23	13
$7^2 = 49$	modulo 23	3
$8^2 = 64$	modulo 23	18
$9^2 = 81$	modulo 23	12
$10^2 = 100$	modulo 23	8
$11^2 = 121$	modulo 23	6
$12^2 = 144$	modulo 23	6
$13^2 = 169$	modulo 23	8
$14^2 = 196$	modulo 23	12
$15^2 = 225$	modulo 23	18
$16^2 = 256$	modulo 23	3
$17^2 = 289$	modulo 23	13
$18^2 = 324$	modulo 23	2
$19^2 = 361$	modulo 23	16
$20^2 = 400$	modulo 23	9
$21^2 = 441$	modulo 23	4
$22^2 = 484$	modulo 23	1
$23^2 = 529$	modulo 23	0

Table 5.1: Compute β

The squares we have are $(1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18)$.

Now we need to find the smallest nonzero squares K whose power gives all of nonzero squares. We try 2:

$2^0 = 1,$

$2^1 = 2,$

$2^2 = 4,$

$2^3 = 8,$

$2^4 = 16,$

$2^5 = 32 = 9,$

$2^6 = 64 = 18,$

$2^7 = 128 = 13,$

$2^8 = 256 = 3,$

$2^9 = 512 = 6,$

$2^{10} = 1024 = 12,$

$\beta : x \to 2x$

$1 \to 2 \to 4 \to 8 \to 16 \to 9 \to 18 \to 13 \to 3 \to 6 \to 12 \to 1$

$(1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12)$

$5 \to 10 \to 20 \to 17 \to 11 \to 22 \to 21 \to 19 \to 15 \to 7 \to 14 \to 5$

$(5, 10, 20, 17, 11, 22, 21, 19, 15, 7, 14)$

Thus, $\beta = (1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12)(5, 10, 20, 17, 11, 22, 21, 19, 15, 7, 14)$.

Now we will compute γ.

$\gamma : x \to \frac{-1}{x} = -x^{-1}$

Thus, we will get $(23, 24)(2, 1, 22, 21, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3)$.

Using magma we will verify our answers by following:

```
S:=Sym(24);
a:=S!(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22);
b:=S!(1,2,4,8,16,9,18,13,3,6,12)(5,10,20,17,11,22,21,19,15,7,14);
g:=S!(1,22)(2,11)(3,15)(4,17)(5,9)(6,19)(7,13)(8,20)(10,16)(12,21)(14,18);
psl 223:=sub<S|a,b,g>;
IsIsomorphic(PSL(2,23),psl 223);

true Homomorphism of GrpPerm: $, Degree 24, $ Order 2^3 * 3 * 11 * 23 into
GrpPerm: psl223, Degree 24, Order 2^3 * 3 * 11 * 23 induced by
(3, 14, 5, 12, 20, 23, 19, 17, 11, 21, 18)
(4, 7, 15, 22, 13, 24, 9, 6, 16, 10, 8)
|--> (1, 4, 17, 12, 21, 14, 22, 3, 5, 6, 18)
(2, 16, 23, 15, 11, 9, 8, 19, 13, 10, 20)
(1, 13, 2)(3, 11, 18)(4, 6, 17)(5, 20, 14)
(7, 16, 23)(8, 21, 9)(10, 15, 12)(19, 24, 22)
```

```
|--> (1, 4, 12)(2, 17, 19)(3, 6, 18)(5, 11, 13)
(7, 23,24)(8, 21, 20)(9, 22, 10)(14, 15, 16)
```

5.0.1 Linear Fractional Maps

We need to compute the linear fractional maps and see what they give us using the induced permutations that magma gives us in the previous isomorphism command.

- $(1, 4, 17, 12, 21, 14, 22, 3, 5, 6, 18)(2, 16, 23, 15, 11, 9, 8, 19, 13, 10, 20)$

- $(1, 4, 12)(2, 17, 19)(3, 6, 18)(5, 11, 13)(7, 23, 24)(8, 21, 20)(9, 22, 10)(14, 15, 16)$

We will use the function $\frac{ax+b}{cx+d}$ to calculate all the equations such as:

First equation

$\frac{a+b}{c+d} \implies a + b = 4c + 4d$

Second equation

$\frac{4a+b}{4c+d} = 17 \implies 4a + b = 68c + 17d$

Third equation

$\frac{17a+b}{17c+d} = 12 \implies 17a + b = 204c + d$

Fourth equation

$\frac{16a+b}{16c+d} = 0 \implies 16a + b = 0$

Solving the system of equations we get the map to be:

$A := (PSL(2, 23).1);$

$B := (PSL(2, 23).2);$

Now we will find linear maps for A and B.

```
A;
(1, 20, 21, 12, 15, 24, 13, 16, 7, 8, 4)
(2, 17, 10, 5, 9, 14, 19, 23, 18, 11, 3)

B;
(1, 19, 10)(2, 11, 4)(3, 22, 6)(5, 8, 23)
(7, 24, 13)(9, 18, 16)(12, 15, 20)(14, 21, 17)
```

First, we will calculate a linear map for A and check our results.

$\frac{ax+b}{cx+d}$

$\frac{x+b}{c+d} = 20 \implies a + b = 20c + 20d$

$\frac{20a+b}{20c+d} = 21 \implies 20a + b = 420c + 21d$

$\frac{21a+b}{21c+d} = 12 \implies 21a + b = 252c + 12d$

$\frac{12a+b}{12c+d} = 15 \implies 12a + b = 180c + 15d$

Solving these equations we get the linear map of A to be:

$\frac{16x+18}{3x+1}$

Now, we will check if the equation works for all elements:

$\frac{16(2)+18}{3(2)+1} = \frac{50}{7} = \frac{4}{7} = 4 * 7^{-1} = 4 * 10 = 40 \equiv 23 = 17$

True

$\frac{16(15)+18}{3(15)+1} = \frac{258}{46} = \frac{5}{0} = \infty = 24$

True

Thus, our map for A is true since it works for all elements.

Then, we will find a linear map for B following the same process.

$\frac{a+b}{c+d} = 19 \implies a + b = 19c + 19d$

$\frac{19a+b}{19c+d} = 10 \implies 19a + b = 190c + 10d$

$\frac{10a+b}{10c+d} = 1 \implies 10a + b = 10c + d.$

Solving these system of equations we get the linear map of B to be:

$\frac{8x+28}{13x+1}$

Let us check if it works for all elements:

$\frac{8(2)+28}{13(2)+1} = \frac{44}{27} = \frac{21}{4} = 21 * 4^{-1} = 21 * 6 = 126 \equiv 23 = 11$

True

$\frac{8(11)+28}{13(11)+1} = \frac{116}{144} = \frac{1}{6} = 1 * 6^{-1} = 1 * 4 = 4$

True

Thus, our map for B is true since it works for all elements.

Chapter 6

Isomorphism Types

Definition 6.1. *If H and K are groups. Then their **direct product**, denoted $H \times K$, is the group with elements all ordered pairs (h, k) where $h \in H$, $k \in K$, and with operation $(h, k)(h', k') = (hh', kk')$ [Rot94].*

Definition 6.2. *A group G is a **semi-direct product** of K by Q, denoted by $K \rtimes Q$ if $K \triangle G$ and K has a complement of $Q_1 \cong Q$. One also says that G splits over K [Rot94].*

Definition 6.3. *If K and Q are groups, then an **extension** of K by Q is a group G having a normal subgroup $K_1 \cong K$ with $\frac{G}{k_1} \cong Q$ [Rot94].*

Definition 6.4. *A **central extension** of K by Q is an extension G of K by Q with $K \leq Z(G)$ [Rot94].*

Definition 6.5. *Let G be a group with $H \leq G$, $N \leq G$, and $N \triangle G$ such that $|G| = |N||H|$. Then G is a **mixed extension** by H, denoted $G \cong N^{\cdot} : H$, if G is formed by both central extension and semi-direct products [Rot94].*

In this chapter, we evaluate each image of the progenitor G that is noted to be faithful and whose number of subgroups generated by x and y are equal to the order of our control group N. We will focus on the composition factors to find the rough shape of the images and regard it as the isomorphism type of the groups. There are four types of extension: direct and semi-direct product, and mixed and central. We will include examples of some types using similar steps and most importantly using composition factors and normal lattice of each group.

6.1 Direct Product $(A5 \times A5 \times A5 \times A5 \times A5) : ((2 \times A5) : 2)$

We are given G is a transitive group on 25 letters, which is generated by

$xx \sim (1, 4, 3, 5)(6, 25, 20, 10, 21, 18, 8, 24, 16, 9, 22, 19)(7, 23, 17)(11, 15, 12, 13)$

and $yy \sim (1, 14, 18)(2, 13, 20, 3, 11, 19)(4, 15, 16, 5, 12, 17)(6, 24, 7, 22)(8, 21, 10, 25)(9, 23)$.

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G as shown below:

```
N:=sub < S|xx,yy >;
#N;
{186624000000}

CompositionFactors(N);
G
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Alternating(5)
    *
    |  Alternating(5)
    *
    |  Alternating(5)
    *
    |  Alternating(5)
    1
```
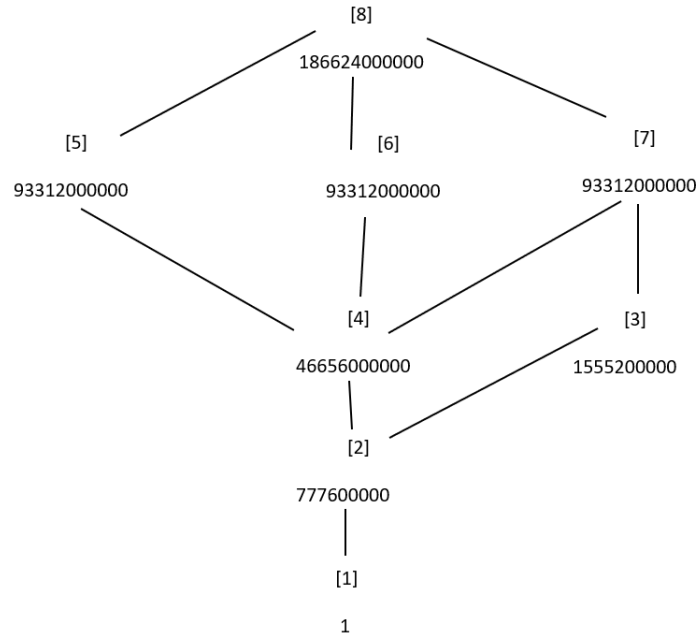
Normal Lattice

Figure 6.1: $(A5 \times A5 \times A5 \times A5 \times A5) : ((2 \times A5) : 2)$

By looking to the composition factors of G it is not clear what the isomorphism type may be. After analyzing the normal lattice of G, we see that $NL[1]$ is the largest normal abelian subgroup. The order of $NL[1]$ is 1 and the order of G is 186624000000. Now $\frac{186624000000}{1} = 186624000000$. However, G has a normal subgroup of order 1. Thus, G is a direct-product. We write G as follows:

$G \cong (A5 \times A5 \times A5 \times A5 \times A5) : ((2 \times A5) : 2)$.

6.2 Direct Product $(A_5 \times A_5) : 4$

We are given G is a transitive group on 25 letters which generated by

$xx \sim (1, 5, 15, 13, 23, 24, 9, 6)(2, 20, 12, 18, 22, 19, 7, 16)(3, 25, 14, 8, 21, 4, 10, 11)$

and $yy \sim (1, 22, 9, 20, 3, 12, 6, 25, 4, 17, 8, 15)(2, 7, 10, 5)(11, 21, 24, 19, 18, 13)(14, 16, 23)$

The presentation of G is:

```
G<x,y,t>:=Group<x,y,t| x^8 ,
y^12 ,
(x * y^-3)^2 ,
x * y^2 * x^4 * y^-5 ,
(x * y * x)^4 ,
(x * y * x^-1 * y^-1 * x^-1 * y^-1)^2 ,
(x * y * x^-1 * y * x^-1 * y^-1)^2 ,
y * x * y * x^-1 * y^-1 * x^3 * y^4 * x^2,
t^2 >;
```

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G:

```
N: = sub < S|xx,yy >;
#N;
{14400}

CompositionFactors(N);
    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Alternating(5)
    1
```

Normal Lattice

Figure 6.2: $(A5 \times A5) : 4$

By looking to the composition factors of G it is not clear what the isomorphism type is. After analyzing the normal lattice of G, we see that $NL[1]$ is the largest normal abelian subgroup. The order of $NL[1]$ is 1 and the order of G is 14400. Now $\frac{14400}{1} = 14400$. However, G has a normal subgroup of order 1. Thus, G is a direct-product. We write G as follows:

$N \cong (A5xA5) : 4.$

6.3 Direct-Product $5^4 \times 5$

Given G a transitive group of 25 letters generated by:

$xx \sim (1, 12, 25, 7, 20, 3, 14, 23, 9, 17, 5, 11, 21, 6, 19, 2, 13, 24, 8, 16, 4, 15, 22, 10, 18)$, and

$yy \sim (1, 16, 10, 24, 15, 4, 19, 8, 21, 13, 2, 17, 6, 23, 11, 5, 20, 9, 25, 14, 3, 18, 7, 22, 12)$.

The presentation of G is:

```
G<x,y,t>:=Group<x,y,t|
y^-1 * x^-1 * y * x^2 * y * x^-1 * y^-1 ,
y * x^5 * y^4 ,
(y^-1 * x^-1)^5 ,
x^-1 * y^-1 * x * y^-1 * x^3 * y * x^-2 * y * x^-1 ,
y * x * y^-1 * x^-1 * y * x^-2 * y^-1 * x^-1 * y^-1 *
x^-1 * y * x^-1,
t^2 >;
```

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G:

```
CompositionFactors(G);
   G
   |  Cyclic(5)
   *
   |  Cyclic(5)
   *
   |  Cyclic(5)
   *
   |  Cyclic(5)
   *
   |  Cyclic(5)
   1
```
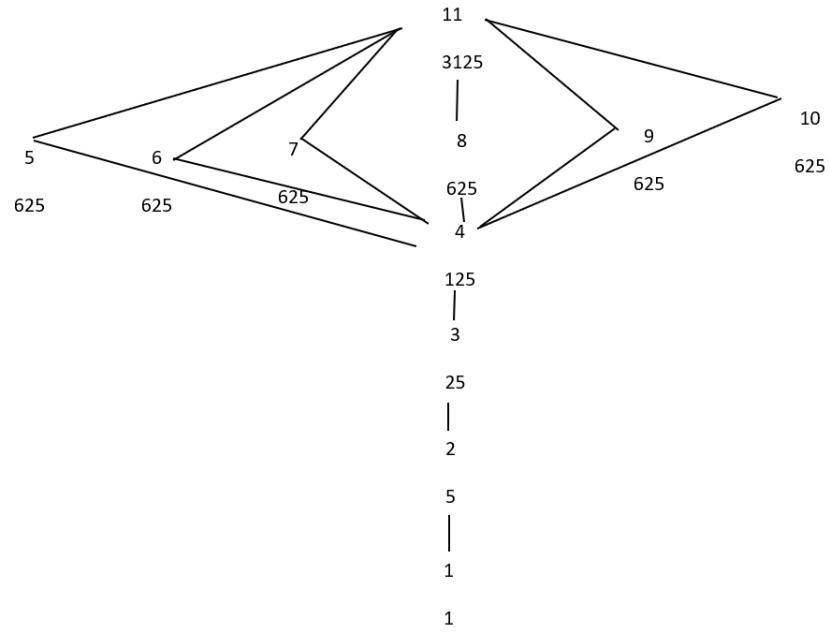
Normal Lattice

Figure 6.3: $5^4 \cdot 5$

After analyzing the normal lattice of G, we see that $NL[6]$ is the largest normal abelian subgroup. The order of $NL[6]$ is 625 and the order of G is 3125. Now $\frac{3125}{625} = 5$. We see that G has a normal subgroup of order 5. Thus, G is a direct product $G \cong 5^4 \times 5$

6.4 Semi-Direct Product $5^2 : 4$

We are given G is a transitive group on 25 letters which generated by

$xx \sim (1, 13, 23, 10, 20)(2, 14, 24, 6, 16)(3, 15, 25, 7, 17)(4, 11, 21, 8, 18)(5, 12, 22, 9, 19)$

and $yy \sim (1, 18, 14, 21)(2, 20, 13, 24)(3, 17, 12, 22)(4, 19, 11, 25)(5, 16, 15, 23)(6, 9, 10, 7)$.

The presentation of G is:

```
G<x,y,t>:=Group<x,y,t|x^5 ,
y^4 ,
y^-2 * x^-1 * y^2 * x^-1 ,
x^-1 * y^-1 * x^-1 * y^-1 * x^-1 * y * x * y,
t^2>;
```

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G:

```
G: = sub < S|xx,yy >;
#G;
100

CompositionFactors(G);
 G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    1
```
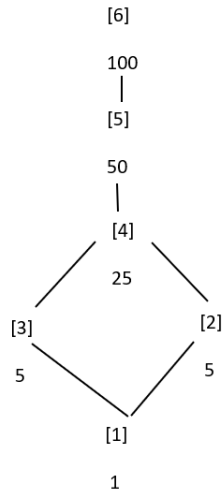
<u>Normal Lattice</u>

Figure 6.4: $5^2 : 4$

By looking to the composition factors of G it is not clear what the isomorphism type is. After analyzing the normal lattice of G, we see that $NL[4]$ is the largest normal abelian subgroup. The order of $NL[4]$ is 25 and the order of G is 100. Now $\frac{100}{25} = 4$. But G does not have a normal subgroup of order 4. Thus, G is an extension of $NL[4]$ by a group $q = \frac{G}{NL[4]}$ but it is not a direct product because q is not isomorphic to a normal subgroup of G.

Now, we need to investigate to see whether G is the semi-direct product (split extension) of $NL[4]$ by q. It is clear that $NL[4]$ is a group with the generator:

$A \sim (1, 21, 16, 12, 7)(2, 22, 17, 13, 8)(3, 23, 18, 14, 9)(4, 24, 19, 15, 10)(5, 25, 20, 11, 6)$

$B \sim (1, 15, 22, 6, 18)(2, 11, 23, 7, 19)(3, 12, 24, 8, 20)(4, 13, 25, 9, 16)(5, 14, 21, 10, 17)$

Now, q is isomorphic to 5^2 and its presentation is given below:

```
IsIsomorphic(NL[4],AbelianGroup(GrpPerm,[5,5]));
True

FPGroup(q);
d^4 ,
    c
```

The group q has two generators $q.1$, and $q.2$ of order $4 = 2^2$. We label them C and d respectively. Our next step is to find the action of the generators C and d on the

generators A and B of $NL[4]$. In order to do so, we need to look at the transversals of $NL[4]$ in G:

```
ff(T[1]) eq q.1;
/*true*/
ff(T[2]) eq q.2;
/*true*/
```

Thus, we see $a^c = b$, $b^c = a^4$. We will include these results in our presentation of H and verify it is isomorphic to G to find the presentation of G.

```
H <a,b,c,d >:= Group <a, b, c,d | a^5, b^5,(a,b),
c, d^4,
a^c=b, b^c=a^4,
a^d=a^4,
b^d=b^4>;

#H;
100

f,H1,k: = CosetAction(H,sub < H|Id(H) >;
IsIsomorphic(H1,G);
True
```

This true isomorphism tells us that we have a semi-direct product. Now we need to consider how to write our progenitor. We do so by looking at the order of $NL[4]$ which was confirmed to us using isomorphism of $NL[4]$ in respects to group permutations to be 5^2. Next we look at the order of q, which is 4.

Thus we have a semi-direct product $G \cong 5^2 : 4$.

6.5 Semi-Direct Product $5^2 : 2^5$

We are given G is a transitive group on 25 letters, which is generated by
$xx \sim (2, 15, 6, 23)(3, 24, 11, 20)(4, 8, 16, 12)(5, 17, 21, 9)(10, 14, 22, 18)$,
$yy \sim (2, 23, 4, 12, 5, 9, 3, 20)(6, 17, 16, 24, 21, 15, 11, 8)(7, 14, 19, 10, 25, 18, 13, 22)$,
and $zz \sim (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)(16, 17, 18, 19, 20)(21, 22, 23, 24, 25)$.
The presentation of G is:

```
G<x,y,z,t>:=Group<x,y,z,t| x^4 ,
y^8 ,
z^5 ,
(x^-1 * y^-1)^2 ,
y * x * y^-1 * x * y^2 ,
x * z^-1 * y * x * z * y ,
y^-1 * x * z^-1 * x^-1 * y * z^-2 ,
z * x^-2 * z^-1 * x^-1 * z^-1 * x^-1 * z,
t^2 >;
```

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G:

```
 CompositionFactors(G);

    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    1
```
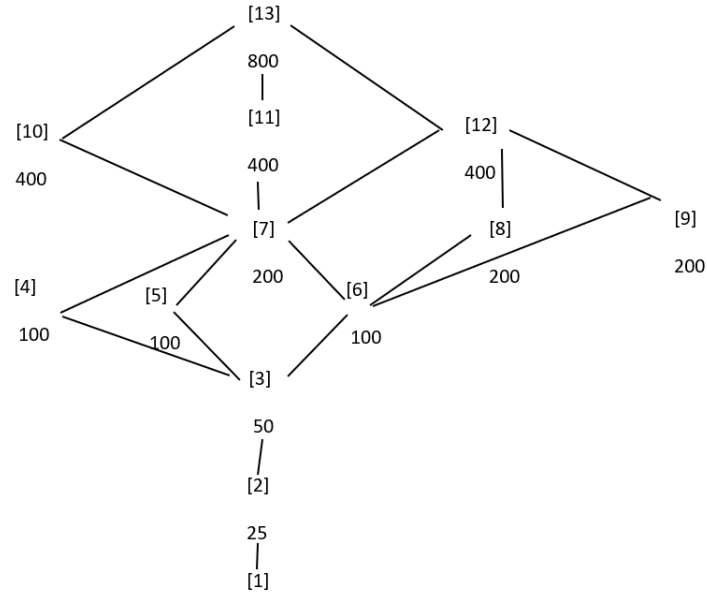
Normal Lattice
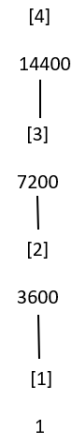

Figure 6.5: $5^2 : 2^5$

By looking to the composition factors of G, it is not clear what the isomorphism type is. After analyzing the normal lattice of G, we see that $NL[2]$ is the largest normal abelian subgroup. The order of $NL[2]$ is 25 and the order of G is 800. Now $\frac{800}{25} = 32$. But G does not have a normal subgroup of order 32. Thus, G is an extension of $NL[2]$ by a group say $q = \frac{G}{NL[2]}$ but it is not a direct product because q is not isomorphic to a normal subgroup of G.

Now, we need to investigate to see whether G is the semi-direct product (split extension) of $NL[2]$ by q. It is clear that $NL[2]$ is a cyclic group of order 25 with the generator:

$A \sim (1, 22, 18, 14, 10)(2, 23, 19, 15, 6)(3, 24, 20, 11, 7)(4, 25, 16, 12, 8)(5, 21, 17, 13, 9)$

$B \sim (1, 6, 11, 16, 21)(2, 7, 12, 17, 22)(3, 8, 13, 18, 23)(4, 9, 14, 19, 24)(5, 10, 15, 20, 25)$.

Now, q is isomorphic to 5^2 and its presentation is given below:

```
IsIsomorphic(NL[2],AbelianGroup(GrpPerm,[5,5]));
True

q;
x^4,
(x^-1 * y^-1)^2,
```

```
y * x * y^-1 * x * y^2,
z>;
```

The group q has two generators $q.1$, and $q.2$, we label them as D and E respectively. Our next step is to find the action of the generators D and E on the generators A and B of $NL[2]$. In order to do so we need to look at the transversals of $NL[2]$ in G:

```
ff(T[2]) eq q.1;
/*true*/
ff(T[3]) eq q.2;
/*true*/
ff(T[1]) eq q.3;
/*true*/
```

Thus, we see $a^d = a^2, b^d = a^2 * b, a^e = a * b^2, b^e = a * b^4$.

We will include these results in our presentation of H and verify it is isomorphic to G to find the presentation of G.

```
H<a,b,d,e>:=Group<a,b,d,e|a^5,b^5,
(a,b), d^4,
(d^-1 * e^-1)^2,
e * d * e^-1 * d * e^2,
a^d=a^2,b^d=a^2*b,a^e=a*b^2,b^e=a*b^4>;

#G;
800

f,H1,k:=CosetAction(H,sub < H|Id(H)>);
IsIsomorphic(H1,G);
True
```

The true isomorphism tells us that we have a semi-direct product. Now we need to consider how to write our isomorphism. We do so by looking at the order of $NL[2]$ which was confirmed to us using isomorphism of $NL[2]$ in respects to group permutation to be 5^2. Next we look at the order of q which is of order $32 = 2^5$.

Thus, we have the following isomorphism $G \cong (5^2) : 2^5$.

6.6 Mixed Extension $3 \,\vdots\, PGL(2 \times 9)$

We are given G is a transitive group on 5 letters, which is generated by:

$xx \sim (1,2,3,4,6)$, and

$yy \sim (1,4)(5,6)$.

The presentation of G is:

```
G<x,y,t>:=Group<x,y,t|x^5,y^2,(y*x^-1)^3,
t^2 >;
```

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G:

```
 CompositionFactors(G);
G
| Cyclic(2)
*
| Alternating(6)
*
| Cyclic(3)
1
```

Normal Lattice

NL [4]

Order 2160

NL [3]

Order1080

NL [2]

Order3

NL [1]
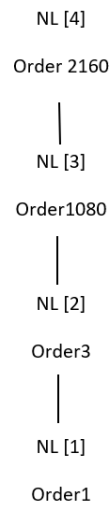
Order1

Figure 6.6: $PGL(2 \times 9)$

From the composition factors of G, we see that G must be one of the following: $3 \cdot S_6$ or $3 \cdot M_{10}$ or $3 \cdot PGL(2,9)$. After analyzing the normal lattice of G. we see that $NL[2]$ is

the largest normal abelian subgroup. The order of $NL[2]$ is 3 and the order of G is 2160. Now $\frac{2160}{3} = 720$. But G does not have a normal subgroup of order 720. Thus, G is an extension of $NL[2]$ by a group $q = \frac{G}{NL[2]}$. But it is not a direct product because q is not isomorphic to a normal subgroup of G.

Now, we need to investigate to see whether G is the semi-direct product (split extension) of $NL[2]$ by q. It is clear that $NL[2]$ is a cyclic group of order 3 with the generator:

$A = (1, 35, 36)(2, 32, 31)(3, 25, 27)(4, 33, 20)$

$(5, 21, 30)(6, 34, 28)(7, 17, 15)(8, 12, 26)$

$(9, 29, 23)(10, 24, 13)(11, 18, 22)(14, 19, 16)$;

Now q is isomorphic to $PGL(2 \times 9)$,

```
 IsIsomorphic (q,PGL(2,9));
True
```

The group q has three generators $q.1$, $q.2$, and $q.3$. We label them as B,C, and D respectively. Our next step is to find the action of the generators B,C, and D on the generator A of $NL[2]$. In order to do so, we need to look at the transversals of $NL[2]$ in G:

```
f(T[2]) eq q.1;
true
ff(T[3]) eq q.2;
true
ff(T[4]) eq q.3;
true
A^{T2} eq A;
true
A^{T3} eq A;
true
A^{T4} eq A^2;
true
```

Thus, we see $a^b = a, a^c = a, a^d = a^2$.

Now, we include these results in our presentation and obtain:

```
H<a,b,c,d>:=Group<a,b,c,d|a^3,b^5,c^2,d^2,
(c * b^-1)^3,
c * d * c * b^2 * d * b^-2,
(b^-1 * d * b * c)^2,
b^-1 * d * b^-1 * d * c * b * d * c * d * c,
a^b=a,a^c=a,a^d=a^2 >;
```

We now find a permutation representation and call it $H1$ to check if it is isomorphic to G,

```
 f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,G);
False
```

Since it is false we conclude it is not a semi-direct product.

There must be elements of q that can be written in terms of the generators of $NL[2]$.

The order of $b^{-1} * d * b^{-1} * d * c * b * d * c * d * c$ has changed. So, we need to rewrite $b^{-1} * d * b^{-1} * d * c * b * d * c * d * c$ in terms of the generators of $NL[2]$.

$T2^{-1} * T4 * T2^{-1} * T4 * T3 * T2 * T4 * T3 * T4 * T3$ eq A^2 ;

True

Thus, We modify the relation $T2^{-1} * T4 * T2^{-1} * T4 * T3 * T2 * T4 * T3 * T4 * T3$ to A^2 to get a presentation of G and check our work in magma.

Now, we know it is mixed extension, and our group presentation is:

```
H<a,b,c,d>:=Group<a,b,c,d|a^3,b^5,c^2,d^2,
(c * b^-1)^3,
c * d * c * b^2 * d * b^-2,
(b^-1 * d * b * c)^2,
b^-1 * d * b^-1 * d * c * b * d * c * d * c = a^2,
a^b = a, a^c = a, a^d = a^2>;

f,H1,K:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,G);
True
```

The true isomorphism tells us that we have a mixed extension $G \cong 3 : PGL(2 \times 9)$.

6.7 Mixed Extension $5^5 : (2^7 : 5)$

We are given G is a transitive group on 25 letters, which is generated by:

$xx \sim (1,2,4,3)(6,22)(7,25,10,24)(8,23,9,21)(11,19,14,17)(12,20,13,16)(15,18),$

$yy \sim (1,10,4,8,2,6,5,9,3,7)(11,22,13,24,15,21,12,23,14,25)(16,18,19,17).$

and $zz \sim (2,5)(3,4)(6,21,7,22)(8,23,10,25)(9,24)(11,20,13,16,15,17,12,18,14,19)$ The presentation of G is:

```
G<x,y,z,t>:=Group<x,y,z,t|x^4 , y^20 , z^20 ,
(x^-1 * z^-1)^4 , (x * z^-1)^4 ,
y^3 * x^-2 * y^4 * x^2 * y ,
(x^-1 * z^-3 * y)^2 ,
z^-1 * y^-1 * x^-2 * y * z^-1 * y * x^2 * y^-1 * z^-2,
z^3 * x^-2 * z^4 * x^2 * z ,
y^-1 * x^-1 * y^-1 * x^-2 * y * x^-1 * y * x^2 * z^-1 * x^2 * z ,
y^-1 * x^-1 * y^-1 * z^-1 * x^-1 * y^-1 * x^-1 * z^2 * y^-1 * z^-1 * x^-1 ,
y * z^-1 * y^-1 * z^-1 * x^-1 * y^-1 * x^-1 * y^2 * z^-2 * y ,
(x * z * y^-1 * z^-1 * x^-1 * y^-1)^2 ,
(x^-1 * z^-1 * y * z * x^-1 * y^-1)^2 ,
y^-1 * z^-1 * y * z * x^-1 * y^-1 * x * y * z^-4 ,
(x^-1 * z * y^-1 * z^-1 * x * y^-1)^2 ,
z * y^2 * z^-1 * x * y^-1 * x^-1 * z^-1 * y^-1 * x * z^-1 * x^-1 ,
z^-1 * y * x^-1 * z * x * y^-1 * x^-1 * z * y^-1 * x * y * z^-1 ,
(x * z^-1 * y * z * x * y^-1)^2 ,
y^-1 * z * y * z * x * y^-1 * x * y^-1 * z * x^-1 * z^-1 * x ,
z * x^-1 * z * x^-1 * y^-2 * x * z^-1 * x * z^-1 * y^-2,
z * x * z^-1 * x * y^-2 * x^-1 * z * y * z * x^-1 * y,
(x^-1 * z^-1 * y * x^-1 * z^-1 * y^-1)^2 ,
z^-1 * y^-1 * z * x^-1 * z^-1 * y^-1 * x * z * y^-1 *
z^-1 * y^-1 * z^-1 ,
(x * z^-1 * y^-1)^4 ,
z^2 * y^-1 * x * z^-1 * y^-1 * x^-1 * z^-3 * y^-2 ,
z * x * z * y * z^-1 * y^-1 * x^-1 * y^-1 * z * y * x^-1 * y^-1 ,
x^-1 * z * y^-1 * z^2 * y^-1 * x^-1 * z * x * z * x^-1 * z^-1>;
```

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G:

```
CompositionFactors(G);
/*
 G
    |  Cyclic(2)
    *
```

```
|  Cyclic(5)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(5)
*
|  Cyclic(2)
*
|  Cyclic(5)
*
|  Cyclic(2)
*
|  Cyclic(5)
*
|  Cyclic(2)
*
|  Cyclic(5)
*
|  Cyclic(2)
*
|  Cyclic(5)
1
```

Normal Lattice
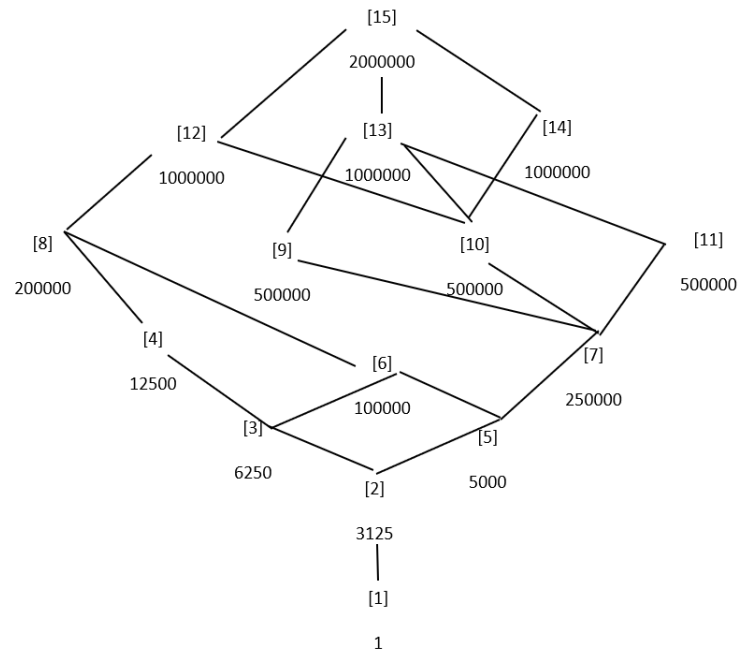
Figure 6.7: $5^5 : (2^7 : 5)$

After analyzing the normal lattice of G, we see that $NL[2]$ is the largest normal abelian subgroup. The order of $NL[2]$ is 3125 and the order of G is 2000000. Now $\frac{2000000}{3125} = 640$. But G does not have a normal subgroup of order 640. Thus, G is an extension of $NL[2]$ by a group $q = \frac{G}{NL[2]}$. But it is not a direct product because q is not isomorphic to a normal subgroup of G.

Now, we need to investigate to see whether G is the semi-direct product (split extension) of $NL[2]$ by q. $NL[2]$ is of order 3125 with the generators:

$A := N!(16, 17, 18, 19, 20)(21, 25, 24, 23, 22)$;

$B := N!(21, 22, 23, 24, 25)$;

$C := N!(11, 14, 12, 15, 13)(16, 18, 20, 17, 19)(21, 22, 23, 24, 25)$;

$D := N!(1, 5, 4, 3, 2)(11, 14, 12, 15, 13)$;

$E := N!(6, 7, 8, 9, 10)(16, 17, 18, 19, 20)$;

Now q is isomorphic to 5^4,

```
IsIsomorphic(q,AbelianGroup(GrpPerm,[5,5,5,5,5]));
True
```

The group q has three generators $q.1, q.2$, and $q.3$. We label them as f, g, and h respectively. Our next step is to find the action of the generators f, g, and h on the generators A, B, C, D, and E of $NL[2]$. In order to do so, we need to look at the transversals of $NL[2]$ in G:

```
ff(T[2]) eq q.1;
/*true*/
ff(T[3]) eq q.2;
/*true*/
ff(T[4]) eq q.3;
/*true*/
```

Thus, we see $a^f = a^2 * b^4 * c^3 * e^2, b^f = a^2 * b^2 * e^3, c^f = a^3 * b^2 * c * e^3, d^f = a^2 * b^4 * c^3 * d^2, e^f = a^4 * b^4 * c^3,$

$a^g = a^2 * b^4 * c^3, b^g = a * b^4 * c^2, c^g = a^2 * b^3 * c^2, d^g = a * b^4 * e^4, e^g = a * c * d^4,$

$a^h = a * b^2 * c^4 * e, b^h = a * b * e^4, c^h = a^4 * b * c^3 * e^4, d^h = a^2 * b * c * d^4, e^h = a^2 * b^4 * c^4 >;$

Now, we include these results in our presentation and obtain:

```
H<a,b,c,d,e,f,g,h>:=Group<a,b,c,d,e,f,g,h|a^5,b^5,c^5,d^5,e^5,(a,b),
(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
f^4 , g^4 ,  h^4 ,
f^-1 * h^-1 * f * h^-1 ,
f^-1 * g^-1 * f^2 * g * f^-1 ,
h^-1 * f^-1 * g^-1 * f^-1 * h * g^-1 ,
h * f^-1 * g^-1 * f * h^-1 * g ,
g^-2 * f^-1 * g^-2 * f^-1 * g^2 * f * g^2 * f ,
g^-1 * f^-1 * g^-1 * f^-1 * g^-1 * f * g^-1 * f * g^-1 *
f^-1,
a^f,b^f,c^f,d^f,e^f,a^g,b^g,c^g,d^g,e^g,a^h,b^h,c^h,d^h,e^h>;
```

We now find a permutation representation and call it $H1$ to check if it is isomorphic to G,

```
 f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,G);
False
```

The false statement tells us that it is not a semi-direct product.

There must be elements of q that can be written in terms of the generators of $NL[2]$.

Thus, We modify some of the relations given in the presentation and have different orders

than before to get a presentation of G and check our work in magma.

Now, we know it is mixed extension, and our group presentation is:

```
H<a,b,c,d,e,f,g,h>:=Group<a,b,c,d,e,f,g,h|a^5,b^5,c^5,d^5,e^5,(a,b),
(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
f^4 , g^4 , h^4 ,
f^-1 * h^-1 * f * h^-1 ,
f^-1 * g^-1 * f^2 * g * f^-1 ,
h^-1 * f^-1 * g^-1 * f^-1 * h * g^-1 ,
h * f^-1 * g^-1 * f * h^-1 * g ,
g^-2 * f^-1 * g^-2 * f^-1 * g^2 * f * g^2 * f ,
g^-1 * f^-1 * g^-1 * f^-1 * g^-1 * f * g^-1 * f * g^-1 *
f^-1,
a^f=a^2*b^4*c^3*e^2,b^f=a^2*b^2*e^3,c^f=a^3*b^2*c*e^3,
d^f=a^2*b^4*c^3*d^2,e^f=a^4*b^4*c^3,
a^g=a^2*b^4*c^3,b^g=a*b^4*c^2,c^g=a^2*b^3*c^2,
d^g=a*b^4*e^4,e^g=a*c*d^4,
a^h=a*b^2*c^4*e,b^h=a*b*e^4,c^h=a^4*b*c^3*e^4,
d^h=a^2*b*c*d^4,e^h=a^2*b^4*c^4>;

f,H1,K:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,G);
True
```

Thus, we have the mixed extension $G \cong 5^5 : (2^7 : 5)$.

Chapter 7

Double Coset Enumeration

7.1 Construction of $3^4 : (2^2 : S_3)$ over $(3^3 : (3 : 2))$

Let $N \cong (3^3 : (3 : 2))$ with $\langle x, y \rangle$ where

$x \sim (1, 9, 5)(2, 7, 6)(3, 8, 4)$, and

$y \sim (1, 6, 2, 4, 3, 5)(7, 8, 9)$

The progenitor $2^{*9} : N$ has symmetric presentation.

```
G<x,y,t>:=Group<x,y,t|
x^3, y^6,
y * x * y^-1 * x^-1 * y^-1 * x * y,
t^2>;
```

We want to verify that the presentation given above is correct.

Using the theorem: $\frac{2^{*n}:N}{t_i, t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$

We will show that $\frac{2^{*9}:N}{((y^2*x^{-1}*y^{-1}*x^{-1})*t^{(x*y*x^{-1}*y*x)})^3} \cong (3^4 : (2^2 : S_3))$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|
x^3, y^6,
y * x * y^-1 * x^-1 * y^-1 * x * y,
t^2,
(t,(y^2 * x^-1 * y^-1)),
(t,(x^-1 * y^2 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-2)),
((y^2 * x^-1 * y^-1 * x^-1)*t^{( x * y * x^-1 * y * x)})^3> ~ (3^4:(2^2:S_3))
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one

representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw. We find our index which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{1944}{162} = 12$. Now we know that we have 12 distinct single cosets.

Expanding Relation We will expand our only relation.

$$((y^2 * x^{-1} * y^{-1} * x^{-1}) * t^{(x*y*x^{-1}*y*x)})^3$$

$$= (y^2 * x^{-1} * y^{-1} * x^{-1})t_3$$
$$= (y^2 * x^{-1} * y^{-1} * x^{-1}) * t_3 * (y^2 * x^{-1} * y^{-1} * x^{-1})t_3 * (y^2 * x^{-1} * y^{-1} * x^{-1}) * t_3$$
$$= (y^2 * x^{-1} * y^{-1} * x^{-1})^3 * t_3^{(y^2*x^{-1}*y^{-1}*x^{-1})^2} * t_3^{(y^2*x^{-1}*y^{-1}*x^{-1})}$$
$$= (y^2 * x^{-1} * y^{-1} * x^{-1})^3 * t_3 * t_4$$
$$\implies (y^2 * x^{-1} * y^{-1} * x^{-1})^3 * t_3 = t_4$$

First Double Coset [∗]

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N. The formula for the number of right coset in N is
$\frac{|N|}{|N|} = \frac{162}{162} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We need only determine the double coset of the right coset Nt_1. Thus nine t_i's extend the double coset $[*]$ to the double coset $Nt_1N = [1]$.
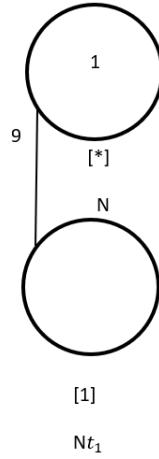
Cayley Diagram

Figure 7.1: Cayley Diagram for G over S_9

Second Double Coset [1]

$Nt_1N = \{Nt_1^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9\}$.

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\} = \langle (4,8,6,7,5,9), (7,8,9) \rangle$

Thus, $N^{(1)} \geq N^1 = \langle (4,8,6,7,5,9), (7,8,9) \rangle$.

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{162}{18} = 9$.

The orbits of $N^{(1)}$ on $X = \{1,2,3,4,5,6,7,8,9\}$ are

$\{1\}, \{2\}, \{3\}, \{4,8,6,9,7,5\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = N \in [*]$.

This means one t_i takes [1] to [*].

Choose 2 from $\{2\}$

$Nt_1t_2 \in [12]$.

This means one t_i takes [1] to [12].

Choose 3 from $\{3\}$

$Nt_1t_3 \in [13]$.

This means one t_i takes [1] to [13].

Choose 4 from $\{4, 8, 6, 9, 7, 5\}$

$Nt_1t_4 \in [1]$.

We have $Nt_1t_4 = y^3 \, t_1$

This means six t_i's take [1] to [1].
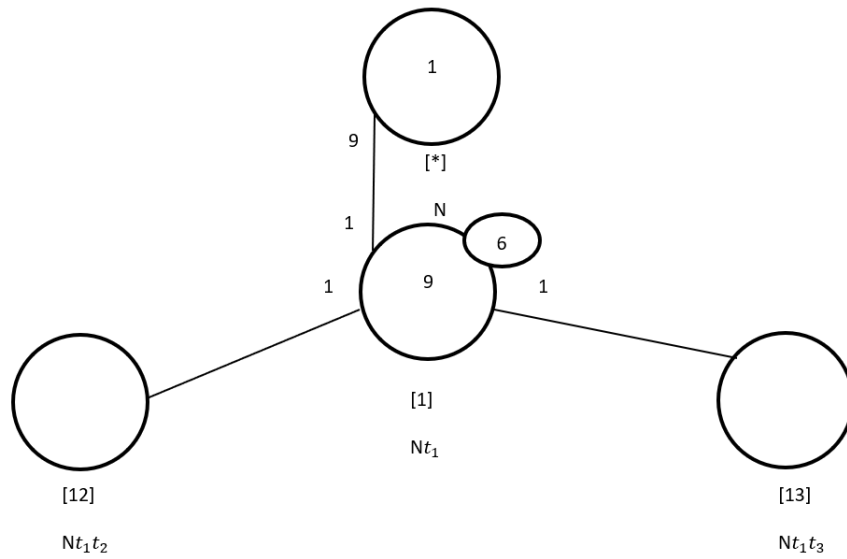
Cayley Diagram

Figure 7.2: Cayley Diagram for G over S_9

Third Double Coset [12]

$Nt_1t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(12)}$. We first find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$

$N^{12} = \langle (4, 8, 6, 7, 5, 9), (7, 8, 9) \rangle$

We find different names of Nt_1t_2. Now $t_1t_2 = (y^-1 * x * y * x)t_7t_8 \implies Nt_1t_2 = Nt_7t_8$.

We have $N(t_1t_2)^{(1,7)(2,8)\cdots} = Nt_7t_8$.

But $Nt_7t_8 = Nt_1t_2$ so, $N(t_1t_2)^{(1,7)(2,8)\cdots} = Nt_1t_2$

$\implies (1, 7, 2, 8, 3, 9)(4, 5, 6), (1, 7, 6, 2, 8, 4, 3, 9, 5), (1, 7)(2, 8)(3, 9),$

$(1, 7, 5, 3, 9, 4, 2, 8, 6), (1, 7, 3, 9, 2, 8)(4, 6, 5), (1, 7, 4)(2, 8, 5)(3, 9, 6), (1, 7, 3, 9, 2, 8)(4, 5, 6),$

$(1, 7, 4, 3, 9, 6, 2, 8, 5), (1, 7, 2, 8, 3, 9), (1, 7, 6)(2, 8, 4)(3, 9, 5),$

$(1, 7)(2, 8)(3, 9)(4, 6, 5), (1, 7, 5, 2, 8, 6, 3, 9, 4), (1, 7)(2, 8)(3, 9)(4, 5, 6),$

$(1, 7, 5)(2, 8, 6)(3, 9, 4), (1, 7, 3, 9, 2, 8), (1, 7, 4, 2, 8, 5, 3, 9, 6),$

$(1, 7, 2, 8, 3, 9)(4, 6, 5), (1, 7, 6, 3, 9, 5, 2, 8, 4) \in N^{(12)}$

Thus $N^{((12)} \geq \langle N^{12}, (1, 7, 2, 8, 3, 9)(4, 5, 6),$

$(1, 7, 6, 2, 8, 4, 3, 9, 5), (1, 7)(2, 8)(3, 9), (1, 7, 5, 3, 9, 4, 2, 8, 6),$

$(1, 7, 3, 9, 2, 8)(4, 6, 5), (1, 7, 4)(2, 8, 5)(3, 9, 6), (1, 7, 3, 9, 2, 8)(4, 5, 6),$

$(1, 7, 4, 3, 9, 6, 2, 8, 5), (1, 7, 2, 8, 3, 9), (1, 7, 6)(2, 8, 4)(3, 9, 5),$

$(1, 7)(2, 8)(3, 9)(4, 6, 5), (1, 7, 5, 2, 8, 6, 3, 9, 4), (1, 7)(2, 8)(3, 9)(4, 5, 6),$

$(1, 7, 5)(2, 8, 6)(3, 9, 4), (1, 7, 3, 9, 2, 8), (1, 7, 4, 2, 8, 5, 3, 9, 6),$

$(1, 7, 2, 8, 3, 9)(4, 6, 5), (1, 7, 6, 3, 9, 5, 2, 8, 4)\rangle$

$= \langle (4, 8, 6, 7, 5, 9), (7, 8, 9), (1, 7, 2, 8, 3, 9)(4, 5, 6),$

$(1, 7, 6, 2, 8, 4, 3, 9, 5), (1, 7)(2, 8)(3, 9), (1, 7, 5, 3, 9, 4, 2, 8, 6),$

$(1, 7, 3, 9, 2, 8)(4, 6, 5), (1, 7, 4)(2, 8, 5)(3, 9, 6), (1, 7, 3, 9, 2, 8)(4, 5, 6),$

$(1, 7, 4, 3, 9, 6, 2, 8, 5), (1, 7, 2, 8, 3, 9), (1, 7, 6)(2, 8, 4)(3, 9, 5),$

$(1, 7)(2, 8)(3, 9)(4, 6, 5), (1, 7, 5, 2, 8, 6, 3, 9, 4), (1, 7)(2, 8)(3, 9)(4, 5, 6),$

$(1, 7, 5)(2, 8, 6)(3, 9, 4), (1, 7, 3, 9, 2, 8), (1, 7, 4, 2, 8, 5, 3, 9, 6),$

$(1, 7, 2, 8, 3, 9)(4, 6, 5), (1, 7, 6, 3, 9, 5, 2, 8, 4)\rangle \cong 3^3 : S_3.$

The number of distinct right cosets in Nt_1t_2N is calculated by the formula.

$\frac{|N|}{|N^{(12)}|} = \frac{162}{162} = 1.$

The orbits of $N^{(12)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is a single orbit

$\{1, 7, 5, 8, 2, 6, 3, 4, 9\}$ We will determine the double cosetes by selecting one representative

from this orbit such as,

Choose 2 from $\{1, 7, 5, 8, 2, 6, 3, 4, 9\}$

$Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1].$

This means nine t_i's take $[12]$ to $[1]$.

Cayley Diagram

Figure 7.3: Cayley Diagram for G over S_9

Fourth Double Coset [13]

$Nt_1t_3 = \{N(t_1t_3)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(13)}$. We first find the point stabilizer of 1 and 3 in N.

$N^{13} = \{n \in N | (13)^n = 13\}$

$N^{13} = \langle (4,8,6,7,5,9), (7,8,9) \rangle$

We find different names of Nt_1t_3. Now $t_1t_3 = (x^-1*y^-1*x^-1*y)t_7t_9 \implies Nt_1t_3 = Nt_7t_9$.

We have $N(t_1t_3)^{(1,7)(2,8)\cdots} = Nt_7t_9$.

But $Nt_7t_9 = Nt_1t_3$ so, $N(t_1t_3)^{(1,7)(2,8)\cdots} = Nt_1t_3$

$\implies (1,7,2,8,3,9)(4,5,6), (1,7,6,2,8,4,3,9,5), (1,7)(2,8)(3,9),$

$(1,7,5,3,9,4,2,8,6), (1,7,3,9,2,8)(4,6,5), (1,7,4)(2,8,5)(3,9,6), (1,7,3,9,2,8)(4,5,6),$

$(1,7,4,3,9,6,2,8,5), (1,7,2,8,3,9), (1,7,6)(2,8,4)(3,9,5),$

$(1,7)(2,8)(3,9)(4,6,5), (1,7,5,2,8,6,3,9,4), (1,7)(2,8)(3,9)(4,5,6),$

$(1,7,5)(2,8,6)(3,9,4), (1,7,3,9,2,8), (1,7,4,2,8,5,3,9,6),$

$(1,7,2,8,3,9)(4,6,5), (1,7,6,3,9,5,2,8,4) \in N^{(13)}$.

Thus $N^{((13)} \geq \langle N^{13}, (1,7,2,8,3,9)(4,5,6),$

$(1,7,6,2,8,4,3,9,5), (1,7)(2,8)(3,9), (1,7,5,3,9,4,2,8,6),$

$(1, 7, 3, 9, 2, 8)(4, 6, 5), (1, 7, 4)(2, 8, 5)(3, 9, 6), (1, 7, 3, 9, 2, 8)(4, 5, 6),$

$(1, 7, 4, 3, 9, 6, 2, 8, 5), (1, 7, 2, 8, 3, 9), (1, 7, 6)(2, 8, 4)(3, 9, 5),$

$(1, 7)(2, 8)(3, 9)(4, 6, 5), (1, 7, 5, 2, 8, 6, 3, 9, 4), (1, 7)(2, 8)(3, 9)(4, 5, 6),$

$(1, 7, 5)(2, 8, 6)(3, 9, 4), (1, 7, 3, 9, 2, 8), (1, 7, 4, 2, 8, 5, 3, 9, 6),$

$(1, 7, 2, 8, 3, 9)(4, 6, 5), (1, 7, 6, 3, 9, 5, 2, 8, 4)\rangle$

$= \langle (4, 8, 6, 7, 5, 9), (7, 8, 9), (1, 7, 2, 8, 3, 9)(4, 5, 6),$

$(1, 7, 6, 2, 8, 4, 3, 9, 5), (1, 7)(2, 8)(3, 9), (1, 7, 5, 3, 9, 4, 2, 8, 6),$

$(1, 7, 3, 9, 2, 8)(4, 6, 5), (1, 7, 4)(2, 8, 5)(3, 9, 6), (1, 7, 3, 9, 2, 8)(4, 5, 6),$

$(1, 7, 4, 3, 9, 6, 2, 8, 5), (1, 7, 2, 8, 3, 9), (1, 7, 6)(2, 8, 4)(3, 9, 5),$

$(1, 7)(2, 8)(3, 9)(4, 6, 5), (1, 7, 5, 2, 8, 6, 3, 9, 4), (1, 7)(2, 8)(3, 9)(4, 5, 6),$

$(1, 7, 5)(2, 8, 6)(3, 9, 4), (1, 7, 3, 9, 2, 8), (1, 7, 4, 2, 8, 5, 3, 9, 6),$

$(1, 7, 2, 8, 3, 9)(4, 6, 5), (1, 7, 6, 3, 9, 5, 2, 8, 4)\rangle \cong 3^3 : S_3.$

The number of right cosets in Nt_1t_3N is calculated by the formula,

$\frac{|N|}{|N^{(13)}|} = \frac{162}{162} = 1.$

The orbits of $N^{(13)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is a single orbit

$\{1, 7, 5, 8, 2, 6, 3, 4, 9\}.$

We will determine the double cosetes by selecting one representative from this orbit such

as,

Choose 3 from {1, 7, 5, 8, 2, 6, 3, 4, 9}

$Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1].$

This means nine t_i's take [13] to [1].

Cayley Diagram

Figure 7.4: Cayley Diagram for G over S_9

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases. Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_2 \cup Nt_1t_3$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} \times |N|)$

$|G| \leq (1 + 9 + 1 + 1 + 1) \times 162 \implies |G| \leq 12 \times 162 \implies |G| \leq 1944.$

G acts on 12 cosets that are given in the cayley diagram.

Let X be the set of these 12 cosets.

Now, $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle \implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|$

But $\#\langle f(x), f(y), f(t) \rangle = 1944.$

So, $|\frac{G}{kerf}| = 1944$

This means $|G| \geq 1944$. We know $|G| \leq 1944$ from cayley diagram.

Therefore, $|G| = 1944.$

From $|G| = 1944 \times |Kerf|$ we find $|Kerf| = 1$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong 3^3 : (3 : 2) \implies G \cong 3^3 : (3 : 2).$

7.2 Construction of $3^4 : (2^3 : S_3)$ over $(3^3 : (3 : 2))$

Let $N \cong (3^3 : (3 : 2))$ with $\langle x, y \rangle$ where

$x \sim (1,9,5)(2,7,6)(3,8,4)$, and

$y \sim (1,6,2,4,3,5)(7,8,9)$

The progenitor $2^{*9} : N$ has symmetric presentation.

```
G<x,y,t>:=Group<x,y,t|
x^3, y^6,
y * x * y^-1 * x^-1 * y^-1 * x * y,
t^2 >;
```

We want to verify that the presentation given above is correct.

Using the theorem: $\frac{2^{*n}:N}{t_i,t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$

We will show that $\frac{2^{*9}:N}{(x^{-1}*y^{-1}*x^{-1}*y)*t^{(x*y^{-2})})^6,((x)*t^{(x*y*x^{-1}*y*x)})^4} \cong (3^4 : (2^3 : 3))$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|
x^3, y^6,
y * x * y^-1 * x^-1 * y^-1 * x * y,
t^2,
(t,(y^2 * x^-1 * y^-1)),
(t,(x^-1 * y^2 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-1 * x * y^-2)),
(x^-1 * y^-1 * x^-1 * y)*t^{( x * y^-2)})^6,
((x)*t^{( x * y * x^-1 * y * x)})^4> ~ (3^4:(2^3:3)).
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{3888}{162} = 24$. Now we know that we have 24 distinct single cosets.

Expanding Relation We will expand our first relation.

$$(x^{-1} * y^{-1} * x^{-1} * y) * t^{(x*y^{-2})})^6$$

$$= (x^{-1} * y^{-1} * x^{-1} * y) * t_7 * (x^{-1} * y^{-1} * x^{-1} * y) * t_7 * (x^{-1} * y^{-1} * x^{-1} * y) * t_7 *$$
$(x^{-1} * y^{-1} * x^{-1} * y) * t_7 * (x^{-1} * y^{-1} * x^{-1} * y) * t_7 * (x^{-1} * y^{-1} * x^{-1} * y) * t_7$
$= (x^{-1}*y^{-1}*x^{-1}*y)^6*t_7^{(x^{-1}*y^{-1}*x^{-1}*y)^5}*t_7^{(x^{-1}*y^{-1}*x^{-1}*y)^4}*t_7^{(x^{-1}*y^{-1}*x^{-1}*y)^3}*t_7^{(x^{-1}*y^{-1}*x^{-1}*y)^2}*$

$$t_7^{(x^{-1}*y^{-1}*x^{-1}*y)} * t_7$$

$$= (x^{-1} * y^{-1} * x^{-1} * y)^6 * t_9 * t_8 * t_7 * t_9 * t_8 * t_7$$

$$\implies (x^{-1} * y^{-1} * x^{-1} * y)^6 * t_9 * t_8 * t_7 = t_7 * t_8 * t_9$$

<u>Expanding Relation</u> we will expand our second relation.

$$((x) * t^{(x*y*x^{-1}*y*x)})^4$$

$$= x * t_3 * x * t_3 * x * t_3 * x * t_3$$

$$= x^4 * t_3^{x^3} * t_3^{x^2} * t_3^x * t_3$$

$$= x^4 * t_3 * t_4 * t_8$$

$$\implies x^4 * t_3 = t_8 * t_4$$

First Double Coset [$*$]

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right coset in N is $\frac{|N|}{|N|} = \frac{162}{162} = 1$.

Since N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

We need only determine the double coset of the right coset Nt_1.

Thus nine t_i's extend the double coset [$*$] to the double coset $Nt_1N = [1]$.

Cayley Diagram

Figure 7.5: Cayley Diagram for G over S_9

Second Double Coset [1]

$Nt_1N = \{Nt_1^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9\}$. We now find the Coset Stabilizer $N^{(1)}$.

We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

$N^1 = \langle (4, 8, 6, 7, 5, 9), (7, 8, 9) \rangle$.

Thus, $N^{(1)} \geq \langle (4, 8, 6, 7, 5, 9), (7, 8, 9) \rangle$

$\implies (4, 8, 6, 7, 5, 9), (7, 8, 9) \in N^{(1)}$.

The number of right cosets in Nt_1N is calculated by the formula, $\frac{|N|}{|N^{(1)}|} = \frac{162}{18} = 9$.

The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are $\{1\}, \{2\}, \{3\}, \{4, 8, 6, 9, 7, 5\}$. We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = N \in [*]$.

This means one t_i takes [1] to [*].

Choose 2 from $\{2\}$

$Nt_1t_2 \in [12]$.

This means one t_i takes [1] to [12].

Choose 3 from $\{3\}$

$Nt_1t_3 \in [13]$.

This means one t_i takes [1] to [13].

Choose 4 from $\{4, 8, 6, 9, 7, 5\}$

$Nt_1t_4 \in [14]$.

This means six t_i's take [1] to [14].

Cayley Diagram

Figure 7.6: Cayley Diagram for G over S_9

Third Double Coset [12]

$Nt_1t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(12)}$. We first find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$

$N^{12} = \langle (4, 8, 6, 7, 5, 9), (7, 8, 9) \rangle$.

We will find different names of Nt_1t_2. Now, $Nt_1t_2 = (y^{-1} * x * y * x)t_7t_8 \implies Nt_1t_2 = Nt_7t_8$

We have $N(t_1t_2)^{(1,2,3)(7,9,8)} = Nt_7t_8$. But $Nt_7t_8 = Nt_1t_2$

So, $N(t_1t_2)^{(1,2,3)(7,9,8)} = Nt_1t_2$

$\implies (1,2,3)(7,9,8) \in N^{(12)}$

Thus, $N^{(12)} \geq \langle N^{12}, (1,2,3)(7,9,8)\rangle = \langle (4,8,6,7,5,9), (7,8,9), (1,2,3)(7,9,8)\rangle$. The number of distinct right cosets in Nt_1t_2N is calculated by the formula.

$\frac{|N|}{|N^{(12)}|} = \frac{162}{162} = 1.$

The orbits of $N^{(12)}$ on $X = \{1,2,3,4,5,6,7,8,9\}$ is a single orbit

$\{1,7,5,8,2,6,3,4,9\}$. We will determine the double cosetes by selecting one representative from this orbit such as,

Choose 2 from $\{1, 7, 5, 8, 2, 6, 3, 4, 9\}$

$Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1].$
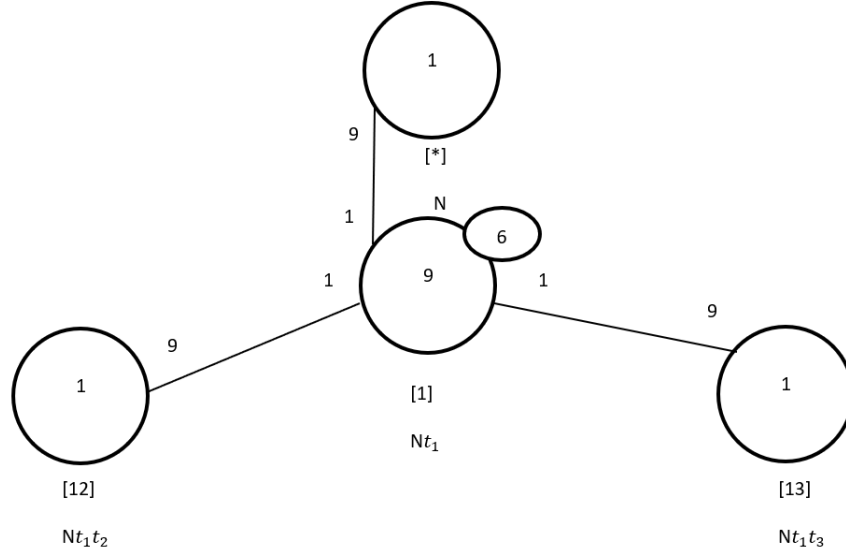
This means nine t_i's take [12] to [1].
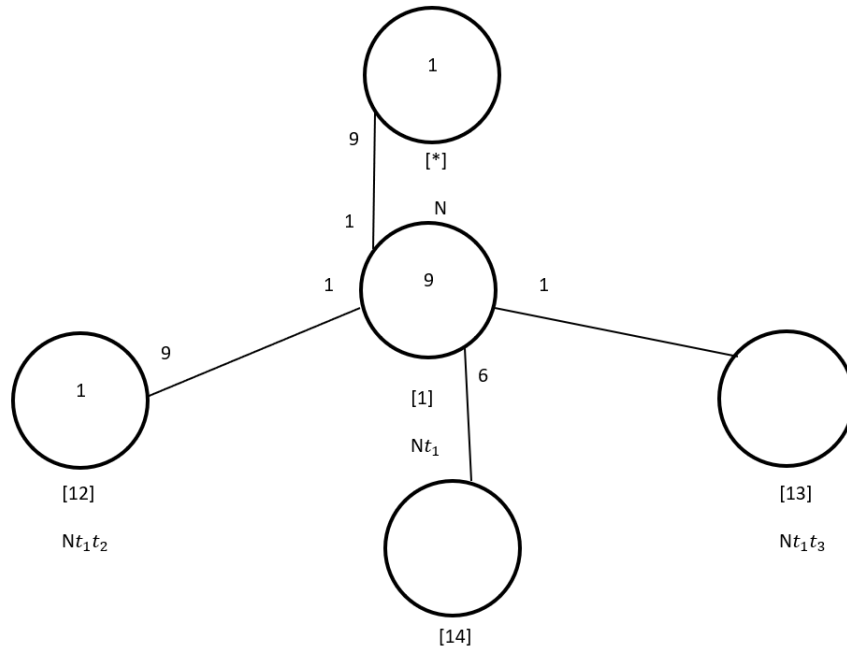
Cayley Diagram

Figure 7.7: Cayley Diagram for G over S_9

Fourth Double Coset [13]

$Nt_1t_3 = \{N(t_1t_3)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(13)}$. We first find the point stabilizer of 1 and 3 in N.

$N^{13} = \{n \in N | (13)^n = 13\}$

$N^{13} = \langle (4,8,6,7,5,9), (7,8,9) \rangle$.

We will find different names of Nt_1t_3. Now, $Nt_1t_3 = (x^-1 * y^-1 * x^-1 * y)t_3t_2 \implies$ $Nt_1t_3 = Nt_3t_2$

We have $N(t_1t_3)^{(1,3,2)(7,8,9)} = Nt_3t_2$. But $Nt_3t_2 = Nt_1t_3$

So, $N(t_1t_3)^{(1,3,2)(7,8,9)} = Nt_1t_3$

$\implies (1,3,2)(7,8,9) \in N^{(13)}$

Thus, $N^{(13)} \geq \langle N^{13}, (1,3,2)(7,8,9) \rangle = \langle (4,8,6,7,5,9), (7,8,9), (1,3,2)(7,8,9) \rangle$ The number of right cosets in Nt_1t_3N is calculated by the formula,

$\frac{|N|}{|N^{(13)}|} = \frac{162}{162} = 1$.

The orbits of $N^{(13)}$ on $X = \{1,2,3,4,5,6,7,8,9\}$ is a single orbit

$\{1,7,5,8,2,6,3,4,9\}$ We will determine the double cosetes by selecting one representative from this orbit such as,

Choose 3 from $\{1, 7, 5, 8, 2, 6, 3, 4, 9\}$

$Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1]$.

This means nine t_i's take [13] to [1].

Cayley Diagram

Figure 7.8: Cayley Diagram for G over S_9

Fifth Double Coset [14]

$Nt_1t_4 = \{N(t_1t_4)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(14)}$. We first find the point stabilizer of 1 and 4 in N.

$N^{14} = \{n \in N | (14)^n = 14\}$

$N^{14} = \langle (7,8,9) \rangle$.

We will find different names of Nt_1t_4. Now, $t_1t_4 = (x * y^- 1 * x * y * x^- 1)t_4t_9 \implies Nt_1t_4 = Nt_4t_9$

We have $N(t_1t_4)^{(1,4,9)(2,5,7)(3,6,8)} = Nt_4t_9$.

But $Nt_4t_9 = Nt_1t_4$. So, $N(t_1t_4)^{(1,4,9)(2,5,7)(3,6,8)} = Nt_1t_4$

$\implies (1,4,9)(2,5,7)(3,6,8) \in N^{(14)}$

Thus $N^{((14)} \geq \langle N^{14}, (1,4,9)(2,5,7)(3,6,8) \rangle$

$= \langle (7,8,9), (1,4,9)(2,5,7)(3,6,8) \rangle$ The number of right cosets in Nt_1t_4N is calculated by the formula,

$\frac{|N|}{|N^{(14)}|} = \frac{162}{18} = 9$.

The orbits of $N^{(14)}$ on $X = \{1,2,3,4,5,6,7,8,9\}$ are

$\{1\}, \{2\}, \{3\}, \{4,9,7,5,6,8\}$

We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_4t_1 \in [141]$. This means one t_i takes [14] to [141].

Choose 2 from $\{2\}$

$Nt_1t_4t_2 \in [142]$.

This means one t_i takes [14] to [142].

Choose 3 from $\{3\}$

$Nt_1t_4t_3 \in [143]$.

This means one t_i takes [14] to [143].

Choose 4 from $\{4, 9, 7, 5, 6, 8\}$

$Nt_1t_4t_4 = Nt_1t_4^2 = Nt_1 \in [1]$.

This means six t_i's take [14] to [1].

Cayley Diagram
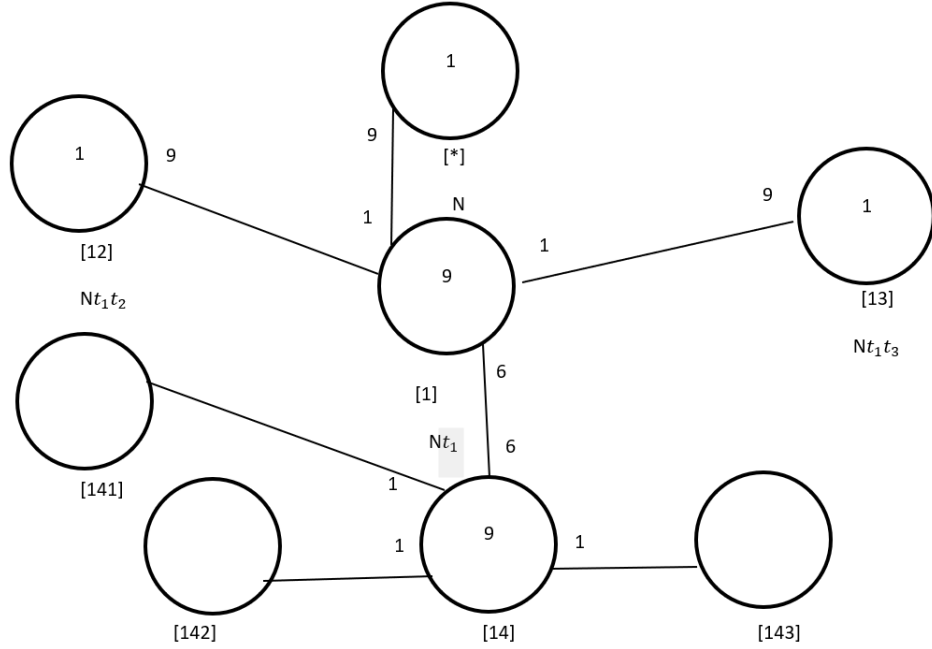
Figure 7.9: Cayley Diagram for G over S_9

Sixth Double Coset [141]

$Nt_1t_4t_1 = \{N(t_1t_4t_1)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(141)}$. We first find the point stabilizer of 1, 4 and 1 in N.

$N^{141} = \{n \in N | (141)^n = 141\}$

$N^{141} = \langle(7,8,9)\rangle$.

We will find different names of $Nt_1t_4t_1$. Now, $t_1t_4t_1 = (x^-1 * y * x * y^-1 * x)t_4t_8t_4 \implies Nt_1t_4t_1 = Nt_4t_8t_4$.

We have $N(t_1t_4t_1)^{(1,4,8)(2,5,9)(3,6,7)} = Nt_4t_8t_4$.

But $Nt_1t_4t_1 = Nt_4t_8t_4$. So, $N(t_1t_4t_1)^{(1,4,8)(2,5,9)(3,6,7)} = Nt_1t_4t_1$

$\implies (1,4,8)(2,5,9)(3,6,7) \in N^{(141)}$ Thus $N^{((141)} \geq \langle N^{141}, (1,4,8)(2,5,9)(3,6,7)\rangle$

$= \langle(7,8,9),(1,4,8)(2,5,9)(3,6,7)\rangle \cong 3^3 : S_3$. The number of right cosets in $Nt_1t_4t_1N$ is calculated by the formula,

$\frac{|N|}{|N^{(141)}|} = \frac{162}{162} = 1$.

The orbits of $N^{(141)}$ on $X = \{1,2,3,4,5,6,7,8,9\}$ is a single orbit

$\{1, 7, 5, 8, 2, 6, 3, 4, 9\}$ We will determine the double cosetes by selecting one representative from this orbit such as,

Choose 1 from $\{1, 6, 7, 8, 3, 2, 9, 5, 4\}$

$Nt_1t_4t_1t_1 = Nt_1t_4t_1^2 = Nt_1t_4 \in [14]$.

This means nine t_i's take $[141]$ to $[14]$.
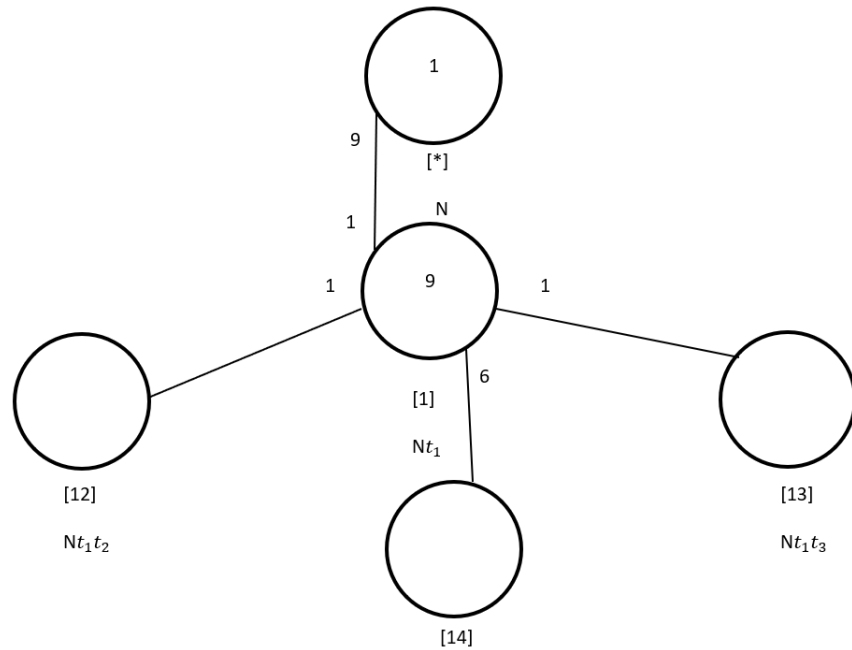
Cayley Diagram
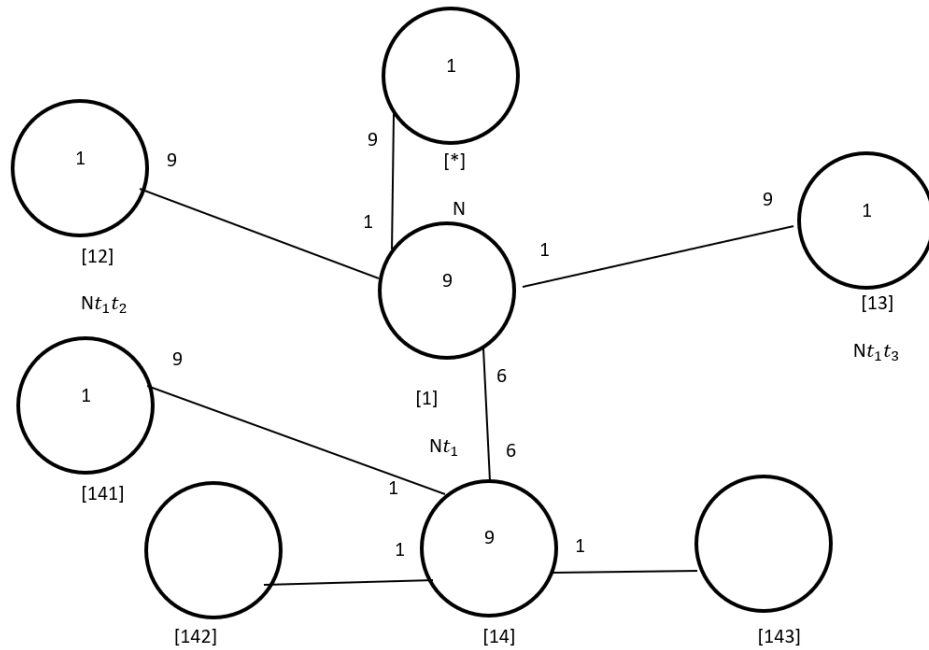
Figure 7.10: Cayley Diagram for G over S_9

Seventh Double Coset $[142]$

$Nt_1t_4t_2 = \{N(t_1t_4t_2)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(142)}$. We first find the point stabilizer of 1, 4 and 2 in N.

$N^{142} = \{n \in N | (142)^n = 142\}$

$N^{142} = \langle (7, 8, 9) \rangle$.

We will find different names of $Nt_1t_4t_2$. Now, $t_1t_4t_2 = (x * y^-1 * x * y * x^-1)t_4t_9t_5 \implies Nt_1t_4t_2 = Nt_4t_9t_5$.

We have $N(t_1t_4t_2)^{(1,4,9)(2,5,7)(3,6,8)} = Nt_4t_9t_5$.

But $Nt_1t_4t_2 = Nt_4t_9t_5$. So, $N(t_1t_4t_2)^{(1,4,9)(2,5,7)(3,6,8)} = Nt_1t_4t_2$

$\implies (1,4,9)(2,5,7)(3,6,8) \in N^{(142)}$ Thus $N^{((142)} \geq \langle N^{142}, (1,4,9)(2,5,7)(3,6,8)\rangle$
$= \langle (7,8,9), (1,4,9)(2,5,7)(3,6,8)\rangle \cong 3^3 : S_3$. The number of right cosets in $Nt_1t_4t_2N$ is calculated by the formula,

$\frac{|N|}{|N^{(142)}|} = \frac{162}{162} = 1.$

The orbits of $N^{(142)}$ on $X = \{1,2,3,4,5,6,7,8,9\}$ is a single orbit

$\{1,7,5,8,2,6,3,4,9\}$. We will determine the double cosetes by selecting one representative from this orbit such as,

Choose 2 from $\{1, 6, 7, 8, 3, 2, 9, 5, 4\}$

$Nt_1t_4t_2t_2 = Nt_1t_4t_2^2 = Nt_1t_4 \in [14].$

This means nine t_i's take [142] to [14].
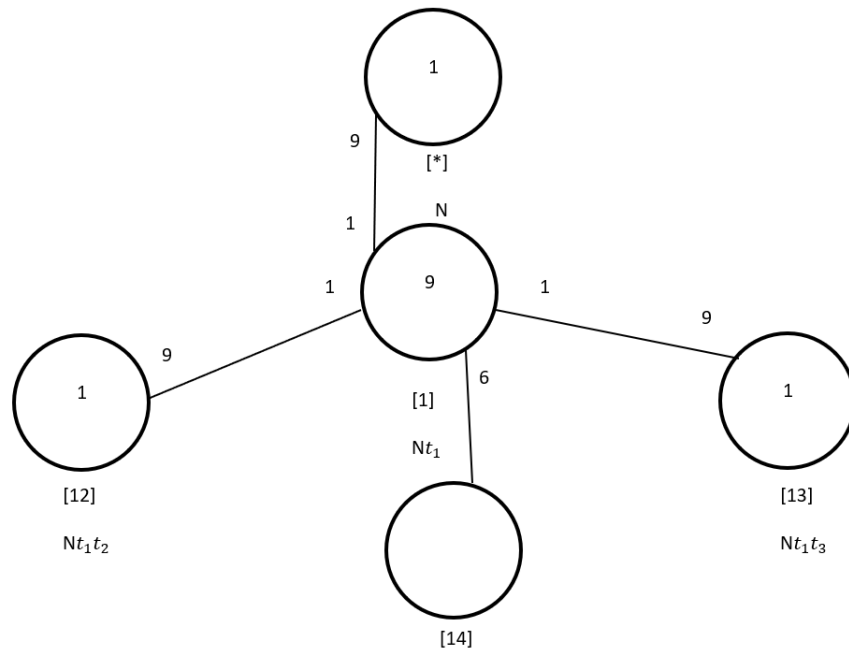
Cayley Diagram
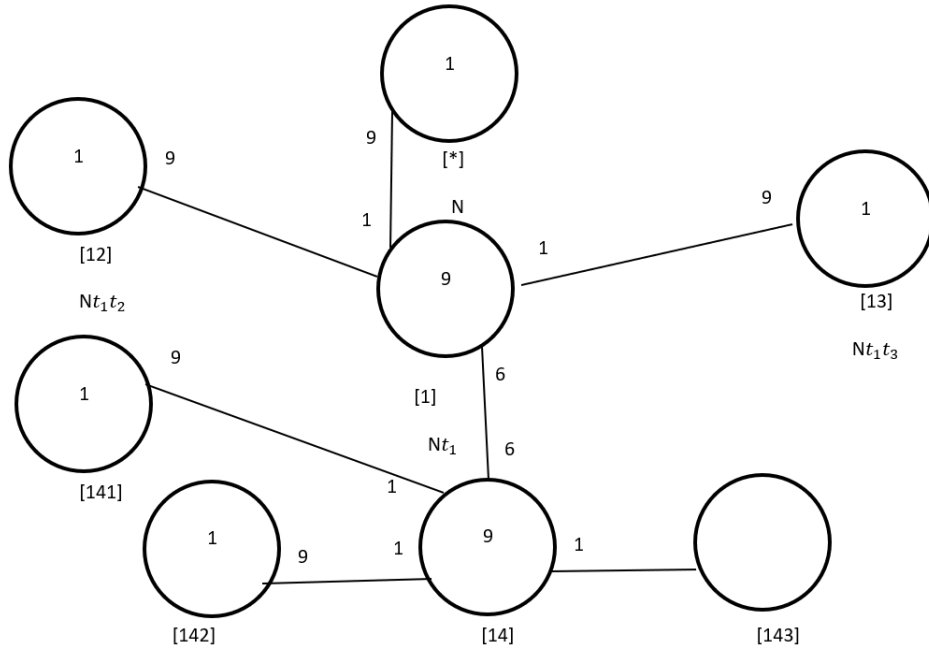
Figure 7.11: Cayley Diagram for G over S_9

Eightieth Double Coset [143]

$Nt_1t_4t_3 = \{N(t_1t_4t_3)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(143)}$. We first find the point stabilizer of 1, 4 and 3 in N.

$N^{143} = \{n \in N | (143)^n = 143\}$

$N^{143} = \langle (7, 8, 9) \rangle$.

We will find different names of $Nt_1t_4t_3$. Now, $t_1t_4t_3 = (y, x^-1)t_7t_3t_9 \implies Nt_1t_4t_3 = Nt_7t_3t_9$

We have $N(t_1t_4t_3)^{(1,7,5)(2,8,6)(3,9,4)} = Nt_7t_3t_9$.

But $Nt_1t_4t_3 = Nt_7t_3t_9$. So, $N(t_1t_4t_3)^{(1,7,5)(2,8,6)(3,9,4)} = Nt_1t_4t_3$

$\implies (1, 7, 5)(2, 8, 6)(3, 9, 4) \in N^{(143)}$. Thus $N^{((143)} \geq \langle N^{143}, (1, 7, 5)(2, 8, 6)(3, 9, 4) \rangle$

$= \langle (7, 8, 9), (1, 7, 5)(2, 8, 6)(3, 9, 4) \rangle \cong 3^3 : S_3$. The number of right cosets in $Nt_1t_4t_3N$ is calculated by the formula,

$\frac{|N|}{|N^{(143)}|} = \frac{162}{162} = 1$.

The orbits of $N^{(143)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is a single orbit

$\{1, 7, 5, 8, 2, 6, 3, 4, 9\}$ We will determine the double cosetes by selecting one representative from this orbit such as,

Choose 3 from $\{1, 6, 7, 8, 3, 2, 9, 5, 4\}$

$Nt_1t_4t_3t_3 = Nt_1t_4t_3^2 = Nt_1t_4 \in [14]$.

This means nine t_i's take [143] to [14].

Cayley Diagram
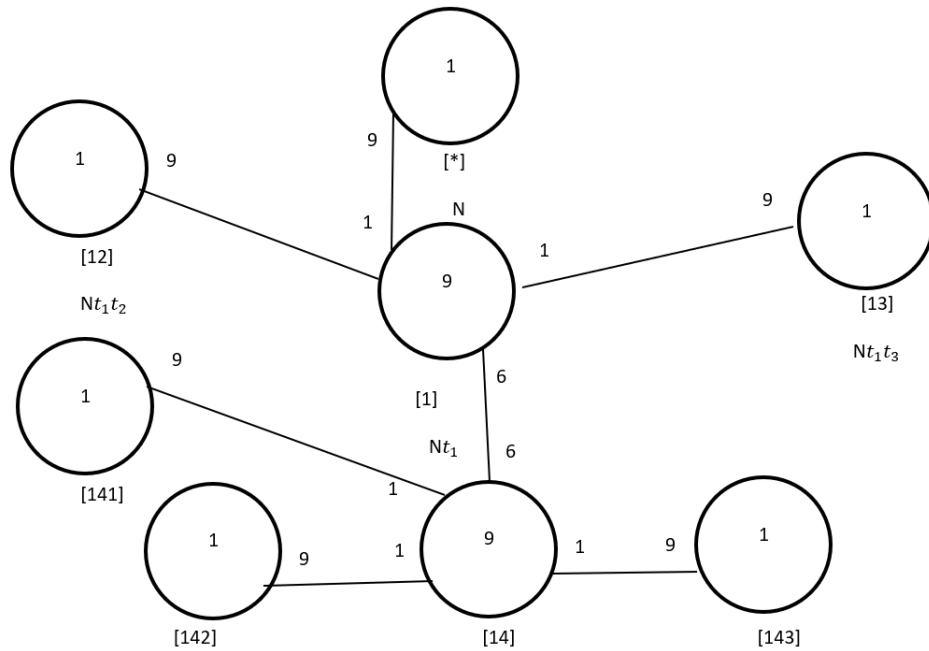
Figure 7.12: Cayley Diagram for G over S_9

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases.

Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_2 \cup Nt_1t_3 \cup Nt_1t_4 \cup Nt_1t_4t_1 \cup Nt_1t_4t_2 \cup Nt_1t_4t_3$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(141)}|} + \frac{|N|}{|N^{(142)}|} + \frac{|N|}{|N^{(143)}|} \times |N|$

$|G| \leq (1 + 9 + 1 + 1 + 9 + 1 + 1 + 1) \times 162$

$\implies |G| \leq 24 \times 162$

$\implies |G| \leq 3888.$

G acts on 24 cosets that are given in the cayley diagram.

Let X be the set of these 24 cosets.

Now, $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|$

But $\#\langle f(x), f(y), f(t) \rangle = 3888.$

So, $|\frac{G}{kerf}| = 3888$

This means $|G| \geq 3888$. We know $|G| \leq 3888$ from cayley diagram.

Therefore, $|G| = 3888.$

From $|G| = 3888 \times |Kerf|$ we find $|Kerf| = 1$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong 3^3 : (3 : 2) \implies G \cong 3^3 : (3 : 2).$

7.3 Construction of $5^3 : (2 \times A_5)$ over $A_5 \times 2$

Let $N \cong A_5 \times 2 = \langle x, y \rangle$ where $x \sim (1, 6, 3)(2, 8, 4)(5, 7, 9)(10, 11, 12)$, and $y \sim (1, 5, 2, 3, 11, 4, 10, 6, 8, 7)(9, 12)$.

The progenitor $2^{*12} : N$ has symmetric presentation

```
G<x,y,t>:=Group<x,y,t|x^3, (x * y^-2)^2,
(x^-1 * y * x^-1 * y^-1)^2,
(x^-1 * y^-3)^2,
t^2>;
```

We want to verify that the presentation given above is correct.

Theorem: $\frac{2^{*n}:N}{t_i, t_j} \cong 2^n : N$ where $1 \le i \le j \le n$.

We will show that $\frac{2^{*12}:N}{((y^3)*t^{(y^2*x*y*x^{-1}*y)}))^5} \cong 5^3 : (2 : A_5)$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^3, (x * y^-2)^2,
(x^-1 * y * x^-1 * y^-1)^2,
(x^-1 * y^-3)^2,
t^2,(t,x^-1 * y^-1 * x * y^2),(t,x * y * x^-1 * y^-1 * x * y),
((y^3)* t^{(y^2 * x * y * x^-1 * y)}))^5> ~ 5^3:(2:A_5)
```

We need to determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N . Hence, $\frac{|G|}{|N|} = \frac{15000}{125} = 120$. Now we know that we have 120 distinct single cosets.

Expanding Relation We expand our only relation.

$$((y^3) * t^{(y^2*x*y*x^{-1}*y)}))^5$$

$$= (y^3) * t_2(y^3) * t_2(y^3) * t_2(y^3) * t_2(y^3) * t_2$$
$$= (y^3)^5 * (t_2^{(y^3)^4}) * (t_2^{(y^3)^3}) * (t_2^{(y^3)^2}) * (t_2^{(y^3)})t_2$$
$$= y^{15} * (t_2)^{y^{12}} * (t_2)^{y^9} * (t_2)^{y^6} * (t_2)^{y^3} * t_2$$

Now since the order of y is 10, we can reduce our relation to be

$y^5 * (t_2)^{y^2} * (t_2)^{y^9} * (t_2)^{y^6} * (t_2)^{y^3} * t_2 \implies y^5 t_{11} t_5 t_8 t_4 t_2 = e \implies y^5 * t_{11} t_5 t_8 = t_2 t_4.$

First Double Coset [∗]

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [∗]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right coset in N is $\frac{|N|}{|N|} = \frac{120}{120} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. We need only determine the double coset of the right coset Nt_1.

Thus 12 t_i's extend the double coset $[∗]$ to the double coset $Nt_1N = [1]$.
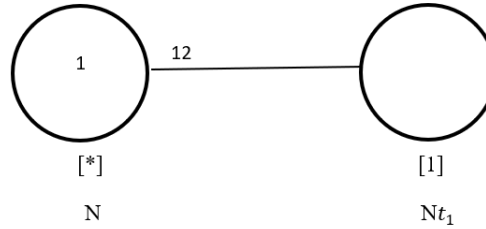
Cayley Diagram

Figure 7.13: Cayley Diagram for G over S_{12}

Second Double Coset [1]

$Nt_1N = \{Nt_1^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}$

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

$= \langle (2, 5, 7, 8, 12)(3, 9, 6, 10, 11), (3, 11)(5, 12)(7, 8)(9, 10) \rangle$.

Thus, $N^{(1)} \geq N^1 = \langle (2, 5, 7, 8, 12)(3, 9, 6, 10, 11), (3, 11)(5, 12)(7, 8)(9, 10) \rangle$

$\implies (2, 5, 7, 8, 12)(3, 9, 6, 10, 11), (3, 11)(5, 12)(7, 8)(9, 10) \in N^{(1)}$.

The number of single right cosets in Nt_1N is calculated by the formula $\frac{|N|}{|N^{(1)}|} = \frac{120}{10} = 12$.

The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{4\}, \{2, 5, 7, 12, 8\}, \{3, 9, 11, 6, 10\}$. We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = N \in [*]$.

This means one t_i takes [1] to [*].

Choose 4 from $\{4\}$

$Nt_1t_4 \in [14]$.

This means one t_i takes [1] to [14].

Choose 2 from $\{2,5,7,12,8\}$

$Nt_1t_2 \in [12]$.

This means five t_i's take [1] to [12].

Choose 3 from $\{3,9,11,6,10\}$

$Nt_1t_3 \in [13]$.

This means five t_i's take [1] to [13].
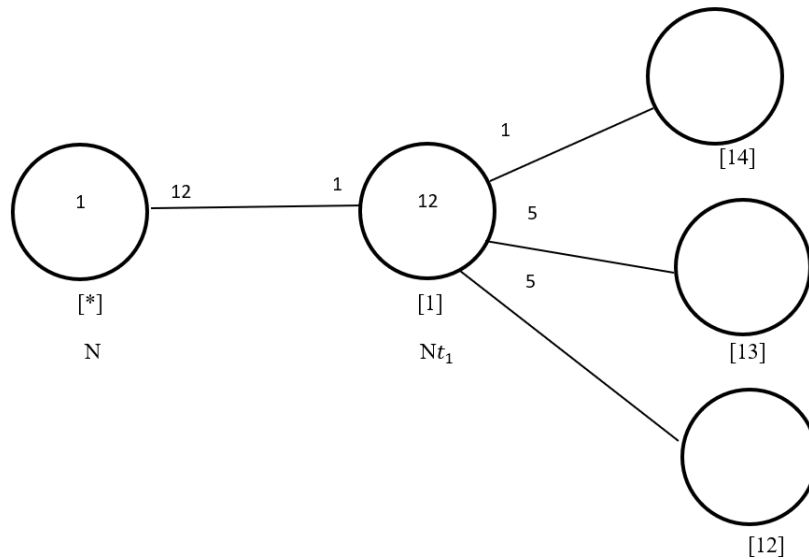
Cayley Diagram

Figure 7.14: Cayley Diagram for G over S_{12}

Third Double Coset [12]

$Nt_1t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(12)}$. We first find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$

$N^{12} = \langle (3,11)(5,12)(7,8)(9,10) \rangle$.

We will find different names of Nt_1t_2. Now $N(t_1t_2)^{(1,6)(2,4)(5,9)(10,12)} = Nt_6t_4$. But $Nt_1t_2 = Nt_6t_4$, so $N(t_1t_2)^{(1,6)(2,4)(5,9)(10,12)} = Nt_1t_2$

$\implies (1,6)(2,4)(5,9)(10,12) \in N^{(12)}$.

Thus $N^{(12)} \geq \langle N^{12}, (1,6)(2,4)(5,9)(10,12) \rangle = \langle (3,11)(5,12)(7,8)(9,10), (1,6)(2,4)(5,9)(10,12) \rangle \cong 2^2$.

The number of distinct right cosets in Nt_1t_2N is calculated by the formula,

$\frac{|N|}{|N^{(12)}|} = \frac{120}{4} = 30$.

The orbits of N^{12} on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1,6\}, \{2,4\}, \{3,11\}, \{7,8\}, \{5,12,10,9\}$.

We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from $\{1,6\}$

$Nt_1t_2t_1 \in [121]$.

This means two t_i's take [12] to [121].

Choose 2 from $\{2,4\}$

$Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$.

This means two t_i's take [12] to [1].

Choose 3 from $\{3,11\}$

$Nt_1t_2t_3 \in [131]$.

We have $Nt_1t_2t_3 = Nt_9t_1t_9$

$= (t_1t_3t_1)^{(x*y^{-1}*x^{-1}*y^{-1})} \in [131]$.

This means two t_i's take [12] to [131].

Choose 5 from $\{5,12,10,9\}$

$Nt_1t_2t_5 \in [12]$.

We have $Nt_1t_2t_5 = y^5(t_8t_{12})$

$= y^5(t_1t_2)^{y^2*x} \in [12]$.

This means four t_i's take [12] to [12].

Choose 7 from $\{7,8\}$

$Nt_1t_2t_7 \in [13]$.

We have $Nt_1t_2t_7 = y^5(t_{12}t_5$

$= y^5(t_1t_3)^{x*y^{-1}*x^{-1}} \in [13]$.

This means two t_i's take [12] to [13].

Cayley Diagram
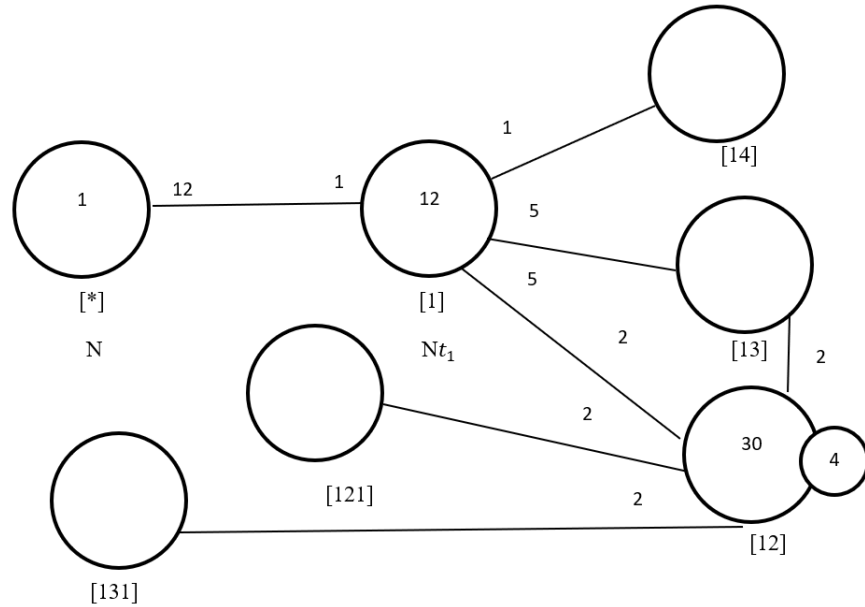
Figure 7.15: Cayley Diagram for G over S_{12}

Fourth Double Coset [13]

$Nt_1t_3 = \{N(t_1t_3)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(13)}$. We first find the point stabilizer of 1 and 3 in N.

$N^{13} = \{n \in N | (13)^n = 13\}$

$N^{13} = \langle (2,5)(6,10)(7,12)(9,11) \rangle$.

We will find different names of Nt_1t_3. Now $N(13)^{(1,8)(2,10)(3,4)(5,6)} = Nt_8t_4$. But $Nt_1t_3 = Nt_8t_4$, so $N(13)^{(1,8)(2,10)(3,4)(5,6)} = Nt_1t_3$

Thus $N^{(13)} \geq \langle N^{13}, (1,8)(2,10)(3,4)(5,6) \rangle = \langle (2,5)(6,10)(7,12)(9,11), (1,8)(2,10)(3,4)(5,6) \rangle \cong 2^2$

The number of distinct right cosets in Nt_1t_3N is calculated by the formula,

$\frac{|N|}{|N^{(13)}|} = \frac{120}{4} = 30$.

The orbits of N^{13} on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1,8\}, \{3,4\}, \{7,12\}, \{9,11\}, \{2,5,10,6\}$. We will determine the double cosetes by select-

ing one representative from each orbit such as,

Choose 1 from {1,8}

$Nt_1t_3t_1 \in [131]$.

This means two t_i's take [13] to [131].

Choose 2 from {2,5,10,6}

$Nt_1t_3t_2 \in [13]$.

We have $Nt_1t_3t_2 = y^5(t_{10}t_7$

$= y^5(t_1t_3)^{x*y^{-1}} \in [13]$

This means four t_i's take [13] to [13].

Choose 3 from {3,4}

$Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1]$.

This means two t_i's take [13] to [1].

Choose 7 from {7,12}

$Nt_1t_3t_7 \in [121]$.

We have $Nt_1t_3t_7 = Nt_9t_7t_9$

$= (t_1t_2t_1)^{y^{-1}*x} \in [121]$

This means nine t_i's take [13] to [121].

Choose 9 from {9,11}

$Nt_1t_3t_9 \in [12]$.

We have $Nt_1t_3t_9 = y^5t_5t_2$

$= y^5(t_1t_2)^{(y*x*y^2)} \in [12]$.

This means two t_i's take [13] to [12].

Cayley Diagram
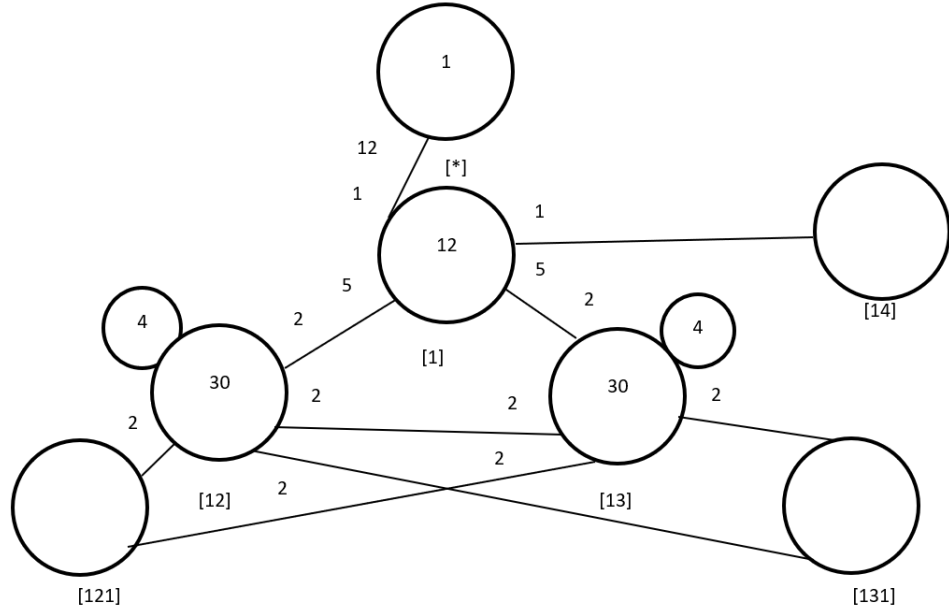
Figure 7.16: Cayley Diagram for G over S_{12}

Fifth Double Coset [14]

$Nt_1t_4 = \{N(t_1t_4)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(14)}$. We first find the point stabilizer of 1 and 4 in N.

$N^{14} = \{n \in N | (14)^n = 14\}$

$N^{14} = \langle (2,5,7,8,12)(3,9,6,10,11), (2,12)(3,10)(5,8)(6,9) \rangle$

Thus, $N^{(14)} \geq N^{14} = \langle (2,5,7,8,12)(3,9,6,10,11), (2,12)(3,10)(5,8)(6,9) \rangle \cong 2 \times 5$.

The number of distinct single cosets in Nt_1t_4N is calculated by the formula,

$\frac{|N|}{|N^{(14)}|} = \frac{120}{10} = 12$.

The Orbits of N^{14} on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{4\}, \{2,5,7,8,12\}, \{3,9,11,6,10\}$.

We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_4t_1 \in [14]$.

We have $Nt_1t_4t_1 = y^5(t_1t_4) \in [14]$

This means one t_i takes [14] to [14].

Choose 2 from {2,5,7,8,12}

$Nt_1t_4t_2 \in [131]$.

We have $Nt_1t_4t_2 = Nt_7t_{12}t_7$

$= (t_1t_3t_1)^{(y*x)} \in [131]$.

This means five t_i's take [14] to [131].

Choose 3 from {3,9,11,6,10}

$Nt_1t_4t_3 \in [121]$.

We have $Nt_1t_4t_3 = Nt_9t_6t_9$

$= (t_1t_2t_1)^{(y*x^{-1})} \in [121]$.

This means five t_i's take [14] to [121].

Choose 4 from {4}

$Nt_1t_4t_4 = Nt_1t_4^2 = Nt_1 \in [1]$

This means one t_i takes [14] to [1].
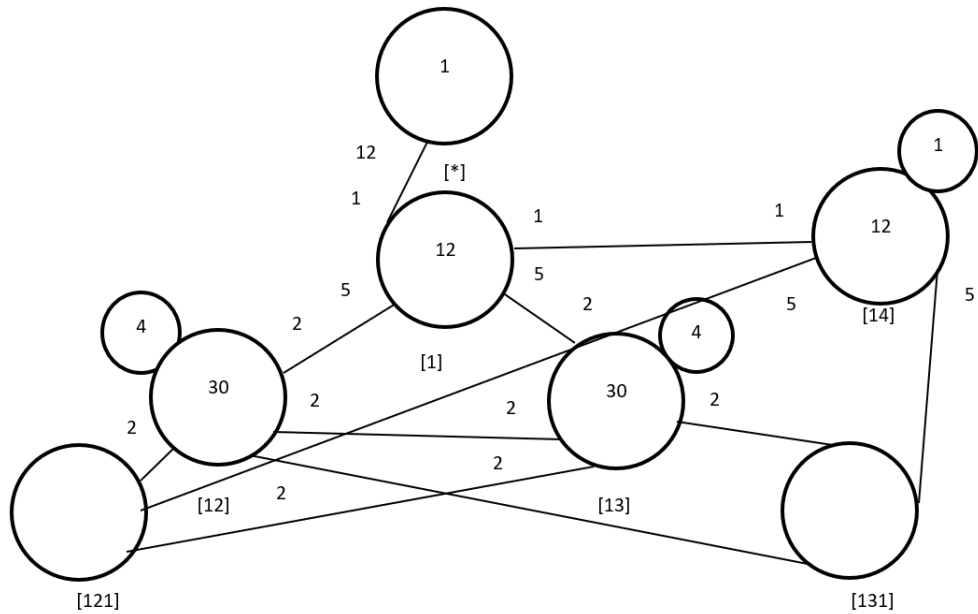
Cayley Diagram

Figure 7.17: Cayley Diagram for G over S_{12}

Sixth Double Coset [121]

$Nt_1t_2t_1 = \{N(t_1t_2t_1)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(121)}$. We first find the point stabilizer of 1,2 and 1 in N.

$N^{121} = \{n \in N | (121)^n = 121\}$

$N^{121} = \langle (3,11)(5,12)(7,8)(9,10) \rangle$.

We will find different names of $Nt_1t_2t_1$. Now $N(t_1t_2t_1)^{(1,9)(2,3)(4,12)(6,8)} = Nt_9t_3t_9$. But $Nt_1t_2t_1 = Nt_9t_3t_9$ so, $N(t_1t_2t_1)^{(1,9)(2,3)(4,12)(6,8)} = Nt_1t_2t_1 \implies (1,9)(2,3)(4,12)(6,8) \in N^{(121)}$

Thus $N^{(12)} \geq \langle N^{12}, (1,9)(2,3)(4,12)(6,8) \rangle = \langle (3,11)(5,12)(7,8)(9,10), (1,9)(2,3)(4,12)(6,8) \rangle \cong S_3$.

The number of single cosets in $Nt_1t_2t_1$ is calculated by the formula $\frac{|N|}{|N^{(121)}|} = \frac{120}{6} = 20$.

The orbits of $N^{(121)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1,9,10\}, \{2,3,11\}, \{4,12,5\}, \{6,8,7\}$.

We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from {1,9,10}

$Nt_1t_2t_1t_1 = Nt_1t_2t_1^2 = Nt_1t_2 \in [12]$.

This means three t_i's takes [121] to [12].

Choose 2 from {2,3,11}

$Nt_1t_2t_1t_2 \in [13]$.

We have $Nt_1t_2t_1t_2 = Nt_3t_7$

$= (t_1t_3)^{(y*x*y*x^{-1})} \in [13]$.

This means three t_i's takes [121] to [13].

Choose 4 from {4,12,5}

$Nt_1t_2t_1t_4 \in [131]$.

We have $Nt_1t_2t_1t_4 = y^5t_7t_{12}t_7$

$= y^5(t_1t_3t_1)^{(y*x)} \in [131]$.

This means three t_i's takes [121] to [131].

Choose 6 from {6,8,7}

$Nt_1t_2t_1t_6 \in [14]$.

We have $Nt_1t_2t_1t_6 = Nt_1t_4 \in [14]$.

This means three t_i's takes [121] to [14].

Cayley Diagram

Figure 7.18: Cayley Diagram for G over S_{12}

Seventh Double Coset [131]

$Nt_1t_3t_1 = \{N(t_1t_3t_1)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(131)}$. We first find the point stabilizer of 1,3 and 1 in N.

$Nt_1t_3t_1 = \{N(t_1t_3t_1)^n | n \in N\}$ $N^{131} = \langle (2,5)(6,10)(7,12)(9,11) \rangle$.

We will find different names of $Nt_1t_3t_1$. Now $N(t_1t_3t_1)^{(1,2,5)(3,7,12)(4,6,10)(8,11,9)} = Nt_2t_7t_2$.

But $t_1t_3t_1 = Nt_2t_7t_2$ so, $N(t_1t_3t_1)^{(1,2,5)(3,7,12)(4,6,10)(8,11,9)} = Nt_1t_3t_1$

$\implies (1,2,5)(3,7,12)(4,6,10)(8,11,9) \in N^{(131)}$.

Thus $N^{(131)} \geq \langle N^{131}, (1,2,5)(3,7,12)(4,6,10)(8,11,9) \rangle$

$= \langle (2,5)(6,10)(7,12)(9,11), (1,2,5)(3,7,12)(4,6,10)(8,11,9) \rangle \cong S_3$.

The number of single cosets in $Nt_1t_3t_1$ is calculated by the formula $\frac{|N|}{|N^{(131)}|} = \frac{120}{6} = 20$.

The orbits of $N^{(131)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1,2,5\}, \{3,7,12\}, \{4,6,10\}, \{8,11,9\}$.

We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from {1,2,5}

$Nt_1t_3t_1t_1 = Nt_1t_3t_1^2 = Nt_1t_3 \in [13]$.

This means three t_i's take [131] to [13].

Choose 3 from {3,7,12}

$Nt_1t_3t_1t_3 \in [12]$.

We have $Nt_1t_3t_1t_3 = Nt_{12}t_{11}$

$= (t_1t_2)^{(x*y*x*y*x^{-1})} \in [12]$

This means three t_i's take [131] to [12].

Choose 4 from {4,6,10}

$Nt_1t_3t_1t_4 \in [121]$.

We have $Nt_1t_3t_1t_4 = y^5t_9t_6t_9$

$= y^5(t_1t_2t_1)^{(y*x^{-1})} \in [121]$

This means three t_i's take [131] to [121].

Choose 8 from {8,11,9}

$Nt_1t_3t_1t_8 \in [14]$.

We have $Nt_1t_3t_1t_8 = Nt_1t_4$

This means three t_i's take [131] to [14].

Cayley Diagram
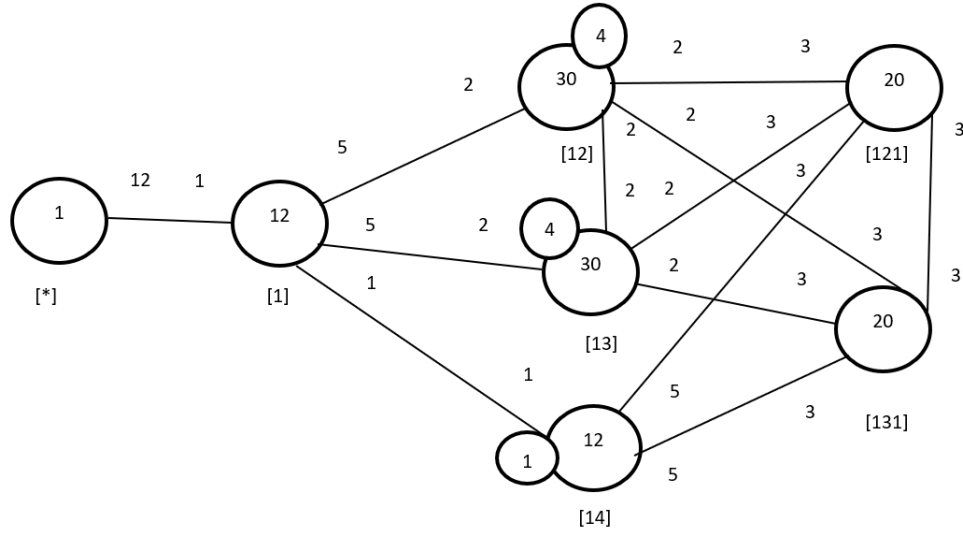
Figure 7.19: Cayley Diagram for G over S_{12}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases.

Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_2 \cup Nt_1t_3 \cup Nt_1t_4 \cup Nt_1t_2t_1 \cup Nt_1t_3t_1$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(121)}|} + \frac{|N|}{|N^{(131)}|} \times |N|)$

$|G| \leq (1 + 12 + 30 + 30 + 12 + 20 + 20) \times 120 \implies |G| \leq 125 \times 120 \implies |G| \leq 15000.$

G acts on 125 cosets that are given in the cayley diagram.

Let X be the set of these 125 cosets. Now, $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|$

But $\#\langle f(x), f(y), f(t) \rangle = 15000$. So, $|\frac{G}{kerf}| = 15000$

This means $|G| \geq 15000$. We know $|G| \leq 15000$ from cayley diagram. Therefore, $|G| = 15000$.

From $|G| = 15000 \times |Kerf|$ we find $|Kerf| = 1$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong A_5 \times 2 \implies G \cong A_5 \times 2.$

7.4 Construction of $A_5 \times A_5$ over $A_5 \times 2$

Let $N \cong A_5 \times 2 = \langle x, y \rangle$ where $x \sim (1,6,3)(2,8,4)(5,7,9)(10,11,12)$, and

$y \sim (1,5,2,3,11,4,10,6,8,7)(9,12)$. The progenitor $2^{*12} : N$ has symmetric presentation

```
G<x,y,t>:=Group<x,y,t|x^3, (x * y^-2)^2,
    (x^-1 * y * x^-1 * y^-1)^2,
    (x^-1 * y^-3)^2,
    t^2>;
```

We want to verify that the presentation given above is correct.

Theorem: $\frac{2^{*n} : N}{t_i, t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$

We will show that $\frac{2^{*12} : N}{((y*x*y^2)*(t^{(y^{-2}*x^{-1}*y)))^3} \cong A_5 \times A_5$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^3, (x * y^-2)^2,
(x^-1 * y * x^-1 * y^-1)^2,
(x^-1 * y^-3)^2,
t^2,(t,x^-1 * y^-1 * x * y^2),(t,x * y * x^-1 * y^-1 * x * y),
((y * x * y^2)*(t^(y^-2 * x^-1 * y))^3 ~ A_5 X 2
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{14400}{120} = 120$. Now we know that we have 120 distinct single cosets.

Expanding Relation We will expand our only relation.

$$((y * x * y^2) * (t^{(y^{-2} * x^{-1} * y)}))^3$$

$= (y * x * y^2) * t_3 (y * x * y^2) * t_3 (y * x * y^2) * t_3$

$= (y * x * y^2)^3 * (t_3^{(y*x*y^2)^2}) * t_3^{(y*x*y^2)}) * t_3$

$\implies (y * x * y^2)^3 t_3 t_{12} t_3 = e$

$\implies (y * x * y^2)^3 t_3 = t_3 t_{12}$

First Double Coset $[*]$

$NeN = \{Ne^n | n \in N\} = [N]$.

The double coset NeN = $[*]$ contains 1 right coset. The coset stabiliser of the coset Ne

is N.

The formula for the number of right cosets in N is $\frac{120}{120} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. We need only determine the double coset of the right cosets Nt_1. Thus 12 t_i's extend the double coset $[*]$ to the double coset $Nt_1N = [1]$.
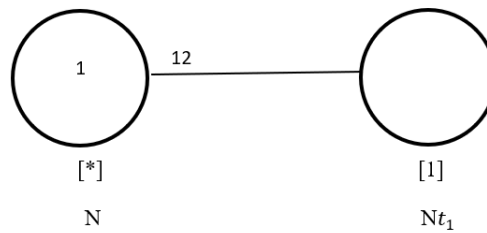
Cayley Diagram

Figure 7.20: Cayley Diagram for G over S_{12}

Second Double Coset [1]

$Nt_1N = \{N(t_1)^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}\}$.

We now find the Coset Stabilier $N^{1)}$. We first find the point stabiliser of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

$N^1 = \langle (2, 5, 7, 8, 12)(3, 9, 6, 10, 11), (3, 11)(5, 12)(7, 8)(9, 10) \rangle$.

Thus, $N^{(1)} \geq N^1 = \langle (2, 5, 7, 8, 12)(3, 9, 6, 10, 11), (3, 11)(5, 12)(7, 8)(9, 10) \rangle$

$\implies (2, 5, 7, 8, 12)(3, 9, 6, 10, 11), (3, 11)(5, 12)(7, 8)(9, 10) \in N^{(1)}$.

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{120}{10} = 12$.

The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are $\{1\}$, $\{4\}$, $\{2, 5, 7, 12, 8\}$, $\{3, 9, 11, 6, 10\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Ne \in [*]$.

This means one t_i takes [1] to [*].

Choose 2 from {2,5,7,12,8}

$Nt_1t_2 \in [12]$

This means five t_i's take [1] to [12].

Choose 3 from {3,9,11,6,10}

$Nt_1t_3 \in [1]$.

We have $Nt_1t_3 = (y * x * y^{-1} * x^{-1} * y)t_1 \in [1]$.

This means five t_i's take [1] to [1].

Choose 4 from {4}

$Nt_1t_4 \in [14]$

This means one t_i takes [1] to [14]. Now we need to investigate the first new double coset.
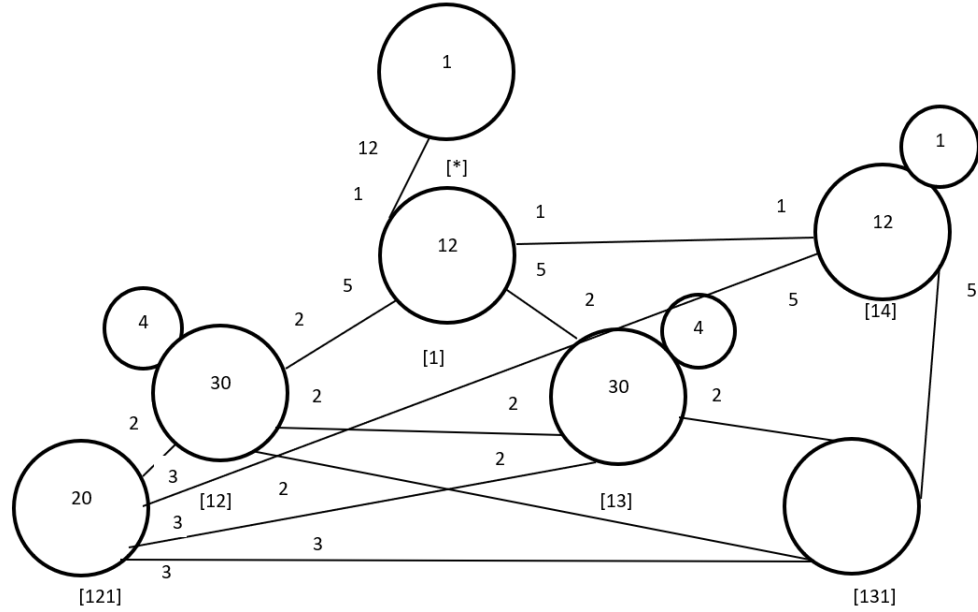
Cayley Diagram
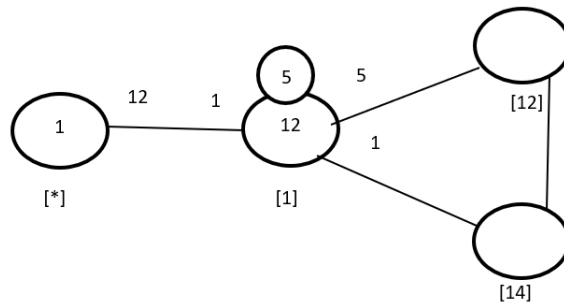
Figure 7.21: Cayley Diagram for G over S_{12}

Third Double Coset [12]

$Nt_1t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the coset stablilizer $N^{(1,2)}$. We first find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$

$N^{12} = \langle (3, 11)(5, 12)(7, 8)(9, 10)\rangle$.

We find different names of Nt_1t_2. Now, $t_1t_2 = (y^-2 * x^-1)t_7t_9 \implies Nt_1t_2 = Nt_7t_9$

$N(t_1t_2)^{(1,7)(2,9)(4,11)(6,12)} = Nt_7t_9$.

But $Nt_7t_9 = Nt_1t_2 \implies N(t_1t_2)^{(1,7)(2,9)(4,11)(6,12)} = Nt_1t_2$.

$\implies (1, 7)(2, 9)(4, 11)(6, 12), (1, 7, 8)(2, 9, 10)(3, 4, 11)(5, 6, 12) \in N^{(12)}$. Thus $N^{(1,2)} \geq$

$\langle N^{1,2}, (1,7)(2,9)(4,11)(6,12), (1,7,8)(2,9,10)(3,4,11)(5,6,12)\rangle$.

$= \langle (3,11)(5,12)(7,8)(9,10), (1,7)(2,9)(4,11)(6,12), (1,7,8)(2,9,10)(3,4,11)(5,6,12)\rangle$.

The number of distinct right cosets in Nt_1t_2N is calculated by the formula

$\frac{|N|}{|N^{(1,2)}|} = \frac{120}{6} = 20$.

The orbits of $N^{1,2}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1,7,8\}, \{2,9,10\}, \{3,11,4\}, \{5,12,6\}$.

We will determine the double cosets by selecting one representative from each orbit such

as

Choose 1 from {1,7,8}

$Nt_1t_2t_1 \in [12]$.

We have $Nt_1t_2t_1 = (x * y^{-1} * x^{-1} * y * x * y)t_1t_2 \in [12]$.

This means three t_i's take [12] to [12].

Choose 2 from {2,9,10}

$Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$.

This means three t_i's take [12] to [1].

Choose 3 from {3,11,4}

$Nt_1t_2t_3 \in [14]$.

We have $Nt_1t_2t_3(y^x)t_5t_{10}$

$= (y^x)(t_1t_4)^y \in [14]$.

This means three t_i's take [12] to [14].

Choose 5 from {5,12,6}

$Nt_1t_2t_5 \in [125]$.

This means three t_i's take [12] to [125].

Cayley Diagram

Figure 7.22: Cayley Diagram for G over S_{12}

Fourth Double Coset [14]

$Nt_1t_4 = \{N(t_1t_4)^n | n \in N\}$.

We now find the coset stabiliser $N^{(1,4)}$. We first find the point stabiliser of 1 and 4 in N.

$N^{14} = \{n \in N | (14)^n = 14\}$

Thus $N^{(14)} \geq N^{14} = \langle (2,5,7,8,12)(3,9,6,10,11)(3,11)(5,12)(7,8)(9,10) \rangle$.

The number of distinct right cosets in Nt_1t_4N is calculated by the formula,

$\frac{|N|}{|N^{1,4}|} = \frac{120}{10} = 12$.

The orbits of N^{14} ox $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{4\}, \{2,5,7,8,12\}, \{3,9,11,6,10\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_4t_1 \in [141]$.

This means one t_i takes [14] to [141].

Choose 2 from $\{2,5,7,8,12\}$

$Nt_1t_4t_2 \in [12]$.

We have $Nt_1t_4t_2$

$= (y * x^{-1} * y^{-1} * x * y)t_6t_4$

$$= (y * x^{-1} * y^{-1} * x * y) * (t_1 t_2)^{(x*y*x^{-1}*y*x^{-1})} \in [12]$$

This means five t_i's take [14] to [12].

Choose 3 from $\{3,9,11,6,10\}$

$Nt_1 t_4 t_3 \in [125]$.

We have $Nt_1 t_4 t_3$

$= (y^{-2}) t_8 t_6 t_7$

$= (y^{-2})(t_1 t_2 t_5)^{(y^{-1}*x^{-1}*y^{-1}*x*y)} \in [125]$

This means five t_i's take [14] to [125].

Choose 4 from $\{4\}$

$Nt_1 t_4 t_4 = Nt_1 t_4^2 = Nt_1 \in [1]$.

This means one t_i takes [14] to [1].

Cayley Diagram

Figure 7.23: Cayley Diagram for G over S_{12}

Fifth Double Coset [125]

$Nt_1 t_2 t_5 = \{N(t_1 t_2 t_5)^n | n \in N\}$.

We now find the coset stabiliser $N^{(125)}$. We first find the point stabilizer of 1, 2, and 5 in N.

$N^{125} = \{n \in N | (125)^n = 125\}$

$N^{125} = \langle e \rangle$

We find different names of $Nt_1 t_2 t_5$. Now $t_1 t_2 t_5 = (x^- 1 * y^- 1 * x^- 1 * y^- 1 * x) t_7 t_6 t_8 \implies Nt_1 t_2 t_5 = Nt_7 t_6 t_8$.

We have $N(t_1t_2t_5)^{(1,7)(2,6)(3,10)(4,11)(5,8)(9,12)} = Nt_7t_6t_8$.

But, $Nt_7t_6t_8 = Nt_1t_2t_5$ so, $N(t_1t_2t_5)^{(1,7)(2,6)(3,10)(4,11)(5,8)(9,12)} = Nt_1t_2t_5$.

$\implies (1,7)(2,6)(3,10)(4,11)(5,8)(9,12) \in N^{(125)}$.

Thus $N^{(125)} \geq \langle N^{125}, (1,7)(2,6)(3,10)(4,11)(5,8)(9,12) \rangle$.

$= \langle e, (1,7)(2,6)(3,10)(4,11)(5,8)(9,12) \rangle$.

The number of distinct right cosets in $Nt_1t_2t_5N$ is calculated by the formula

$\frac{|N|}{|N^{(125)}|} = \frac{120}{4} = 30$.

The orbits of $N^{(125)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1, 7\}, \{3, 10\}, \{4, 11\}, \{5, 8\}, \{2, 6, 9, 12\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1,7\}$

$Nt_1t_2t_5t_1 \in [141]$.

We have $Nt_1t_2t_5t_1$

$= (x * y^3)t_{11}t_7t_{11}$

$= (x * y^3)(t_1t_4t_1)^{(x*y^{-1}*x)} \in [141]$

This means two t_i's take [125] to [141].

Choose 2 from $\{2,6,9,12\}$

$Nt_1t_2t_5t_2 \in [125]$.

We have $Nt_1t_2t_5t_2$

$= (x * y^3)t_1t_2t_5 \in [125]$

This means four t_i's take [125] to [125].

Choose 3 from $\{3,10\}$

$Nt_1t_2t_5t_3 \in [1253]$.

This means two t_i's take [125] to [1253].

Choose 4 from $\{4,11\}$

$Nt_1t_2t_5t_4 \in [14]$.

We have $Nt_1t_2t_5t_4$

$= (x * y^{-1} * x^{-1} * y^{-1} * x^{-1})t_7t_{11}$

$= (x * y^{-1} * x^{-1} * y^{-1} * x^{-1})(t_1t_4)^{(y^{-1})} \in [14]$

This means two t_i's take [125] to [14].

Choose 5 from $\{5,8\}$

$Nt_1t_2t_5t_5 = Nt_1t_2t_5^2 = Nt_1t_2 \in [12]$.

This means two t_i's take $[125]$ to $[12]$.

Cayley Diagram
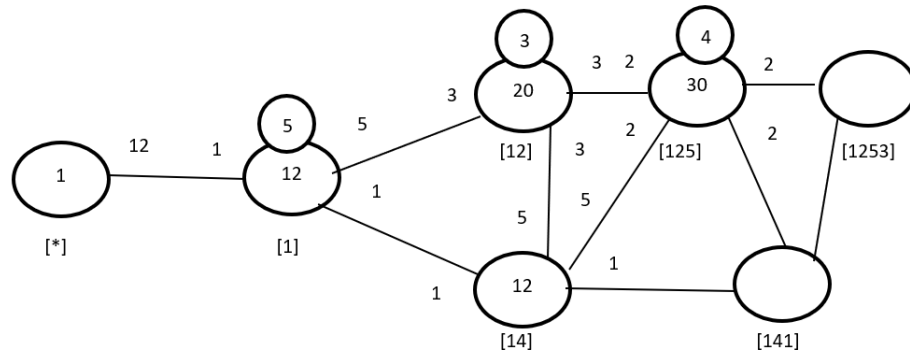
Figure 7.24: Cayley Diagram for G over S_{12}

Sixth Double Coset $[141]$

$Nt_1t_4t_1 = \{N(t_1t_4t_1)^n | n \in N\}$.

We now find the coset stabiliser $N^{(141)}$. We first find the point stabiliser of 1, 4, and 1 in N.

$N^{141} = \{n \in N | (141)^n = 141\}$

Thus, $N^{(141)} \geq N^{141} = \langle (2,5,7,8,12)(3,9,6,10,11)(3,11)(5,12)(7,8)(9,10) \rangle$.

The number of distinct single cosets in $Nt_1t_4t_1N$ is calculated by the formula

$\frac{|N|}{|N^{(141)}|} = \frac{120}{10} = 12$.

The orbits of $N^{(141)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{4\}, \{2,5,7,8,12\}, \{3,9,11,6,10\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_4t_1t_1 = Nt_1t_4t_1^2 = Nt_1t_4 \in [14]$

This means one t_i takes $[141]$ to $[14]$.

Choose 2 from $\{2,5,7,8,12\}$

$Nt_1t_4t_1t_2 \in [1253]$.

We have $Nt_1t_4t_1t_2$

$= xt_{11}t_{12}t_2t_1$

$= x(t_1t_2t_5t_3)^{(y*x*y*x^{-1}*y)} \in [1253]$

This means five t_i's take [141] to [1235].

Choose 3 from {3,9,11,6,10}

$Nt_1t_4t_1t_3 \in [125]$.

We have $Nt_1t_4t_1t_3$

$= (y*x^{-1}*y^{-1}*x*y)t_3t_5t_9$

$= (y*x^{-1}*y^{-1}*x*y)(t_1t_2t_5)^{(y^2*x*y^{-1}*x)} \in [125]$

This means five t_i's take [141] to [125].

Choose 4 from {4}

$Nt_1t_4t_1t_4 \in [1414]$

This means one t_i takes [141] to [1414].

Cayley Diagram
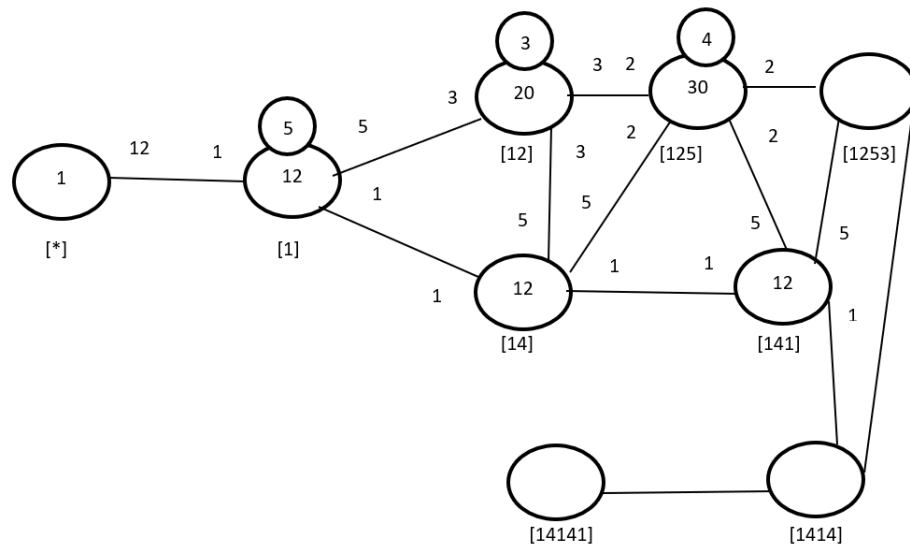
Figure 7.25: Cayley Diagram for G over S_{12}

Seventh Double Coset [1414]

$Nt_1t_4t_1t_4 = \{N(t_1t_4t_1t_4)^n | n \in N\}$.

We now find the coset stabiliser $N^{(1414)}$. We first find the point stabiliser of 1, 4, 1, and 4 in N.

$N^{1414} = \{n \in N | (1414)^n = 1414\}$

Thus $N^{(1414)} \geq N^{1414} = \langle(2,5,7,8,12)(3,9,6,10,11)(3,11)(5,12)(7,8)(9,10)\rangle$.

The number of distinct single right cosets in $Nt_1t_4t_1t_4N$ is calculated by the formula,

$\frac{|N|}{|N^{(1414)}|} = \frac{120}{10} = 12$.

The orbits of $N^{(1414)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{4\}, \{2,5,7,8,12\}, \{3,9,11,6,10\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_4t_1t_4t_1 \in [14141]$

This means one t_i takes [1414] to [14141].

Choose 2 from $\{2,5,7,8,12\}$

$Nt_1t_4t_1t_4t_2 \in [1253]$.

We have $Nt_1t_4t_1t_4t_2$

$= (x * y * x^{-1} * y^{-1} * x * y)t_{11}t_{10}t_4t_6$

$= (x * y * x^{-1} * y^{-1} * x * y)(t_1t_2t_5t_3)^{(y^4)} \in [1253]$

This means five t_i's take [1414] to [1253].

Choose 3 from $\{3,9,11,6,10\}$

$Nt_1t_4t_1t_4t_3 \in [1414]$.

We have $Nt_1t_4t_1t_4t_3$

$= (y * x * y^{-1} * x^{-1} * y)t_1t_4t_1t_4 \in [1414]$

This means five t_i's take [1414] to [1414].

Choose 4 from $\{4\}$

$Nt_1t_4t_1t_4t_4 = Nt_1t_4t_1t_4^2 = Nt_1t_4t_1 \in [141]$.

This means one t_i takes [1414] to [141].

Cayley Diagram
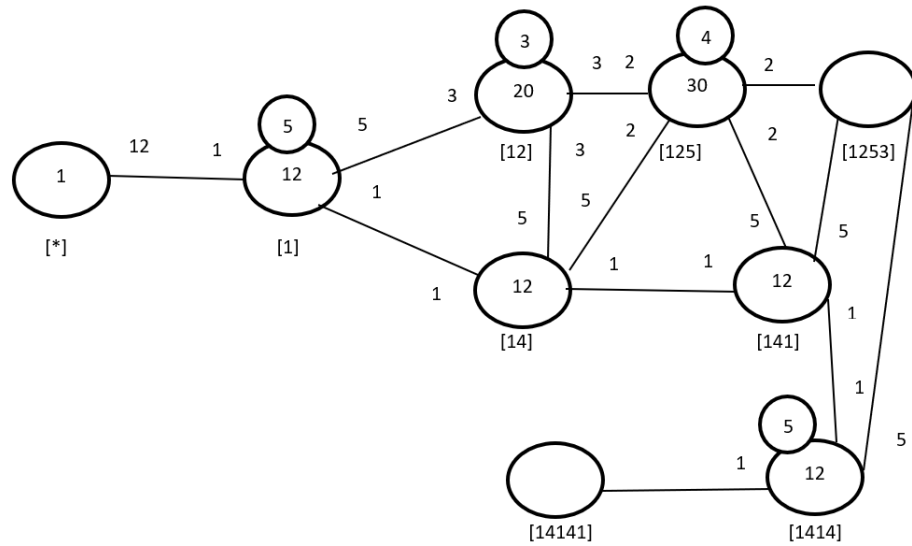
Figure 7.26: Cayley Diagram for G over S_{12}

Eightieth Double Coset [14141]

$Nt_1t_4t_1t_4t_1 = \{N(t_1t_4t_1t_4t_1)^n | n \in N\}$.

We now find the coset stabiliser $N^{(1,4,1,4,1)}$. We first find the point stabiliser of 1, 4, 1, 4, and 1 in N.

$N^{14141} = \{n \in N | (14141)^n = 14141\}$ $N^{14141} = \langle (2,5,7,8,12)(3,9,6,10,11)(3,11)(5,12)(7,8)(9,10) \rangle$

Now, $t_1t_4t_1t_4t_1 = t_9t_{12}t_9t_{12}t_9$. We have $N(t_1t_4t_1t_4t_1)^{(1,9,11,8,5,4,12,7,3,10)(2,6)\cdots} = t_9t_{12}t_9t_{12}t_9$.

But $Nt_1t_4t_1t_4t_1 = Nt_9t_{12}t_9t_{12}t_9$so,$N(t_1t_4t_1t_4t_1)^{(1,9,11,8,5,4,12,7,3,10)(2,6)\cdots} = Nt_1t_4t_1t_4t_1$

$\implies (1,9,11,8,5,4,12,7,3,10)(2,6)\ldots \in N^{(14141)}$

Thus, $N^{(14141)} \geq \langle N^{14141}, (1,9,11,8,5,4,12,7,3,10)(2,6)\ldots \rangle$.

$= \langle (2,5,7,8,12)(3,9,6,10,11)(3,11)(5,12)(7,8)(9,10), (1,9,11,8,5,4,12,7,3,10)(2,6)\ldots \rangle$.

The number of distinct single right cosets in $Nt_1t_4t_1t_4t_1N$ is calculated by the formula

$\frac{|N|}{|N^{(14141)}|} = \frac{120}{120} = 1$

The orbit of N^{14141} on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ is the single orbit

$\{1,2,3,4,5,6,7,8,9,10,11,12\}$.

We will determine the double cosets by selecting one representative from the orbit such

as,

Choose 1 from $\{1,9,6,10,11,2,8,3,5,4,7,12\}$

$Nt_1t_4t_1t_4t_1t_1$

$= Nt_1t_4t_1t_4t_1^2$

$= Nt_1t_4t_1t_4 \in [1414]$

This means twelve t_i's take $[14141]$ to $[1414]$.

Cayley Diagram
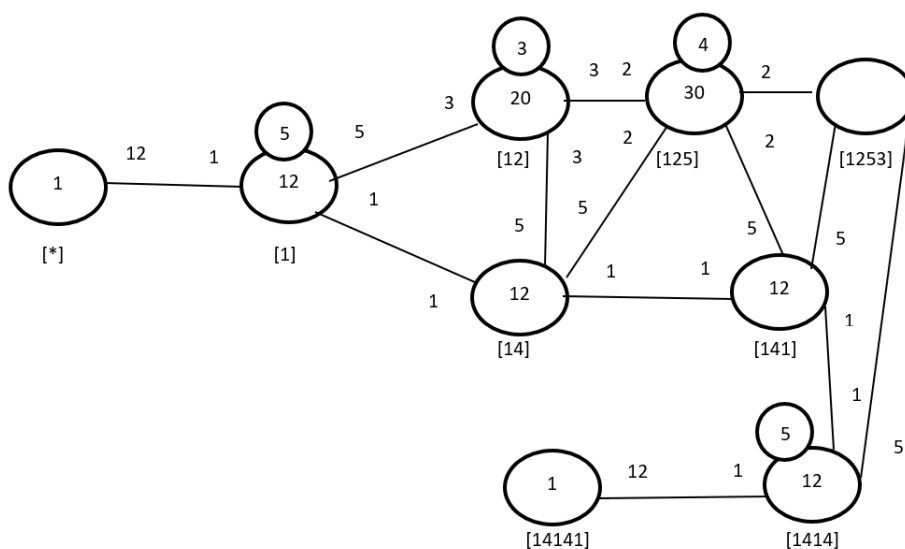
Figure 7.27: Cayley Diagram for G over S_{12}

Nineth Double Coset $[1253]$

$Nt_1t_2t_5t_3 = \{N(t_1t_2t_5t_3)^n | n \in N\}$.

WE now find the coset stabiliser $N^{(1253)}$. We first find the point stabiliser of 1, 2, 5, and 3 in N.

$N^{1253} = \{n \in N | (1253)^n = 1253\}$

Thus, $N^{(1253)} \geq N^{1253} = \langle (1,5,7)(2,9,8)(3,6,12)(4,10,11) \rangle$.

The number of distinct single cosets in $Nt_1t_2t_5t_3N$ is calculated by the formula

$\frac{|N|}{|N^{(1253)}|} = \frac{120}{6} = 20$.

The orbits of $N^{(1253)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1, 5, 7\}, \{2, 8, 9\}, \{3, 6, 12\}, \{4, 10, 11\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1,5,7\}$

$Nt_1t_2t_5t_3t_1 \in [141].$

We have $Nt_1t_2t_5t_3t_1$

$= (x^{-1} * y * x^{-1} * y * x)t_{12}t_9t_{12}$

$= (x^{-1} * y * x^{-1} * y * x)(t_1t_4t_1)^{(y*x^{-1}*y)} \in [141]$

This means three t_i's take [1253] to [141].

Choose 2 from $\{2,8,9\}$

$Nt_1t_2t_5t_3t_2 \in [1414].$

We have $Nt_1t_2t_5t_3t_2$

$= (x * y^2 * x * y * x^{-1})t_{11}t_7t_{11}t_7$

$= (x * y^2 * x * y * x^{-1})(t_1t_4t_1t_4)^{(x*y^{-1}*x)} \in [1414$

This means three t_i's take [1253] to [1414].

Choose 3 from $\{3,6,12\}$

$Nt_1t_2t_5t_3t_3 = Nt_1t_2t_5t_3^2 \in [125]$

This means three t_i's take [1253] to [125].

Choose 4 from $\{4,10,11\}$

$Nt_1t_2t_5t_3t_4 \in [1253].$

We have $Nt_1t_2t_5t_3t_4$

$= (y^{-1} * x^{-1})t_5t_2t_1t_{12}$

$= (y^{-1} * x^{-1})(t_1t_2t_5t_3)^{(y*x*y^2)} \in [1253]$

This means three t_i's take [1253] to [1253].

Cayley Diagram
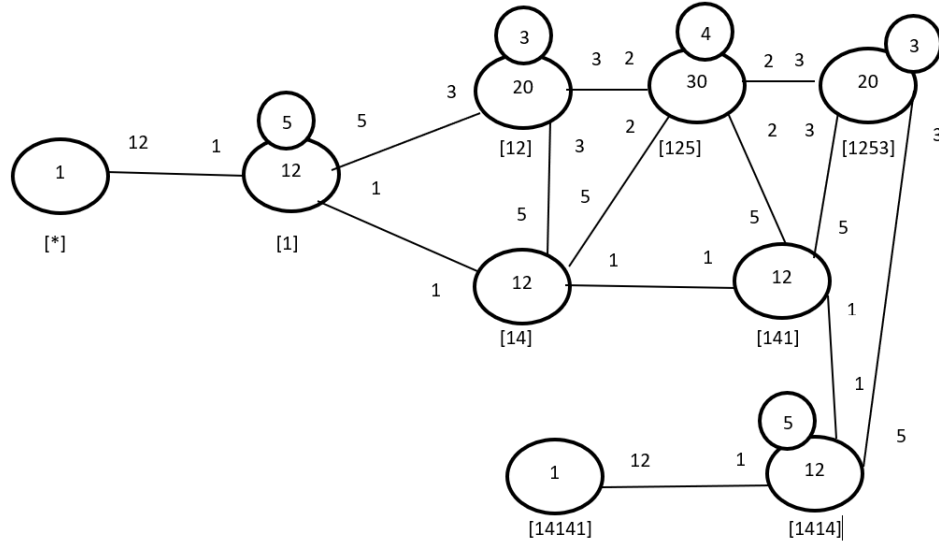
Figure 7.28: Cayley Diagram for G over S_{12}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^w|}$ decreases. Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_2 \cup Nt_1t_4 \cup Nt_1t_2t_5 \cup Nt_1t_4t_1 \cup Nt_1t_4t_1t_4 \cup Nt_1t_4t_1t_4t_1 \cup Nt_1t_2t_5t_3$

$|G| \leq (|N| + |\frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(125)}|} + \frac{|N|}{|N^{(141)}|} + \frac{|N|}{|N^{(1414)}|} + \frac{|N|}{|N^{(14141)}|} + \frac{|N|}{|N^{(1253)}|} \times |N|)$

$G \leq (1 + 12 + 20 + 12 + 12 + 12 + 30 + 1 + 20) \times 120$

$\implies |G| \leq 120 \times 120 \implies |G| \leq 14400.$

G acts on 120 cosets that are given in the cayley diagram. Let X be the set of these 120 cosets. Now $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong \text{Imf}$ (First Isomorphsim Theorem) $\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = 14400.$

This means $|G| \geq 14400$. We know $|G| \leq 14400$ from the cayley diagram. Therefore, $|G| = 14400.$

From $|G| = 14400 \times |Kerf|$ we find $|kerf| = 1$.

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong A_5 \times 2 \implies G \cong A_5 \times 2.$

7.5 Construction of $(2 : A_5 \times A_5)$ over $(A_5 \times 2)$

Let $N \cong A_5 \times 2 = \langle x, y \rangle$ where $x \sim (1, 6, 3)(2, 8, 4)(5, 7, 9)(10, 11, 12)$, and $y \sim (1, 5, 2, 3, 11, 4, 10, 6, 8, 7)(9, 12)$. The progenitor $2^{*12} : N$ has symmetric presentation

```
G<x,y,t>:=Group<x,y,t|x^3, (x * y^-2)^2,
(x^-1 * y * x^-1 * y^-1)^2,
(x^-1 * y^-3)^2,
t^2>;
```

We want to verify that the presentation given above is correct.

Theorem: $\frac{2^{*n}:N}{t_i, t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$

We will show that $\frac{2^{*12}:N}{((y^5)*(t^{(y^2*x*y*x^{-1}*y)})^5, ((y*x*y^2)*(t^{(y^{-2}*x^{-1}*y)})^3)} \cong 2 : (A_5 \times A_5)$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^3, (x * y^-2)^2,
(x^-1 * y * x^-1 * y^-1)^2,
(x^-1 * y^-3)^2,
t^2,(t,x^-1 * y^-1 * x * y^2),
(t,x * y * x^-1 * y^-1 * x * y),
((y^5)*(t^(y^2 * x * y * x^-1 * y))^5,
((y * x * y^2)*(t^(y^-2 * x^-1 * y))^3) ~ A_5 X 2
```

We will determine the order of G. We perform manual double coset enumeration of G over N. We need to determine all distinct double cosets and find the number of right cosets in each double coset NwN. It suffices to find the double coset of Nwt_i for one representative t_i for each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw. So we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{7200}{120} = 60$. Now we know that we have 60 distinct single cosets.

Expanding First Relation We will expand our first relation.

$$(((y^5) * (t^{(y^2*x*y*x^{-1}*y)}))^5$$

$= (y^5) * t_2 * (y^5) * t_2 * (y^5) * t_2 * (y^5) * t_2 * (y^5) * t_2$

$= (y^5)^5 * (t_2^{(y^5)^4}) * (t_2^{(y^5)^3}) * (t_2^{(y^5)^2}) * (t_2^{(y^5)}) * t_2$

$= y^{25} * (t_2^{(y^{20})}) * (t_2^{(y^{15})}) * (t_2^{(y^{10})}) * (t_2^{(y^5)}) * t_2$

Since the order of y is 10 we can reduce the relation such as,

$$y^5 * t_2 t_6 t_2 t_6 t_2 = e \implies y^5 * t_2 t_6 = t_2 t_6 t_2$$

<u>Expanding Second Relation</u> We will expand our second relation.

$$((y * x * y^2) * (t^{(y^{-2} * x^{-1} * y)}))^3$$

$$= (y * x * y^2) * t_3 (y * x * y^2) * t_3 (y * x * y^2) * t_3$$
$$= (y * x * y^2)^3 * (t_3^{(y*x*y^2)^2}) * (t_3^{(y*x*y^2)}) * t_3$$

Since the order of $(y * x * y^2)$ is 2 we can reduce the relation such as,

$$(y * x * y^2) * t_3 t_{12} t_3 = e \implies (y * x * y^2) * t_3 = t_3 t_{12}$$

First Double Coset [∗]

$NeN = \{Ne^n | n \in N] = [N]$.

The double coset NeN $= [∗]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right cosets in N is $\frac{120}{120} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. We need only determine the double coset of the right cosets Nt_1. Thus 12 t_i's extend the double coset $[∗]$ to the double coset $Nt_1 N = [1]$.

<u>Cayley Diagram</u>

Figure 7.29: Cayley Diagram for G over S_{12}

Second Double Coset [1]

$Nt_1 N = \{N(t_1)^n | n \in N\}$
$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}\}$.

We now find the Coset Stabilier $N^{1)}$. We first find the point stabiliser of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

Thus, $N^{(1)} \geq \langle N^1 = (2,5,7,8,12)(3,9,6,10,11), (3,11)(5,12)(7,8)(9,10) \rangle$

$\implies (2,5,7,8,12)(3,9,6,10,11), (3,11)(5,12)(7,8)(9,10) \in N^{(1)}$.

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{120}{10} = 12$.

The orbits of $N^{(1)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}$, $\{4\}$, $\{2,5,7,12,8\}$, $\{3,9,11,6,10\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Ne \in [*]$.

This means one t_i takes [1] to [*].

Choose 2 from $\{2,5,7,12,8\}$

$Nt_1t_2 \in [12]$.

This means five t_i's take [1] to [12].

Choose 3 from $\{3,9,11,6,10\}$

$Nt_1t_3 \in [1]$.

We have $Nt_1t_3 = (y * x * y^{-1} * x^{-1} * y)t_1 \in [1]$

This means five t_i's take [1] to [1].

Choose 4 from $\{4\}$ $Nt_1t_4 \in [14]$.

This means one t_i takes [1] to [14].

Cayley Diagram

Figure 7.30: Cayley Diagram for G over S_{12}

Third Double Coset [12]

$Nt_1t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the coset stablilizer $N^{(12)}$. We first find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$

$N^{12} = \langle (3,11)(5,12)(7,8)(9,10) \rangle$.

We find different names of Nt_1t_2. Now, $t_1t_2 = (y^-2 * x^-1)t_7t_9 \implies Nt_1t_2 = Nt_7t_9$

$N(t_1t_2)^{(1,7)(2,9)(4,11)(6,12)} = Nt_7t_9$.

But $Nt_7t_9 = Nt_1t_2 \implies N(t_1t_2)^{(1,7)(2,9)(4,11)(6,12)} = Nt_1t_2$.

$\implies (1,7)(2,9)(4,11)(6,12), (1,7,8)(2,9,10)(3,4,11)(5,6,12) \in N^{(12)}$. Thus $N^{(12)} \geq \langle N^{12}, (1,7)(2,9)(4,11)(6,12), (1,7,8)(2,9,10)(3,4,11)(5,6,12) \rangle$.

$= \langle (3,11)(5,12)(7,8)(9,10), (1,7)(2,9)(4,11)(6,12), (1,7,8)(2,9,10)(3,4,11)(5,6,12) \rangle$.

The number of single right cosets in Nt_1t_2N is calculated by the formula,

$\frac{|N|}{|N^{(12)}|} = \frac{120}{6} = 20$.

The orbits of $N^{(12)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1,7,8\}, \{2,9,10\}, \{3,11,4\}, \{5,12,6\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from {1,7,8}

$Nt_1t_2t_1 \in [12]$.

We have $Nt_1t_2t_1 = (x * y^{-1} * x^{-1} * y * x * y)t_1t_2 \in [12]$

This means three t_i's take [12] to [12].

Choose 2 from {2,9,10}

$Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$.

This means three t_i's take [12] to [1].

Choose 3 from {3,11,4}

$Nt_1t_2t_3 \in [14]$.

We have $Nt_1t_2t_3 = (y^x)t_5t_{10}$

$= (y^x)(t_1t_4)^y \in [14]$

This means three t_i's take [12] to [14].

Choose 5 from {5,12,6}

$Nt_1t_2t_5 \in [125]$.

This means three t_i's take [12] to [125].

Cayley Diagram

Figure 7.31: Cayley Diagram for G over S_{12}

Fourth Double Coset [14]

$Nt_1t_4 = \{N(t_1t_4)^n | n \in N\}$.

We now find the coset stablilizer $N^{(14)}$. We first find the point stabilizer of 1 and 4 in N.

$N^{14} = \{n \in N | (14)^n = 14\}$ Thus $N^{(14)} \geq N^{14} = \langle (2,5,7,8,12)(3,9,6,10,11)(3,11)(5,12)(7,8)(9,10)\rangle$.

The number of single right cosets in Nt_1t_2N is calculated by the formula,

$\frac{|N|}{|N^{(14)}|} = \frac{120}{10} = 12$.

The orbits of $N^{(14)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{4\}, \{2,5,7,8,12\}, \{3,9,11,6,10\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1\}$

$Nt_1t_4t_1 \in [14]$.

We have $Nt_1t_4t_1 = y^5t_1t_4 \in [14]$

This means one t_i takes [14] to [14].

Choose 2 from $\{2,5,7,8,12\}$

$Nt_1t_4t_2 \in [12]$.

We have $Nt_1t_4t_2 = (y * x^{-1} * y^{-1} * x * y)t_6t_4$

$= (y * x^{-1} * y^{-1} * x * y)(t_1t_2)^{(x*y*x^{-1}*y*x^{-1})} \in [12]$

This means five t_i's take [14] to [12].

Choose 3 from $\{3,9,11,6,10\}$

$Nt_1t_4t_3 \in [125]$.

We have $Nt_1t_4t_3 = (y^{-2})t_8t_6t_7$

$= (y^{-2})(t_1t_2t_5)^{(y^{-1}*x^{-1}*y^{-1}*x*y)} \in [125]$

This means five t_i's take [14] to [125].

Choose 4 from $\{4\}$

$Nt_1t_4t_4 = Nt_1t_4^2 = Nt_1 \in [1]$.

This means one t_i takes [14] to [1].

Cayley Diagram

Figure 7.32: Cayley Diagram for G over S_{12}

Fifth Double Coset [125]

$Nt_1t_2t_5 = \{N(t_1t_2t_5)^n | n \in N\}$.

We now find the coset stablilizer $N^{(125)}$. We first find the point stabilizer of 1,2 and 5 in N.

$N^{125} = \{n \in N | (125)^n = 125\}$

$N^{125} = \langle e \rangle$.

We find different names of $Nt_1t_2t_5$. Now, $t_1t_2t_5 = (x^{-1} * y * x * y * x^{-1})t_{11}t_{12}t_{10} \implies$

$Nt_1t_2t_5 = Nt_{11}t_{12}t_{10}$

$N(t_1t_2t_5)^{(1,11)(2,12)(3,8)(4,7)(5,10)(6,9)} = Nt_{11}t_{12}t_{10}$.

But $Nt_{11}t_{12}t_{10} = Nt_1t_2t_5 \implies N(t_1t_2t_5)^{(1,11)(2,12)(3,8)(4,7)(5,10)(6,9)} = Nt_1t_2t_5$.

$\implies (1,11)(2,12)(3,8)(4,7)(5,10)(6,9) \in N^{(125)}$. Thus $N^{(125)} \geq \langle N^{125}, (1,11)(2,12)(3,8)(4,7)(5,10)(6$

$= \langle e, (1,11)(2,12)(3,8)(4,7)(5,10)(6,9) \rangle$.

The number of single right cosets in $Nt_1t_2t_5N$ is calculated by the formula,

$\frac{|N|}{|N^{(125)}|} = \frac{120}{8} = 15$.

The orbits of $N^{(125)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1, 7, 4, 11\}, \{2, 6, 9, 12\}, \{3, 10, 5, 8\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1,7,4,11\}$

$Nt_1t_2t_5t_1 \in [141].$

We have $Nt_1t_2t_5t_1 = (x * y^3)t_{11}t_7t_{11}$

$= (x * y^3)(t_1t_4t_1)^{(x*y^{-1}*x)} \in [141]$

This means four t_i's take $[125]$ to $[141]$.

Choose 2 from $\{2,6,9,12\}$

$Nt_1t_2t_5t_2 \in [125].$

We have $Nt_1t_2t_5t_2 = (y * x * y^{-2})t_4t_9t_3$

$= (y * x * y^{-2})(t_1t_2t_5)^{(y*x*y^{-1}*x)} \in [125]$

This means four t_i's take $[125]$ to $[125]$.

Choose 3 from $\{3,10\}$

$Nt_1t_2t_5t_3 \in [12].$

We have $Nt_1t_2t_5t_3 = (x * y * x^{-1} * y^{-1} * x)t_{11}t_2$

$= (x * y * x^{-1} * y^{-1} * x)(t_1t_2)^{((y*x^{-1})^2)} \in [12]$

This means four t_i's take $[125]$ to $[12]$.

Cayley Diagram
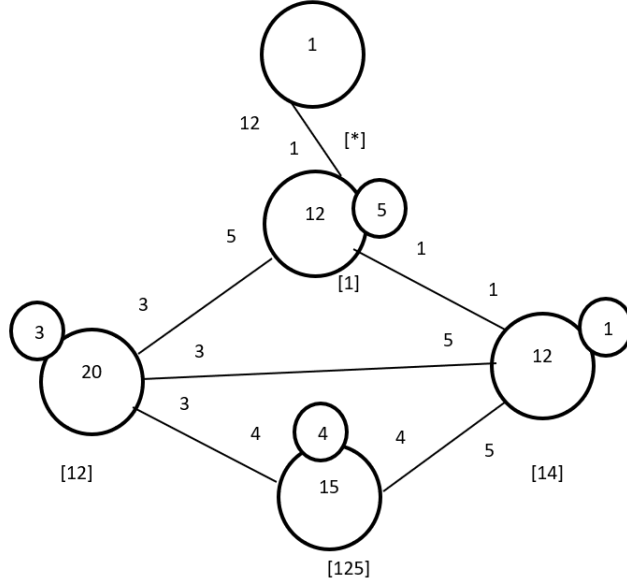
Figure 7.33: Cayley Diagram for G over S_{12}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^w|}$ decreases. Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_2 \cup Nt_1t_4 \cup Nt_1t_2t_5$

$|G| \le (|N| + |\frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(125)}|} \times |N|$

$|G| \le (1 + 12 + 20 + 12 + 15) \times 120 \implies |G| \le 60 \times 120 \implies |G| \le 7200.$

G acts on 60 cosets that are given in the cayley diagram. Let X be the set of these 60 cosets. Now $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong \text{Imf}$ (First Isomorphsim Theorem) $\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = 7200.$

This means $|G| \ge 7200$. We know $|G| \le 7200$ from the cayley diagram. Therefore, $|G| = 7200.$

From $|G| = 7200 \times |Kerf|$ we find $|kerf| = 1.$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong A_5 \times 2 \implies G \cong A_5 \times 2.$

7.6   Construction of $J1$ over $(2 \times A_5)$

Let $N \cong A_5 \times 2 = \langle x, y \rangle$ where $x \sim (1, 6, 3)(2, 8, 4)(5, 7, 9)(10, 11, 12)$, and

$y \sim (1, 5, 2, 3, 11, 4, 10, 6, 8, 7)(9, 12)$. The progenitor $2^{*12} : N$ has symmetric presentation

```
G<x,y,t>:=Group<x,y,t|x^3, (x * y^-2)^2,
    (x^-1 * y * x^-1 * y^-1)^2,
    (x^-1 * y^-3)^2,
    t^2>;
```

We want to verify that the presentation given above is correct.

Using the theorem: $\frac{2^{*n}:N}{t_i, t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$,

we will show that $\frac{2^{*12}:N}{((y^5)*(t^{(y^2*x*y*x^{-1}*y)})^3, ((x)*(t^{(y^{-2}*x^{-1}*y)})^5} \cong J1$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^3, (x * y^-2)^2,
(x^-1 * y * x^-1 * y^-1)^2,
(x^-1 * y^-3)^2,
t^2,
(t,x^-1 * y^-1 * x * y^2),
(t,x * y * x^-1 * y^-1 * x * y),
((y^5)*(t^{(y^2 * x * y * x^-1 * y)}))^3,
((x)*(t^{(y^-2 * x^-1 * y)}))^5 ~ J1
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{175560}{120} = 1463$. Now we know that we have 1463 distinct single cosets.

Expanding First Relation We will expand our first relation.

$$((y^5) * (t^{(y^2*x*y*x^{-1}*y)}))^3$$

$= y^5 * t_2 * y^5 * t_2 * y^5 * t_2$

$= (y^5)^3 * (t_2^{(y^5)^2}) * (t_2^{(y^5)}) * t_2$

Since the order of y is 10 we can reduce our relation such as,

$y^5 * t_2 t_6 t_2 = e \implies y^5 * t_2 = t_2 t_6$

Expanding Second Relation We will expand our second relation.

$$((x) * (t^{(y^{-2}*x^{-1}*y)}))^5$$

$$= x * t_3 * x * t_3 * x * t_3 * x * t_3 * x * t_3$$

$$= x^5 * (t_3^{(x^4)}) * (t_3^{(x^3)}) * (t_3^{(x^2)}) * (t_3^x) * t_3$$

$$\implies x^5 * t_1 * t_3 * t_6 * t_1 * t_3 = e$$

$$\implies x^5 * t_1 t_3 t_6 = t_3 t_1$$

First Double Coset [*]

$NeN = \{Ne^n | n \in N] = [N]$.

The double coset NeN = [*] contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right cosets in N is $\frac{120}{120} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. We need only determine the double coset of the right cosets Nt_1. Thus 12 t_i's extend the double coset [*] to the double coset $Nt_1N = [1]$.
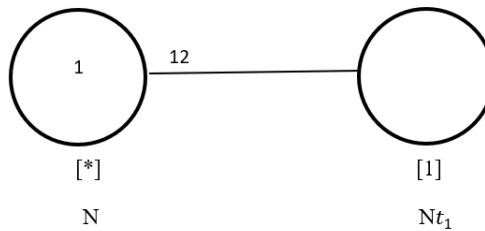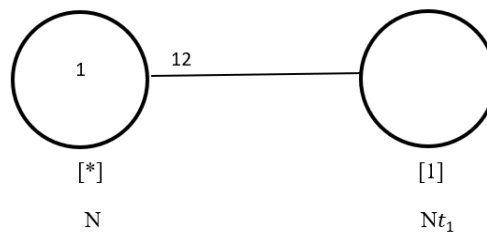
Cayley Diagram

Figure 7.34: Cayley Diagram for G over S_{12}

Second Double Coset [1]

$Nt_1N = \{N(t_1)^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}\}$.

We now find the Coset Stabilier $N^{1)}$. We first find the point stabiliser of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

Thus, $N^{(1)} \geq \langle N^1 = (2, 5, 7, 8, 12)(3, 9, 6, 10, 11), (3, 11)(5, 12)(7, 8)(9, 10) \rangle$

$\implies (2, 5, 7, 8, 12)(3, 9, 6, 10, 11), (3, 11)(5, 12)(7, 8)(9, 10) \in N^{(1)}$.

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{120}{10} = 12$.

The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{4\}, \{2, 5, 7, 12, 8\}, \{3, 9, 11, 6, 10\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = Ne \in [*]$.

This means one t_i takes [1] to [*].

Choose 4 from $\{4\}$

$Nt_1t_4 \in [1]$.

we have $Nt_1t_4 = y^5t_1 \in [1]$

This means one t_i takes [1] to [1].

Choose 2 from $\{2,5,7,12,8\}$

$Nt_1t_2 \in [12]$.

This means five t_i's take [1] to [12].

Choose 3 from $\{3,9,11,6,10\}$

$Nt_1t_3 \in [13]$.

This means five t_i's take [1] to [13].
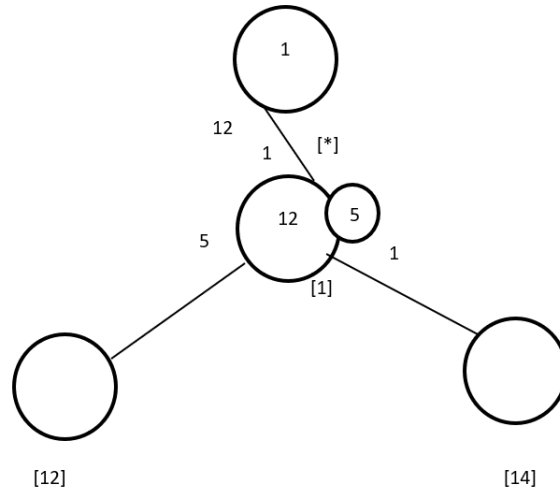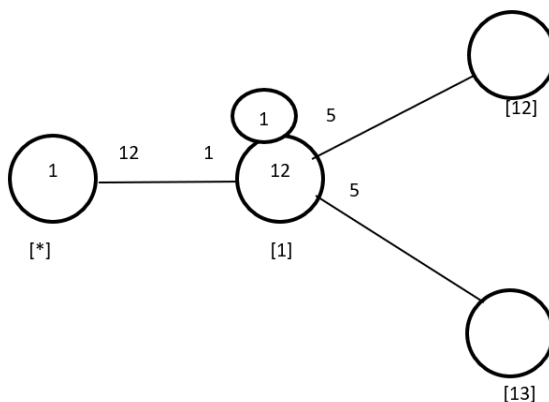
Cayley Diagram

Figure 7.35: Cayley Diagram for G over S_{12}

Third Double Coset [12]

$Nt_1t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the coset stablilizer $N^{(12)}$. We first find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$

Thus $N^{(12)} \geq N^{12} = \langle (3, 11)(5, 12)(7, 8)(9, 10) \rangle$.

The number of single right cosets in Nt_1t_2N is calculated by the formula,

$\frac{|N|}{|N^{(12)}|} = \frac{120}{2} = 60$.

The orbits of $N^{(12)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{4\}, \{6\}, \{3, 11\}, \{5, 12\}, \{7, 8\}, \{9, 10\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1\}$

$Nt_1t_2t_1 \in [12]$.

we have $Nt_1t_2t_1 = (y^2 * x * y * x^{-1} * y)t_1t_2 \in [12]$

This means one t_i takes $[12]$ to $[12]$.

Choose 2 from $\{2\}$

$Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$.

This means one t_i takes [12] to [1].

Choose 3 from {3,11}

$Nt_1t_2t_3 \in [123]$.

This means two t_i's take [12] to [123].

Choose 4 from {4}

$Nt_1t_2t_4 \in [124]$.

This means one t_i takes [12] to [124].

Choose 5 from {5,12}

$Nt_1t_2t_5 \in [125]$.

This means two t_i's take [12] to [125].

Choose 6 from {6}

$Nt_1t_2t_6 \in [13]$.

We have $Nt_1t_2t_6 = y^5t_4t_2$

$= \mathrm{y}^5 \, (\mathrm{t_1t_3})^{y*x^{-1}*y*x*y^{-1}} \in [13]$

This means one t_i takes [12] to [13].

Choose 7 from {7,8}

$Nt_1t_2t_7 \in [127]$.

This means two t_i's take [12] to [127].

Choose 9 from {9,10}

$Nt_1t_2t_9 \in [129]$.

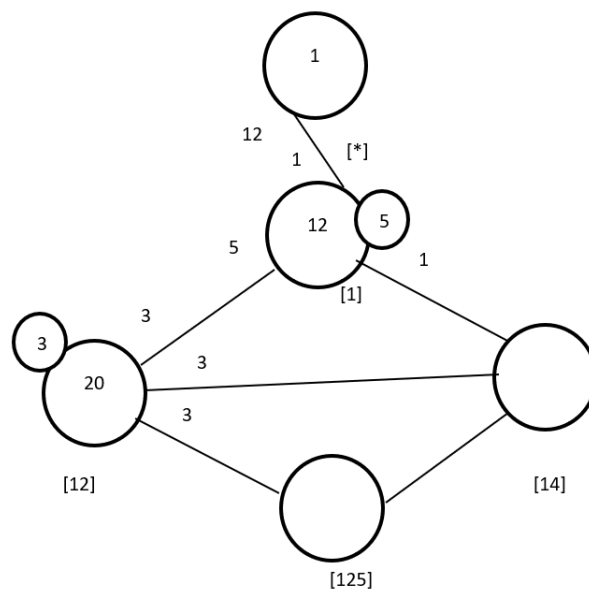This means two t_i's take [12] to [129].

Cayley Diagram
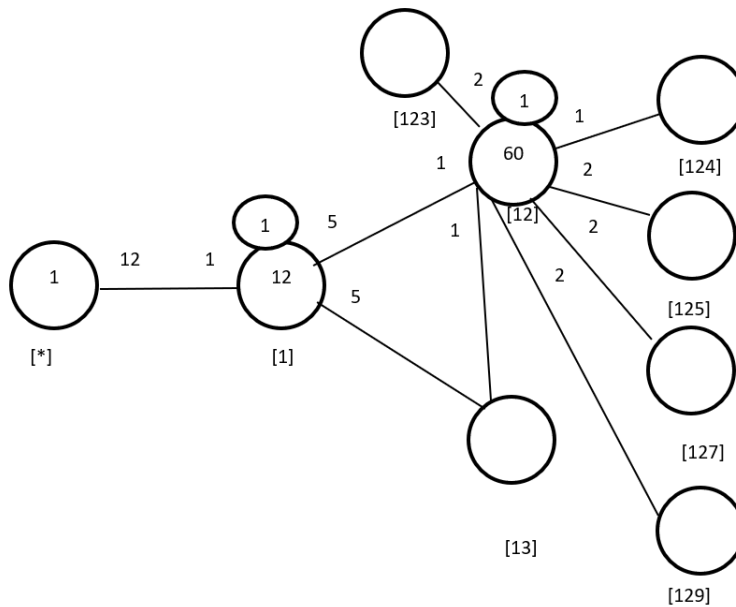
Figure 7.36: Cayley Diagram for G over S_{12}

Fourth Double Coset [13]

$Nt_1t_3 = \{N(t_1t_3)^n | n \in N\}$.

We now find the coset stablilizer $N^{(13)}$. We first find the point stabilizer of 1 and 3 in N.

$N^{13} = \{n \in N | (13)^n = 13\}$

Thus $N^{(13)} \geq N^{13} = \langle (2,5)(6,10)(7,12)(9,11) \rangle$.

The number of single right cosets in Nt_1t_3N is calculated by the formula,

$\frac{|N|}{|N^{(13)}|} = \frac{120}{2} = 60$.

The orbits of $N^{(13)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{3\}, \{4\}, \{8\}, \{2,5\}, \{6,10\}, \{7,12\}, \{9,11\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1\}$

$Nt_1t_3t_1 \in [13]$.

We have $Nt_1t_3t_1 = (y * x * y^{-1} * x^{-1} * y)t_1t_3 \in [13]$

This means one t_i takes [13] to [13].

Choose 2 from {2,5}

$Nt_1t_3t_2 \in [132]$.

This means two t_i's take [13] to [132].

Choose 3 from {3}

$Nt_1t_3t_3 \in [1]$.

This means one t_i takes [13] to [1].

Choose 4 from {4}

$Nt_1t_3t_4 \in [134]$.

This means one t_i takes [13] to [134].

Choose 6 from {6,10}

$Nt_1t_3t_6 \in [13]$.

We have $Nt_1t_3t_6 = xt_3t_1$

$= x(t_1t_3)^{(x^{-1}*y*x^{-1}*y*x)} \in [13]$

This means two t_i's take [13] to [13].

Choose 7 from {7,12}

$Nt_1t_3t_7 \in [137]$.

This means two t_i's take [13] to [137].

Choose 8 from {8}

$Nt_1t_3t_8 \in [12]$.

We have $Nt_1t_3t_8 = y^5t_4t_3$

$= y^5(t_1t_2)^{(y*x^{-1}*y*x^{-1}*y)} \in [12]$ This means one t_i takes [13] to [12].

Choose 9 from {9,11}

$Nt_1t_3t_9 \in [139]$.

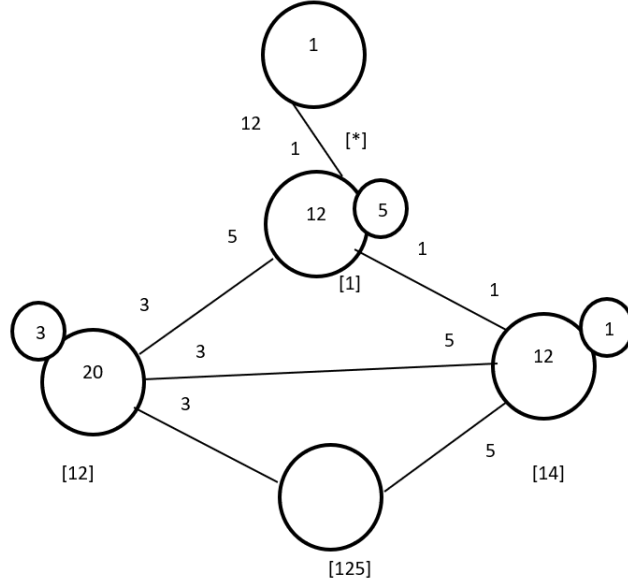This means two t_i's take [13] to [139].

Cayley Diagram

Figure 7.37: Cayley Diagram for G over S_{12}

Fifth Double Coset [123]

$Nt_1t_2t_3 = \{N(t_1t_2t_3)^n | n \in N\}$.

We now find the coset stablilizer $N^{(123)}$. We first find the point stabilizer of 1,2 and 3 in N.

$N^{123} = \{n \in N | (123)^n = 123\}$

$N^{123} = \langle e \rangle$.

We find different names of $Nt_1t_2t_3$. Now, $Nt_1t_2t_3 = Nt_8t_6t_4$

$N(t_1t_2t_3)^{(1,8)(2,6)(3,4)(5,10)(7,12)(9,11)} = Nt_8t_6t_4$.

But $Nt_8t_6t_4 = Nt_1t_2t_3 \implies N(t_1t_2t_3)^{(1,8)(2,6)(3,4)(5,10)(7,12)(9,11)} = Nt_1t_2t_3$.

$\implies (1,8)(2,6)(3,4)(5,10)(7,12)(9,11) \in N^{(123)}$.

Thus $N^{(123)} \geq \langle N^{123}, (1,8)(2,6)(3,4)(5,10)(7,12)(9,11) \rangle$.

$= \langle e, (1,8)(2,6)(3,4)(5,10)(7,12)(9,11) \rangle$.

The number of single right cosets in $Nt_1t_2t_3N$ is calculated by the formula,

$\frac{|N|}{|N^{(123)}|} = \frac{120}{2} = 60$.

The orbits of $N^{(123)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_2t_3t_1 \in [127].$

We have $Nt_1t_2t_3t_1 = y^5 t_3 t_2 t_4$

$= y^5 (t_1t_2t_7)^{x*y*x^{-1}*y} \in [127]$

This means one t_i takes $[123]$ to $[127]$.

Choose 2 from $\{2\}$

$Nt_1t_2t_3t_2 \in [132].$

We have $Nt_1t_2t_3t_2 = (y*x*y^{-1}*x^{-1}*y)*(t_6t_1t_8)$

$= (y*x*y^{-1}*x^{-1}*y)*(t_1t_3t_2)^x \in [132]$

This means one t_i takes $[123]$ to $[132]$.

Choose 3 from $\{3\}$

$Nt_1t_2t_3t_3 \in [12].$

This means one t_i takes $[123]$ to $[12]$.

Choose 4 from $\{4\}$

$Nt_1t_2t_3t_4 \in [12].$

We have $Nt_1t_2t_3t_4 = Nt_8t_6$

$= (t_1t_2)^{(y*x*y^2*x*y^{-1})} \in [12]$

This means one t_i takes $[123]$ to $[12]$.

Choose 5 from $\{5\}$

$Nt_1t_2t_3t_5 \in [1235].$

This means one t_i takes $[123]$ to $[1235]$.

Choose 6 from $\{6\}$

$Nt_1t_2t_3t_6 \in [132].$

We have $Nt_1t_2t_3t_6 = (y^2*x*y*x^{-1}*y)(t_3t_6t_4)$

$= (y^2*x*y*x^{-1}*y)*(t_1t_3t_2)^{(x^{-1})} \in [132]$

This means one t_i takes $[123]$ to $[132]$.

Choose 7 from $\{7\}$

$Nt_1t_2t_3t_7 \in [1237].$

This means one t_i takes $[123]$ to $[1237]$.

Choose 8 from {8}

$Nt_1t_2t_3t_8 \in [127]$.

We have $Nt_1t_2t_3t_8 = y^5(t_4t_6t_3)$

$= y^5(t_1t_2t_7)^{(x^{-1}*y^2)} \in [127]$

This means one t_i takes [123] to [127].

Choose 9 from {9}

$Nt_1t_2t_3t_9 \in [1239]$.

This means one t_i takes [123] to [1239].

Choose 10 from {10}

$Nt_1t_2t_3t_{10} \in [1235]$.

We have $Nt_1t_2t_3t_{10} = Nt_8t_6t_4t_{10}$

$= (t_1t_2t_3t_5)^{(y*x*y^2*x*y^{-1})} \in [1235]$

This means one t_i takes [123] to [1235].

Choose 11 from {11}

$Nt_1t_2t_3t_{11} \in [1239]$.

We have $Nt_1t_2t_3t_{11} = Nt_8t_6t_4t_{11}$

$= (t_1t_2t_3t_9)^{(y*x*y^2*x*y^{-1})} \in [1239]$

This means one t_i takes [123] to [1239].

Choose 12 from {12}

$Nt_1t_2t_3t_{12} \in [1237]$.

We have $Nt_1t_2t_3t_{12} = Nt_8t_6t_4t_{12}$

$= (t_1t_2t_3t_7)^{(y*x*y^2*x*y^{-1})} \in [1237]$

This means one t_i takes [123] to [1237].

Cayley Diagram

Figure 7.38: Cayley Diagram for G over S_{12}

Sixth Double Coset [124]

$Nt_1t_2t_4 = \{N(t_1t_2t_4)^n | n \in N\}$.

We now find the coset stablilizer $N^{(124)}$. We first find the point stabilizer of 1,2 and 4 in N.

$N^{124} = \{n \in N | (124)^n = 124\}$

$N^{124} = \langle (3,11)(5,12)(7,8)(9,10) \rangle$.

We find different names of $Nt_1t_2t_4$. Now, $t_1t_2t_4 = (y^2 * x * y * x^{-1} * y)t_6t_4t_2 \implies Nt_1t_2t_4 = Nt_6t_4t_2$

$N(t_1t_2t_4)^{(1,6)(2,4)(3,11)(5,10)(7,8)(9,12)} = Nt_6t_4t_2$.

But $Nt_6t_4t_2 = Nt_1t_2t_4 \implies N(t_1t_2t_4)^{(1,6)(2,4)(3,11)(5,10)(7,8)(9,12)} = Nt_1t_2t_4$.

$\implies (1,6)(2,4)(3,11)(5,10)(7,8)(9,12) \in N^{(124)}$.

Thus $N^{(124)} \geq \langle N^{124}, (1,6)(2,4)(3,11)(5,10)(7,8)(9,12) \rangle$.

$= \langle (3,11)(5,12)(7,8)(9,10), (1,6)(2,4)(3,11)(5,10)(7,8)(9,12) \rangle$.

The number of single right cosets in $Nt_1t_2t_4N$ is calculated by the formula,

$\frac{|N|}{|N^{(124)}|} = \frac{120}{8} = 15$.

The orbits of $N^{(124)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1, 6, 2, 4\}, \{3, 11, 8, 7\}, \{5, 12, 10, 9\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from {1,6,2,4}

$Nt_1t_2t_4t_1 \in [12]$.

We have $Nt_1t_2t_4t_1 = (y * x^{-1} * y^{-1} * x * y)t_4t_6$

$= (y * x^{-1} * y^{-1} * x * y)(t_1t_2)^{y^5} \in [12]$

This means four t_i's take [124] to [12].

Choose 3 from {3,11,8,7}

$Nt_1t_2t_4t_3 \in [124]$.

We have $Nt_1t_2t_4t_3 = (y * x * y * x * y)t_4t_6t_1$

$= (y * x * y * x * y) * (t_1t_2t_4)^{y^5} \in [124]$

This means four t_i's take [124] to [124].

Choose 5 from {5,12,10,9}

$Nt_1t_2t_4t_5 \in [1245]$.

This means four t_i's take [124] to [1245].

Cayley Diagram
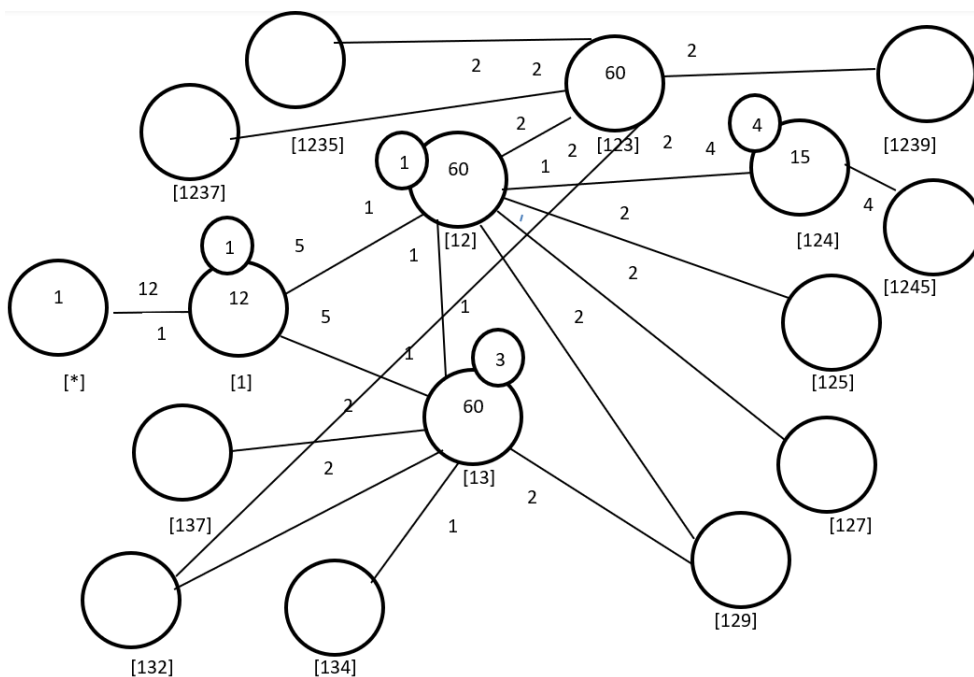
Figure 7.39: Cayley Diagram for G over S_{12}

Seventh Double Coset [125]

$Nt_1t_2t_5 = \{N(t_1t_2t_5)^n | n \in N\}$.

We now find the coset stablilizer $N^{(125)}$. We first find the point stabilizer of 1,2 and 5 in N.

$N^{125} = \{n \in N | (125)^n = 125\}$

Thus, $N^{(125)} \geq N^{125} = \langle e \rangle$.

The number of single right cosets in $Nt_1t_2t_5N$ is calculated by the formula,

$\frac{|N|}{|N^{(125)}|} = \frac{120}{1} = 120$.

The orbits of $N^{(125)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_2t_5t_1 \in [1251]$.

This means one t_i takes $[125]$ to $[125]$.

Choose 2 from $\{2\}$

$Nt_1t_2t_5t_2 \in [125]$.

We have $Nt_1t_2t_5t_2 = (x^{-1}*y*x^{-1}*y*x)t_3t_2t_5$

$= (x^{-1}*y*x^{-1}*y*x)(t_1t_2t_5)^{(y*x*y^{-1}*x^{-1}*y)} \in [125]$

This means one t_i takes $[125]$ to $[125]$.

Choose 3 from $\{3\}$

$Nt_1t_2t_5t_3 \in [1237]$.

We have $Nt_1t_2t_5t_3 = (y*x^{-1}*y)t_6t_{10}t_{11}t_9$

$= (y*x^{-1}*y)*(t_1t_2t_3t_7)^{(y*x^{-1}*y*x*y)} \in [1237]$

This means one t_i takes $[125]$ to $[1237]$.

Choose 4 from $\{4\}$

$Nt_1t_2t_5t_4 \in [125]$.

We have $Nt_1t_2t_5t_4 = (x^{-1}*y^{-1}*x^{-1})t_1t_{12}t_8$

$= (x^{-1}*y^{-1}*x^{-1})*(t_1t_2t_5)^{(x^{-1}*y^{-1}*x*y^2)} \in [125]$

This means one t_i takes $[125]$ to $[125]$.

Choose 5 from $\{5\}$

$Nt_1t_2t_5t_5 \in [12]$.

This means one t_i takes $[125]$ to $[12]$.

Choose 6 from $\{6\}$

$Nt_1t_2t_5t_6 \in [1256]$.

This means one t_i takes $[125]$ to $[1256]$.

Choose 7 from $\{7\}$

$Nt_1t_2t_5t_7 \in [1235]$.

We have $Nt_1t_2t_5t_7 = (x^{-1}*y^{-1}*x^{-1}*y^{-1}*x)t_4t_{11}t_2t_3$

$= (x^{-1}*y^{-1}*x^{-1}*y^{-1}*x)*(t_1t_2t_3t_5)^{(y*x^{-1}*y*x*y^{-1})} \in [1235]$

This means one t_i takes $[125]$ to $[1235]$.

Choose 8 from $\{8\}$

$Nt_1t_2t_5t_8 \in [1245]$.

We have $Nt_1t_2t_5t_8 = (y^2*x*y^{-1}*x)t_{11}t_3t_7t_4$

$= (y^2*x*y^{-1}*x)*(t_1t_2t_4t_5)^{(x*y^{-1}*x)} \in [1245]$

This means one t_i takes $[125]$ to $[1245]$.

Choose 9 from {9}

$Nt_1t_2t_5t_9 \in [1239]$.

We have $Nt_1t_2t_5t_9 = t_1t_{12}t_{10}t_6$

$= (t_1t_2t_3t_9)^{x^{-1}*y^{-1}*x*y^2)} \in [1239]$

This means one t_i takes [125] to [1239].

Choose 10 from {10}

$Nt_1t_2t_5t_{10} \in [129]$.

We have $Nt_1t_2t_5t_{10} = y^5t_4t_6t_5$

$= y^5(t_1t_2t_9)^{(x^{-1}*y^2)} \in [129]$

This means one t_i takes [125] to [129].

Choose 11 from {11}

$Nt_1t_2t_5t_{11} \in [1,2,5,11]$.

This means one t_i takes [125] to [125,11].

Choose 12 from {12}

$Nt_1t_2t_5t_{12} \in [1,2,5,12]$.

This means one t_i takes [125] to [125,12].

Cayley Diagram
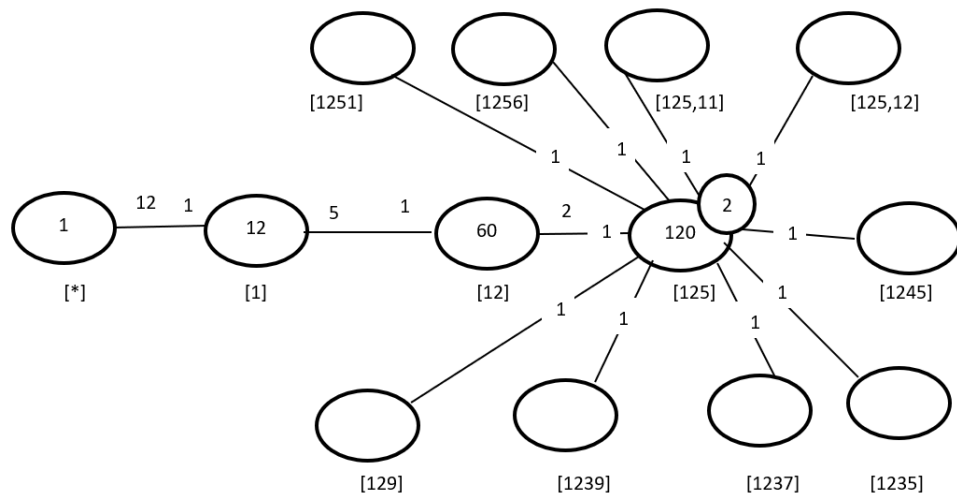
Figure 7.40: Cayley Diagram for G over S_{12}

Eightieth Double Coset [127]

$Nt_1t_2t_7 = \{N(t_1t_2t_7)^n | n \in N\}$.

We now find the coset stablilizer $N^{(127)}$. We first find the point stabilizer of 1,2 and 7 in N.

$N^{127} = \{n \in N | (127)^n = 127\}$

$N^{127} = \langle e \rangle$.

We find different names of $Nt_1t_2t_7$. Now, $t_1t_2t_7 = (y*x)t_{11}t_2t_4 \implies Nt_1t_2t_7 = Nt_{11}t_2t_4$

$N(t_1t_2t_7)^{(1,11)(3,5)(4,7)(8,10)} = Nt_{11}t_2t_4$.

But $Nt_{11}t_2t_4 = Nt_1t_2t_7 \implies N(t_1t_2t_7)^{(1,11)(3,5)(4,7)(8,10)} = Nt_1t_2t_7$.

$\implies (1,11)(3,5)(4,7)(8,10) \in N^{(127)}$.

Thus $N^{(127)} \geq \langle N^{127}, (1,11)(3,5)(4,7)(8,10) \rangle$.

$= \langle e, (1,11)(3,5)(4,7)(8,10) \rangle$.

The number of single cosets in $Nt_1t_2t_7$ is calculated by the formula,

$\frac{|N|}{|N^{(127)}|} = \frac{120}{2} = 60$.

The orbits of $N^{(127)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1,11\}, \{2\}, \{3,5\}, \{4,7\}, \{6\}, \{8,10\}, \{9\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from {1}

$Nt_1t_2t_7t_1 \in [123]$.

We have $Nt_1t_2t_7t_1 = ((x^{-1}*y^{-1})^2)t_7t_6t_4$

$= ((x^{-1}*y^{-1})^2)(t_1t_2t_3)^{(y^{-1}*x^{-1}*y*x*y)} \in [123]$

This means one t_i takes [127] to [123].

Choose 2 from {2}

$Nt_1t_2t_7t_2 \in [127]$.

We have $Nt_1t_2t_7t_2 = (x*y*x^{-1}*y^{-2})t_1t_2t_7 \in [127]$.

This means one t_i takes [127] to [127].

Choose 3 from {3}

$Nt_1t_2t_7t_3 \in [125,11]$.

We have $Nt_1t_2t_7t_3 = (y*x^{-1}*y*x*y^{-1})t_8t_7t_6t_5$

$= (y*x^{-1}*y*x*y^{-1})(t_1t_2t_5t_{11})^{(y*x*y^{-1})} \in [125,11]$

This means one t_i takes [127] to [125,11].

Choose 4 from {4}

$Nt_1t_2t_7t_4 \in [12]$.

We have $Nt_1t_2t_7t_4 = (y*x)t_{11}t_2$

$= (y*x)(t_1t_2)^{(y*x)} \in [12]$

This means one t_i takes [127] to [12].

Choose 5 from {5}

$Nt_1t_2t_7t_5 \in [125, 11]$.

We have $Nt_1t_2t_7t_5 = (x*y*x*y^{-1})(t_{10}t_4t_6t_3)$

$= (x*y*x*y^{-1})*(t_1t_2t_5t_{11})^{(x*y^{-1}*x*y^2)} \in [1,2,5,11]$

This means one t_i takes [127] to [125,11].

Choose 6 from {6}

$Nt_1t_2t_7t_6 \in [134]$.

We have $Nt_1t_2t_7t_6 = (y^2*x*y*x^{-1}*y)t_1t_{11}t_4$

$= (y^2*x*y*x^{-1}*y)(t_1t_3t_4)^{(y^{-1}*x^{-1}*y^{-1})} \in [134]$

This means one t_i takes [127] to [134].

Choose 7 from {7}

$Nt_1t_2t_7t_7 = Nt_1t_2t_7^2 = Nt_1t_2 \in [12]$.

This means one t_i takes [127] to [12].

Choose 8 from {8}

$Nt_1t_2t_7t_8 \in [137]$.

We have $Nt_1t_2t_7t_8 = (y*x^{-1})t_{12}t_4t_2$

$= (y*x^{-1})(t_1t_3t_7)^{(y*x^{-1}*y^{-1})}$

This means one t_i takes [127] to [137].

Choose 9 from {9}

$Nt_1t_2t_7t_9 \in [125, 12]$.

We have $Nt_1t_2t_7t_9 = (x*y^{-1}*x*y^2)t_9t_7t_6t_5$

$= (x*y^{-1}*x*y^2)(t_1t_2t_5t_{12})^{(y^{-1}*x)}$

This means one t_i takes [127] to [125,12].

Choose 10 from {10}

$Nt_1t_2t_7t_{10} \in [137]$.

We have $Nt_1t_2t_7t_{10} = (y*x^{-1}*y^{-1})t_{12}t_7t_2$

$= (y*x^{-1}*y^{-1})(t_1t_3t_7)^{(x*y^2*x*y)}$

This means one t_i takes [127] to [137].

Choose 11 from {11}

$Nt_1t_2t_7t_{11} \in [123]$.

We have $Nt_1t_2t_7t_{11} = y^5(t_4t_6t_7)$

$= y^5(t_1t_2t_3)^{(x^{-1}*y^2)}$

This means one t_i takes [127] to [123].

Choose 12 from {12}

$Nt_1t_2t_7t_{12} \in [1245]$.

We have $Nt_1t_2t_7t_{12} = ((y*x)^2)t_5t_3t_{10}t_2$

$= ((y*x)^2)(t_1t_2t_4t_5)^y$

This means one t_i takes [127] to [1245].
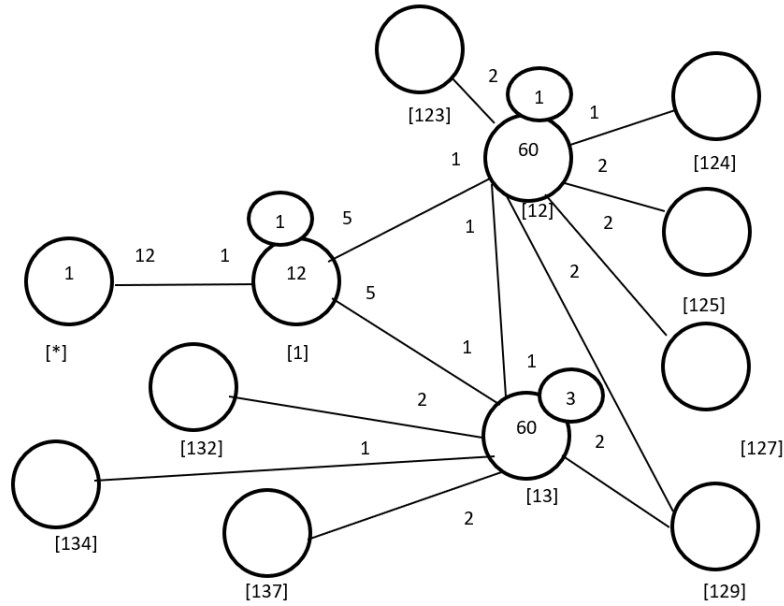
Cayley Diagram
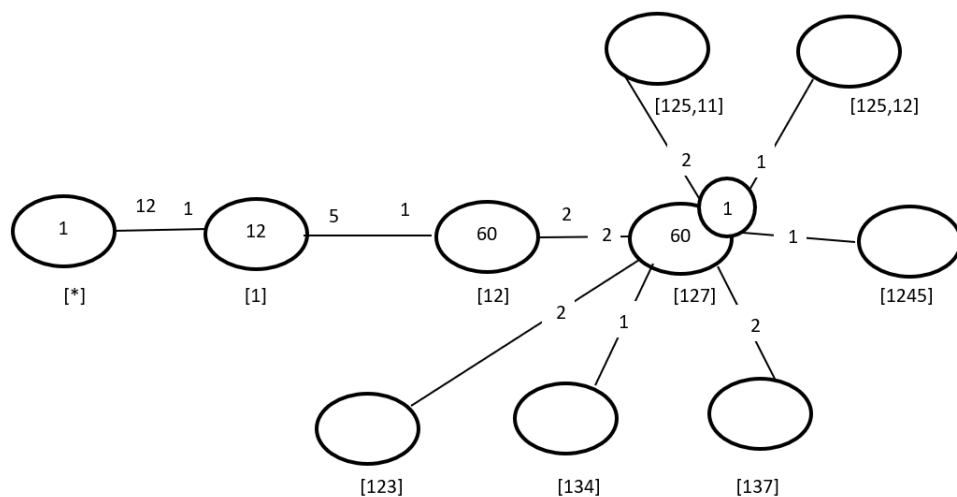
Figure 7.41: Cayley Diagram for G over S_{12}

Ninth Double Coset [129]

$Nt_1t_2t_9 = \{N(t_1t_2t_9)^n | n \in N\}$.

We now find the coset stablilizer $N^{(129)}$. We first find the point stabilizer of 1,2 and 9 in N.

$N^{129} = \{n \in N | (129)^n = 129\}$

Thus, $N^{(129)} \geq N129 = \langle \text{ e } \rangle$.

The number of single cosets in $Nt_1t_2t_9$ is calculated by the formula,

$\frac{|N|}{|N^{(129)}|} = \frac{120}{1} = 120$.

The orbits of $N^{(129)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1\}$

$Nt_1t_2t_9t_1 \in [125, 12]$.

We have $Nt_1t_2t_9t_1 = (y * x^{-1})t_{11}t_4t_{10}t_3$

$= (y * x^{-1})(t_1t_2t_5t_{12})^{x*y^{-3})} \in [125, 12]$

This means one t_i takes [129] to [125,12].

Choose 2 from $\{2\}$

$Nt_1t_2t_9t_2 \in [137]$.

We have $Nt_1t_2t_9t_2 = (y^2 * x * y)t_7t_2t_9$

$= (y^2 * x * y)(t_1t_3t_7)^{(x*y^2)} \in [137]$

This means one t_i takes [129] to [137].

Choose 3 from $\{3\}$

$Nt_1t_2t_9t_3 \in [1256]$.

We have $Nt_1t_2t_9t_3 = (y^{-3})t_{12}t_{10}t_8t_5$

$= (y^{-3})(t_1t_2t_5t_6)^{(x*y^2*x*y^{-1})} \in [1256]$

This means one t_i takes [129] to [1256].

Choose 4 from $\{4\}$

$Nt_1t_2t_9t_4 \in [1237]$.

We have $Nt_1t_2t_9t_4 = (x * y^4)t_5t_7t_8t_2$

$= (x * y^4)(t_1t_2t_3t_7)^{(x*y^{-1}*x^{-1}*y*x)} \in [1237]$

This means one t_i takes [129] to [1237].

Choose 5 from $\{5\}$

$Nt_1t_2t_9t_5 \in [137]$.

We have $Nt_1t_2t_9t_5 = (y^{-3})t_{12}t_3t_{10}$

$= (y^{-3})(t_1t_3t_7)^{(x*y^{-1}*x^{-1}*y^2)} \in [137]$

This means one t_i takes [129] to [137].

Choose 6 from $\{6\}$

$Nt_1t_2t_9t_6 \in [1296]$.

This means one t_i takes [129] to [1296].

Choose 7 from $\{7\}$

$Nt_1t_2t_9t_7 \in [1237]$.

We have $Nt_1t_2t_9t_7 = (x^{-1} * y^{-1} * x^{-1})t_5t_7t_6t_3$

$= (x^{-1} * y^{-1} * x^{-1})(t_1t_2t_3t_7)^{(y^2*x*y*x^{-1})} \in [1237]$

This means one t_i takes [129] to [1237].

Choose 8 from $\{8\}$

$Nt_1t_2t_9t_8 \in [129]$.

We have $Nt_1t_2t_9t_8 = (x^{-1} * y * x * y * x^{-1})t_7t_9t_2$

$= (x^{-1} * y * x * y * x^{-1})(t_1t_2t_9)^{(y^2*x*y)} \in [129]$

This means one t_i takes [129] to [129].

Choose 9 from $\{9\}$

$Nt_1t_2t_9t_9 \in [12]$.

This means one t_i takes [129] to [12].

Choose 10 from $\{10\}$

$Nt_1t_2t_9t_{10} \in [13]$.

We have $Nt_1t_2t_9t_{10} = (y^2)t_9t_8$

$= (y^2)(t_1t_3)^{(y^{-1}*x)} \in [13]$

This means one t_i takes [129] to [13].

Choose 11 from $\{11\}$

$Nt_1t_2t_9t_{11} \in [1235]$.

We have $Nt_1t_2t_9t_{11} = ((y^{-1} * x^{-1})^2)t_{12}t_8t_7t_1$

$= ((y^{-1} * x^{-1})^2)(t_1t_2t_3t_5)^{(x*y^2*x*y)} \in [1235]$

This means one t_i takes [129] to [1235].

Choose 12 from $\{12\}$

$Nt_1t_2t_9t_{12} \in [125]$.

We have $Nt_1t_2t_9t_{12} = (y^5)t_4t_6t_9$

$= (y^5)(t_1t_2t_5)^{(x^{-1}*y^2)} \in [125]$

This means one t_i takes [129] to [125].
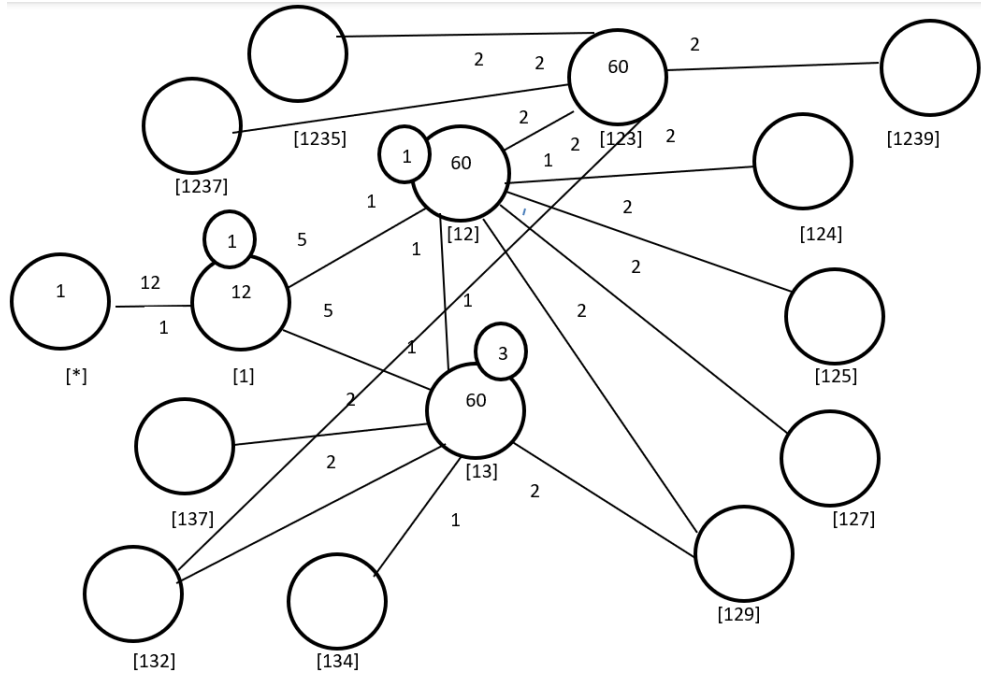
Cayley Diagram
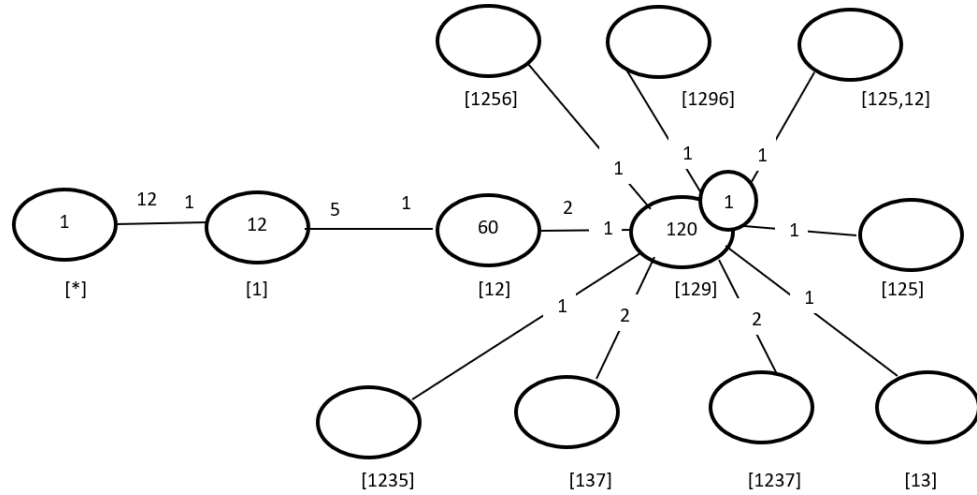
Figure 7.42: Cayley Diagram for G over S_{12}

Tenth Double Coset [132]

$Nt_1t_3t_2 = \{N(t_1t_3t_2)^n | n \in N\}$.

We now find the coset stablilizer $N^{(132)}$. We first find the point stabilizer of 1,3 and 2 in N.

$N^{132} = \{n \in N | (132)^n = 132\}$

$N^{132} = \langle e \rangle$.

We find different names of $Nt_1t_3t_2$. Now, $t_1t_3t_2 = (y^-1 * x^-1 * y^-1 * x^-1 * y^-1)t_8t_4t_6 \implies$

$Nt_1t_3t_2 = Nt_8t_4t_6$

$N(t_1t_3t_2)^{(1,8)(2,6)(3,4)(5,10)(7,12)(9,11)} = Nt_8t_4t_6$.

But $Nt_8t_4t_6 = Nt_1t_3t_2 \implies N(t_1t_3t_2)^{(1,8)(2,6)(3,4)(5,10)(7,12)(9,11)} = Nt_1t_3t_2$.

$\implies (1,8)(2,6)(3,4)(5,10)(7,12)(9,11) \in N^{(132)}$.

Thus $N^{(132)} \geq \langle N^{132}, (1,8)(2,6)(3,4)(5,10)(7,12)(9,11) \rangle$.

$= \langle e, (1,8)(2,6)(3,4)(5,10)(7,12)(9,11) \rangle$.

The number of single cosets in $Nt_1t_3t_2$ is calculated by this formula,

$\frac{|N|}{|N^{(132)}|} = \frac{120}{2} = 60$.

The orbits of $N^{(132)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

<u>Choose 1 from {1}</u>

$Nt_1t_3t_2t_1 \in [132]$.

We have $Nt_1t_3t_2t_1 = (y^5)t_1t_6t_8$

$= (y^5)(t_1t_3t_2)^{(x*y*x^{-1}*y^{-1}*x*y)} \in [132]$

This means one t_i takes [132] to [132].

<u>Choose 2 from {2}</u>

$Nt_1t_3t_2t_2 = Nt_1t_3t_2^2 = Nt_1t_3 \in [13]$.

This means one t_i takes [132] to [13].

<u>Choose 3 from {3}</u>

$Nt_1t_3t_2t_3 \in [123]$.

We have $Nt_1t_3t_2t_3 = (x*y^2*x*y*x^{-1}*y)t_6t_8t_1$

$= (x*y^2*x*y*x^{-1}*y)(t_1t_2t_3)^x \in [123]$

This means one t_i takes [132] to [123].

<u>Choose 4 from {4}</u>

$Nt_1t_3t_2t_4 \in [123]$.

We have $Nt_1t_3t_2t_4 = (x*y*x^{-1}*y^{-1}*x*y)(t_3t_4t_6)$

$= (x*y*x^{-1}*y^{-1}*x*y)*(t_1t_2t_3)^{(x^{-1})} \in [123]$

This means one t_i takes [132] to [123].

<u>Choose 5 from {5}</u>

$Nt_1t_3t_2t_5 \in [1256]$.

We have $Nt_1t_3t_2t_5 = (y*x*y)t_{12}t_8t_1t_3$

$= (y*x*y)(t_1t_2t_5t_6)^{(x*y^2*x*y)} \in [1256]$

This means one t_i takes [132] to [1256].

<u>Choose 6 from {6}</u>

$Nt_1t_3t_2t_6 \in [13]$.

We have $Nt_1t_3t_2t_6 = (y^{-1}*x^{-1}*y^{-1}*x^{-1}*y^{-1})*(t_8t_4)$

$= (y^{-1}*x^{-1}*y^{-1}*x^{-1}*y^{-1})*(t_1t_3)^{(x*y*x^{-1}*y*x^{-1}*y)} \in [13]$

This means one t_i takes [132] to [13].

<u>Choose 7 from {7}</u>

$Nt_1t_3t_2t_7 \in [1237]$.

We have $Nt_1t_3t_2t_7 = (y^3)t_6t_{10}t_{11}t_9$

$= (y^3)(t_1t_2t_3t_7)^{(y*x^{-1}*y*x*y)} \in [1237]$

This means one t_i takes [132] to [1237].

<u>Choose 8 from {8}</u>

$Nt_1t_3t_2t_8 \in [132]$.

We have $Nt_1t_3t_2t_8 = x * (t_8t_2t_1)$

$= x * (t_1t_3t_2)^{(y*x*y*x*y)} \in [132]$

This means one t_i takes [132] to [132].

<u>Choose 9 from {9}</u>

$Nt_1t_3t_2t_9 \in [125, 11]$.

We have $Nt_1t_3t_2t_9 = (x * y^{-1} * x)t_{11}t_{12}t_2t_8$

$= (x * y^{-1} * x)(t_1t_2t_5t_{11})^{(y*x*y*x^{-1}*y)} \in [125, 11]$

This means one t_i takes [132] to [125,11].

<u>Choose 10 from {10}</u>

$Nt_1t_3t_2t_{10} \in [1256]$.

We have $Nt_1t_3t_2t_{10} = y * (t_7t_1t_8t_4)$

$= y * (t_1t_2t_5t_6)^{(y*x)} \in [1256]$

This means one t_i takes [132] to [1256].

<u>Choose 11 from {11}</u>

$Nt_1t_3t_2t_{11} \in [1, 2, 5, 11]$.

We have $Nt_1t_3t_2t_{11} = (y^{-1} * x^{-1} * y^{-1})(t_9t_7t_6t_1)$

$= (y^{-1} * x^{-1} * y^{-1}) * (t_1t_2t_5t_{11})^{(y^{-1}*x)} \in [125, 11]$

This means one t_i takes [132] to [125,11].

<u>Choose 12 from {12}</u>

$Nt_1t_3t_2t_{12} \in [1237]$.

We have $Nt_1t_3t_2t_{12} = (y^{-2})(t_2t_5t_9t_{11})$

$= (y^{-2}) * (t_1t_2t_3t_7)^{(x*y*x^{-1})} \in [1237]$

This means one t_i takes [132] to [1237].

Cayley Diagram
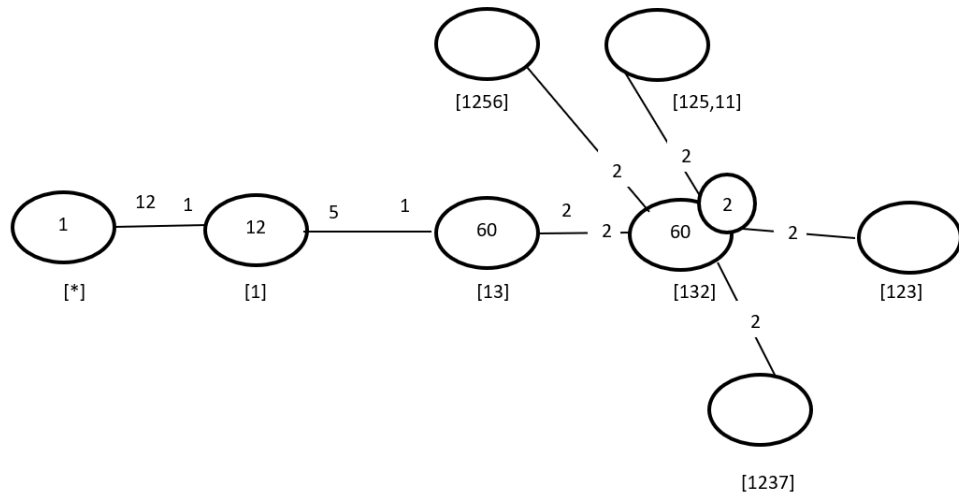
Figure 7.43: Cayley Diagram for G over S_{12}

Eleventh Double Coset [134]

$Nt_1t_3t_4 = \{N(t_1t_3t_4)^n | n \in N\}$.

We now find the coset stablilizer $N^{(134)}$. We first find the point stabilizer of 1,3 and 4 in N.

$N^{134} = \{n \in N | (134)^n = 134\}$

$N^{134} = \langle(2,5)(6,10)(7,12)(9,11)\rangle$.

We find different names of $Nt_1t_3t_4$. Now, $t_1t_3t_4 = (y * x * y^-1 * x^-1 * y)t_8t_4t_3 \implies Nt_1t_3t_4 = Nt_8t_4t_3$

$N(t_1t_3t_4)^{(1,8)(2,6)(3,4)(5,10)(7,12)(9,11)} = Nt_8t_4t_3$.

But $Nt_8t_4t_3 = Nt_1t_3t_4 \implies N(t_1t_3t_4)^{(1,8)(2,6)(3,4)(5,10)(7,12)(9,11)} = Nt_1t_3t_4$.

$\implies (1,8)(2,6)(3,4)(5,10)(7,12)(9,11) \in N^{(134)}$.

Thus $N^{(134)} \geq \langle N^{134}, (1,8)(2,6)(3,4)(5,10)(7,12)(9,11)\rangle$.

$= \langle(2,5)(6,10)(7,12)(9,11), (1,8)(2,6)(3,4)(5,10)(7,12)(9,11)\rangle$.

The number of single cosets in $Nt_1t_3t_4$ is calculated by this formula,

$\frac{|N|}{|N^{(134)}|} = \frac{120}{8} = 15$.

The orbits of $N^{(134)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1, 4, 3, 8\}, \{2, 5, 6, 10\}, \{7, 12, 11, 9\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from {1,4,3,8}

$Nt_1t_3t_4t_1 \in [13]$.

We have $Nt_1t_3t_4t_1 = (y * x * y^2 * x * y^{-1})t_4t_8$

$= (y * x * y^2 * x * y^{-1})(t_1t_3)^{(x*y^{-2})} \in [13]$

This means four t_i's take [134] to [13].

Choose 2 from {2,5,6,10}

$Nt_1t_3t_4t_2 \in [127]$.

We have $Nt_1t_3t_4t_2 = (x^{-1})t_4t_6t_3$

$= (x^{-1})(t_1t_2t_7)^{(x^{-1}*y^2)} \in [127]$

This means four t_i's take [134] to [127].

Choose 7 from {7,12,11,9}

$Nt_1t_3t_4t_7 \in [125, 12]$.

We have $Nt_1t_3t_4t_7 = (x * y * x)t_{12}t_2t_{11}t_1$

$= (x * y * x)(t_1t_2t_5t_{12})^{(x*y^{-1}*x^{-1}*y^2)} \in [125, 12]$

This means four t_i's take [134] to [125,12].

Cayley Diagram
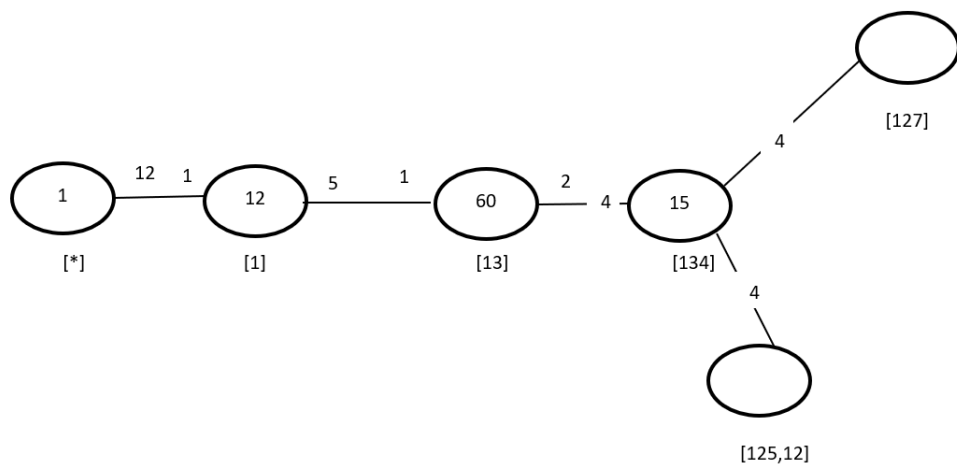
Figure 7.44: Cayley Diagram for G over S_{12}

Twelveth Double Coset [137]

$Nt_1t_3t_7 = \{N(t_1t_3t_7)^n | n \in N\}$.

We now find the coset stablilizer $N^{(137)}$. We first find the point stabilizer of 1,3 and 7 in N.

$N^{137} = \{n \in N | (137)^n = 137\}$. Thus, $N^{(137)} \geq N137 = \langle e \rangle$.

The number of single cosets in $Nt_1t_3t_7$ is calculated by this formula,

$\frac{|N|}{|N^{(137)}|} = \frac{120}{1} = 120$.

The orbits of $N^{(137)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_3t_7t_1 \in [1256]$.

We have $Nt_1t_3t_7t_1 = (x * y)t_1t_{12}t_2t_9$

$= (x * y)(t_1t_2t_5t_6)^{(y^{-1}*x^{-1}*y^{-1})} \in [1256]$

This means one t_i takes [137] to [1256].

Choose 2 from $\{2\}$

$Nt_1t_3t_7t_2 \in [125, 11]$.

We have $Nt_1t_3t_7t_2 = (y * x^{-1})t_1t_5t_7t_3$

$= (y * x^{-1})(t_1t_2t_5t_{11})^{(y*x*y)} \in [125, 11]$

This means one t_i takes [137] to [125,11].

Choose 3 from $\{3\}$

$Nt_1t_3t_7t_3 \in [129]$.

We have $Nt_1t_3t_7t_3 = (x * y^{-1} * x^{-1} * y * x * y)t_2t_3t_7$

$= (x * y^{-1} * x^{-1} * y * x * y)(t_1t_2t_9)^{(y^{-2}*x^{-1})} \in [129]$

This means one t_i takes [137] to [129].

Choose 4 from $\{4\}$

$Nt_1t_3t_7t_4 \in [1239]$.

We have $Nt_1t_3t_7t_4 = (y * x * y * x^{-1} * y)t_9t_7t_1t_2$

$= (y * x * y * x^{-1} * y)(t_1t_2t_3t_9)^{(x*y^{-1}*x^{-1}*y^{-1})} \in [1239]$

This means one t_i takes [137] to [1239].

Choose 5 from $\{5\}$

$Nt_1t_3t_7t_5 \in [1296]$.

We have $Nt_1t_3t_7t_5 = (x * y^{-1})t_{12}t_1t_6t_4$

$= (x * y^{-1})(t_1t_2t_9t_6)^{(x^y)} \in [1296]$

This means one t_i takes [137] to [1296].

Choose 6 from $\{6\}$

$Nt_1t_3t_7t_6 \in [1235]$.

We have $Nt_1t_3t_7t_6 = (x * y^{-1} * x^{-1} * y^{-1} * x^{-1})t_9t_4t_{11}t_3$

$= (x * y^{-1} * x^{-1} * y^{-1} * x^{-1})(t_1t_2t_3t_5)^{(y*x^{-1}*y^{-2})} \in [1235]$

This means one t_i takes [137] to [1235].

Choose 7 from $\{7\}$

$Nt_1t_3t_7t_7 = Nt_1t_3t_7^2 = Nt_1t_3 \in [13]$.

This means one t_i takes [137] to [13].

Choose 8 from $\{8\}$

$Nt_1t_3t_7t_8 \in [1296]$.

We have $Nt_1t_3t_7t_8 = (x * y * x^{-1} * y * x^{-1})t_6t_8t_{11}t_3$

$= (x * y * x^{-1} * y * x^{-1})(t_1t_2t_9t_6)^{(y*x*y^{-2})} \in [1296]$

This means one t_i takes [137] to [1296].

Choose 9 from $\{9\}$

$Nt_1t_3t_7t_9 \in [125, 11]$.

We have $Nt_1t_3t_7t_9 = (y^{-2} * x^{-1} * y)t_9t_5t_7t_2$

$= (y^{-2} * x^{-1} * y)(t_1t_2t_5t_{11})^{(y^{-1}*x*y^2)} \in [125, 11]$

This means one t_i takes [137] to [125,11].

Choose 10 from $\{10\}$

$Nt_1t_3t_7t_{10} \in [137]$.

We have $Nt_1t_3t_7t_{10} = (y^{-1} * x^{-1} * y)t_2t_7t_3$

$= (y^{-1} * x^{-1} * y)(t_1t_3t_7)^{(x*y^{-1}*x^{-1}*y*x*y)} \in [137]$

This means one t_i takes [137] to [137].

Choose 11 from $\{11\}$

$Nt_1t_3t_7t_{11} \in [129]$.

We have $Nt_1t_3t_7t_{11} = (x * y * x)t_{12}t_2t_4$

$= (x * y * x)(t_1t_2t_9)^{(x*y^{-1}*x^{-1}*y^2)} \in [129]$

This means one t_i takes [137] to [129].

Choose 12 from {12}

$Nt_1t_3t_7t_{12} \in [127]$.

We have $Nt_1t_3t_7t_{12} = (y^{-1} * x^{-1} * y^2)t_8t_7t_{10}$

$= (y^{-1} * x^{-1} * y^2)(t_1t_2t_7)^{(y*x*y^{-1})} \in [127]$

This means one t_i takes [137] to [127].

Cayley Diagram
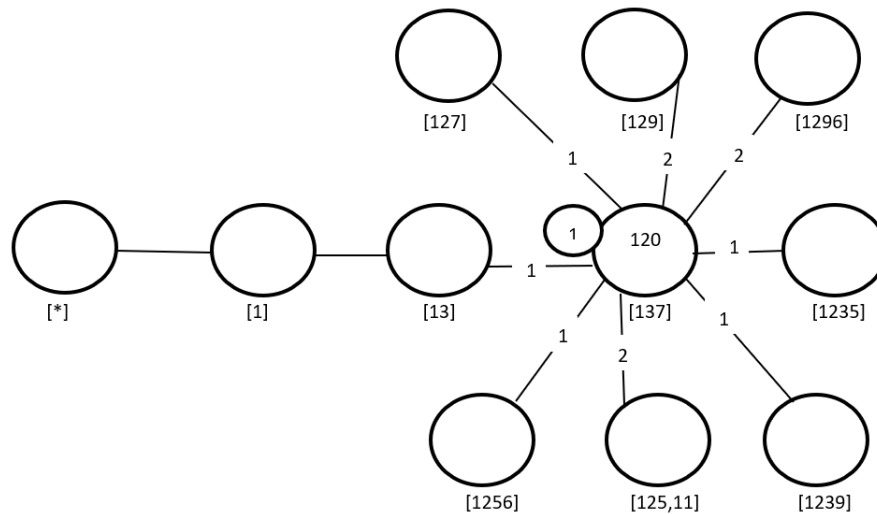
Figure 7.45: Cayley Diagram for G over S_{12}

Thirteenth Double Coset [1235]

$Nt_1t_2t_3t_5 = \{N(t_1t_2t_3t_5)^n | n \in N\}$.

We now find the coset stablilizer $N^{(1235)}$. We first find the point stabilizer of 1,2,3 and 5 in N. $N^{(1235)} \geq N^{1235} = \langle e \rangle$.

The number of single cosets in $Nt_1t_2t_3t_5$ is calculated by this formula,

$\frac{|N|}{|N^{(1235)}|} = \frac{120}{1} = 120$.

The orbits of $N^{(1235)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from {1}

$Nt_1t_2t_3t_5t_1 \in [12351]$.

This means one t_i takes [1235] to [12351].

Choose 2 from {2}

$Nt_1t_2t_3t_5t_2 \in [125, 11]$.

We have $Nt_1t_2t_3t_5t_2 = (y^{-1} * x)t_1t_8t_{12}t_6$

$= (y^{-1} * x)(t_1t_2t_5t_{11})^{(x*y*x^{-1}*y*x)} \in [125, 11]$

This means one t_i takes [1235] to [125,11]

Choose 3 from {3}

$Nt_1t_2t_3t_5t_3 \in [1237]$.

We have $Nt_1t_2t_3t_5t_3 = (y^{-1} * x^{-1} * y^{-1} * x)t_{12}t_8t_7t_2$

$= (y^{-1} * x^{-1} * y^{-1} * x)(t_1t_2t_3t_7)^{(x*y^2*x*y)} \in [1237]$

This means one t_i takes [1235] to [1237]

Choose 4 from {4}

$Nt_1t_2t_3t_5t_4 \in [1235]$.

We have $Nt_1t_2t_3t_5t_4 = (x^{-1} * y^{-1} * x^{-1})t_1t_{12}t_{10}t_8$

$= (x^{-1} * y^{-1} * x^{-1})(t_1t_2t_3t_5)^{(x^{-1}*y^{-1}*x*y^2)} \in [1235]$

This means one t_i takes [1235] to [1235]

Choose 5 from {5}

$Nt_1t_2t_3t_5t_5 = Nt_1t_2t_3t_5^2 = Nt_1t_2t_3 \in [123]$.

This means one t_i takes [1235] to [123]

Choose 6 from {6}

$Nt_1t_2t_3t_5t_6 \in [125]$.

We have $Nt_1t_2t_3t_5t_6 = (x * y * x * y * x^{-1})t_4t_3t_9$

$= (x * y * x * y * x^{-1})(t_1t_2t_5)^{(y*x^{-1}*y*x^{-1}*y)} \in [125]$

This means one t_i takes [1235] to [125]

Choose 7 from {7}

$Nt_1t_2t_3t_5t_7 \in [1239]$.

We have $Nt_1t_2t_3t_5t_7 = (y^2 * x)t_2t_5t_9t_4$

$= (y^2 * x)(t_1t_2t_3t_9)^{(x*y*x^{-1})} \in [1239]$

This means one t_i takes [1235] to [1239]

Choose 8 from {8}

$Nt_1t_2t_3t_5t_8 \in [129]$.

We have $Nt_1t_2t_3t_5t_8 = ((y,x))t_5t_7t_4$

$= ((y,x))(t_1t_2t_9)^{(y^2*x*y*x^{-1})} \in [129]$

This means one t_i takes [1235] to [129]

Choose 9 from {9}

$Nt_1t_2t_3t_5t_9 \in [125,11]$.

We have $Nt_1t_2t_3t_5t_9 = (x)t_2t_{11}t_3t_{10}$

$= (x)(t_1t_2t_5t_{11})^{(y^2)} \in [125,11]$

This means one t_i takes [1235] to [125,11]

Choose 10 from {10}

$Nt_1t_2t_3t_5t_{10} \in [1235]$.

We have $Nt_1t_2t_3t_5t_{10} = (y^5)t_3t_2t_1t_5$

$= (y^5)(t_1t_2t_3t_5)^{(y*x*y^{-1}*x^{-1}*y)} \in [1235]$

This means one t_i takes [1235] to [1235]

Choose 11 from {11}

$Nt_1t_2t_3t_5t_{11} \in [1239]$.

We have $Nt_1t_2t_3t_5t_{11} = (x*y*x)t_{10}t_4t_3t_2$

$= (x*y*x)(t_1t_2t_3t_9)^{(y*x^{-1}*y*x)} \in [1239]$

This means one t_i takes [1235] to [1239]

Choose 12 from {12}

$Nt_1t_2t_3t_5t_{12} \in [137]$.

We have $Nt_1t_2t_3t_5t_{12} = (x*y^2)t_6t_5t_8$

$= (x*y^2)(t_1t_3t_7)^{(y^2*x*y^{-1})} \in [137]$

This means one t_i takes [1235] to [137]

Cayley Diagram
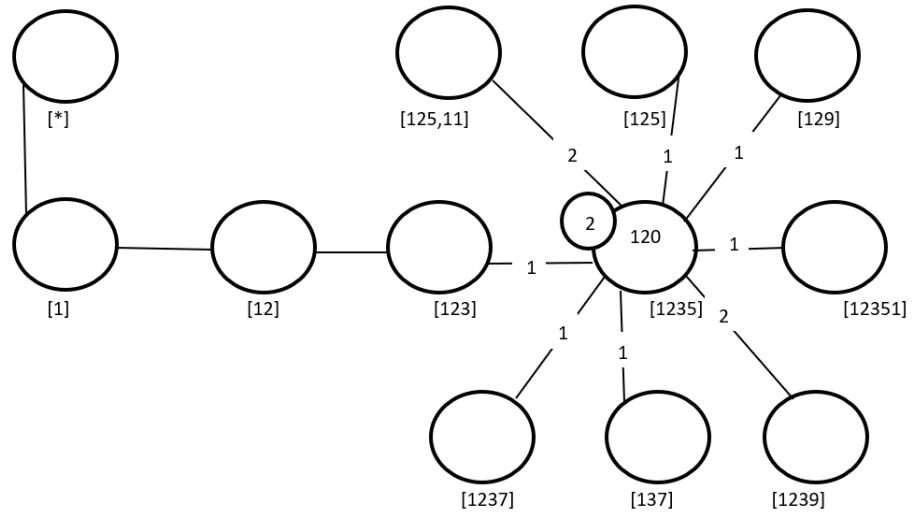
Figure 7.46: Cayley Diagram for G over S_{12}

Fourteenth Double Coset $[1237]$

$Nt_1t_2t_3t_7 = \{N(t_1t_2t_3t_7)^n | n \in N\}$.

We now find the coset stablilizer $N^{(1237)}$. We first find the point stabilizer of 1,2,3 and 7 in N. $N^{(1237)} \geq N^{1237} = \langle e \rangle$.

The number of single cosets in $Nt_1t_2t_3t_7$ is calculated by the formula,

$\frac{|N|}{|N^{(1237)}|} = \frac{120}{1} = 120$.

The orbits of $N^{(1237)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_2t_3t_7t_1 \in [1296]$.

We have $Nt_1t_2t_3t_7t_1 = (y * x)t_9t_7t_2t_{11}$

$= (y * x)(t_1t_2t_9t_6)^{(x*y^{-1}*x^{-1}*y^{-1})} \in [1296]$

This means one t_i takes $[1237]$ to $[1296]$

Choose 2 from $\{2\}$

$Nt_1t_2t_3t_7t_2 \in [129]$.

We have $Nt_1t_2t_3t_7t_2 = (y^3)t_{12}t_8t_5$

$= (y^3)(t_1t_2t_9)^{(x*y^2*x*y)} \in [129]$

This means one t_i takes [1237] to [129]

Choose 3 from $\{3\}$

$Nt_1t_2t_3t_7t_3 \in [125, 11]$.

We have $Nt_1t_2t_3t_7t_3 = (x)t_{11}t_3t_4t_5$

$= (x)(t_1t_2t_5t_{11})^{(x*y^{-1}*x)} \in [125, 11]$

This means one t_i takes [1237] to [125,11]

Choose 4 from $\{4\}$

$Nt_1t_2t_3t_7t_4 \in [125, 11]$.

We have $Nt_1t_2t_3t_7t_4 = (x * y^2 * x * y * x^{-1})t_{10}t_6t_4t_7$

$= (x * y^2 * x * y * x^{-1})(t_1t_2t_5t_{11})^{(x^{-1}*y^{-1}*x^{-1}*y)} \in [125, 11]$

This means one t_i takes [1237] to [125,11]

Choose 5 from $\{5\}$

$Nt_1t_2t_3t_7t_5 \in [125, 12]$.

We have $Nt_1t_2t_3t_7t_5 = (y * x * y * x * y)t_2t_{12}t_1t_{11}$

$= (y * x * y * x * y)(t_1t_2t_5t_{12})^{(y*x*y*x^{-1}*y^{-1})} \in [125, 12]$

This means one t_i takes [1237] to [125,12]

Choose 6 from $\{6\}$

$Nt_1t_2t_3t_7t_6 \in [1235]$.

We have $Nt_1t_2t_3t_7t_6 = ((x^{-1} * y)^2)t_5t_7t_6t_9$

$= ((x^{-1} * y)^2)(t_1t_2t_3t_5)^{(y^2*x*y*x^{-1})} \in [1235]$

This means one t_i takes [1237] to [1235]

Choose 7 from $\{7\}$

$Nt_1t_2t_3t_7t_7 = Nt_1t_2t_3t_7^2 = Nt_1t_2t_3 \in [123]$.

This means one t_i takes [1237] to [123]

Choose 8 from $\{8\}$

$Nt_1t_2t_3t_7t_8 \in [132]$.

We have $Nt_1t_2t_3t_7t_8 = (y^{-1} * x^{-1} * y^2)t_{10}t_9t_4$

$= (y^{-1} * x^{-1} * y^2)(t_1t_3t_2)^{(x*y^{-1}*x*y^2)} \in [132]$

This means one t_i takes [1237] to [132]

Choose 9 from {9}

$Nt_1t_2t_3t_7t_9 \in [125]$.

We have $Nt_1t_2t_3t_7t_9 = (x*y^2*x)t_{10}t_4t_6$

$= (x*y^2*x)(t_1t_2t_5)^{(x*y^{-1}*x*y^2)} \in [125]$

This means one t_i takes $[1237]$ to $[125]$

Choose 10 from {10}

$Nt_1t_2t_3t_7t_{10} \in [129]$.

We have $Nt_1t_2t_3t_7t_{10} = (x^{-1}*y^{-2})t_5t_7t_{12}$

$= (x^{-1}*y^{-2})(t_1t_2t_9)^{(x*y^{-1}*x^{-1}*y*x)} \in [129]$

This means one t_i takes $[1237]$ to $[129]$

Choose 11 from {11}

$Nt_1t_2t_3t_7t_{11} \in [1239]$.

We have $Nt_1t_2t_3t_7t_{11} = (y^5)t_3t_2t_1t_7$

$= (y^5)(t_1t_2t_3t_9)^{(y*x*y^{-1}*x^{-1}*y)} \in [1239]$

This means one t_i takes $[1237]$ to $[1239]$

Choose 12 from {12}

$Nt_1t_2t_3t_7t_{12} \in [1245]$.

We have $Nt_1t_2t_3t_7t_{12} = (x*y)t_{12}t_8t_9t_{10}$

$= (x*y)(t_1t_2t_4t_5)^{(y*x^{-1}*y)} \in [1245]$

This means one t_i takes $[1237]$ to $[1245]$

Cayley Diagram
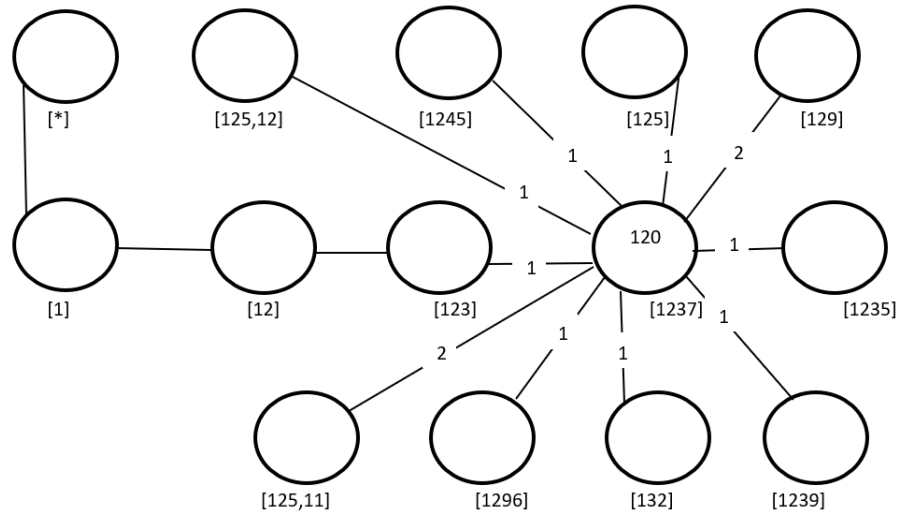
Figure 7.47: Cayley Diagram for G over S_{12}

Fifteenth Double Coset $[1239]$

$Nt_1t_2t_3t_9 = \{N(t_1t_2t_3t_9)^n | n \in N\}$.

We now find the coset stablilizer $N^{(1239)}$. We first find the point stabilizer of 1,2,3 and 9 in N. $N^{(1239)} \geq N^{1239} = \langle e \rangle$.

The number of single right cosets in $Nt_1t_2t_3t_9$ is calculated by the formula,

$\frac{|N|}{|N^{(1239)}|} = \frac{120}{1} = 120$.

The orbits of $N^{(1239)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_2t_3t_9t_1 \in [1296]$.

We have $Nt_1t_2t_3t_9t_1 = (x * y^{-1} * x * y^2)t_4t_{11}t_5t_7$

$= (x * y^{-1} * x * y^2)(t_1t_2t_9t_6)^{(y*x^{-1}*y*x*y^{-1})} \in [1296]$

This means one t_i takes $[1239]$ to $[1296]$

Choose 2 from $\{2 \}$

$Nt_1t_2t_3t_9t_2 \in [125, 12]$.

We have $Nt_1t_2t_3t_9t_2 = (y^5)t_4t_9t_3t_6$

$= (y^5)(t_1t_2t_5t_{12})^{(y*x*y^{-1}*x)} \in [125, 12]$

This means one t_i takes [1239] to [125,12]

Choose 3 from { 3}

$Nt_1t_2t_3t_9t_3 \in [1256]$.

We have $Nt_1t_2t_3t_9t_3 = (x^{-1} * y^{-1})t_8t_1t_{12}t_4$

$= (x^{-1} * y^{-1})(t_1t_2t_5t_6)^{(y*x*y*x*y)} \in [1256]$

This means one t_i takes [1239] to [1256]

Choose 4 from {4}

$Nt_1t_2t_3t_9t_4 \in [1239]$.

We have $Nt_1t_2t_3t_9t_4 = (y * x^{-1} * y)t_1t_8t_6t_9$

$= (y * x^{-1} * y)(t_1t_2t_3t_9)^{(x*y*x^{-1}*y^{-1}*x*y)} \in [1239]$

This means one t_i takes [1239] to [1239]

Choose 5 from { 5}

$Nt_1t_2t_3t_9t_5 \in [1235]$.

We have $Nt_1t_2t_3t_9t_5 = (x * y^{-1})t_6t_9t_3t_4$

$= (x * y^{-1})(t_1t_2t_3t_5)^{((x^{-1}*y)^2)} \in [1235]$

This means one t_i takes [1239] to [1235]

Choose 6 from {6}

$Nt_1t_2t_3t_9t_6 \in [125]$.

We have $Nt_1t_2t_3t_9t_6 = t_1t_{12}t_8$

$= (t_1t_2t_5)^{(x^{-1}*y^{-1}*x*y^2)} \in [125]$

This means one t_i takes [1239] to [125]

Choose 7 from {7}

$Nt_1t_2t_3t_9t_7 \in [1239]$.

We have $Nt_1t_2t_3t_9t_7 = (x * y^{-1} * x^{-1} * y^{-1} * x^{-1})t_2t_{12}t_{10}t_4$

$= (x * y^{-1} * x^{-1} * y^{-1} * x^{-1})(t_1t_2t_3t_9)^{(y^{-1}*x^{-1}*y)} \in [1239]$

This means one t_i takes [1239] to [1239]

Choose 8 from {8}

$Nt_1t_2t_3t_9t_8 \in [137]$.

We have $Nt_1t_2t_3t_9t_8 = (x * y^2 * x * y)t_3t_7t_2$

$= (x * y^2 * x * y)(t_1 t_3 t_7)^{(y*x*y*x^{-1})} \in [137]$

This means one t_i takes [1239] to [137]

Choose 9 from $\{9\}$

$Nt_1 t_2 t_3 t_9 t_9 = Nt_1 t_2 t_3 t_9^2 = Nt_1 t_2 t_3 \in [123].$

This means one t_i takes [1239] to [123]

Choose 10 from $\{10\}$

$Nt_1 t_2 t_3 t_9 t_{10} \in [1239].$

We have $Nt_1 t_2 t_3 t_9 t_{10} = (x^{-1} * y^{-2}) t_{12} t_1 t_7 t_6$

$= (x^{-1} * y^{-2})(t_1 t_2 t_3 t_9)^{(x^y)} \in [1239]$

This means one t_i takes [1239] to [1239]

Choose 11 from $\{11 \}$

$Nt_1 t_2 t_3 t_9 t_{11} \in [1235].$

We have $Nt_1 t_2 t_3 t_9 t_{11} = (x^{-1} * y^{-2}) t_{12} t_1 t_5 t_2$

$= (x^{-1} * y^{-2})(t_1 t_2 t_3 t_5)^{(x*y^{-1}*x^{-1})} \in [1235]$

This means one t_i takes [1239] to [1235]

Choose 12 from $\{ 12\}$

$Nt_1 t_2 t_3 t_9 t_{12} \in [1237].$

We have $Nt_1 t_2 t_3 t_9 t_{12} = (y^5) t_3 t_2 t_1 t_9$

$= (y^5)(t_1 t_2 t_3 t_7)^{(y*x*y^{-1}*x^{-1}*y)} \in [1237]$

This means one t_i takes [1239] to [1237]

Cayley Diagram
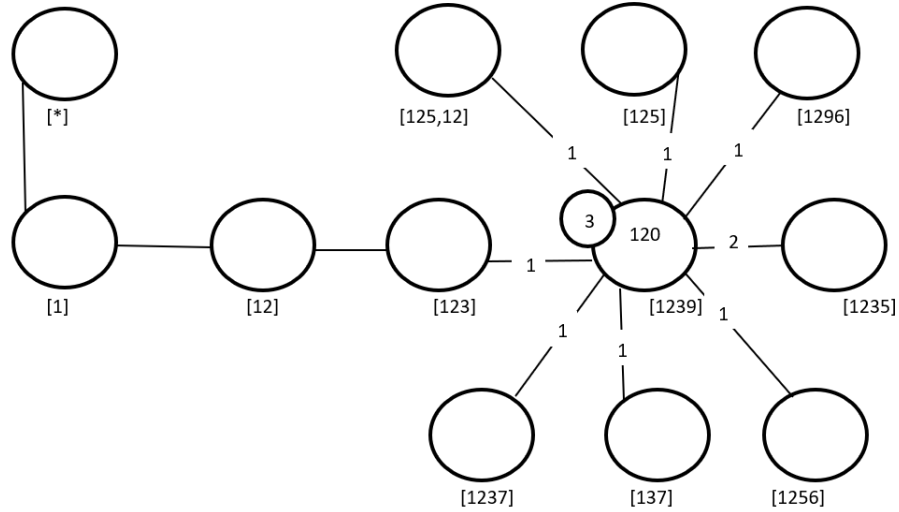
Figure 7.48: Cayley Diagram for G over S_{12}

Sixteenth Double Coset $[1, 2, 5, 1]$

$Nt_1t_2t_5t_1 = \{N(t_1t_2t_5t_1)^n | n \in N\}$.

We now find the coset stablilizer $N^{(1251)}$. We first find the point stabilizer of 1,2,5 and 1 in N.

$N^{1251} = \{n \in N | (1251)^n = 1251\}$

$N^{1251} = \langle e \rangle$.

We find different names of $Nt_1t_2t_5t_1$. Now, $t_1t_2t_5t_1 = (x*y^-1*x^-1*y^-1*x^-1)t_5t_7t_1t_5 \implies$

$Nt_1t_2t_5t_1 = Nt_5t_7t_1t_5$

$N(t_1t_2t_5t_1)^{(1,5)(2,7)(3,8)(4,10)(6,11)(9,12)} = Nt_5t_7t_1t_5$.

But $Nt_5t_7t_1t_5 = Nt_1t_2t_5t_1 \implies N(t_1t_2t_5t_1)^{(1,5)(2,7)(3,8)(4,10)(6,11)(9,12)} = Nt_1t_2t_5t_1$.

$\implies (1,5)(2,7)(3,8)(4,10)(6,11)(9,12) \in N^{(1251)}$.

Thus $N^{(1251)} \geq \langle N^{1251}, (1,5)(2,7)(3,8)(4,10)(6,11)(9,12) \rangle$.

$= \langle e, (1,5)(2,7)(3,8)(4,10)(6,11)(9,12) \rangle$.

The number of single cosets in $Nt_1t_2t_5t_1$ is calculated by this formula,

$\frac{|N|}{|N^{(1251)}|} = \frac{120}{6} = 20$.

The orbits of $N^{(1251)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1, 8, 3, 5, 4, 10\}, \{2, 6, 11, 7, 9, 12\}$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1,8,3,5,4,10\}$

$Nt_1t_2t_5t_1t_1 = Nt_1t_2t_5t_1^2 = Nt_1t_2t_5 \in [125]$.

This means six t_i's take [1251] to [125]

Choose 2 from $\{2,6,11,7,9,12\}$

$Nt_1t_2t_5t_1t_2 \in [125, 11]$.

We have $Nt_1t_2t_5t_1t_2 = (x*y*x)t_8t_{12}t_1t_{11}$

$= (x*y*x)(t_1t_2t_5t_{11})^{(y^2*x)} \in [125, 11]$

This means six t_i's take [1251] to [125,11]

Cayley Diagram
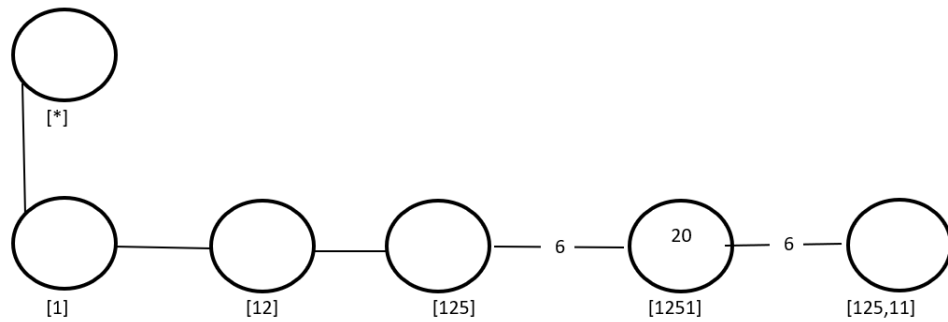
Figure 7.49: Cayley Diagram for G over S_{12}

Seventeenth Double Coset [1256]

$Nt_1t_2t_5t_6 = \{N(t_1t_2t_5t_6)^n | n \in N\}$.

We now find the coset stablilizer $N^{(1256)}$. We first find the point stabilizer of 1,2,5 and 6 in N.

$N^{1256} = \{n \in N | (1256)^n = 1256\}$

$N^{1256} = \langle e \rangle$.

We find different names of $Nt_1t_2t_5t_6$. Now, $t_1t_2t_5t_6 = (x*y*x^-1*y*x^-1*y)t_8t_6t_{10}t_2 \implies$

$Nt_1t_2t_5t_6 = Nt_8t_6t_{10}t_2$

$N(t_1t_2t_5t_6)^{(1,8)(2,6)(3,4)(5,10)(7,12)(9,11)} = Nt_8t_6t_{10}t_2$.

But $Nt_8t_6t_{10}t_2 = Nt_1t_2t_5t_6 \implies N(t_1t_2t_5t_6)^{(1,8)(2,6)(3,4)(5,10)(7,12)(9,11)} = Nt_1t_2t_5t_6.$

$\implies (1,8)(2,6)(3,4)(5,10)(7,12)(9,11) \in N^{(1256)}.$

Thus $N^{(1256)} \geq \langle N^{1256}, (1,8)(2,6)(3,4)(5,10)(7,12)(9,11)\rangle.$

$= \langle e, (1,8)(2,6)(3,4)(5,10)(7,12)(9,11)\rangle.$

The number of single cosets in $Nt_1t_2t_5t_6$ is calculated by this formula,

$\frac{|N|}{|N^{(1256)}|} = \frac{120}{2} = 60.$

The orbits of $N^{(1256)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1,8\},\{2,6\},\{3,4\},\{5,10\},\{7,12\},\{9,11\}$

We will determine the double cosets by selecting one representative from each orbit such

as,

<u>Choose 1 from {1,8}</u>

$Nt_1t_2t_5t_6t_1 \in [137].$

We have $Nt_1t_2t_5t_6t_1 = (y * x * y^2 * x)t_1t_9t_8$

$= (y * x * y^2 * x)(t_1t_3t_7)^{(y*x*y)}$

This means two t_i's take $[1256]$ to $[137]$

<u>Choose 2 from {2,6}</u>

$Nt_1t_2t_5t_6t_2 \in [125].$

We have $Nt_1t_2t_5t_6t_2 = (x * y * x^{-1} * y * x^{-1} * y)t_8t_6t_{10}$

$= (x * y * x^{-1} * y * x^{-1} * y)(t_1t_2t_5)^{(y*x*y^2*x*y^{-1})} \in [125]$

This means two t_i's take $[1256]$ to $[125]$

<u>Choose 3 from {3,4}</u>

$Nt_1t_2t_5t_6t_3 \in [1239].$

We have $Nt_1t_2t_5t_6t_3 = (x * y^{-1} * x^{-1} * y^{-1} * x)t_6t_4t_3t_5$

$= (x * y^{-1} * x^{-1} * y^{-1} * x)(t_1t_2t_3t_9)^{(y*x^{-1}*y^{-1}*x*y)} \in [1239]$

This means two t_i's take $[1256]$ to $[1239]$

<u>Choose 5 from {5,10}</u>

$Nt_1t_2t_5t_6t_5 \in [129].$

We have $Nt_1t_2t_5t_6t_5 = (x * y * x^{-1} * y)t_{12}t_1t_3$

$= (x * y * x^{-1} * y)(t_1t_2t_9)^{(x*y^{-1}*x^{-1})} \in [129]$

This means two t_i's take $[1256]$ to $[129]$

<u>Choose 7 from {7,12}</u>

$Nt_1t_2t_5t_6t_7 \in [1256].$

We have $Nt_1t_2t_5t_6t_7 = (x*y*x^{-1}*y^{-2})t_1t_5t_2t_{10} = (x*y*x^{-1}*y^{-2})(t_1t_2t_5t_6)^{(x*y^3)} \in [1256]$

This means two t_i's take [1256] to [1256]

Choose 9 from {9,11}

$Nt_1t_2t_5t_6t_9 \in [132]$.

We have $Nt_1t_2t_5t_6t_9 = (y^{-2}*x^{-1})t_5t_6t_7$

$= (y^{-2}*x^{-1})(t_1t_3t_2)^{y^2*x*y*x^{-1}} \in [132]$

This means two t_i's take [1256] to [132]

Cayley Diagram
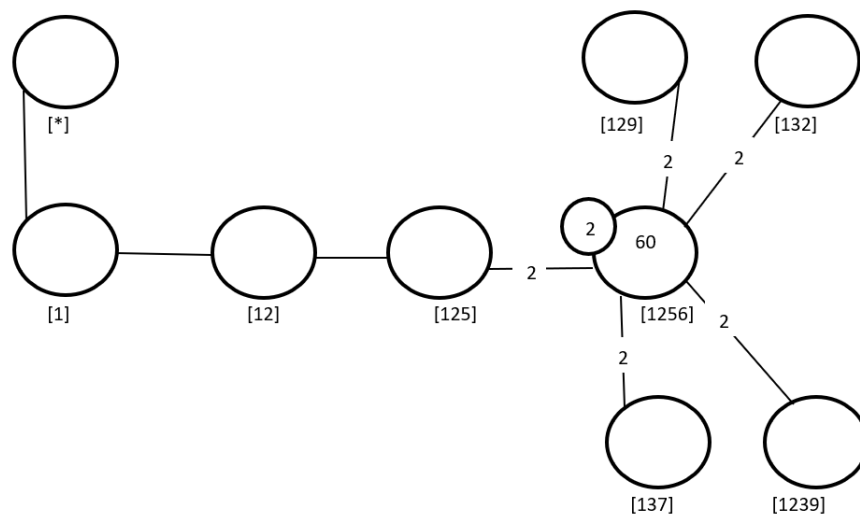
Figure 7.50: Cayley Diagram for G over S_{12}

Eighteenth Double Coset $[1, 2, 5, 11]$

$Nt_1t_2t_5t_{11} = \{N(t_1t_2t_5t_{11})^n | n \in N\}$.

We now find the coset stablilizer $N^{(125,11)}$. We first find the point stabilizer of 1,2,5 and 11 in N. $N^{(125,11)} \geq N^{125,11} = \langle e \rangle$.

The number of single right cosets in $Nt_1t_2t_5t_{11}$ is calculated by the formula,

$\frac{|N|}{|N^{(12511)}|} = \frac{120}{1} = 120$.

The orbits of $N^{(12511)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_2t_5t_{11}t_1 \in [137]$.

We have $Nt_1t_2t_5t_{11}t_1 = (y^{-1} * x * y^2)t_3t_{12}t_5$

$= (y^{-1} * x * y^2)(t_1t_3t_7)^{(y^{-2}*x^{-1}*y)} \in [137]$

This means one t_i takes [125,11] to [137]

Choose 2 from $\{\ 2\ \}$

$Nt_1t_2t_5t_{11}t_2 \in [1237]$.

We have $Nt_1t_2t_5t_{11}t_2 = ((x,y))t_{10}t_{12}t_2t_4$

$= ((x,y))(t_1t_2t_3t_7)^{(x^{-1}*y*x^{-1})} \in [1237]$

This means one t_i takes [125,11] to [1237]

Choose 3 from $\{3\}$

$Nt_1t_2t_5t_{11}t_3 \in [1251]$.

We have $Nt_1t_2t_5t_{11}t_3 = (y * x * y^{-2})t_5t_3t_9t_5$

$= (y * x * y^{-2})(t_1t_2t_5t_1)^{(x^{-1}*y^{-2})} \in [1251]$

This means one t_i takes [125,11] to [1251]

Choose 4 from $\{4\ \}$

$Nt_1t_2t_5t_{11}t_4 \in [127]$.

We have $Nt_1t_2t_5t_{11}t_4 = (y^{-1})t_{12}t_{10}t_2$

$= (y^{-1})(t_1t_2t_7)^{(y*x^{-1}*y^{-1})} \in [127]$

This means one t_i takes [125,11] to [127]

Choose 5 from $\{5\}$

$Nt_1t_2t_5t_{11}t_5 \in [1237]$.

We have $Nt_1t_2t_5t_{11}t_5 = (x * y^{-1} * x^{-1} * y^2)t_{10}t_6t_9t_{11}$

$= (x * y^{-1} * x^{-1} * y^2)(t_1t_2t_3t_7)^{(x^{-1}*y^{-1}*x^{-1}*y)} \in [1237]$

This means one t_i takes [125,11] to [1237]

Choose 6 from $\{\ 6\ \}$

$Nt_1t_2t_5t_{11}t_6 \in [132]$.

We have $Nt_1t_2t_5t_{11}t_6 = (x^{-1} * y^{-2})t_{11}t_8t_{10}$

$= (x^{-1} * y^{-2})(t_1t_3t_2)^{(x^{-1}*y)} \in [132]$

This means one t_i takes [125,11] to [132]

Choose 7 from $\{7\}$

$Nt_1t_2t_5t_{11}t_7 \in [1235]$.

We have $Nt_1t_2t_5t_{11}t_7 = (y * x^{-1})t_1t_7t_6t_8$

$= (y * x^{-1})(t_1t_2t_3t_5)^{(x^{-1}*y*x^{-1}*y*x^{-1})} \in [1235]$

This means one t_i takes [125,11] to [1235]

<u>Choose 8 from {8}</u>

$Nt_1t_2t_5t_{11}t_8 \in [125, 11]$.

We have $Nt_1t_2t_5t_{11}t_8 = (y^{-1} * x)t_{10}t_{11}t_4t_2$

$= (y^{-1} * x)(t_1t_2t_5t_{11})^{(x*y^2*x*y^{-1}*x)} \in [125, 11]$

This means one t_i takes [125,11] to [125,11]

<u>Choose 9 from {9}</u>

$Nt_1t_2t_5t_{11}t_9 \in [1235]$.

We have $Nt_1t_2t_5t_{11}t_9 = ((y^{-1} * x^{-1})^2)t_8t_1t_5t_7$

$= ((y^{-1} * x^{-1})^2)(t_1t_2t_3t_5)^{(y^{-2})} \in [1235]$

This means one t_i takes [125,11] to [1235]

<u>Choose 10 from {10}</u>

$Nt_1t_2t_5t_{11}t_{10} \in [1296]$.

We have $Nt_1t_2t_5t_{11}t_{10} = (x * y * x^{-1} * y * x^{-1} * y)t_8t_7t_2t_{11}$

$= (x * y * x^{-1} * y * x^{-1} * y)(t_1t_2t_9t_6)^{(x*y)} \in [1296]$

This means one t_i takes [125,11] to [1296]

<u>Choose 11 from {11}</u>

$Nt_1t_2t_5t_{11}t_{11} = Nt_1t_2t_5t_{11}^2 = Nt_1t_2t_5 \in [125]$.

This means one t_i takes [125,11] to [125]

<u>Choose 12 from {12}</u>

$Nt_1t_2t_5t_{11}t_{12} \in [137]$.

We have $Nt_1t_2t_5t_{11}t_{12} = (y^{-3})t_1t_{11}t_5$

$= (y^{-3})(t_1t_3t_7)^{(y^{-1}*x^{-1}*y^{-1})} \in [137]$

This means one t_i takes [125,11] to [137]

Cayley Diagram
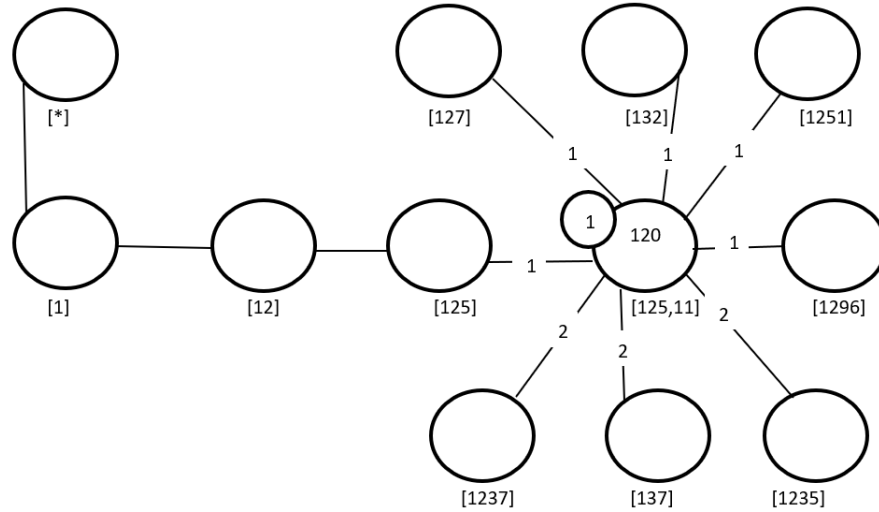
Figure 7.51: Cayley Diagram for G over S_{12}

Nineteenth Double Coset $[1, 2, 5, 12]$

$Nt_1t_2t_5t_{12} = \{N(t_1t_2t_5t_{12})^n | n \in N\}$.

We now find the coset stablilizer $N^{(125,12)}$. We first find the point stabilizer of 1,2,5 and 12 in N.

$N^{125,12} = \{n \in N | (125, 12)^n = 125, 12\}$

$N^{125,12} = \langle e \rangle$.

We find different names of $Nt_1t_2t_5t_{12}$. Now, $t_1t_2t_5t_{12} = (x*y^-3)t_1t_7t_5t_8 \implies Nt_1t_2t_5t_{12} = Nt_1t_7t_5t_8$

$N(t_1t_2t_5t_{12})^{(2,7)(3,9)(6,11)(8,12)} = Nt_1t_7t_5t_8$.

But $Nt_1t_7t_5t_8 = Nt_1t_2t_5t_{12} \implies N(t_1t_2t_5t_{12})^{(2,7)(3,9)(6,11)(8,12)} = Nt_1t_2t_5t_{12}$.

$\implies (2, 7)(3, 9)(6, 11)(8, 12) \in N^{(125,12)}$.

Thus $N^{(125,12)} \geq \langle N^{125,12}, (2, 7)(3, 9)(6, 11)(8, 12) \rangle$.

$= \langle e, (2, 7)(3, 9)(6, 11)(8, 12) \rangle$.

The number of single right cosets in $Nt_1t_2t_5t_{12}$ is calculated by the formula,

$\frac{|N|}{|N^{(12512)}|} = \frac{120}{2} = 60$.

The orbits of $N^{(12512)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from {1}

$Nt_1t_2t_5t_{12}t_1 \in [127]$.

We have $Nt_1t_2t_5t_{12}t_1 = (y * x * y * x * y)t_{11}t_{10}t_2$

$= (y * x * y * x * y)(t_1t_2t_7)^{(x^{-1}*y)}$

This means one t_i takes [125,12] to [127]

Choose 2 from {2}

$Nt_1t_2t_5t_{12}t_2 \in [1237]$.

We have $Nt_1t_2t_5t_{12}t_2 = ((x^{-1} * y)^2)t_5t_1t_{12}t_3$

$= ((x^{-1} * y)^2)(t_1t_2t_3t_7)^{(x^{-1},y^{-1})} \in [1237]$

This means one t_i takes [125,12] to [1237]

Choose 3 from {3}

$Nt_1t_2t_5t_{12}t_3 \in [1239]$.

We have $Nt_1t_2t_5t_{12}t_3 = (x * y^2)t_4t_3t_5t_7$

$= (x * y^2)(t_1t_2t_3t_9)^{(y*x^{-1}*y*x^{-1}*y)} \in [1239]$

This means one t_i takes [125,12] to [1239]

Choose 4 from {4}

$Nt_1t_2t_5t_{12}t_4 \in [1245]$.

We have $Nt_1t_2t_5t_{12}t_4 = (x * y^2 * x * y)t_3t_9t_8t_5$

$= (x * y^2 * x * y)(t_1t_2t_4t_5)^{(y*x*y*x^{-1})} \in [1245]$

This means one t_i takes [125,12] to [1245]

Choose 5 from {5}

$Nt_1t_2t_5t_{12}t_5 \in [125, 12]$.

We have $Nt_1t_2t_5t_{12}t_5 = (y^3 * x)t_1t_2t_5t_{12} \in [125, 12]$

This means one t_i takes [125,12] to [125,12]

Choose 6 from {6}

$Nt_1t_2t_5t_{12}t_6 \in [129]$.

We have $Nt_1t_2t_5t_{12}t_6 = (x^{-1} * y)t_6t_8t_{11}$

$= (x^{-1} * y)(t_1t_2t_9)^{(y*x*y^{-2})} \in [129]$

This means one t_i takes [125,12] to [129]

Choose 7 from {7}

$Nt_1t_2t_5t_{12}t_7 \in [1237]$.

We have $Nt_1t_2t_5t_{12}t_7 = (x*y)(t_5t_1t_8t_9)$

$= (x*y)*(t_1t_2t_3t_7)^{(y*x*y^{-1}*x^{-1}*y^{-1})} \in [1237]$

This means one t_i takes [125,12] to [1237]

Choose 8 from {8}

$Nt_1t_2t_5t_{12}t_8 \in [125]$.

We have $Nt_1t_2t_5t_{12}t_8 = (x*y^{-3})t_1t_7t_5$

$= (x*y^{-3})(t_1t_2t_5)^{(x*y*x^{-1}*y^{-2})} \in [125]$

This means one t_i takes [125,12] to [125]

Choose 9 from {9}

$Nt_1t_2t_5t_{12}t_9 \in [1239]$.

We have $Nt_1t_2t_5t_{12}t_9 = y^5(t_4t_9t_5t_2)$

$= y^5(t_1t_2t_3t_9)^{(y*x*y^{-1}*x)} \in [1239]$

This means one t_i takes [125,12] to [1239]

Choose 10 from {10}

$Nt_1t_2t_5t_{12}t_{10} \in [134]$.

We have $Nt_1t_2t_5t_{12}t_{10} = (y^2)t_3t_{12}t_8$

$= (y^2)(t_1t_3t_4)^{(x*y*x^{-1}*y)} \in [134]$

This means one t_i takes [125,12] to [134]

Choose 11 from {11}

$Nt_1t_2t_5t_{12}t_{11} \in [129]$.

We have $Nt_1t_2t_5t_{12}t_{11} = (x*y^4)(t_{11}t_{12}t_6)$

$= (x*y^4)*(t_1t_2t_9)^{(x*y*x*y^{-1})} \in [129]$

This means one t_i takes [125,12] to [129]

Choose 12 from {12}

$Nt_1t_2t_5t_{12}t_{12} = Nt_1t_2t_5t_{12}^2 = Nt_1t_2t_5 \in [125]$

This means one t_i takes [125,12] to [125]

Cayley Diagram

Figure 7.52: Cayley Diagram for G over S_{12}

Twenty Double Coset [1245]

$Nt_1t_2t_4t_5 = \{N(t_1t_2t_4t_5)^n | n \in N\}.$

We now find the coset stablilizer $N^{(1245)}$. We first find the point stabilizer of 1,2,4 and 5 in N.

$N^{1245} = \{n \in N | (1245)^n = 1245\}$

$N^{1245} = \langle e \rangle.$

We find different names of $Nt_1t_2t_4t_5$. Now, $t_1t_2t_4t_5 = (y^5)t_2t_1t_6t_5 \implies Nt_1t_2t_4t_5 = Nt_2t_1t_6t_5$

$N(t_1t_2t_4t_5)^{(1,2)(3,7)(4,6)(8,11)} = Nt_2t_1t_6t_5.$

But $Nt_2t_1t_6t_5 = Nt_1t_2t_4t_5 \implies N(t_1t_2t_4t_5)^{(1,2)(3,7)(4,6)(8,11)} = Nt_1t_2t_4t_5.$

$\implies (1,2)(3,7)(4,6)(8,11) \in N^{(1245)}.$

Thus $N^{(1245)} \geq \langle N^{1245}, (1,2)(3,7)(4,6)(8,11) \rangle.$

$= \langle e, (1,2)(3,7)(4,6)(8,11) \rangle.$

The number of single cosets in $Nt_1t_2t_4t_5$ is calculated by the formula,

$\frac{|N|}{|N^{(1245)}|} = \frac{120}{2} = 60.$

The orbits of $N^{(1245)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_2t_4t_5t_1 \in [1237]$.

We have $Nt_1t_2t_4t_5t_1 = (y*x*y*x^{-1}*y)t_{12}t_8t_6t_{11}$

$= (y*x*y*x^{-1}*y)(t_1t_2t_3t_7)^{(y*x^{-1}*y)} \in [1237]$

This means one t_i takes [1245] to [1237]

Choose 2 from $\{2\}$

$Nt_1t_2t_4t_5t_2 \in [1237]$.

We have $Nt_1t_2t_4t_5t_2 = (x, y^{-1})(t_{12}t_{11}t_4t_8)$

$= (x, y^{-1}) * (t_1t_2t_3t_7)^{(y^x)} \in [1237]$

This means one t_i takes [1245] to [1237]

Choose 3 from $\{3\}$

$Nt_1t_2t_4t_5t_3 \in [12351]$.

We have $Nt_1t_2t_4t_5t_3 = (x*y^3)t_{12}t_2t_3t_{11}t_{12}$

$= (x*y^3)(t_1t_2t_3t_5t_1)^{(x*y^{-1}*x^{-1}*y^2)} \in [12351]$

This means one t_i takes [1245] to [12351]

Choose 4 from $\{4\}$

$Nt_1t_2t_4t_5t_4 \in [125]$.

We have $Nt_1t_2t_4t_5t_4 = (x^{-1}, y^{-1})t_{10}t_{12}t_8$

$= (x^{-1}, y^{-1})(t_1t_2t_5)^{((x*y)^2)} \in [125]$

This means one t_i takes [1245] to [125]

Choose 5 from $\{5\}$

$Nt_1t_2t_4t_5t_5 = Nt_1t_2t_4t_5^2 = Nt_1t_2t_4 \in [124]$.

This means one t_i takes [1245] to [124]

Choose 6 from $\{6\}$

$Nt_1t_2t_4t_5t_6 \in [125]$.

We have $Nt_1t_2t_4t_5t_6 = (y*x^{-1}*y*x*y)(t_{10}t_{12}t_{11})$

$= (y*x^{-1}*y*x*y) * (t_1t_2t_5)^{(x^{-1}*y*x^{-1})} \in [125]$

This means one t_i takes [1245] to [125]

Choose 7 from $\{7\}$

$Nt_1t_2t_4t_5t_7 \in [12351]$.

We have $Nt_1t_2t_4t_5t_7 = (x * y^4)t_1t_2t_3t_5t_1 \in [12351]$

This means one t_i takes $[1245]$ to $[12351]$

Choose 8 from $\{8\}$

$Nt_1t_2t_4t_5t_8 \in [1245]$.

We have $Nt_1t_2t_4t_5t_8 = (x^{-1} * y)t_{12}t_2t_9t_{11}$

$= (x^{-1} * y)(t_1t_2t_4t_5)^{(x*y^{-1}*x^{-1}*y^2)} \in [1245]$

This means one t_i takes $[1245]$ to $[1245]$

Choose 9 from $\{9\}$

$Nt_1t_2t_4t_5t_9 \in [127]$.

We have $Nt_1t_2t_4t_5t_9 = (y^{-1} * x^{-1} * y)t_7t_5t_8$

$= (y^{-1} * x^{-1} * y)(t_1t_2t_7)^{(y^{-1})} \in [127]$

This means one t_i takes $[1245]$ to $[127]$

Choose 10 from $\{10\}$

$Nt_1t_2t_4t_5t_{10} \in [1245]$.

We have $Nt_1t_2t_4t_5t_{10} = (y^2 * x * y * x^{-1} * y)t_1t_2t_4t_5 \in [1245]$

This means one t_i takes $[1245]$ to $[1245]$

Choose 11 from $\{11\}$

$Nt_1t_2t_4t_5t_{11} \in [1245]$.

We have $Nt_1t_2t_4t_5t_{11} = (x^{-1} * y^{-2})(t_{12}t_1t_9t_8)$

$= (x^{-1} * y^{-2}) * (t_1t_2t_4t_5)^{(x^y)} \in [1245]$

This means one t_i takes $[1245]$ to $[1245]$

Choose 12 from $\{12\}$

$Nt_1t_2t_4t_5t_{12} \in [125, 12]$.

We have $Nt_1t_2t_4t_5t_{12} = (y * x * y * x^{-1} * y)t_9t_7t_5t_6$

$= (y * x * y * x^{-1} * y)(t_1t_2t_5t_{12})^{(x*y^{-1}*x^{-1}*y^{-1})} \in [125, 12]$

This means one t_i takes $[1245]$ to $[125,12]$

Cayley Diagram
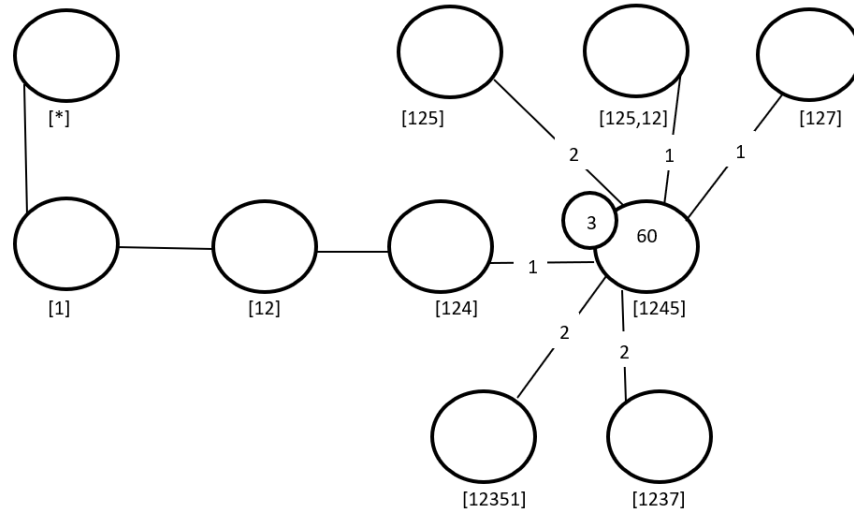
Figure 7.53: Cayley Diagram for G over S_{12}

Twenty-one Double Coset $[1, 2, 9, 6]$

$Nt_1t_2t_9t_6 = \{N(t_1t_2t_9t_6)^n | n \in N\}$.

We now find the coset stablilizer $N^{(1296)}$. We first find the point stabilizer of 1,2,9 and 6 in N.

$N^{1296} = \{n \in N | (1296)^n = 1296\}$

$N^{1296} = \langle e \rangle$.

We find different names of $Nt_1t_2t_9t_6$. Now, $t_1t_2t_9t_6 = (y^5)t_4t_9t_2t_{12} \implies Nt_1t_2t_9t_6 = Nt_4t_9t_2t_{12}$

$N(t_1t_2t_9t_6)^{(1,4)(2,9)(3,5)(6,12)(7,11)(8,10)} = Nt_4t_9t_2t_{12}$.

But $Nt_4t_9t_2t_{12} = Nt_1t_2t_9t_6 \implies N(t_1t_2t_9t_6)^{(1,4)(2,9)(3,5)(6,12)(7,11)(8,10)} = Nt_1t_2t_9t_6$.

$\implies (1,4)(2,9)(3,5)(6,12)(7,11)(8,10) \in N^{(1296)}$.

Thus $N^{(1296)} \geq \langle N^{1296}, (1,4)(2,9)(3,5)(6,12)(7,11)(8,10) \rangle$.

$= \langle e, (1,4)(2,9)(3,5)(6,12)(7,11)(8,10) \rangle$.

The number of single cosets in $Nt_1t_2t_9t_6$ is calculated by the formula,

$\frac{|N|}{|N^{(1296)}|} = \frac{120}{2} = 60$.

The orbits of $N^{(1296)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_2t_9t_6t_1 \in [1239]$.

We have $Nt_1t_2t_9t_6t_1 = (x^{-1}, y^{-1})t_1t_5t_3t_{11}$

$= (x^{-1}, y^{-1})(t_1t_2t_3t_9)^{(x*y^3)} \in [1239]$

This means one t_i takes $[1296]$ to $[1239]$

Choose 2 from $\{2\}$

$Nt_1t_2t_9t_6t_2 \in [137]$.

We have $Nt_1t_2t_9t_6t_2 = (x*y*x*y^{-1})t_{11}t_6t_{12}$

$= (x*y*x*y^{-1})(t_1t_3t_7)^{(x*y^{-3})} \in [137]$

This means one t_i takes $[1296]$ to $[137]$

Choose 3 from $\{3\}$

$Nt_1t_2t_9t_6t_3 \in [1237]$.

We have $Nt_1t_2t_9t_6t_3 = (x*y^{-1}*x)t_3t_9t_7t_2$

$= (x*y^{-1}*x)*(t_1t_2t_3t_7)^{(y*x*y*x^{-1})} \in [1237]$

This means one t_i takes $[1296]$ to $[1237]$

Choose 4 from $\{4\}$

$Nt_1t_2t_9t_6t_4 \in [1239]$.

We have $Nt_1t_2t_9t_6t_4 = (y^{-2}*x^{-1}*y)(t_4t_3t_5t_7)$

$= (y^{-2}*x^{-1}*y)*(t_1t_2t_3t_9)^{(y*x^{-1}*y*x^{-1}*y)} \in [1239]$

This means one t_i takes $[1296]$ to $[1239]$

Choose 5 from $\{5\}$

$Nt_1t_2t_9t_6t_5 \in [1237]$.

We have $Nt_1t_2t_9t_6t_5 = (x, y)(t_5t_2t_{11}t_9)$

$= (x, y)*(t_1t_2t_3t_7)^{(x*y^4)} \in [1237]$

This means one t_i takes $[1296]$ to $[1237]$

Choose 6 from $\{6\}$

$Nt_1t_2t_9t_6t_6 = Nt_1t_2t_9t_6^2 = Nt_1t_2t_9 \in [129]$.

This means one t_i takes $[1296]$ to $[129]$

Choose 7 from $\{7\}$

$Nt_1t_2t_9t_6t_7 \in [137]$.

We have $Nt_1t_2t_9t_6t_7 = (x*y^4)t_9t_8t_5$

$= (x*y^4)*(t_1t_3t_7)^{(x*y^2*x)} \in [137]$

This means one t_i takes [1296] to [137]

Choose 8 from {8}

$Nt_1t_2t_9t_6t_8 \in [125, 11]$.

We have $Nt_1t_2t_9t_6t_8 = (y*x*y^2)t_5t_9t_3t_6$

$= (y*x*y^2)*(t_1t_2t_5t_{11})^{(y^{-1}*x^{-1})} \in [125, 11]$

This means one t_i takes [1296] to [125,11]

Choose 9 from {9}

$Nt_1t_2t_9t_6t_9 \in [137]$.

We have $Nt_1t_2t_9t_6t_9 = (y^2*x*y)(t_7t_{12}t_6)$

$= (y^2*x*y)*(t_1t_3t_7)^{(y*x)} \in [137]$

This means one t_i takes [1296] to [137]

Choose 10 from {10}

$Nt_1t_2t_9t_6t_{10} \in [125, 11]$.

We have $Nt_1t_2t_9t_6t_{10} = (x^{-1}*y*x^{-1}*y*x)(t_3t_2t_5t_{12})$

$= (x^{-1}*y*x^{-1}*y*x)*(t_1t_2t_5t_{11})^{(x^{-1}*y*x^{-1}*y*x)} \in [125, 11]$

This means one t_i takes [1296] to [125,11]

Choose 11 from {11}

$Nt_1t_2t_9t_6t_{11} \in [137]$.

We have $Nt_1t_2t_9t_6t_{11} = y^3(t_2t_{10}t_3)$

$= y^3(t_1t_3t_7)^{(y^{-1}*x^{-1}*y)} \in [137]$

This means one t_i takes [1296] to [137]

Choose 12 from {12}

$Nt_1t_2t_9t_6t_{12} \in [129]$.

We have $Nt_1t_2t_9t_6t_{12} = y^5(t_4t_9t_2)$

$= y^5*(t_1t_2t_9)^{(y*x*y^{-1}*x)} \in [129]$

This means one t_i takes [1296] to [129]

Cayley Diagram
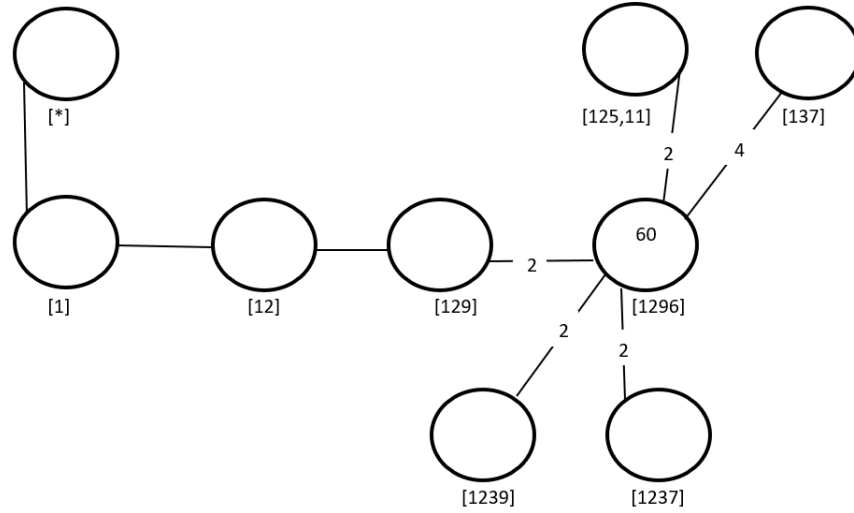
Figure 7.54: Cayley Diagram for G over S_{12}

Twenty-two Double Coset $[1, 2, 3, 5, 1]$

$Nt_1t_2t_3t_5t_1 = \{N(t_1t_2t_3t_5t_1)^n | n \in N\}$.

We now find the coset stablilizer $N^{(12351)}$. We first find the point stabilizer of 1,2,3,5 and 1 in N.

$N^{12351} = \{n \in N | (12351)^n = 12351\}$

$N^{12351} = \langle e \rangle$.

We find different names of $Nt_1t_2t_3t_5t_1$. Now, $t_1t_2t_3t_5t_1 = (x^-1*y*x^-1*y*x)t_2t_{12}t_{10}t_{11}t_2 \implies$

$Nt_1t_2t_3t_5t_1 = Nt_2t_{12}t_{10}t_{11}t_2$

$N(t_1t_2t_3t_5t_1)^{(1,2,12)(3,10,7)(4,6,9)(5,11,8)} = Nt_2t_{12}t_{10}t_{11}t_2$.

But $Nt_2t_{12}t_{10}t_{11}t_2 = Nt_1t_2t_3t_5t_1 \implies N(t_1t_2t_3t_5t_1)^{(1,2,12)(3,10,7)(4,6,9)(5,11,8)} = Nt_1t_2t_3t_5t_1$.

$\implies (1, 2, 12)(3, 10, 7)(4, 6, 9)(5, 11, 8) \in N^{(12351)}$.

Thus $N^{(12351)} \geq \langle N^{12351}, (1, 2, 12)(3, 10, 7)(4, 6, 9)(5, 11, 8) \rangle$.

$= \langle e, (1, 2, 12)(3, 10, 7)(4, 6, 9)(5, 11, 8) \rangle$.

The number of single right cosets in $Nt_1t_2t_3t_5t_1$ is calculated by the formula,

$\frac{|N|}{|N^{(12351)}|} = \frac{120}{6} = 20$.

The orbits of $N^{(12351)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are

$\{1, 12, 6, 2, 9, 4\}, \{3, 7, 11, 10, 8, 5\}$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1,12,6,2,9,4\}$

$Nt_1t_2t_3t_5t_1t_1 = Nt_1t_2t_3t_5t_1^2 = Nt_1t_2t_3t_5 \in [1235]$.

This means six t_i's take $[12351]$ to $[1235]$

Choose 3 from $\{3,7,11,10,8,5\}$

$Nt_1t_2t_3t_5t_1t_3 \in [1245]$.

We have $Nt_1t_2t_3t_5t_1t_3 = (x * y^2 * x * y^{-1} * x)t_{12}t_2t_9t_{11}$

$= (x * y^2 * x * y^{-1} * x) * (t_1t_2t_4t_5)^{(x*y^{-1}*x^{-1}*y^2)} \in [1245]$

This means six t_i's take $[12351]$ to $[1245]$

Cayley Diagram
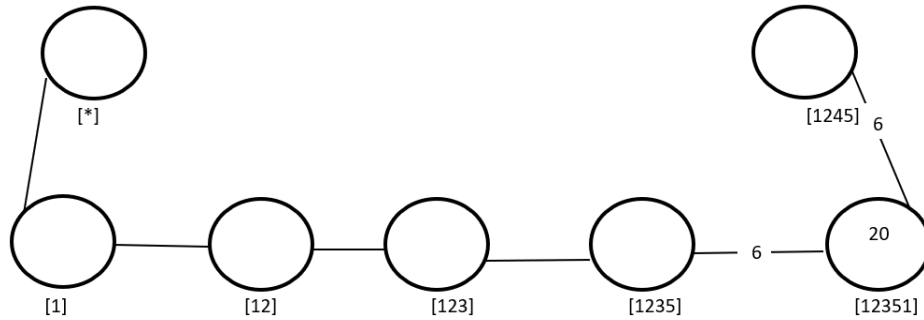
Figure 7.55: Cayley Diagram for G over S_{12}

Now, we have Completed the Double Coset Enumeration since all the single right Cosets go back and there is no new Cosets. This is the whole picture in Figure 4.44.

Cayley Diagram
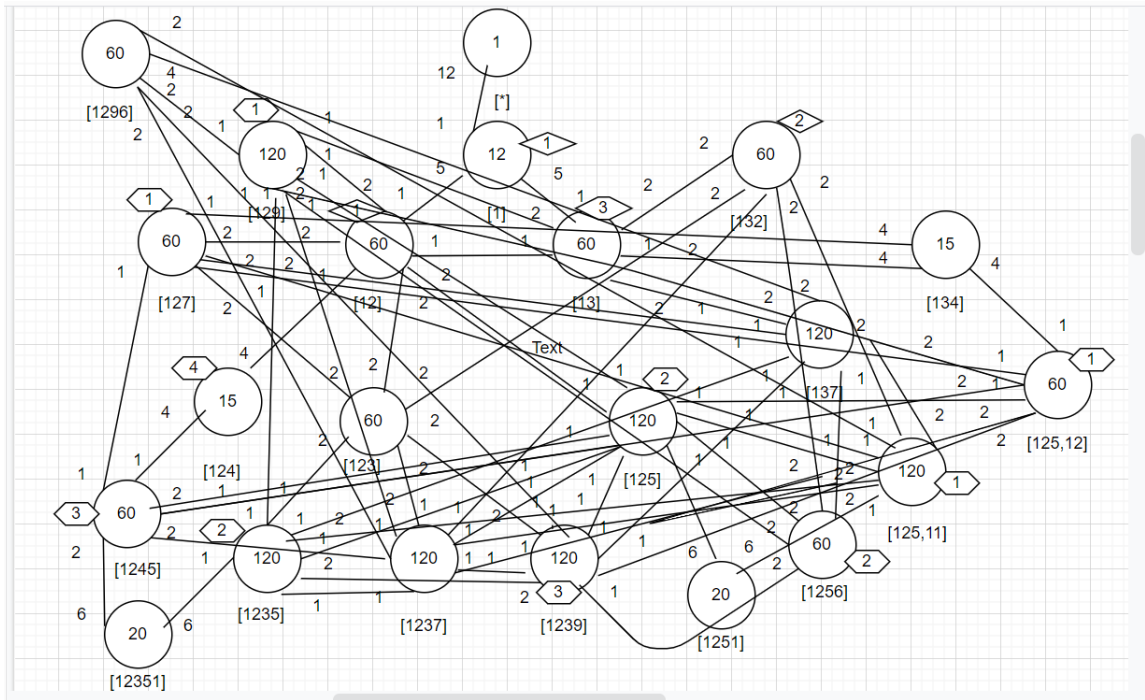
Figure 7.56: Cayley Diagram for G over S_{12}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^w|}$ decreases. Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_2 \cup Nt_1t_3 \cup Nt_1t_2t_3 \cup Nt_1t_2t_4 \cup Nt_1t_2t_5 \cup Nt_1t_2t_7$

$\cup Nt_1t_2t_9 \cup Nt_1t_3t_2 \cup Nt_1t_3t_4 \cup Nt_1t_3t_7 \cup Nt_1t_2t_3t_5 \cup Nt_1t_2t_3t_7$

$\cup Nt_1t_2t_3t_9 \cup Nt_1t_2t_5t_1 \cup Nt_1t_2t_5t_6 \cup Nt_1t_2t_5t_{11}$

$\cup Nt_1t_2t_5t_{12} \cup Nt_1t_2t_4t_5 \cup Nt_1t_2t_9t_6 \cup Nt_1t_2t_3t_5t_1$

$|G| \leq (|N| + |\frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(123)}|} + \frac{|N|}{|N^{(124)}|}$

$+ \frac{|N|}{|N^{(125)}|} + \frac{|N|}{|N^{(127)}|} + \frac{|N|}{|N^{(129)}|} + \frac{|N|}{|N^{(132)}|} + \frac{|N|}{|N^{134}|} + \frac{|N|}{|N^{(137)}|} + \frac{|N|}{|N^{(1235)}|} + \frac{|N|}{|N^{(1237)}|} + \frac{|N|}{|N^{(1239)}|} + \frac{|N|}{|N^{(1251)}|}$

$+ \frac{|N|}{|N^{(1256)}|} + \frac{|N|}{|N^{(12511)}|} + \frac{|N|}{|N^{(12512)}|} + \frac{|N|}{|N^{(1245)}|} + \frac{|N|}{|N^{(1296)}|} + \frac{|N|}{|N^{(12351)}|} X|N|)$

$G \leq (1 + 12 + 60 + 60 + 60 + 15 + 120 + 60 + 120 + 60 + 15 + 120 + 120 + 120 + 120 + 20 + 60 + 120 + 60 + 60 + 60 + 20)$

$\implies |G| \leq 1463 X 120 \implies |G| \leq 175560.$

G acts on 1463 cosets that are given in the cayley diagram. Let X be the set of these 1463 cosets. Now $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong \text{Imf}$ (First Isomorphsim Theorem) $\implies$ $\frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies$ $|\frac{G}{Kerf}| = 175560.$

This means $|G| \geq 175560$. We know $|G| \leq 175560$ from the cayley diagram. Therefore, $|G| = 175560.$

7.7 Construction of $(3 : 2^4) : (S_5 : 2)$ over S_5

Let $N \cong S_5\langle x, y \rangle$ where $x \sim (1, 20, 6, 10)(2, 19, 8, 12)(3, 13)(4, 15)(5, 16, 11, 17)(7, 14, 9, 18)$, and

$y \sim (1, 15)(2, 13)(3, 11, 8, 18)(4, 9, 6, 17)(5, 16, 19, 10)(7, 14, 20, 12)$. The progenitor 2^{*20} : N has symmetric presentation

```
G<x,y,t>:=Group<x,y,t|x^4 , y^4 ,
(y^-1 * x)^3 ,
x^-2 * y^-2 * x^-1 * y^2 * x^2 * y^-1,
t^2>;
```

We want to verify that the presentation given above is correct.

Using the theorem: $\frac{2^{*n} : N}{t_i, t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$,

we will show that $\frac{2^{*20} : N}{((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x * y * x * y^2)})^3} \cong (3 : 2^4) : (S_5 : 2)$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^4 , y^4 ,
(y^-1 * x)^3 ,x^-2 * y^-2 * x^-1 * y^2 * x^2 * y^-1,
t^2,
(t,y^2),(t,(y^-1*x*y)^2),
(t,t^{(y*x)}),(t,t^{(x^2*y^-1)}),
(t,t^{(x^2*y*x^-1)}),
(t,t^{(x*y*x)}),(t,t^{(x^2*y*x^-1*y^-1)})> ~ (3:2^4):(S_5:2)
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{5760}{120} = 48$. Now we know that we have 48 distinct single cosets.

Expanding Relation We will expand our only relation.

$$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x * y * x * y^2)})^3$$

$$= (x^2 * y^{-1} * x^{-1} * y^{-1}) * t_2 * (x^2 * y^{-1} * x^{-1} * y^{-1}) * t_2 * (x^2 * y^{-1} * x^{-1} * y^{-1}) * t_2$$
$$= (x^2 * y^{-1} * x^{-1} * y^{-1})^3 * (t_2^{(x^2 * y^{-1} * x^{-1} * y^{-1})^2}) * (t_2^{(x^2 * y^{-1} * x^{-1} * y^{-1})}) * t_2$$
$$\implies (x^2 * y^{-1} * x^{-1} * y^{-1})^3 * t_2 * t_5 t_2 = e \implies (x^2 * y^{-1} * x^{-1} * y^{-1})^3 * t_2 = t_2 * t_5$$

First Double Coset [∗]

$NeN = \{Ne^n | n \in N] = [N]$.

The double coset NeN = [∗] contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right cosets in N is $\frac{120}{120} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$. We need only determine the double coset of the right cosets Nt_1. Thus 20 t_i's extend the double coset [∗] to the double coset $Nt_1N = [1]$.

Cayley Diagram

captionCayley Diagram for G over S_{20}

Second Double Coset [1]

$Nt_1N = \{Nt_1^n | n \in N\}$
$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10},$
$Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}$

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$
$= \langle (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14)$
$(3,20)(4,19)(5,12)(6,14)(7,10)(8,16)(13,18)(15,17)\rangle$.

Thus, $N^{(1)} \geq N^1 = \langle (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14)$
$(3,20)(4,19)(5,12)(6,14)(7,10)(8,16)(13,18)(15,17)\rangle$

The number of single right cosets in Nt_1N is calculated by the formula $\frac{|N|}{|N^{(1)}|} = \frac{120}{6} = 20$. The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ are $\{1\}, \{2\}, \{3, 8, 20, 16, 7, 10\}, \{4, 6, 19, 14, 5, 12\}, \{9, 17, 15\}, \{11, 18, 13\}$. We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = N \in [*]$

This means one t_i takes $[1]$ to $[*]$

Choose 2 from $\{2\}$

$Nt_1t_2 \in [12]$

This means one t_i takes $[1]$ to $[12]$

Choose 3 from $\{3, 8, 20, 16, 7, 10\}$

$Nt_1t_3 \in [1]$.

We have $Nt_1t_3 = (x * y * x * y * x)t_1 \in [1]$

This means six t_i's take $[1]$ to $[1]$

Choose 4 from $\{4, 6, 19, 14, 5, 12\}$

$Nt_1t_4 \in [14]$

This means six t_i's take $[1]$ to $[14]$

Choose 9 from $\{9,17,15\}$

$Nt_1t_9 \in [19]$

This means three t_i's take $[1]$ to $[19]$

Choose 11 from $\{11, 18, 13\}$

$Nt_1t_{11} \in [1, 11]$

This means three t_i's take $[1]$ to $[1,11]$

Cayley Diagram
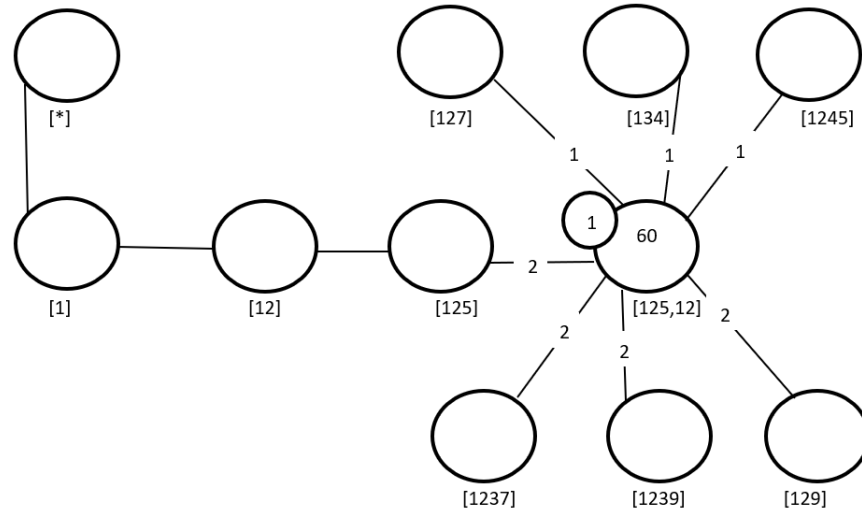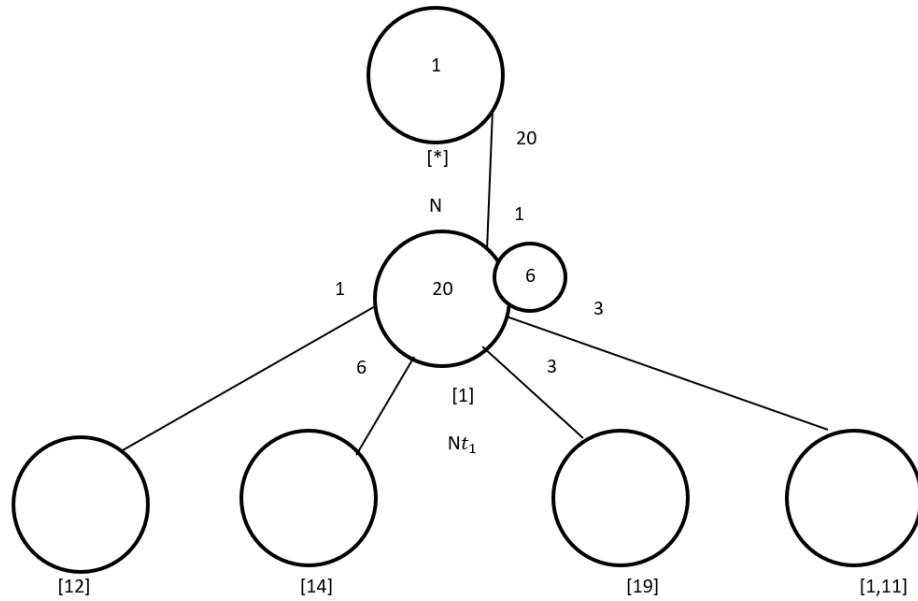

Figure 7.57: Cayley Diagram for G over S_{20}

Third Double Coset [12]

$Nt_1t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the coset stablilizer $N^{(12)}$. We first find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$

$N^{12} = \langle (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14)$

$(3,20)(4,19)(5,12)(6,14)(7,10)(8,16)(13,18)(15,17)\rangle$.

We find different names of Nt_1t_2. Now, $Nt_1t_2 = Nt_6t_8$

$N(t_1t_2)^{(1,6)(2,8)(5,11)(7,9)(10,20)(12,19)(14,18)(16,17)} = Nt_6t_8$.

But $Nt_6t_8 = Nt_1t_2 \implies N(t_1t_2)^{(1,6)(2,8)(5,11)(7,9)(10,20)(12,19)(14,18)(16,17)} = Nt_1t_2$.

$\implies (1,6)(2,8)(5,11)(7,9)(10,20)(12,19)(14,18)(16,17) \in N^{(12)}$.

Thus $N^{(12)} \geq \langle N^{12}, (1,6)(2,8)(5,11)(7,9)(10,20)(12,19)(14,18)(16,17)\rangle$.

$= \langle (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14)$

$(3,20)(4,19)(5,12)(6,14)(7,10)(8,16)(13,18)(15,17),$

$(1,6)(2,8)(5,11)(7,9)(10,20)(12,19)(14,18)(16,17)\rangle$.

The number of single right cosets in Nt_1t_2N is calculated by the formula,

$\frac{|N|}{|N^{(12)}|} = \frac{120}{60} = 2.$

The orbits of $N^{(12)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$ are

$\{1,6,4,14,18,12,19,11,13,5\}, \{2,8,3,16,17,10,20,9,15,7\}$ We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1, 6, 4, 14, 18, 12, 19, 11, 13, 5\}$

$Nt_1t_2t_1 \in [1]$.

We have $Nt_1t_2t_1 = (x*y*x*y*x)t_1t_4 \in [14]$

This means ten t_i's take [1] to [14]

Choose 2 from $\{2, 8, 3, 16, 17, 10, 20, 9, 15, 7\}$

$Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$

This means ten t_i's take [1] to [1]

Cayley Diagram
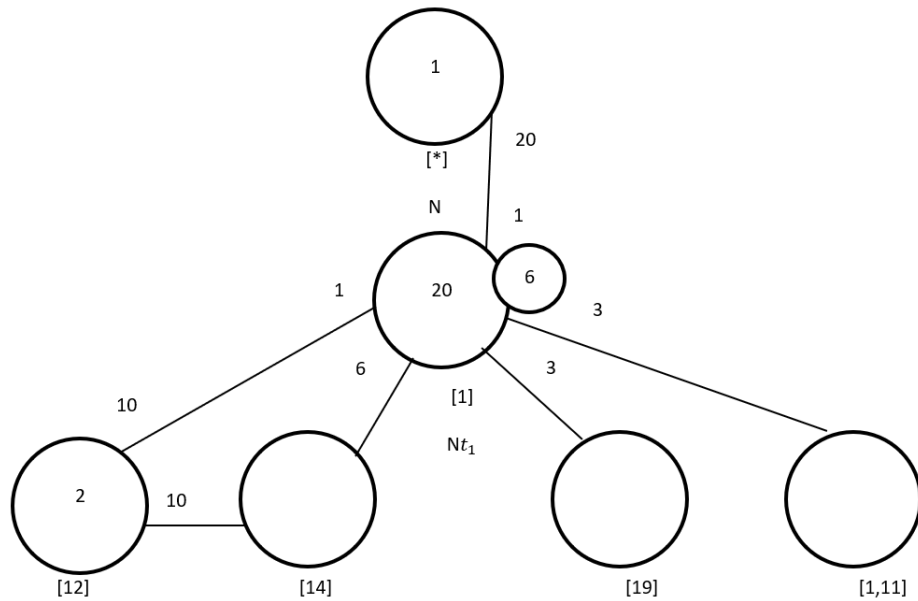
Figure 7.58: Cayley Diagram for G over S_{20}

Fourth Double Coset [14]

$Nt_1t_4 = \{N(t_1t_4)^n | n \in N\}.$

We now find the coset stablilizer $N^{(14)}$. We first find the point stabilizer of 1 and 4 in N.

$N^{14} = \{n \in N | (14)^n = 14\}$

$N^{14} = \langle e \rangle$.

We find different names of Nt_1t_4. Now, $Nt_1t_4 = (x^2 * y^2)t_1t_6 \implies Nt_1t_4 = Nt_1t_6$

$N(t_1t_4)^{(3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14)} = Nt_1t_6$.

But $Nt_1t_6 = Nt_1t_4 \implies N(t_1t_4)^{(3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14)} = Nt_1t_4$.

$\implies (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14) \in N^{(14)}$.

Thus $N^{(14)} \geq \langle N^{14}, (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14) \rangle$.

$= \langle e, (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14) \rangle$.

The number of single right cosets in Nt_1t_4N is calculated by the formula,

$\frac{|N|}{|N^{(14)}|} = \frac{120}{12} = 10$.

The orbits of N^{14} on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$ are

$\{1,2\}, \{3,20,14,7,6,8,4,10,5,16,12,19\}, \{9,11,17,18,15,13\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1,2\}$

$Nt_1t_4t_1 \in [12]$.

We have $Nt_1t_4t_1 = (x * y * x * y * x)t_1t_2 \in [12]$

This means two t_i's take [14] to [12]

Choose 3 from $\{3, 20, 14, 7, 6, 8, 4, 10, 5, 16, 12, 19\}$

$Nt_1t_4t_3 \in [1]$.

We have $Nt_1t_4t_3 = Nt_2$

$= (t_1)^{(x*y*x)} \in [1]$

This means twelve t_i's take [14] to [1]

Choose 9 from $\{9, 11, 17, 18, 15, 13\}$

$Nt_1t_4t_9 \in [19]$.

We have $Nt_1t_4t_9 = (x * y * x * y^{-1} * x)t_8t_{14}$

$= (x * y * x * y^{-1} * x) * (t_1t_9)^{(y*x^{-1}*y^{-1}*x^{-1}*y)} \in [19]$

This means six t_i's take [14] to [19]

Cayley Diagram

Figure 7.59: Cayley Diagram for G over S_{20}

Fifth Double Coset [19]

$Nt_1t_9 = \{N(t_1t_9)^n | n \in N\}$.

We now find the coset stablilizer $N^{(19)}$. We first find the point stabilizer of 1 and 9 in N.

$N^{19} = \{n \in N | (19)^n = 19\}$

$N^{19} = \langle(3, 20)(4, 19)(5, 12)(6, 14)(7, 10)(8, 16)(13, 18)(15, 17)\rangle$.

We find different names of Nt_1t_9. Now, $t_1t_9 = ((x^-1 * y * x)^2)t_6t_{16} \implies Nt_1t_9 = Nt_6t_{16}$

$N(t_1t_9)^{(1,6,4)(2,8,3)(5,12,18)(7,10,17)(9,16,20)(11,14,19)} = Nt_6t_{16}$.

But $Nt_6t_{16} = Nt_1t_9 \implies N(t_1t_9)^{(1,6,4)(2,8,3)(5,12,18)(7,10,17)(9,16,20)(11,14,19)} = Nt_1t_9$.

$\implies (1, 6, 4)(2, 8, 3)(5, 12, 18)(7, 10, 17)(9, 16, 20)(11, 14, 19)$

$(1, 6, 19)(2, 8, 20)(3, 9, 16)(4, 11, 14)(5, 18, 13)(7, 17, 15) \in N^{(19)}$.

Thus $N^{(19)} \geq \langle N^{19}, (1, 6, 4)(2, 8, 3)(5, 12, 18)(7, 10, 17)(9, 16, 20)(11, 14, 19)$

$(1, 6, 19)(2, 8, 20)(3, 9, 16)(4, 11, 14)(5, 18, 13)(7, 17, 15)\rangle$.

$= \langle(3, 20)(4, 19)(5, 12)(6, 14)(7, 10)(8, 16)(13, 18)(15, 17),$

$(1, 6, 4)(2, 8, 3)(5, 12, 18)(7, 10, 17)(9, 16, 20)(11, 14, 19)$

$(1, 6, 19)(2, 8, 20)(3, 9, 16)(4, 11, 14)(5, 18, 13)(7, 17, 15)\rangle$.

The number of single right cosets in Nt_1t_9N is calculated by the formula,

$\frac{|N|}{|N^{(19)}|} = \frac{120}{12} = 10$.

The orbits of N^{19} on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ are

$\{1, 6, 14, 4, 19, 11\}, \{2, 8, 16, 3, 20, 9\}, \{5, 12, 18, 13\}, \{7, 10, 17, 15\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1, 6, 14, 4, 19, 11\}$

$Nt_1t_9t_1 \in [14]$.

We have $Nt_1t_9t_1 = (x * y * x * y^2)t_9t_{15}$

$= (x * y * x * y^2)(t_1t_4)^{(x*y^2*x^2)} \in [14]$

This means six t_i's take [19] to [14]

Choose 2 from $\{2, 8, 16, 3, 20, 9\}$

$Nt_1t_9t_2 \in [1]$.

We have $Nt_1t_9t_2 = Nt_{11}$

$= (t_1)^{((x^{-1}, y^{-1}))} \in [1]$

This means six t_i's take [19] to [1]

Choose 5 from $\{5, 12, 18, 13\}$

$Nt_1t_9t_5 \in [1, 11]$.

We have $Nt_1t_9t_5 = (x^2 * y^{-1} * x^{-1} * y^{-1})t_1t_{11} \in [1, 11]$

This means four t_i's take [19] to [1,11]

Choose 7 from $\{7, 10, 17, 15\}$

$Nt_1t_9t_7 \in [19]$.

We have $Nt_1t_9t_7 = (x^2 * y^{-1} * x^{-1} * y^{-1})t_1t_9 \in [19]$

This means six t_i's take [19] to [19]
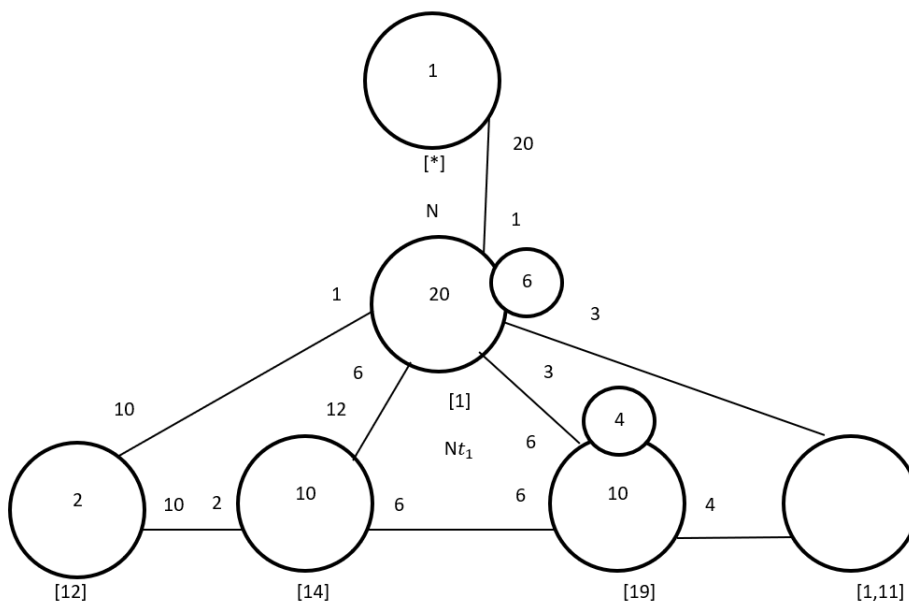
Cayley Diagram

Figure 7.60: Cayley Diagram for G over S_{20}

Sixth Double Coset $[1, 11]$

$Nt_1t_{11}N = \{Nt_1t_{11}^n | n \in N\}$.

We now find the coset stablilizer $N^{(1,11)}$. We first find the point stabilizer of 1 and 11 in N.

$N^{1,11} = \{n \in N | (1, 11)^n = 1, 11\}$

$N^{1,11} = \langle (3, 20)(4, 19)(5, 12)(6, 14)(7, 10)(8, 16)(13, 18)(15, 17) \rangle$.

We find different names of Nt_1t_{11}. Now, $Nt_1t_{11} = Nt_2t_9$

$N(t_1t_{11})^{(1,2)(3,14)(4,16)(5,7)(6,20)(8,19)(9,11)(10,12)(13,17)(15,18)} = Nt_2t_9$.

But $Nt_2t_9 = Nt_1t_{11} \implies N(t_1t_{11})^{(1,2)(3,14)(4,16)(5,7)(6,20)(8,19)(9,11)(10,12)(13,17)(15,18)} = Nt_1t_{11}$.

$\implies (1, 2)(3, 14)(4, 16)(5, 7)(6, 20)(8, 19)(9, 11)(10, 12)(13, 17)(15, 18)$,

$(1, 2)(3, 6)(4, 8)(5, 10)(7, 12)(9, 11)(13, 15)(14, 20)(16, 19)(17, 18) \in N^{(1,11)}$.

Thus $N^{(1,11)} \geq \langle N^{1,11}, (1, 2)(3, 14)(4, 16)(5, 7)(6, 20)(8, 19)(9, 11)(10, 12)(13, 17)(15, 18) \rangle$.

$= \langle (3, 20)(4, 19)(5, 12)(6, 14)(7, 10)(8, 16)(13, 18)(15, 17)$,

$(1, 2)(3, 14)(4, 16)(5, 7)(6, 20)(8, 19)(9, 11)(10, 12)(13, 17)(15, 18)$,

$(1,2)(3,6)(4,8)(5,10)(7,12)(9,11)(13,15)(14,20)(16,19)(17,18)\rangle.$

The number of single right cosets in $Nt_1t_{11}N$ is calculated by the formula,

$\frac{|N|}{|N^{(1,11)}|} = \frac{120}{24} = 5.$

The orbits of $N^{1,11}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$ are

$\{1,20,2,3,11,6,14,19,9,8,16,4], [5,12,15,10,7,17,18,13\}.$

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1, 20, 2, 3, 11, 6, 14, 19, 9, 8, 16, 4\}$

$Nt_1t_{11}t_1 \in [1].$

We have $Nt_1t_{11}t_1 = Nt_{11}$

$= (t_1)^{((x^{-1},y^{-1}))} \in [1]$

This means twelve t_i's take [1,11] to [1]

Choose 5 from $\{5, 12, 15, 10, 7, 17, 18, 13\}$

$Nt_1t_{11}t_5 \in [19].$

We have $Nt_1t_{11}t_5 = (x^2 * y^{-1} * x^{-1} * y^{-1})t_1t_9$

This means twelve t_i's take [1,11] to [19]

Cayley Diagram
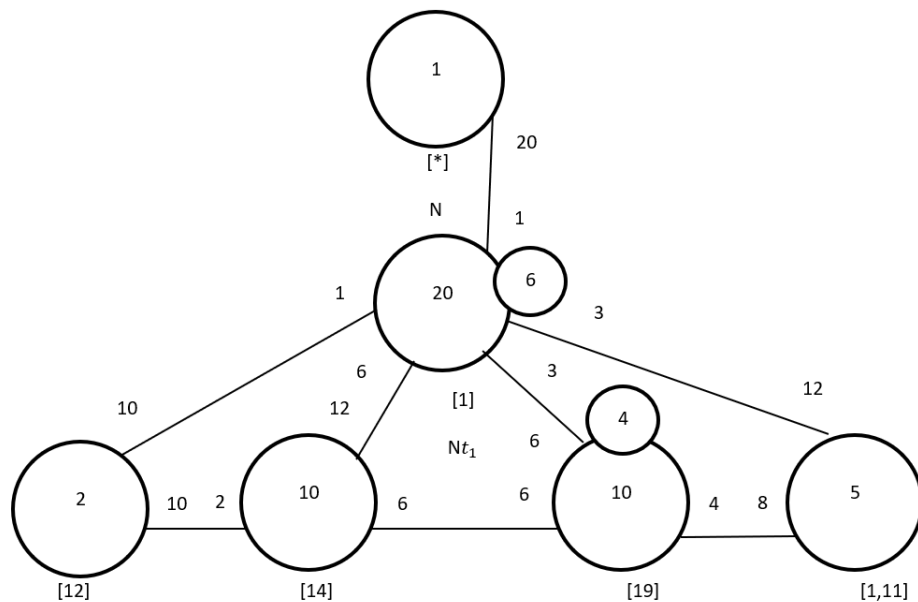
Figure 7.61: Cayley Diagram for G over S_{20}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^w|}$ decreases. Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_2 \cup Nt_1t_4 \cup Nt_1t_9 \cup Nt_1t_{11}$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(14)}|} + \frac{|N|}{|N^{(19)}|} + \frac{|N|}{|N^{(111)}|}) \times |N|$

$|G| \leq (1 + 20 + 2 + 10 + 10 + 5) \times 120 \implies |G| \leq 48 \times 120 \implies |G| \leq 5760.$

G acts on 48 cosets that are given in the cayley diagram. Let X be the set of these 48 cosets. Now $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong \text{Im}f$ (First Isomorphsim Theorem)

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = 5760.$

This means $|G| \geq 5760$.

We know $|G| \leq 5760$ from the cayley diagram.

Therefore, $|G| = 5760.$

Construction of $(2^4) : (S_5 : 2)$ over S_5

Let $N \cong S_5 \langle x, y \rangle$ where $x \sim (1, 20, 6, 10)(2, 19, 8, 12)(3, 13)(4, 15)(5, 16, 11, 17)(7, 14, 9, 18)$,

and

$y \sim (1, 15)(2, 13)(3, 11, 8, 18)(4, 9, 6, 17)(5, 16, 19, 10)(7, 14, 20, 12)$. The progenitor 2^{*20} :

N has symmetric presentation

```
G<x,y,t>:=Group<x,y,t|x^4 , y^4 ,
(y^-1 * x)^3 ,
x^-2 * y^-2 * x^-1 * y^2 * x^2 * y^-1,
t^2>;
```

We want to verify that the presentation given above is correct.

Using the theorem: $\frac{2^{*n}:N}{t_i, t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$,

we will show that $\frac{2^{*20}:N}{((x^2*y^{-1}*x^{-1}*y^{-1})*t^{(y*x^{-1}*y^{-1}*x^{-1}*y)})^3, ((x^2*y^{-1}*x^{-1}*y^{-1})*t^{(x^2*y*x*y^{-1})})^2} \cong (2^4)$:

$(S_5 : 2)$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^4 , y^4 ,
(y^-1 * x)^3 ,x^-2 * y^-2 * x^-1 * y^2 * x^2 * y^-1,
t^2,
(t,y^2),(t,(y^-1*x*y)^2),
(t,t^{(y*x)}),(t,t^{(x^2*y^-1)}),
(t,t^{(x^2*y*x^-1)}),
(t,t^{(x*y*x)}),(t,t^{(x^2*y*x^-1*y^-1)})
((x ^ 2 * y^-1 * x^-1 * y^-1)*t^(y * x^-1 * y^-1 * x^-1 * y))^3,
((x ^ 2 * y^-1 * x^-1 * y^-1)*t^(x^2 * y * x * y^-1))^2> ~ (2^4):(S_5:2)
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{1920}{120} = 16$. Now we know that we have 16 distinct single cosets.

Expanding RelationWe will expand our first relation.

$$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(}y * x^{-1} * y^{-1} * x^{-1} * y))^3$$

$= (x^2 * y^{-1} * x^{-1} * y^{-1})t_8 * (x^2 * y^{-1} * x^{-1} * y^{-1}) * t_8(x^2 * y^{-1} * x^{-1} * y^{-1}) * t_8$

$= (x^2 * y^{-1} * x^{-1} * y^{-1})^3 * t_8^{(x^2*y^{-1}*x^{-1}*y^{-1})^2} * t_8^{(x^2*y^{-1}*x^{-1}*y^{-1})} * t_8$

$= (x^2 * y^{-1} * x^{-1} * y^{-1})^3 * t_8 t_{18} t_8$

$$\implies (x^2 * y^{-1} * x^{-1} * y^{-1})^3 * t_8 = t_8 t_{18}$$

Expanding RelationWe will expand our second relation.

$$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x^2 * y * x * y^{-1})})^2$$

$$= (x^2 * y^{-1} * x^{-1} * y^{-1}) * t_{10} * (x^2 * y^{-1} * x^{-1} * y^{-1}) * t_{10}$$
$$= (x^2 * y^{-1} * x^{-1} * y^{-1})^2 * t_{10}^{(x^2 * y^{-1} * x^{-1} * y^{-1})} * t_{10}$$
$$= (x^2 * y^{-1} * x^{-1} * y^{-1})^2 * t_{12} t_{10}$$
$$\implies (x^2 * y^{-1} * x^{-1} * y^{-1})^2 * t_{12} = t_{10}$$

First Double Coset $[*]$

$NeN = \{Ne^n | n \in N] = [N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right cosets in N is $\frac{120}{120} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.

We need only determine the double coset of the right cosets Nt_1. Thus 20 t_i's extend the double coset $[*]$ to the double coset $Nt_1N = [1]$.

Cayley Diagram

captionCayley Diagram for G over S_{20}

Second Double Coset $[1]$

$Nt_1N = \{Nt_1^n | n \in N\}$
$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10},$
$Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}$

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

$= \langle (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14)$

$(3,20)(4,19)(5,12)(6,14)(7,10)(8,16)(13,18)(15,17) \rangle$.

Thus, $N^{(1)} \geq N^1 = \langle (3,8)(4,6)(5,19)(7,20)(9,17)(10,16)(11,18)(12,14)$

$(3,20)(4,19)(5,12)(6,14)(7,10)(8,16)(13,18)(15,17) \rangle$

The number of single right cosets in Nt_1N is calculated by the formula $\frac{|N|}{|N^{(1)}|} = \frac{120}{6} = 20$.

The orbits of $N^{(1)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$ are

$\{1\}, \{2\}, \{3,8,20,16,7,10\}, \{4,6,19,14,5,12\}, \{9,17,15\}, \{11,18,13\}$. We will determine

the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = N \in [*]$

This means one t_i takes [1] to [*]

Choose 2 from $\{2\}$

$Nt_1t_2 \in [1]$

This means one t_i takes [1] to [*]

Choose 3 from $\{3, 8, 20, 16, 7, 10\}$

$Nt_1t_3 \in [1]$.

We have $Nt_1t_3 = (x*y*x*y*x)t_1 \in [1]$

This means six t_i's take [1] to [1]

Choose 4 from $\{4, 6, 19, 14, 5, 12\}$

$Nt_1t_4 \in [1]$

We have $Nt_1t_4 = (x*y*x*y*x)t_1 \in [1]$

This means six t_i's take [1] to [1]

Choose 9 from $\{9,17,15\}$

$Nt_1t_9 \in [19]$

This means three t_i's take [1] to [19]

Choose 11 from $\{11, 18, 13\}$

$Nt_1t_{11} \in [19]$

This means three t_i's take [1] to [19]

Cayley Diagram
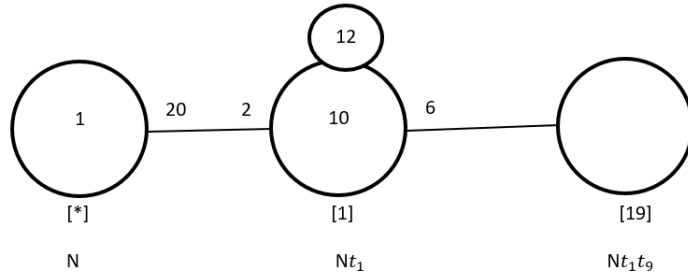
Figure 7.62: Cayley Diagram for G over S_{20}

Third Double Coset [19]

$Nt_1t_9 = \{N(t_1t_9)^n | n \in N\}$.

We now find the coset stablilizer $N^{(19)}$. We first find the point stabilizer of 1 and 9 in N.

$N^{19} = \{n \in N | (19)^n = 19\}$

$N^{19} = \langle (3, 20)(4, 19)(5, 12)(6, 14)(7, 10)(8, 16)(13, 18)(15, 17) \rangle$.

We find different names of Nt_1t_9. Now, $t_1t_9 = ((x^-1 * y * x)^2)t_8t_{14} \implies Nt_1t_9 = Nt_8t_{14}$

$N(t_1t_9)^{(1,8,11,16)(2,6,9,14)(3,19)(4,20)(5,10,18,15)(7,12,17,13)} = Nt_8t_{14}$.

But $Nt_6t_{14} = Nt_1t_9 \implies N(t_1t_9)^{(1,8,11,16)(2,6,9,14)(3,19)(4,20)(5,10,18,15)(7,12,17,13)} = Nt_1t_9$.

$\implies (1, 8, 11, 16)(2, 6, 9, 14)(3, 19)(4, 20)(5, 10, 18, 15)(7, 12, 17, 13)$,

$(1, 8)(2, 6)(3, 4)(5, 17)(7, 18)(9, 14)(10, 12)(11, 16)(13, 15)(19, 20) \in N^{(19)}$.

Thus $N^{(19)} \geq \langle N^{19}, (1, 8, 11, 16)(2, 6, 9, 14)(3, 19)(4, 20)(5, 10, 18, 15)(7, 12, 17, 13)$,

$(1, 8)(2, 6)(3, 4)(5, 17)(7, 18)(9, 14)(10, 12)(11, 16)(13, 15)(19, 20) \rangle$.

$= \langle (3, 20)(4, 19)(5, 12)(6, 14)(7, 10)(8, 16)(13, 18)(15, 17)$,

$(1, 8, 11, 16)(2, 6, 9, 14)(3, 19)(4, 20)(5, 10, 18, 15)(7, 12, 17, 13)$,

$(1, 8)(2, 6)(3, 4)(5, 17)(7, 18)(9, 14)(10, 12)(11, 16)(13, 15)(19, 20) \rangle$.

The number of single right cosets in Nt_1t_9N is calculated by the formula,

$\frac{|N|}{|N^{(19)}|} = \frac{120}{24} = 5$.

The orbits of N^{19} on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ are

$\{1, 6, 2, 14, 4, 19, 20, 3, 8, 11, 16, 9\}, \{5, 12, 18, 7, 10, 13, 15, 17\}$

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1, 6, 2, 14, 4, 19, 20, 3, 8, 11, 16, 9\}$

$Nt_1t_9t_1 \in [1]$.

We have $Nt_1t_9t_1 = t_9 = (t_1)^{x^2*y^{-1}} \in [1]$

This means twelve t_i's take [19] to [1]

Choose 5 from $\{5, 12, 18, 7, 10, 13, 15, 17\}$

$Nt_1t_9t_5 \in [19]$.

We have $Nt_1t_9t_5 = (x^2 * y^{-1} * x^{-1} * y^{-1})t_1t_9 \in [19]$

This means eight t_i's take [19] to [19]

Cayley Diagram
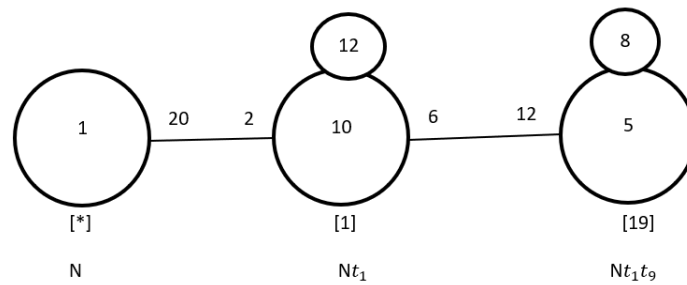
Figure 7.63: Cayley Diagram for G over S_{20}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^w|}$ decreases. Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_9$

$|G| \le (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(19)}|}) \times |N|$

$|G| \le (1 + 10 + 5) \times 120 \implies |G| \le 16 \times 120 \implies |G| \le 1920.$

G acts on 16 cosets that are given in the cayley diagram. Let X be the set of these 16 cosets. Now $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong \text{Imf}$ (First Isomorphsim Theorem)

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = 1920.$

This means $|G| \ge 1920$.

We know $|G| \le 1920$ from the cayley diagram.

Therefore, $|G| = 1920$.

7.8 Construction of $(2^2 : 3) : (2^4 : (7 : 3))$ over $(21 \times (S_6)$

Let $N \cong 21 \times S_6 = \langle x, y \rangle$ where $x \sim (1, 21, 3, 10, 14, 16)(2, 20, 5, 13, 8, 17)(4, 19, 9)(6, 18)(7, 15, 11)$

, and

$y \sim (1, 19, 6, 15, 2, 9, 5, 4, 14, 12, 8, 7, 3, 11)(10, 21, 18, 20, 13, 16, 17)$

The progenitor $2^{*21} : N$ has symmetric presentation

```
   G<x,y,t>:=Group<x,y,t|x^6,
  (y^-1 * x^-1)^3,
   (y^-1 * x)^3,
  x^-1 * y^-1 * x^3 * y^-1 * x * y,
t^2>;
```

We want to verify that the presentation given above is correct.

Theorem: $\frac{2^{*n}:N}{t_i, t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$

We will show that $\frac{2^{*21}:N}{((y*x^2)*(t^{(x*y*x^{-1})^2)})^3} \cong (2^2 : 3) : (2^4 : (7 : 3)$ Thus, we will prove

```
   G<x,y,t>:=Group<x,y,t|x^6,
  (y^-1 * x^-1)^3,
  (y^-1 * x)^3,
  x^-1 * y^-1 * x^3 * y^-1 * x * y,
t^2,
(t,x * y * x^-1 * y^-2),(t,x^3*y*x^-1),
(t,t^(y^x)), (t,t^(x^2)),
(t,t^(y*x^-1)),(t,t^(y^-3)),
(t,t^(y*x)),
((y * x^2)*(t^{(x * y * x^-1)^2)})^3> ~ (2^2:3):(2^4:(7:3).
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{4032}{126} = 32$. Now we know that we have 32 distinct single cosets.

Expanding Relation We expand our only relation:

$$((y * x^2) * (t^{(}x * y * x^{-1})^2))^3$$

$$= (y*x^2)*t_2*(y*x^2)*t_2*(y*x^2)*t_2$$
$$= (y*x^2)^3*(t_2^{(y*x^2)^2})*(t_2^{(y*x^2)})*t_2$$
$$\implies (y*x^2)^3*t_6t_{19}t_2 = e$$
$$\implies (y*x^2)^3*t_6 = t_2t_{19}$$

First Double Coset [*]

$NeN = \{Ne^n|n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right coset in N is $\frac{|N|}{|N|} = \frac{126}{126} = 1$.

Since N is transitive on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21\}$.

We need only determine the double coset of the right coset Nt_1.

Thus 21 t_i's extend the double coset $[*]$ to the double coset $Nt_1N = [1]$.
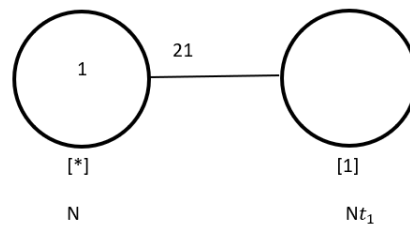
Cayley Diagram

Figure 7.64: Cayley Diagram for G over S_{21}

Second Double Coset [1]

$Nt_1N = \{Nt_1^n|n \in N\}$
$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11},$
$Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}\}$.

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N|1^n = 1\}$

Thus, $N^{(1)} \geq N^1 = \langle(2,6,14)(3,5,8)(4,10)(7,18,19,13,12,21)(9,17,15,16,11,20)\rangle$.

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{126}{6} = 21.$

The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

are

$\{1\}, \{4, 10\}, \{2, 6, 14\}, \{3, 5, 8\}, \{7, 18, 19, 13, 12, 21\}, \{9, 17, 15, 16, 11, 20\}.$

We will determine the double cosetes by selecting one representative from each orbit such

as,

Choose 1 from $\{1\}$

$Nt_1t_1 = t_1^2 = e \in [*].$

This means one t_i takes $[1]$ to $[*]$.

Choose 2 from $\{2,6,14\}$

$Nt_1t_2 \in [13].$

We have $Nt_1t_2 = (t_1t_3)^y \in [13].$

This means three t_i's take $[1]$ to $[13]$.

Choose 3 from $\{3,5,8\}$

$Nt_1t_3 \in [13]$

This means three t_i's take $[1]$ to $[13]$.

Choose 4 from $\{4,10\}$

$Nt_1t_4 \in [14].$

This means two t_i's take $[1]$ to $[14]$.

Choose 7 from $\{7,18,19,13,12,21\}$

$Nt_1t_7 \in [1].$

We have $Nt_1t_7 = (y * x^{-1} * y^{-2} * x)t_3$

$= (y * x^{-1} * y^{-2} * x)(t_1)^{y^{-2}} \in [1].$

This means six t_i's take $[1]$ to $[1]$.

Choose 9 from $\{9,17,15,16,11,20\}$

$Nt_1t_9 \in [1].$

We have $Nt_1t_9 = (y * x^{-1} * y^{-2} * x)t_2$

$= (y * x^{-1} * y^{-2} * x)(t_1)^{(y^{-1}*x^{-1}*y)} \in [1].$

This means six t_i's take $[1]$ to $[1]$.
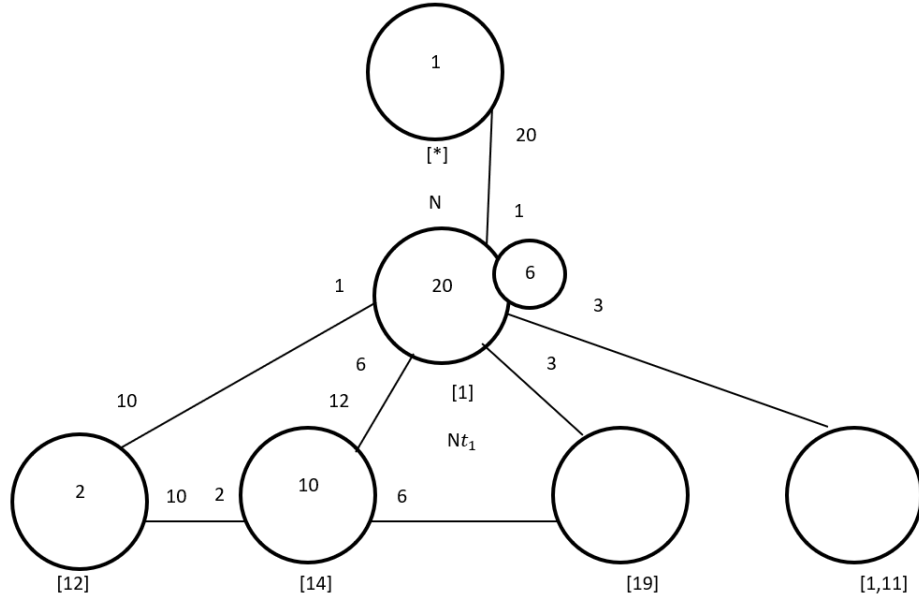
Cayley Diagram

Figure 7.65: Cayley Diagram for G over S_{21}

Third Double Coset [13]

$Nt_1t_3 = \{N(t_1t_3)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(13)}$. We first find the point stabilizer of 1 and 3 in N.

$N^{13} = \{n \in N | (13)^n = 13\}$

$N^{13} = \langle (4,10)(7,13)(9,16)(11,17)(12,18)(15,20)(19,21) \rangle$ We will find different names of Nt_1t_3N. $Nt_1t_3 = Nt_{15}t_{11}$.

$N(t_1t_3)^{(1,15,18)(2,7,13)(3,11,21)(4,20,6)(5,19,16)(8,12,10)(9,17,14)} = Nt_{15}t_{11}$.

But, $Nt_1t_3 = Nt_{15}t_{11} \implies N(t_1t_3)^{(1,15,18)(2,7,13)(3,11,21)(4,20,6)(5,19,16)(8,12,10)(9,17,14)} = Nt_1t_3$

$\implies (1,15,18)(2,7,13)(3,11,21)(4,20,6)(5,19,16)(8,12,10)(9,17,14)$,

$(1,15,6,4,8,12)(2,7)(3,11,14,9,5,19)(10,20,18)(16,17,21) \in N^{(13)}$

Thus $N^{(13)} \geq \langle N^{13}, (1,15,6,4,8,12)(2,7)(3,11,14,9,5,19)(10,20,18)(16,17,21)$,

$(1,15,18)(2,7,13)(3,11,21)(4,20,6)(5,19,16)(8,12,10)(9,17,14) \rangle$

$= \langle (4,10)(7,13)(9,16)(11,17)(12,18)(15,20)(19,21)$,

$(1,15,6,4,8,12)(2,7)(3,11,14,9,5,19)(10,20,18)(16,17,21)$,

$(1,15,18)(2,7,13)(3,11,21)(4,20,6)(5,19,16)(8,12,10)(9,17,14) \rangle \cong 3 \times S_3$.

The number of distinct right cosets in Nt_1t_3N is calculated by the formula,

$\frac{|N|}{|N^{(13)}|} = \frac{126}{18} = 7$.

The orbits of N^{13} on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21\}$

are

$\{2, 7, 13\}, \{1, 6, 8, 15, 4, 12, 20, 10, 18\}, \{3, 14, 5, 11, 9, 19, 17, 16, 21\}.$

We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from $\{1,6,8,15,4,12,20,10,18\}$

$Nt_1t_3t_1 \in [1].$

We have $Nt_1t_3t_1 = t_3$

$= (t_1)^{y^{-2}} \in [1].$

This means nine t_i's take [13] to [1].

Choose 2 from $\{2,7,13\}$

$Nt_1t_3t_2 \in [13].$

We have $Nt_1t_3t_2 = (y * x^{-1} * y^{-2} * x)t_1t_4 \in [14].$

This means three t_i's take [13] to [14].

Choose 3 from $\{3,14,5,11,9,19,17,16,21\}$

$Nt_1t_3t_3 = Nt_1^2 = Nt_1 \in [1].$

This means nine t_i's take [13] to [1].

Cayley Diagram

Figure 7.66: Cayley Diagram for G over S_{21}

Fourth Double Coset [14]

$Nt_1t_4 = \{N(t_1t_4)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(14)}$. We first find the point stabilizer of 1 and 4 in N.

$N^{14} = \{n \in N | (14)^n = 14\}$

$N^{14} = \langle (2, 6, 14)(3, 5, 8)(7, 12, 19)(9, 11, 15)(13, 18, 21)(16, 17, 20) \rangle$

We will find different names of Nt_1t_4N

$t_1t_4 = (x * y^2 * x * y^{-1} * x)t_8t_{20} \implies Nt_1t_4 = Nt_8t_{20}$

$N(t_1t_4)^{(1,8,14)(2,3,6)(4,20,19,10,15,21)(7,16,12,13,9,18)(11,17)} = Nt_8t_{20}$.

But, $Nt_8t_{20} = Nt_1t_4 \implies$

$N(t_1t_4)^{(1,8,14)(2,3,6)(4,20,19,10,15,21)(7,16,12,13,9,18)(11,17)} = Nt_1t_4$.

$\implies (1, 8, 14)(2, 3, 6)(4, 20, 19, 10, 15, 21)(7, 16, 12, 13, 9, 18)(11, 17) \in N^{(14)}$

Thus, $N^{(14)} \geq \langle N^{14}, (1, 8, 14)(2, 3, 6)(4, 20, 19, 10, 15, 21)(7, 16, 12, 13, 9, 18)(11, 17),$

$(1, 8, 5, 6, 3, 14, 2)(4, 20, 11, 18, 9, 21, 7, 10, 15, 17, 12, 16, 19, 13),$

$(1, 8, 6)(3, 5, 14)(4, 20, 12, 10, 15, 18)(7, 13)(9, 17, 19, 16, 11, 21) \rangle$.

$= \langle (2, 6, 14)(3, 5, 8)(7, 12, 19)(9, 11, 15)(13, 18, 21)(16, 17, 20),$

$(1, 8, 14)(2, 3, 6)(4, 20, 19, 10, 15, 21)(7, 16, 12, 13, 9, 18)(11, 17),$

$(1, 8, 5, 6, 3, 14, 2)(4, 20, 11, 18, 9, 21, 7, 10, 15, 17, 12, 16, 19, 13),$

$(1, 8, 6)(3, 5, 14)(4, 20, 12, 10, 15, 18)(7, 13)(9, 17, 19, 16, 11, 21) \rangle \cong 14 : 3$.

The number of distinct single cosets in Nt_1t_4N is calculated by the formula,

$\frac{|N|}{|N^{(1,4)}|} = \frac{126}{42} = 3$.

The Orbits of $N^{(14)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

are

$\{1, 8, 3, 14, 5, 6, 2\}, \{4, 20, 16, 19, 11, 12, 17, 7, 10, 13, 15, 18, 21, 9\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from {1,8,3,14,5,6,2}

$Nt_1t_4t_1 \in [13]$.

We have $Nt_1t_4t_1 = (y * x^{-1} * y^{-2} * x)t_9t_7$

$= (y * x^{-1} * y^{-2} * x)(t_1t_3)^{(y*x)} \in [13]$.

This means seven t_i's take [14] to [13].

Choose 4 from {4,20,16,19,11,12,17,7,10,13,15,18,21,9}

$Nt_1t_4t_4 = Nt_1t_4^2 = Nt_1 \in [1]$.

This means fourteen t_i's take [14] to [1].

Cayley Diagram
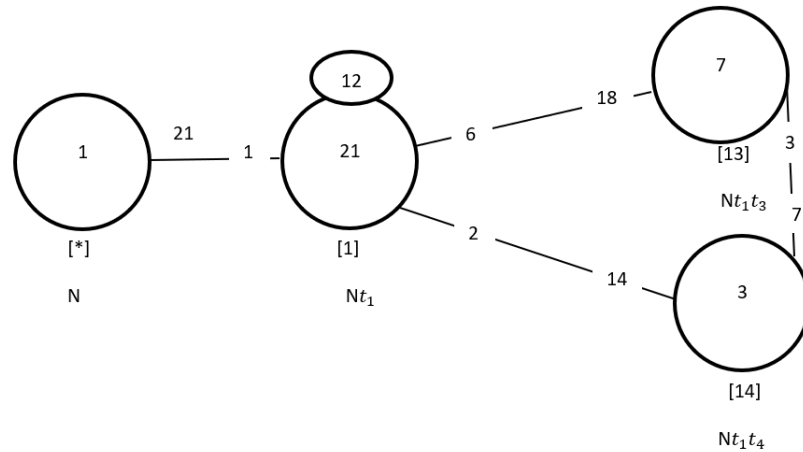
Figure 7.67: Cayley Diagram for G over S_{21}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases.

Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_3 \cup Nt_1t_4$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(14)}|} \times |N|$

$|G| \leq (1 + 21 + 7 + 3) \times 126$

$\implies |G| \leq 32 \times 126$

$\implies |G| \leq 4032.$

G acts on 32 cosets that are given in the cayley diagram.

Let X be the set of these 32 cosets.

Now, $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|$

But $\#\langle f(x), f(y), f(t) \rangle = 4032.$

So, $|\frac{G}{kerf}| = 4032$

This means $|G| \geq 4032$. We know $|G| \leq 4032$ from cayley diagram.

Therefore, $|G| = 4032$.

From $|G| = 4032 \times |Kerf|$ we find $|Kerf| = 1$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong (2^2 : 3)(2^4 : 7 : 3) \implies G \cong (2^2 : 3)(2^4 : 7 : 3)$.

7.9 Construction of $(2^6 : S_6)(7 : 3)$ over $(21 \times (S_6))$

Let $N \cong (21 \times S_6) = \langle x, y \rangle$ where $x \sim (1, 21, 3, 10, 14, 16)(2, 20, 5, 13, 8, 17)(4, 19, 9)(6, 18)(7, 15, 11)$, and

$y \sim (1, 19, 6, 15, 2, 9, 5, 4, 14, 12, 8, 7, 3, 11)(10, 21, 18, 20, 13, 16, 17)$.

The progenitor $2^{*21} : (21 \times S_6)$ has symmetric presentation

```
G<x,y,t>:=Group<x,y,t|x^6,
(y^-1 * x^-1)^3,
(y^-1 * x)^3,
x^-1 * y^-1 * x^3 * y^-1 * x * y,
t^2>;
```

We want to verify that the presentation given above is correct.

Theorem: $\frac{2^{*n}:N}{t_i t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$.

We will show that $\frac{2^{*21}:(21 \times S_6)}{((x^2*y^2*x*y^{-1})*t)^3, ((x^2*y^2)*(t^{(x*y*x^{-1})^2}))^3} \cong (2^6 : S_6)(7 : 3)$.

Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^6,
(y^-1 * x^-1)^3,
(y^-1 * x)^3,
x^-1 * y^-1 * x^3 * y^-1 * x * y,
t^2,
(t,x * y * x^-1 * y^-2),(t,x^3*y*x^-1),
((x^2 * y^2 * x * y^-1)*t)^3,
((x^2 * y^2)*(t^{(x * y * x^-1)^2}))^3)> ~ (2^6:S_6)(7:3).
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double cosets of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{8064}{126} = 64$. Now we know that we have 64 distinct single cosets.

<u>Expanding First Relation</u> We will expand our first relation.

$$(x^2 * y^2 * x * y^{-1}) * t)^3$$

$$= (x^2 * y^2 * x * y^{-1}) * t_1 * (x^2 * y^2 * x * y^{-1}) * t_1 * (x^2 * y^2 * x * y^{-1}) * t_1$$

$$= (x^2 * y^2 * x * y^{-1})^3 * (t_1^{(x^2*y^2*x*y^{-1})^2}) * (t_1^{(x^2*y^2*x*y^{-1})}) * t_1$$

$$\implies (x^2 * y^2 * x * y^{-1})^3 * t_4 t_{10} t_1 = e$$

$$\implies (x^2 * y^2 * x * y^{-1})^3 t_4 = t_1 t_{10}$$

Expanding Second Relation We will expand our second relation.

$$(x^2 * y^2) * (t^{(}x * y * x^{-1})^2))^3$$

$$= (x^2 * y^2) * t_2 * (x^2 * y^2) * t_2 * (x^2 * y^2) * t_2$$
$$= (x^2 * y^2)^3 * (t_2^{(x^2*y^2)^2}) * (t_2^{(x^2*y^2)}) * t_2$$
$$\implies (x^2 * y^2)^3 * t_6 t_{14} t_2 = e$$
$$\implies (x^2 * y^2)^3 * t_6 = t_2 t_{14}$$

First Double Coset [*]

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right coset in N is $\frac{|N|}{|N|} = \frac{126}{126} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$.

We need only determine the double coset of the right coset Nt_1.

Thus 21 t_i's extend the double coset $[*]$ to the double coset $Nt_1 N = [1]$.
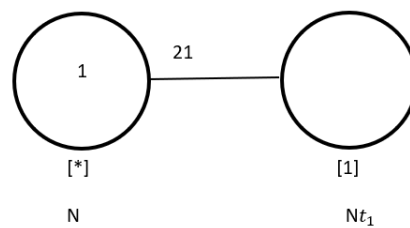
Cayley Diagram

Figure 7.68: Cayley Diagram for G over S_{21}

Second Double Coset [1]

$Nt_1N = \{Nt_1^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11},$

$Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}\}.$

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

Thus, $N^{(1)} \geq N^1 = \langle (2,6,14)(3,5,8)(4,10)(7,18,19,13,12,21)(9,17,15,16,11,20),$

$(2,6,14)(3,5,8)(4,10)(7,18,19,13,12,21)(9,17,15,16,11,20)\rangle.$

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{126}{6} = 21.$

The orbits of $N^{(1)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21\}$

are

$\{1\}, \{4,10\}, \{2,6,14\}, \{3,5,8\}, \{7,18,19,13,12,21\}, \{9,17,15,16,11,20\}.$

We will determine the double cosetes by selecting one representative from each orbit such

as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = N \in [*].$

This means one t_i takes [1] to [*].

Choose 2 from {2,6,14}

$Nt_1t_2 \in [1].$

We have $Nt_1t_2 = t_3$

$= (t_1)^{y^{-2}} \in [1]$

This means three t_i's take [1] to [1].

Choose 3 from {3,5,8}

$Nt_1t_3 \in [1].$

We have $Nt_1t_3 = t_2$

$= (t_1)^{(y^{-1}*x^{-1}*y)} \in [1]$

This means three t_i's take [1] to [1].

Choose 4 from {4,10}

$Nt_1t_4 \in [1].$

We have $Nt_1t_4 = t_{10}$

$= (t_1)^{(y^{-2}*x)} \in [1]$

This means two t_i's take $[1]$ to $[1]$.

Choose 7 from $\{7,18,19,13,12,21\}$

$Nt_1t_7 \in [17]$.

This means six t_i's take $[1]$ to $[17]$.

Choose 9 from $\{9,17,15,16,11,20\}$

$Nt_1t_9 \in [17]$.

We have $Nt_1t_9 = t_9t_1$

$= (t_1t_7)^{(y*x^{-2})} \in [17]$

This means six t_i's take $[1]$ to $[17]$.
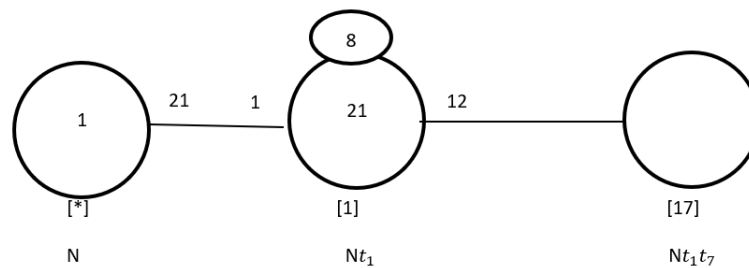
Cayley Diagram

Figure 7.69: Cayley Diagram for G over S_{21}

Third Double Coset [17]

$Nt_1t_7 = \{N(t_1t_7)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(17)}$. We first find the point stabilizer of 1 and 7 in N.

$N^{17} = \{n \in N | (17)^n = 17\}$

$N^{17} = \langle e \rangle$

We will find different names of Nt_1t_7N

$Nt_1t_7 = Nt_{13}t_3$

$N(t_1t_7)^{(1,13,9)(2,16,4)(3,10,7)(5,18,19)(6,21,11)(8,20,15)(12,14,17)} = Nt_{13}t_3$.

But, $Nt_{13}t_3 = Nt_1t_7 \implies$

$N(t_1t_7)^{(1,13,9)(2,16,4)(3,10,7)(5,18,19)(6,21,11)(8,20,15)(12,14,17)} = Nt_1t_7$.

$\implies (1,13,9)(2,16,4)(3,10,7)(5,18,19)(6,21,11)(8,20,15)(12,14,17) \in N^{(17)}$

Thus, $N^{(17)} \geq \langle N^{17}, (1,13,9)(2,16,4)(3,10,7)(5,18,19)(6,21,11)(8,20,15)(12,14,17) \rangle$.

$= \langle e, (1,13,9)(2,16,4)(3,10,7)(5,18,19)(6,21,11)(8,20,15)(12,14,17) \rangle$.

The number of distinct cosets in Nt_1t_7N is calculated by the formula,

$\frac{|N|}{|N^{(17)}|} = \frac{126}{3} = 42$.

The orbits of $N^{(17)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21\}$ are

$\{1,13,9\}, \{2,16,4\}, \{3,10,7\}, \{5,18,19\}, \{6,21,11\}, \{8,20,15\}, \{12,14,17\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1,13,9\}$

$Nt_1t_7t_1 \in [1]$.

We have $Nt_1t_7t_1 = t_7$

$= (t_1)^{(y^{-1}*x)} \in [1]$

This means three t_i's take [17] to [1].

Choose 2 from $\{2,16,4\}$

$Nt_1t_7t_2 \in [17]$.

We have $Nt_1t_7t_2 = t_1t_{13}$

$= (t_1t_7)^{(x*y*x^{-1}*y^{-2})} \in [17]$

This means three t_i's take [17] to [17].

Choose 3 from $\{3,10,7\}$

$Nt_1t_7t_3 \in [1]$.

We have $Nt_1t_7t_3 = t_{13}$

$= (t_1)^{(x*y^3)} \in [1]$

This means three t_i's take [17] to [1].

Choose 5 from $\{5,18,19\}$

$Nt_1t_7t_5 \in [17]$.

We have $Nt_1t_7t_5 = t_{13}t_{14}$

$= (t_1t_7)^{((y^{-1},x))} \in [17]$

This means three t_i's take [17] to [17].

Choose 6 from $\{6,21,11\}$

$Nt_1t_7t_6 \in [17]$.

We have $Nt_1t_7t_6 = t_{17}t_{15}$

$= (t_1 t_7)^{(x^2 * y^{-2} * x)} \in [17]$

This means three t_i's take [17] to [17].

Choose 8 from {8,20,15}

$N t_1 t_7 t_8 \in [17]$.

We have $N t_1 t_7 t_8 = t_{21} t_{12}$

$= (t_1 t_7)^{(y^{-2} * x^{-1})} \in [17]$

This means three t_i's take [17] to [17].

Choose 12 from {12,14,17}

$N t_1 t_7 t_{12} \in [17]$.

We have $N t_1 t_7 t_{12} = t_{21} t_8$

$= (t_1 t_7)^{(y^2 * x * y^{-1})} \in [17]$

This means three t_i's take [17] to [17].

Cayley Diagram
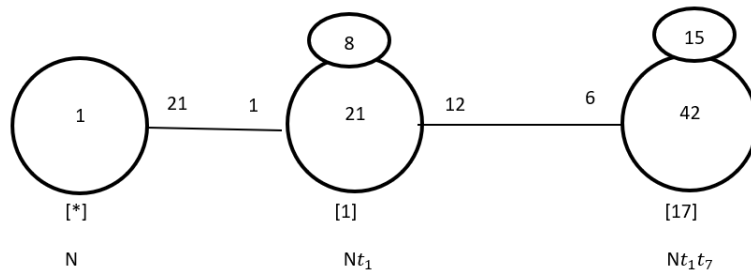
Figure 7.70: Cayley Diagram for G over S_{21}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases.

Our cayley diagram shows that

$G = N \cup N t_1 \cup N t_1 t_7$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(17)}|}) \times |N|$

$|G| \leq (1 + 21 + 42) \times 126$

$\implies |G| \leq 64 \times 126$

$\implies |G| \leq 8064.$

G acts on 64 cosets that are given in the cayley diagram.

Let X be the set of these 64 cosets.

Now, $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|$

But $\#\langle f(x), f(y), f(t) \rangle = 8064$.

So, $|\frac{G}{kerf}| = 8064$

This means $|G| \geq 8064$. We know $|G| \leq 8064$ from cayley diagram.

Therefore, $|G| = 8064$.

From $|G| = 8064 \times |Kerf|$ we find $|Kerf| = 1$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong (2^6 : S_6)(7 : 3) \implies G \cong (2^6 : S_6)(7 : 3)$.

7.10 Construction of $7^3 : S_3 : 2$ over $(21 \times (S_6)$

Let $N \cong (21 \times S_6) = \langle x, y \rangle$ where $x \sim (1, 21, 3, 10, 14, 16)(2, 20, 5, 13, 8, 17)(4, 19, 9)(6, 18)(7, 15, 11)$,

and

$y \sim (1, 19, 6, 15, 2, 9, 5, 4, 14, 12, 8, 7, 3, 11)(10, 21, 18, 20, 13, 16, 17)$.

The progenitor $2^{*21} : (21 \times S_6)$ has symmetric presentation

```
G<x,y,t>:=Group<x,y,t|x^6, (y^-1 * x^-1)^3,
(y^-1 * x)^3,
x^-1 * y^-1 * x^3 * y^-1 * x * y,
t^2>.
```

We want to verify that the presentation given above is correct.

Theorem: $\frac{2^{*n}:N}{t_i t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$.

We will show that $\frac{2^{*21}:(21 \times S_6)}{((y*x^{-1}*y^{-2}*x)*t)^3, ((x^2*y^{-1}*x)*(t^{(x*y*x^{-1})^2})^2)^4} \cong (7^3 : S_3 : 2)$.

Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^6, (y^-1 * x^-1)^3,
(y^-1 * x)^3,
x^-1 * y^-1 * x^3 * y^-1 * x * y,
t^2,
(t,x * y * x^-1 * y^-2),(t,x^3*y*x^-1),
((y * x^-1 * y^-2 * x)*t)^3,
((x^2 * y^-1 * x)*(t^{(x * y * x^-1)^2})^4)> ~ (7^3:S_3:2)
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double cosets of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{24696}{126} = 196$. Now we know that we have 196 distinct single cosets.

<u>Expanding First Relation</u> We will expand our first relation.

$$((y * x^{-1} * y^{-2} * x) * t)^3$$

$= (y * x^{-1} * y^{-2} * x) * t_1 * (y * x^{-1} * y^{-2} * x) * t_1 * (y * x^{-1} * y^{-2} * x) * t_1$

$= (y * x^{-1} * y^{-2} * x)^3 * (t_1^{(y*x^{-1}*y^{-2}*x)^2}) * (t_1^{(y*x^{-1}*y^{-2}*x)}) * t_1$

$\implies (y * x^{-1} * y^{-2} * x)^3 * t_1 * t_4 * t_1$

$\implies (y * x^{-1} * y^{-2} * x)^3 * t_1 = t_1 * t_4$

Expanding Second Relation We will expand our second relation.

$$((x^2 * y^{-1} * x) * (t^{(x * y * x^{-1})^2}))^4$$

$$= (x^2 * y^{-1} * x) * t_2 * (x^2 * y^{-1} * x) * t_2 * (x^2 * y^{-1} * x) * t_2 * (x^2 * y^{-1} * x) * t_2$$
$$= (x^2 * y^{-1} * x)^4 * (t_2^{(x^2 * y^{-1} * x)^3}) * (t_2^{(x^2 * y^{-1} * x)^2}) * (t_2^{(x^2 * y^{-1} * x)}) * t_2$$
$$\implies (x^2 * y^{-1} * x)^4 * t_5 * t_{20} * t_4 * t_2$$
$$\implies (x^2 * y^{-1} * x)^4 * t_5 * t_{20} = t_2 * t_4$$

First Double Coset [*]

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right coset in N is $\frac{|N|}{|N|} = \frac{126}{126} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$.

We need only determine the double coset of the right coset Nt_1.

Thus 21 t_i's extend the double coset [*] to the double coset $Nt_1N = [1]$.
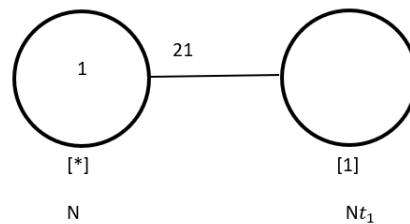
Cayley Diagram

Figure 7.71: Cayley Diagram for G over S_{21}

Second Double Coset [1]

$Nt_1N = \{Nt_1^n | n \in N\}$
$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11},$

$Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}\}$.

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

Thus, $N^{(1)} \geq N^1 = \langle (2, 6, 14)(3, 5, 8)(4, 10)(7, 18, 19, 13, 12, 21)(9, 17, 15, 16, 11, 20),$

$(2, 6, 14)(3, 5, 8)(4, 10)(7, 18, 19, 13, 12, 21)(9, 17, 15, 16, 11, 20) \rangle$.

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{126}{6} = 21$.

The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

are

$\{1\}, \{4, 10\}, \{2, 6, 14\}, \{3, 5, 8\}, \{7, 12, 18, 19, 21, 13\}, \{9, 11, 17, 15, 20, 16\}$.

We will determine the double cosetes by selecting one representative from each orbit such

as,

<u>Choose 1 from {1}</u>

$Nt_1t_1 = Nt_1^2 = N \in [*]$.

This means one t_i takes $[1]$ to $[*]$.

<u>Choose 2 from {2,6,14}</u>

$Nt_1t_2 \in [12]$.

This means three t_i's take $[1]$ to $[12]$.

<u>Choose 3 from {3,5,8}</u>

$Nt_1t_3 \in [13]$.

This means three t_i's take $[1]$ to $[13]$.

<u>Choose 4 from {4,10}</u>

$Nt_1t_4 \in [1]$.

We have $Nt_1t_4 = (y * x^{-1} * y^{-2} * x)t_1 \in [1]$

This means two t_i's take $[1]$ to $[1]$.

<u>Choose 7 from {7,12,18,19,21,13}</u>

$Nt_1t_7 \in [17]$.

This means six t_i's take $[1]$ to $[17]$.

<u>Choose 9 from {9,11,17,15,20,16}</u>

$Nt_1t_9 \in [17]$.

We have $Nt_1t_9 = t_{14}t_{20}$

$= (x^{-1}, y)(t_1t_7)^{(y*x*y^3)} \in [17]$

This means six t_i's take [1] to [17].
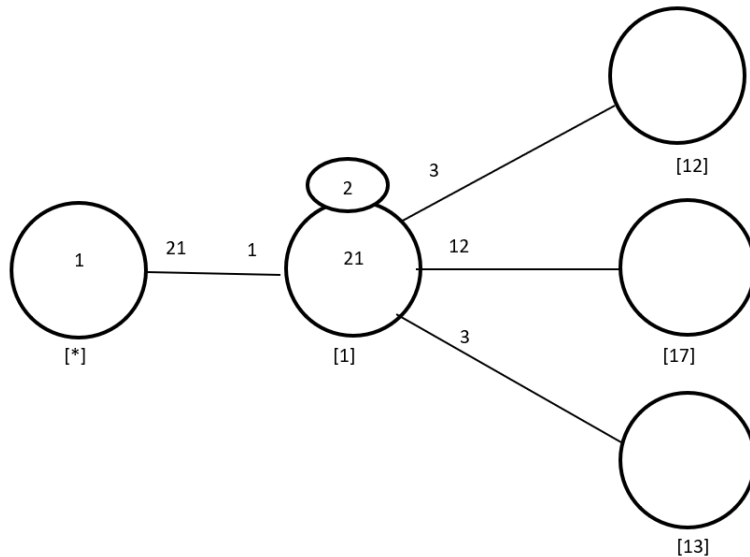
Cayley Diagram

Figure 7.72: Cayley Diagram for G over S_{21}

Third Double Coset [12]

$Nt_1t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the coset stabilizer $N^{(12)}$. We first find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$

$N^{12} = \langle (4, 10)(7, 13)(9, 16)(11, 17)(12, 18)(15, 20)(19, 21) \rangle$

We will find different names of Nt_1t_2N. $Nt_1t_2 = Nt_{14}t_3$.

$N(t_1t_2)^{(1,14,6,8,2,3,5)(4,19,12,15,7,9,11)(10,21,18,20,13,16,17)} = Nt_{14}t_3$.

But, $Nt_{14}t_3 = Nt_1t_2 \implies$

$N(t_1t_2)^{(1,14,6,8,2,3,5)(4,19,12,15,7,9,11)(10,21,18,20,13,16,17)} = Nt_1t_2$.

Thus $N^{(12)} \geq \langle N^{12}, (1, 14, 6, 8, 2, 3, 5)(4, 19, 12, 15, 7, 9, 11)(10, 21, 18, 20, 13, 16, 17)$

$(1, 14, 6, 8, 2, 3, 5)(4, 21, 12, 20, 7, 16, 11, 10, 19, 18, 15, 13, 9, 17) \rangle$.

$= (4, 10)(7, 13)(9, 16)(11, 17)(12, 18)(15, 20)(19, 21),$

$(1, 14, 6, 8, 2, 3, 5)(4, 19, 12, 15, 7, 9, 11)(10, 21, 18, 20, 13, 16, 17),$

$(1, 14, 6, 8, 2, 3, 5)(4, 21, 12, 20, 7, 16, 11, 10, 19, 18, 15, 13, 9, 17)$

The number of distinct right cosets in Nt_1t_2N is calculated by the formula,

$\frac{|N|}{|N^{(12)}|} = \frac{126}{14} = 9.$

The orbits of $N^{(12)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

are

$\{1, 14, 6, 8, 2, 3, 5\}, \{4, 10, 19, 21, 12, 18, 15, 20, 7, 13, 9, 16, 11, 17\}.$

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1, 14, 6, 8, 2, 3, 5\}$

$Nt_1t_2t_1 \in [1].$

We have $Nt_1t_2t_1 = Nt_8$

$= (t_1)^{(x*y^{-1}*x^{-1})^2} \in [1]$

This means seven t_i's take $[12]$ to $[1]$.

Choose 4 from $\{4, 10, 19, 21, 12, 18, 15, 20, 7, 13, 9, 16, 11, 17\}$

$Nt_1t_2t_4 \in [17].$

We have $Nt_1t_2t_4 = (y * x^{-1} * y^{-2} * x)t_{15}t_1$

$= (y * x^{-1} * y^{-2} * x)(t_1t_7)^{y^3} \in [17]$

This means fourteen t_i's take $[12]$ to $[17]$.
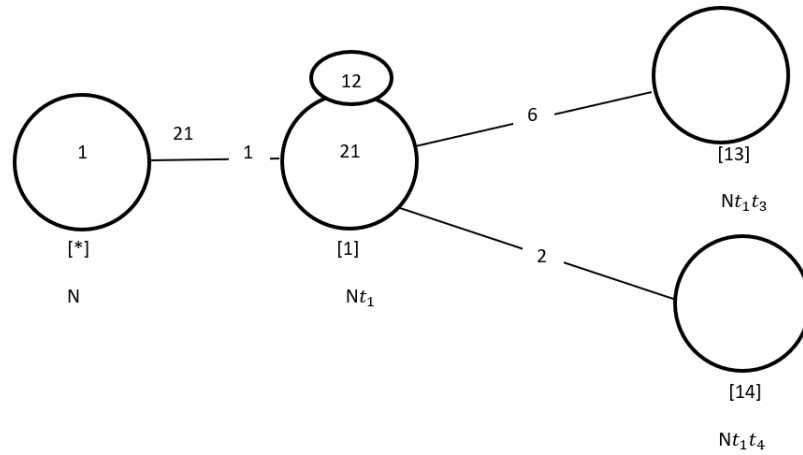
Cayley Diagram
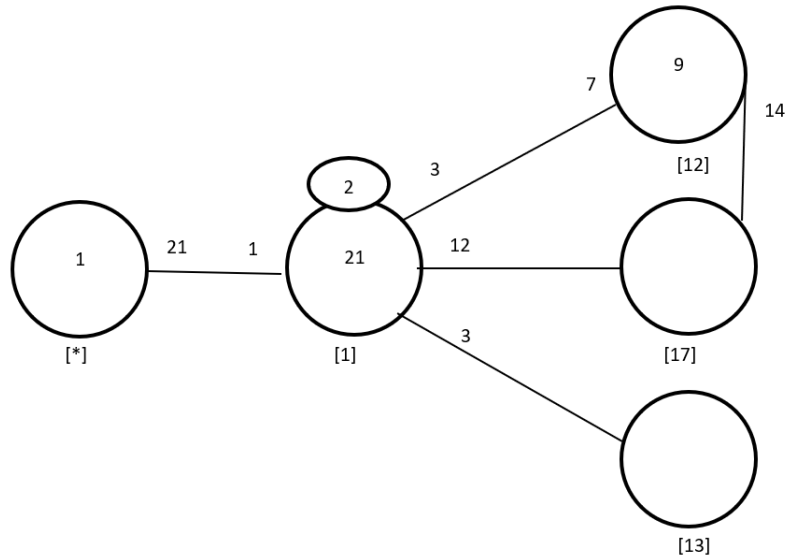
Figure 7.73: Cayley Diagram for G over S_{21}

Fourth Double Coset [13]

$Nt_1t_3 = \{N(t_1t_3)^n | n \in N\}$.

We now find the coset stabilizer $N^{(13)}$. We first find the point stabilizer of 1 and 3 in N.

$N^{13} = \{n \in N | (13)^n = 13\}$

$N^{13} = \langle (4, 10)(7, 13)(9, 16)(11, 17)(12, 18)(15, 20)(19, 21) \rangle$

We will find different names of Nt_1t_3N. $Nt_1t_2 = Nt_5t_2$.

$N(t_1t_3)^{(1,5,3,2,8,6,14)(4,11,9,7,15,12,19)(10,17,16,13,20,18,21))} = Nt_5t_2$.

But, $Nt_5t_2 = Nt_1t_3 \implies$

$N(t_1t_3)^{(1,5,3,2,8,6,14)(4,11,9,7,15,12,19)(10,17,16,13,20,18,21)} = Nt_1t_3$.

Thus $N^{(13)} \geq \langle N^{13}, (1, 5, 3, 2, 8, 6, 14)(4, 11, 9, 7, 15, 12, 19)(10, 17, 16, 13, 20, 18, 21)$

$(1, 5, 3, 2, 8, 6, 14)(4, 17, 9, 13, 15, 18, 19, 10, 11, 16, 7, 20, 12, 21) \rangle$.

$= (4, 10)(7, 13)(9, 16)(11, 17)(12, 18)(15, 20)(19, 21)$,

$(1, 5, 3, 2, 8, 6, 14)(4, 11, 9, 7, 15, 12, 19)(10, 17, 16, 13, 20, 18, 21)$

$(1, 5, 3, 2, 8, 6, 14)(4, 17, 9, 13, 15, 18, 19, 10, 11, 16, 7, 20, 12, 21)$

The number of distinct right cosets in Nt_1t_3N is calculated by the formula,

$\frac{|N|}{|N^{(13)}|} = \frac{126}{14} = 9.$

The orbits on $N^{(13)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

are

$\{1, 5, 3, 2, 8, 6, 14\}, \{4, 10, 11, 17, 9, 16, 7, 13, 15, 20, 12, 18, 19, 21\}$. We will determine the

double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1, 5, 3, 2, 8, 6, 14\}$

$Nt_1t_3t_1 \in [1]$.

We have $Nt_1t_3t_1 = Nt_6$

$= (t_1)^{y^2} \in [1]$

This means seven t_i's take $[13]$ to $[1]$.

Choose 4 from $\{4, 10, 11, 17, 9, 16, 7, 13, 15, 20, 12, 18, 19, 21\}$

$Nt_1t_3t_4 \in [17]$.

We have $Nt_1t_3t_4 = ((x * y * x^{-1})^3)t_{15}t_{16}$

$= ((x * y * x^{-1})^3) * (t_1t_7)^{(x^2*y*x^{-1})} \in [17]$

This means fourteen t_i's take $[13]$ to $[17]$.
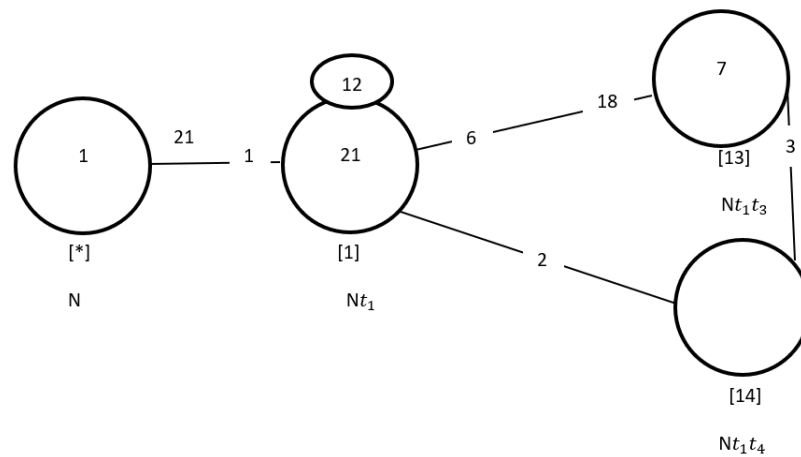
Cayley Diagram
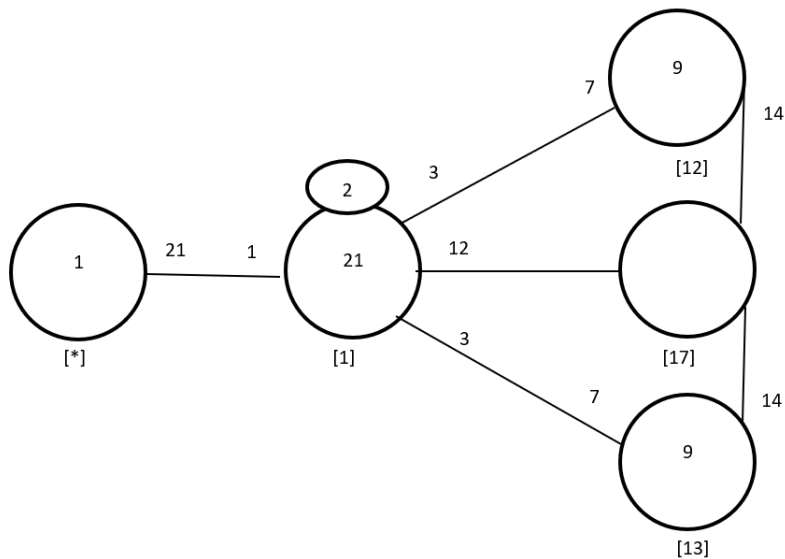
Figure 7.74: Cayley Diagram for G over S_{21}

Fifth Double Coset [17]

$Nt_1t_7 = \{N(t_1t_7)^n | n \in N\}$.

We now find the coset stabilizer $N^{(17)}$. We ind the point stabilizer of 1 and 7 in N.

Thus $N^{(17)} \geq N^{17}\langle e \rangle$

The number of distinct right cosets in Nt_1t_7N is calculated by the formula,

$\frac{|N|}{|N^{(17)}|} = \frac{126}{1} = 126$.

The orbits of $N^{(17)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\},$

$\{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1\}$

$Nt_1t_7t_1 \in [171]$.

This means one t_i takes [17] to [171].

Choose 2 from $\{2\ \}$

$Nt_1t_7t_2 \in [12]$.

We have $Nt_1t_7t_2 = (y * x^{-1} * y^{-2} * x)t_{19}t_9$

$= (y * x^{-1} * y^{-2} * x)(t_1t_2)^y \in [12]$

This means one t_i takes [17] to [12].

Choose 3 from $\{3\}$

$Nt_1t_7t_3 \in [171]$.

We have $Nt_1t_7t_3 = (x^{-2} * y * x^{-1})t_{18}t_5t_{18}$

$= (x^{-2} * y * x^{-1})(t_1t_7t_1)^{(x*y*x^2)} \in [171]$

This means one t_i takes [17] to [171].

Choose 4 from $\{4\}$

$Nt_1t_7t_4 \in [17]$.

We have $Nt_1t_7t_4 = (x^3 * y^{-2})t_6t_{17}$

$= (x^3 * y^{-2})(t_1t_7)^{(x*y*x^{-1})} \in [17]$

This means one t_i takes [17] to [17].

Choose 5 from $\{5\}$

$Nt_1t_7t_5 \in [171]$.

We have $Nt_1t_7t_5 = (x^3 * y)t_{18}t_7t_{18}$

$= (x^3 * y) * (t_1t_7t_1)^{(y^2*x)} \in [171]$

This means one t_i takes [17] to [171].

Choose 6 from {6}

$Nt_1t_7t_6 \in [176]$.

This means one t_i takes [17] to [176].

Choose 7 from {7}

$Nt_1t_7t_7 = Nt_1t_7^2 \in [1]$.

This means one t_i takes [17] to [1].

Choose 8 from {8}

$Nt_1t_7t_8 \in [178]$.

This means one t_i takes [17] to [178].

Choose 9 from {9}

$Nt_1t_7t_9 \in [17]$.

We have $Nt_1t_7t_9 = (y * x^{-1} * y^{-2} * x)t_1t_{19}$

$= (y * x^{-1} * y^{-2} * x) * (t_1t_7)^{(x^2*y^2)} \in [17]$

This means one t_i takes [17] to [17].

Choose 10 from {10}

$Nt_1t_7t_{10} \in [17]$.

We have $Nt_1t_7t_{10} = (x * y^{-1} * x^{-1})t_6t_{15}$

$= (x * y^{-1} * x^{-1}) * (t_1t_7)^{(y^2*x^2)} \in [17]$

This means one t_i takes [17] to [17].

Choose 11 from {11}

$Nt_1t_7t_{11} \in [17]$.

We have $Nt_1t_7t_{11} = (y * x^{-1} * y^{-2} * x)t_1t_{12}$

$= (y * x^{-1} * y^{-2} * x) * (t_1t_7)^{(y^{-2}*x^{-2})} \in [17]$

This means one t_i takes [17] to [17].

Choose 12 from {12}

$Nt_1t_7t_{12} \in [17]$.

We have $Nt_1t_7t_{12} = (x^2 * y^{-2} * x)t_{14}t_{20}$

$= (x^2 * y^{-2} * x) * (t_1t_7)^{(y*x*y^3)} \in [17]$

This means one t_i takes [17] to [17].

Choose 13 from {13}

$Nt_1t_7t_{13} \in [17]$.

We have $Nt_1t_7t_{13} = (x * y * x^{-1} * y^{-2})t_1t_7 \in [17]$ This means one t_i takes [17] to [17].

Choose 14 from {14}

$Nt_1t_7t_{14} \in [13]$.

We have $Nt_1t_7t_{14} = (x * y^{-1} * x * y * x)t_{18}t_{16}$

$= (x * y^{-1} * x * y * x) * (t_1t_3)^{(y^2 * x^{-1})} \in [13]$

This means one t_i takes [17] to [13].

Choose 15 from {15}

$Nt_1t_7t_{15} \in [17]$.

We have $Nt_1t_7t_{15} = (x * y * x^2 * y)t_2t_{16}$

$= (x * y * x^2 * y) * (t_1t_7)^{(y^{-1} * x^{-1} * y)} \in [17]$

This means one t_i takes [17] to [17].

Choose 16 from {16}

$Nt_1t_7t_{16} \in [17]$.

We have $Nt_1t_7t_{16} = (x * y^{-1} * x * y * x)t_3t_{18}$

$= (x * y^{-1} * x * y * x) * (t_1t_7)^{(x * y^{-1} * x^{-1})} \in [17]$

This means one t_i takes [17] to [17].

Choose 17 from {17}

$Nt_1t_7t_{17} \in [17]$.

We have $Nt_1t_7t_{17} = ((x * y^{-1} * x^{-1})^3)t_5t_{19}$

$= ((x * y^{-1} * x^{-1})^3) * (t_1t_7)^{(y * x^{-2} * y)} \in [17]$

This means one t_i takes [17] to [17].

Choose 18 from {18}

$Nt_1t_7t_{18} \in [17]$.

We have $Nt_1t_7t_{18} = (x * y^{-1} * x * y * x)t_3t_{17}$

$= (x * y^{-1} * x * y * x) * (t_1t_7)^{(y^2 * x * y^{-1} * x)} \in [17]$

This means one t_i takes [17] to [17].

Choose 19 from {19}

$Nt_1t_7t_{19} \in [17]$.

We have $Nt_1t_7t_{19} = (y * x^{-1} * y^{-2} * x)t_1t_7 \in [17]$

This means one t_i takes [17] to [17].

Choose 20 from {20}

$Nt_1t_7t_{20} \in [17]$.

We have $Nt_1t_7t_{20} = (x * y^{-1} * x * y * x)t_3t_{10}$

$= (x * y^{-1} * x * y * x) * (t_1t_7)^{(x^y)} \in [17]$

This means one t_i takes [17] to [17].

Choose 21 from {21}

$Nt_1t_7t_{21} \in [1]$.

We have $Nt_1t_7t_{21} = (x^{-2} * y * x^{-1})t_3$

$= (x^{-2} * y * x^{-1}) * (t_1)^{(y^{-2})} \in [1]$

This means one t_i takes [17] to [1].

Cayley Diagram
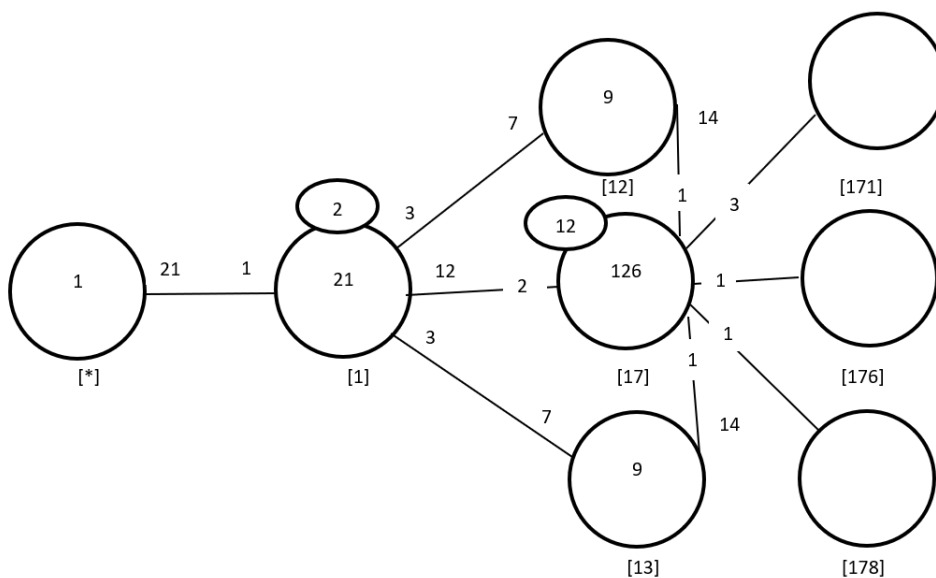
Figure 7.75: Cayley Diagram for G over S_{21}

Sixth Double Coset [171]

$Nt_1t_7t_1 = \{N(t_1t_7t_1)^n | n \in N\}$.

We now find the coset stabilizer N$^{(171)}$. We first find the point stabilizer of 1,7 and 1 in N.

$N^{171} = \{n \in N | (171)^n = 171\}$

$N^{171} = \langle e \rangle$

We will find different names of $Nt_1t_7t_1N$. $Nt_1t_7t_1 = Nt_2t_{19}t_2$.

$N(t_1t_7t_1)^{(1,2,14,3,6,5,8)(4,7,19,9,12,11,15)(10,13,21,16,18,17,20)} = Nt_2t_{19}t_2$.

But, $Nt_2t_{19}t_2 = Nt_1t_7t_1 \implies$

$N(t_1t_7t_1)^{(1,2,14,3,6,5,8)(4,7,19,9,12,11,15)(10,13,21,16,18,17,20)} = Nt_1t_7t_1$.

$\implies (1,2,14,3,6,5,8)(4,7,19,9,12,11,15)(10,13,21,16,18,17,20) \in Nt_1t_7t_1N$

Thus $N^{(171)} \geq \langle N^{171}, (1,2,14,3,6,5,8)(4,7,19,9,12,11,15)(10,13,21,16,18,17,20) \rangle$.

$= e, (1,2,14,3,6,5,8)(4,7,19,9,12,11,15)(10,13,21,16,18,17,20)$

The number of distinct right cosets in $Nt_1t_7t_1N$ is calculated by the formula,

$\frac{|N|}{|N^{(1,7,1)}|} = \frac{126}{7} = 18$.

The orbits of $N^{(171)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21\}$

are

$\{1,2,14,3,6,5,8\}, \{4,7,19,9,12,11,15\}, \{10,13,21,16,18,17,20\}$.

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from {1, 2, 14, 3, 6, 5, 8}

$Nt_1t_7t_1t_1 = t_1t_7t_1^2 = t_1t_7 \in [17]$.

This means seven t_i's take [171] to [17].

Choose 4 from {4, 7, 19, 9, 12, 11, 15}

$Nt_1t_7t_1t_4 \in [17]$.

We have $Nt_1t_7t_1t_4 = (x * y^{-1} * x^2)t_{12}t_{17}$

$= (x * y^{-1} * x^2) * (t_1t_7)^{(x^{-2}*y*x^{-1})} \in [17]$

This means seven t_i's take [171] to [17].

Choose 10 from {10, 13, 21, 16, 18, 17, 20}

$Nt_1t_7t_1t_{10} \in [17]$.

We have $Nt_1t_7t_1t_{10} = (x * y^{-1} * x^2)t_{18}t_{15}$

$= (x * y^{-1} * x^2) * (t_1t_7)^{(y^2*x^{-1})} \in [17]$

This means seven t_i's take [171] to [17].

Cayley Diagram
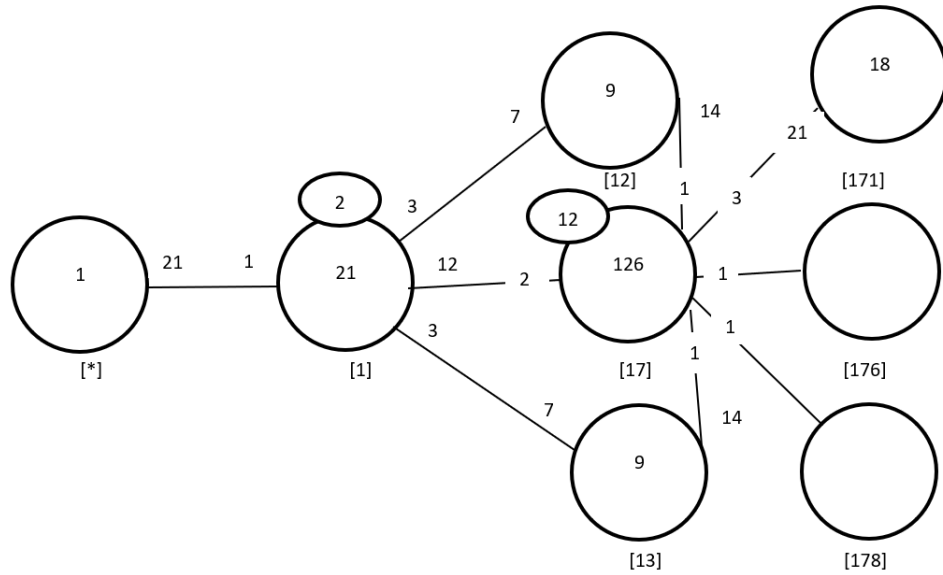
Figure 7.76: Cayley Diagram for G over S_{21}

Seventh Double Coset [176]

$Nt_1t_7t_6 = \{N(t_1t_7t_6)^n | n \in N\}$.

We now find the coset stabilizer $N^{(176)}$. We first find the point stabilizer of 1,7 and 6 in N.

$N^{176} = \{n \in N | (176)^n = 176\}$

$N^{176} = \langle e \rangle$

We will find different names of $Nt_1t_7t_6N$. $t_1t_7t_6 = (x^{-2} * y * x^{-1})t_{20}t_2t_{10} \implies Nt_1t_7t_6 = Nt_{20}t_2t_{10}$.

$N(t_1t_7t_6)^{(1,20,12)(2,13,7)(3,17,19)(4,8,18)(5,21,9)(6,10,15)(11,14,16)} = Nt_{20}t_2t_{10}$.

But, $Nt_{20}t_2t_{10} = Nt_1t_7t_6 \implies$

$N(t_1t_7t_6)^{(1,20,12)(2,13,7)(3,17,19)(4,8,18)(5,21,9)(6,10,15)(11,14,16)} = Nt_1t_7t_6$.

$\implies (1, 20, 12)(2, 13, 7)(3, 17, 19)(4, 8, 18)(5, 21, 9)(6, 10, 15)(11, 14, 16) \in Nt_1t_7t_6N$

Thus $N^{(176)} \geq \langle N^{176}, (1, 20, 12)(2, 13, 7)(3, 17, 19)(4, 8, 18)(5, 21, 9)(6, 10, 15)(11, 14, 16)\rangle$.

$= e, (1, 20, 12)(2, 13, 7)(3, 17, 19)(4, 8, 18)(5, 21, 9)(6, 10, 15)(11, 14, 16)$

The number of distinct right cosets in $Nt_1t_7t_6$ is calculated by the formula

$\frac{|N|}{|N^{(1,7,6)}|} = \frac{126}{21} = 6$.

The orbits of $N^{(1,7,6)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\},$

$\{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}.$

We will determine the double cosets by selecting one represebtative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1 t_7 t_6 t_1 \in [17]$.

We have $Nt_1 t_7 t_6 t_1 = Nt_3 t_{12}$

$= (t_1 t_7)^{(y^{-2})} \in [17]$

This means one t_i takes [176] to [17].

Choose 2 from $\{2\}$

$Nt_1 t_7 t_6 t_2 \in [17]$.

We have $Nt_1 t_7 t_6 t_2 = Nt_6 t_{11}$

$= (t_1 t_7)^{(y^2)} \in [17]$

This means one t_i takes [176] to [17].

Choose 3 from $\{3\}$

$Nt_1 t_7 t_6 t_3 \in [17]$.

We have $Nt_1 t_7 t_6 t_3 = Nt_8 t_4$

$= (t_1 t_7)^{((x*y^{-1}*x^{-1})^2)} \in [17]$

This means one t_i takes [176] to [17].

Choose 4 from $\{4\}$

$Nt_1 t_7 t_6 t_4 \in [17]$.

We have $Nt_1 t_7 t_6 t_4 = (x^3 * y^{-1}) t_{11} t_{21}$

$= (x^3 * y^{-1}) * (t_1 t_7)^{(x^2*y^{-1}*x^{-1})} \in [17]$

This means one t_i takes [176] to [17].

Choose 5 from $\{5\}$

$Nt_1 t_7 t_6 t_5 \in [17]$.

We have $Nt_1 t_7 t_6 t_5 = Nt_2 t_{19}$

$= (t_1 t_7)^{((x*y*x^{-1})^2)} \in [17]$

This means one t_i takes [176] to [17].

Choose 6 from $\{6\}$ $Nt_1 t_7 t_6 t_6 = Nt_1 t_7 t_6^2 = Nt_1 t_7 \in [17]$.

This means one t_i takes [176] to [17].

Choose 7 from {7}

$Nt_1t_7t_6t_7 \in [17]$.

We have $Nt_1t_7t_6t_7 = (x^3 * y^{-1})t_{15}t_{16}$

$= (x^3 * y^{-1}) * (t_1t_7)^{(x^2*y*x^{-1})} \in [17]$

This means one t_i takes [176] to [17].

Choose 8 from {8}

$Nt_1t_7t_6t_8 \in [17]$.

We have $Nt_1t_7t_6t_8 = Nt_{14}t_9$

$= (t_1t_7)^{((x^{-1}*y*x)^2)} \in [17]$

This means one t_i takes [176] to [17].

Choose 9 from {9}

$Nt_1t_7t_6t_9 \in [17]$.

We have $Nt_1t_7t_6t_9 = (x^3 * y^{-1})t_7t_{17}$

$= (x^3 * y^{-1}) * (t_1t_7)^{(y^{-1}*x)} \in [17]$

This means one t_i takes [176] to [17].

Choose 10 from {10}

$Nt_1t_7t_6t_{10} \in [17]$.

We have $Nt_1t_7t_6t_{10} = (x^{-2} * y * x^{-1})t_{20}t_2$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(y*x*y^{-1}*x)} \in [17]$

This means one t_i takes [176] to [17].

Choose 11 from {11}

$Nt_1t_7t_6t_{11} \in [17]$.

We have $Nt_1t_7t_6t_{11} = (x^3 * y^{-1})t_9t_{10}$

$= (x^3 * y^{-1}) * (t_1t_7)^{(y*x)} \in [17]$

This means one t_i takes [176] to [17].

Choose 12 from {12}

$Nt_1t_7t_6t_{12} \in [17]$.

We have $Nt_1t_7t_6t_{12} = (x^3 * y^{-1})t_{19}t_{20}$

$= (x^3 * y^{-1}) * (t_1t_7)^{(x^2*y*x^{-1}*y^{-2})} \in [17]$

This means one t_i takes [176] to [17].

Choose 13 from {13}

$Nt_1t_7t_6t_{13} \in [17]$.

We have $Nt_1t_7t_6t_{13} = (x^{-2} * y * x^{-1})t_{10}t_{14}$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(y^{-1}*x*y*x)} \in [17]$

This means one t_i takes [176] to [17].

Choose 14 from {14}

$Nt_1t_7t_6t_{14} \in [17]$.

We have $Nt_1t_7t_6t_{14} = Nt_5t_{15}$

$= (t_1t_7)^{((x^{-1}*y^{-1}*x)^2)} \in [17]$

This means one t_i takes [176] to [17].

Choose 15 from {15}

$Nt_1t_7t_6t_{15} \in [17]$.

We have $Nt_1t_7t_6t_{15} = (x^3 * y^{-1})t_{12}t_{13}$

$= (x^3 * y^{-1}) * (t_1t_7)^{(x^{-1}*y*x^{-1}*y^{-1})} \in [17]$

This means one t_i takes [176] to [17].

Choose 16 from {16}

$Nt_1t_7t_6t_{16} \in [17]$.

We have $Nt_1t_7t_6t_{16} = (x^{-2} * y * x^{-1})t_{21}t_6$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x*y^{-1}*x^{-2})} \in [17]$

This means one t_i takes [176] to [17].

Choose 17 from {17}

$Nt_1t_7t_6t_{17} \in [17]$.

We have $Nt_1t_7t_6t_{17} = (x^{-2} * y * x^{-1})t_{18}t_8$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x*y*x^{-2})} \in [17]$

This means one t_i takes [176] to [17].

Choose 18 from {18}

$Nt_1t_7t_6t_{18} \in [17]$.

We have $Nt_1t_7t_6t_{18} = (x^{-2} * y * x^{-1})t_{16}t_5$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x*y^2*x^{-2}*y^{-1})} \in [17]$

This means one t_i takes [176] to [17].

Choose 19 from {19}

$Nt_1t_7t_6t_{19} \in [17]$.

We have $Nt_1t_7t_6t_{19} = (x^3 * y^{-1})t_4t_{18}$

$= (x^3 * y^{-1}) * (t_1t_7)^{(y*x*y^2)} \in [17]$

This means one t_i takes [176] to [17].

Choose 20 from {20}

$Nt_1t_7t_6t_{20} \in [17]$.

We have $Nt_1t_7t_6t_{20} = (x^{-2} * y * x^{-1})t_{17}t_1$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x^{-1}*y)} \in [17]$

This means one t_i takes [176] to [17].

Choose 21 from {21}

$Nt_1t_7t_6t_{21} \in [17]$.

We have $Nt_1t_7t_6t_{21} = (x^{-2} * y * x^{-1})t_{13}t_3$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x^{-1}*y^{-1})} \in [17]$

This means one t_i takes [176] to [17].

Cayley Diagram
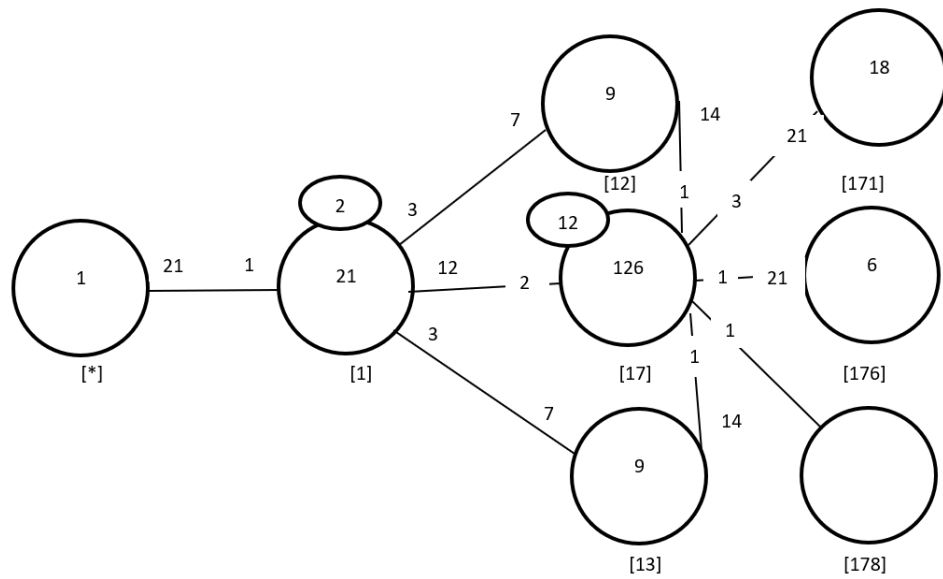
Figure 7.77: Cayley Diagram for G over S_{21}

Eightieth Double Coset [178]

$Nt_1t_7t_8 = \{N(t_1t_7t_8)^n | n \in N\}$.

We now find the coset stabilizer $N^{(178)}$. We first find the point stabilizer of 1,7 and 8 in

N.

$N^{178} = \{n \in N | (178)^n = 178\}$

$N^{178} = \langle e \rangle$

We will find different names of $Nt_1t_7t_8N$. $t_1t_7t_8 = (x^{-2} * y * x^{-1})t_{21}t_8t_{17} \implies Nt_1t_7t_8 = Nt_{21}t_8t_{17}$.

$N(t_1t_7t_8)^{(1,21,9)(2,20,11)(3,10,19)(4,14,16)(5,13,15)(6,18,12)(7,8,17)} = Nt_{21}t_8t_{17}$.

But, $Nt_{21}t_8t_{17} = Nt_1t_7t_8 \implies$

$N(t_1t_7t_8)^{(1,21,9)(2,20,11)(3,10,19)(4,14,16)(5,13,15)(6,18,12)(7,8,17)} = Nt_1t_7t_8$.

$\implies (1,21,9)(2,20,11)(3,10,19)(4,14,16)(5,13,15)(6,18,12)(7,8,17) \in Nt_1t_7t_8N$

Thus $N^{(178)} \geq \langle N^{178}, \rangle$.

$= e, (1,21,9)(2,20,11)(3,10,19)(4,14,16)(5,13,15)(6,18,12)(7,8,17)$

The number of distinct right cosets in $Nt_1t_7t_8N$ is calculated by the formula,

$\frac{|N|}{|N^{(1,7,8)}|} = \frac{126}{21} = 6$.

The orbits of $N^{(178)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\},$

$\{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

<u>Choose 1 from $\{1\}$</u>

$Nt_1t_7t_8t_1 \in [17]$.

We have $Nt_1t_7t_8t_1 = Nt_2t_{19}$

$= (t_1t_7)^{((x*y*x^{-1})^2)} \in [17]$

This means one t_i takes $[178]$ to $[17]$.

<u>Choose 2 from $\{2\}$</u>

$Nt_1t_7t_8t_2 \in [17]$.

We have $Nt_1t_7t_8t_2 = Nt_{14}t_9$

$= (t_1t_7)^{((x^{-1}*y*x)^2)} \in [17]$

This means one t_i takes $[178]$ to $[17]$.

<u>Choose 3 from $\{3\}$</u>

$Nt_1t_7t_8t_3 \in [17]$.

We have $Nt_1t_7t_8t_3 = Nt_6t_{11}$

$= (t_1 t_7)^{(y^2)} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 4 from $\{4\}$</u>

$N t_1 t_7 t_8 t_4 \in [17]$.

We have $N t_1 t_7 t_8 t_4 = (y * x * y * x^{-1}) t_{19} t_{18}$

$= (y * x * y * x^{-1}) * (t_1 t_7)^{(x*y*x*y^{-1})} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 5 from $\{5\}$</u>

$N t_1 t_7 t_8 t_5 \in [17]$.

We have $N t_1 t_7 t_8 t_5 = N t_8 t_4$

$= (t_1 t_7)^{((x*y^{-1}*x^{-1})^2)} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 6 from $\{6\}$</u>

$N t_1 t_7 t_8 t_6 \in [17]$.

We have $N t_1 t_7 t_8 t_6 = N t_5 t_{15}$

$= (t_1 t_7)^{((x^{-1}*y^{-1}*x)^2)} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 7 from $\{7\}$</u>

$N t_1 t_7 t_8 t_7 \in [17]$.

We have $N t_1 t_7 t_8 t_7 = (y * x * y * x^{-1}) t_9 t_{17}$

$= (y * x * y * x^{-1}) * (t_1 t_7)^{(y*x^{-1}*y^{-2})} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 8 from $\{8\}$</u>

$N t_1 t_7 t_8 t_8 = N t_1 t_7 t_8^2 = N t_1 t_7 \in [17]$.

This means one t_i takes [178] to [17].

<u>Choose 9 from $\{9\}$</u>

$N t_1 t_7 t_8 t_9 \in [17]$.

We have $N t_1 t_7 t_8 t_9 = (y * x * y * x^{-1}) t_{11} t_{10}$

$= (y * x * y * x^{-1}) * (t_1 t_7)^{(y^{-3}*x^{-1})} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 10 from $\{10\}$</u>

$N t_1 t_7 t_8 t_{10} \in [17]$.

We have $Nt_1t_7t_8t_{10} = (x^{-2} * y * x^{-1})t_{18}t_2$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x*y)} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 11 from $\{11\}$</u>

$Nt_1t_7t_8t_{11} \in [17]$.

We have $Nt_1t_7t_8t_{11} = (y * x * y * x^{-1})t_4t_{21}$

$= (y * x * y * x^{-1}) * (t_1t_7)^{(y*x^{-1})} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 12 from $\{12\}$</u>

$Nt_1t_7t_8t_{12} \in [17]$.

We have $Nt_1t_7t_8t_{12} = (y * x * y * x^{-1})t_{15}t_{13}$

$= (y * x * y * x^{-1}) * (t_1t_7)^{(y^{-1}*x^{-1})} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 13 from $\{13\}$</u>

$Nt_1t_7t_8t_{13} \in [17]$.

We have $Nt_1t_7t_8t_{13} = (x^{-2} * y * x^{-1})t_{17}t_{14}$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x^{-1}*y^{-1}*x^2)} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 14 from $\{14\}$</u>

$Nt_1t_7t_8t_{14} \in [17]$.

We have $Nt_1t_7t_8t_{14} = Nt_3t_{12}$

$= (t_1t_7)^{(y^{-2})} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 15 from $\{15\}$</u>

$Nt_1t_7t_8t_{15} \in [17]$.

We have $Nt_1t_7t_8t_{15} = (y * x * y * x^{-1})t_7t_{16}$

$= (y * x * y * x^{-1}) * (t_1t_7)^{(y^3*x^{-1})} \in [17]$

This means one t_i takes [178] to [17].

<u>Choose 16 from $\{16\}$</u>

$Nt_1t_7t_8t_{16} \in [17]$.

We have $Nt_1t_7t_8t_{16} = (x^{-2} * y * x^{-1})t_{10}t_6$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x*y^{-1})} \in [17]$

This means one t_i takes [178] to [17].

Choose 17 from {17}

$Nt_1t_7t_8t_{17} \in [17]$.

We have $Nt_1t_7t_8t_{17} = (x^{-2} * y * x^{-1})t_{21}t_8$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(y^2*x*y^{-1})} \in [17]$

This means one t_i takes [178] to [17].

Choose 18 from {18}

$Nt_1t_7t_8t_{18} \in [17]$.

We have $Nt_1t_7t_8t_{18} = (x^{-2} * y * x^{-1})t_{13}t_5$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x*y^3)} \in [17]$

This means one t_i takes [178] to [17].

Choose 19 from {19}

$Nt_1t_7t_8t_{19} \in [17]$.

We have $Nt_1t_7t_8t_{19} = (y * x * y * x^{-1})t_{12}t_{20}$

$= (y * x * y * x^{-1}) * (t_1t_7)^{(x*y^{-1}*x*y)} \in [17]$

This means one t_i takes [178] to [17].

Choose 20 from {20}

$Nt_1t_7t_8t_{20} \in [17]$.

We have $Nt_1t_7t_8t_{20} = (x^{-2} * y * x^{-1})t_{16}t_1$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(x*y^{-3})} \in [17]$

This means one t_i takes [178] to [17].

Choose 21 from {21}

$Nt_1t_7t_8t_{21} \in [17]$.

We have $Nt_1t_7t_8t_{21} = (x^{-2} * y * x^{-1})t_{20}t_3$

$= (x^{-2} * y * x^{-1}) * (t_1t_7)^{(y^2*x*y)} \in [17]$

This means one t_i takes [178] to [17].

Cayley Diagram
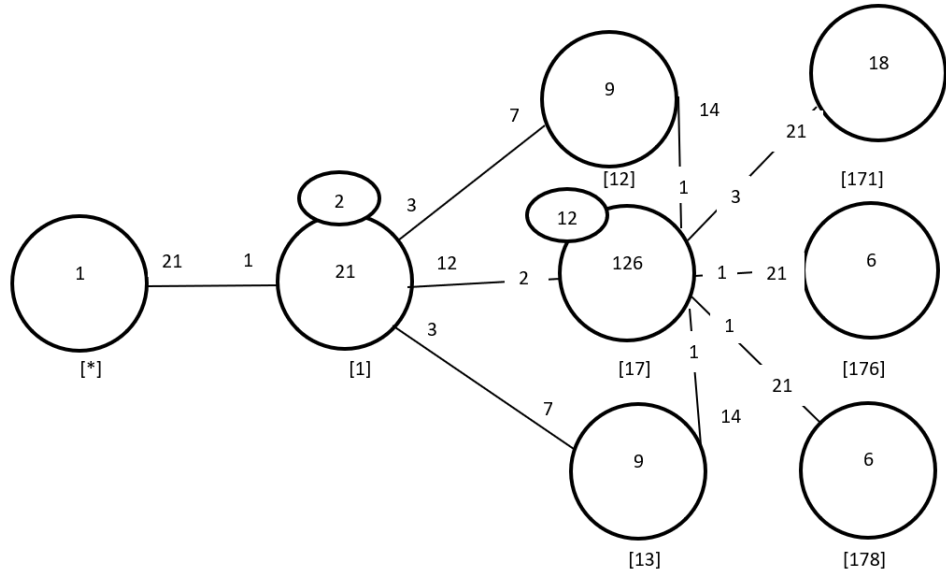
Figure 7.78: Cayley Diagram for G over S_{21}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases.

Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_2 \cup Nt_1t_3 \cup Nt_1t_7 \cup Nt_1t_7t_1 \cup Nt_1t_7t_6 \cup Nt_1t_7t_8$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(13)}|} + \frac{|N|}{|N^{(17)}|} + \frac{|N|}{|N^{(171)}|} + \frac{|N|}{|N^{(176)}|} + \frac{|N|}{|N^{(178)}|}) \times |N|$

$|G| \leq (1 + 21 + 9 + 9 + 126 + 18 + 6 + 6) \times 126$

$\implies |G| \leq 196 \times 126$

$\implies |G| \leq 24696.$

G acts on 196 cosets that are given in the cayley diagram.

Let X be the set of these 196 cosets.

Now, $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$

$\implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|$

But $\#\langle f(x), f(y), f(t) \rangle = 24696.$

So, $|\frac{G}{kerf}| = 24696$

This means $|G| \geq 24696$. We know $|G| \leq 24696$ from cayley diagram.

Therefore, $|G| = 24696$.

From $|G| = 24696 \times |Kerf|$ we find $|Kerf| = 1$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong 7^3 : S_3 : 2 \implies G \cong 7^3 : S_3 : 2$.

7.11 Construction of $2^6 : PSL(2,7)$ over $PSL(2,7)$

Let $N \cong PSL(2,7)$ where

$x \sim (1,2)(3,5)(4,7)(6,10)(8,13)(9,15)(11,18)(12,19)(14,22)(16,24)$

$(17,26)(20,29)(21,31)(23,33)(25,30)(27,36)(28,37)(32,40)(34,42)(35,43)(38,47)(39,48)$

$(41,51)(44,49)(45,55)(46,57)(50,61)(52,58)(53,64)(54,66)(56,65)(59,69)(60,71)$

$(62,70)(63,73)(68,76)(75,80)(78,82)(79,83)(81,84)$, and

$y \sim (1,3,6,11)(2,4,8,14)(5,9,16,25)(7,12,20,30)(10,17,13,21)(15,23,19,28)$

$(18,27,24,34)(22,32,29,38)(26,35,44,54)(31,39,49,60)(33,41,52,63)$

$(36,45,56,48)(37,46,58,68)(40,50,62,43)(42,53,65,57)(47,59,70,51)(55,67,71,78)(61,72,66,75)$

$(64,74,76,81)(69,77,73,79)(80,84)(82,83)$.

The progenitor $2^{*84} : PSL(2,7)$ has symmetric presentation

```
 G<x,y>:=Group<x,y,t| x^2, y^4,
 y^-2 * x * y^2 * x * y^2 * x,
 (x * y^-1)^7,
 t^2 >.
```

We want to verify that the presentation given above is correct.

Theorem: $\frac{2^{*n}:N}{t_i t_j} \cong 2^n : N$ where $1 \le i \le j \le n$.

We will show that $\frac{2^6 : PSL(2,7)}{((y)*t^{(y*x*y^-1*x*y*x*y^-1)})^3, ((y)*t^{(x*y*x*y^-1*x*y*x)})^3} \cong PSL(2,7)$.

Thus, we will prove

```
 G<x,y>:=Group<x,y,t| x^2, y^4, y^-2 * x * y^2 * x * y^2 * x,
 (x * y^-1)^7,
 t^2,
 (t,(x * y * x * y^-1)^2 ),
 (t,t^{(y^x)}), (t,t^{(x^2)}),(t,t^{(y*x^-1)}),
 (t,t^{(y^-3)}),(t,t^{(y*x)}),
 ((y)*t^(y * x * y^-1 * x * y * x * y^-1))^3,
 ((y)*t^(x * y * x * y^-1 * x * y * x))^3> ~ PSL(2,7).
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double cosets of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{10752}{168} = 64$.

Now we know that we have 64 distinct single cosets.

<u>Expanding First Relation</u> We will expand our first relation.

$$((y) * t^{(y*x*y^{-1}*x*y*x*y^{-1})})^3$$

$$=y * t_2 * y * t_2 * y * t_2$$
$$= y^3 * (t_2^{(y^2)}) * (t_2^{(y)}) * t_2$$
$$\implies y^3 * t_8 t_4 t_2 = e$$
$$\implies y^3 * t_8 = t_2 * t_4$$

<u>Expanding Second Relation</u>We will expand our second relation.

$$((y) * t^{(x*y*x*y^{-1}*x*y*x)})^3$$

$$=y * t_3 * y * t_3 * y * t_3$$
$$= y^3 * (t_3^{(y^2)}) * (t_3^{(y)}) * t_3$$
$$\implies y^3 * t_{11} * t_6 * t_3 = e$$
$$\implies y^3 * t_{11} = t_3 * t_6$$

First Double Coset [*]

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right coset in N is $\frac{|N|}{|N|} = \frac{168}{168} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,$

$12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$

$31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$

$51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,$

$71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84\}$.

We need only determine the double coset of the right coset Nt_1.

Thus 84 t_i's extend the double coset [*] to the double coset $Nt_1N = [1]$.

<u>Cayley Diagram</u>

Figure 7.79: Cayley Diagram for G over S_{84}

Second Double Coset [1]

$Nt_1N = \{Nt_1^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11},$

$Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21},$

$Nt_{22}, Nt_{23}, Nt_{24}, Nt_{25}, Nt_{26}, Nt_{27}, Nt_{28}, Nt_{29}, Nt_{30},$

$Nt_{31}, Nt_{32}, Nt_{33}, Nt_{34}, Nt_{35}, Nt_{36}, Nt_{37}, Nt_{38}, Nt_{39}, Nt_{40},$

$Nt_{41}, Nt_{42}, Nt_{43}, Nt_{44}, Nt_{45}, Nt_{46}, Nt_{47}, Nt_{48}, Nt_{49}, Nt_{50},$

$Nt_{51}, Nt_{52}, Nt_{53}, Nt_{54}, Nt_{55}, Nt_{56}, Nt_{57}, Nt_{58}, Nt_{59}, Nt_{60},$

$Nt_{61}, Nt_{62}, Nt_{63}, Nt_{64}, Nt_{65}, Nt_{66}, Nt_{67}, Nt_{68}, Nt_{69}, Nt_{70},$

$Nt_{71}, Nt_{72}, Nt_{73}, Nt_{74}, Nt_{75}, Nt_{76}, Nt_{77}, Nt_{78}, Nt_{79}, Nt_{80}, Nt_{81}, Nt_{82}, Nt_{83}, Nt_{84}\}.$

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

Thus, $N^{(1)} \geq N^1 = \langle(3, 65)(4, 70)(5, 56)(6, 84)(7, 62)(8, 83)(9, 82)(10, 81)(11, 35)(12, 80),$

$(13, 79)(14, 39)(15, 78)(16, 41)(17, 33)(18, 43)(19, 75)(20, 46)(21,$

$37)(22, 48)(23, 26)(24, 51)(27, 73)(28, 31)(29, 57)(32, 76)(34, 66)(36, 63)(38, 71)(40, 68),$

$(42, 54)(44, 49)(45, 53)(47, 60)(50, 59)(52, 58)(55, 64)(61, 69)(67, 72)(74, 77)\rangle.$

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{168}{84} = 2$. The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,$

$13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$

$31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$

$51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,$

$71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84\}$ are

$\{1\}, \{2\}, \{25\}, \{30\}, \{3, 65\}, \{4, 70\}, \{5, 56\}, \{6, 84\}, \{7, 62\},$

$\{8, 83\}, \{9, 82\}, \{10, 81\}, \{11, 35\}, \{12, 80\}, \{13, 79\}, \{14, 39\}, \{15, 78\},$

$\{16, 41\}, \{17, 33\}, \{18, 43\}, \{19, 75\},$

$\{20, 46\}, \{21, 37\}, \{22, 48\}, \{23, 26\}, \{24, 51\}, \{27, 73\},$

$\{28, 31\}, \{29, 57\}, \{32, 76\}, \{34, 66\}, \{36, 63\}, \{38, 71\}, \{40, 68\},$

$\{42, 54\}, \{44, 49\}, \{45, 53\}, \{47, 60\}, \{50, 59\}, \{52, 58\}, \{55, 64\}, \{61, 69\}, \{67, 72\}, \{74, 77\}.$

We will determine the double cosetes by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = N \in [*].$

This means one t_i takes [1] to [*].

Choose 2 from $\{2 \}$

$Nt_1t_2 \in [*].$

This means one t_i takes [1] to [*].

Choose 25 from $\{25\}$

$Nt_1t_{25} \in [1].$

We have $Nt_1t_{25} = Nt_{44}$

$= (t_1)^{(y^2*x*y*x*y^2)} \in [1]$

This means one t_i takes [1] to [1].

Choose 30 from $\{30\}$

$Nt_1t_{30} \in [1].$

We have $Nt_1t_{30} = Nt_{44}$

$= (t_1)^{(y^2*x*y*x*y^2)} \in [1]$

This means one t_i takes [1] to [1].

Choose 3 from $\{3,65\}$

$Nt_1t_3 \in [1].$

We have $Nt_1t_3 = (y^{-1})t_6$

$= (y^{-1})(t_1)^{(y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 4 from $\{4,70\}$

$Nt_1t_4 \in [1].$

We have $Nt_1t_4 = (y^{-1})t_6$

$= (y^{-1}) * (t_1)^{(y^2)} \in [1]$

This means two t_i's take [1] to [1].

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Choose 5 from {5,56}

$Nt_1t_5 \in [1]$.

We have $Nt_1t_5 = (x*y^{-1}*x)t_{10}$

$= (x*y^{-1}*x)*(t_1)^{(y^2*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 6 from {6,84}

$Nt_1t_6 \in [16]$.

This means two t_i's take [1] to [16].

Choose 7 from {7,62}

$Nt_1t_7 \in [1]$.

We have $Nt_1t_7 = (x*y^{-1}*x)t_{10}$

$= (x*y^{-1}*x)*(t_1)^{(y^2*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 8 from {8,83}

$Nt_1t_8 \in [16]$.

We have $Nt_1t_8 = Nt_1t_6$.

This means two t_i's take [1] to [16].

Choose 9 from {9,82}

$Nt_1t_9 \in [19]$.

This means two t_i's take [1] to [19].

Choose 10 from {10,81}

$Nt_1t_{10} \in [16]$.

We have $Nt_1t_{10} = Nt_{10}t_2$

$= (t_1t_6)^{(y^2*x)} \in [16]$

This means two t_i's take [1] to [16].

Choose 11 from {11,35}

$Nt_1t_{11} \in [1]$.

We have $Nt_1t_{11} = y*t_6 = y*(t_1)^{(y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 12 from {12,80}

$Nt_1t_{12} \in [19]$.

We have $Nt_1t_{12} = Nt_1t_9$.

This means two t_i's take [1] to [19].

Choose 13 from {13,79}

$Nt_1t_{13} \in [16]$.

We have $Nt_1t_{13} = Nt_{10}t_2$

$= (t_1t_6)^{(y^2*x)} \in [16]$

This means two t_i's take [1] to [16].

Choose 14 from {14,39}

$Nt_1t_{14} \in [1]$.

We have $Nt_1t_{14} = yt_6$

$= y(t_1)^{(y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 15 from {15,78}

$Nt_1t_{15} \in [19]$.

We have $Nt_1t_{15} = Nt_2t_{15}$

$= (t_1t_9)^x \in [19]$

This means two t_i's take [1] to [19].

Choose 16 from {16,41}

$Nt_1t_{16} \in [19]$.

We have $Nt_1t_{16} = (y^{-1} * x * y^2)t_{32}t_{31}$

$= (y^{-1} * x * y^2) * (t_1t_9)^{(y*x*y^{-1}*x*y^{-1}*x*y^{-1})} \in [19]$

This means two t_i's take [1] to [19].

Choose 17 from {17,33}

$Nt_1t_{17} \in [19]$.

We have $Nt_1t_{17} = Nt_1t_{82}$

$= (t_1t_9)^{((x*y*x*y^{-1})^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 18 from {18,43}

$Nt_1t_{18} \in [1]$.

We have $Nt_1t_{18} = (x * y * x)t_{10}$

$= (x * y * x) * (t_1)^{(y^2*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 19 from {19,75}

$Nt_1t_{19} \in [19]$.

We have $Nt_1t_{19} = Nt_2t_{15}$

$= (t_1t_9)^x \in [19]$

This means two t_i's take [1] to [19].

Choose 20 from {20,46}

$Nt_1t_{20} \in [19]$.

We have $Nt_1t_{20} = (y^{-1} * x * y^2)t_{32}t_{31}$

$= (y^{-1} * x * y^2) * (t_1t_9)^{(y*x*y^{-1}*x*y^{-1}*x*y^{-1})} \in [19]$

This means two t_i's take [1] to [19].

Choose 21 from {21,37}

$Nt_1t_{21} \in [19]$.

We have $Nt_1t_{21} = Nt_1t_{82}$

$= (t_1t_9)^{((x*y*x*y^{-1})^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 22 from {22,48}

$Nt_1t_{22} \in [1]$.

We have $Nt_1t_{22} = (x * y * x)t_{10}$

$= (x * y * x) * (t_1)^{(y^2*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 23 from {23,26}

$Nt_1t_{23} \in [19]$.

We have $Nt_1t_{23} = Nt_2t_{15}$

$= (t_1t_9)^x \in [19]$

This means two t_i's take [1] to [19].

Choose 24 from {24,51}

$Nt_1t_{24} \in [19]$.

We have $Nt_1t_{24} = (y^{-1} * x)^3)t_7t_{33}$

$= (y^{-1} * x)^3) * (t_1t_9)^{(x*y*x)} \in [19]$

This means two t_i's take [1] to [19].

Choose 27 from {27,73}

$Nt_1t_{27} \in [1]$.

We have $Nt_1t_{27} = (y^2 * x * y * x * y)t_{10}$

$= (y^2 * x * y * x * y) * (t_1)^{(y^2 * x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 28 from {28,31}

$Nt_1 t_{28} \in [19].$

We have $Nt_1 t_{28} = Nt_2 t_{15}$

$= (t_1 t_9)^x \in [19]$

This means two t_i's take [1] to [19].

Choose 29 from {29,57}

$Nt_1 t_{29} \in [19].$

We have $Nt_1 t_{29} = (x * y * x * y^2) t_{40} t_{21}$

$= (x * y * x * y^2) * (t_1 t_9)^{(y*x*y^{-1}*x*y^{-1}*x*y^{-1}*x)} \in [19]$

This means two t_i's take [1] to [19].

Choose 32 from {32,76}

$Nt_1 t_{32} \in [1].$

We have $Nt_1 t_{32} = (y^2 * x * y * x * y) t_{10}$

$= (y^2 * x * y * x * y) * (t_1)^{(y^2 * x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 34 from {34,66}

$Nt_1 t_{34} \in [19].$

We have $Nt_1 t_{34} = x t_{60} t_{49}$

$= x(t_1 t_9)^{(y^{-1}*x*y*x*y^{-1}*x*y^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 36 from {36,63}

$Nt_1 t_{36} \in [1].$

We have $Nt_1 t_{36} = (y^2 * x * y^{-1} * x * y * x) t_6$

$= (y^2 * x * y^{-1} * x * y * x) * (t_1)^{(y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 38 from {38,71}

$Nt_1 t_{38} \in [19].$

We have $Nt_1 t_{38} = x t_{60} t_{49}$

$= x(t_1 t_9)^{(y^{-1}*x*y*x*y^{-1}*x*y^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 40 from {40,68}

$Nt_1t_{40} \in [1]$.

We have $Nt_1t_{40} = (y^2 * x * y^{-1} * x * y * x)t_6$

$= (y^2 * x * y^{-1} * x * y * x) * (t_1)^{(y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 42 from {42,54}

$Nt_1t_{42} \in [19]$.

We have $Nt_1t_{42} = xt_{60}t_{49}$

$= x(t_1t_9)^{(y^{-1}*x*y*x*y^{-1}*x*y^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 44 from {44,49}

$Nt_1t_{44} \in [1]$.

We have $Nt_1t_{44} = Nt_{25}$

$= (t_1)^{(y*x*y^{-1})} \in [1]$

This means two t_i's take [1] to [1].

Choose 45 from {45,53}

$Nt_1t_{45} \in [1]$.

We have $Nt_1t_{45} = (x * y^{-1} * x * y * x * y^2)t_6$

$= (x * y^{-1} * x * y * x * y^2) * (t_1)^{(y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 47 from {47,60}

$Nt_1t_{47} \in [19]$.

We have $Nt_1t_{47} = xt_{60}t_{49}$

$= x(t_1t_9)^{(y^{-1}*x*y*x*y^{-1}*x*y^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 50 from {50,59}

$Nt_1t_{50} \in [1]$.

We have $Nt_1t_{50} = (x * y^{-1} * x * y * x * y^2)t_6$

$= (x * y^{-1} * x * y * x * y^2) * (t_1)^{(y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 52 from {52,58}

$Nt_1t_{52} \in [1]$.

We have $Nt_1t_{52} = Nt_{44}$

$= (t_1)^{(y^2*x*y*x*y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 55 from {55,64}

$Nt_1t_{55} \in [1]$.

We have $Nt_1t_{55} = (y^{-1}*x*y^{-1}*x*y^2)t_{10}$

$= (y^{-1}*x*y^{-1}*x*y^2)*(t_1)^{(y^2*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 61 from {61,69}

$Nt_1t_{61} \in [1]$.

We have $Nt_1t_{61} = (y^{-1}*x*y^{-1}*x*y^2)t_{10}$

$= (y^{-1}*x*y^{-1}*x*y^2)*(t_1)^{(y^2*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 67 from {67,72}

$Nt_1t_{67} \in [1]$.

We have $Nt_1t_{67} = Nt_{25}$

$= (t_1)^{(y*x*y^{-1})} \in [1]$

This means two t_i's take [1] to [1].

Choose 74 from {74,77}

$Nt_1t_{74} \in [*]$.

This means two t_i's take [1] to [*].

Cayley Diagram
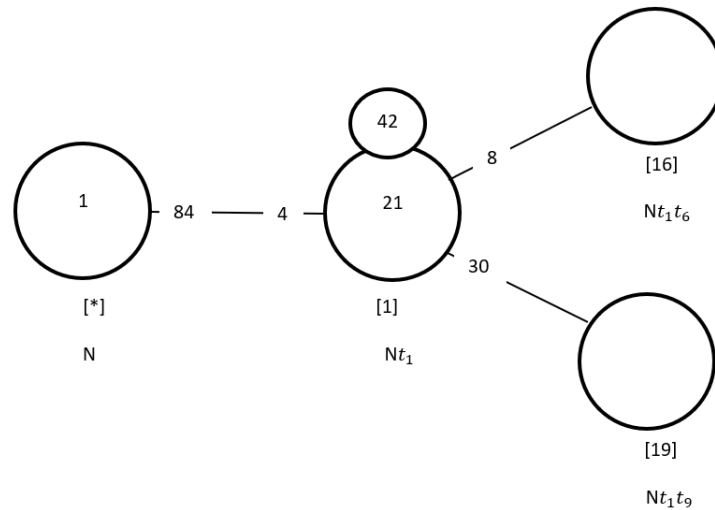
Figure 7.80: Cayley Diagram for G over S_{84}

Third Double Coset [16]

$Nt_1t_6 = \{N(t_1t_6)^n | n \in N\}$.

We now find the coset stabilizer $N^{(16)}$. We first find the point stabilizer of 1 and 6 in N.

$N^{16} = \{n \in N | (16)^n = 16\}$

$N^{16} = \langle e \rangle$

We will find different names of Nt_1t_6N. $Nt_1t_6 = Nt_{81}t_2$.

$N(t_1t_6)^{(1,81,77,8),(2,79,74,6)} = Nt_{81}t_2$.

But, $Nt_{81}t_2 = Nt_1t_6 \implies$

$N(t_1t_6)^{(1,81,77,8),(2,79,74,6)} = Nt_1t_6$.

$\implies (1,81,77,8)(2,79,74,6)(3,43,69,36)(4,48,64,40)(5,51,61,42)(7,57,55,47)$

$(9,30,75,67)(10,83,13,84)(11,56,73,50)(12,25,78,72)(14,62,76,45)(15,80,19,82)$

$(16,65,66,59)(17,28)(18,41,27,54)(20,70,71,53)(21,23)(22,46,32,60)$

$(24,35,34,63)(26,44,33,52)(29,39,38,68)(31,49,37,58) \in Nt_1t_6N$

Thus $N^{(16)} \geq \langle N^{16}, (1,81,77,8)(2,79,74,6)(3,43,69,36)(4,48,64,40)$

$(5,51,61,42)(7,57,55,47)(9,30,75,67)(10,83,13,84)$

$(11, 56, 73, 50)(12, 25, 78, 72)(14, 62, 76, 45)(15, 80, 19, 82)(16, 65, 66, 59)(17, 28)$

$(18, 41, 27, 54)(20, 70, 71, 53)(21, 23)(22, 46, 32, 60)$

$(24, 35, 34, 63)(26, 44, 33, 52)(29, 39, 38, 68)(31, 49, 37, 58)\rangle.$

$= \langle e, (1, 81, 77, 8)(2, 79, 74, 6)(3, 43, 69, 36)(4, 48, 64, 40)(5, 51, 61, 42)(7, 57, 55, 47)$

$(9, 30, 75, 67)(10, 83, 13, 84)(11, 56, 73, 50)(12, 25, 78, 72)(14, 62, 76, 45)(15, 80, 19, 82)$

$(16, 65, 66, 59)(17, 28)(18, 41, 27, 54)(20, 70, 71, 53)(21, 23)(22, 46, 32, 60)$

$(24, 35, 34, 63)(26, 44, 33, 52)(29, 39, 38, 68)(31, 49, 37, 58)\rangle$

The number of distinct right cosets in Nt_1t_6N is calculated by the formula,

$\frac{|N|}{|N^{(1,6)}|} = \frac{168}{24} = 7$.

The orbits of $N^{(16)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,$

$13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$

$31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$

$51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,$

$71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84\}$ are

$\{10, 82, 13, 17, 28, 80, 23, 83, 21, 19, 15, 84\}, \{18, 33, 27, 52, 41, 26, 63, 44, 24, 54, 35, 34\},$

$\{22, 37, 32, 58, 46, 31, 68, 49, 29, 60, 39, 38\},$

$\{1, 76, 77, 64, 3, 36, 14, 56, 81, 4, 73, 43, 50, 74, 62, 69, 40, 6, 45, 2, 79, 48, 8, 11\},$

$\{5, 70, 61, 47, 9, 67, 53, 78, 51, 57, 72, 30, 12, 59, 20, 75, 7, 16, 71, 65, 66, 55, 42, 25\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1, 76, 77, 64, 3, 36, 14, 56, 81, 4, 73,$

$43, 50, 74, 62, 69, 40, 6, 45, 2, 79, 48, 8, 11\}$

$Nt_1t_6t_1 \in [1]$.

We have $Nt_1t_6t_1 = Nt_6$

$= (t_1)^{(y^2)} \in [1]$

This means twenty t_i's take $[16]$ to $[1]$.

Choose 5 from $\{5, 70, 61, 47, 9, 67, 53, 78, 51, 57, 72, 30, 12, 59, 20,$

$75, 7, 16, 71, 65, 66, 55, 42, 25\}$

$Nt_1t_6t_5 \in [19]$.

We have $Nt_1t_6t_5 = (y * x * y * x * y^{-1} * x * y^{-1})t_1t_9 \in [19]$

This means twenty t_i's take $[16]$ to $[19]$.

Choose 10 from $\{10, 82, 13, 17, 28, 80, 23, 83, 21, 19, 15, 84\}$

$Nt_1t_6t_{10} \in [191]$.

We have $Nt_1t_6t_{10} = (y * x * y^{-1} * x * y * x * y^{-1})t_{20}t_{52}t_{20}$

$= (y * x * y^{-1} * x * y * x * y^{-1}) * (t_1t_9t_1)^{(x*y*x*y^2)} \in [191]$

This means twelve t_i's take [16] to [191].

Choose 18 from $\{18, 33, 27, 52, 41, 26, 63, 44, 24, 54, 35, 34\}$

$Nt_1t_6t_{18} \in [19]$.

We have $Nt_1t_6t_{18} = (x * y * x * y^{-1} * x * y^{-1} * x * y * x * y)t_1t_9 \in [19]$

This means twelve t_i's take [16] to [19].

Choose 22 from $\{22, 37, 32, 58, 46, 31, 68, 49, 29, 60, 39, 38\}$

$Nt_1t_6t_{22} \in [19]$.

We have $Nt_1t_6t_{22} = (y^2 * x)t_{14}t_{71}$

$= (y^2 * x) * (t_1t_9)^{(y*x*y^{-1}*x*y*x*y^2)} \in [19]$

This means twelve t_i's take [16] to [19].

Cayley Diagram
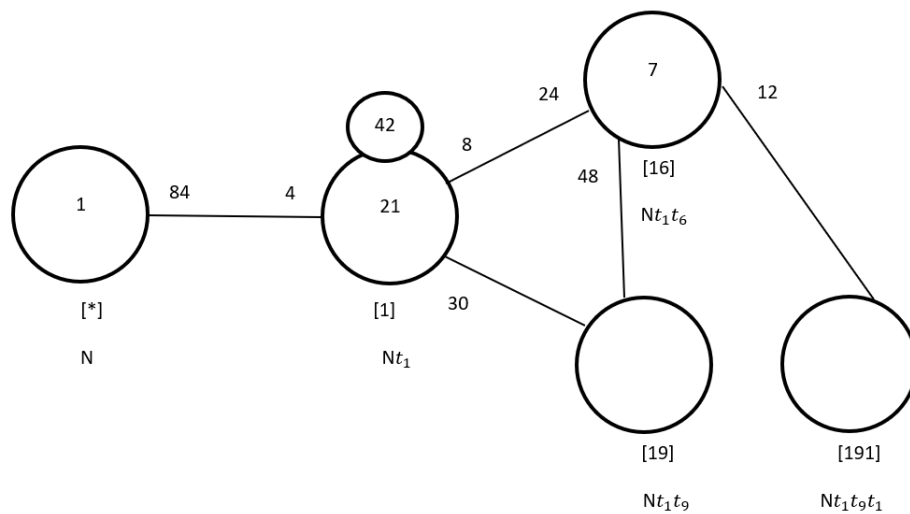
Figure 7.81: Cayley Diagram for G over S_{84}

Fourth Double Coset [19]

$Nt_1t_9 = \{N(t_1t_9)^n | n \in N\}$.

We now find the coset stabilizer $N^{(19)}$. We first find the point stabilizer of 1 and 9 in N.

$N^{19} = \{n \in N | (19)^n = 19\}$

$N^{19} = \langle e \rangle$

We will find different names of Nt_1t_9N. $Nt_1t_9 = Nt_{74}t_{12}$.

$N(t_1t_9)^{(1,74),(9,12)} = Nt_{74}t_{12}$.

But, $Nt_{74}t_{12} = Nt_1t_9 \implies$

$N(t_1t_9)^{(1,74),(9,12)} = Nt_1t_9$.

$\implies (1,74)(2,77)(3,50)(4,45)(5,59)(6,8)(7,53)(9,12)(10,83)(11,36)$

$(13,84)(14,40)(15,80)(16,42)(17,23)(18,63)(19,82)(20,47)(21,28)(22,68)(24,54)$

$(25,67)(27,35)(29,60)(30,72)(32,39)(34,41)(38,46)(43,73)(44,52)$

$(48,76)(49,58)(51,66)(55,70)(56,69)(57,71)(61,65)(62,64)(75,78)(79,81) \in Nt_1t_9N$

Thus $N^{(19)} \geq \langle N^{19}, (1,74)(2,77)(3,50)(4,45)(5,59)(6,8)$

$(7,53)(9,12)(10,83)(11,36)(13,84)(14,40)(15,80)(16,42)$

$(17,23)(18,63)(19,82)(20,47)(21,28)(22,68)(24,54)(25,67)(27,35)$

$(29,60)(30,72)(32,39)(34,41)(38,46)(43,73)(44,52)(48,76)(49,58)$

$(51,66)(55,70)(56,69)(57,71)(61,65)(62,64)(75,78)(79,81)\rangle$.

$= \langle e, (1,74)(2,77)(3,50)(4,45)(5,59)(6,8)(7,53)(9,12)(10,83)(11,36)$

$(13,84)(14,40)(15,80)(16,42)(17,23)(18,63)(19,82)(20,47)(21,28)(22,68)(24,54)(25,67)$

$(27,35)(29,60)(30,72)(32,39)(34,41)(38,46)(43,73)(44,52)(48,76)$

$(49,58)(51,66)(55,70)(56,69)(57,71)(61,65)(62,64)(75,78)(79,81)\rangle$

The number of distinct right cosets in Nt_1t_9N is calculated by the formula,

$\frac{|N|}{|N^{(1,9)}|} = \frac{168}{6} = 28$.

The orbits of $N^{(19)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,$

$16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35,$

$36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55,$

$56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75,$

$76, 77, 78, 79, 80, 81, 82, 83, 84\}$ are

$\{1, 69, 74, 56, 14, 40\}, \{2, 64, 77, 62, 11, 36\},$

$\{3, 79, 50, 81, 45, 4\}, \{5, 75, 59, 78, 53, 7\},$

$\{6, 73, 8, 43, 76, 48\}, \{9, 66, 12, 51, 71, 57\},$

$\{10, 17, 83, 23, 80, 15\}, \{13, 21, 84, 28, 82, 19\},$

$\{16, 72, 42, 30, 70, 55\}, \{18, 63, 26\},$

$\{20, 67, 47, 25, 65, 61], [22, 68, 31\}, \{24, 54, 33\},$

$\{27, 34, 35, 41, 52, 44\}, \{29, 60, 37\}, \{32, 38, 39, 46, 58, 49\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1, 69, 74, 56, 14, 40\}$

$Nt_1t_9t_1 \in [191].$

This means six t_i's take [19] to [191].

Choose 2 from $\{2, 64, 77, 62, 11, 36\}$

$Nt_1t_9t_2 \in [191].$

We have $Nt_1t_9t_2 = Nt_1t_9t_1.$

This means six t_i's take [19] to [191].

Choose 3 from $\{3, 79, 50, 81, 45, 4\}$

$Nt_1t_9t_3 \in [19].$

We have $Nt_1t_9t_3 = (y * x * y * x * y^{-1} * x * y)t_{11}t_5$

$= (y * x * y * x * y^{-1} * x * y) * (t_1t_9)^{(y^{-1})} \in [19]$

This means six t_i's take [19] to [19].

Choose 5 from $\{5, 75, 59, 78, 53, 7\}$

$Nt_1t_9t_5 \in [16].$

We have $Nt_1t_9t_5 = (y * x * y * x * y^{-1} * x * y)t_3t_{11}$

$= (y * x * y * x * y^{-1} * x * y) * (t_1t_6)^y \in [16]$

This means six t_i's take [19] to [16].

Choose 6 from $\{6, 73, 8, 43, 76, 48\}$

$Nt_1t_9t_6 \in [19].$

We have $Nt_1t_9t_6 = (y^2 * x * y * x * y^{-1} * x)t_3t_{16}$

$= (y^2 * x * y * x * y^{-1} * x) * (t_1t_9)^y \in [19]$

This means six t_i's take [19] to [19].

Choose 9 from $\{9, 66, 12, 51, 71, 57\}$

$Nt_1t_9t_9 = Nt_1t_9^2 = Nt_1 \in [1].$

This means six t_i's take [19] to [1].

Choose 10 from $\{10, 17, 83, 23, 80, 15\}$

$Nt_1t_9t_{10} \in [1]$.

We have $Nt_1t_9t_{10} = (y^{-1} * x * y^{-1} * x * y * x * y)t_{26}$

$= (y^{-1} * x * y^{-1} * x * y * x * y) * (t_1)^{(y^2*x*y*x)} \in [1]$

This means six t_i's take [19] to [1].

Choose 13 from $\{13, 21, 84, 28, 82, 19\}$

$Nt_1t_9t_{13} \in [1]$.

We have $Nt_1t_9t_{13} = (y^{-1} * x * y^{-1} * x * y * x * y)t_{26}$

$= (y^{-1} * x * y^{-1} * x * y * x * y) * (t_1)^{(y^2*x*y*x)} \in [1]$

This means six t_i's take [19] to [1].

Choose 16 from $\{16, 72, 42, 30, 70, 55 \}$

$Nt_1t_9t_{16} \in [19]$.

We have $Nt_1t_9t_{16} = (y^2 * x * y * x * y^{-1} * x)t_{25}t_{73}$

$= (y^2 * x * y * x * y^{-1} * x) * (t_1t_9)^{(x*y*x*y^{-1}*x)} \in [19]$

This means six t_i's take [19] to [19].

Choose 18 from $\{18, 63, 26\}$

$Nt_1t_9t_{18} \in [16]$.

We have $Nt_1t_9t_{18} = (y * x * y * x * y^{-1} * x * y * x)t_3t_{11}$

$= (y * x * y * x * y^{-1} * x * y * x) * (t_1t_6)^y \in [16]$

This means six t_i's take [19] to [16].

Choose 20 from $\{20, 67, 47, 25, 65, 61\}$

$Nt_1t_9t_{20} \in [19]$.

We have $Nt_1t_9t_{20} = (y^2*x*y*x*y^{-1}*x)t_{25}t_{73} = (y^2*x*y*x*y^{-1}*x)*(t_1t_9)^{(x*y*x*y^{-1}*x)} \in [19]$

This means six t_i's take [19] to [19].

Choose 22 from $\{22, 68, 31\}$

$Nt_1t_9t_{22} \in [16]$.

We have $Nt_1t_9t_{22} = (y * x * y * x * y^{-1} * x * y * x)t_3t_{11}$

$= (y * x * y * x * y^{-1} * x * y * x) * (t_1t_6)^y \in [16]$

This means three t_i's take [19] to [16].

Choose 24 from $\{24, 54, 33\}$

$Nt_1t_9t_{24} \in [1]$.

We have $Nt_1t_9t_{24} = (y * x * y^{-1} * x * y^{-1} * x * y * x)t_{36}$

$$= (y * x * y^{-1} * x * y^{-1} * x * y * x) * (t_1)^{(y^{-1} * x * y * x)} \in [1]$$

This means three t_i's take [19] to [1].

Choose 27 from $\{27, 34, 35, 41, 52, 44\}$

$Nt_1 t_9 t_{27} \in [19]$.

We have $Nt_1 t_9 t_{27} = (y^{-1} * x * y * x * y) t_{20} t_{48}$

$= (y^{-1} * x * y * x * y) * (t_1 t_9)^{(y * x * y^{-1} * x * y^{-1})} \in [19]$ This means six t_i's take [19] to [19].

Choose 29 from $\{29, 60, 37\}$

$Nt_1 t_9 t_{29} \in [1]$.

We have $Nt_1 t_9 t_{29} = (y * x * y^{-1} * x * y^{-1} * x * y * x) t_{36}$

$= (y * x * y^{-1} * x * y^{-1} * x * y * x) * (t_1)^{(y^{-1} * x * y * x)} \in [1]$

This means three t_i's take [19] to [1].

Choose 32 from $\{32, 38, 39, 46, 58, 49\}$

$Nt_1 t_9 t_{32} \in [19]$.

We have $Nt_1 t_9 t_{32} = (y^{-1} * x * y * x * y) t_{20} t_{48}$

$= (y^{-1} * x * y * x * y) * (t_1 t_9)^{(y * x * y^{-1} * x * y^{-1})} \in [19]$

This means six t_i's take [19] to [19].
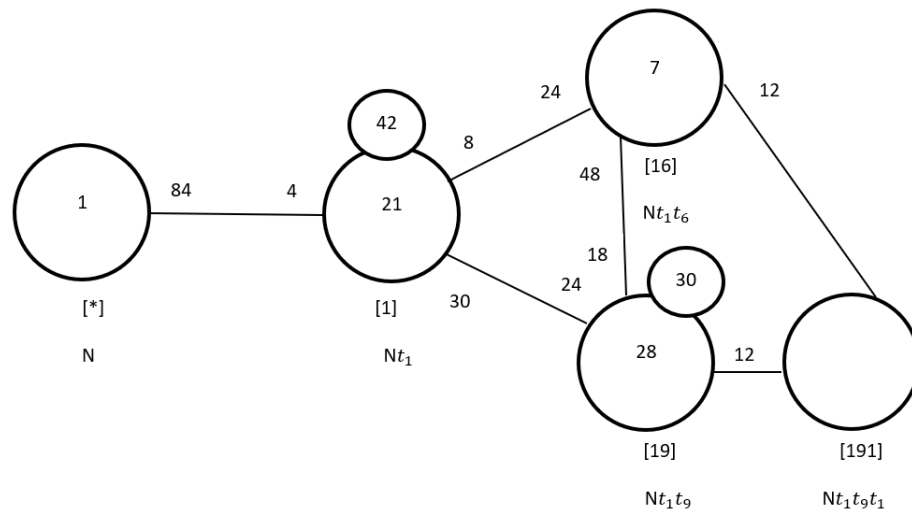
Cayley Diagram

Figure 7.82: Cayley Diagram for G over S_{84}

Fifth Double Coset [191]

$Nt_1t_9t_1 = \{N(t_1t_9t_1)^n | n \in N\}$.

We now find the coset stabilizer $N^{(191)}$. We first find the point stabilizer of 1,9 and 1 in N.

$N^{191} = \{n \in N | (191)^n = 191\}$

$N^{19} = \langle e \rangle$

We will find different names of $Nt_1t_9t_1N$. $Nt_1t_9t_1 = Nt_{13}t_{12}t_{13}$.

$N(t_1t_9t_1)^{(1,13),(9,12)} = Nt_{13}t_{12}t_{13}$.

But, $Nt_{13}t_{12}t_{13} = Nt_1t_9t_1 \implies$

$N(t_1t_9t_1)^{(1,13),(9,12)} = Nt_1t_9t_1$.

$\implies (1,13)(2,10)(3,24)(4,29)(5,18)(6,8)(7,22)(9,12)(11,16)(14,20)(15,30)(17,49)$
$(19,25)(21,44)(23,58)(26,31)(27,65)(28,52)(32,70)(33,37)(34,56)(35,61)(36,42)$
$(38,62)(39,55)(40,47)(41,69)(43,66)(45,60)(46,64)(48,71)(50,54)(51,73)(53,68)$
$(57,76)(59,63)(67,82)(72,80)(74,84)(77,83) \in Nt_1t_9t_1N$

Thus $N^{(191)} \geq \langle N^{191}, (1,13)(2,10)(3,24)(4,29)(5,18)(6,8)(7,22)(9,12)(11,16)(14,20)(15,30)$
$(17,49)(19,25)(21,44)(23,58)(26,31)(27,65)(28,52)(32,70)(33,37)(34,56)(35,61)(36,42)$
$(38,62)(39,55)(40,47)(41,69)(43,66)(45,60)(46,64)(48,71)(50,54)(51,73)(53,68)(57,76)$
$(59,63)(67,82)(72,80)(74,84)(77,83)\rangle$.

$= \langle e, (1,13)(2,10)(3,24)(4,29)(5,18)(6,8)(7,22)(9,12)(11,16)(14,20)(15,30)(17,49)(19,25)$
$(21,44)(23,58)(26,31)(27,65)(28,52)(32,70)(33,37)(34,56)(35,61)(36,42)(38,62)(39,55)$
$(40,47)(41,69)(43,66)(45,60)(46,64)(48,71)(50,54)(51,73)(53,68)(57,76)(59,63)$
$(67,82)(72,80)(74,84)(77,83)\rangle$

The number of distinct right cosets in $Nt_1t_9t_1N$ is calculated by the formula,

$\frac{|N|}{|N^{(1,9,1)}|} = \frac{168}{24} = 7$.

The orbits of $N^{(191)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,$
$13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,$
$31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,$
$51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,$
$71,72,73,74,75,76,77,78,79,80,81,82,83,84\}$ are

$\{3,24,33,45,37,60,29,54,81,4,50,79\}$,

$\{5,18,26,53,31,68,22,63,78,7,59,75\}$,

$\{6,8,73,76,51,9,57,12,48,66,43,71\}$,

$\{1, 13, 82, 14, 67, 61, 20, 65, 28, 35, 34, 56, 27, 52, 41, 47, 44, 25, 69, 40, 74, 21, 19, 84\},$

$\{2, 10, 80, 11, 72, 55, 16, 70, 23, 39, 38, 62, 32, 58, 46, 42, 49, 30, 64, 36, 77, 17, 15, 83\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1, 13, 82, 14, 67, 61, 20, 65, 28, 35, 34, 56, 27,$
$52, 41, 47, 44, 25, 69, 40, 74, 21, 19, 84\}$

$Nt_1t_9t_1t_1 = Nt_1t_9t_1^2 = Nt_1t_9 \in [19].$

This means twenty t_i's take [191] to [19].

Choose 2 from $\{2, 10, 80, 11, 72, 55, 16, 70, 23, 39, 38, 62, 32, 58, 46, 42, 49,$
$30, 64, 36, 77, 17, 15, 83\}$

$Nt_1t_9t_1t_2 \in [19].$

We have $Nt_1t_9t_1t_2 = Nt_1t_9.$

This means twenty t_i's take [191] to [19].

Choose 3 from $\{3, 24, 33, 45, 37, 60, 29, 54, 81, 4, 50, 79\}$

$Nt_1t_9t_1t_3 \in [191].$

We have $Nt_1t_9t_1t_3 = (y^2 * x * y * x * y^{-1} * x * y^2)t_{11}t_5t_{11}$
$= (y^2 * x * y * x * y^{-1} * x * y^2) * (t_1t_9t_1)^{(y^{-1})} \in [191]$

This means twelve t_i's take [191] to [191].

Choose 5 from $\{5, 18, 26, 53, 31, 68, 22, 63, 78, 7, 59, 75\}$

$Nt_1t_9t_1t_5 \in [16].$

We have $Nt_1t_9t_1t_5 = (y * x * y^{-1} * x * y^{-1})t_{14}t_{21}$
$= (y * x * y^{-1} * x * y^{-1}) * (t_1t_6)^{(x*y^{-1})} \in [16]$

This means twelve t_i's take [191] to [16].

Choose 6 from $\{6, 8, 73, 76, 51, 9, 57, 12, 48, 66, 43, 71\}$

$Nt_1t_9t_1t_6 \in [191].$

We have $Nt_1t_9t_1t_6 = Nt_2t_{78}t_2$
$= (t_1t_9t_1)^{(y*x*y^{-1}*x*y*x*y^{-1})} \in [191]$

This means twelve t_i's take [191] to [191].

Cayley Diagram
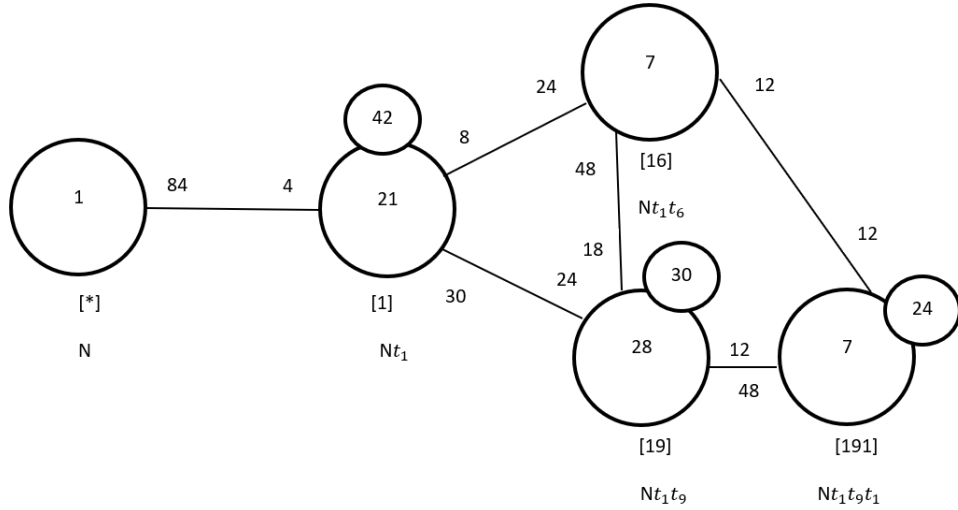
Figure 7.83: Cayley Diagram for G over S_{84}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases.

Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_6 \cup Nt_1t_9 \cup Nt_1t_9t_1$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(16)}|} + \frac{|N|}{|N^{(19)}|} + \frac{|N|}{|N^{(191)}|} \times |N|$

$|G| \leq (1 + 21 + 7 + 28 + 7) \times 168 \implies |G| \leq 64 \times 168 \implies |G| \leq 10752.$

G acts on 168 cosets that are given in the cayley diagram.

Let X be the set of these 168 cosets.

Now, $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle \implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|$

But $\#\langle f(x), f(y), f(t) \rangle = 10752$. So, $|\frac{G}{kerf}| = 10752$

This means $|G| \geq 10752$. We know $|G| \leq 10752$ from cayley diagram.

Therefore, $|G| = 10752$.

From $|G| = 10752 \times |Kerf|$ we find $|Kerf| = 1$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong 2^6 : PSL(2,7) \implies G \cong 2^6 : PSL(2,7).$

Construction of $2^7 : PSL(2,7)$ over $PSL(2,7)$

Let $N \cong PSL(2,7)$ where $x \sim (1,2)(3,5)(4,7)(6,10)(8,13)(9,15)(11,18)$
$(12,19)(14,22)(16,24)(17,26)(20,29)(21,31)(23,33)(25,30)(27,36)(28,37)(32,40)$
$(34,42)(35,43)(38,47)(39,48)(41,51)(44,49)(45,55)(46,57)(50,61)(52,58)(53,64)$
$(54,66)(56,65)(59,69)(60,71)(62,70)(63,73)(68,76)(75,80)(78,82)(79,83)(81,84)$,and
$y \sim (1,3,6,11)(2,4,8,14)(5,9,16,25)(7,12,20,30)(10,17,13,21)(15,23,19,28)$
$(18,27,24,34)(22,32,29,38)(26,35,44,54)(31,39,49,60)(33,41,52,63)(36,45,56,48)$
$(37,46,58,68)(40,50,62,43)(42,53,65,57)(47,59,70,51)(55,67,71,78)(61,72,66,75)$
$(64,74,76,81)(69,77,73,79)(80,84)(82,83)$.

The progenitor $2^{*84} : N$ has symmetric presentation

```
G<x,y>:=Group<x,y,t| x^2, y^4, y^-2 * x * y^2 * x * y^2 * x,
(x * y^-1)^7,t^2>;
```

We want to verify that the presentation given above is correct.

Using the theorem: $\frac{2^{*n}:N}{t_i t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$,

we will show that

$$\frac{2^7 : PSL(2,7)}{((y^2)*t^{(y*x*y^{-1}*x*y*x*y^{-1})})^3, ((y^2)*t^{(x*y*x*y^{-1}*x*y*x)})^3, ((y*x*y)*t^{(x*y)})^4, ((y*x*y)*t^{(x*y*x*y^{-1}*x*y)})^3} \cong PSL(2,7).$$

Thus, we will prove

```
G<x,y>:=Group<x,y,t| x^2, y^4, y^-2 * x * y^2 * x * y^2 * x,
(x * y^-1)^7,
t^2,
(t,(x * y * x * y^-1)^2 ),
(t,t^{(y^x)}), (t,t^{(x^2)}),(t,t^{(y*x^-1)}),
(t,t^{(y^-3)}),(t,t^{(y*x)}),
((y^2)*t^{(y * x * y^-1 * x * y * x * y^-1)})^3
((y^2)*t^{(x * y * x * y^-1 * x * y * x)})^3,
((y * x * y)*t^{(x * y)})^4,
 ((y * x * y)*t^{(x * y * x * y^-1 * x * y)})^3> ~ PSL(2,7).
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double cosets of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{21504}{168} = 128$.

Now we know that we have 128 distinct single cosets.

Expanding First Relation We will expand our first relation.

$$((y^2) * t^{(y*x*y^{-1}*x*y*x*y^{-1})})^3$$

$$= y^2 * t_2 * y^2 * t_2 * y^2 * t_2$$
$$= (y^2)^3 * (t_2^{(y^2)^2}) * (t_2^{(y^2)}) * t_2$$
$$= (y^6) * (t_2^{(y^4)}) * (t_2^{(y^2)}) * t_2$$
$$\implies y^6 * t_2 * t_8 * t_2 = e$$
$$\implies y^6 * t_2 = t_2 * t_8$$

Expanding Second Relation We will expand our second relation.

$$((y^2) * t^{(x*y*x*y^{-1}*x*y*x)})^3$$

$$= y^2 * t_3 * y^2 * t_3 * y^2 * t_3$$
$$= (y^2)^3 * (t_3^{(y^2)^2}) * (t_3^{(y^2)}) * t_3$$
$$= (y^6) * (t_3^{(y^4)}) * (t_3^{(y^2)}) * t_3$$
$$\implies y^6 * t_3 t_{11} * t_3 = e \implies y^6 * t_3 = t_3 * t_{11}$$

Expanding Third Relation We will expand our third relation.

$$((y * x * y) * t^{(x*y)})^4$$

$$= (y * x * y) * t_4 * (y * x * y) * t_4 * (y * x * y) * t_4 * (y * x * y) * t_4$$
$$= (y * x * y)^4 * (t_4^{(y*x*y)^3}) * (t_4^{(y*x*y)^2}) * (t_4^{(y*x*y)}) * t_4$$
$$\implies (y * x * y)^4 * t_4 * t_{11} * t_{21} * t_4 = e \implies (y * x * y)^4 * t_4 * t_{11} = t_4 * t_{21}$$

Expanding Fourth Relation We will expand our fourth relation.

$$((y * x * y) * t^{(x*y*x*y^{-1}*x*y)})^3$$

$$= (y * x * y) * t_5 * (y * x * y) * t_5 * (y * x * y) * t_5$$
$$= (y * x * y)^3 * (t_5^{(y*x*y)^2}) * (t_5^{(y*x*y)}) * t_5$$
$$\implies (y * x * y)^3 * t_{20} * t_{23} t_5 = e \implies (y * x * y)^3 * t_{20} = t_5 * t_{23}$$

First Double Coset [∗]

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right coset in N is $\frac{|N|}{|N|} = \frac{168}{168} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,$

$12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$

$31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$

$51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,$

$71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84\}$.

We need only determine the double coset of the right coset Nt_1.

Thus 84 t_i's extend the double coset $[*]$ to the double coset $Nt_1N = [1]$.

Cayley Diagram

Figure 7.84: Cayley Diagram for G over S_{84}

Second Double Coset [1]

$Nt_1N = \{Nt_1^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11},$

$Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21},$

$Nt_{22}, Nt_{23}, Nt_{24}, Nt_{25}, Nt_{26}, Nt_{27}, Nt_{28}, Nt_{29}, Nt_{30},$

$Nt_{31}, Nt_{32}, Nt_{33}, Nt_{34}, Nt_{35}, Nt_{36}, Nt_{37}, Nt_{38}, Nt_{39}, Nt_{40},$

$Nt_{41}, Nt_{42}, Nt_{43}, Nt_{44}, Nt_{45}, Nt_{46}, Nt_{47}, Nt_{48}, Nt_{49}, Nt_{50},$

$Nt_{51}, Nt_{52}, Nt_{53}, Nt_{54}, Nt_{55}, Nt_{56}, Nt_{57}, Nt_{58}, Nt_{59}, Nt_{60},$

$Nt_{61}, Nt_{62}, Nt_{63}, Nt_{64}, Nt_{65}, Nt_{66}, Nt_{67}, Nt_{68}, Nt_{69}, Nt_{70},$

$Nt_{71}, Nt_{72}, Nt_{73}, Nt_{74}, Nt_{75}, Nt_{76}, Nt_{77}, Nt_{78}, Nt_{79}, Nt_{80}, Nt_{81}, Nt_{82}, Nt_{83}, Nt_{84}\}.$

We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

Thus, $N^{(1)} \geq N^1 = \langle (3, 65)(4, 70)(5, 56)(6, 84)(7, 62)(8, 83)(9, 82)(10, 81)(11, 35)(12, 80),$

$(13, 79)(14, 39)(15, 78)(16, 41)(17, 33)(18, 43)(19, 75)(20, 46)(21,$

$37)(22, 48)(23, 26)(24, 51)(27, 73)(28, 31)(29, 57)(32, 76)(34, 66)(36, 63)(38, 71)(40, 68),$

$(42, 54)(44, 49)(45, 53)(47, 60)(50, 59)(52, 58)(55, 64)(61, 69)(67, 72)(74, 77)\rangle.$

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{168}{84} = 2$. The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,$

$13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$

$31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$

$51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,$

$71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84\}$ are

$\{1\}, \{2\}, \{25\}, \{30\}, \{3, 65\}, \{4, 70\}, \{5, 56\}, \{6, 84\}, \{7, 62\},$

$\{8, 83\}, \{9, 82\}, \{10, 81\}, \{11, 35\}, \{12, 80\}, \{13, 79\}, \{14, 39\}, \{15, 78\}, \{16, 41\},$

$\{17, 33\}, \{18, 43\}, \{19, 75\}, \{20, 46\}, \{21, 37\}, \{22, 48\}, \{23, 26\}, \{24, 51\}, \{27, 73\},$

$\{28, 31\}, \{29, 57\}, \{32, 76\}, \{34, 66\}, \{36, 63\}, \{38, 71\}, \{40, 68\}, \{42, 54\}, \{44, 49\},$

$\{45, 53\}, \{47, 60\}, \{50, 59\}, \{52, 58\}, \{55, 64\}, \{61, 69\}, \{67, 72\}, \{74, 77\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1\}$

$Nt_1t_1 = Nt_1^2 = N \in [*].$

This means one t_i takes [1] to [*].

Choose 2 from $\{2\}$

$Nt_1t_2 \in [*].$

This means one t_i takes [1] to [*].

Choose 25 from $\{25\}$

$Nt_1t_{25} \in [1, 25].$

This means one t_i takes [1] to [1,25].

Choose 30 from $\{30\}$

$Nt_1t_{30} \in [1, 25].$

We have $Nt_1t_{30} = Nt_1t_{25} \in [1, 25]$

This means one t_i takes [1] to [1,25].

Choose 3 from {3,65}

$Nt_1t_3 \in [1]$.

We have $Nt_1t_3 = (y^{-1} * x * y^{-1} * x * y * x * y^2)t_{74}$

$= (y^{-1} * x * y^{-1} * x * y * x * y^2)(t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y)} \in [1]$

This means two t_i's take [1] to [1].

Choose 4 from {4,70}

$Nt_1t_4 \in [1]$.

We have $Nt_1t_4 = (y^{-1} * x * y^{-1} * x * y * x * y^2)t_{74}$

$= (y^{-1} * x * y^{-1} * x * y * x * y^2)(t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y)} \in [1]$

This means two t_i's take [1] to [1].

Choose 5 from {5,56}

$Nt_1t_5 \in [1]$.

We have $Nt_1t_5 = (y * x * y * x * y * x * y^{-1})t_{74}$

$= (y * x * y * x * y * x * y^{-1})(t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y)} \in [1]$

This means two t_i's take [1] to [1].

Choose 6 from {6,84}

$Nt_1t_6 \in [1]$.

We have $Nt_1t_6 = y^2t_1 \in [1]$

This means two t_i's take [1] to [1].

Choose 7 from {7,62}

$Nt_1t_7 \in [1]$.

We have $Nt_1t_7 = (y * x * y * x * y * x * y^{-1})t_{74}$

$= (y * x * y * x * y * x * y^{-1})(t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y)} \in [1]$

This means two t_i's take [1] to [1].

Choose 8 from {8,83}

$Nt_1t_8 \in [1]$.

We have $Nt_1t_8 = y^2t_1 \in [1]$

This means two t_i's take [1] to [1].

Choose 9 from {9,82}

$Nt_1t_9 \in [19]$.

This means two t_i's take [1] to [19].

Choose 10 from {10,81}

$Nt_1t_{10} \in [1]$.

We have $Nt_1t_{10} = (x * y^2 * x)t_1 \in [1]$

This means two t_i's take [1] to [1].

Choose 11 from {11,35}

$Nt_1t_{11} \in [1]$.

We have $Nt_1t_{11} = (y^{-1} * x * y^{-1} * x * y * x)t_{45}$

$= (y^{-1} * x * y^{-1} * x * y * x)(t_1)^{(y^2 * x * y^{-1} * x * y * x * y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 12 from {12,80}

$Nt_1t_{12} \in [19]$.

We have $Nt_1t_{12} = Nt_1t_9 \in [19]$.

This means two t_i's take [1] to [19].

Choose 13 from {13,79}

$Nt_1t_{13} \in [1]$.

We have $Nt_1t_{13} = (x * y^2 * x)t_1 \in [1]$

This means two t_i's take [1] to [1].

Choose 14 from {14,39}

$Nt_1t_{14} \in [1]$.

We have $Nt_1t_{14} = y^{-1} * x * y^{-1} * x * y * x)t_{45}$

$= y^{-1} * x * y^{-1} * x * y * x)(t_1)^{(y^2 * x * y^{-1} * x * y * x * y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 15 from {15,78}

$Nt_1t_{15} \in [19]$.

We have $Nt_1t_{15} = Nt_2t_{15}$

$= (t_1t_9)^x \in [19]$

This means two t_i's take [1] to [19].

Choose 16 from {16,41}

$Nt_1t_{16} \in [19]$.

We have $Nt_1t_{16} = (y^{-1} * x * y^{-1})t_8t_{19}$

$= (y^{-1} * x * y^{-1})(t_1t_9)^{(x * y^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 17 from {17,33}

$Nt_1t_{17} \in [19]$.

We have $Nt_1t_{17} = (y^2 * x * y * x * y^{-1} * x * y * x)t_{24}t_{11}$

$= (y^2 * x * y * x * y^{-1} * x * y * x)(t_1t_9)^{(y^{-1}*x*y^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 18 from {18,43}

$Nt_1t_{18} \in [1]$.

We have $Nt_1t_{18} = (x * y^{-1} * x * y^{-1} * x * y)t_{55}$

$= (x * y^{-1} * x * y^{-1} * x * y)(t_1)^{(y^2*x*y^{-1}*x*y^{-1}*x*y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 19 from {19,75}

$Nt_1t_{19} \in [19]$.

We have $Nt_1t_{19} = Nt_2t_{15}$

$= (t_1t_9)^x \in [19]$

This means two t_i's take [1] to [19].

Choose 20 from {20,46}

$Nt_1t_{20} \in [19]$.

We have $Nt_1t_{20} = (y^{-1} * x * y^{-1})t_8t_{19}$

$= (y^{-1} * x * y^{-1}) * (t_1t_9)^{(x*y^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 21 from {21,37}

$Nt_1t_{21} \in [19]$.

We have $Nt_1t_{21} = (x * y * x * y^{-1} * x * y^{-1} * x * y * x * y)t_{47}t_{76}$

$= (x * y * x * y^{-1} * x * y^{-1} * x * y * x * y) * (t_1t_9)^{(x*y^{-1}*x*y^{-1}*x)} \in [19]$

This means two t_i's take [1] to [19].

Choose 22 from {22,48}

$Nt_1t_{22} \in [1]$.

We have $Nt_1t_{22} = (x * y^{-1} * x * y^{-1} * x * y)t_{55}$

$= (x * y^{-1} * x * y^{-1} * x * y)(t_1)^{(y^2*x*y^{-1}*x*y^{-1}*x*y^2)} \in [1]$

This means two t_i's take [1] to [1].

Choose 23 from {23,26}

$Nt_1t_{23} \in [19]$.

We have $Nt_1t_{23} = (y * x * y^{-1} * x * y * x * y)t_{81}t_{15}$

$= (y * x * y^{-1} * x * y * x * y)(t_1t_9)^{(y*x*y*x*y^{-1}*x*y^{-1}*x*y)} \in [19]$

This means two t_i's take [1] to [19].

Choose 24 from {24,51}

$Nt_1t_{24} \in [19]$.

We have $Nt_1t_{24} = (x * y^{-1} * x * y^{-1} * x)t_{13}t_{12}$

$= (x * y^{-1} * x * y^{-1} * x)(t_1t_9)^{(x*y^2*x)} \in [19]$

This means two t_i's take [1] to [19].

Choose 27 from {27,73}

$Nt_1t_{27} \in [1]$.

We have $Nt_1t_{27} = Nt_{63}$

$= (t_1)^{(y*x*y*x*y*x*y^{-1})} \in [1]$

This means two t_i's take [1] to [1].

Choose 28 from {28,31}

$Nt_1t_{28} \in [19]$.

We have $Nt_1t_{28} = ((y * x)^2)t_{41}t_{43}$

$= ((y * x)^2)(t_1t_9)^{(x*y*x*y*x*y^{-1}*x*y)} \in [19]$

This means two t_i's take [1] to [19].

Choose 29 from {29,57}

$Nt_1t_{29} \in [19]$.

We have $Nt_1t_{29} = (x * y * x * y^{-1} * x * y^{-1} * x)t_{66}t_{27}$

$= (x * y * x * y^{-1} * x * y^{-1} * x)(t_1t_9)^{(x*y^{-1}*x*y*x*y*x*y^2)} \in [19]$

This means two t_i's take [1] to [19].

Choose 32 from {32,76}

$Nt_1t_{32} \in [1]$.

We have $Nt_1t_{32} = Nt_{63}$

$= (t_1)^{(y*x*y*x*y*x*y^{-1})} \in [1]$

This means two t_i's take [1] to [1].

Choose 34 from {34,66}

$Nt_1t_{34} \in [19]$.

We have $Nt_1t_{34} = (x * y * x * y^{-1} * x * y^{-1} * x * y)t_{83}t_{75}$

$= (x*y*x*y^{-1}*x*y^{-1}*x*y)(t_1t_9)^{(x*y^{-1}*x*y^{-1}*x*y*x*y^{-1}*x)} \in [19]$

This means two t_i's take [1] to [19].

Choose 36 from {36,63}

$Nt_1t_{36} \in [1]$.

We have $Nt_1t_{36} = Nt_{73}$

$= (t_1)^{(y*x*y*x*y*x*y^{-1}*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 38 from {38,71}

$Nt_1t_{38} \in [19]$.

We have $Nt_1t_{38} = (x*y*x*y^{-1}*x*y^{-1}*x*y)t_{83}t_{75}$

$= (x*y*x*y^{-1}*x*y^{-1}*x*y)(t_1t_9)^{(x*y^{-1}*x*y^{-1}*x*y*x*y^{-1}*x)} \in [19]$

This means two t_i's take [1] to [19].

Choose 40 from {40,68}

$Nt_1t_{40} \in [1]$.

We have $Nt_1t_{40} = Nt_{73}$

$= (t_1)^{(y*x*y*x*y*x*y^{-1}*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 42 from {42,54}

$Nt_1t_{42} \in [19]$.

We have $Nt_1t_{42} = (x*y^{-1}*x*y^{-1}*x*y*x*y^{-1}*x)t_{57}t_{14}$

$= (x*y^{-1}*x*y^{-1}*x*y*x*y^{-1}*x)(t_1t_9)^{(y^{-1}*x*y^{-1}*x*y^{-1})} \in [19]$

This means two t_i's take [1] to [19].

Choose 44 from {44,49}

$Nt_1t_{44} \in [1, 25]$.

We have $Nt_1t_{44} = ((x*y^{-1}*x*y)^2)t_3t_5$

$= ((x*y^{-1}*x*y)^2)(t_1t_{25})^y \in [1, 25]$

This means two t_i's take [1] to [1,25].

Choose 45 from {45,53}

$Nt_1t_{45} \in [1]$.

We have $Nt_1t_{45} = (y^{-1}*x*y^{-1}*x*y*x)t_{74}$

$= (y^{-1}*x*y^{-1}*x*y*x)(t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 47 from {47,60}

$Nt_1t_{47} \in [19]$.

We have $Nt_1t_{47} = (x * y^{-1} * x * y^{-1} * x * y * x * y^{-1} * x)t_{57}t_{14}$

$= (x * y^{-1} * x * y^{-1} * x * y * x * y^{-1} * x)(t_1t_9)^{(y^{-1}*x*y^{-1}*x*y^{-1})} \in [19]$

This means two t_i's take [1] to [19].

Choose 50 from {50,59}

$Nt_1t_{50} \in [1]$.

We have $Nt_1t_{50} = (y^{-1} * x * y^{-1} * x * y * x)t_{74}$

$= (y^{-1} * x * y^{-1} * x * y * x)(t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 52 from {52,58}

$Nt_1t_{52} \in [1, 52]$.

This means two t_i's take [1] to [1,52].

Choose 55 from {55,64}

$Nt_1t_{55} \in [1]$.

We have $Nt_1t_{55} = (x * y^{-1} * x * y^{-1} * x * y)t_{74}$

$= (x * y^{-1} * x * y^{-1} * x * y)(t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 61 from {61,69}

$Nt_1t_{61} \in [1]$.

We have $Nt_1t_{61} = (x * y^{-1} * x * y^{-1} * x * y)t_{74}$

$= (x * y^{-1} * x * y^{-1} * x * y)(t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y*x)} \in [1]$

This means two t_i's take [1] to [1].

Choose 67 from {67,72}

$Nt_1t_{67} \in [1, 52]$.

We have $Nt_1t_{67} = (y * x * y^{-1} * x * y^{-1} * x * y^{-1} * x)t_3t_{63}$

$= (y * x * y^{-1} * x * y^{-1} * x * y^{-1} * x)(t_1t_{52})^y \in [1, 52]$

This means two t_i's take [1] to [1,52].

Choose 74 from {74,77}

$Nt_1t_{74} \in [1, 74]$.

This means two t_i's take [1] to [1,74].

Cayley Diagram
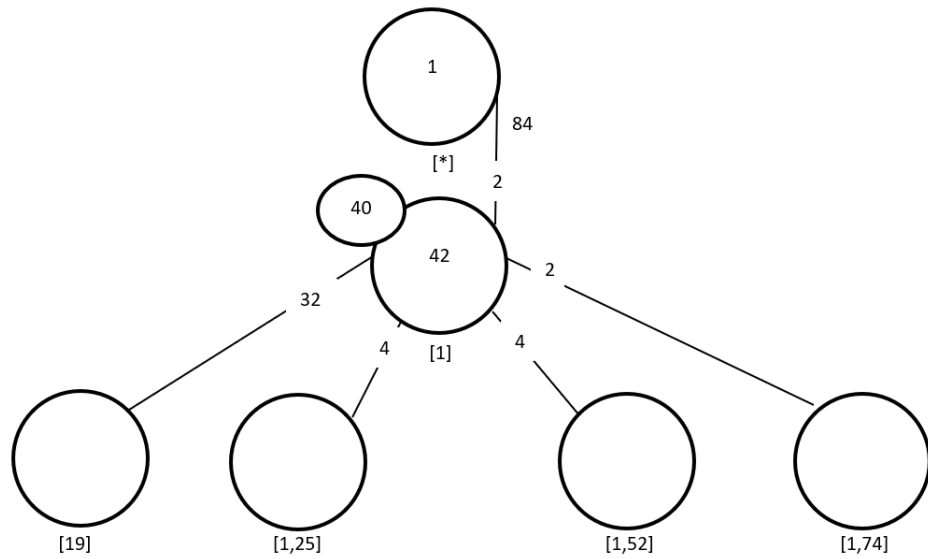
Figure 7.85: Cayley Diagram for G over S_{84}

Third Double Coset [19]

$Nt_1t_9 = \{N(t_1t_9)^n | n \in N\}.$

We now find the coset stabilizer $N^{(19)}$. We first find the point stabilizer of 1 and 9 in N.

$N^{19} = \{n \in N | (19)^n = 19\}$

$N^{19} = \langle e \rangle$

We will find different names of Nt_1t_9N. $Nt_1t_9 = Nt_4t_{55}$.

$N(t_1t_9)^{(1,4,43),(9,55,65)} = Nt_4t_{55}.$

But, $Nt_4t_{55} = Nt_1t_9 \implies$

$N(t_1t_9)^{(1,4,43),(9,55,65)} = Nt_1t_9.$

$\implies (1, 4, 43)(2, 3, 48)(5, 57, 30)(6, 64, 56)(7, 51, 25)(8, 69, 62)(9, 55, 65)$

$(10, 80, 28)(11, 40, 81)(12, 61, 70)(13, 82, 23)(14, 36, 79)(15, 83, 21)(16, 47, 78)(17, 19, 84)$

$(18, 44, 54)(20, 42, 75)(22, 49, 60)(24, 52, 63)(26, 41, 34)(27, 33, 35)(29, 58, 68)(31, 46, 38)$

$(32, 37, 39)(45, 74, 73)(50, 77, 76)(53, 67, 66)(59, 72, 71) \in Nt_1t_9N$

Thus $N^{(19)} \geq \langle N^{19}, (1, 4, 43)(2, 3, 48)(5, 57, 30)(6, 64, 56)(7, 51, 25)(8, 69, 62)(9, 55, 65)$

$(10, 80, 28)(11, 40, 81)(12, 61, 70)(13, 82, 23)(14, 36, 79)(15, 83, 21)(16, 47, 78)(17, 19, 84)$

$(18, 44, 54)(20, 42, 75)(22, 49, 60)(24, 52, 63)(26, 41, 34)(27, 33, 35)(29, 58, 68)(31, 46, 38)$

$(32, 37, 39)(45, 74, 73)(50, 77, 76)(53, 67, 66)(59, 72, 71)\rangle.$

$= \langle e, (1, 4, 43)(2, 3, 48)(5, 57, 30)(6, 64, 56)(7, 51, 25)(8, 69, 62)(9, 55, 65)$

$(10, 80, 28)(11, 40, 81)(12, 61, 70)(13, 82, 23)(14, 36, 79)(15, 83, 21)(16, 47, 78)(17, 19, 84)$

$(18, 44, 54)(20, 42, 75)(22, 49, 60)(24, 52, 63)(26, 41, 34)(27, 33, 35)(29, 58, 68)$

$(31, 46, 38)(32, 37, 39)(45, 74, 73)(50, 77, 76)(53, 67, 66)(59, 72, 71)\rangle$

The number of distinct right cosets in Nt_1t_9N is calculated by the formula,

$\frac{|N|}{|N^{(1,9)}|} = \frac{168}{6} = 28.$

The orbits of $N^{(19)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,$

$16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35,$

$36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55,$

$56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75,$

$76, 77, 78, 79, 80, 81, 82, 83, 84\}$ are

$\{1, 4, 43\}, \{2, 3, 48\}, \{5, 57, 30\}, \{6, 64, 6\}, \{7, 51, 25\}, \{8, 69, 62\},$

$\{9, 55, 65\}, \{10, 80, 28\}, \{11, 40, 81\}, \{12, 61, 70\}, \{13, 82, 23\}, \{14, 36, 79\}, \{15, 83, 21\}, \{16, 47, 78\},$

$\{17, 19, 84\}, \{18, 44, 54\}, \{20, 42, 75\}, \{22, 49, 60\}, \{24, 52, 63\}, \{26, 41, 34\}, \{27, 33, 35\},$

$\{29, 58, 68\}, \{31, 46, 38\}, \{32, 37, 39\}, \{45, 74, 73\}, \{50, 77, 76\}, \{53, 67, 66\}, \{59, 72, 71\}.$

We will determine the double cosets by selecting one representative from each orbit such

as,

Choose 1 from $\{1, 4, 43\}$

$Nt_1t_9t_1 \in [1].$

We have $Nt_1t_9t_1 = y^2t_{23}$

$= y^2(t_1)^{(y*x*y*x*y)} \in [1]$

This means three t_i's take [19] to [1].

Choose 2 from $\{2, 3, 48\}$

$Nt_1t_9t_2 \in [1].$

We have $Nt_1t_9t_2 = y^2t_{23}$

$= y^2(t_1)^{(y*x*y*x*y)} \in [1]$

This means three t_i's take [19] to [1].

Choose 5 from $\{5, 57, 30\}$

$Nt_1t_9t_5 \in [19].$

We have $Nt_1t_9t_5 = (y^{-1} * x * y^{-1})t_{27}t_6$

$= (y^{-1} * x * y^{-1})(t_1t_9)^{(x^y)} \in [19]$

This means three t_i's take [19] to [1].

Choose 6 from {6, 64, 56}

$Nt_1t_9t_6 \in [19]$.

We have $Nt_1t_9t_6 = y^2t_1t_9$

This means three t_i's take [19] to [19].

Choose 7 from {7, 51, 25}

$Nt_1t_9t_7 \in [19]$.

We have $Nt_1t_9t_7 = (y^{-1} * x * y^{-1})t_{27}t_6$

$= (y^{-1} * x * y^{-1})(t_1t_9)^{(x^y)} \in [19]$

This means three t_i's take [19] to [19].

Choose 8 from {8, 69, 62}

$Nt_1t_9t_8 \in [19]$.

We have $Nt_1t_9t_8 = y^2t_1t_9$

This means three t_i's take [19] to [19].

Choose 9 from {9, 55, 65}

$Nt_1t_9t_9 = Nt_1t_9^2 = Nt_1 \in [1]$.

This means three t_i's take [19] to [1].

Choose 10 from {10, 80, 28}

$Nt_1t_9t_{10} \in [1, 52]$.

We have $Nt_1t_9t_{10} = (x * y^2)t_1t_{52}$

This means three t_i's take [19] to [1,52].

Choose 11 from {11, 40, 81}

$Nt_1t_9t_{11} \in [19]$.

We have $Nt_1t_9t_{11} = (y^{-1} * x * y * x * y^{-1} * x * y)t_1t_9 \in [19]$

This means three t_i's take [19] to [19].

Choose 12 from {12, 61, 70}

$Nt_1t_9t_{12} \in [1]$.

We have $Nt_1t_9t_{12} = Nt_1$

This means three t_i's take [19] to [1].

Choose 13 from {13, 82, 23}

$Nt_1t_9t_{13} \in [1, 52]$.

We have $Nt_1t_9t_{13} = (x * y^2)t_1t_{52}$

This means three t_i's take [19] to [1,52].

Choose 14 from {14, 36, 79}

$Nt_1t_9t_{14} \in [19]$.

We have $Nt_1t_9t_{14} = (y^{-1} * x * y * x * y^{-1} * x * y)t_1t_9 \in [19]$

This means three t_i's take [19] to [19].

Choose 15 from {15, 83, 21}

$Nt_1t_9t_{15} \in [1, 25]$.

We have $Nt_1t_9t_{15} = Nt_1t_{25}$.

This means three t_i's take [19] to [1,25].

Choose 16 from {16, 47, 78}

$Nt_1t_9t_{16} \in [1, 9, 16]$.

This means three t_i's take [19] to [1,9,16].

Choose 17 from {17, 19, 84}

$Nt_1t_9t_{17} \in [1, 25]$.

We have $Nt_1t_9t_{17} = (y^{-1} * x * y^{-1} * x * y * x * x * y^{-1})t_6t_9$

$= (y^{-1} * x * y^{-1} * x * y * x * x * y^{-1})(t_1t_{25})^{y^2} \in [1, 25]$

This means three t_i's take [19] to [1,25].

Choose 18 from {18, 44, 54}

$Nt_1t_9t_{18} \in [19]$.

We have $Nt_1t_9t_{18} = (x * y * x * y^{-1} * x * y * x)t_{45}t_{70}$

$= (x * y * x * y^{-1} * x * y * x)(t_1t_9)^{(y^2*x*y^{-1}*x*y*x*y^2)} \in [19]$

This means three t_i's take [19] to [19].

Choose 20 from {20, 42, 75}

$Nt_1t_9t_{20} \in [1, 9, 16]$.

We have $Nt_1t_9t_{20} = Nt_1t_9t_{16}$.

This means three t_i's take [19] to [1,9,16].

Choose 22 from {22, 49, 60}

$Nt_1t_9t_{22} \in [19]$.

We have $Nt_1t_9t_{22} = (x * y * x * y^{-1} * x * y * x)t_{45}t_{70}$

$= (x * y * x * y^{-1} * x * y * x)(t_1t_9)^{(y^2*x*y^{-1}*x*y*x*y^2)} \in [19]$

This means three t_i's take [19] to [19].

Choose 24 from {24, 52, 63}

$Nt_1t_9t_{24} \in [19]$.

We have $Nt_1t_9t_{24} = (y^{-1} * x * y^{-1} * x * y * x * y^2)t_{74}t_{12}$

$= (y^{-1} * x * y^{-1} * x * y * x * y^2)(t_1t_9)^{(y^{-1}*x*y^{-1}*x*y*x*y)} \in [19]$

This means three t_i's take [19] to [19].

Choose 26 from $\{26, 41, 34\}$

$Nt_1t_9t_{26} \in [1, 9, 16]$.

We have $Nt_1t_9t_{26} = (y^{-1} * x * y * x * y * x * y)t_5t_{24}t_{30}$

$= (y^{-1} * x * y * x * y * x * y)(t_1t_9t_{16})^{(y*x)} \in [1, 9, 16]$

This means three t_i's take [19] to [1,9,16].

Choose 27 from $\{27, 33, 35\}$

$Nt_1t_9t_{27} \in [1]$.

We have $Nt_1t_9t_{27} = (y * x * y^{-1})t_{15}$

$= (y * x * y^{-1})(t_1)^{((y*x)^2)} \in [1]$

This means three t_i's take [19] to [1].

Choose 29 from $\{29, 58, 68\}$

$Nt_1t_9t_{29} \in [19]$.

We have $Nt_1t_9t_{29} = (y^{-1} * x * y^{-1} * x * y * x * y^2)t_{74}t_{12}$

$= (y^{-1} * x * y^{-1} * x * y * x * y^2)(t_1t_9)^{(y^{-1}*x*y^{-1}*x*y*x*y)} \in [19]$

This means three t_i's take [19] to [19].

Choose 31 from $\{31, 46, 38\}$

$Nt_1t_9t_{31} \in [1, 9, 16]$.

We have $Nt_1t_9t_{31} = (y^{-1} * x * y * x * y * x * y)t_5t_{24}t_{30}$

$= (y^{-1} * x * y * x * y * x * y)(t_1t_9t_{31})^{(y*x)} \in [1, 9, 16]$

This means three t_i's take [19] to [1,9,16].

Choose 32 from $\{32, 37, 39\}$

$Nt_1t_9t_{32} \in [1]$.

We have $Nt_1t_9t_{32} = (y * x * y^{-1})t_{15}$

$= (y * x * y^{-1})(t_1)^{((y*x)^2)} \in [1]$

This means three t_i's take [19] to [1].

Choose 45 from $\{45, 74, 73\}$

$Nt_1t_9t_{45} \in [1]$.

We have $Nt_1t_9t_{45} = (y^2 * x * y^{-1} * x * y^{-1})t_{10}$

$= (y^2 * x * y^{-1} * x * y^{-1})(t_1)^{(y^2*x)} \in [1]$

This means three t_i's take [19] to [1].

Choose 50 from {50, 77, 76}

$Nt_1t_9t_{50} \in [1]$.

We have $Nt_1t_9t_{50} = (y^2 * x * y^{-1} * x * y^{-1})t_{10}$

$= (y^2 * x * y^{-1} * x * y^{-1})(t_1)^{(y^2*x)} \in [1]$

This means three t_i's take [19] to [1].

Choose 53 from {53, 67, 66}

$Nt_1t_9t_{53} \in [19]$.

We have $Nt_1t_9t_{53} = (y * x * y^{-1} * x * y^{-1})t_{61}t_6$

$= (y * x * y^{-1} * x * y^{-1})(t_1t_9)^{(y^2*x*y*x*y^{-1}*x*y^2)} \in [19]$

This means three t_i's take [19] to [19].

Choose 59 from {59, 72, 71}

$Nt_1t_9t_{59} \in [19]$.

We have $Nt_1t_9t_{59} = (x * y * x * y^{-1} * x * y^{-1} * x * y * x * y)t_{54}t_2$

$= (x * y * x * y^{-1} * x * y^{-1} * x * y * x * y)(t_1t_9)^{((y^2*x*y^{-1})^2)} \in [19]$

This means three t_i's take [19] to [19].

Cayley Diagram
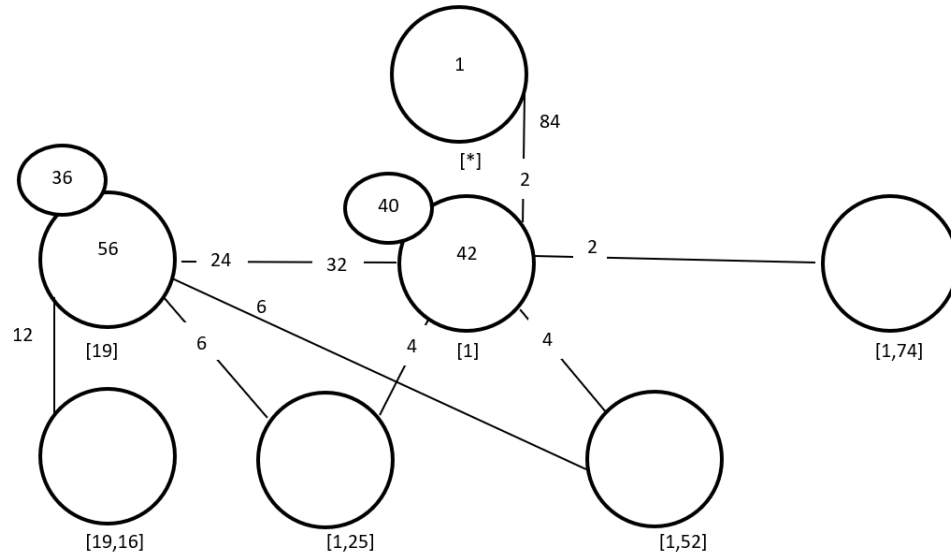
Figure 7.86: Cayley Diagram for G over S_{84}

Fourth Double Coset $[1, 25]$

$Nt_1t_{25} = \{N(t_1t_{25})^n | n \in N\}$.

We now find the coset stabilizer $N^{(1,25)}$. We first find the point stabilizer of 1 and 25 in N.

$N^{1,25} = \{n \in N | (1, 25)^n = 1, 25\}$

$N^{1,25} = \langle (3, 65)(4, 70)(5, 56)(6, 84)(7, 62)(8, 83)(9, 82)(10, 81)(11, 35)(12, 80)(13, 79)$

$(14, 39)(15, 78)(16, 41)(17, 33)(18, 43)(19, 75)(20, 46)(21, 37)(22, 48)(23, 26)$

$(24, 51)(27, 73)(28, 31)(29, 57)(32, 76)(34, 66)(36, 63)(38, 71)(40, 68)$

$(42, 54)(44, 49)(45, 53)(47, 60)(50, 59)(52, 58)(55, 64)(61, 69)(67, 72)(74, 77)\rangle$

We will find different names of $Nt_1t_{25}N$. $Nt_1t_{25} = Nt_2t_{30}$.

$N(t_1t_{25})^{(1,2),(25,30)} = Nt_2t_{30}$.

But, $Nt_2t_{30} = Nt_1t_{25} \implies N(t_1t_{25})^{(1,2),(25,30)} = Nt_1t_{25}$.

$\implies (1, 2)(3, 56)(4, 62)(5, 65)(6, 81)(7, 70)(8, 79)(9, 78)(10, 84)(11, 43)(12, 75)(13, 83)$

$(14, 48)(15, 82)(16, 51)(17, 23)(18, 35)(19, 80)(20, 57)(21, 28)(22, 39)(24, 41)(25, 30)$

$(26, 33)(27, 63)(29, 46)(31, 37)(32, 68)(34, 54)(36, 73)(38, 60)(40, 76)(42, 66)(45, 64)$

$(47, 71)(50, 69)(53, 55)(59, 61)(67, 72)(74, 77), (1, 2)(3, 5)(4, 7)(6, 10)(8, 13)(9, 15)(11, 18)$

$(12, 19)(14, 22)(16, 24)(17, 26)(20, 29)(21, 31)(23, 33)(25, 30)(27, 36)(28, 37)(32, 40)$

$(34, 42)(35, 43)(38, 47)(39, 48)(41, 51)(44, 49)(45, 55)(46, 57)(50, 61)(52, 58)(53, 64)$

$(54, 66)(56, 65)(59, 69)(60, 71)(62, 70)(63, 73)(68, 76)(75, 80)(78, 82)(79, 83)(81, 84) \in N t_1 t_{25} N$

Thus $N^{(1,25)} \geq \langle N^{1,25}, (1, 2)(3, 56)(4, 62)(5, 65)(6, 81)(7, 70)(8, 79)(9, 78)(10, 84)$

$(11, 43)(12, 75)(13, 83)(14, 48)(15, 82)(16, 51)(17, 23)(18, 35)(19, 80)(20, 57)(21, 28)(22, 39)$

$(24, 41)(25, 30)(26, 33)(27, 63)(29, 46)(31, 37)(32, 68)(34, 54)(36, 73)(38, 60)(40, 76)(42, 66)$

$(45, 64)(47, 71)(50, 69)(53, 55)(59, 61)(67, 72)(74, 77), (1, 2)(3, 5)(4, 7)(6, 10)(8, 13)$

$(9, 15)(11, 18)(12, 19)(14, 22)(16, 24)(17, 26)(20, 29)(21, 31)(23, 33)(25, 30)(27, 36)(28, 37)$

$(32, 40)(34, 42)(35, 43)(38, 47)(39, 48)(41, 51)(44, 49)(45, 55)(46, 57)(50, 61)(52, 58)$

$(53, 64)(54, 66)(56, 65)(59, 69)(60, 71)(62, 70)(63, 73)(68, 76)(75, 80)(78, 82)(79, 83)(81, 84)\rangle.$

$= \langle (3, 65)(4, 70)(5, 56)(6, 84)(7, 62)(8, 83)(9, 82)(10, 81)(11, 35)(12, 80)(13, 79)(14, 39)(15, 78)$

$(16, 41)(17, 33)(18, 43)(19, 75)(20, 46)(21, 37)(22, 48)(23, 26)(24, 51)(27, 73)(28, 31)(29, 57)$

$(32, 76)(34, 66)(36, 63)(38, 71)(40, 68)(42, 54)(44, 49)(45, 53)(47, 60)(50, 59)(52, 58)$

$(55, 64)(61, 69)(67, 72)(74, 77), (1, 2)(3, 56)(4, 62)(5, 65)(6, 81)(7, 70)(8, 79)(9, 78)(10, 84)$

$(11, 43)(12, 75)(13, 83)(14, 48)(15, 82)(16, 51)$

$(17, 23)(18, 35)(19, 80)(20, 57)(21, 28)(22, 39)(24, 41)(25, 30)(26, 33)(27, 63)(29, 46)(31, 37)$

$(32, 68)(34, 54)(36, 73)(38, 60)(40, 76)(42, 66)(45, 64)(47, 71)(50, 69)(53, 55)(59, 61)(67, 72)$

$(74, 77), (1, 2)(3, 5)(4, 7)(6, 10)(8, 13)(9, 15)(11, 18)(12, 19)(14, 22)(16, 24)(17, 26)(20, 29)(21, 31)$

$(23, 33)(25, 30)(27, 36)(28, 37)(32, 40)(34, 42)(35, 43)(38, 47)(39, 48)(41, 51)(44, 49)$

$(45, 55)(46, 57)(50, 61)(52, 58)(53, 64)(54, 66)(56, 65)(59, 69)(60, 71)$

$(62, 70)(63, 73)(68, 76)(75, 80)(78, 82)(79, 83)(81, 84)\rangle$

The number of distinct right cosets in $N t_1 t_{25} N$ is calculated by the formula,

$\frac{|N|}{|N^{(1,25)}|} = \frac{168}{24} = 7.$

The orbits of $N^{(1,25)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,$

$13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$

$31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$

$51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,$

$71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84\}$ are

$\{1, 50, 2, 59, 55, 64, 69, 61, 45, 25, 53, 30\},$

$\{11, 35, 52, 74, 43, 18, 58, 22, 77, 48, 14, 39\},$

$\{16, 41, 44, 67, 51, 24, 49, 29, 72, 57, 20, 46\},$

$\{3, 65, 71, 8, 56, 5, 38, 68, 75, 47, 60, 83, 19, 40, 79, 13, 12, 76, 37, 32, 21, 80, 31, 28\},$

$\{4, 70, 66, 6, 62, 7, 34, 63, 78, 42, 54, 84, 15, 36, 81, 10, 9, 73, 33, 27, 17, 82, 26, 23\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1, 50, 2, 59, 55, 64, 69, 61, 45, 25, 53, 30\}$

$Nt_1t_{25}t_1 \in [1].$

We have $Nt_1t_{25}t_1 = Nt_{25}$

$= (t_1)^{(y*x*y^{-1})} \in [1]$

This means twelve t_i's take [1,25] to [1].

Choose 3 from $\{3, 65, 71, 8, 56, 5, 38, 68, 75, 47, 60, 83, 19, 40,$

$79, 13, 12, 76, 37, 32, 21, 80, 31, 28\}$

$Nt_1t_{25}t_3 \in [19].$

We have $Nt_1t_{25}t_3 = (y*x*y^{-1})t_{59}t_{81}$

$= (y*x*y^{-1})(t_1t_9)^{(x*y^{-1}*x*y^{-1}*x*y)} \in [19]$

This means twenty-four t_i's take [1,25] to [19].

Choose 4 from $\{4, 70, 66, 6, 62, 7, 34, 63, 78, 42, 54, 84, 15, 36, 81$

$10, 9, 73, 33, 27, 17, 82, 26, 23\}$

$Nt_1t_{25}t_4 \in [19].$

We have $Nt_1t_{25}t_4 = (y*x*y^{-1})t_{59}t_{81}$

$= (y*x*y^{-1})(t_1t_9)^{(x*y^{-1}*x*y^{-1}*x*y)} \in [19]$

This means twenty-four t_i's take [1,25] to [19].

Choose 11 from $\{11, 35, 52, 74, 43, 18, 58, 22, 77, 48, 14, 39\}$

$Nt_1t_{25}t_{11} \in [1].$

We have $Nt_1t_{25}t_{11} = ((x*y*x*y^{-1})^2)t_{35}$

$= ((x*y*x*y^{-1})^2)(t_1)^{(x*y^{-1}*x*y*x*y^{-1}*x)} \in [1]$

This means twelve t_i's take [1,25] to [1].

Choose 16 from $\{16, 41, 44, 67, 51, 24, 49, 29, 72, 57, 20, 46\}$

$Nt_1t_{25}t_{16} \in [1, 9, 16].$

We have $Nt_1t_{25}t_{16} = (y^2*x*y^{-1}*x*y^{-1}*x*y*x)t_{84}t_{25}t_{56}$

$= (y^2*x*y^{-1}*x*y^{-1}*x*y*x)(t_1t_9t_{16})^{(y^{-1}*x*y^{-1}*x*y*x*y^{-1}*x)} \in [1]$

This means twelve t_i's take [1,25] to [1].

Cayley Diagram
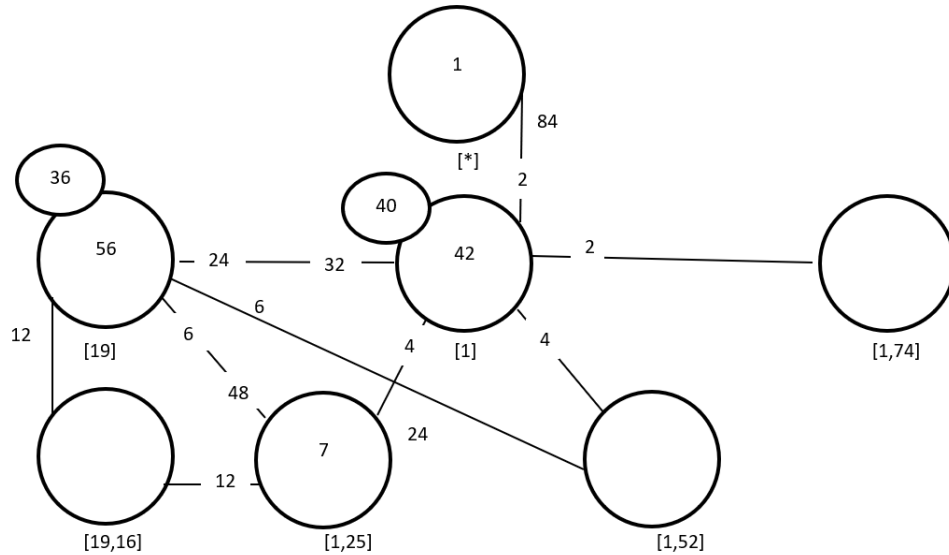
Figure 7.87: Cayley Diagram for G over S_{84}

Fifth Double Coset $[1, 52]$

$Nt_1t_{52} = \{N(t_1t_{52})^n | n \in N\}$.

We now find the coset stabilizer $N^{(1,52)}$. We first find the point stabilizer of 1 and 52 in N.

$N^{1,52} = \{n \in N | (1, 52)^n = 1, 52\}$

$N^{1,52} = \langle e \rangle$

We will find different names of $Nt_1t_{52}N$. $Nt_1t_{52} = Nt_2t_{52}$.

$N(t_1t_{52})^{(1,2)} = Nt_2t_{52}$.

But, $Nt_2t_{52} = Nt_1t_{52} \implies N(t_1t_{52})^{(1,2)} = Nt_1t_{52}$.

$\implies (1, 2)(3, 56)(4, 62)(5, 65)(6, 81)(7, 70)(8, 79)(9, 78)(10, 84)(11, 43)$

$(12, 75)(13, 83)(14, 48)(15, 82)(16, 51)(17, 23)(18, 35)$

$(19, 80)(20, 57)(21, 28)(22, 39)(24, 41)(25, 30)(26, 33)(27, 63)(29, 46)(31, 37)(32, 68)$

$(34, 54)(36, 73)(38, 60)(40, 76)$

$(42, 66)(45, 64)(47, 71)(50, 69)(53, 55)(59, 61)(67, 72)(74, 77) \in Nt_1t_{52}N$

Thus $N^{(1,52)} \geq \langle N^{1,52}, (1, 2)(3, 56)(4, 62)(5, 65)(6, 81)(7, 70)(8, 79)(9, 78)$

$(10, 84)(11, 43)(12, 75)(13, 83)(14, 48)(15, 82)(16, 51)(17, 23)(18, 35)$

$(19, 80)(20, 57)(21, 28)(22, 39)(24, 41)(25, 30)(26, 33)(27, 63)(29, 46)(31, 37)(32, 68)$

$(34, 54)(36, 73)(38, 60)(40, 76)$

$(42, 66)(45, 64)(47, 71)(50, 69)(53, 55)(59, 61)(67, 72)(74, 77)\rangle.$

$= \langle e, (1, 2)(3, 56)(4, 62)(5, 65)(6, 81)(7, 70)(8, 79)(9, 78)(10, 84)(11, 43)$

$(12, 75)(13, 83)(14, 48)(15, 82)(16, 51)(17, 23)(18, 35)(19, 80)(20, 57)$

$(21, 28)(22, 39)(24, 41)(25, 30)(26, 33)(27, 63)(29, 46)(31, 37)(32, 68)(34, 54)(36, 73)(38, 60)$

$(40, 76)(42, 66)(45, 64)(47, 71)(50, 69)(53, 55)(59, 61)(67, 72)(74, 77)\rangle.$

The number of distinct right cosets in $Nt_1t_{52}N$ is calculated by the formula,

$\frac{|N|}{|N^{(1,52)}|} = \frac{168}{24} = 7.$

The orbits of $N^{(1,25)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13,$

$14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$

$31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,$

$51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70,$

$71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84\}$ are

$\{1, 69, 59, 25, 2, 45, 64, 61, 50, 53, 30, 55\},$

$\{11, 52, 77, 39, 43, 14, 18, 74, 58, 48, 22, 35\},$

$\{16, 44, 72, 46, 51, 20, 24, 67, 49, 57, 29, 41\},$

$\{3, 47, 83, 37, 56, 76, 75, 13, 71, 32, 38, 28, 31, 19, 8, 12, 40, 68, 5, 80, 60, 65, 21, 79\},$

$\{4, 42, 84, 33, 62, 73, 78, 10, 66, 27, 34, 23, 26, 15, 6, 9, 36, 63, 7, 82, 54, 70, 17, 81\}.$

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1, 69, 59, 25, 2, 45, 64, 61, 50, 53, 30, 55\}$

$Nt_1t_{52}t_1 \in [1].$

We have $Nt_1t_{52}t_1 = (t_1)^{(y*x*y*x*y^{-1}*x*y^2)} \in [1]$

This means twelve t_i's take [1,52] to [1].

Choose 3 from $\{3, 47, 83, 37, 56, 76, 75, 13, 71, 32, 38, 28,$
$31, 19, 8, 12, 40, 68, 5, 80, 60, 65, 21, 79\}$

$Nt_1t_{52}t_3 \in [19].$

We have $Nt_1t_{52}t_3 = (y^{-1} * x * y * x)t_{42}t_2$

$= (y^{-1} * x * y * x)(t_1t_9)^{((y^{-1}*x)^2)} \in [19]$

This means twenty-four t_i's take [1,52] to [19].

Choose 4 from {4, 42, 84, 33, 62, 73, 78, 10, 66, 27, 34, 23, 26, 15, 6, 9, 36, 63, 7, 82, 54, 70, 17, 81}

$Nt_1t_{52}t_4 \in [19]$.

We have $Nt_1t_{52}t_4 = (y^{-1} * x * y^{-1} * x * y * x)t_{18}t_{79}$

$= (y^{-1} * x * y^{-1} * x * y * x)(t_1t_9)^{(x*y*x*y^{-1}*x*y^{-1}*x*y^2)} \in [19]$

This means twenty-four t_i's take [1,52] to [19].

Choose 11 from {11, 52, 77, 39, 43, 14, 18, 74, 58, 48, 22, 35}

$Nt_1t_{52}t_{11} \in [1]$.

We have $Nt_1t_{52}t_{11} = (x * y^{-1} * x * y^{-1} * x * y)t_{55}$

$= (x * y^{-1} * x * y^{-1} * x * y)(t_1)^{(y^2*x*y^{-1}*x*y^{-1}*x*y^2)} \in [1]$

This means twelve t_i's take [1,52] to [1].

Choose 16 from {16, 44, 72, 46, 51, 20, 24, 67, 49, 57, 29, 41}

$Nt_1t_{52}t_{16} \in [1, 9, 16]$.

We have $Nt_1t_{52}t_{16} = (x * y^2)t_6t_{25}t_5$

$= (x * y^2)(t_1t_9t_{16})^{y^2} \in [1, 9, 16]$

This means twelve t_i's take [1,52] to [1,9,16].

Cayley Diagram
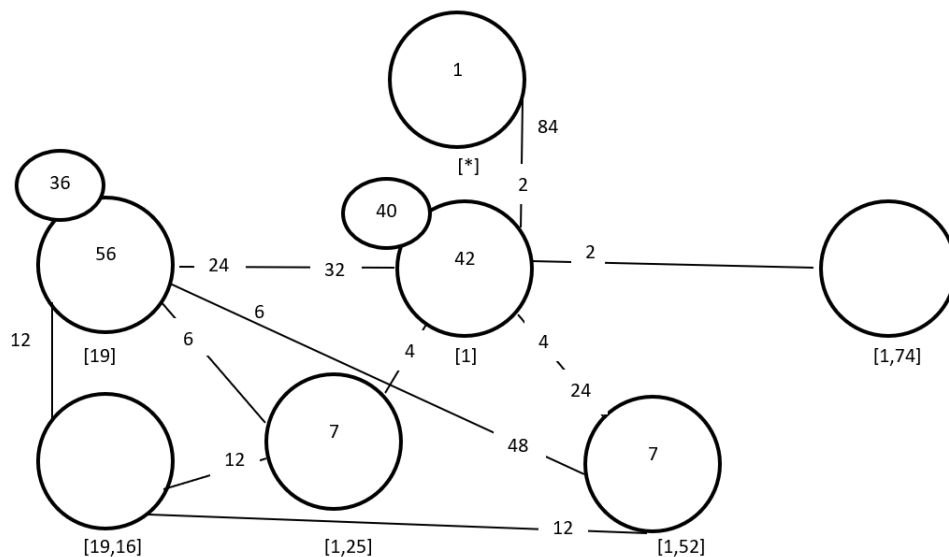
Figure 7.88: Cayley Diagram for G over S_{84}

Sixth Double Coset $[1, 74]$

$Nt_1t_{74} = \{N(t_1t_{74})^n | n \in N\}$.

We now find the coset stabilizer $N^{(1,74)}$. We first find the point stabilizer of 1 and 74 in N.

$N^{1,74} = \{n \in N | (1, 74)^n = 1, 74\}$

$N^{1,74} = \langle e \rangle$

We will find different names of $Nt_1t_{74}N$. $Nt_1t_{74} = Nt_{64}t_{14}$.

$N(t_1t_{74})^{(1,64,59),(14,22,74)} = Nt_{64}t_{14}$.

But, $Nt_{64}t_{14} = Nt_1t_{74} \implies N(t_1t_{74})^{(1,64,59),(14,22,74)} = Nt_1t_{74}$.

$\implies (1, 64, 59)(2, 69, 53)(3, 75, 83)(4, 78, 84)(5, 79, 80)(6, 17, 63)(7, 81, 82)$
$(8, 21, 68)(9, 23, 54)(10, 73, 26)(11, 18, 77)(12, 28, 60)(13, 76, 31)(14, 22, 74)(15, 66, 33)(16, 24, 72)$
$(19, 71, 37)(20, 29, 67)(25, 55, 50)(27, 62, 42)(30, 61, 45)(32, 56, 47)$
$(34, 70, 36)(35, 48, 58)(38, 65, 40)(39, 43, 52)(41, 57, 49)(44, 46, 51) \in Nt_1t_{74}N$

Thus $N^{(1,74)} \geq \langle N^{1,74}, (1, 64, 59)(2, 69, 53)(3, 75, 83)(4, 78, 84)(5, 79, 80)(6, 17, 63)(7, 81, 82)(8, 21, 68)$
$(9, 23, 54)(10, 73, 26)(11, 18, 77)(12, 28, 60)(13, 76, 31)(14, 22, 74)(15, 66, 33)$
$(16, 24, 72)(19, 71, 37)(20, 29, 67)(25, 55, 50)(27, 62, 42)(30, 61, 45)$
$(32, 56, 47)(34, 70, 36)(35, 48, 58)(38, 65, 40)(39, 43, 52)(41, 57, 49)(44, 46, 51)\rangle$.

$= \langle e, (1, 64, 59)(2, 69, 53)(3, 75, 83)(4, 78, 84)(5, 79, 80)(6, 17, 63)(7, 81, 82)$
$(8, 21, 68)(9, 23, 54)(10, 73, 26)(11, 18, 77)(12, 28, 60)(13, 76, 31)$
$(14, 22, 74)(15, 66, 33)(16, 24, 72)(19, 71, 37)(20, 29, 67)(25, 55, 50)(27, 62, 42)(30, 61, 45)$
$(32, 56, 47)(34, 70, 36)(35, 48, 58)(38, 65, 40)(39, 43, 52)(41, 57, 49)(44, 46, 51)\rangle$.

The number of distinct right cosets in $Nt_1t_{74}N$ is calculated by the formula,

$\frac{|N|}{|N^{(1,74)}|} = \frac{168}{168} = 1$.

The orbits of $N^{(1,25)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,$
$13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32,$
$33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51,$
$52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71,$
$72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84\}$ are
$\{1, 26, 82, 57, 68, 4, 61, 43, 41, 78, 14, 37, 23, 34, 30, 10, 40, 29, 5, 65, 54,$
$71, 46, 70, 83, 11, 67, 74, 50, 16, 36, 52, 39, 79, 8, 13, 58, 64, 80, 81, 28, 21, 24, 62, 38, 72, 9, 19,$
$42, 18, 59, 76, 63, 12, 32, 3, 55, 22, 66, 73, 35, 77, 45, 47,$
$49, 2, 20, 53, 69, 75, 56, 33, 6, 15, 17, 27, 31, 84, 44, 7, 48, 60, 25, 51\}$. We will determine the

double cosets by selecting one representative from each orbit such as,

Choose 1 from {1, 26, 82, 57, 68, 4, 61, 43, 41, 78, 14, 37, 23, 34, 30, 10, 40, 29,

5, 65, 54, 71, 46, 70, 83, 11, 67, 74, 50, 16, 36, 52, 39, 79, 8, 13, 58, 64, 80, 81, 28,

21, 24, 62, 38, 72, 9, 19, 42, 18, 59, 76, 63, 12, 32, 3, 55, 22, 66, 73, 35, 77, 45, 47,

49, 2, 20, 53, 69, 75, 56, 33, 6, 15, 17, 27, 31, 84, 44, 7, 48, 60, 25, 51}

$Nt_{74}t_1 \in [1]$.

We have $Nt_1t_{74}t_1 = Nt_{74}$

$= (t_1)^{(y^{-1}*x*y^{-1}*x*y*x*y*x)}$

This means eighty-four t_i's take [1,74] to [1].

Cayley Diagram

Figure 7.89: Cayley Diagram for G over S_{84}

Seventh Double Coset $[1, 9, 16]$

$Nt_1t_9t_{16} = \{N(t_1t_9t_{16})^n | n \in N\}$.

We now find the coset stabilizer $N^{(1,9,16)}$. We first find the point stabilizer of 1, 9 and 16

in N.

$N^{1,9,16} = \{n \in N | (1, 9, 16)^n = 1, 9, 16\}$

$N^{1,9,16} = \langle e \rangle$

We will find different names of $Nt_1t_9t_{16}N$. $Nt_1t_9t_{16} = Nt_{44}t_{48}t_{83}$.

$N(t_1t_9t_{16})^{(1,44),(9,48),(16,83)} = Nt_{44}t_{48}t_{83}$.

But, $Nt_{44}t_{48}t_{83} = Nt_1t_9t_{16} \implies N(t_1t_9t_{16})^{(1,44),(9,48),(16,83)} = Nt_1t_9t_{16}$.

$\implies (1,44)(2,49)(6,57)(8,51)(9,48)(10,46)(11,80)(12,43)(13,41)(14,82)$

$(15,39)(16,83)(17,64)(18,75)(19,35)(20,84)(21,69)(22,78)(23,55)(24,79)$

$(25,52)(26,53)(27,40)(28,61)(29,81)(30,58)(31,59)(32,36)(33,45)(34,47)(37,50)$

$(38,42)(54,60)(56,65)(62,70)(63,68)(66,71)(67,74)(72,77)(73,76) \in Nt_1t_9t_{16}N$

Thus $N^{(1,9,16)} \geq \langle N^{1,9,16}, (1,44)(2,49)(6,57)(8,51)(9,48)(10,46)(11,80)$

$(12,43)(13,41)(14,82)(15,39)(16,83)(17,64)(18,75)(19,35)(20,84)(21,69)(22,78)$

$(23,55)(24,79)(25,52)(26,53)(27,40)(28,61)(29,81)$

$(30,58)(31,59)(32,36)(33,45)(34,47)(37,50)$

$(38,42)(54,60)(56,65)(62,70)(63,68)(66,71)(67,74)(72,77)(73,76)\rangle$.

$= \langle e, (1,44)(2,49)(6,57)(8,51)(9,48)(10,46)(11,80)(12,43)(13,41)(14,82)$

$(15,39)(16,83)(17,64)(18,75)(19,35)(20,84)(21,69)(22,78)(23,55)(24,79)(25,52)(26,53)$

$(27,40)(28,61)(29,81)(30,58)(31,59)(32,36)(33,45)(34,47)(37,50)$

$(38,42)(54,60)(56,65)(62,70)(63,68)(66,71)(67,74)(72,77)(73,76)\rangle$.

The number of distinct right cosets in $Nt_1t_9t_{16}N$ is calculated by the formula,

$\frac{|N|}{|N^{(1,9,16)}|} = \frac{168}{12} = 14$.

The orbits of $N^{(1,9,16)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,$

$13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,$

$31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,$

$51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,$

$71,72,73,74,75,76,77,78,79,80,81,82,83,84\}$ are

$\{1,50,29,55,83,16,23,27,40,44,81,37\}$,

$\{2,45,24,61,84,20,28,32,36,49,79,33\}$,

$\{3,71,68,63,4,66\}, \{5,76,60,54,7,73\}$,

$\{6,15,46,70,77,72,62,9,48,57,10,39\}$,

$\{8,19,41,65,74,67,56,12,43,51,13,35\}$,

$\{11,52,47,22,59,31,78,17,64,80,34,25\}$,

$\{14,58,42,18,53,26,75,21,69,82,38,30\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from $\{1, 50, 29, 55, 83, 16, 23, 27, 40, 44, 81, 37\}$

$Nt_1t_9t_{16}t_1 \in [19]$.

We have $Nt_1t_9t_{16}t_1 = (y^{-1} * x * y^{-1} * x * y^2)t_{37}t_{46}$

$= (y^{-1} * x * y^{-1} * x * y^2)(t_1t_9)^{(y*x*y*x*y^{-1}*x)} \in [19]$

This means twelve t_i's take [1,9,16] to [19].

Choose 2 from $\{2, 45, 24, 61, 84, 20, 28, 32, 36, 49, 79, 33\}$

$Nt_1t_9t_{16}t_2 \in [19]$.

We have $Nt_1t_9t_{16}t_2 = (y^{-1} * x * y^{-1} * x * y^2)t_{37}t_{46}$

$= (y^{-1} * x * y^{-1} * x * y^2)(t_1t_9)^{(y*x*y*x*y^{-1}*x)} \in [19]$

This means twelve t_i's take [1,9,16] to [19].

Choose 3 from $\{3, 71, 68, 63, 4, 66\}$

$Nt_1t_9t_{16}t_3 \in [1, 25]$.

We have $Nt_1t_9t_{16}t_3 = (y^2 * x * y * x * y^{-1} * x * y^{-1})t_{35}t_{41}$

$= (y^2 * x * y * x * y^{-1} * x * y^{-1})(t_1t_{25})^{(x*y^{-1}*x*y*x*y^{-1}*x)} \in [1, 25]$

This means six t_i's take [1,9,16] to [1,25].

Choose 5 from $\{5, 76, 60, 54, 7, 73\}$

$Nt_1t_9t_{16}t_5 \in [1, 52]$.

We have $Nt_1t_9t_{16}t_5 = (x * y^2)t_6t_{33}$

$= (x * y^2)(t_1t_{52})^{y^2} \in [1, 52]$

This means six t_i's take [1,9,16] to [1,52].

Choose 6 from $\{6, 15, 46, 70, 77, 72, 62, 9, 48, 57, 10, 39\}$

$Nt_1t_9t_{16}t_6 \in [19]$.

We have $Nt_1t_9t_{16}t_6 = ((y^{-1} * x)^2)t_{58}t_{32}$

$= ((y^{-1} * x)^2)(t_1t_9)^{(y^{-1}*x*y^{-1}*x*y^{-1}*x*y)} \in [19]$

This means twelve t_i's take [1,9,16] to [19].

Choose 8 from $\{8, 19, 41, 65, 74, 67, 56, 12, 43, 51, 13,35\}$

$Nt_1t_9t_{16}t_8 \in [19]$.

We have $Nt_1t_9t_{16}t_8 = ((y^{-1} * x)^2)t_{58}t_{32}$

$= ((y^{-1} * x)^2)(t_1t_9)^{(y^{-1}*x*y^{-1}*x*y^{-1}*x*y)} \in [19]$

This means twelve t_i's take [1,9,16] to [19].

Choose 11 from $\{11, 52, 47, 22, 59, 31, 78, 17, 64, 80, 34,25\}$

$Nt_1t_9t_{16}t_{11} \in [1, 9, 16]$.

We have $Nt_1t_9t_{16}t_{11} = (y * x * y^{-1})t_{81}t_{15}t_{27}$

$= (y * x * y^{-1})(t_1t_9t_{16})^{(y*x*y*x*y^{-1}*x*y^{-1}*x*y)} \in [1, 9, 16]$

This means twelve t_i's take [1,9,16] to [1,9,16].

Choose 14 from $\{14, 58, 42, 18, 53, 26, 75, 21, 69, 82, 38, 30\}$

$Nt_1t_9t_{16}t_{14} \in [1, 9, 16]$.

We have $Nt_1t_9t_{16}t_{14} = ((y^{-1} * x)^2)t_{37}t_{46}t_1$

$= ((y^{-1} * x)^2)(t_1t_9t_{16})^{(y*x*y*x*y^{-1}*x)} \in [1, 9, 16]$

This means twelve t_i's take [1,9,16] to [1,9,16].

Cayley Diagram
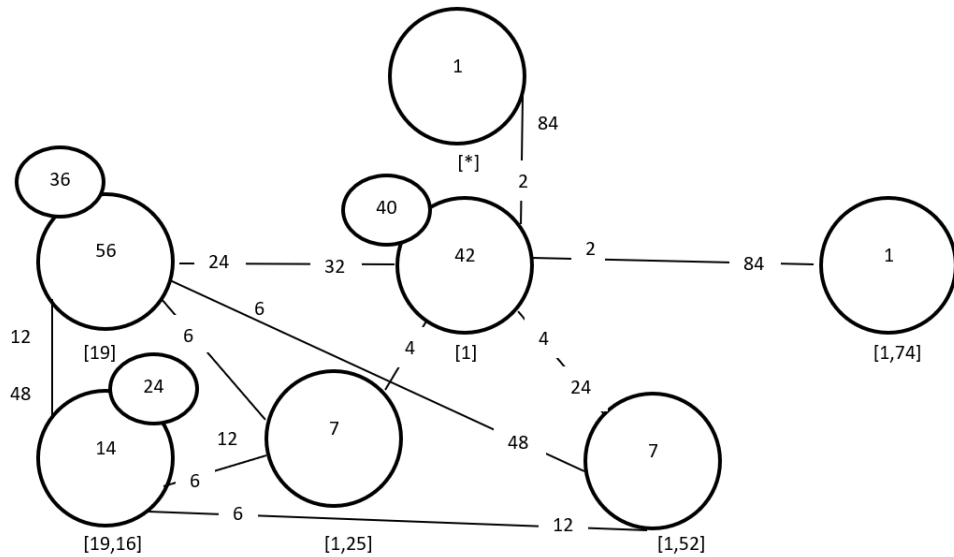
Figure 7.90: Cayley Diagram for G over S_{84}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases.

Our cayley diagram shows that

$G = N \cup Nt_1 \cup Nt_1t_9 \cup Nt_1t_{25} \cup Nt_1t_{52} \cup Nt_1t_{74} \cup Nt_1t_9t_{16}$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(19)}|} + \frac{|N|}{|N^{(1,25)}|} + \frac{|N|}{|N^{(1,52)}|} + \frac{|N|}{|N^{(1,74)}|} + \frac{|N|}{|N^{(1,9,16)}|}) \times |N|$

$|G| \leq (1 + 42 + 56 + 7 + 7 + 1 + 14) \times 168 \implies |G| \leq 128 \times 168 \implies |G| \leq 21504.$

G acts on 168 cosets that are given in the cayley diagram.

Let X be the set of these 168 cosets.

Now, $f : G \to S_x$ is a homomorphism.

$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).

$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle \implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|$

But $\#\langle f(x), f(y), f(t) \rangle = 21504$. So, $|\frac{G}{kerf}| = 21504$

This means $|G| \geq 21504$. We know $|G| \leq 21504$ from cayley diagram.

Therefore, $|G| = 21504$.

From $|G| = 21504 \times |Kerf|$ we find $|Kerf| = 1$

$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong 2^7 : PSL(2,7) \implies G \cong 2^7 : PSL(2,7)$.

7.12   Lemma 3.3 in Construction of $2^{*7} : (7 : 3))$ over $(2^3 : (7 : 3)$

Let $N \cong 2^3 : (7 : 3) = \langle x, y \rangle$ where $x \sim (1, 11, 13)(2, 12, 10)(3, 7, 9)(5, 6, 14)$, and $y \sim (1, 13, 2)(3, 9, 6)(4, 7, 12, 8, 11, 14)(5, 10)$. The progenitor $2^{*14} : N$ has symmetric presentation.

```
G<x,y,t>:=Group<x,y,t|x^3, y^6,
(y^-1 * x^-1)^3,
 x * y * x^-1 * y^-1 * x^-1 * y * x^-1 * y^-2,
 t^2>
```

We want to verify that the presentation given above is correct.

Using the theorem: $\frac{2^{*n} : N}{t_i, t_j} \cong 2^n : N$ where $1 \leq i \leq j \leq n$,

we will show that $\frac{2^{*14} : N}{((y^2) * t^{(y^{-1})})^3} \cong 2^7 : (7 : 3)$ Thus, we will prove

```
G<x,y,t>:=Group<x,y,t|x^3, y^6, (y^-1 * x^-1)^3,
 x * y * x^-1 * y^-1 * x^-1 * y * x^-1 * y^-2,
 t^2,
 (t,y^3),(t,x^-1*y^-1),
(t,t^{(y^-1)}),(t,t^{(x * y^2)}),
(t,t^{(x * y * x^-1 * y^-1 * x)}),
((y^2)*t^(y^{-1}))^3> ~ (2^7:(7:3))
```

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find our index, which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{2688}{168} = 16$. Now we know that we have 16 distinct single cosets.

<u>Expanding Relation</u> We will expand our only relation.

$$((y^2) * t^{(y^{-1})})^3$$

$$= y^2 * t_2 * y^2 * t_2 * y^2 * t_2$$
$$= (y^2)^3 * (t_2^{(y^2)^2}) * (t_2^{(y^2)}) * t_2$$
$$= (y^6) * (t_2^{(y^4)}) * (t_2^{(y^2)}) * t_2$$
$$\implies (y^6) * t_1 t_{13} t_2 = e \implies (y^6) * t_1 t_{13} = t_2$$

First Double Coset $[*]$

$NeN = \{Ne^n | n \in N\}$. The double coset NeN $= [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The formula for the number of right cosets in N is $\frac{168}{168} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. We need only determine the double coset of the right cosets Nt_1. Thus 14 t_i's extend the double coset $[*]$ to the double coset $Nt_1 N = [1]$.
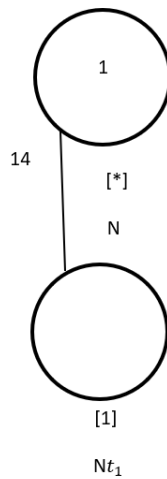
Cayley Diagram

Figure 7.91: Cayley Diagram for G over S_{14}

Second Double Coset $[1]$

$Nt_1 N = \{N(t_1)^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}\}$.

We now find the Coset Stabilier $N^{1)}$. We first find the point stabiliser of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

Thus, $N^{(1)} \geq \langle N^1 = (2, 11, 10)(4, 13, 12)(5, 6, 7)(8, 9, 14)(2, 6)(7, 11)(9, 13)(12, 14)\rangle$

The number of right cosets in $Nt_1 N$ is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{168}{14} = 12$.

The orbits of N^1 on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ are

$\{1\}, \{3\}, \{2, 7, 6, 10, 11, 5\}, \{4, 13, 14, 9, 12, 8\}$.

We will determine the double cosets by selecting one representative from each orbit such as,

Choose 1 from {1}

$Nt_1t_1 = Nt_1^2 = N \in [*]$

This means one t_i takes [1] to [*]

Choose 3 from {3}

$Nt_1t_3 \in [13]$

This means one t_i takes [1] to [13]

Choose 2 from {2,7,6,10,11,5}

$Nt_1t_2 \in [1]$.

We have $Nt_1t_2 = Nt_{13}$

$= (t_1)^y$

This means six t_i's take [1] to [1]

Choose 4 from {4, 13, 14, 9, 12, 8}

$Nt_1t_4 \in [1]$.

We have $Nt_1t_4 = Nt_{10}$

$= (t_1)^{(y^{-1}*x^{-1})}$

This means six t_i's take [1] to [1]
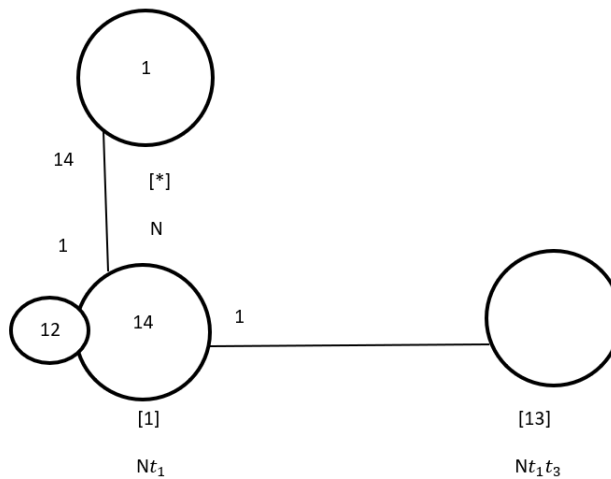
Cayley Diagram

Figure 7.92: Cayley Diagram for G over S_{14}

Third Double Coset [13]

$Nt_1t_3 = \{N(t_1t_3)^n | n \in N\}$.

We now find the Coset Stabilier $N^{13)}$. We first find the point stabiliser of 1 and 3 in N.

$N^{13} = \{n \in N | 13^n = 13\}$

$N^{13} = \langle(2,6)(7,11)(9,13)(12,14)(2,11,5)(4,9,14)(6,7,10)(8,13,12)\rangle$.

Now we will find different names of Nt_1t_3. We have $Nt_1t_3 = Nt_8t_4$ Thus, $N^{(13)} \geq \langle N^{13} = (2,6)(7,11)(9,13)(12,14)(2,11,5)(4,9,14)(6,7,10)(8,13,12)\rangle$

The number of single cosets in Nt_1t_3 is calculated by the formula, $\frac{|N|}{|N^{(13)}|} = \frac{168}{168} = 1$

The orbits of N^{13} on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$ are

$\{1,8,13,12,10,2,14,5,6,9,4,11,3,7\}$.

Now, we choose one representative from this orbit,

Choose 1 from $\{1, 8, 13, 12, 10, 2, 14, 5, 6, 9, 4, 11, 3, 7 \}$

$1 \in \{1,8,13,12,10,2,14,5,6,9,4,11,3,7\} \implies Nt_1t_3t_1 \in [1]$.

We have $Nt_1t_3t_1 = Nt_3$

$= (t_1)^{x*y*x^{-1}*y^{-1}*x}$

$= (t_1)^{(1,3)(2,10,11,6,5,7)(4,14,9)(8,12,13)} \in [1]$.

This means fourteen t_i's take [13] to [1].

Cayley Diagram
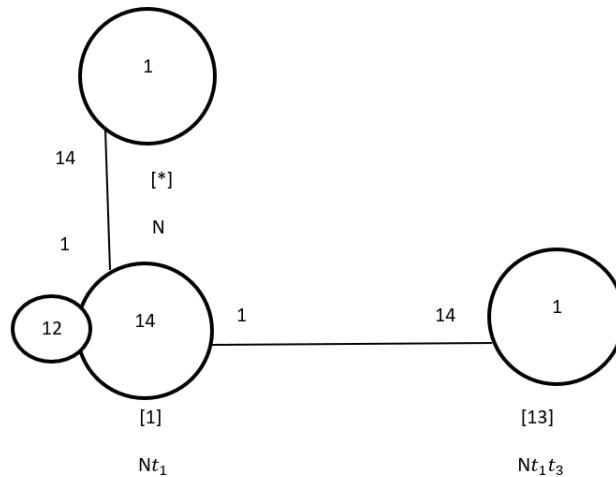
Figure 7.93: Cayley Diagram for G over S_{14}

Now, all elements being addressed to their double cosets, we calculate.

$G = N \cup Nt_1 \cup Nt_1t_3$

$|G| \leq (|N| + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(13)}|} \times |N| \implies |G| \leq (1 + 14 + 1) \times 168 \implies |G| \leq 16 \times 168 \implies$
$|G| \leq 2688$. But from Cayley Diagram we know that $|G| \geq 2688$. Therefore, $|G| = 2688$

_____

Chapter 8

Double Coset Enumeration Involving Maximal Subgroup

As stated in previous chapter, a double coset enumeration is performed using an algorithm. In some cases, double coset enumerations are more difficult than others. It depends on the group itself and the control group N. It has been discovered that performing the double coset enumeration over a maximal subgroup produces the same outcome. In the next example we will show the process of double coset enumerations over a maximal subgroup H.

8.1 S_5 over $2^4 : A_4$ and $2^4 : 3$

Consider H to be a subgroup of G such that $N \leq H \leq G$. We can write G as a set of single cosets in H. We perform double coset enumeration of G over $H = 2^4 : A_4 = \langle x, y, t_1 t_2 t_3 t_1 \rangle = (1, 2, 3)$ and $2^4 : 3 = N$. It should be noted that the order of H is 24. We will prove that the progenitor $2^{*4} : (2^2 : 3)$, where $2^{*4} : (2^2 : 3) = \langle x, y \rangle$ giving $x \sim (1, 2)(3, 4)$ and $y \sim (1, 2, 3)$ factored by one relation is isomorphic to S_5.

Let $G \cong \frac{2^{*4}(2^2:2)}{t_1 t_2 t_3 t_1 = (1,2,3)}$. Thus we show that $G \sim S_5$.

First Double Coset $[*]$

$HeN = \{H(e)^n | n \in N\} = \{H | n \in N\}$

The coset stabiliser He is H. The number of single right cosets in the double coset

$HeN = [*]$ is given by

$\frac{|H|}{|H|} = \frac{24}{24} = 1$.

The orbits of H on $X = \{1, 2, 3, 4\}$ is $\{1, 2, 3, 4\}$. We will now choose an orbit representative and multiply the representative by H on the right and determine its double coset.

Choose 1 from $\{1,2,3,4\}$

$Ht_1 \in [1]$

Second Double Coset

$Ht_1 N = \{H(t_1)^n | n \in N\}$

$= \{Ht_1, Ht_2, Ht_3, Ht_4\}$

The coset stabiliser $N^{(1)} \geq N^1 = \langle (2, 3, 4) \rangle$

The number of single right cosets in the double coset $Ht_1 N = [1]$ is given by the formula,

$\frac{|N|}{|N^{(1)}|}| = \frac{24}{6} = 4$

The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4\}$ are $\{1\}, \{2, 3, 4\}$. We will now choose a representative of each orbit and multiply Ht_1 on the right and determine its double coset.

Choose 1 from $\{1\}$

$Ht_1 t_1 = Ht_1^2 = H \in [*]$.

This means one t_i takes $[1]$ to $[*]$.

Choose 2 from $\{2, 3, 4\}$

$Ht_1 t_2 \in [12]$.

This means three t_i's take $[1]$ to $[12]$.

Third Double Coset

$Ht_1 t_2 N = \{H(t_1 t_2)^n | n \in N\}$

$= \{Ht_1 t_2, Ht_2 t_3, \ldots, Ht_2 t_1\}$.

We now find the Coset Stabilier $N^{12)}$. We first find the point stabiliser of 1 and 2 in N.

The point-stabiliser N^{12} is given by $\langle e \rangle$

The coset stabiliser $N^{(12)} = \langle e, (2, 3, 4) \rangle$

The orbits of $N^{(12)}$ on $X = \{1, 2, 3, 4\}$ are $\{1\}, \{2\}, \{3\}, \{4\}$. We will now choose a representative of each orbit and multiply $Ht_1 t_2$ on the right and determine its double coset.

Choose 1 from $\{1\}$

$Nt_1t_2t_1 \in [121]$

Choose 2 from $\{2\}$

$Nt_1t_2t_2 \in [*]$

Choose 3 from $\{3\}$

$Nt_1t_2t_3 \in [121]$

Choose 4 from $\{4\}$

$Nt_1t_2t_4 \in [121]$

Fourth Double Coset

$Ht_1t_2t_1N = \{H(t_1t_2t_1)^n | n \in N\}$

$= \{Ht_1t_2t_1, Ht_2t_3t_1, \ldots, Ht_2t_1t_2\}.$

We now find the Coset Stabilier $N^{121)}$. We first find the point stabiliser of 1,2 and 1 in

N. The point-stabiliser N^{121} is given by $\langle e \rangle$

The coset stabiliser $N^{(121)} = \langle e, (2,3,4) \rangle$

The orbits of $N^{(121)}$ on $X = \{1,2,3,4\}$ is $\{1,3,4,2\}$. We will now choose a representative

of this orbit and multiply $Ht_1t_2t_1$ on the right and determine its double coset.

Using our relation (see below) we have $12 \sim 13 \sim 14$.

$121 = (1,2,4)141$

$121 = 1(213)3 = (2,1,3)323 \implies (2,3,4), (1,3,4) \in N^{(121)}$. Thus $N^{(121)} \geq \langle e, (2,3,4), (1,3,4) \rangle$.

$\implies Nt_1t_2t_1 = Nt_1t_3t_1 = Nt_1t_4t_1$. This means all the symmetric generators are in the

same double coset.

$\frac{|N|}{|N^{(121)}|} = \frac{12}{12} = 1$

Thus, $G = H \cup Ht_1 \cup Ht_1t_2 \cup Ht_1t_2t_1$

Now $H = \langle N, t_1t_2t_1 \rangle$, which is double the order of N, then $H \cup Nt_1t_2t_1$ since the order

of H is 24 and the order of N is 12. So,

$G = N \cup Nt_1t_2t_1$ which gives

$G = N \cup Nt_1t_2t_1 \cup (N \cup Nt_1t_2t_1)t_1 \cup (N \cup Nt_1t_2t_1)t_2 \cup (N \cup Nt_1t_2t_1)t_3 \cup (N \cup Nt_1t_2t_1)t_4$

$\implies G = N \cup Nt_1t_2t_1 \cup Nt_1 \cup Nt_1t_2t_1t_1 \cup Nt_2 \cup Nt_1t_2t_1t_2 \cup Nt_3 \cup Nt_1t_2t_1t_3 \cup Nt_4 \cup Nt_1t_2t_1t_4$

Our goal is to get G similar to our DCE $2^{*4} : (2^4 : 3)$ which is $G = N \cup Nt_1 \cup Nt_2 \cup Nt_3 \cup$

$Nt_4 \cup Nt_1t_2 \cup Nt_1t_3 \cup Nt_1t_4 \cup Nt_2t_3 \cup Nt_2t_4 \cup Nt_3t_4.$

We need to investigate further, so we are going to take the following elements of our G,

and use our relation to see where they belong:

$Ht_1t_2t_1$. Now $Nt_1t_2 = Nt_1t_3$ (Relation) $\implies (2,3,4) \in N^{(12)}$.

Thus $N^{(12)} = \langle (2,3,4) \rangle$.

$Nt_1t_2 = Nt_1t_3 = Nt_1t_4$

From Magma we got:

```
{[1,2]}^N;
/*
GSet{@
    {
        [ 1, 3 ],
        [ 1, 2 ],
        [ 1, 4 ]
    },
    {
        [ 2, 4 ],
        [ 2, 3 ],
        [ 2, 1 ]
    },
    {
        [ 3, 4 ],
        [ 3, 2 ],
        [ 3, 1 ]
    },
    {
        [ 4, 1 ],
        [ 4, 3 ],
        [ 4, 2 ]
    }
@}
*/
```

That means, $13 \sim 12 \sim 14$, $24 \sim 23 \sim 21$, $34 \sim 32 \sim 31$, and $41 \sim 43 \sim 42$

$\frac{|N|}{|N^{(12)}|} = \frac{12}{3} = 4$

Now $Nt_1t_2t_1 = Nt_1t_3t_1 = Nt_1t_4t_1$

Given relation: $12 = (1,2,3)13$.

using conjugation by elements of N we get,

$(Nt_1t_2)^{(12)(34)} = (Nt_1t_3)^{(12)(34)} \implies 21 \sim 24$

$(Nt_1t_2)^{(132)} = (Nt_1t_3)^{(132)} \implies 31 \sim 32$

$(Nt_1t_2)^{(124)} = (Nt_1t_3)^{(124)} \implies 24 \sim 23$

$(Nt_1t_2)^{(13)(24)} = (Nt_1t_3)^{(13)(24)} \implies 34 \sim 31$

$(Nt_1t_2)^{(143)} = (Nt_1t_3)^{(143)} \implies 42 \sim 41$

$(Nt_1t_2)^{(134)} = (Nt_1t_3)^{(134)} \implies 32 \sim 34$

$(Nt_1t_2)^{(142)} = (Nt_1t_3)^{(142)} \implies 41 \sim 43$

$(Nt_1t_2)^{(14)(23)} = (Nt_1t_3)^{(14)(23)} \implies 43 \sim 42$

$(Nt_1t_2)^{(243)} = (Nt_1t_3)^{(243)} \implies 14 \sim 12$

$(Nt_1t_2)^{(123)} = (Nt_1t_3)^{(123)} \implies 23 \sim 21$

$(Nt_1t_2)^{(234)} = (Nt_1t_3)^{(234)} \implies 13 \sim 14$

Now, we write G as a union of the right Cosets,

$G = N \cup Nt_1 \cup Nt_2 \cup Nt_3 \cup Nt_4 \cup Nt_1t_2 \cup Nt_2t_3 \cup Nt_3t_1 \cup Nt_4t_3 \cup Nt_1t_2t_1$

$|G| \leq \frac{|N|}{|N|} + \frac{|N|}{|N^{(1)}|} + \frac{|N|}{|N^{(12)}|} + \frac{|N|}{|N^{(121)}|} \times |N|$

$= 1 + 4 + 4 + 1 \times 12$

$= 10 \times 12 = 120$

The second Part is

DCE of $G = \frac{2^{*4}:A_4}{1231=(123)}$ over $H = < A_4, t_1, t_2, t_1 >$ and $N = S_4$

$N^{(1)} = \langle e \rangle$ Orbits are $\{1\}, \{2\}, \{3\}, \{4\}$

$Ht_1t_1 \in [*]$

$H = \langle S_4, t_1, t_2, t_1 \rangle$

$\implies t_1t_2t_1 \in H \implies Ht_1t_2t_1 = H$

$\implies Ht_1t_2 = Ht_1 \in [1]$.

Our relation is $12 = (123)13$

$Ht_1t_2 = Ht_1t_3 \implies Ht_1t_3 \in [1]$

$G = H \cup Ht_1 \cup Ht_2 \cup Ht_3 \cup Ht_4$

Replace H we get,

$G = N \cup Nt_1t_2t_1 \cup Nt_1 \cup Nt_1t_2t_1t_1 \cup Nt_2 \cup Nt_1t_2t_1t_2 \cup Nt_3 \cup Nt_1t_2t_1t_3 \cup Nt_4 \cup Nt_1t_2t_1t_4$

if we conjugate, $H = \langle N, t_1t_2t_1 \rangle$

$t_1t_2t_1 \in H, t_2t_3t_2 \in H, t_3t_1t_3 \in H, t_4t_3t_4 \in H$

$\implies Ht_1t_2t_1 = Ht_2t_3t_2 = Ht_3t_1t_3 = Ht_4t_3t_4$

Now we have,

$G = N \cup Nt_1 \cup Nt_2 \cup Nt_3 \cup Nt_4 \cup Nt_1t_2 \cup Nt_2t_3t_2t_2 = Nt_2t_3 \cup Nt_3t_1t_3t_3 = Nt_3t_1 \cup$ $Nt_4t_3t_4t_4 = Nt_4t_3 \cup Nt_1t_2t_1$.

Thus is the same decomposition as in the Case of DCE G over N.

Chapter 9

Monomial Representations

In this chapter we find monomial presentations for several groups. We will use Magma to view the character table of each group with the goal of writing the monomial representation of each respective control group, G. In our control group's character table we look for characters with degrees greater than 1 ,say m, that are also faithful. Our group is guaranteed a faithful irreducible monomial representation if the group has a subgroup, H, with an index m and if H has a linear character that will induce up to the degree m character of G. After obtaining the faithful irreducible monomial representation of the given group, we determine the progenitor. We then convert those matrices into permutations and construct a permutation representation of G.

Theorem 9.1. *The number of irreducible characters of G is equal to the number of conjugacy classes of G [Led77].*

Definition 9.2. *commutator Let G be a group with $a, b \in G$. Then the commutator of a and b, denoted $[a, b]$, is $[a, b] = aba^{-1}b^{-1}$ [Rot94]*

Definition 9.3. *normalizer Let G be a group with $H \leq G$. Then the normalizer of H in G is the set $\{a \in G | aHa^{-1} = H\}$ [Rot94]*

Definition 9.4. *Let G be a group, the **Monomial Representation** is a map $M : G \xrightarrow{Hom} GL(n, F)$ that provide which $M(x)$ and $M(y)$ are monomial matrices [Led77].*

Definition 9.5. *Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m. We consider the characteristic polynomial of $A(x)$, namely.*

$$det(\lambda I - A(x)) = \begin{bmatrix} \Lambda - a_{11}(x) & \Lambda - a_{12}(x) & ... & \Lambda - a_{1m}(x) \\ \Lambda - a_{21}(x) & \Lambda - a_{22}(x) & ... & \Lambda - a_{2m}(x) \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \Lambda - a_{m1}(x) & \Lambda - a_{m2}(x) & ... & \Lambda - a_{mm}(x) \end{bmatrix}$$ *This is polyno-*

mial of degree m in λ, with the coefficient of $-\lambda^{m-1}$ is

$\alpha(x) = a_{11}(x) + a_{22}(x) + ... + a_{mm}(x).$

*It is customary to call the right-hand side of this equation the trace of $A(x)$ abbreviated to $trA(x)$, so that $\alpha(x) = trA(x)$. We regard $\alpha(x)$ as a function on G with values in field K, and we call it **the character** of $A(x)$ [Led77].*

Definition 9.6. *The character of $A(x)$, which is called the **induced character** of ϕ denoted by ϕ^G. Thus,*

$$\phi^G = trA(x) = \sum_{i=1}^{n} \phi(t_i x t_i^{-1})$$

. The formula of induced character is given by,

$$\phi \uparrow_{\alpha}^{G} = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} = \alpha(w), \alpha = 1, 2, 3, ..., m$$

[Led77].

9.1   First Monomial Representations of $37^{*2} :_m (2^3 : S_3)$

We are given G, a transitive group on 24 letters, which is generated by

```
xx ~ (1, 16, 17, 8, 9, 23)(2, 15, 18, 7, 10, 24)
(3, 6, 19, 21, 11, 13)(4, 5, 20, 22, 12, 14).
yy ~ (1, 20)(2, 19)(3, 18)(4, 17)(5, 15)(6, 16)
(7, 14)(8, 13)(9, 12)(10, 11)(21, 23)(22, 24).
zz ~ (1, 21, 18, 14, 9, 6, 2, 22, 17, 13, 10, 5)
(3, 24, 20, 16, 11, 7, 4, 23, 19, 15, 12, 8).
G:=sub<G|xx,yy,zz>;
#G;
48
```

We note that $G \cong 2^3 : S_3$ We will write a presentation for the monomial progenitor $37^{*2} : (2^3 : S_3)$ note that 2^{*3} represents 2 t's of order 3. We now use the character table of $2^3 : S_3$ to determine if it has any faithful characters of degree greater than 1. We can see that it has 2 characters with degree 3, but only 1 of them is faithful. We consider this character $CG[16]$. We note that in order for $2^3 : S_3$ to have a faithful irreducible monomial representation of degree 3 there must exist a subgroup, H, of G with index 3 which has a linear character that induces up to $CG[16]$. We label the classes of $2^3 : S_3$ below, and then find the suitable subgroup H. We need to induce a non-trivial linear character from a subgroup H of G such that $\frac{|G|}{|H|} = 2$ which is the number of t's in our presentation. $|G| = 48$ therefore we must have $|H| = 24$. Here, the conjugacy classes of G are:

$C_1 = \{e\}$

$C_2 = \{(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)\}$

$C_3 = \{(1,8)(2,7)(3,21)(4,22)(5,12)(6,11)(9,16)(10,15)(13,19)(14,20)(17,23)(18,24)\}$

$C_4 = \{(1,20)(2,19)(3,18)(4,17)(5,15)(6,16)(7,14)(8,13)(9,12)(10,11)(21,23)(22,24)\}$

$C_5 = \{(1,2)(3,11)(4,12)(5,21)(6,22)(9,18)(10,17)(13,14)(15,24)(16,23)\}$

$C_6 = \{(1,17,9)(2,18,10)(3,19,11)(4,20,12)(5,22,14)(6,21,13)(7,24,15)(8,23,16)\}$

$C_7 = \{(1,14,2,13)(3,16,4,15)(5,18,6,17)(7,19,8,20)(9,22,10,21)(11,23,12,24)\}$

$C_8 = \{(1,13,2,14)(3,15,4,16)(5,17,6,18)(7,20,8,19)(9,21,10,22)(11,24,12,23)\}$

$C_9 = \{(1,20,2,19)(3,9,4,10)(5,24,6,23)(7,13,8,14)(11,17,12,18)(15,21,16,22)\}$

$C_{10} = \{(1,6,2,5)(3,16,4,15)(7,11,8,12)(9,21,10,22)(13,18,14,17)(19,23,20,24)\}$

$C_{11} = \{(1,24,2,23)(3,21,4,22)(5,19,6,20)(7,18,8,17)(9,15,10,16)(11,13,12,14)\}$

$C_{12} = \{(1,16,17,8,9,23)(2,15,18,7,10,24)(3,6,19,21,11,13)(4,5,20,22,12,14)\}$

$C_{13} = \{(1,23,9,8,17,16)(2,24,10,7,18,15)(3,13,11,21,19,6)(4,14,12,22,20,5)\}$

$C_{14} = \{(1,18,9,2,17,10)(3,20,11,4,19,12)(5,21,14,6,22,13)(7,23,15,8,24,16)\}$

$$C_{15} = \{(1, 21, 18, 14, 9, 6, 2, 22, 17, 13, 10, 5)(3, 24, 20, 16, 11, 7, 4, 23, 19, 15, 12, 8)\}$$

$$C_{16} = \{(1, 22, 18, 13, 9, 5, 2, 21, 17, 14, 10, 6)(3, 23, 20, 15, 11, 8, 4, 24, 19, 16, 12, 7)\}$$

$$C_{17} = \{(1, 11, 10, 20, 17, 3, 2, 12, 9, 19, 18, 4)(5, 16, 13, 24, 22, 8, 6, 15, 14, 23, 21, 7)\}$$

$$C_{18} = \{(1, 3, 18, 20, 9, 11, 2, 4, 17, 19, 10, 12)(5, 8, 21, 24, 14, 16, 6, 7, 22, 23, 13, 15)\}$$

Consider the subgroup H of G given below.

```
H = <(1, 20, 2, 19)(3, 9, 4, 10)(5, 24, 6, 23)
(7, 13, 8, 14)(11, 17, 12, 18)(15, 21, 16, 22)
(1, 16, 17, 8, 9, 23)(2, 15, 18, 7, 10, 24)
(3, 6, 19, 21, 11, 13)(4, 5, 20,22, 12, 14)
(1, 10, 17, 2, 9, 18)(3, 12, 19, 4, 11, 20)
(5, 13, 22, 6, 14, 21)(7, 16, 24,8, 15, 23)
(1, 17, 9)(2, 18, 10)(3, 19, 11)(4, 20, 12)
(5, 22, 14)(6, 21, 13)(7, 24,15)(8, 23, 16)>
```

The conjugacy classes of H are

$$D_1 = \{Id(H)\}$$

$$D_2 = \{(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)\}$$

$$D_3 = \{(1, 8)(2, 7)(3, 21)(4, 22)(5, 12)(6, 11)(9, 16)(10, 15)(13, 19)(14, 20)(17, 23)(18, 24)\}$$

$$D_4 = \{(1, 7)(2, 8)(3, 22)(4, 21)(5, 11)(6, 12)(9, 15)(10, 16)(13, 20)(14, 19)(17, 24)(18, 23)\}$$

$$D_5 = \{(1, 17, 9)(2, 18, 10)(3, 19, 11)(4, 20, 12)(5, 22, 14)(6, 21, 13)(7, 24, 15)(8, 23, 16)\}$$

$$D_6 = \{(1, 9, 17)(2, 10, 18)(3, 11, 19)(4, 12, 20)(5, 14, 22)(6, 13, 21)(7, 15, 24)(8, 16, 23)\}$$

$$D_7 = \{(1, 20, 2, 19)(3, 9, 4, 10)(5, 24, 6, 23)(7, 13, 8, 14)(11, 17, 12, 18)(15, 21, 16, 22)\}$$

$$D_8 = \{(1, 19, 2, 20)(3, 10, 4, 9)(5, 23, 6, 24)(7, 14, 8, 13)(11, 18, 12, 17)(15, 22, 16, 21)\}$$

$$D_9 = \{(1, 13, 2, 14)(3, 15, 4, 16)(5, 17, 6, 18)(7, 20, 8, 19)(9, 21, 10, 22)(11, 24, 12, 23)\}$$

$$D_{10} = \{(1, 14, 2, 13)(3, 16, 4, 15)(5, 18, 6, 17)(7, 19, 8, 20)(9, 22, 10, 21)(11, 23, 12, 24)\}$$

$$D_{11} = \{(1, 16, 17, 8, 9, 23)(2, 15, 18, 7, 10, 24)(3, 6, 19, 21, 11, 13)(4, 5, 20, 22, 12, 14)\}$$

$$D_{12} = \{(1, 23, 9, 8, 17, 16)(2, 24, 10, 7, 18, 15)(3, 13, 11, 21, 19, 6)(4, 14, 12, 22, 20, 5)\}$$

$$D_{13} = \{(1, 10, 17, 2, 9, 18)(3, 12, 19, 4, 11, 20)(5, 13, 22, 6, 14, 21)(7, 16, 24, 8, 15, 23)\}$$

$$D_{14} = \{(1, 18, 9, 2, 17, 10)(3, 20, 11, 4, 19, 12)(5, 21, 14, 6, 22, 13)(7, 23, 15, 8, 24, 16)\}$$

$$D_{15} = \{(1, 24, 9, 7, 17, 15)(2, 23, 10, 8, 18, 16)(3, 14, 11, 22, 19, 5)(4, 13, 12, 21, 20, 6)\}$$

$$D_{16} = \{(1, 15, 17, 7, 9, 24)(2, 16, 18, 8, 10, 23)(3, 5, 19, 22, 11, 14)(4, 6, 20, 21, 12, 13)\}$$

$$D_{17} = \{(1, 22, 18, 13, 9, 5, 2, 21, 17, 14, 10, 6)(3, 23, 20, 15, 11, 8, 4, 24, 19, 16, 12, 7)\}$$

$$D_{18} = \{(1, 5, 10, 13, 17, 22, 2, 6, 9, 14, 18, 21)(3, 8, 12, 15, 19, 23, 4, 7, 11, 16, 20, 24)\}$$

$$D_{19} = \{(1, 21, 18, 14, 9, 6, 2, 22, 17, 13, 10, 5)(3, 24, 20, 16, 11, 7, 4, 23, 19, 15, 12, 8)\}$$

$$D_{20} = \{(1, 6, 10, 14, 17, 21, 2, 5, 9, 13, 18, 22)(3, 7, 12, 16, 19, 24, 4, 8, 11, 15, 20, 23)\}$$

$$D_{21} = \{(1, 3, 18, 20, 9, 11, 2, 4, 17, 19, 10, 12)(5, 8, 21, 24, 14, 16, 6, 7, 22, 23, 13, 15)\}$$

$$D_{22} = \{(1, 11, 10, 20, 17, 3, 2, 12, 9, 19, 18, 4)(5, 16, 13, 24, 22, 8, 6, 15, 14, 23, 21, 7)\}$$

$$D_{23} = \{(1, 4, 18, 19, 9, 12, 2, 3, 17, 20, 10, 11)(5, 7, 21, 23, 14, 15, 6, 8, 22, 24, 13, 16)\}$$

$$D_{24} = \{(1, 12, 10, 19, 17, 4, 2, 11, 9, 20, 18, 3)(5, 15, 13, 23, 22, 7, 6, 16, 14, 24, 21, 8)\}$$

We verify. Induction(CH[15],G) eq CG[16];

true

We know that the character table of H is given by

```
   Character Table of Group H
   --------------------------
```

Class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Size	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Order	1	2	2	2	3	3	4	4	4	4	6	6	6	6	6	6	12	12	12

| p = 2 | 1 | 1 | 1 | 1 | 6 | 5 | 2 | 2 | 2 | 2 | 5 | 6 | 5 | 6 | 6 | 5 | 14 | 13 | 14 |
| p = 3 | 1 | 2 | 3 | 4 | 1 | 1 | 8 | 7 | 10 | 9 | 3 | 3 | 2 | 2 | 4 | 4 | 9 | 9 | 10 |

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	+	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	1
X.3	0	1	-1	-1	1	1	1	-I	I	-I	I	-1	-1	-1	-1	1	1	I	I	-I
X.4	0	1	-1	1	-1	1	1	I	-I	-I	I	1	1	-1	-1	-1	-1	I	I	-I
X.5	+	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1
X.6	+	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	-1
X.7	0	1	-1	-1	1	1	1	I	-I	I	-I	-1	-1	-1	-1	1	1	-I	-I	I
X.8	0	1	-1	1	-1	1	1	-I	I	I	-I	1	1	-1	-1	-1	-1	-I	-I	I
X.9	0	1	1	1	1	J-1-J	1	1	1	1-1-J	J-1-J	J	J-1-J	-1-J	J	-1-J
X.10	0	1	1	-1	-1	J-1-J	-1	-1	1	1	1+J	-J-1-J	J	-J	1+J	-1-J	J	-1-J
X.11	0	1	-1	-1	1	J-1-J	-I	I	-I	I	1+J	-J	1+J	-J	J-1-J	Z1	Z1#5	-Z1
X.12	0	1	-1	1	-1	J-1-J	I	-I	-I	I-1-J	J	1+J	-J	-J	1+J	Z1	Z1#5	-Z1
X.13	0	1	1	1	1	J-1-J	-1	-1	-1	-1-1-J	J-1-J	J	J-1-J	1+J	-J	1+J
X.14	0	1	1	-1	-1	J-1-J	1	1	-1	-1	1+J	-J-1-J	J	-J	1+J	1+J	-J	1+J
X.15	0	1	-1	-1	1	J-1-J	I	-I	I	-I	1+J	-J	1+J	-J	J-1-J	-Z1	-Z1#5	Z1
X.16	0	1	-1	1	-1	J-1-J	-I	I	I	-I-1-J	J	1+J	-J	-J	1+J	-Z1	-Z1#5	Z1
X.17	0	1	1	1	1-1-J	J	1	1	1	1	J-1-J	J-1-J-1-J	J	J	-1-J	J
X.18	0	1	1	-1	-1-1-J	J	-1	-1	1	1	-J	1+J	J-1-J	1+J	-J	J	-1-J	J
X.19	0	1	-1	-1	1-1-J	J	-I	I	-I	I	-J	1+J	-J	1+J-1-J	J	Z1#5	Z1-Z1#5
X.20	0	1	-1	1	-1-1-J	J	I	-I	-I	I	J-1-J	-J	1+J	1+J	-J	Z1#5	Z1-Z1#5
X.21	0	1	1	1	1-1-J	J	-1	-1	-1	-1	J-1-J	J-1-J-1-J	J	-J	1+J	-J
X.22	0	1	1	-1	-1-1-J	J	1	1	-1	-1	-J	1+J	J-1-J	1+J	-J	-J	1+J	-J
X.23	0	1	-1	-1	1-1-J	J	I	-I	I	-I	-J	1+J	-J	1+J-1-J	J-Z1#5	-Z1	Z1#5
X.24	0	1	-1	1	-1-1-J	J	-I	I	I	-I	J-1-J	-J	1+J	1+J	-J-Z1#5	-Z1	Z1#5

```
   --------------------------------
```

Class	20	21	22	23	24
Size	1	1	1	1	1

```
Order |    12   12   12   12   12
--------------------------------
p  =  2     13   14   13   14   13
p  =  3     10    7    7    8    8
--------------------------------
X.1   +      1    1    1    1    1
X.2   +      1   -1   -1   -1   -1
X.3   0     -I    I    I   -I   -I
X.4   0     -I   -I   -I    I    I
X.5   +     -1   -1   -1   -1   -1
X.6   +     -1    1    1    1    1
X.7   0      I   -I   -I    I    I
X.8   0      I    I    I   -I   -I
X.9   0      J -1-J    J -1-J    J
X.10  0      J  1+J   -J  1+J   -J
X.11  0  -Z1#5   Z1 Z1#5  -Z1-Z1#5
X.12  0  -Z1#5 -Z1-Z1#5   Z1 Z1#5
X.13  0     -J  1+J   -J  1+J   -J
X.14  0     -J -1-J    J -1-J    J
X.15  0   Z1#5 -Z1-Z1#5   Z1 Z1#5
X.16  0   Z1#5   Z1 Z1#5  -Z1-Z1#5
X.17  0   -1-J    J -1-J    J -1-J
X.18  0   -1-J   -J  1+J   -J  1+J
X.19  0    -Z1 Z1#5   Z1-Z1#5  -Z1
X.20  0  -Z1-Z1#5  -Z1 Z1#5    Z1
X.21  0    1+J   -J  1+J   -J  1+J
X.22  0    1+J    J -1-J    J -1-J
X.23  0   Z1-Z1#5  -Z1 Z1#5    Z1
X.24  0     Z1 Z1#5   Z1-Z1#5  -Z1
```

We verify by hand that we have a monomial representation of $2^3 : S_3$ by inducing $CH[15]$ up to $CG[16]$. Consider the irreducible character ϕ of H and ϕ^G of G given below.

Irreducible Character of ϕ			
Class	Size	Representation	ϕ
D_1	1	$Id(H)$	1
D_2	1	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$	-1
D_3	1	$(1,8)(2,7)(3,21)(4,22)(5,12)(6,11)(9,16)(10,15)(13,19)(14,20)(17,23)(18,24)$	-1
D_4	1	$(1,7)(2,8)(3,22)(4,21)(5,11)(6,12)(9,15)(10,16)(13,20)(14,19)(17,24)(18,23)$	1
D_5	1	$(1,17,9)(2,18,10)(3,19,11)(4,20,12)(5,22,14)(6,21,13)(7,24,15)(8,23,16)$	3
D_6	1	$(1,9,17)(2,10,18)(3,11,19)(4,12,20)(5,14,22)(6,13,21)(7,15,24)(8,16,23)$	33
D_7	1	$(1,20,2,19)(3,9,4,10)(5,24,6,23)(7,13,8,14)(11,17,12,18)(15,21,16,22)$	9
D_8	1	$(1,19,2,20)(3,10,4,9)(5,23,6,24)(7,14,8,13)(11,18,12,17)(15,22,16,21)$	28
D_9	1	$(1,13,2,14)(3,15,4,16)(5,17,6,18)(7,20,8,19)(9,21,10,22)(11,24,12,23)$	9
D_{10}	1	$(1,14,2,13)(3,16,4,15)(5,18,6,17)(7,19,8,20)(9,22,10,21)(11,23,12,24)$	28
D_{11}	1	$(1,16,17,8,9,23)(2,15,18,7,10,24)(3,6,19,21,11,13)(4,5,20,22,12,14)$	4
D_{12}	1	$(1,23,9,8,17,16)(2,24,10,7,18,15)(3,13,11,21,19,6)(4,14,12,22,20,5)$	34
D_{13}	1	$(1,10,17,2,9,18)(3,12,19,4,11,20)(5,13,22,6,14,21)(7,16,24,8,15,23)$	4
D_{14}	1	$(1,18,9,2,17,10)(3,20,11,4,19,12)(5,21,14,6,22,13)(7,23,15,8,24,16)$	34
D_{15}	1	$(1,24,9,7,17,15)(2,23,10,8,18,16)(3,14,11,22,19,5)(4,13,12,21,20,6)$	3
D_{16}	1	$(1,15,17,7,9,24)(2,16,18,8,10,23)(3,5,19,22,11,14)(4,6,20,21,12,13)$	33
D_{17}	1	$(1,22,18,13,9,5,2,21,17,14,10,6)(3,23,20,15,11,8,4,24,19,16,12,7)$	1
D_{18}	1	$(1,5,10,13,17,22,2,6,9,14,18,21)(3,8,12,15,19,23,4,7,11,16,20,24)$	10
D_{19}	1	$(1,21,18,14,9,6,2,22,17,13,10,5)(3,24,20,16,11,7,4,23,19,15,12,8)$	1
D_{20}	1	$(1,6,10,14,17,21,2,5,9,13,18,22)(3,7,12,16,19,24,4,8,11,15,20,23)$	27
D_{21}	1	$(1,3,18,20,9,11,2,4,17,19,10,12)(5,8,21,24,14,16,6,7,22,23,13,15)$	36
D_{22}	1	$(1,11,10,20,17,3,2,12,9,19,18,4)(5,16,13,24,22,8,6,15,14,23,21,7)$	10
D_{23}	1	$(1,4,18,19,9,12,2,3,17,20,10,11)(5,7,21,23,14,15,6,8,22,24,13,16)$	1
D_{24}	1	$(1,12,10,19,17,4,2,11,9,20,18,3)(5,15,13,23,22,7,6,16,14,24,21,8)$	27

Table 9.1: Irreducible Character of ϕ

We now induce the character ϕ of H up to G to obtain the character ϕ^G of G, $\phi \uparrow_H^G$
Using

$$\phi \uparrow_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} = \phi(w)$$

where $n = \frac{|G|}{|H|} = \frac{24}{12} = 2$

$$\phi \uparrow_1^G = \frac{n}{h_\alpha} \sum_{w \in C_1 \cap H} = 2(\phi Id(H)) = 2(1) = 2$$

$\phi \uparrow_2^G = \frac{n}{h_\alpha} \sum_{w \in C_2 \cap H} = \frac{2}{1}(\phi(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,2$

$= (2)(-1) = -2$

$\phi \uparrow_3^G = \frac{n}{h_\alpha} \sum_{w \in C_3 \cap H} = \frac{2}{2}(\phi(1,8)(2,7)(3,21)(4,22)(5,12)(6,11)(9,16)(10,15)(13,19)(14,20)(17,23)(18,2$

$= (1)(-1) = -1$

$\phi \uparrow_4^G = \frac{n}{h_\alpha} \sum_{w \in C_4 \cap H} = \frac{2}{6}(\phi(1,7)(2,8)(3,22)(4,21)(5,11)(6,12)(9,15)(10,16)(13,20)(14,19)(17,24)(18,2$

$= \frac{1}{3}(1) = \frac{1}{3}$

$\phi \uparrow_5^G = \frac{n}{h_\alpha} \sum_{w \in C_5 \cap H} = \frac{2}{6}(\phi(1,17,9)(2,18,10)(3,19,11)(4,20,12)(5,22,14)(6,21,13)(7,24,15)(8,23,16))$

$= \frac{1}{3}(3) = 1$

$\phi \uparrow_6^G = \frac{n}{h_\alpha} \sum_{w \in C_6 \cap H} = \frac{2}{2}(\phi(1,9,17)(2,10,18)(3,11,19)(4,12,20)(5,14,22)(6,13,21)(7,15,24)(8,16,23))$

$= (1)(33) = 33$

$\phi \uparrow_7^G = \frac{n}{h_\alpha} \sum_{w \in C_7 \cap H} = \frac{2}{1}(\phi(1,20,2,19)(3,9,4,10)(5,24,6,23)(7,13,8,14)(11,17,12,18)(15,21,16,22))$

$= (2)(9) = 18$

$\phi \uparrow_8^G = \frac{n}{h_\alpha} \sum_{w \in C_8 \cap H} = \frac{2}{1}(\phi(1,19,2,20)(3,10,4,9)(5,23,6,24)(7,14,8,13)(11,18,12,17)(15,22,16,21))$

$= (2)(28) = 56 = 19$

$\phi \uparrow_9^G = \frac{n}{h_\alpha} \sum_{w \in C_9 \cap H} = \frac{2}{2}(\phi(1,13,2,14)(3,15,4,16)(5,17,6,18)(7,20,8,19)(9,21,10,22)(11,24,12,23))$

$$= (1)(9) = 9$$

$$\phi \uparrow^G_{10} = \frac{n}{h_\alpha} \sum_{w \in C_{10} \cap H} = \frac{2}{6}(\phi(1,14,2,13)(3,16,4,15)(5,18,6,17)(7,19,8,20)(9,22,10,21)(11,23,12,24))$$

$$= \tfrac{1}{3}(28) = \tfrac{28}{3}$$

$$\phi \uparrow^G_{11} = \frac{n}{h_\alpha} \sum_{w \in C_{11} \cap H} = \frac{2}{6}(\phi(1,16,17,8,9,23)(2,15,18,7,10,24)(3,6,19,21,11,13)(4,5,20,22,12,14))$$

$$= \tfrac{1}{3}(4) = \tfrac{4}{3}$$

$$\phi \uparrow^G_{12} = \frac{n}{h_\alpha} \sum_{w \in C_{12} \cap H} = \frac{2}{2}(\phi(1,23,9,8,17,16)(2,24,10,7,18,15)(3,13,11,21,19,6)(4,14,12,22,20,5))$$

$$= (1)(34) = 34$$

$$\phi \uparrow^G_{13} = \frac{n}{h_\alpha} \sum_{w \in C_{13} \cap H} = \frac{2}{2}(\phi(1,10,17,2,9,18)(3,12,19,4,11,20)(5,13,22,6,14,21)(7,16,24,8,15,23))$$

$$= (1)(4) = 4$$

$$\phi \uparrow^G_{14} = \frac{n}{h_\alpha} \sum_{w \in C_{14} \cap H} = \frac{2}{2}(\phi(1,18,9,2,17,10)(3,20,11,4,19,12)(5,21,14,6,22,13)(7,23,15,8,24,16))$$

$$= (1)(34) = 34$$

$$\phi \uparrow^G_{15} = \frac{n}{h_\alpha} \sum_{w \in C_{15} \cap H} = \frac{2}{2}(\phi(1,24,9,7,17,15)(2,23,10,8,18,16)(3,14,11,22,19,5)(4,13,12,21,20,6))$$

$$= (1)(3) = 3$$

$$\phi \uparrow^G_{16} = \frac{n}{h_\alpha} \sum_{w \in C_{16} \cap H} = \frac{2}{2}(\phi(1,15,17,7,9,24)(2,16,18,8,10,23)(3,5,19,22,11,14)(4,6,20,21,12,13))$$

$$= (1)(33) = 33$$

$$\phi \uparrow^G_{17} = \frac{n}{h_\alpha} \sum_{w \in C_{17} \cap H} = \frac{2}{2}(\phi(1,22,18,13,9,5,2,21,17,14,10,6)(3,23,20,15,11,8,4,24,19,16,12,7))$$

$$= (1)(1) = 1$$

$$\phi \uparrow^G_{18} = \frac{n}{h_\alpha} \sum_{w \in C_{18} \cap H} = \frac{2}{2}(\phi(1,5,10,13,17,22,2,6,9,14,18,21)(3,8,12,15,19,23,4,7,11,16,20,24))$$

$$= (1)(10) = 10$$

```
 CH[15];
( 1, -1, -1, 1, zeta(12)_3, -zeta(12)_3 - 1,
zeta(12)_4, -zeta(12)_4,zeta(12)_4, -zeta(12)_4,
zeta(12)_3 + 1, zeta(12)_3, zeta(12)_3 + 1,
-zeta(12)_3, zeta(12)_3, -zeta(12)_3 - 1,
zeta(12)_4*zeta(12)_3 + zeta(12)_4, -zeta(12)_4*zeta(12)_3,
-zeta(12)_4*zeta(12)_3 - zeta(12)_4,
zeta(12)_4*zeta(12)_3, zeta(12)_4*zeta(12)_3 + zeta(12)_4,
-zeta(12)_4*zeta(12)_3, -zeta(12)_4*zeta(12)_3 - zeta(12)_4,
zeta(12)_4*zeta(12)_3 )
*/
```

```
 CG[16];
( 2, -2, 0, 0, 0, -1, -2*zeta(12)_4, 2*zeta(12)_4, 0, 0, 0,
2*zeta(12)_3 + 1, -2*zeta(12)_3 - 1, 1, -zeta(12)_4, zeta(12)_4,
-2*zeta(12)_4*zeta(12)_3 -zeta(12)_4,
2*zeta(12)_4*zeta(12)_3 + zeta(12)_4 )
```

Thus, $CH[15] \uparrow_H^G = CG[16]$. This ensures that our group has a faithful irreducible monomial representation.

We now find an irreducible monomial representation of $2^3 : S_3$. From Magma, we can find the transversals of G and H. Note that the number of transversals equals the order of G divided by the order of H. Below is the code:

```
    T:=Transversal(G,H);
#T;
2
 T;
{}Id(G),(1, 20)(2, 19)(3, 18)(4, 17)(5, 15)(6, 16)(7, 14)
(8, 13)(9, 12)(10, 11)(21, 23)(22, 24)}
```

So, $G = He \cup H((1,20)(2,19)(3,18)(4,17)(5,15)(6,16)(7,14)(8,13)(9,12)(10,11)(21,23)(22,24)$

Let $t_1 = e, t_2 = (1,20)(2,19)(3,18)(4,17)(5,15)(6,16)(7,14)(8,13)(9,12)(10,11)(21,23)(22,24$

Then,

$$A(xx) = \begin{bmatrix} \phi(t_1 x x t_1^{-1}) & \phi(t_1 x x t_2^{-1}) \\ \phi(t_2 x x t_1^{-1}) & \phi(t_2 x x t_2^{-1}) \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & 26 \end{bmatrix}$$

$$A(yy) = \begin{bmatrix} \phi(t_1yyt_1^{-1}) & \phi(t_1yyt_2^{-1}) \\ \phi(t_2yyt_1^{-1}) & \phi(t_2yyt_2^{-1}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A(zz) = \begin{bmatrix} \phi(t_1zzt_1^{-1}) & \phi(t_1zzt_2^{-1}) \\ \phi(t_2zzt_1^{-1}) & \phi(t_2zzt_2^{-1}) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 29 \end{bmatrix}$$

Let us now verify if this representation is faithful.

```
IsIsomorphic(G,sub<GG|GG!A,GG!B,GG!D>);
true
```

Hence $\langle A(xx), A(yy), A(zz) \rangle$ is a faithful monomial representation of $2^3 : 3$.

We aim to write a monomial progenitor for $2^3 : 3$. We first need to find a permutation representation using the field order and the degree of representation. By Euler's Formula, the primitive square root of unity is $e^{\frac{i2\pi}{2}} = cos(\frac{2\pi}{2}) + isin(\frac{2\pi}{2}) = cos(\pi) + isin(\pi) = -1$ The field order will be the smallest finite field that has square roots of unity. This will be a cyclic group of order $p - 1$ where $2|p - 1$. So the field order will be Z_2. Since the dimension of $A(xx)$, $A(yy)$ and $A(zz)$ is 2, then the degree of the representation is 2. Thus the monomial progenitor is $3^2 :_m (2^3 : 3)$. We will use the matrices we created in order to label the automorphisms of t_i's.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{27}	t_2^{26}	t_1^{17}	t_2^{15}	t_1^7	t_2^4	t_1^{34}	t_2^{30}	t_1^{24}	t_2^{19}	t_1^{14}	t_2^8	t_1^4	t_2^{34}	t_1^{31}	t_2^{23}	t_1^{21}	t_2^{12}	t_1^{11}	t_2	t_1	t_2^{27}	t_1^{28}	t_2^{16}
53	52	33	30	13	8	67	60	47	38	27	16	7	68	61	46	41	24	21	2	1	54	55	32

25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}	t_2^{15}	t_1^{16}	t_2^{16}	t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}	t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}	t_1^{23}	t_2^{23}	t_1^{24}	t_2^{24}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{18}	t_2^5	t_1^8	t_2^{31}	t_1^{35}	t_2^{20}	t_1^{25}	t_2^9	t_1^{15}	t_2^{35}	t_1^5	t_2^{24}	t_1^{32}	t_2^{13}	t_1^{22}	t_2^2	t_1^{12}	t_2^{28}	t_1^2	t_2^{17}	t_1^{29}	t_2^6	t_1^{19}	t_2^{32}
32	35	10	15	62	69	40	49	18	29	70	9	48	63	26	43	4	23	56	3	34	57	12	37

49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}	t_2^{30}	t_1^{31}	t_2^{31}	t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^9	t_2^{21}	t_1^{36}	t_2^{10}	t_1^{26}	t_2^{36}	t_1^{16}	t_2^{25}	t_1^6	t_2^{14}	t_1^{33}	t_2^3	t_1^{23}	t_2^{29}	t_1^{13}	t_2^{18}	t_1^3	t_2^7	t_1^{30}	t_2^{33}	t_1^{20}	t_2^{22}	t_1^{10}	t_2^{11}
17	42	71	20	51	72	31	50	11	28	65	6	45	58	25	36	5	14	59	66	39	44	19	22

Thus,

$xx : (1, 53, 51, 71, 19, 21)(2, 52, 20)(3, 33, 29, 69, 39, 43)(4, 30, 40)$

$(5, 13, 7, 67, 59, 65)(6, 8, 60)(9, 47, 37, 63, 25, 35)(10, 38, 26)$

$(11, 27, 15, 61, 45, 57)(12, 16, 46)(14, 68, 66)(17, 41, 23, 55, 31, 49)$

$(18, 24, 32)(22, 54, 72)(36, 48, 64)(28, 62, 58)(42, 56, 50)(44, 34, 70)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_1	t_2^2	t_1^2	t_2^3	t_1^3	t_2^4	t_1^4	t_2^5	t_1^5	t_2^6	t_1^6	t_2^7	t_1^7	t_2^8	t_1^8	t_2^9	t_1^9	t_2^{10}	t_1^{10}	t_2^{11}	t_1^{11}	t_2^{12}	t_1^{12}
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23

25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}	t_2^{15}	t_1^{16}	t_2^{16}	t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}	t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}	t_1^{23}	t_2^{23}	t_1^{24}	t_2^{24}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2^{13}	t_1^{13}	t_2^{14}	t_1^{14}	t_2^{15}	t_1^{15}	t_2^{16}	t_1^{16}	t_2^{17}	t_1^{17}	t_2^{18}	t_1^{18}	t_2^{19}	t_1^{19}	t_2^{20}	t_1^{20}	t_2^{21}	t_1^{21}	t_2^{22}	t_1^{22}	t_2^{23}	t_1^{23}	t_2^{24}	t_1^{24}
26	25	28	27	30	29	32	31	34	32	36	35	38	37	40	39	42	41	44	43	46	45	48	47

49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}	t_2^{30}	t_1^{31}	t_2^{31}	t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2^{25}	t_1^{25}	t_2^{26}	t_1^{26}	t_2^{27}	t_1^{27}	t_2^{28}	t_1^{28}	t_2^{29}	t_1^{29}	t_2^{30}	t_1^{30}	t_2^{31}	t_1^{31}	t_2^{32}	t_1^{32}	t_2^{33}	t_1^{33}	t_2^{34}	t_1^{34}	t_2^{35}	t_1^{35}	t_2^{36}	t_1^{36}
50	49	52	51	54	53	56	55	58	57	60	59	62	61	64	63	66	65	68	67	70	69	72	71

Thus,

$yy : (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)$

$(21, 22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)$

$(35, 36)(37, 38)(39, 40)(41, 42)(43, 44)(45, 46)(47, 48)(49, 50)$

$(51, 52)(53, 54)(55, 56)(57, 58)(59, 60)(61, 62)(63, 64)(65, 66)(67, 68)(69, 70)(71, 72)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{14}	t_2^{29}	t_1^{28}	t_2^{21}	t_1^5	t_2^{13}	t_1^{19}	t_2^5	t_1^{33}	t_2^{34}	t_1^{10}	t_2^{26}	t_1^{24}	t_2^{18}	t_1	t_2^{10}	t_1^{15}	t_2^2	t_1^{29}	t_2^{31}	t_1^6	t_2^{23}	t_1^{20}	t_2^{15}
27	58	55	42	9	26	37	10	65	68	19	52	47	36	1	20	29	4	57	62	11	46	39	30

25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}	t_2^{15}	t_1^{16}	t_2^{16}	t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}	t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}	t_1^{23}	t_2^{23}	t_1^{24}	t_2^{24}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{34}	t_2^7	t_1^{11}	t_2^{36}	t_1^{25}	t_2^{28}	t_1^2	t_2^{20}	t_1^{16}	t_2^{12}	t_1^{30}	t_2^4	t_1^7	t_2^{33}	t_1^{21}	t_2^{25}	t_1^{35}	t_2^{17}	t_1^{12}	t_2^9	t_1^{26}	t_2	t_1^3	t_2^{30}
67	14	21	72	49	56	3	40	31	24	59	8	13	66	41	50	69	34	23	18	51	2	5	60

49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}	t_2^{30}	t_1^{31}	t_2^{31}	t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{17}	t_2^{22}	t_1^{31}	t_2^{14}	t_1^8	t_2^6	t_1^{22}	t_2^{35}	t_1^{36}	t_2^{27}	t_1^{13}	t_2^{19}	t_1^{27}	t_2^{11}	t_1^4	t_2^3	t_1^{18}	t_2^{32}	t_1^{32}	t_2^{24}	t_1^9	t_2^{16}	t_1^{23}	t_2^8
33	44	61	28	15	12	43	70	71	54	25	38	53	22	7	6	35	64	63	48	17	32	45	16

Thus,

$zz : (1, 27, 21, 11, 19, 57, 71, 45, 51, 61, 53, 15)$

$(2, 58, 54, 12, 52, 28, 72, 16, 20, 62, 22, 46)$

$(3, 55, 43, 23, 39, 41, 69, 17, 29, 49, 33, 31)$

$(4, 42, 34, 24, 30, 56, 70, 32, 40, 50, 44, 18)$

$(5, 9, 65, 35, 59, 25, 67, 63, 7, 37, 13, 47)$

$(6, 26, 14, 36, 8, 10, 68, 48, 60, 38, 66, 64)$

We need to verify that this permutation representation is faithful.

```
S:=Sym(72);
xx:= S!(1,53,51,71,19,21)(2,52,20)(3,33,29,69,39,43)(4,30,40)
(5,13,7,67,59,65)(6,8,60) (9,47,37,63,25,35)(10,38,26)
(11,27,15,61,45,57)(12,16,46)(14,68,66)(17,41,23,55,31,49)
(18,24,32)(22,54,72)(36,48,64)(28,62,58)(42,56,50)(44,34,70);

yy:= S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)
(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32)
(33,34)(35,36)(37,38)(39,40)(41,42)(43,44)
(45,46)(47,48)(49,50)(51,52)(53,54)(55,56)(57,58)
(59,60)(61,62)(63,64)(65,66)(67,68)(69,70)(71,72);
```

```
zz:= S!(1,27,21,11,19,57,71,45,51,61,53,15)
(2,58,54,12,52,28,72,16,20,62,22,46)
(3,55,43,23,39,41,69,17,29,49,33,31)
(4,42,34,24,30,56,70,32,40,50,44,18)
(5,9,65,35,59,25,67,63,7,37,13,47)
(6,26,14,36,8,10,68,48,60,38,66,64);

N:=sub<S|xx,yy,zz>;
#N;
/*48*/
IsIsomorphic(G,N);
true
```

Now, we write the symmetric presentation of the monomial progenitor $3^2 :_m (2^3 : 3)$ by including the presentation of $2^3 : 3$, the order of our t's, and the Normalizer of $< t >$. We got this,

```
G<x,y,z,t>:=Group<x,y,z,t|y^2, (x, z), x^6,
(x * y * x)^2,
z * y * x^2 * z^-1 * y, z^2 * x^2 * z^2, y * x^-1 * y * x * z^-2, t^37,
t^(x^3)=t^(36), t^(x * z^2 * y * z * y)=t^29>;
```

We will test our progenitor to see if it is correct by adding first order relation.

```
G<x,y,z,t>:=Group<x,y,z,t|y^2, (x, z), x^6,
(x * y * x)^2,
z * y * x^2 * z^-1 * y, z^2 * x^2 * z^2, y * x^-1 * y * x * z^-2,
t^37,
t^(x^3)=t^(36), t^(x * z^2 * y * z * y)=t^29,(t,t^y)>;
Index(G,sub<G|x,y,z>);
1369
37^2;
1369

#G;
65712

37^2*48;
65712
```

Thus, our progenitor is correct.

9.2 Second Monomial Representations of $37^{*2} :_m 2^3 : S_3$

We are given G, a transitive group on 24 letters, which is generated by

```
xx ~ (1, 16, 17, 8, 9, 23)(2, 15, 18, 7, 10, 24)
(3, 6, 19, 21, 11, 13)(4, 5, 20, 22, 12, 14).
yy ~ (1, 20)(2, 19)(3, 18)(4, 17)(5, 15)(6, 16)
(7, 14)(8, 13)(9, 12)(10, 11)(21, 23)(22, 24).
zz ~ (1, 21, 18, 14, 9, 6, 2, 22, 17, 13, 10, 5)
(3, 24, 20, 16, 11, 7, 4, 23, 19, 15, 12, 8).
G:=sub<G|xx,yy,zz>;
#G;
48
```

We will use the information from last section to apply it here. We verify.

$Induction(CH[23], G)eqCG[15];$

$/ * true * /$

From Magma, we can find the transversals of G and H. Note that the number of transversals equals the order of G divided by the order of H. Below is the code:

```
T:=Transversal(G,H);
#T;
2
T;
{Id(G),(1, 20)(2, 19)(3, 18)(4, 17)(5, 15)(6, 16)(7, 14)
(8, 13)(9, 12)(10, 11)(21, 23)(22, 24)}
```

So, $G = He \cup H((1,20)(2,19)(3,18)(4,17)(5,15)(6,16)(7,14)(8,13)(9,12)(10,11)(21,23)(22,24)$

Let $t_1 = e, t_2 = (1,20)(2,19)(3,18)(4,17)(5,15)(6,16)(7,14)(8,13)(9,12)(10,11)(21,23)(22,24$

Then,

$$A(xx) = \begin{bmatrix} \phi(t_1 xx t_1^{-1}) & \phi(t_1 xx t_2^{-1}) \\ \phi(t_2 xx t_1^{-1}) & \phi(t_2 xx t_2^{-1}) \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A(yy) = \begin{bmatrix} \phi(t_1 yy t_1^{-1}) & \phi(t_1 yy t_2^{-1}) \\ \phi(t_2 yy t_1^{-1}) & \phi(t_2 yy t_2^{-1}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A(zz) = \begin{bmatrix} \phi(t_1 zz t_1^{-1}) & \phi(t_1 zz t_2^{-1}) \\ \phi(t_2 zz t_1^{-1}) & \phi(t_2 zz t_2^{-1}) \end{bmatrix} = \begin{bmatrix} 29 & 0 \\ 0 & 14 \end{bmatrix}$$

Let us now verify if this representation is faithful.

```
IsIsomorphic(G,sub<GG|GG!A,GG!B,GG!D>);
true
```

Hence $\langle A(xx), A(yy), A(zz) \rangle$ is a faithful monomial representation of $2^3 : S_3$. We aim to write a monomial progenitor for $2^3 : S_3$. We first need to find a permutation representation using the field order and the degree of representation. By Euler's Formula, the primitive square root of unity is $e^{\frac{i2\pi}{2}} = cos(\frac{2\pi}{2}) + isin(\frac{2\pi}{2}) = cos(\pi) + isin(\pi) = -1$ The field order will be the finite field that has square roots of unity. This will be a cyclic group of order $p-1$ where $2|p-1$. So the field order will be Z_2. Since the dimension of $A(xx)$, $A(yy)$ and $A(zz)$ is 2, then the degree of the representation is 2. Thus the monomial progenitor is $3^2 :_m (2^3 : S_3)$. We will use the matrices we created in order to label the automorphisms of t_i's.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{11}	t_2^{10}	t_1^{22}	t_2^{20}	t_1^{33}	t_2^{30}	t_1^7	t_2^3	t_1^{18}	t_2^{13}	t_1^{29}	t_2^{23}	t_1^3	t_2^{33}	t_1^{14}	t_2^6	t_1^{25}	t_2^{16}	t_1^{36}	t_2^{26}	t_1^{10}	t_2^{36}	t_1^{21}	t_2^9
21	20	43	40	65	60	13	6	35	26	57	46	5	66	27	12	49	32	71	52	19	72	41	18

25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}	t_2^{15}	t_1^{16}	t_2^{16}	t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}	t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}	t_1^{23}	t_2^{23}	t_1^{24}	t_2^{24}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{32}	t_2^{19}	t_1^6	t_2^{29}	t_1^{17}	t_2^2	t_1^{28}	t_2^{12}	t_1^2	t_2^{22}	t_1^{13}	t_2^{32}	t_1^{24}	t_2^5	t_1^{35}	t_2^{15}	t_1^9	t_2^{25}	t_1^{20}	t_2^{35}	t_1^{31}	t_2^8	t_1^5	t_2^{18}
63	38	11	58	33	4	55	24	3	44	25	64	47	10	69	30	17	50	39	70	61	16	9	36

49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}	t_2^{30}	t_1^{31}	t_2^{31}	t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{16}	t_2^{28}	t_1^{27}	t_2	t_1	t_2^{11}	t_1^{12}	t_2^{21}	t_1^{23}	t_2^{31}	t_1^{34}	t_2^4	t_1^8	t_2^{14}	t_1^{19}	t_2^{24}	t_1^{30}	t_2^{34}	t_1^4	t_2^7	t_1^{15}	t_2^{17}	t_1^{26}	t_2^{27}
31	56	53	2	1	22	23	42	45	62	67	8	15	28	37	48	59	68	7	14	29	34	51	54

Thus,

$xx : (1, 21, 19, 71, 51, 53)(2, 20, 52)(3, 43, 39, 69, 29, 33)(4, 40, 30)$

$(5, 65, 59, 67, 7, 13)(6, 60, 8)(9, 35, 25, 63, 37, 47)(10, 26, 38)(11, 57, 45, 61, 15, 27)$

$(12, 46, 16)(14, 66, 68)(17, 49, 31, 55, 23, 41)(18, 32, 24)$

$(22, 72, 54)(28, 58, 62)(34, 44, 70)(36, 64, 48)(42, 50, 56)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2	t_1	t_2^2	t_1^2	t_2^3	t_1^3	t_2^4	t_1^4	t_2^5	t_1^5	t_2^6	t_1^6	t_2^7	t_1^7	t_2^8	t_1^8	t_2^9	t_1^9	t_2^{10}	t_1^{10}	t_2^{11}	t_1^{11}	t_2^{12}	t_1^{12}
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23

25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}	t_2^{15}	t_1^{16}	t_2^{16}	t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}	t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}	t_1^{23}	t_2^{23}	t_1^{24}	t_2^{24}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2^{13}	t_1^{13}	t_2^{14}	t_1^{14}	t_2^{15}	t_1^{15}	t_2^{16}	t_1^{16}	t_2^{17}	t_1^{17}	t_2^{18}	t_1^{18}	t_2^{19}	t_1^{19}	t_2^{20}	t_1^{20}	t_2^{21}	t_1^{21}	t_2^{22}	t_1^{22}	t_2^{23}	t_1^{23}	t_2^{24}	t_1^{24}
26	25	28	27	30	29	32	31	34	32	36	35	38	37	40	39	42	41	44	43	46	45	48	47

49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}	t_2^{30}	t_1^{31}	t_2^{31}	t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2^{25}	t_1^{25}	t_2^{26}	t_1^{26}	t_2^{27}	t_1^{27}	t_2^{28}	t_1^{28}	t_2^{29}	t_1^{29}	t_2^{30}	t_1^{30}	t_2^{31}	t_1^{31}	t_2^{32}	t_1^{32}	t_2^{33}	t_1^{33}	t_2^{34}	t_1^{34}	t_2^{35}	t_1^{35}	t_2^{36}	t_1^{36}
50	49	52	51	54	53	56	55	58	57	60	59	62	61	64	63	66	65	68	67	70	69	72	71

Thus,

$yy : (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)$

$(21, 22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)$

$(35, 36)(37, 38)(39, 40)(41, 42)(43, 44)(45, 46)(47, 48)(49, 50)$

$(51, 52)(53, 54)(55, 56)(57, 58)(59, 60)(61, 62)(63, 64)(65, 66)(67, 68)(69, 70)(71, 72)$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6	t_1^7	t_2^7	t_1^8	t_2^8	t_1^9	t_2^9	t_1^{10}	t_2^{10}	t_1^{11}	t_2^{11}	t_1^{12}	t_2^{12}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{29}	t_2^{14}	t_1^{21}	t_2^{28}	t_1^{13}	t_2^5	t_1^5	t_2^{19}	t_1^{34}	t_2^{33}	t_1^{26}	t_2^{10}	t_1^{18}	t_2^{24}	t_1^{10}	t_2	t_1^2	t_2^{15}	t_1^{31}	t_2^{29}	t_1^{23}	t_2^6	t_1^{15}	t_2^{20}
57	28	41	56	25	10	9	38	67	66	51	20	35	48	19	2	3	30	61	58	45	12	29	40

25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
t_1^{13}	t_2^{13}	t_1^{14}	t_2^{14}	t_1^{15}	t_2^{15}	t_1^{16}	t_2^{16}	t_1^{17}	t_2^{17}	t_1^{18}	t_2^{18}	t_1^{19}	t_2^{19}	t_1^{20}	t_2^{20}	t_1^{21}	t_2^{21}	t_1^{22}	t_2^{22}	t_1^{23}	t_2^{23}	t_1^{24}	t_2^{24}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^7	t_2^{34}	t_1^{36}	t_2^{11}	t_1^{28}	t_2^{25}	t_1^{20}	t_2^2	t_1^{12}	t_2^{16}	t_1^4	t_2^{30}	t_1^{33}	t_2^7	t_1^{25}	t_2^{21}	t_1^{17}	t_2^{35}	t_1^9	t_2^{12}	t_1	t_2^{26}	t_1^{30}	t_2^3
13	68	71	22	55	50	39	4	23	32	7	60	65	14	49	42	33	70	17	24	1	52	59	6

49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
t_1^{25}	t_2^{25}	t_1^{26}	t_2^{26}	t_1^{27}	t_2^{27}	t_1^{28}	t_2^{28}	t_1^{29}	t_2^{29}	t_1^{30}	t_2^{30}	t_1^{31}	t_2^{31}	t_1^{32}	t_2^{32}	t_1^{33}	t_2^{33}	t_1^{34}	t_2^{34}	t_1^{35}	t_2^{35}	t_1^{36}	t_2^{36}
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^{22}	t_2^{17}	t_1^{14}	t_2^{31}	t_1^6	t_2^8	t_1^{35}	t_2^{22}	t_1^{27}	t_2^{36}	t_1^{19}	t_2^{13}	t_1^{11}	t_2^{27}	t_1^3	t_2^4	t_1^{32}	t_2^{18}	t_1^{24}	t_2^{32}	t_1^{16}	t_2^9	t_1^8	t_2^{23}
43	34	27	62	11	16	69	44	53	72	37	26	21	54	5	8	63	36	47	64	31	18	15	46

Thus,

$zz : (1, 57, 53, 11, 51, 27, 71, 15, 19, 61, 21, 45)$

$(2, 28, 22, 12, 20, 58, 72, 46, 52, 62, 54, 16)$

$(3, 41, 33, 23, 29, 55, 69, 31, 39, 49, 43, 17)$

$(4, 56, 44, 24, 40, 42, 70, 18, 30, 50, 34, 32)$

$(5, 25, 13, 35, 7, 9, 67, 47, 59, 37, 65, 63)$

$(6, 10, 66, 36, 60, 26, 68, 64, 8, 38, 14, 48)$

We need to verify that this permutation representation is faithful.

```
S:=Sym(72);
xx:= S!(1,21,19,71,51,53)(2,20,52)(3,43,39,69,29,33)
(4,40,30)(5,65,59,67,7,13)(6,60,8)(9,35,25,63,37,47)
(10,26,38)(11,57,45,61,15,27)(12,46,16)(14,66,68)
(17,49,31,55,23,41)(18,32,24)(22,72,54)(28,58,62)
(34,44,70)(36,64,48)(42,50,56);

yy:= S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)
(17,18)(19,20)(21,22)(23,24)(25,26)(27,28)(29,30)(31,32)
(33,34)(35,36)(37,38)(39,40)(41,42)(43,44)(45,46)(47,48)
(49,50)(51,52)(53,54)(55,56)(57,58)(59,60)(61,62)
(63,64)(65,66)(67,68)(69,70)(71,72);
```

```
zz:= S!(1,57,53,11,51,27,71,15,19,61,21,45)
(2,28,22,12,20,58,72,46,52,62,54,16)
(3,41,33,23,29,55,69,31,39,49,43,17)
(4,56,44,24,40,42,70,18,30,50,34,32)
(5,25,13,35,7,9,67,47,59,37,65,63)
(6,10,66,36,60,26,68,64,8,38,14,48);

N:=sub<S|xx,yy,zz>;
#N;
/*48*/
IsIsomorphic(G,N);
true
```

Now, we write the symmetric presentation of the monomial progenitor $3^2 :_m (2^3 : S_3)$ by including the presentation of $2^3 : S_3$, the order of our t's, and the Normalizer of $< t >$. We got this,

```
G<x,y,z,t>:=Group<x,y,z,t|y^2, (x, z), x^6, (x * y * x)^2,
z * y * x^2 * z^-1 * y, z^2 * x^2 * z^2, y * x^-1 * y * x * z^-2, t^37,
t^(x^3)=t^(36), t^(x * z^2 * y * z * y)=t^14>;
```

We will test our progenitor to see if it is correct by adding first order relation.

```
G<x,y,z,t>:=Group<x,y,z,t|y^2, (x, z), x^6, (x * y * x)^2,
z * y * x^2 * z^-1 * y, z^2 * x^2 * z^2, y * x^-1 * y * x * z^-2,
 t^37, t^(x^3)=t^(36),
 t^(x * z^2 * y * z * y)=t^14,(t,t^y)>;

Index(G,sub<G|x,y,z>);
1369

37^2;
1369

#G;
65712

37^2*48;
65712
```

Thus, our progenitor is correct.

9.3 Linear Lifting Characters

Definition 9.7. *Lifting Process Let N be a normal subgroup of G and suppose that $A_{00}(Nx)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(Nx)$ defines a representation of G/N lifted from G/N. If $\alpha_0(Nx)$ is a character of $A_0(Nx)$, then $\alpha(x) = \alpha_0(Nx)$ is the **lifted character** of $A(x)$. Also, if $u \in N$, then $A(u) = Im, \alpha(u) = m = \alpha(1)$. The lifting process preserves irreducibility [Led77].*

Definition 9.8. *A representation $\rho : G \xrightarrow{Rep} GL(n, F)$ is given by $g\rho = F_n$ is a **Trivial Representation** of G*

- *$\chi(x) = \chi(y)$ if x and y are conjugates.*

- *Equivelent representation have the same character.*

- *The number of irreducible character of G is equal to the number of the conjugacy classes of G [Rot94].*

Definition 9.9. *The **Trivial Character** is the character χ of the trivial representation, where $\chi : G \to T$ given by $\chi(x) = 1, \forall g \in G$ [Rot94].*

9.3.1 Linear Lifting Characters of $2^{*6} :_m (Sym(5))$

Let $H = $ ISOM TYPE $= \langle (2, 5, 6, 3), (2, 6)(3, 5), (2, 5, 4, 3, 6) \rangle$ where the order of H is 20. Then $H' = $ ISOM TYPE $= \langle (2, 5, 4, 3, 6) \rangle$ and the order of H' is 5. Then $H/H' = \{(1, 2, 3, 4)\}$. The classes of H' are

$C_1 = \{e\}$

$C_2 = \{(2, 3)(4, 5), (2, 6)(3, 5), (3, 4)(5, 6), (2, 4)(3, 6), (2, 5)(4, 6)\}$

$C_3 = \{(2, 6, 5, 4), (2, 5, 6, 3), (2, 4, 3, 5), (3, 6, 4, 5), (2, 3, 4, 6)\}$

$C_4 = \{(2, 3, 6, 5), (3, 5, 4, 6), (2, 5, 3, 4), (2, 6, 4, 3), (2, 4, 5, 6)\}$

$C_5 = \{(2, 5, 4, 3, 6), (2, 3, 5, 6, 4), (2, 4, 6, 5, 3), (2, 6, 3, 4, 5)\}$

We know that the character table of H/H' is given by

Character Table				
	e	a=T[2]	a²=b=T[3]	a³=c=T[4]
χ.1	1	1	1	1
χ.2	1	-1	i	-i
χ.3	1	1	-1	-1
χ.4	1	-1	-i	i

Table 9.2: Character Table H/H'

We lift the characters of H/H' to H. To calculate the lift of χ of character χ of H/H', we note that

$\chi(C[1][3]) = \chi(1) = 1,$

$\chi(C[2][3]) = \chi T[3],$

$\chi(C[3][3]) = \chi T[2],$

$\chi(C[4][3]) = \chi T[4].$

Lifted-Character Table				
Class Representative	e	$a = T[2]$	$a^2 = b = T[3]$	$a^3 = c = T[4]$
χ.1	1	1	1	1
χ.2	1	-1	i	-1
χ.3	1	1	-1	-1
χ.4	1	-1	$-\mathbb{Z}_4$	$\mathbb{Z}_4$

Table 9.3: Lifted-Character Table H/H'

Thus χ.1, χ.2, χ.3, χ.4 are irreducible characters of H since χ.1, χ.2, χ.3, χ.4 are irreducible characters of H/H'.

We consider $G = $ ISOM TYPE $= \langle (1, 5, 3)(2, 6, 4), (1, 5, 4, 2, 3, 6) \rangle$. We want to find the monomial representative of G, if possible. In order to see this we must first look to the

character table of G.

```
Character Table of Group G
----------------------------
Class |   1  2  3  4  5  6  7
Size  |   1 10 15 20 30 24 20
Order |   1  2  2  3  4  5  6
----------------------------
p  =  2   1  1  1  4  3  6  4
p  =  3   1  2  3  1  5  6  2
p  =  5   1  2  3  4  5  1  7
----------------------------
X.1   +   1  1  1  1  1  1  1
X.2   +   1 -1  1  1 -1  1 -1
X.3   +   4 -2  0  1  0 -1  1
X.4   +   4  2  0  1  0 -1 -1
X.5   +   5  1  1 -1 -1  0  1
X.6   +   5 -1  1 -1  1  0 -1
X.7   +   6  0 -2  0  0  1  0
```

Now the character table of G has characters whose degree is greater than one. It should be noted that all characters of degree larger than one have degree four, five, and six. Since there are two characters of degree four and five and one character of degree six, it is possible to have five different monomial representations. All of which will be irreducible monomial representations. So we need to determine which of the five irreducible characters of G are faithful. We note that there is one faithful character namely $\chi.7 = (6, 0, -2, 0, 0, 1, 0)$. In order for G to have an irreducible faithful monomial representation, G is required to have a subgroup of H to the index 6 in G and that such H must have a linear character that induces it to character $\chi.7$ of G.

We find $H = \langle (2, 3, 4, 6), (3, 6, 4, 5) \rangle$ has an index 6 and that of H induces up to the character $\chi.7$ of G.

Therefore, G has a faithful irreducible monomial representation of degree six. The field entries of the representation is determined by the character values ϕ of H that is being induced which brings our focus to $(1, -1, -\mathbb{Z}_4, \mathbb{Z}_4, 1)$ has $\mathbb{Z}_4$ since the character values are fourth root of unity, the field of entries is the cycloatomicfield of 4th roots of unity. Thus, explicitly this representation is

$\rho : G \longrightarrow GL(C)$, where C=CyclotomicField(4),

$\rho\,(\mathrm{xx}) =$

$T[1]xxT[1]^{-1}$ $T[1]xxT[2]^{-1}$ $T[1]xxT[3]^{-1}$ $T[1]xxT[4]^{-1}$ $T[1]xxT[5]^{-1}$ $T[1]xxT[6]^{-1}$
$T[2]xxT[1]^{-1}$ $T[2]xxT[2]^{-1}$ $T[2]xxT[3]^{-1}$ $T[2]xxT[4]^{-1}$ $T[2]xxT[5]^{-1}$ $T[2]xxT[6]^{-1}$
$T[3]xxT[1]^{-1}$ $T[3]xxT[2]^{-1}$ $T[3]xxT[3]^{-1}$ $T[3]xxT[4]^{-1}$ $T[3]xxT[5]^{-1}$ $T[3]xxT[6]^{-1}$
$T[4]xxT[1]^{-1}$ $T[4]xxT[2]^{-1}$ $T[4]xxT[3]^{-1}$ $T[4]xxT[4]^{-1}$ $T[4]xxT[5]^{-1}$ $T[4]xxT[6]^{-1}$
$T[5]xxT[1]^{-1}$ $T[5]xxT[2]^{-1}$ $T[5]xxT[3]^{-1}$ $T[5]xxT[4]^{-1}$ $T[5]xxT[5]^{-1}$ $T[5]xxT[6]^{-1}$

$\rho\,(yy) =$
$T[1]yyT[1]^{-1}$ $T[1]yyT[2]^{-1}$ $T[1]yyT[3]^{-1}$ $T[1]yyT[4]^{-1}$ $T[1]yyT[5]^{-1}$ $T[1]yyT[6]^{-1}$
$T[2]yyT[1]^{-1}$ $T[2]yyT[2]^{-1}$ $T[2]yyT[3]^{-1}$ $T[2]yyT[4]^{-1}$ $T[2]yyT[5]^{-1}$ $T[2]yyT[6]^{-1}$
$T[3]yyT[1]^{-1}$ $T[3]yyT[2]^{-1}$ $T[3]yyT[3]^{-1}$ $T[3]yyT[4]^{-1}$ $T[3]yyT[5]^{-1}$ $T[3]yyT[6]^{-1}$
$T[4]yyT[1]^{-1}$ $T[4]yyT[2]^{-1}$ $T[4]yyT[3]^{-1}$ $T[4]yyT[4]^{-1}$ $T[4]yyT[5]^{-1}$ $T[4]yyT[6]^{-1}$
$T[5]yyT[1]^{-1}$ $T[5]yyT[2]^{-1}$ $T[5]yyT[3]^{-1}$ $T[5]yyT[4]^{-1}$ $T[5]yyT[5]^{-1}$ $T[5]yyT[6]^{-1}$

where $G = \langle xx, yy \rangle$ and $G = HT[1] \cup HT[2] \cup HT[3] \cup HT[4] \cup HT[5] \cup HT[6]$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -\mathbb{Z}_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & \mathbb{Z}_4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

where $A = \rho(xx)$ and $B = \rho(yy)$.

Therefore, the generators of the faithful irreducible monomial representation are A and B. Now the smallest finite field that contains the fourth root of unity (elements of order 5) is $\mathbb{Z}_5$. The elements of order four in $\mathbb{Z}_5$ are generators of $\mathbb{Z}_5$-$\{0\}$.

We now find a permutation representation of our monomial representation. we denote the permutation representative of A, B and $A(xx), B(yy)$ respectively. Our progenitor is: 2^{*6}. We have six t_i's and those are $t_1, t_2, t_3, t_4, t_5, t_6$. We now interpret the automorphisms given by the two matrices A and B by using the formula $a_{ij} \Leftrightarrow t_i \to t_j$.

First we will consider the A matirix,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

So, $a_{12} = 1$, $a_{23} = 1$, $a_{31} = 1$, $a_{46} = -1$, $a_{54} = -1$, $a_{65} = 1$.

Then we have,

$t_1 \rightarrow t_2$

$t_2 \rightarrow t_3$

$t_3 \rightarrow t_1$

$t_4 \rightarrow t_6^{-1}$

$t_5 \rightarrow t_4$

$t_6 \rightarrow t_5$

We have two distinct powers of each of the six t_i's and to simplify we will use modulo 5. We label $t_1, t_2, t_3, t_4, t_5, t_6$ by $1, 2, 3, 4, 5, 6$ respectively. Apply $t_1 \rightarrow t_1^2$ and $t_2 \rightarrow t_3$ to form the permutation.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
t_1	t_2	t_3	t_4	t_5	t_6	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_1^3	t_2^3	t_3^3	t_4^3	t_5^3	t_6^3	t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_6^4
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_1^2	t_2	t_1	t_2^{12}	t_2^{11}	t_1^3	t_2^4	t_1^5	t_1^4	t_2^9	t_2^8	t_1^6	t_2^7	t_1^8	t_1^7	t_2^6	t_2^5	t_1^9	t_2^{10}	t_1^{11}	$t_1 10$	t_2^3	t_2^2	t_1^{12}
2	3	1	24	22	5	8	9	7	18	16	11	14	15	13	12	10	17	20	21	19	6	4	23

Therefore, $A(xx) = (1, 2, 3)(4, 24, 23)(5, 22, 6)(7, 8, 9)(10, 18, 17)(11, 16, 12)(13, 14, 15)(19, 20, 21)$.

Next we consider $B = \begin{bmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & i \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

So,

$a_{12} = -i$

$a_{24} = 1$

$a_{35} = -1$

$a_{46} = -i$

$a_{51} = 1$

$a_{63} = 1$

Then we have,

$t_1 \rightarrow t_2^{-i} = t_2^3$

$t_2 \rightarrow t_4$

$t_3 \rightarrow t_5^{-1}$

$t_4 \rightarrow t_6^{-i} = t_6^3$

$t_5 \rightarrow t_1$

$t_6 \rightarrow t_3$

We label $t_1, t_2, t_3, t_4, t_5, t_6$ by $1, 2, 3, 4, 5, 6$ respectively. Apply $t_1 \rightarrow t_2^3, t_2 \rightarrow t_4$ to form the permutation. Thus the permutation for $B(yy) = (1, 14, 16, 24, 21, 5)(2, 4, 18, 15, 11, 7)(3, 23, 19, 8, 10, 6)($

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
t_1	t_2	t_3	t_4	t_5	t_6	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_1^3	t_2^3	t_3^3	t_4^3	t_5^3	t_6^3	t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_6^4
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
14	4	23	18	1	3	2	10	17	6	7	9	20	16	11	24	13	15	8	22	5	12	19	21

Now we want to find a symmetric presentation for our progenitor. The stabilizer $(N, \{1, 7, 13, 19\}$ where, $< t_1 > = \{t_1, t_1^2, ..., t_1^{24}\}$ and $\{g \epsilon N | < t >^g = < t >\}$ is called the normalizer of $< t >$ in G. This tells us the number of different conjugates of $< t_1 >$ is twenty-four. Thus, our presentation of the progenitor $2^{*6} : Sym(5)$ is

$< x, y, t | x^3, y^6, (x * y^{-2})^2, (x * y^{-1} * x^{-1} * y^{-1})^2, t^5 >$

We check if our progenitor is correct using Grindstaff's lemma. To do so, we will begin by looking at the orbits of N with respects to 1 which we will denote as $N1$. We find that $N1$ has the following orbits $\{1\}, \{7\}, \{13\}, \{19\}, \{2, 3, 12, 10, 11\}, \{4, 5, 14, 15, 6\}, \{8, 9, 24, 22, 23\}, \{16, 17, 20$ Next we will choose an orbit representative of each orbit and search for different combinations of x and y that takes 1 to the chosen representative. We will choose the following orbit representatives: $7, 13, 19, 2, 4, 8, 16$. Therefore we have

Next we will add the following to our progenitor $t^{(xy^5)}$, $t^{(yx^{-1})}$, $t^{(x^2y^2)}$, t^x, $t^{(xy)}$, $t^{(xy)}$, $t^{(x^2y^3)}$, $t^{(y^2)}$ and then apply Grindstaff's lemma by finding the index of G with respects to sub$< G | x, y >$. In magma we compute the following

```
S:=Sym(24);
xx:=S!(1,2,3)(4,24,23)(5,22,6)(7,8,9)(10,18,17)(11,16,12)(13,14,15)(19,20,21);
yy:=S!(1,14,16,24,21,5)(2,4,18,15,11,7)(3,23,19,8,10,6)(9,17,13,20,22,12);
N:=sub<S|xx,yy>;
#N;
/*120*/
G<x,y,t>:=Group<x,y,t|x^3 ,
y^6 ,
(x * y^-2)^2 ,
(x * y^-1 * x^-1 * y^-1)^2,
t^5,
(t,(y*x*y^3*x^-1)),(t,(y*x^-1)),(t,t^(x*y^5)),
```

$N1$ Orbit representatative table		
Orbit Representative	Permutation	Written as
7	xy^5	$t^{(xy^5)}$
13	yx^{-1}	$t^{(yx^{-1})}$
19	x^2y^2	$t^{(x^2y^2)}$
2	x	t^x
4	xy	$t^{(xy)}$
8	x^2y^3	$t^{(x^2y^3)}$
16	y^2	$t^{(y^2)}$

Table 9.4: $N1$ Orbit Representatative Table

```
(t,t^(y*x^-1)),(t,t^(x^2*y^2)),(t,t^(x)),
(t,t^(x*y)),(t,t^(x^2*y^3)),(t,t^(y^2)) >;
Index(G,sub<G|x,y>);
/*15625*/
```

Next we want to find our first order relations and to do so, we must investigate the five conjugacy classes of $PSL(6,5)$. In the table below we list the conjugacy classes and their representatives.

Conjugacy Classes and First Order Relations		
Class	Class Representative	Elements of the form πt_i
2	$(y^3) = (1, 24)(2, 15)(3, 8)(4, 11)(5, 16)(6, 19)(7, 18)(9, 20)(10, 23)(12, 13)(14, 21)(17, 22)$	$(y^3)t$, $(y^3)t_2$
3	$(x*y*x^{-1}*y) = (1, 19)(2, 21)(3, 20)(4, 22)(5, 24)(6, 23)(7, 13)(8, 15)(9, 14)(10, 16)(11, 18)(12, 17)$	$(x*y*x^{-1}*y)t$, $(x*y*x^{-1}*y)t_2$, $(x * y * x^{-1} * y)t_8$
4	$x = (1, 2, 3)(4, 24, 23)(5, 22, 6)(7, 8, 9)(10, 18, 17)(11, 16, 12)(13, 14, 15)(19, 20, 21)$	xt, xt_5, xt_7, xt_{11}
5	$(x * y^{-1}) = (1, 7, 19, 13)(2, 6, 21, 23)(3, 5, 20, 24)(4, 16, 22, 10)(8, 12, 15, 17)(9, 11, 14, 18)$	$(x * y^{-1})t$, $(x * y^{-1})t_2$, $(x * y^{-1})t_3$, $(x*y^{-1})t_4$, $(x*y^{-1})t_8$, $(x * y^{-1})t_9$
6	$(y * x * y * x * y^{-2}) = (2, 3, 12, 10, 11)(4, 5, 14, 15, 6)(8, 9, 24, 22, 23)(16, 17, 20, 21, 18)$	$(y * x * y * x * y^{-2})t$, $(y * x * y * x * y^{-2})t_7$, $(y * x * y * x * y^{-2})t_{13}$, $(y*x*y*x*y^{-2})t_{19}$, $(y*x*y*x*y^{-2})t_2$, $(y*x*y*x*y^{-2})t_4$, $(y*x*y*x*y^{-2})t_8$, $(y * x * y * x * y^{-2})t_{16}$
7	$y = (1, 14, 16, 24, 21, 5)(2, 4, 18, 15, 11, 7)(3, 23, 19, 8, 10, 6)(9, 17, 13, 20, 22, 12)$	yt, yt_2, yt_3, yt_9

Table 9.5: Conjugacy Classes of $PSL(6, 5)$

9.3.2 Linear Lifting Characters of $2^{*10} :_m (Alt(6))$

Let $H =$ ISOM TYPE $= \langle(1,5,6,2)(3,4),(1,6)(2,5),(2,4,5),(1,6,3)\rangle$ with the following character table for H.

```
Character Table of Group H
--------------------------
Class |  1  2  3  4  5  6
Size  |  1  9  4  4  9  9
Order |  1  2  3  3  4  4
--------------------------
p  =  2  1  1  3  4  2  2
p  =  3  1  2  1  1  6  5
--------------------------
X.1   +  1  1  1  1  1  1
X.2   +  1  1  1  1 -1 -1
X.3   0  1 -1  1  1  I -I
X.4   0  1 -1  1  1 -I  I
X.5   +  4  0  1 -2  0  0
X.6   +  4  0 -2  1  0  0
Explanation of Character Value Symbols
--------------------------------------
I = RootOfUnity(4)
```

We consider our $G =$ ISOM TYPE $= \langle(1,2)(3,4,5,6),(1,2,3)\rangle$. We want to find the monomial representative of G, if possible. In order to see if this may be possible we first must consider the character table for G.

```
Character Table of Group G
--------------------------------
Class |   1  2  3  4  5   6    7
Size  |   1 45 40 40 90  72   72
Order |   1  2  3  3  4   5    5
--------------------------------
p  =  2   1  1  3  4  2   7    6
p  =  3   1  2  1  1  5   7    6
p  =  5   1  2  3  4  5   1    1
--------------------------------
X.1   +   1  1  1  1  1   1    1
X.2   +   5  1  2 -1 -1   0    0
X.3   +   5  1 -1  2 -1   0    0
X.4   +   8  0 -1 -1  0  Z1 Z1#2
X.5   +   8  0 -1 -1  0 Z1#2   Z1
X.6   +   9  1  0  0  1  -1   -1
```

```
X.7   +   10 -2  1  1  0   0   0
Explanation of Character Value Symbols
--------------------------------------
# denotes algebraic conjugation, that is,
#k indicates replacing the root of unity w by w^k
Z1 = (CyclotomicField(5: Sparse := true)) ! [ RationalField() | 1, 0, 1, 1 ]
```

Now the character table for G has characters with degree greater than one. Those characters larger than one being of degree five, eight, nine and ten. Since we have two characters of degree five, two characters of degree eight, one character of degree nine and one character of degree ten we have a possible six different monomial interpretations to write for G. All of these will be irreducible monomial representations, so we need to determine which of the six irreducible characters of G are faithful. We consider that there is one faithful character namely $\chi.10$, where $\chi.10 = (10, -2, 1, 1, 0, 0, 0)$. In order for G to have an irreducible faithful monomial representation, G is required to have a subgroup H to the index ten and H must be a linear character that induces up to the character $\chi.10$ of G.

We find $H = \langle (1, 5, 6, 2)(3, 4), (1, 6)(2, 5), (2, 4, 5), (1, 6, 3) \rangle$ has an index of ten in G and induces up to the character $\chi.10$ of G.

Therefore, G has a faithful monomial representation of degree ten. The field entries of the representation is determined by the character values ϕ of H that is being induced, which brings our attention to $\mathbb{Z}_4$ since the character values are fourth root of unity. Therefore, explicitly the representation is $\rho : G \longrightarrow GL(C)$, where C=CyclotomicField(4).

$$\rho(xx) = \begin{bmatrix} T[1]xxT[1]^{-1} & \ldots & T[1]xxT[10]^{-1} \\ \vdots & \ddots & \vdots \\ T[10]xxT[1]^{-1} & \ldots & T[10]xxT[10]^{-1} \end{bmatrix}$$

and

$$\rho(yy) = \begin{bmatrix} T[1]yyT[1]^{-1} & \ldots & T[1]yyT[10]^{-1} \\ \vdots & \ddots & \vdots \\ T[10]yyT[1]^{-1} & \ldots & T[10]yyT[10]^{-1} \end{bmatrix}$$

where $G = \langle xx, yy \rangle$ and $G = HT[1] \cup HT[2] \cup HT[3] \cup HT[4] \cup HT[5] \cup HT[6] \cup HT[7] \cup HT[8] \cup HT[9] \cup HT[10]$

$$A = \begin{bmatrix} 0 & -\mathbb{Z}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{Z}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

So,

$a_{1,2} = -\mathbb{Z}_4 = -i = 3$

$a_{2,3} = 1$

$a_{3,4} = \mathbb{Z}_4 = i = 2$

$a_{4,6} = 1$

$a_{5,8} = 1$

$a_{6,1} = 1$

$a_{7,5} = 1$

$a_{8,9} = 1$

$a_{9,10} = 1$

$a_{10,7} = 1$

Then we have,

$a_{1,2}, t_1 \rightarrow t_2^3$

$a_{2,3}, t_2 \rightarrow t_3$

$a_{3,4}, t_3 \rightarrow t_4^2$

$a_{4,6}, t_4 \rightarrow t_6$

$a_{5,8}, t_5 \rightarrow t_8$

$a_{6,1}, t_6 \rightarrow t_1$

$a_{7,5}, t_7 \rightarrow t_5$

$a_{8,9}, t_8 \rightarrow t_9$

$a_{9,10}, t_9 \rightarrow t_{10}$

$a_{10,7}, t_{10} \rightarrow t_7$

We have nine distinct powers of the ten t_i's and to simplify we will use modulo 5. We label $t_1, t_2, \ldots, t_{10}$ as $1, 2, \ldots, 10$ respectively. Apply $t_1 \to t_2^{-i}$, $t_2 \to t_3$ and so on to form the permutation.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_7^2	t_8^2	t_9^2	t_{10}^2
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2^3	t_3	t_4^2	t_6	t_8	t_1	t_5	t_9	t_{10}	t_7	t_2	t_3^2	t_4^4	t_6^2	t_8^2	t_1^2	t_5^2	t_9^2	t_{10}^2	t_7^2
22	3	14	6	8	1	5	9	10	7	2	13	34	16	18	11	15	19	20	17

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
t_1^3	t_2^3	t_3^3	t_4^3	t_5^3	t_6^3	t_7^3	t_8^3	t_9^3	t_{10}^3	t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_6^4	t_7^4	t_8^4	t_9^4	t_{10}^4
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2^4	t_3^3	t_4	t_6^3	t_8^3	t_1^3	t_5^3	t_9^3	t_{10}^3	t_7^3	t_2^2	t_3^4	t_4^3	t_6^4	t_8^4	t_1^4	t_5^4	t_9^4	t_{10}^4	t_7^4
32	23	4	26	28	21	25	29	30	27	12	33	24	36	38	31	35	39	40	37

Therefore the permutation for $A(xx)$ is:

$(1, 22, 23, 4, 6)(2, 3, 14, 16, 11)(5, 8, 9, 10, 7)(12, 13, 34, 36, 31)$

$(15, 18, 19, 20, 17)(21, 32, 33, 24, 26)(25, 28, 29, 30, 27)(35, 38, 39, 40, 37)$

Next we will consider $B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathbb{Z}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbb{Z}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbb{Z}_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}_4 \end{bmatrix}$

Then,

$a_{1,2} = 1$

$a_{2,3} = -\mathbb{Z}_4 = -i = 3$

$a_{3,5} = 1$

$a_{4,7} = 1$

$a_{5,1} = \mathbb{Z}_4 = i = 2$

$a_{6,4} = 1$

$a_{7,8} = 1$

$a_{8,9} = 1$

$a_{9,9} = -\mathbb{Z}_4 = -i = 3$

$a_{10,10} = \mathbb{Z}_4 = i = 2$

Then we have,

$a_{1,2},\ t_1 \rightarrow t_2$

$a_{2,3},\ t_2 \rightarrow t_3^3$

$a_{3,5},\ t_3 \rightarrow t_5$

$a_{4,7},\ t_4 \rightarrow t_7$

$a_{5,1},\ t_5 \rightarrow t_1^2$

$a_{6,4},\ t_6 \rightarrow t_4$

$a_{7,8},\ t_7 \rightarrow t_8$

$a_{8,6},\ t_8 \rightarrow t_6$

$a_{9,9},\ t_9 \rightarrow t_9^3$

$a_{10,10},\ t_{10} \rightarrow t_{10}^2$

We label $t_1, t_2, \ldots, t_{10}$ as $1, 2, \ldots, 10$ respectively. Apply $t_1 \rightarrow t_2$, $t_2 \rightarrow t_3^3$ and so on to form the permutation.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_7^2	t_8^2	t_9^2	t_{10}^2
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2	t_3^3	t_5	t_7	t_1^2	t_4	t_8	t_6	t_9^3	t_{10}^2	t_2^2	t_3	t_5^2	t_7^2	t_1^4	t_4^2	t_8^2	t_6^2	t_9	t_{10}^4
2	23	5	7	11	4	8	6	29	20	12	3	15	17	31	14	18	16	9	40

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
t_1^3	t_2^3	t_3^3	t_4^3	t_5^3	t_6^3	t_7^3	t_8^3	t_9^3	t_{10}^3	t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_6^4	t_7^4	t_8^4	t_9^4	t_{10}^4
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2^3	t_3^4	t_5^3	t_7^3	t_1	t_4^3	t_8^3	t_6^3	t_9^4	t_{10}	t_2^4	t_3^2	t_5^4	t_7^4	t_1^3	t_4^4	t_8^4	t_6^4	t_9^2	t_{10}^3
22	33	25	27	1	24	28	26	39	10	32	13	35	37	21	34	38	36	19	30

Thus we have the permutation for $B(yy)$ to be

$B(yy) = (1, 2, 23, 25)(3, 5, 11, 12)(4, 7, 8, 6)(9, 29, 39, 19)(10, 20, 40, 30)$

$(14, 17, 18, 16)(13, 15, 31, 32)(21, 22, 33, 35)(24, 27, 28, 26)(34, 37, 38, 36)$.

Now we want to find a symmetric presentation for our progenitor. The stabiliser $(N, \{1, 11, 21, 31\})$ where, $\langle t_1 \geq \{t_1, t_1^4\} \rangle$ and $\{g \in N | t^g = t\}$ is called the normalizer of $\langle t \rangle$ in G. Thus tells us the number of different conjugates of $\langle t_1 \rangle$ is four. Thus our presentation of the progenitor is:

$G\langle x, y, t \rangle := \text{Group } \langle x, y, t | x^5, y^4, (y^{-1}x^{-1})^3, (x^{-1}, y^{-1})^2,$
$t^5, (t, (y^2 x^{-2})), (t, (x^{-2}y^{-1}x)), t^{(yx^{-1})} = t^2 \rangle.$

We check using Grindstaff's lemma, if we add $(t_1, t_{11}), (t_1, t_{21}), (t_1, t_{31}), (t_1, t_2), (t_1, t_3), (t_1, t_4), (t_1, t_5)$ to the presentation of G then the order of $|G| = 9765625$. Now $t_1 = t, t_{11} = t^{(yx^4)}, t_{21} = t^{(xy^{-1})}, t_{31} = t^{(yxy^3xy^2)}, t_2 = t^{(y)}, t_3 = t^{(yx)}, t_4 = t^{(x^3)}, t_5 = t^{(yxy)}$ then we have,

```
G < x,y,t >:= Group < x,y,t| x^5 ,
y^4 ,
(y^-1  x^-1)^3 ,(x^-1, y^-1)^2,
t^5, (t,(y^2  x^-2)),
(t,(x^-2 y^-1  x)),
t^(y  x^-1)= t^2,
(t,t^(yx^4)),
(t,t^(xy^-1)),
(t,t^(yxy^3xy^2)),
(t,t^(y)),
(t,t^(yx)),
(t,t^(x^3)),
(t,t^(yxy))>;

#G;
0
```

Since our G is zero, this means that the number for our G may either be too large for magma to compute or we have an incorrect progenitor. Therefore we will input into

magma the following:

```
print Index (G,sub<G|x,y>: CosetLimit:=9^{10}, Hard:=true, Print:=2);

CP: a=9739415 r=9774 h=6216204 n=9740071; l=19481 c=+0.29; m=9739415 t=9740070

CP: a=9749413 r=9784 h=6348237 n=9750071; l=19501 c=+0.27; m=9749413 t=9750070

CP: a=9759413 r=9794 h=6535478 n=9760071; l=19521 c=+0.25; m=9759413 t=9760070

INDEX = 9765625 (a=9765625 r=9801 h=9766283 n=9766283; l=19536 c=170.25;
m=9765625 t=9766282)
```

Thus, our progenitor is correct since we found an index of 9765625

Chapter 10

Composition Charts

In this chapter, we will show the isomorphic images for most of the composition factors we discovered.

10.1 $2^{*9} : (3^3 : (3 : 2))$

We have the following information.

S:=Sym(9), we are working with 9 letters.

$x \sim (1,9,5)(2,7,6)(3,8,4)$

$y \sim (1,6,2,4,3,5)(7,8,9)$

The order of $|N| = 162$.

$2^{*9} : (3^3 : (3 : 2))$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	0	0	0	3	$3^4 : S_4$
0	0	0	0	0	0	9	4	0	9	$PSL(2,19)$
0	0	0	0	0	0	9	4	0	10	$PSL(2,19) \times S_3$
0	0	0	0	0	0	8	0	0	10	$Sym(6) : 2$
0	0	0	0	0	2	5	6	6	0	$A_5 : 2$
0	0	0	0	0	6	3	6	6	0	$2^2 : (6 : 3^3)$
0	0	0	0	0	6	4	0	0	0	$3^5 : (2^3 : 2)$
0	0	0	0	0	6	5	0	0	0	$(3^6 : A_5) : 2$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^3, y^6, y * x * y^{-1} * x^{-1} * y^{-1} * x * y, t^2,$

$(t, (y^2 * x^{-1} * y^{-1})),$

$(t, (x^{-1} * y^2 * x * y^{-1} * x * y^{-1} * x * y^{-1} * x * y^{-2})),$

$((y^3) * t^{(x*y^{-2}*x^{-1}*y^{-1})})^a,$

$((x * y * x^-1 * y * x * y^{-2}) * t^{(x)})^b,$

$((y^{-2}) * t^{(x*y*x^{-1}*y*x)})^c,$

$(((y * x^{-1})^2) * t^{(y^2)})^e,$

$((x^{-1} * y^{-1} * x^{-1} * y) * t^{(x*y^{-2})})^f,$

$((x) * t^{(x*y*x^{-1}*y*x)})^g,$

$((x * y^{-1}) * t^{(y*x*y^{-2}*x^{-1})})^h,$

$((y * x^{-1}) * t^{(x*y^{-2}*x^{-1}*y^{-1})})^i,$

$((y^2 * x^{-1} * y^{-1} * x^{-1}) * t^{(x*y*x^{-1}*y*x)})^j >$

$2^{*9} : (3^3 : (3 : 2))$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	2	6	10	10	10	$5^2 : (6 : 2)$
0	0	0	0	0	2	7	6	6	9	$PSL(2,8)$
0	0	0	0	0	2	7	10	8	9	$PSL(2,8)$
0	0	0	0	0	2	8	6	6	6	$4^2 : (6 : 2)$
0	0	0	0	0	2	8	6	6	7	$PSL(2,7) : 2$
0	0	0	0	0	2	8	6	6	8	$(PSL(2,7) : 2) : 2$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, y^6, y * x * y^{-1} * x^{-1} * y^{-1} * x * y, t^2,$

$(t, (y^2 * x^{-1} * y^{-1})),$

$(t, (x^{-1} * y^2 * x * y^{-1} * x * y^{-1} * x * y^{-1} * x * y^{-1} * x * y^{-2})),$

$((y^3) * t^{(x*y^{-2}*x^{-1}*y^{-1})})^a,$

$((x * y * x^{-1} * y * x * y^{-2}) * t^{(x)})^b,$

$((y^{-2}) * t^{(x*y*x^{-1}*y*x)})^c,$

$(((y * x^{-1})^2) * t^{(y^2)})^e,$

$((x^{-1} * y^{-1} * x^{-1} * y) * t^{(x*y^{-2})})^f,$

$((x) * t^{(x*y*x^{-1}*y*x)})^g,$

$((x * y^{-1}) * t^{(y*x*y^{-2}*x^{-1})})^h,$

$((y * x^{-1}) * t^{(x*y^{-2}*x^{-1}*y^{-1})})^i,$

$((y^2 * x^{-1} * y^{-1} * x^{-1}) * t^{(x*y*x^{-1}*y*x)})^j >$

10.2 $2^{*12} : (2 \times A_5)$

We have the following information

S:=Sym(12), we are working with 12 letters.

$x \sim (1, 6, 3)(2, 8, 4)(5, 7, 9)(10, 11, 12)$

$y \sim (1, 5, 2, 3, 11, 4, 10, 6, 8, 7)(9, 12)$

The order of $|N| = 120$.

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
0	3	0	0	0	0	0	0	0	0	$(2 : A_5) : A_5$
5	3	0	0	0	0	0	0	0	0	$A_5 : A_5$
3	4	0	0	0	0	0	0	0	0	$S(6, 2) \times (A_5 \times 2)$
2	8	4	0	0	0	0	0	0	0	$2^6 : A_5$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2, (x^{-1} * y * x^{-1} * y^{-1})^2,$

$(x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y)$

$((y^5) * t^{(y^2 * x * y * x^{-1} * y)})^a,$

$((y * x * y^2) * t^{(y^{-2} * x^{-1} * y)})^b,$

$((x^{-1} * y * x^{-1} * y * x) * t^{(x * y^{-2})})^c,$

$((x) * t^{(y * x * y^{-1} * x^{-1} * y^{-1})})^d,$

$((y^2) * t^{(x)})^e,$

$((y * x * y * x^{-1}) * t^{(y^{-1} * x^{-1} * y * x * y)})^f,$

$((x * y) * t^{(y * x^{-1})})^g,$

$((y) * t^{(x * y^{-1})})^h,$

$((y^3) * t^{(x * y * x * y^{-1})})^i,$

$((y) * t^{(x * y^2 * x * y^{-1})})^j >$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
0	5	0	0	0	0	0	0	0	0	$PSL(2, 19)$
9	5	0	0	0	0	0	0	0	0	$PSL(2, 19)$
4	8	0	0	0	0	0	0	0	0	$(6 : A_6 \times A_6) : 2^2$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2, (x^{-1} * y * x^{-1} * y^{-1})^2,$

$(x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y)$

$((y * x * y * x^{-1}) * t^{(y^2 * x * y * x^{-1} * y)}))^a,$

$((x * y) * t^{(y^{-2} * x^{-1} * y)})^b,$

$((y * x * y * x^{-1}) * t^{(x * y^{-2})})^c,$

$((x) * t^{(y * x * y^{-1} * x^{-1} * y^{-1})})^d,$

$((y^3) * t^{(x)})^e,$

$((y * x * y * x^{-1}) * t^{(y^{-1} * x^{-1} * y * x * y)})^f,$

$((y) * t^{(y * x^{-1})})^g,$

$((y^3) * t^{(x * y^{-1})})^h,$

$((x) * t^{(x * y * x * y^{-1})})^i,$

$((x * y) * t^{(x * y^2 * x * y^{-1})})^j >$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
0	4	0	0	0	0	0	0	0	0	$(2 : A_5 \times A_5) : 2^2$
10	0	6	0	0	0	0	0	0	0	$PSL(3, 4)$
8	8	8	0	0	0	0	0	0	0	$M12$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2, (x^{-1} * y * x^{-1} * y^{-1})^2,$

$(x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y)$

$((x * y) * t^{(y^2 * x * y * x^{-1} * y)})^a,$

$((x) * t^{(y^{-2} * x^{-1} * y)})^b,$

$((y * x * y * x^{-1}) * t^{(x * y^{-2})})^c,$

$$((x * y) * t^{(y*x*y^{-1}*x^{-1}*y^{-1})})d,$$

$$((y) * t^{(x)})e,$$

$$((y * x * y * x^{-1}) * t^{(y^{-1}*x^{-1}*y*x*y)})f,$$

$$((y^5) * t^{(y*x^{-1})})g,$$

$$((x) * t^{(x*y^{-1})})h,$$

$$((y^3) * t^{(x*y*x*y^{-1})})i,$$

$$((x) * t^{(x*y^2*x*y^{-1})})j >$$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
6	4	0	0	0	0	0	0	0	0	$PSL(2,25):2$
0	4	6	0	0	0	0	0	0	0	$PSL(2,121):2$

$G\langle x,y,t\rangle := Group\langle x,y,t | x^3, (x*y^{-2})^2, (x^{-1}*y*x^{-1}*y^{-1})^2,$

$(x^{-1}*y^{-3})^2, t^2,$

$(t, x^{-1}*y^{-1}*x*y^2), (t, x*y*x^{-1}*y^{-1}*x*y)$

$((x)*t^{(y^2*x*y*x^{-1}*y)})^a,$

$((y^2)*t^{(y^{-2}*x^{-1}*y)})^b,$

$((x^{-1}*y*x^{-1}*y*x)*t^{(x*y^{-2})})^c,$

$((x)*t^{(y*x*y^{-1}*x^{-1}*y^{-1})})^d,$

$((y^3)*t^{(x)})^e,$

$((x)*t^{(y^{-1}*x^{-1}*y*x*y)})^f,$

$((x^{-1}*y*x^{-1}*y*x)*t^{(y*x^{-1})})^g,$

$((x*y)*t^{(x*y^{-1})})^h,$

$((x^{-1}*y*x^{-1}*y*x)*t^{(x*y*x*y^{-1})})^i,$

$((y)*t^{(x*y^2*x*y^{-1})})^j >$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
5	6	0	0	0	0	0	0	0	0	$J1$
0	0	4	0	0	0	0	0	0	0	$PSL(2,11):2$
6	0	6	0	0	0	0	0	0	0	$(A_7:2):3$

$G\langle x,y,t\rangle := Group\langle x,y,t | x^3, (x*y^{-2})^2, (x^{-1}*y*x^{-1}*y^{-1})^2,$

$(x^{-1}*y^{-3})^2, t^2,$

$(t, x^{-1}*y^{-1}*x*y^2), (t, x*y*x^{-1}*y^{-1}*x*y)$

$((x^{-1}*y*x^{-1}*y*x)*t^{(y^2*x*y*x^{-1}*y)})^a,$

$((x*y)*t^{(y^{-2}*x^{-1}*y)})^b,$

$((y)*t^{(x*y^{-2})})^c,$

$((x)*t^{(y*x*y^{-1}*x^{-1}*y^{-1})})^d,$

$((y^5)*t^{(x)})^e,$

$((x * y) * t^{(y^{-1}*x^{-1}*y*x*y)}))f,$

$((y^3) * t^{(y*x^{-1})})g,$

$((x) * t^{(x*y^{-1})})h,$

$((y) * t^{(x*y*x*y^{-1})})i,$

$((x^{-1} * y * x^{-1} * y * x) * t^{(x*y^2*x*y^{-1})})j >$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
3	5	0	0	0	0	0	0	0	0	$J1$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2, (x^{-1} * y * x^{-1} * y^{-1})^2,$

$(x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y)$

$((y^5) * t^{(y^2*x*y*x^{-1}*y)})a,$

$((x) * t^{(y^{-2}*x^{-1}*y)})b,$

$((y) * t^{(x*y^{-2})})c,$

$((x^{-1} * y * x^{-1} * y * x) * t^{(y*x*y^{-1}*x^{-1}*y^{-1})})d,$

$((y^2) * t^{(x)})e,$

$((x^{-1} * y * x^{-1} * y * x) * t^{(y^{-1}*x^{-1}*y*x*y)})f,$

$((y^2) * t^{(y*x^{-1})})g,$

$((y^3) * t^{(x*y^{-1})})h,$

$((y) * t^{(x*y*x*y^{-1})})i,$

$((x) * t^{(x*y^2*x*y^{-1})})j >$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	0	0	2	7	$PSL(2, 41)$
0	0	0	0	0	0	0	0	3	4	$PSL(2, 16) : 2$
0	0	0	0	0	0	0	7	2	7	$PSL(2, 41)$
0	0	0	0	0	0	0	10	4	4	$(2^{11} : PSL(2 : 11)) : 2$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2, (x^{-1} * y * x^{-1} * y^{-1})^2,$

$(x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y)$

$((y^2) * t^{(y^2 * x * y * x^{-1} * y)})^a,$

$((x * y) * t^{(y^{-2} * x^{-1} * y)})^b,$

$((y^5) * t^{(x * y^{-2})})^c,$

$((x * y) * t^{(y * x * y^{-1} * x^{-1} * y^{-1})})^d,$

$((y^3) * t^{(x)})^e,$

$((y * x * y * x^{-1}) * t^{(y^{-1} * x^{-1} * y * x * y)})^f,$

$((y) * t^{(y * x^{-1})})^g,$

$((x * y) * t^{(x * y^{-1})})^h,$

$((y^5) * t^{(x * y * x * y^{-1})})^i,$

$((y * x * y * x^{-1}) * t^{(x * y^2 * x * y^{-1})})^j >$

$2^{*12} : (2xA_5)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	8	0	2	10	$(3 : 2^9)(A_6 : 2)$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2, (x^{-1} * y * x^{-1} * y^{-1})^2,$

$(x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y)$

$((y) * t^{(y^2 * x * y * x^{-1} * y)})^a,$

$((x * y) * t^{(y^{-2} * x^{-1} * y)})^b,$

$((y^3) * t^{(x * y^{-2})})^c,$

$((y * x * y * x^{-1}) * t^{(y * x * y^{-1} * x^{-1} * y^{-1})})^d,$

$((x) * t^{(x)})^e,$

$((x) * t^{(y^{-1} * x^{-1} * y * x * y)})^f,$

$((y * x * y * x^{-1}) * t^{(y * x^{-1})})^g,$

$((y^3) * t^{(x * y^{-1})})^h,$

$((y^5) * t^{(x * y * x * y^{-1})})^i,$

$((x * y) * t^{(x * y^2 * x * y^{-1})})^j >$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	0	8	2	6	$PSL(3,4):2^2$
0	0	0	0	0	0	0	8	0	4	$(6:A_6 \times A_6):2^2$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2, (x^{-1} * y * x^{-1} * y^{-1})^2,$

$(x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y)$

$((y^2) * t^{(y^2 * x * y * x^{-1} * y)})^a,$

$((x * y) * t^{(y^{-2} * x^{-1} * y)})^b,$

$((y^5) * t^{(x * y^{-2})})^c,$

$((x * y) * t^{(y * x * y^{-1} * x^{-1} * y^{-1})})^d,$

$((y^3) * t^{(x)})^e,$

$((y * x * y * x^{-1}) * t^{(y^{-1} * x^{-1} * y * x * y)})^f,$

$((y) * t^{(y * x^{-1})})^g,$

$((x * y) * t^{(x * y^{-1})})^h,$

$((y^5) * t^{(x * y * x * y^{-1})})^i,$

$((y * x * y * x^{-1}) * t^{(x * y^2 * x * y^{-1})})^j >;$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
4	0	0	0	0	0	0	0	0	0	$PSL(2:11):2$
5	0	0	0	0	0	0	0	0	0	$(5^3:2):A_5$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2, (x^{-1} * y * x^{-1} * y^{-1})^2,$

$(x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y)$

$((y^3) * t^{(y^2 * x * y * x^{-1} * y)})^a,$

$((x * y) * t^{(y^{-2} * x^{-1} * y)})^b,$

$((y^3) * t^{(x * y^{-2})})^c,$

$((x * y) * t^{(y * x * y^{-1} * x^{-1} * y^{-1})})^d,$

$((y^3) * t^{(x)})^e,$

$((x * y) * t^{(y^{-1} * x^{-1} * y * x * y)})^f,$

$((x * y) * t^{(y * x^{-1})})^g,$

$((y^3) * t^{(x * y^{-1})})^h,$

$((y^3) * t^{(x * y * x * y^{-1})})^i,$

$((x * y) * t^{(x * y^2 * x * y^{-1})})^j >;$

$2^{*12} : (2 \times A_5)$										
a	b	c	d	e	f	g	h	i	j	G
0	5	0	0	0	0	0	0	0	0	$PSL(2:19)$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, (x * y^{-2})^2,$

$(x^{-1} * y * x^{-1} * y^{-1})^2, (x^{-1} * y^{-3})^2, t^2,$

$(t, x^{-1} * y^{-1} * x * y^2), (t, x * y * x^{-1} * y^{-1} * x * y),$

$((y^5) * t^{(y^2 * x * y * x^{-1} * y)})^a,$

$((x * y) * t^{(y^{-2} * x^{-1} * y)})^b,$

$((y^5) * t^{(x * y^{-2})})^c,$

$((x) * t^{(y * x * y^{-1} * x^{-1} * y^{-1})})^d,$

$((y^3) * t^{(x)})^e,$

$((x) * t^{(y^{-1} * x^{-1} * y * x * y)})^f,$

$((x) * t^{(y * x^{-1})})^g,$

$((y^5) * t^{(x * y^{-1})})^h,$

$((x) * t^{(x * y * x * y^{-1})})^i,$

$((x * y) * t^{(x * y^2 * x * y^{-1})})^j >;$

10.3 $2^{*14} : (2^3 : 7)$

We have the following information

S:=Sym(14), we are working with 14 letters.

$x \sim (1, 11, 13)(2, 12, 10)(3, 7, 9)(5, 6, 14)$

$y \sim (1, 13, 2)(3, 9, 6)(4, 7, 12, 8, 11, 14)(5, 10)$

The order of $|N| = 168$.

	a	b	c	d	e	f	g	h	i	j	k	l	m	G
							$2^{*14} : (2^3 : 7)$							
	3	0	0	0	0	0	0	0	0	0	0	0	0	$(2^4 : 2^3) : (7 : 3)$
	11	0	5	0	0	0	0	0	0	0	0	0	0	$M22$
	3	2	6	0	0	0	0	0	0	0	0	0	0	$(2^3 : 7) : 3$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, y^6,$

$(y^{-1} * x^{-1})^3, x * y * x^{-1} * y^{-1} * x^{-1} * y * x^{-1} * y^{-2}, t^2,$

$(t, y^3), (t, x^{-1} * y^{-1}),$

$((y^2) * t^{(y^{-1})})^a,$

$((y^3) * t^{(x*y*x^{-1}*y)})^b,$

$((y) * t^{(x^{-1}*y^{-1}*x*y^2)})^c,$

$((y^{-2}) * t^{(x*y^{-2}*x*y)})^d,$

$((y^{-1}) * t^{(y^{-2}*x^{-1}*y*x^{-1})})^e,$

$((y * x^{-1}) * t^{(y^{-1}*x*y^{-1})})^f,$

$(((y * x^{-1})^3) * t^{(y^2*x*y)})^g,$

$((y^2) * t^{(x*y*x^{-1}*y*x^{-1})})^h,$

$((y^3) * t^{(x^{-1}*y^{-2}*x^{-1})})^i,$

$((y^{-2}) * t^{(y*x^{-1})})^j,$

$((y^{-1}) * t^{(y^x)})^k,$

$(((y * x^{-1})^3) * t^{(x*y^{-2}*x^{-1}*y*x^{-1})})^l,$

$((y^2) * t^{(x*y^2*x*y^{-1})})^m >$

$2^{*14} : (2^3 : 7)$													
a	b	c	d	e	f	g	h	i	j	k	l	m	G
8	0	5	0	0	0	0	0	0	0	0	0	0	$M22 : 2$
7	6	6	0	0	0	0	0	0	0	0	0	0	$(2^{10} : 2^3) : (7 : 3)$
6	7	7	14	12	0	0	0	0	0	0	0	0	A_8
6	10	6	14	12	0	0	0	0	0	0	0	0	$(3 : A_7) : 2$
6	0	6	14	12	0	0	0	0	0	0	0	0	$(21 : 3) : (A_7 : 2)$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, y^6,$

$(y^{-1} * x^{-1})^3, x * y * x^{-1} * y^{-1} * x^{-1} * y * x^{-1} * y^{-2}, t^2,$

$(t, y^3), (t, x^{-1} * y^{-1}),$

$(((y * x^{-1})^3) * t^{(y^{-1})})^a,$

$((y) * t^{(x*y*x^{-1}*y)})^b,$

$((y^{-2}) * t^{(x^{-1}*y^{-1}*x*y^2)})^c,$

$((y^3) * t^{(x*y^{-2}*x*y)})^d,$

$((y * x^{-1}) * t^{(y^{-2}*x^{-1}*y*x^{-1})})^e,$

$((y^3) * t^{(y^{-1}*x*y^{-1})})^f,$

$((y) * t^{(y^2*x*y)})^g,$

$((y^{-1}) * t^{(x*y*x^{-1}*y*x^{-1})})^h,$

$((y^{-2}) * t^{(x^{-1}*y^{-2}*x^{-1})})^i,$

$((y^3) * t^{(y*x^{-1})})^j,$

$((y) * t^{(y^x)})^k,$

$((y^{-2}) * t^{(x*y^{-2}*x^{-1}*y*x^{-1})})^l,$

$((y^{-1}) * t^{(x*y^2*x*y^{-1})})^m >$

$2^{*14} : (2^3 : 7)$													
a	b	c	d	e	f	g	h	i	j	k	l	m	G
0	0	0	0	0	0	0	6	10	6	6	7	0	$(2^6 : 7) : 3$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, y^6,$

$(y^{-1} * x^{-1})^3, x * y * x^{-1} * y^{-1} * x^{-1} * y * x^{-1} * y^{-2}, t^2,$

$(t, y^3), (t, x^{-1} * y^{-1}),$

$((y) * t^{(y^{-1})})^a,$

$((y^{-2}) * t^{(x*y*x^{-1}*y)})^b,$

$((y^{-1}) * t^{(x^{-1}*y^{-1}*x*y^2)})^c,$

$((y) * t^{(x*y^{-2}*x*y)})^d,$

$((y^{-2}) * t^{(y^{-2}*x^{-1}*y*x^{-1})})^e,$

$((y * x^{-1}) * t^{(y^{-1}*x*y^{-1})})^f,$

$((y) * t^{(y^2*x*y)})^g,$

$((y^{-2}) * t^{(x*y*x^{-1}*y*x^{-1})})^h,$

$((y^3) * t^{(x^{-1}*y^{-2}*x^{-1})})^i,$

$((y^2) * t^{(y*x^{-1})})^j,$

$((y) * t^{(y^x)})^k,$

$(((y * x^{-1})^3) * t^{(x*y^{-2}*x^{-1}*y*x^{-1})})^l,$

$((y^{-2}) * t^{(x*y^2*x*y^{-1})})^m >$

$2^{*14} : (2^3 : 7)$													
a	b	c	d	e	f	g	h	i	j	k	l	m	G
0	0	0	0	0	0	0	0	0	0	6	6	8	$(2:3):(7:3)(2^6:A_7:2)$
0	0	0	0	0	0	0	0	0	0	6	6	12	$(3:7):(3:2^6):(A_7:2)$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, y^6,$

$(y^{-1} * x^{-1})^3, x * y * x^{-1} * y^{-1} * x^{-1} * y * x^{-1} * y^{-2}, t^2,$

$(t, y^3), (t, x^{-1} * y^{-1}),$

$((y) * t^{(y^{-1})})^a,$

$((y^{-2}) * t^{(x*y*x^{-1}*y)})^b,$

$((y^{-1}) * t^{(x^{-1}*y^{-1}*x*y^2)})^c,$

$((y^3) * t^{(x*y^{-2}*x*y)})^d,$

$((y * x^{-1}) * t^{(y^{-2}*x^{-1}*y*x^{-1})})^e,$

$((y^3) * t^{(y^{-1}*x*y^{-1})})^f,$

$((y^{-1}) * t^{(y^2*x*y)})^g,$

$((y^2) * t^{(x*y*x^{-1}*y*x^{-1})})^h,$

$((y^{-2}) * t^{(x^{-1}*y^{-2}*x^{-1})})^i,$

$((y^3) * t^{(y*x^{-1})})^j,$

$(((y * x^{-1})^3) * t^{(y^x)})^k,$

$((y) * t^{(x*y^{-2}*x^{-1}*y*x^{-1})})^l,$

$((y^3) * t^{(x*y^2*x*y^{-1})})^m >$

$2^{*14} : (2^3 : 7)$													
a	b	c	d	e	f	g	h	i	j	k	l	m	G
0	0	0	0	0	0	0	0	0	0	6	6	10	$(3 : 2^7) : (A_7 : 2)$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^3, y^6,$

$(y^{-1} * x^{-1})^3, x * y * x^{-1} * y^{-1} * x^{-1} * y * x^{-1} * y^{-2}, t^2,$

$(t, y^3), (t, x^{-1} * y^{-1}),$

$((y^2) * t^{(y^{-1})})^a,$

$((y^{-1}) * t^{(x*y*x^{-1}*y)})^b,$

$((y^{-2}) * t^{(x^{-1}*y^{-1}*x*y^2)})^c,$

$((y) * t^{(x*y^{-2}*x*y)})^d,$

$((y^3) * t^{(y^{-2}*x^{-1}*y*x^{-1})})^e,$

$(((y * x^{-1})^3) * t^{(y^{-1}*x*y^{-1})})^f,$

$((y^2) * t^{(y^2*x*y)})^g,$

$((y^{-1}) * t^{(x*y*x^{-1}*y*x^{-1})})^h,$

$((y^3) * t^{(x^{-1}*y^{-2}*x^{-1})})^i,$

$((y^{-2}) * t^{(y*x^{-1})})^j,$

$(((y * x^{-1})^3) * t^{(y^x)})^k,$

$((y^{-1}) * t^{(x*y^{-2}*x^{-1}*y*x^{-1})})^l,$

$((y^{-2}) * t^{(x*y^2*x*y^{-1})})^m >$

10.4 $2^{*20} : (S_5)$

We have the following information

S:=Sym(20), we are working with 20 letters.

$x \sim (1, 20, 6, 10)(2, 19, 8, 12)(3, 13)(4, 15)(5, 16, 11, 17)(7, 14, 9, 18)$

$y \sim (1, 15)(2, 13)(3, 11, 8, 18)(4, 9, 6, 17)(5, 16, 19, 10)(7, 14, 20, 12)$

The order of $|N| = 120$.

$2^{*20} : (S_5)$									
a	b	c	d	e	f	g	h	i	G
3	0	0	0	0	0	0	0	0	$(3 : 2^4) : (A_5 : 2)$
0	0	0	0	0	0	3	0	2	$(2^4 : A_5) : 2$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^4, y^4, (y^{-1} * x)^3, x^{-2} * y^{-2} * x^{-1} * y^2 * x^2 * y^{-1}, t^2,$
$(t, y^2), (t, (y^{-1} * x * y)^2),$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x*y*x*y^2)})^a,$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x^2*y*x^{-1}*y^{-1})})^b,$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(y*x^{-1})})^c,$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x^{-1}*y^2*x^{-1})})^d,$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x*y^{-1}*x*y)})^e,$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(y*x*y*x^{-1}*y^{-1})})^f,$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(y*x^{-1}*y^{-1}*x^{-1}*y)})^g,$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x*y^2*x^2)})^h,$
$((x^2 * y^{-1} * x^{-1} * y^{-1}) * t^{(x^2*y*x*y^{-1})})^i >$

10.5 $2^{*21} : (2 \times 21)$

We have the following information

S:=Sym(21), we are working with 21 letters.

$x \sim (1, 21, 3, 10, 14, 16)(2, 20, 5, 13, 8, 17)(4, 19, 9)(6, 18)(7, 15, 11)$

$y \sim (1, 19, 6, 15, 2, 9, 5, 4, 14, 12, 8, 7, 3, 11)(10, 21, 18, 20, 13, 16, 17)$

The order of $|N| = 126$.

																				$2^{*21} : (2 \times 21)$
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v	G
2	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^6 : 7) : (3 : 2)$
3	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(7^3 : 3 : 2^2)(3 : 2)$
2	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^6 : 7)(3 : 2)$
0	6	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^4 : 2^{14}) : (3^2 : 7)$
4	6	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^{14} : 2^2) : (3^2 : 7)$
5	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(7^4 : 3) : A_5$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^6, (y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x * y, t^2,$

$(t, x * y * x^{-1} * y^{-2}), (t, x^2 * y^{-1} * x^{-1} * y),$

$((y * x^{-1} * y^{-2} * x) * t)^a,$

$((x^2 * y^{-1} * x) * t^{(x*y*x^{-1})^2)})^b,$

$((y^3) * t^{(x*y^{-1}*x^{-1})})^c,$

$((x^2 * y^{-1} * x) * t^{(y*x*y^2)})^d,$

$((y) * t^{(y*x^{-2}*y)})^e,$

$((y^{-2} * x^{-2}) * t^{(y^2*x^2)})^f,$

$((y * x^{-1} * y^{-2} * x) * t^{(y^3*x^{-1})})^g,$

$((x^2 * y^{-1} * x) * t^{(x^{-1}*y^{-1}*x)})^h,$

$((y) * t^{(y*x^{-1}*y^{-2})})^i,$

$((x^2 * y^{-1} * x) * t^{(x*y^{-1})})^j,$

$((y) * t^{(x^2*y)})^k,$

$((y * x) * t^{(x*y^{-1}*x*y)})^l,$

$((x^2 * y^{-1} * x) * t^{(x*y^3)})^m,$

$((y^3) * t^{((x*y^{-1}*x^{-1})^3)})^n,$

$$((x^2 * y^{-1} * x) * t^{(y^3)})^o,$$

$$((y) * t^{(x*y^2*x^{-2}*y^{-1})})^p,$$

$$((x^2 * y^{-1} * x) * t^{(x*y^2*x^{-2})})^q,$$

$$((y^3) * t^{(x*y*x^2)})^r,$$

$$((x^2 * y^{-1} * x) * t^{(x*y*x*y^{-1})})^s,$$

$$((y) * t^{(x*y*x^2*y)})^u,$$

$$((x^2 * y^{-1} * x) * t^{(x^2*y^2*x)})^v >$$

$2^{*21} : (2 \times 21)$																					
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v	G
3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^6 : 3) : (2 : 7 : 3)$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^6, (y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x * y, t^2,$

$(t, x * y * x^{-1} * y^{-2}), (t, x^2 * y^{-1} * x^{-1} * y),$

$((x^2 * y^2 * x * y^{-1}) * t)^a,$

$((x^2 * y^2) * t^{(x*y*x^{-1})^2})^b,$

$((y^{-2} * x^{-2}) * t^{(x*y^{-1}*x^{-1})})^c,$

$((x^2 * y^2 * x * y^{-1}) * t^{(y*x*y^2)})^d,$

$((x^2 * y^2) * t^{(y*x^{-2}*y)})^e,$

$((y^{-2} * x^{-2}) * t^{(y^2*x^2)})^f,$

$((x^2 * y^2 * x * y^{-1}) * t^{(y^3*x^{-1})})^g,$

$((y * x) * t^{(x^{-1}*y^{-1}*x)})^h,$

$((y^{-2} * x^{-2}) * t^{(y*x^{-1}*y^{-2})})^i,$

$((x^2 * y^2 * x * y^{-1}) * t^{(x*y^{-1})})^j,$

$((x^2 * y^2) * t^{(x^2*y)})^k,$

$((y * x) * t^{(x*y^{-1}*x*y)})^l,$

$((x^2 * y^2 * x * y^{-1}) * t^{(x*y^3)})^m,$

$((y^{-2} * x^{-2}) * t^{(x*y^{-1}*x^{-1})^3})^n,$

$((x^2 * y^2) * t^{(y^3)})^o,$

$((x^2 * y^2 * x * y^{-1}) * t^{(x*y^2*x^{-2}*y^{-1})})^p,$

$((y * x) * t^{(x*y^2*x^{-2})})^q,$

$((y^{-2} * x^{-2}) * t^{(x*y*x^2)})^r,$

$((x^2 * y^2 * x * y^{-1}) * t^{(x*y*x*y^{-1})})^s,$

$((y * x) * t^{(x*y*x^2*y)})^u,$

$((y^{-2} * x^{-2}) * t^{(x^2*y^2*x)})^v >$

$2^{*21} : (2 \times 21)$																					
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v	G
4	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(3 : A_{10}) : 2$
21	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^5 : 2) : (3^2 : 7)$
14	3	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^3 : 7) : 3$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^6, (y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x * y, t^2,$

$(t, x * y * x^{-1} * y^{-2}), (t, x^2 * y^{-1} * x^{-1} * y),$

$((y^3) * t)^a,$

$((x^2 * y^2) * t^{(x*y*x^{-1})^2})^b,$

$((y * x^2) * t^{(x*y^{-1}*x^{-1})})^c,$

$((x * y * x^2) * t^{(y*x*y^2)})^d,$

$((y) * t^{(y*x^{-2}*y)})^e,$

$((x^2 * y^2 * x * y^{-1}) * t^{(y^2*x^2)})^f,$

$((y^2) * t^{(y^3*x^{-1})})^g,$

$((x * y * x^2) * t^{(x^{-1}*y^{-1}*x)})^h,$

$((y) * t^{(y*x^{-1}*y^{-2})})^i,$

$((y * x^2) * t^{(x*y^{-1})})^j,$

$((y) * t^{(x^2*y)})^k,$

$((x * y * x^2) * t^{(x*y^{-1}*x*y)})^l,$

$((y^-2) * t^{(x*y^3)})^m,$

$((x^2 * y^2) * t^{((x*y^{-1}*x^{-1})^3)})^n,$

$((y * x^2) * t^{(y^3)})^o,$

$((x^2 * y^2 * x * y^-1) * t^{(x*y^2*x^{-2}*y^{-1})})^p,$

$((y * x^2) * t^{(x*y^2*x^{-2})})^q,$

$((y) * t^{(x*y*x^2)})^r,$

$((y * x^2) * t^{(x*y*x*y^{-1})})^s,$

$((y^3) * t^{(x*y*x^2*y)})^u,$

$((y) * t^{(x^2*y^2*x)})^v >$

$2^{*21} : (2 \times 21)$																					
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v	G
6	8	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^3 : 3) : (A_{10} : 2)$

$G\langle x,y,t\rangle := Group\langle x,y,t | x^6, (y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x * y, t^2,$

$(t, x * y * x^{-1} * y^{-2}), (t, x^2 * y^{-1} * x^{-1} * y),$

$((y^{-2}) * t)^a,$

$((x^{-2} * y^{-1}) * t^{(x*y*x^{-1})^2)})^b,$

$((y^3) * t^{(x*y^{-1}*x^{-1})})^c,$

$((x^2 * y^2) * t^{(y*x*y^2)})^d,$

$((y^2) * t^{(y*x^{-2}*y)})^e,$

$((x^2 * y^2) * t^{(y^2*x^2)})^f,$

$((x * y * x^2) * t^{(y^3*x^{-1})})^g,$

$((y) * t^{(x^{-1}*y^{-1}*x)})^h,$

$((x^{-2} * y^{-1}) * t^{(y*x^{-1}*y^{-2})})^i,$

$((y^3) * t^{(x*y^{-1})})^j,$

$((y * x) * t^{(x^2*y)})^k,$

$((x^{-2} * y^{-1}) * t^{(x*y^{-1}*x*y)})^l,$

$((y^3) * t^{(x*y^3)})^m,$

$((y * x^2) * t^{(x*y^{-1}*x^{-1})^3)})^n,$

$((x^{-2} * y^{-1}) * t^{(y^3)})^o,$

$((x * y * x^2) * t^{(x*y^2*x^{-2}*y^{-1})})^p,$

$((y * x^2) * t^{(x*y^2*x^{-2})})^q,$

$((y) * t^{(x*y*x^2)})^r,$

$((x^{-2} * y^{-1}) * t^{(x*y*x*y^{-1})})^s,$

$((y^{-2}) * t^{(x*y*x^2*y)})^u,$

$((x * y * x^2) * t^{(x^2*y^2*x)})^v >$

	2*21 : (2 × 21)																				
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v	G
6	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^{19}:3):(2:7:3)$
6	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^{18}:3):(2:7:3)$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^6, (y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x * y, t^2,$

$(t, x * y * x^{-1} * y^{-2}), (t, x^2 * y^{-1} * x^{-1} * y),$

$((y * x) * t)^a,$

$((x^2 * y^2) * t^{(x*y*x^{-1})^2})^b,$

$((y^{-2} * x^{-2}) * t^{(x*y^{-1}*x^{-1})})^c,$

$((y) * t^{(y*x*y^2)})^d,$

$((x * y * x^2) * t^{(y*x^{-2}*y)})^e,$

$((y^{-2}) * t^{(y^2*x^2)})^f,$

$((y * x) * t^{(y^3*x^{-1})})^g,$

$((y) * t^{(x^{-1}*y^{-1}*x)})^h,$

$((y^3) * t^{(y*x^{-1}*y^{-2})})^i,$

$((x^2 * y^2 * x * y^{-1}) * t^{(x*y^{-1})})^j,$

$((x^2 * y^2) * t^{(x^2*y)})^k,$

$((y * x) * t^{(x*y^{-1}*x*y)})^l,$

$((x * y * x^2) * t^{(x*y^3)})^m,$

$((y^{-2} * x^{-2}) * t^{(x*y^{-1}*x^{-1})^3})^n,$

$((x^2 * y^2) * t^{(y^3)})^o,$

$((x^2 * y^2 * x * y^{-1}) * t^{(x*y^2*x^{-2}*y^{-1})})^p,$

$((y) * t^{(x*y^2*x^{-2})})^q,$

$((y^{-2} * x^{-2}) * t^{(x*y*x^2)})^r,$

$((x * y * x^2) * t^{(x*y*x*y^{-1})})^s,$

$((y^3) * t^{(x*y*x^2*y)})^u,$

$((y^{-2}) * t^{(x^2*y^2*x)})^v >$

											$2^{*21} : (2 \times 21)$										
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v	G
4	4	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2 : A_{10}) : 2$
2	6	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(A_7 : 2) : (3 : 7 : 3)$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^6, (y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x * y, t^2,$

$(t, x * y * x^{-1} * y^{-2}), (t, x^2 * y^{-1} * x^{-1} * y),$

$((x^2 * y^2 * x * y^{-1}) * t)^a,$

$((y) * t^{(x*y*x^{-1})^2)})^b,$

$((y * x^2) * t^{(x*y^{-1}*x^{-1})})^c,$

$((x * y * x^2) * t^{(y*x*y^2)})^d,$

$((y^3) * t^{(y*x^{-2}*y)})^e,$

$((x^2 * y^2 * x * y^{-1}) * t^{(y^2*x^2)})^f,$

$((y^2) * t^{(y^3*x^{-1})})^g,$

$((x * y * x^2) * t^{(x^{-1}*y^{-1}*x)})^h,$

$((y) * t^{(y*x^{-1}*y^{-2})})^i,$

$((y * x^2) * t^{(x*y^{-1})})^j,$

$((y) * t^{(x^2*y)})^k,$

$((x * y * x^2) * t^{(x*y^{-1}*x*y)})^l,$

$((y^{-2}) * t^{(x*y^3)})^m,$

$((x^2 * y^2) * t^{(x*y^{-1}*x^{-1})^3)})^n,$

$((y * x^2) * t^{(y^3)})^o,$

$((x^2 * y^2 * x * y^{-1}) * t^{(x*y^2*x^{-2}*y^{-1})})^p,$

$((y^{-2}) * t^{(x*y^2*x^{-2})})^q,$

$((x * y * x^2) * t^{(x*y*x^2)})^r,$

$((y^2) * t^{(x*y*x*y^{-1})})^s,$

$((y) * t^{(x*y*x^2*y)})^u,$

$((x * y * x^2) * t^{(x^2*y^2*x)})^v >$

$2^{*21} : (2 \times 21)$																					
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v	G
4	14	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$(2^4 : 3) : (A_{10} : 2)$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^6, (y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x * y, t^2,$

$(t, x * y * x^{-1} * y^{-2}), (t, x^2 * y^{-1} * x^{-1} * y),$

$((y^3) * t)^a$

$((x^2 * y^2) * t^{(x*y*x^{-1})^2)})^b,$

$((y * x^2) * t^{(x*y^{-1}*x^{-1})})^c,$

$((x * y * x^2) * t^{(y*x*y^2)})^d,$

$((y) * t^{(y*x^{-2}*y)})^e,$

$((x^2 * y^2 * x * y^{-1}) * t^{(y^2*x^2)})^f,$

$((y^2) * t^{(y^3*x^{-1})})^g,$

$((x * y * x^2) * t^{(x^{-1}*y^{-1}*x)})^h,$

$((y) * t^{(y*x^{-1}*y^{-2})})^i,$

$((y * x^2) * t^{(x*y^{-1})})^j,$

$((y) * t^{(x^2*y)})^k,$

$((x * y * x^2) * t^{(x*y^{-1}*x*y)})^l,$

$((y^{-2}) * t^{(x*y^3)})^m,$

$((x^2 * y^2) * t^{(x*y^{-1}*x^{-1})^3)})^n,$

$((y * x^2) * t^{(y^3)})^o,$

$((x^2 * y^2 * x * y^{-1}) * t^{(x*y^2*x^{-2}*y^{-1})})^p,$

$((y * x^2) * t^{(x*y^2*x^{-2})})^q,$

$((y) * t^{(x*y*x^2)})^r,$

$((y * x^2) * t^{(x*y*x*y^{-1})})^s,$

$((y^3) * t^{(x*y*x^2*y)})^u,$

$((y) * t^{(x^2*y^2*x)})^v >$

10.6 $2^{*30} : (S_6)$

We have the following information

S:=Sym(30), we are working with 30 letters.

$x \sim (1, 2, 4, 7, 10)(3, 6, 9, 14, 18)(5, 8, 12, 11, 16)(13, 17, 19, 24, 21)$

$(15, 20, 25, 29, 23)(22, 27, 30, 26, 28)$

$y \sim (1, 3)(2, 5)(4, 6)(7, 11)(8, 13)(9, 15)(10, 12)(14, 19)(16, 21)(17, 22)$

$(18, 23)(20, 26)(24, 28)(25, 30)(27, 29)$

The order of $|N| = 720$.

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	2	0	6	2	$A_7 : 2$
0	0	0	0	0	0	2	2	0	6	$Sym(6) : 2$
0	0	0	0	0	0	2	2	4	0	2^3
0	0	0	0	0	0	2	2	8	0	2^4
0	0	0	0	0	0	4	2	2	0	$(2 : 3^4)(A_6 : 2)$

$G\langle x, y, t \rangle := Group < x, y, t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((x) * t^{(x^{-1}*y*x^{-1}*y*x^2*y*x*y)})^a,$

$((y) * t^{(x^2*y*x^{-1}*y*x*y)})^b,$

$(((x*y)^3) * t^{(x^{-1}*y*x^{-1}*y*x^2*y*x)})^c,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x^2*y*x^{-2}*y*x^{-2}*y*x*y*x^{-1})})^d,$

$((x * y * x * y * x) * t^{(x*y*x^2*y)})^e,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(y*x^2*y*x^2*y*x^{-1}*y)})^f,$

$(((x*y)^2) * t^{(y*x*y*x^{-2}*y*x^{-2})})^g,$

$((x * y * x^{-1} * y) * t^{(x^{-2}*y*x*y)})^h,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x^2*y*x^{-1})})^i,$

$((x * y) * t^{(y*x*y*x^2*y*x^2)})^j;$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	0	3	0	2	$(2^9 : A_6) : 2$

$G\langle x,y,t \rangle := Group < x,y,t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((x*y) * t^{(x^2*y*x^{-1}*y*x*y)})^a,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(x^2*y*x^{-2}*y*x*y)})^b,$

$((y) * t^{(x*y*x^2*y)})^c,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(y*x*y*x*y*x^2*y*x)})^d,$

$(((x*y)^2) * t^{(x*y*x*y*x^{-2}*y)})^e,$

$((x * y * x^{-1} * y) * t^{(x*y*x^{-1}*y*x^{-1}*y*x)})^f,$

$((x) * t^{(x^{-1}*y*x^{-1}*y*x^2*y*x*y)})^g,$

$((x * y * x * y * x) * t^{(y*x^2*y*x^2*y*x^{-2}*y*x^{-1})})^h,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x*y*x^2*y)})^i,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(y*x^2*y*x^2*y*x^{-1}*y)})^j \text{¿}$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	0	2	0	2	$Cyclic(2)$
0	0	0	0	0	0	0	3	0	2	$(2^6 : A_7) : 2$

$G\langle x,y,t \rangle := Group < x,y,t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((y) * t^{(x*y*x^2*y)})^a,$

$(((x*y)^3) * t^{(x^2*y*x^{-1}*y*x*y)})^b,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(y*x^{-1}*y*x*y*x^{-2}*y*x)})^c,$

$(((x*y)^2) * t^{(x^2*y*x^{-2}*y*x^{-2}*y*x*y*x^{-1})})^d,$

$((x * y * x^{-1} * y) * t^{(x*y*x^{-1}*y*x*y*x)})^e,$

$((x * y * x * y * x) * t^{(x^{-2}*y*x*y)})^f,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(y*x*y*x*y*x^2*y*x)})^g,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(y*x*y*x^2*y*x^2)})^h,$

$((x) * t^{(x^2*y*x^{-1}*y*x*y)})^i,$

$((x * y) * t^{(x^2*y*x^{-1}*y*x^{-1}*y*x)})^j \text{¿}$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	0	3	2	3	$(2^5 : A_6) : 2$

$G\langle x,y,t\rangle := Group < x,y,t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((x*y)*t^{(y*x*y*x^2*y*x^2)})^a,$

$((y)*t^{(y*x^{-1}*y*x*y*x^{-2}*y*x)})^b,$

$((x*y)*t^{(x^{-1}*y*x^{-1}*y*x^2*y*x*y)})^c,$

$(((x*y)^3)*t^{(x^2*y*x^{-1})})^d,$

$((x^2*y*x*y*x^{-1}*y*x^{-2}*y)*t^{(y*x*y*x*y*x^2*y*x)})^e,$

$(((x*y)^2)*t^{(x^2*y*x^{-1}*y*x*y)})^f,$

$((x*y*x^{-1}*y)*t^{(x*y*x^{-1}*y*x^{-2}*y*x^{-2})})^g,$

$((x*y*x*y*x)*t^{(x*y*x^2*y*x^{-1})})^h,$

$((x*y*x^{-1}*y*x^{-1}*y)*t^{(x^2*y*x^{-2}*y*x*y)})^i,$

$((x)*t^{(x*y*x^2*y)})^j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	2	2	0	3	$(3 : A_8) : 2$
0	0	0	0	0	0	2	2	4	3	$A_8 : 2$

$G\langle x,y,t\rangle := Group < x,y,t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((x^2*y*x^{-1}*y*x*y)*t^{(y*x^2*y*x^2*y)})^a,$

$((x*y)*t^{(y*x^2*y*x^2*y*x^{-1}*y)})^b,$

$((x)*t^{(x*y*x*y*x^{-2}*y)})^c,$

$((x*y*x^{-1}*y*x^{-1}*y)*t^{(x*y*x^{-1}*y*x^{-1}*y*x^2)})^d,$

$((x*y*x*y*x)*t^{(x^{-2}*y*x^{-2}*y*x*y*x^{-2})})^e,$

$((x*y*x^{-1}*y)*t^{(x^{-2}*y*x*y)})^f,$

$(((x*y)^2)*t^{(y*x^{-1}*y*x*y*x*y*x^{-2}*y*x)})^g,$

$((x^2*y*x*y*x^{-1}*y*x^{-2}*y)*t^{(x^2*y*x^{-1})})^h,$

$(((x*y)^3)*t^{(x*y*x^{-1}*y*x^{-2}*y*x^{-2})})^i,$

$((y) * t^{(x*y*x^{-1}*y*x^{-1}*y*x)})^j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	2	0	7	2	$(7:2):A_9$

$G\langle x, y, t \rangle := Group < x, y, t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$
$((y) * t^{(x^{-2}*y*x*y)})^a,$
$((x) * t^{(x^{-1}*y*x^{-1}*y*x^2*y*x*y)})^b,$
$((x^2 * y * x^{-1} * y * x * y) * t^{(y*x^2*y*x^2*y*x^{-1}*y)})^c,$
$(((x * y)^3) * t^{(y*x*y*x*y*x^2*y*x)})^d,$
$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(y*x^{-1}*y*x*y*x^{-2}*y*x)})^e,$
$(((x * y)^2) * t^{(x*y*x^2*y)})^f,$
$((x * y * x^{-1} * y) * t^{(x^2*y*x^{-1}*y*x^{-1}*y*x)})^g,$
$((x * y * x * y * x) * t^{(x^2*y*x^{-2}*y*x*y)})^h,$
$((x * y * x^{-1} * y * x^{-1} * y) * t^{(y*x*y*x^2*y*x^2)})^i,$
$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x^2*y*x^{-2}*y*x^{-2}*y*x*y*x^{-1})})^j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	2	8	2	0	$(2^6:A_6):2^2$
0	0	0	0	0	0	3	0	3	2	$(3^5:A_6):2$

$G\langle x, y, t \rangle := Group < x, y, t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$
$((y) * t^{(y*x^{-1}*y*x*y*x^{-2}*y*x)})^a,$
$(((x * y)^3) * t^{(x^2*y*x^{-1})})^b,$
$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(y*x*y*x*y*x^2*y*x)})^c,$
$(((x * y)^2) * t^{(x^2*y*x^{-1}*y*x*y)})^d,$
$((x * y * x^{-1} * y) * t^{(x*y*x^{-1}*y*x^{-2}*y*x^{-2})})^e,$
$((x * y * x * y * x) * t^{(x*y*x^2*y*x^{-1})})^f,$
$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x*y*x^{-2}*y*x^{-2}*y*x)})^g,$
$((x) * t^{(x^2*y*x^{-2}*y*x*y)})^h,$

$((x * y) * t^{(x*y*x^2*y)})^i,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(x^2*y*x^{-1}*y*x^{-1}*y*x)})^j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	2	0	2	4	$(2^{10} : A_6) : 2^2$
0	0	0	0	0	0	2	2	6	3	$(3 : PSL(4,3)) : 2$

$G\langle x,y,t \rangle := Group < x,y,t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(x*y*x*y*x^{-2}*y)})^a,$

$((x * y) * t^{(x*y*x^{-1}*y*x^{-1}*y*x^2)})^b,$

$((x) * t^{(x^{-2}*y*x^{-2}*y*x*y*x^{-2})})^c,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x*y*x^{-1}*y*x^2*y*x^2*y)})^d,$

$((x * y * x * y * x) * t^{(y*x^{-1}*y*x^2*y*x*y)})^e,$

$((x * y * x^{-1} * y) * t^{(x^{-1}*y*x^{-1}*y*x^2*y*x*y)})^f,$

$(((x * y)^2) * t^{(x^{-2}*y*x*y)})^g,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x*y*x*y*x^{-2}*y*x^{-2}*y*x^{-1})})^h,$

$(((x * y)^3) * t^{(x^2*y*x^2*y*x^{-1})})^i,$

$((y) * t^{(x*y*x^2*y*x^{-1})})^j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	0	2	2	3	$A_7 : 2$

$G\langle x,y,t \rangle := Group < x,y,t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((y) * t^{(x^2*y*x^{-1}*y*x*y)})^a,$

$(((x * y)^3) * t^{(x*y*x^{-1}*y*x^{-2}*y*x^{-2})})^b,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x*y*x^2*y*x^{-1})})^c,$

$(((x * y)^2) * t^{(x*y*x^{-2}*y*x^{-2}*y*x)})^d,$

$((x * y * x^{-1} * y) * t^{(x^2*y*x^{-2}*y*x*y)})^e,$

$((x * y * x * y * x) * t^{(x*y*x^2*y)})^f,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x*y*x^{-1}*y*x*y*x)})^g,$

$((x) * t^{(x*y*x*y*x^{-2}*y)})h,$

$((x * y) * t^{(x^{-2}*y*x^{-2}*y*x*y*x^{-2})})i,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(y*x^2*y*x^2*y)})j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	3	2	3	0	$(3 : A_7) : 2$

$G\langle x, y, t \rangle := Group < x, y, t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((y) * t^{(x^2*y*x^{-1}*y*x^{-1}*y*x)})a,$

$(((x * y)^3) * t^{(x*y*x^2*y)})b,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x^2*y*x^{-2}*y*x*y)})c,$

$(((x * y)^2) * t^{(x*y*x^{-2}*y*x^{-2}*y*x)})d,$

$((x * y * x^{-1} * y) * t^{(x*y*x^2*y*x^{-1})})e,$

$((x * y * x * y * x) * t^{(x*y*x^{-1}*y*x^{-2}*y*x^{-2})})f,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x^2*y*x^{-1}*y*x*y)})g,$

$((x) * t^{(y*x*y*x*y*x^2*y*x)})h,$

$((x * y) * t^{(x^2*y*x^{-1})})i,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(y*x^{-1}*y*x*y*x^{-2}*y*x)})j >$

$2^{*30} : (S_6)$									
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	2	3	8	2	$(2^{10} : A_6) : 2$

$G\langle x,y,t\rangle := Group < x,y,t|x^5,y^2,(x^{-1}*y)^6,(x^{-1}*y*x*y)^3,(x*y*x*y*x^{-1}*y*x^{-1}*y)^2,t^2,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(x^{-}2*y*x*y)})^a,$

$((x * y) * t^{(x*y*x^{-1}*y*x^{-1}*y*x)})^b,$

$((x) * t^{(x^{-1}*y*x^{-1}*y*x^2*y*x*y)})^c,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(y*x^{-1}*y*x^2*y*x*y)})^d,$

$((x * y * x * y * x) * t^{(x*y*x^{-1}*y*x^2*y*x^2*y)})^e,$

$((x * y * x^{-1} * y) * t^{(x^{-2}*y*x^{-2}*y*x*y*x*x^{-2})})^f,$

$(((x * y)^2) * t^{(x*y*x^{-1}*y*x^{-1}*y*x^2)})^g,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x*y*x*y*x^{-2}*y)})^h,$

$(((x * y)^3) * t^{(y*x^2*y*x^2*y*x^{-1}*y)})^i,$

$((y) * t^{(y*x^2*y*x^2*y)})^j >$

$2^{*30} : (S_6)$									
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	2	3	5	2	$(2^4 : 3^4)(A_6 : 2)$

$G\langle x,y,t\rangle := Group < x,y,t|x^5,y^2,(x^{-1}*y)^6,(x^{-1}*y*x*y)^3,(x*y*x*y*x^{-1}*y*x^{-1}*y)^2,t^2,$

$((y) * t^{(y*x*y*x*y*x^2*y*x)})^a,$

$(((x * y)^2) * t^{(x*y*x^2*y*x^{-1})})^b,$

$((x * y) * t^{(x*y*x^{-1}*y*x*y*x)})^c,$

$((x) * t^{(x^2*y*x^{-1}*y*x^{-1}*y*x)})^d,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x*y*x^{-1}*y*x^{-2}*y*x^{-2})})^e,$

$((x * y * x * y * x) * t^{(x^2*y*x^{-2}*y*x*y)})^f,$

$((x * y * x^{-1} * y) * t^{(x*y*x^{-2}*y*x^{-2}*y*x)})^g,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x*y*x^2*y)})^h,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(y*x^2*y*x^2*y*x^{-2}*y*x^{-1})})^i,$

$(((x * y)^3) * t^{(x^2*y*x^{-1}*y*x*y)})^j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
0	0	0	0	0	0	0	2	2	5	$(5 : A_7) : 2$
0	0	0	0	0	0	0	2	2	6	$(A_8 : 2)(3 : 2)$

$G\langle x, y, t \rangle := Group < x, y, t | x^5, y^2, (x^{-1} * y)^6, (x^{-1} * y * x * y)^3, (x * y * x * y * x^{-1} * y * x^{-1} * y)^2, t^2,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(y * x^2 * y * x^{-2} * y * x^{-1} * y * x)})^a,$

$((x * y) * t^{(x^{-1} * y * x^{-1} * y * x^2 * y * x)})^b,$

$((x) * t^{(x^2 * y * x^2 * y * x^{-1})})^c,$

$((x * y * x^{-1} * y) * t^{(y * x * y * x^{-2} * y * x^{-2})})^d,$

$(((x * y)^3) * t^{(x^2 * y * x^{-1} * y * x^{-1} * y * x)})^e,$

$((y) * t^{(x * y * x^2 * y)})^f,$

$((x * y * x * y * x) * t^{(x^2 * y * x^{-2} * y * x^{-2} * y * x * y * x^{-1})})^g,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x * y * x^{-1} * y * x * y * x)})^h,$

$(((x * y)^2) * t^{(y * x^2 * y * x^2 * y * x^{-2} * y * x^{-1})})^i,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(y * x * y * x^2 * y * x^2)})^j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
2	4	0	2	0	0	0	0	0	0	$(2^2 : A_7) : 2$

$G\langle x, y, t \rangle := Group < x, y, t | x^5, y^2, (x^{-1} * y)^6, (x^{-1} * y * x * y)^3, (x * y * x * y * x^{-1} * y * x^{-1} * y)^2, t^2,$

$((x) * t^{(y * x^2 * y * x^2 * y)})^a,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(y * x^{-1} * y * x * y * x^{-2} * y * x)})^b,$

$((x * y) * t^{(x * y * x^{-1} * y * x^2 * y * x^2 * y)})^c,$

$((x) * t^{(x^2 * y * x^{-1} * y * x^{-1} * y * x)})^d,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x * y * x * y * x^{-2} * y * x^{-2} * y * x^{-1})})^e,$

$((x * y * x * y * x) * t^{(x * y * x * y * x^{-2} * y)})^f,$

$((x * y * x^{-1} * y) * t^{(x * y * x^2 * y)})^g,$

$(((x * y)^2) * t^{(y * x^2 * y * x^2 * y * x^{-2} * y * x^{-1})})^h,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(x * y * x * y * x^{-2} * y)})^i,$

$(((x * y)^3) * t^{(x * y * x^{-1} * y * x * y * x)})^j >$

$2^{*30} : (S_6)$										
a	b	c	d	e	f	g	h	i	j	G
2	2	4	0	0	0	0	0	0	0	$(2^5 : A_6) : 2^2$

$G\langle x, y, t \rangle := Group < x, y, t | x^5, y^2, (x^{-1}*y)^6, (x^{-1}*y*x*y)^3, (x*y*x*y*x^{-1}*y*x^{-1}*y)^2, t^2,$

$((x^2 * y * x^{-1} * y * x * y) * t^{(y*x^2*y*x^2*y)})^a,$

$((y) * t^{(y*x^{-1}*y*x*y*x^{-2}*y*x)})^b,$

$(((x * y)^3) * t^{(x^2*y*x^2*y*x^{-1})})^c,$

$((x^2 * y * x * y * x^{-1} * y * x^{-2} * y) * t^{(y*x*y*x^2*y*x^2)})^d,$

$(((x * y)^2) * t^{(x^2*y*x^{-1}*y*x*y)})^e,$

$((x * y * x^{-1} * y) * t^{(x*y*x^{-1}*y*x^{-2}*y*x^{-2})})^f,$

$((x * y * x * y * x) * t^{(x*y*x*y*x^{-2}*y*x^{-2}*y*x^{-1})})^g,$

$((x * y * x^{-1} * y * x^{-1} * y) * t^{(x^2*y*x^{-2}*y*x^{-2}*y*x*y*x^{-1})})^h,$

$((x) * t^{(x*y*x^2*y)})^i,$

$((x * y) * t^{(x^{-2}*y*x*y)})^j >$

10.7 $2^{*72} : ((2^3 : 3) : 2)$

We have the following information

S:=Sym(72), we are working with 72 letters.

$x \sim (1, 21, 19, 71, 51, 53)(2, 20, 52)(3, 43, 39, 69, 29, 33)(4, 40, 30)$

$(5, 65, 59, 67, 7, 13)(6, 60, 8)(9, 35, 25, 63, 37, 47)(10, 26, 38)(11, 57, 45, 61, 15, 27)$

$(12, 46, 16)(14, 66, 68)(17, 49, 31, 55, 23, 41)(18, 32, 24)(22, 72, 54)$

$(28, 58, 62)(34, 44, 70)(36, 64, 48)(42, 50, 56)$

$y \sim (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)$

$(17, 18)(19, 20)(21, 22)(23, 24)(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)$

$(35, 36)(37, 38)(39, 40)(41, 42)(43, 44)(45, 46)(47, 48)(49, 50)(51, 52)(53, 54)$

$(55, 56)(57, 58)(59, 60)(61, 62)(63, 64)(65, 66)(67, 68)(69, 70)(71, 72);$

$z \sim (1, 57, 53, 11, 51, 27, 71, 15, 19, 61, 21, 45)(2, 28, 22, 12, 20, 58, 72, 46, 52, 62, 54, 16)$

$(3, 41, 33, 23, 29, 55, 69, 31, 39, 49, 43, 17)(4, 56, 44, 24, 40, 42, 70, 18, 30, 50, 34, 32)$

$(5, 25, 13, 35, 7, 9, 67, 47, 59, 37, 65, 63)(6, 10, 66, 36, 60, 26, 68, 64, 8, 38, 14, 48)$

The order of $|N| = 48$.

							2^{*72}						
a	b	c	d	e	f	g	h	i	j	k	l	m	G
0	4	0	0	0	0	0	0	0	0	0	0	0	$(37^2 : 3) : 2^4$

$G\langle x, y, t\rangle := Group\langle x, y, t | y^2, (x, z), x^6, (x * y * x)^2, z * y * x^2 * z^{-1} * y, z^2 * x^2 * z^2, y * x^{-1} *$
$y * x * z^{-2}, t^3 7, t^{(x^3)} = t^{(36)}, t^{(x * z^2 * y * z * y)} = t^1 4$

$((x^2 * z^{-2}) * t^{(x^2 * y * x^{-1})})^a,$

$((x * y) * t^{(z * x^{-1})})^b,$

$((x^2 * z^{-2}) * t^{(x * y * z)})^c,$

$((z) * t^{(x^3 * z)})^d,$

$((x^2 * z^{-2}) * t^{(x^{-1} * y * z)})^e,$

$((x * y) * t^{(x * z^{-2})})^f,$

$((x^2 * z^{-2}) * t^{(x^2 * y)})^g,$

$((z) * t^{(x)})^h,$

$((x^2 * z^{-2}) * t^{(x * y)})^i,$

$((x) * t^{(x^3 * z^{-1})})^j,$

$((x^2 * z^{-2}) * t^{(x * y * z^{-1})})^k,$

$$((x * y) * t^{(z^{-1})})^l,$$

$$((x^2 * z^{-2}) * t^{(z^{-1} * y)})^m >$$

10.8 $2^{*84} : PSL(2,7)$

We have the following information

S:=Sym(84), we are working with 84 letters.

$x \sim (1,2)(3,5)(4,7)(6,10)(8,13)(9,15)(11,18)(12,19)(14,22)(16,24)$

$(17,26)(20,29)(21,31)(23,33)(25,30)(27,36)(28,37)(32,40)(34,42)(35,43)(38,47)$

$(39,48)(41,51)(44,49)(45,55)(46,57)(50,61)(52,58)(53,64)(54,66)$

$(56,65)(59,69)(60,71)(62,70)(63,73)(68,76)(75,80)(78,82)(79,83)(81,84)$

$y \sim (1,3,6,11)(2,4,8,14)(5,9,16,25)(7,12,20,30)(10,17,13,21)(15,23,19,28)$

$(18,27,24,34)(22,32,29,38)(26,35,44,54)(31,39,49,60)(33,41,52,63)$

$(36,45,56,48)(37,46,58,68)(40,50,62,43)(42,53,65,57)$

$(47,59,70,51)(55,67,71,78)(61,72,66,75)(64,74,76,81)(69,77,73,79)(80,84)(82,83)$

The order of $|N| = 168$.

$2^{*84} : PSL(2,7)$									
a	b	c	d	e	f	g	h	i	G
3	3	0	0	0	0	0	0	0	$PSL(2,7)$

$G\langle x,y,t\rangle := Group\langle x,y,t | x^2, y^4, y^{-2}*x*y^2*x*y^2*x, (x*y^{-1})^7, t^2, (t,(x*y*x*y^{-1})^2)$

$((y)*t^{(y*x*y^{-1}*x*y*x*y^{-1})})^a,$

$((y)*t^{(x*y*x*y^{-1}*x*y*x)})^b,$

$((y)*t^{(x*y)})^c,$

$((y)*t^{(x*y*x*y^{-1}*x*y)})^d,$

$((y)*t^{(y^2)})^e,$

$((y)*t^{(y*x*y^{-1}*x*y)})^f,$

$((y)*t^{(y*x*y^{-1}*x*y*x*y)})^g,$

$((y)*t^{(y*x*y)})^h, ((y)*t^{(x*y*x*y^{-1}*x*y*x*y*x)})^i >$

10.9 $2^{*160} : (J1)$

We have the following information

S:=Sym(160), we are working with 160 letters.

$x \sim (1, 2, 6, 19, 5, 16, 47, 119, 52, 50)(3, 10, 33, 35, 77, 48, 93, 98, 137, 140)$

$(4, 13, 41, 100, 97, 49, 106, 150, 139, 89)(7, 22, 65, 38, 64, 120, 53, 129, 141, 108)$

$(8, 25, 71, 107, 80, 121, 157, 61, 30, 86)(9, 29, 82, 148, 123, 122, 158, 136, 135, 11)$

$(12, 37, 63, 81, 152, 87, 28, 78, 102, 20)(14, 26, 75, 73, 56, 124, 159, 105, 94, 111)$

$(15, 45, 113, 149, 145, 125, 115, 39, 103, 118)(17, 51, 66, 142, 155, 126, 96, 130, 43, 34)$

$(18, 55, 91, 32, 85, 67, 60, 59, 133, 127)(21, 44, 114, 110, 154, 83, 92, 58, 72, 146)$

$(23, 46, 31, 88, 95, 128, 151, 143, 70, 57)(24, 69, 117, 132, 42, 109, 138, 116, 160, 156)$

$(27, 68, 144, 101, 36, 99, 134, 104, 79, 62)(40, 76, 74, 147, 131, 112, 54, 84, 153, 90)$

$y \sim (1, 3, 11, 33, 91, 93)(2, 7)(4, 14)(5, 17, 52, 95, 85, 142)(6, 20, 60, 90, 136, 71)$

$(8, 26, 76, 24, 70, 72)(9, 30, 32, 23, 68, 103)(10, 16, 48, 123, 98, 59)(12, 38, 78, 129, 80, 146)$

$(13, 42, 100, 114, 150, 64)(15, 46, 102, 153, 157, 96)(18, 56, 29, 83, 36, 69)$

$(19, 57, 127, 43, 50, 126)(21, 62, 138, 67, 111, 158)(22, 66, 44, 115, 116, 31)$

$(25, 51, 125, 151, 81, 147)(27, 77)(28, 79, 113, 144, 74, 148)(34, 94, 112, 75, 118, 132)(35, 97)$

$(37, 101, 39, 104, 84, 135)(40, 105, 145, 160, 155, 73)(41, 108, 106, 156, 139, 58)$

$(45, 117, 143, 53, 130, 92)(47, 120)(49, 124)(54, 109, 88, 110, 121, 159)$

$(55, 131, 82, 61, 119, 152)(63, 65, 86, 154, 87, 141)(89, 137)(99, 140)(107, 133, 128, 134, 149, 122)$

The order of $|N| = 1920$.

<table>
<tr><th colspan="15">$2^{*160} : (J1)$</th></tr>
<tr><th>a</th><th>b</th><th>c</th><th>d</th><th>e</th><th>f</th><th>g</th><th>h</th><th>i</th><th>j</th><th>k</th><th>l</th><th>m</th><th>n</th><th>G</th></tr>
<tr><td>5</td><td>2</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>$2^{10} : A_5$</td></tr>
<tr><td>7</td><td>2</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>$J1$</td></tr>
<tr><td>5</td><td>2</td><td>0</td><td>6</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>$2^5 : A_5$</td></tr>
</table>

$G\langle x, y, t \rangle := Group\langle x, y, t | x^{1}0, y^6, (x * y^{-2} * x)^2, (x * y^2 * x^2)^2, (y^{-1} * x^{-1})^5,$

$(x * y^2 * x^{-1} * y^{-1})^2, x^{-1} * y^{-1} * x^5 * y * x^{-4},$

$y * x^{-2} * y^{-1} * x^3 * y * x * y^3 * x^{-1}, t^2, (t, (x * y^{-1} * x^{-1})^2), (t, (y * x * y)^2)$

$((x) * t^{(x)})^a,$

$$((y^3) * t^{(x^2*y^2*x*y*x^{-1})})^b,$$

$$((x^5) * t^{(y*x*y^{-2}*x*y*x)})^c,$$

$$((y) * t^{(x^{-2}*y^{-2})})^d,$$

$$((y * x^{-1}) * t^{(y^{-1}*x^{-1}*y^{-1}*x^{-3})})^e,$$

$$((x * y^{-1} * x * y^{-2} * x * y) * t^{(y*x*y^2*x*y*x*y)})^f,$$

$$((x^5) * t^{(x^{-1}*y*x*y*x^{-1}*y^{-1}*x^{-1}*y^{-1})})^g,$$

$$((x) * t^{(y*x^2*y^{-1}*x)})^h,$$

$$((y) * t^{(y*x)})^i,$$

$$((y * x^{-1}) * t^{(y^2)})^j,$$

$$((x) * t^{(x*y*x^2*y^{-1}*x^{-2})})^k,$$

$$((x * y^{-1} * x * y^{-2} * x * y) * t^{(y^{-1}*x^{-1}*y^{-1}*x*y^{-1}*x^{-1}*y)})^l,$$

$$((y * x^{-1}) * t^{(y^{-1}*x*y*x^{-2}*y*x)})^m,$$

$$((y^3) * t^{(x^{-3}*y^{-1}*x^{-1}*y^2)})^n >$$

$2^{*160} : (J1)$														
a	b	c	d	e	f	g	h	i	j	k	l	m	n	G
3	0	0	0	0	0	0	0	0	0	0	0	0	0	$(3^5 : 2^4) : A_5$
0	2	0	0	0	0	0	0	0	0	0	0	0	0	$Cyclic(2)$
0	3	0	0	0	0	0	0	0	0	0	0	0	0	$PSL(6,2)$
3	12	0	0	0	0	0	0	0	0	0	0	0	0	$(3^5 : 2^4) : A_5$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^10, y^6, (x * y^{-2} * x)^2, (x * y^2 * x^2)^2, (y^{-1} * x^{-1})^5,$

$(x * y^2 * x^{-1} * y^{-1})^2, x^{-1} * y^{-1} * x^5 * y * x^{-4},$

$y * x^{-2} * y^{-1} * x^3 * y * x * y^3 * x^{-1}, t^2, (t, (x * y^{-1} * x^{-1})^2), (t, (y * x * y)^2)$

$((y * x^{-1}) * t^{(x)})^a,$

$((y * x^3 * y) * t^{(x^2 * y^2 * x * y * x^{-1})})^b,$

$((y * x^{-1}) * t^{(y * x * y^{-2} * x * y * x)})^c,$

$((y * x^3 * y) * t^{(x^{-2} * y^{-2})})^d,$

$((y^2) * t^{(y^{-1} * x^{-1} * y^{-1} * x^{-3})})^e,$

$((y * x^3 * y) * t^{(y * x * y^2 * x * y * x * y)})^f,$

$((y^2) * t^{(x^{-1} * y * x * y * x * x^{-1} * y^{-1} * x^{-1} * y^{-1})})^g,$

$((y * x^3 * y) * t^{(y * x^2 * y^{-1} * x)})^h,$

$((y * x^{-1}) * t^{(y * x)})^i,$

$((x * y * x * y^2 * x * y) * t^{(y^2)})^j,$

$((x) * t^{(x * y * x^2 * y^{-1} * x^{-2})})^k,$

$((y^2) * t^{(y^{-1} * x^{-1} * y^{-1} * x * y^{-1} * x^{-1} * y)})^l,$

$((y) * t^{(y^{-1} * x * y * x^{-2} * y * x)})^m,$

$((x * y * x * y^2 * x * y) * t^{(x^{-3} * y^{-1} * x^{-1} * y^2)})^n >$

$2^{*160} : (J1)$														
a	b	c	d	e	f	g	h	i	j	k	l	m	n	G
0	6	2	6	0	0	0	0	0	0	0	0	0	0	$A_5 : (3^4 : 2)$
0	12	2	6	0	0	0	0	0	0	0	0	0	0	$(2^4 : 3^4)(2 : A_5)$
0	10	2	6	0	0	0	0	0	0	0	0	0	0	$(5^4 : A_5) : 2$
0	8	2	6	0	0	0	0	0	0	0	0	0	0	$2^{14} : A_5$
0	14	2	6	0	0	0	0	0	0	0	0	0	0	$A_5 : (7^4 : 2)$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^10, y^6, (x * y^{-2} * x)^2, (x * y^2 * x^2)^2, (y^{-1} * x^{-1})^5,$

$(x * y^2 * x^{-1} * y^{-1})^2, x^{-1} * y^{-1} * x^5 * y * x^{-4},$

$y * x^{-2} * y^{-1} * x^3 * y * x * y^3 * x^{-1}, t^2, (t, (x * y^{-1} * x^{-1})^2), (t, (y * x * y)^2)$

$(((x^{-1} * y)^3) * t^{(x)})^a,$

$((y * x^3 * y) * t^{(x^2*y^2*x*y*x^{-1})})^b,$

$(((x^{-1} * y)^3) * t^{(y*x*y^{-2}*x*y*x)})^c,$

$((y) * t^{(x^{-2}*y^{-2})})^d,$

$(((x^{-1} * y)^3) * t^{(y^{-1}*x^{-1}*y^{-1}*x^{-3})})^e,$

$((x) * t^{(y*x*y^2*x*y*x*y)})^f,$

$(((x^{-1} * y)^3) * t^{(x^{-1}*y*x*y*x^{-1}*y^{-1}*x^{-1}*y^{-1})})^g,$

$((y * x^3 * y) * t^{(y*x^2*y^{-1}*x)})^h,$

$(((x^{-1} * y)^3) * t^{(y*x)})^i,$

$((y) * t^{(y^2)})^j,$

$((x^5) * t^{(x*y*x^2*y^{-1}*x^{-2})})^k,$

$((y * x^3 * y) * t^{(y^{-1}*x^{-1}*y^{-1}*x*y^{-1}*x^{-1}*y)})^l,$

$((x^3) * t^{(y^{-1}*x*y*x^{-2}*y*x)})^m,$

$((y * x^3 * y) * t^{(x^{-3}*y^{-1}*x^{-1}*y^2)})^n \dot{\iota}$

$2^{*160} : (J1)$														
a	b	c	d	e	f	g	h	i	j	k	l	m	n	G
0	3	0	0	0	0	0	0	0	0	0	0	0	0	$J2$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^10, y^6, (x * y^{-2} * x)^2, (x * y^2 * x^2)^2, (y^{-1} * x^{-1})^5,$

$(x * y^2 * x^{-1} * y^{-1})^2, x^{-1} * y^{-1} * x^5 * y * x^{-4},$

$y * x^{-2} * y^{-1} * x^3 * y * x * y^3 * x^{-1}, t^2, (t, (x * y^{-1} * x^{-1})^2), (t, (y * x * y)^2)$

$((y^2) * t^{(x)})^a,$

$((x * y^{-1} * x * y^{-2} * x * y) * t^{(x^2 * y^2 * x * y * x^{-1})})^b,$

$((y^2) * t^{(y * x * y^{-2} * x * y * x)})^c,$

$((y * x^{-1}) * t^{(x^{-2} * y^{-2})})^d,$

$((y^2) * t^{(y^{-1} * x^{-1} * y^{-1} * x^{-3})})^e,$

$(((x^{-1} * y)^3) * t^{(y * x * y^2 * x * y * x * y)})^f,$

$((y^2) * t^{(x^{-1} * y * x * y * x^{-1} * y^{-1} * x^{-1} * y^{-1})})^g,$

$(((x^{-1} * y)^3) * t^{(y * x^2 * y^{-1} * x)})^h,$

$((y^2) * t^{(y * x)})^i,$

$((x) * t^{(y^2)})^j,$

$(((x^{-1} * y)^3) * t^{(x * y * x^2 * y^{-1} * x^{-2})})^k,$

$((y) * t^{(y^{-1} * x^{-1} * y^{-1} * x * y^{-1} * x^{-1} * y)})^l,$

$((y * x^{-1}) * t^{(y^{-1} * x * y * x^{-2} * y * x)})^m,$

$((x * y^{-1} * x * y^{-2} * x * y) * t^{(x^{-3} * y^{-1} * x^{-1} * y^2)})^n >;$

$2^{*160} : (J1)$														
a	b	c	d	e	f	g	h	i	j	k	l	m	n	G
4	2	6	8	0	0	0	0	0	0	0	0	0	0	$2^{19} : A_5$
4	2	0	6	0	0	0	0	0	0	0	0	0	0	$(2^4 : 3^5)(2^5 : A_5)$
8	10	6	2	0	0	0	0	0	0	0	0	0	0	$2^9 : A_5$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^{1}0, y^6, (x * y^{-2} * x)^2, (x * y^2 * x^2)^2, (y^{-1} * x^{-1})^5,$

$(x * y^2 * x^{-1} * y^{-1})^2, x^{-1} * y^{-1} * x^5 * y * x^{-4},$

$y * x^{-2} * y^{-1} * x^3 * y * x * y^3 * x^{-1}, t^2, (t, (x * y^{-1} * x^{-1})^2), (t, (y * x * y)^2)$

$((y * x^3 * y) * t^{(x)})^a,$

$(((x^{-1} * y)^3) * t^{(x^2*y^2*x*y*x^{-1})})^b,$

$((y * x^{-1}) * t^{(y*x*y^{-2}*x*y*x)})^c,$

$(((x^{-1} * y)^3) * t^{(x^{-2}*y^{-2})})^d,$

$((x^2) * t^{(y^{-1}*x^{-1}*y^{-1}*x^{-3})})^e,$

$(((x^{-1} * y)^3) * t^{(y*x*y^2*x*y*x*y)})^f,$

$((x) * t^{(x^{-1}*y*x*y*x^{-1}*y^{-1}*x^{-1}*y^{-1})})^g,$

$(((x^{-1} * y)^3) * t^{(y*x^2*y^{-1}*x)})^h,$

$((x^2) * t^{(y*x)})^i,$

$((y) * t^{(y^2)})^j,$

$(((x * y^{-1} * x^{-1})^2) * t^{(x*y*x^2*y^{-1}*x^{-2})})^k,$

$((y) * t^{(y^{-1}*x^{-1}*y^{-1}*x*y^{-1}*x^{-1}*y)})^l,$

$(((x * y^{-1} * x^{-1})^2) * t^{(y^{-1}*x*y*x^{-2}*y*x)})^m,$

$((y * x^{-1}) * t^{(x^{-3}*y^{-1}*x^{-1}*y^2)})^n >$

$2^{*160} : (J1)$														
a	b	c	d	e	f	g	h	i	j	k	l	m	n	G
6	10	2	0	0	0	0	0	0	0	0	0	0	0	$A_5 : (3^4 : 2)$
6	12	2	8	0	0	0	0	0	0	0	0	0	0	$(2^9 : A_6) : 2$
10	10	2	6	0	0	0	0	0	0	0	0	0	0	$(5^4 : A_5) : 2$
14	10	2	6	0	0	0	0	0	0	0	0	0	0	$(7^4 : 2) : A_5$

$G\langle x, y, t \rangle := Group\langle x, y, t | x^10, y^6, (x * y^{-2} * x)^2, (x * y^2 * x^2)^2, (y^{-1} * x^{-1})^5,$

$(x * y^2 * x^{-1} * y^{-1})^2, x^{-1} * y^{-1} * x^5 * y * x^{-4},$

$y * x^{-2} * y^{-1} * x^3 * y * x * y^3 * x^{-1}, t^2, (t, (x * y^{-1} * x^{-1})^2), (t, (y * x * y)^2)$

$((x * y^{-1} * x * y^{-2} * x * y) * t^{(x)})^a,$

$((x) * t^{(x^2 * y^2 * x * y * x^{-1})})^b,$

$((x * y^{-1} * x * y^{-2} * x * y) * t^{(y * x * y^{-2} * x * y * x)})^c,$

$((y) * t^{(x^{-2} * y^{-2})})^d,$

$((x * y^{-1} * x * y^{-2} * x * y) * t^{(y^{-1} * x^{-1} * y^{-1} * x^{-3})})^e,$

$((y * x^3 * y) * t^{(y * x * y^2 * x * y * x * y)})^f,$

$((x * y^{-1} * x * y^{-2} * x * y) * t^{(x^{-1} * y * x * y * x^{-1} * y^{-1} * x^{-1} * y^{-1})})^g,$

$((y * x^3 * y) * t^{(y * x^2 * y^{-1} * x)})^h,$

$((x * y^{-1} * x * y^{-2} * x * y) * t^{(y * x)})^i,$

$((y) * t^{(y^2)})^j,$

$((x^3) * t^{(x * y * x^2 * y^{-1} * x^{-2})})^k,$

$((y^3) * t^{(y^{-1} * x^{-1} * y^{-1} * x * y^{-1} * x^{-1} * y)})^l,$

$((y * x^3 * y) * t^{(y^{-1} * x * y * x^{-2} * y * x)})^m,$

$((x) * t^{(x^{-3} * y^{-1} * x^{-1} * y^2)})^n >$

$2^{*160} : (J1)$														
a	b	c	d	e	f	g	h	i	j	k	l	m	n	G
3	6	0	0	0	0	0	0	0	0	0	0	0	0	$PSL(4,3)$

$G\langle x, y, t\rangle := Group\langle x, y, t | x^{1}0, y^6, (x * y^{-2} * x)^2, (x * y^2 * x^2)^2, (y^{-1} * x^{-1})^5,$

$(x * y^2 * x^{-1} * y^{-1})^2, x^{-1} * y^{-1} * x^5 * y * x^{-4},$

$y * x^{-2} * y^{-1} * x^3 * y * x * y^3 * x^{-1}, t^2, (t, (x * y^{-1} * x^{-1})^2), (t, (y * x * y)^2)$

$((x * y * x * y^2 * x * y) * t^{(x)})^a,$

$((x) * t^{(x^2 * y^2 * x * y * x^{-1})})^b,$

$((x * y * x * y^2 * x * y) * t^{(y * x * y^{-2} * x * y * x)})^c,$

$(((x^{-1} * y)^3) * t^{(x^{-2} * y^{-2})})^d,$

$((x * y * x * y^2 * x * y) * t^{(y^{-1} * x^{-1} * y^{-1} * x^{-3})})^e,$

$((y) * t^{(y * x * y^2 * x * y * x * y)})^f,$

$((x * y * x * y^2 * x * y) * t^{(x^{-1} * y * x * y * x^{-1} * y^{-1} * x^{-1} * y^{-1})})^g,$

$((y^2) * t^{(y * x^2 * y^{-1} * x)})^h,$

$((x * y * x * y^2 * x * y) * t^{(y * x)})^i,$

$((x * y^{-2} * x * y) * t^{(y^2)})^j,$

$((x^5) * t^{(x * y * x^2 * y^{-1} * x^{-2})})^k,$

$((x) * t^{(y^{-1} * x^{-1} * y^{-1} * x * y * x^{-1} * y)})^l,$

$((x * y^{-2} * x * y) * t^{(y^{-1} * x * y * x^{-2} * y * x)})^m,$

$(((x^{-1} * y)^3) * t^{(x^{-3} * y^{-1} * x^{-1} * y^2)})^n >$

Chapter 11

Unsuccessful Progenitors

Although most of the progenitors give us images that are isomorphic to the groups we try, some progenitors fail to give us any for many reasons. We will include some unsuccessful progenitors and explain why they fail.

11.1 $Sym(84)$

This progenitor has very huge double coset number and even we try to do the maximal double cosets were huge as well.

```
a:=3;b:=3;c:=6;d:=3;e:=0;f:=0;g:=0;h:=0;i:=0;
G<x,y,t>:=Group<x,y,t|x^2, y^4,
y^-2 * x * y^2 * x * y^2 * x,
(x * y^-1)^7,t^2,(t,(x * y * x * y^-1)^2),
((y^2)*t^(y * x * y^-1 * x * y * x * y^-1))^a,
((y^2)*t^(x * y * x * y^-1 * x * y * x))^b,
((y * x * y)*t^(x * y))^c,
((y * x * y)*t^(x * y * x * y^-1 * x * y))^d,
((y)*t^(y^2))^e,
((y)*t^( y * x * y^-1 * x * y))^f,
((y * x )*t^(y * x * y^-1 * x * y * x * y))^g,
((y * x )*t^(y * x * y))^h,
((y * x )*t^(x * y * x * y^-1 * x * y * x * y * x))^i>;
#G;
/* 1102248 */
S:=Sym(84);

xx:=S!(1, 2)(3, 5)(4, 7)(6, 10)(8, 13)(9, 15)(11, 18)
```

```
(12, 19)(14, 22)(16, 24)(17,26)(20, 29)(21, 31)(23, 33)
(25, 30)(27, 36)(28, 37)(32, 40)(34, 42)(35,43)(38, 47)
(39, 48)(41, 51)(44, 49)(45, 55)(46, 57)(50, 61)(52,58)
(53,64)(54, 66)(56, 65)(59, 69)(60, 71)(62, 70)
(63, 73)(68, 76)(75, 80)(78,82)(79, 83)(81, 84);

yy:=S!(1, 3, 6, 11)(2, 4, 8, 14)(5, 9, 16, 25)
(7, 12, 20, 30)(10, 17, 13, 21)(15,23, 19, 28)
(18, 27, 24, 34)(22, 32, 29, 38)(26, 35, 44, 54)
(31, 39, 49,60)(33, 41, 52, 63)(36, 45, 56, 48)
(37, 46, 58, 68)(40, 50, 62, 43)(42,53, 65, 57)
(47, 59, 70, 51)(55, 67, 71, 78)(61, 72, 66, 75)
(64, 74, 76,81)(69, 77, 73, 79)(80, 84)(82, 83);

N:=sub<S|xx,yy>;
#N;
/* 168 */
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*1*/
IN:=sub<G1|f(x),f(y)>;
L<u,v>:=Group<u,v|u^2, v^4,
v^-2 * u * v^2 * u * v^2 * u, (u * v^-1)^7>;
Sch:=SchreierSystem(L,sub<L|Id(L)>);
h:=hom<L->N|u->xx,v->yy>;
g:=hom<IN->N|f(x)->xx,f(y)->yy>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;

CompositionFactors(G1);
/*
  G
    |  A(1, 7)                      = L(2, 7)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
```

```
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
*/
NL:=NormalLattice(G1);
NL;
/*
Normal subgroup lattice
-----------------------

[4]  Order 1102248  Length 1  Maximal Subgroups: 3
---
[3]  Order 6561     Length 1  Maximal Subgroups: 2
---
[2]  Order 3        Length 1  Maximal Subgroups: 1
---
[1]  Order 1        Length 1  Maximal Subgroups:

*/
word:=function(A)
Sch:=SchreierSystem(G,sub<G|Id(G)>);
B:=Id(G);
for i in [2..#G] do
P:=[Id(G1): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^-1; end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;
for i in [0..2] do for j in [0..1] do  printf "%o", i;j,xx^i*yy^j; end for;end for;
/*
00 Id(S)
```

```
01 (1, 3, 6, 11)(2, 4, 8, 14)(5, 9, 16, 25)(7, 12, 20, 30)
(10, 17, 13, 21)(15,23, 19, 28)(18, 27, 24, 34)
(22, 32, 29, 38)(26, 35, 44, 54)(31, 39, 49,60)
(33, 41, 52, 63)(36, 45, 56, 48)(37, 46, 58, 68)
(40, 50, 62, 43)(42, 53,65, 57)(47, 59, 70, 51)
(55, 67, 71, 78)(61, 72, 66, 75)(64, 74, 76, 81)
(69,77, 73, 79)(80, 84)(82, 83)

10 (1, 2)(3, 5)(4, 7)(6, 10)(8, 13)(9, 15)(11, 18)
(12, 19)(14, 22)(16, 24)(17,26)(20, 29)(21, 31)
(23, 33)(25, 30)(27, 36)(28, 37)(32, 40)(34, 42)
(35, 43)(38, 47)(39, 48)(41, 51)(44, 49)(45, 55)
(46, 57)(50, 61)(52, 58)(53,64)(54, 66)(56, 65)
(59, 69)(60, 71)(62, 70)(63, 73)(68, 76)(75, 80)
(78,82)(79, 83)(81, 84)

11 (1, 4, 12, 28, 46, 42, 18)(2, 3, 9, 23, 41, 47, 22)
(5, 6, 17, 35, 40, 29, 30)(7, 8, 21, 39, 36, 24, 25)
(10, 11, 27, 45, 67, 71, 31)(13, 14, 32, 50,72, 66, 26)
(15, 16, 34, 53, 74, 76, 37)(19, 20, 38, 59, 77, 73, 33)
(43, 44, 60, 78, 83, 69, 70)(48, 49, 54, 75, 84, 64, 65)
(51, 52, 68, 81, 80, 61, 62)(55, 56, 57, 58, 63, 79, 82)

20 Id(S)

21 (1, 3, 6, 11)(2, 4, 8, 14)(5, 9, 16, 25)(7, 12, 20, 30)
(10, 17, 13, 21)(15,23, 19, 28)(18, 27, 24, 34)
(22, 32, 29, 38)(26, 35, 44, 54)(31, 39, 49,60)
(33, 41, 52, 63)(36, 45, 56, 48)(37, 46, 58, 68)
(40, 50, 62, 43)(42, 53,65, 57)(47, 59, 70, 51)
(55, 67, 71, 78)(61, 72, 66, 75)(64, 74, 76, 81)
(69,77, 73, 79)(80, 84)(82, 83)
*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/* 63 */
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 84] ];
ts[1]:=f(t);
ts[2]:= f(t^(y * x * y^-1 * x * y * x * y^-1));
ts[3]:= f(t^(x * y * x * y^-1 * x * y * x));
ts[4]:= f(t^(x * y));
ts[5]:= f(t^(x * y * x * y^-1 * x * y));
ts[6]:= f(t^(y^2));
```

```
ts[7]:= f(t^(y * x * y^-1 * x * y));
ts[8]:= f(t^( y * x * y^-1 * x * y * x * y));
ts[9]:= f(t^(y * x * y));
ts[10]:= f(t^(x * y * x * y^-1 * x * y * x * y * x));
ts[11]:= f(t^(x * y * x * y^-1 * x * y * x * y^2));
ts[12]:= f(t^((x * y)^2));
ts[13]:= f(t^(y * x * y^-1 * x * y * x * y * x));
ts[14]:= f(t^( y * x * y^-1 * x * y * x * y^2));
ts[15]:= f(t^(x * y * x * y * x * y^2));
ts[16]:= f(t^(x * y * x * y^-1 * x * y^-1));
ts[17]:= f(t^(y^2 * x * y));
ts[18]:= f(t^(y^-1 * x));
ts[19]:= f(t^(x * y * x * y * x));
ts[20]:= f(t^(x * y * x * y^2));
ts[21]:= f(t^(y^2 * x * y^-1));
ts[22]:= f(t^(y * x * y^-1 * x * y^-1 * x * y^2));
ts[23]:= f(t^(y * x * y * x * y));
ts[24]:= f(t^(x * y * x * y^-1 * x * y^-1 * x));
ts[25]:= f(t^(x * y * x * y^-1 * x));
ts[26]:= f(t^(x * y^-1 * x * y * x * y^-1 * x * y^-1));
ts[27]:= f(t^(x * y * x * y^-1 * x * y^-1 * x * y^-1));
ts[28]:= f(t^((x * y)^3));
ts[29]:= f(t^(x * y^-1 * x * y^2));
ts[30]:= f(t^(y * x * y^-1 * x));
ts[31]:= f(t^(y^-1 * x * y * x * y^-1 * x * y^-1));
ts[32]:= f(t^( x * y^-1 * x * y));
ts[33]:= f(t^((y * x)^3));
ts[34]:= f(t^(y^-1 * x * y^-1));
ts[35]:= f(t^(x * y^-1 * x * y * x * y^-1 * x));
ts[36]:= f(t^(y^2 * x * y^-1 * x * y * x * y));
ts[37]:= f(t^(x * y * x * y * x * y * x));
ts[38]:= f(t^((x * y^-1)^2));
ts[39]:= f(t^(y^-1 * x * y * x * y^-1 * x));
ts[40]:= f(t^(x * y^-1 * x * y * x));
ts[41]:= f(t^(x * y^-1 * x * y^-1 * x * y^-1 * x));
ts[42]:= f(t^(x * y * x * y^-1 * x * y^-1 * x * y * x));
ts[43]:= f(t^(y^2 * x * y * x * y * x));
ts[44]:= f(t^((x * y^-1 * x * y)^2));
ts[45]:= f(t^(y^-1 * x * y * x * y));
ts[46]:= f(t^(y * x * y * x * y^-1 * x * y));
ts[47]:= f(t^(x * y^-1 * x * y^-1 * x));
ts[48]:= f(t^(y^2 * x * y^-1 * x * y * x));
ts[49]:= f(t^(y^2 * x * y^-1 * x * y^2));
ts[50]:= f(t^(x * y^-1 * x * y * x * y));
```

```
ts[51]:= f(t^((x * y^-1)^3));
ts[52]:= f(t^(x * y^-1 * x * y^-1 * x * y^-1 * x * y));
ts[53]:= f(t^(x * y * x * y^-1 * x * y^-1 * x * y * x * y));
ts[54]:= f(t^(x * y^-1 * x * y * x * y^-1 * x * y^2));
ts[55]:= f(t^(y^-1 * x * y * x * y * x));
ts[56]:= f(t^((y^-1 * x * y^2)^2));
ts[57]:= f(t^(y^-1 * x * y^-1 * x * y^-1));
ts[58]:= f(t^(y * x * y * x * y^-1 * x * y^2));
ts[59]:= f(t^(x * y^-1 * x * y^-1 * x * y));
ts[60]:= f(t^(y^2 * x * y^-1 * x * y^-1));
ts[61]:= f(t^(x * y^-1 * x * y * x * y * x));
ts[62]:= f(t^((y^2 * x * y^-1)^3));
ts[63]:= f(t^(y * x * y * x * y * x * y^-1));
ts[64]:= f(t^(y^-1 * x * y^-1 * x * y * x));
ts[65]:= f(t^(y^-1 * x * y^-1 * x * y^2));
ts[66]:= f(t^(y^2 * x * y * x * y^-1 * x));
ts[67]:= f(t^(y^2 * x * y^-1 * x * y^-1 * x * y^-1));
ts[68]:= f(t^(x * y * x * y * x * y * x * y^-1));
ts[69]:= f(t^(x * y^-1 * x * y^-1 * x * y * x * y * x * y^-1));
ts[70]:= f(t^(x * y^-1 * x * y^-1 * x * y^2));
ts[71]:= f(t^((y^-1 * x * y^2)^3));
ts[72]:= f(t^( y^2 * x * y * x * y^-1 * x * y^-1));
ts[73]:= f(t^( x * y * x * y * x * y^-1 * x * y^-1 * x));
ts[74]:= f(t^(y^-1 * x * y^-1 * x * y * x * y));
ts[75]:= f(t^(y^2 * x * y * x * y^-1 * x * y));
ts[76]:= f(t^(y * x * y * x * y^-1 * x * y^-1 * x));
ts[77]:= f(t^(x * y^-1 * x * y^-1 * x * y * x * y * x));
ts[78]:= f(t^(y^2 * x * y^-1 * x * y^-1 * x * y));
ts[79]:= f(t^(x * y^-1 * x * y^-1 * x * y * x * y^-1));
ts[80]:= f(t^(x * y^-1 * x * y * x * y * x * y^-1 * x));
ts[81]:= f(t^(y * x * y * x * y^-1 * x * y^-1 * x * y));
ts[82]:= f(t^(y^2 * x * y^-1 * x * y^-1 * x * y * x));
ts[83]:= f(t^(x * y^-1 * x * y^-1 * x * y * x * y^-1 * x));
ts[84]:= f(t^(y^-1 * x * y^-1 * x * y * x * y^-1 * x));
 DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
/*
{ <GrpFP, Id(G), GrpFP>,
<GrpFP, t * x * y^-1 * x * y^-1 * t * y^-1 * x * y^-1 *t * y^-1 * t, GrpFP>,
<GrpFP, t * x * y * x * y * t * y * t * y * x * y * t,GrpFP>,
<GrpFP, t * y^-1 * x * y^-1 * t * y^-1 * x * y^-1 * t * y^-1 * t,GrpFP>,
<GrpFP, t * y^-1 * x * y * x * y^2 * t * y^-1 * x * t, GrpFP>,
<GrpFP, t* y * x * y * t * y * x * y * t * y * t, GrpFP>,
<GrpFP, t * x * y * t * y^2 * x * y^-1 * x * y * t, GrpFP>,
<GrpFP, t * y * t * y * t * x * y * x * y * t,GrpFP>,
```

```
<GrpFP, t * y * t * y * x * t * y * t * y * t, GrpFP>,
<GrpFP, t * y^2 * x * y * t * y^-1 * x * y^-1 * t, GrpFP>,
<GrpFP, t * y * x * y * t * y^-1 * x * y^2 * t, GrpFP>,
<GrpFP, t * y * x * y * t * y^2 * x * y * t, GrpFP>,
<GrpFP, t * x * y * x * y * t * y * x * y * t, GrpFP>,
<GrpFP, t * y^-1 * t * y^-1 * x * y^-1 * t * y^-1 * t, GrpFP>,
<GrpFP, t * y * t * y^-1 * x * t * y^-1 * x * t,GrpFP>,
<GrpFP, t * y * t * y * t * y * t * y * t, GrpFP>,
<GrpFP, t * y * t * y * t * y * x * y * t, GrpFP>,
<GrpFP, t * y * t * y * x * y * t * y * t, GrpFP>,
<GrpFP, t * y * x * y^-1 * t * y^-1 * t * y * t, GrpFP>,
<GrpFP, t * y * x * y * t * y * t * y * t, GrpFP>,
<GrpFP, t * y * x * y * t * y * x * y * t, GrpFP>,
<GrpFP, t * y * x * t * y * x * t, GrpFP>,
<GrpFP, t * y * x * y * t * y^-1 * t, GrpFP>,
<GrpFP, t * y * x * y * t * y * t, GrpFP>,
<GrpFP, t * x * y^2 * t * y * t, GrpFP>,
<GrpFP, t * x * y * x * t * y * t, GrpFP>,
<GrpFP, t * y * t * y * x * t, GrpFP>,
<GrpFP, t * y^2 * x * y * t, GrpFP>,
<GrpFP, t * y * x * t * y * t,GrpFP>,
<GrpFP, t * x * y^-1 * t * y^-1 * t, GrpFP>,
<GrpFP, t * x * y^-1 * t * y * t, GrpFP>,
<GrpFP, t * x * y * x * y^-1 * t, GrpFP>,
<GrpFP, t * y^-1 * t * y^-1 * t, GrpFP>,
<GrpFP, t, GrpFP>,
<GrpFP, t * y * t, GrpFP>,
<GrpFP, t * y^-1 * t, GrpFP>,
<GrpFP, t * x * y * t, GrpFP>,
<GrpFP, t * y * x * t, GrpFP>,
<GrpFP, t * x * y^2 * t, GrpFP>,
<GrpFP, t * y * x * y * t, GrpFP>,
<GrpFP, t * y * x * y^-1 * t, GrpFP>,
<GrpFP, t * y^2 * x * t, GrpFP>,
<GrpFP, t * y * t * y * t, GrpFP>,
<GrpFP, t * y^-1 * x * y * t, GrpFP>,
<GrpFP, t * y^-1 * x * y^-1 * t, GrpFP>,
<GrpFP, t * y^-1 * t * y * t, GrpFP>,
<GrpFP, t * y * x * y^-1 * t * y^-1 * t, GrpFP>,
<GrpFP, t * y^2 * x * t * y * t, GrpFP>,
<GrpFP, t * y * t * y * x * y^-1 * t, GrpFP>,
<GrpFP, t * y * t * y * t * y * t, GrpFP>,
<GrpFP, t * y * t * y^-1 * x * y^-1 * t, GrpFP>,
<GrpFP, t * y * t * y^-1 * t * y^-1 * t,GrpFP>,
```

```
<GrpFP, t * y^-1 * x * y * t * y * t, GrpFP>,
<GrpFP, t * y^-1 * t * y^-1 * x * y * t, GrpFP>,
<GrpFP, t * x * y * x * y * t * y^-1 * t, GrpFP>,
<GrpFP, t * x * y * t * y * x * y * t, GrpFP>,
<GrpFP, t * x * y * t * y^-1 * x * y * t, GrpFP>,
<GrpFP, t * y * x * y^2 * t * y * t, GrpFP>,
<GrpFP, t * y * x * t * y * t * y * t, GrpFP>,
<GrpFP, t * y^2 * x * y * t * y^-1 * t, GrpFP>,
<GrpFP, t * y * t * y * t * x * y * t, GrpFP>,
<GrpFP, t * y^-1 * x * y^-1 * t * y^-1 * x * t, GrpFP>,
<GrpFP, t * x * y * t * x * y * t * y^-1 * t, GrpFP> }
*/
DC:=[ f( Id(G)),
f(t),
f(t * y^-1 * t),
f(t * y * x * t),
f(t * x * y * t),
f(t * y * t),
f(t * y^2 * x * t),
f(t * y^-1 * x * y^-1 * t),
f(t * x * y^2 * t),
f(t * y * t * y * t),
f( t * y^-1 * t * y^-1 * t),
f(t * y * x * t * y * t),
f(t * y * x * y * t),
f(t * y^2 * x * y * t),
f(t * y * x * y^-1 * t),
f(t * y^-1 * x * y * t),
f(t * x * y * x * y^-1 * t),
f(t * y^-1 * t * y * t),
f(t * x * y * t * y^-1 * x * y * t),
f(t * x * y^-1 * t * y * t),
f( t * x * y^-1 * x * y^-1 * t * y^-1 * x * y^-1 *t * y^-1 * t),
f( t * x * y * x * y * t * y * t * y * x * y * t),
f(t * y^-1 * x * y^-1 * t * y^-1 * x * y^-1 * t * y^-1 * t),
f(t * y^-1 * x * y * x * y^2 * t * y^-1 * x * t),
f(t* y * x * y * t * y * x * y * t * y * t),
f(t * x * y * t * y^2 * x * y^-1 * x * y * t),
f(t * y * t * y * t * x * y * x * y * t),
f(t * y * t * y * x * t * y * t * y * t),
f(t * y^2 * x * y * t * y^-1 * x * y^-1 * t),
f(t * y * x * y * t * y^-1 * x * y^2 * t),
f(t * y * x * y * t * y^2 * x * y * t),
f(t * x * y * x * y * t * y * x * y * t),
```

```
f(t * y^-1 * t * y^-1 * x * y^-1 * t * y^-1 * t),
f(t * y * t * y^-1 * x * t * y^-1 * x * t),
f(t * y * t * y * t * y * t * y * t),
f(t * y * t * y * t * y * x * y * t),
f(t * y * t * y * x * y * t * y * t),
f(t * y * x * y^-1 * t * y^-1 * t * y * t),
f(t * y * x * y * t * y * t * y * t),
f(t * y * x * y * t * y * x * y * t),
f(t * y * x * t * y * x * t),
f(t * y * x * y * t * y^-1 * t),
f(t * y * x * y * t * y * t),
f(t * x * y^2 * t * y * t),
f(t * x * y * x * t * y * t),
f(t * y * t * y * x * t),
 f(t * x * y^-1 * t * y^-1 * t),
f(t * y * x * y^-1 * t * y^-1 * t),
f(t * y^2 * x * t * y * t),
f(t * y * t * y * x * y^-1 * t),
f(t * y * t * y * t * y * t),
f(t * y * t * y^-1 * x * y^-1 * t),
f(t * y * t * y^-1 * t * y^-1 * t),
f(t * y^-1 * x * y * t * y * t),
f(t * y^-1 * t * y^-1 * x * y * t),
f(t * x * y * x * y * t * y^-1 * t),
f( t * x * y * t * y * x * y * t),
f(t * y * x * y^2 * t * y * t),
f(t * y * x * t * y * t * y * t),
f(t * y^2 * x * y * t * y^-1 * t),
f(t * y * t * y * t * x * y * t),
f(t * y^-1 * x * y^-1 * t * y^-1 * x * t),
f(t * x * y * t * x * y * t * y^-1 * t)];
Index(G1,IN);
/* 6561 */
cst := [null : i in [1 .. Index(G1,IN)]] where null is [Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
    v := v^(Q[i]);
   end for;
return v;
end function;
for i := 1 to 84 do
    cst[prodim(1, ts, [i])] := [i];
     end for;
```

```
m:=0; for i in [1..6561] do if cst[i] ne [] then m:=m+1; end if; end for;m;
/* 84 */

Orbits(N);
/*
[
 GSet{@ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,
20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38,
39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57,
58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76,
77, 78, 79, 80, 81, 82, 83, 84 @}
]
*/
for i in [1..#DC] do for m,n in IN do if ts[1] eq m*(DC[i])^n then i; break;end if; e
/*2*/
N1:=Stabiliser(N,1);
#N1;
/* 2 */
Orbits(N1);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 25 @},
    GSet{@ 30 @},
    GSet{@ 3, 65 @},
    GSet{@ 4, 70 @},
    GSet{@ 5, 56 @},
    GSet{@ 6, 84 @},
    GSet{@ 7, 62 @},
    GSet{@ 8, 83 @},
    GSet{@ 9, 82 @},
    GSet{@ 10, 81 @},
    GSet{@ 11, 35 @},
    GSet{@ 12, 80 @},
    GSet{@ 13, 79 @},
    GSet{@ 14, 39 @},
    GSet{@ 15, 78 @},
    GSet{@ 16, 41 @},
    GSet{@ 17, 33 @},
    GSet{@ 18, 43 @},
    GSet{@ 19, 75 @},
    GSet{@ 20, 46 @},
    GSet{@ 21, 37 @},
```

```
        GSet{@ 22, 48 @},
        GSet{@ 23, 26 @},
        GSet{@ 24, 51 @},
        GSet{@ 27, 73 @},
        GSet{@ 28, 31 @},
        GSet{@ 29, 57 @},
        GSet{@ 32, 76 @},
        GSet{@ 34, 66 @},
        GSet{@ 36, 63 @},
        GSet{@ 38, 71 @},
        GSet{@ 40, 68 @},
        GSet{@ 42, 54 @},
        GSet{@ 44, 49 @},
        GSet{@ 45, 53 @},
        GSet{@ 47, 60 @},
        GSet{@ 50, 59 @},
        GSet{@ 52, 58 @},
        GSet{@ 55, 64 @},
        GSet{@ 61, 69 @},
        GSet{@ 67, 72 @},
        GSet{@ 74, 77 @}
]
*/
M:=MaximalSubgroups(G1);
M;
/*
Conjugacy classes of subgroups
------------------------------

[1]     Order 504        Length 2187
        Permutation group acting on a set of cardinality 6561
        Order = 504 = 2^3 * 3^2 * 7
[2]     Order 504        Length 2187
        Permutation group acting on a set of cardinality 6561
        Order = 504 = 2^3 * 3^2 * 7
[3]     Order 504        Length 2187
        Permutation group acting on a set of cardinality 6561
        Order = 504 = 2^3 * 3^2 * 7
[4]     Order 137781     Length 8
        Permutation group acting on a set of cardinality 6561
        Order = 137781 = 3^9 * 7
[5]     Order 157464     Length 7
        Permutation group acting on a set of cardinality 6561
        Order = 157464 = 2^3 * 3^9
```

```
[6]     Order 157464      Length 7
        Permutation group acting on a set of cardinality 6561
        Order = 157464 = 2^3 * 3^9
*/
IN:=sub<G1|f(x),f(y)>;                                M1:=M[1]`subgroup;
f(x) in M1;
/* true */
T:={{M1}};
for n in G1 do T:=T join {M1^n}; end for;
#T;
/* 1 */
TT:=Setseq(T);
TT[1];
/*
{
    Permutation group M1 acting on a set of cardinality 6561
    Order = 504 = 2^3 * 3^2 * 7
}
*/
f(y) in M1;
/* false */
[ 504, 42336, 84672, 14112, 14112, 28224, 84672, 84672, 28224, 21168, 28224,
28224, 7056, 84672, 42336, 7056, 42336, 84672, 84672, 84672, 84672, 28224,
28224, 28224, 28224, 4032, 4032 ]
#DoubleCosetRepresentatives(G1,DD[1],sub<G1|f(x),f(y)>);
/*27*/
```
Since we got a huge number of Double Cosets, we will not be able to do them.

11.2 $Sym(160)$

This progenitor never give us any good numbers for G

```
S:=Sym(160);
xx:=S!(1, 2, 6, 19, 5, 16, 47, 119, 52, 50)
(3, 10, 33, 35, 77, 48, 93, 98, 137,140)
(4, 13, 41, 100, 97, 49, 106, 150, 139, 89)
(7, 22, 65, 38, 64, 120,53, 129, 141, 108)
(8, 25, 71, 107, 80, 121, 157, 61, 30, 86)
(9, 29, 82,148, 123, 122, 158, 136, 135, 11)
(12, 37, 63, 81, 152, 87, 28, 78, 102,20)
(14, 26, 75, 73, 56, 124, 159, 105, 94, 111)
(15, 45, 113, 149, 145,125, 115, 39, 103, 118)
(17, 51, 66, 142, 155, 126, 96, 130, 43, 34)
(18,55, 91, 32, 85, 67, 60, 59, 133, 127)
(21, 44, 114, 110, 154, 83, 92, 58,72, 146)
(23, 46, 31, 88, 95, 128, 151, 143, 70, 57)
(24, 69, 117, 132,42, 109, 138, 116, 160, 156)
(27, 68, 144, 101, 36, 99, 134, 104, 79,62)
(40, 76, 74, 147, 131, 112, 54, 84, 153, 90);

 yy:=S!(1, 3, 11, 33, 91, 93)(2, 7)(4, 14)
 (5, 17, 52, 95, 85, 142)(6, 20, 60, 90,136, 71)
 (8, 26, 76, 24, 70, 72)(9, 30, 32, 23, 68, 103)
 (10, 16, 48, 123,98, 59)(12, 38, 78, 129, 80, 146)
 (13, 42, 100, 114, 150, 64)(15, 46,102, 153, 157, 96)
 (18, 56, 29, 83, 36, 69)(19, 57, 127, 43, 50, 126)
 (21,62, 138, 67, 111, 158)(22, 66, 44, 115, 116, 31)
 (25, 51, 125, 151, 81,147)(27, 77)
 (28, 79, 113, 144, 74, 148)
 (34, 94, 112, 75, 118, 132)(35,97)
 (37, 101, 39, 104, 84, 135)(40, 105, 145, 160, 155, 73)
 (41, 108, 106,156, 139, 58)(45, 117, 143, 53, 130, 92)
 (47, 120)(49, 124)(54, 109, 88,110, 121, 159)
 (55, 131, 82, 61, 119, 152)(63, 65, 86, 154, 87, 141)
 (89,137)(99, 140)(107, 133, 128, 134, 149, 122);

N:=sub<S|xx,yy>;
 #N;
/* 1920 */
   N1:=Stabiliser(N,1);
   Generators(N1);
/*
{
```

```
A:=  (2, 158, 67)(3, 93, 77)(4, 106, 41)(5, 60, 148)
(6, 52, 133)(7, 138, 21)(8,63, 145)(9, 127, 101)
(10, 140, 48)(11, 27, 91)(12, 54, 115)(13, 150,49)
(14, 58, 156)(15, 96, 151)(17, 74, 128)(18, 47, 29)
(19, 32, 119)(20,107, 28)(22, 129, 110)(23, 126, 84)
(24, 154, 73)(25, 153, 102)(26, 105,117)(30, 37, 152)
(31, 112, 80)(34, 66, 88)(38, 44, 132)(39, 57, 61)
(40,86, 143)(42, 124, 114)(43, 103, 131)(45, 87, 76)
(46, 125, 51)(50, 55,135)(53, 65, 72)(59, 123, 99)
(68, 104, 82)(69, 83, 120)(70, 155,130)(71, 113, 95)
(75, 116, 159)(78, 118, 121)(79, 122, 85)(81, 157,147)
(90, 142, 149)(92, 160, 141)(94, 109, 146)(134, 144, 136),

B:= (2, 150, 130)(3, 142, 107)(4, 18, 54)(5, 53, 15)
(6, 153, 141)(7, 10, 152)(8,100, 145)(9, 79, 146)
(11, 27, 89)(12, 88, 106)(13, 87, 70)(14, 35,156)
(19, 132, 134)(20, 120, 93)(21, 103, 140)(22, 125, 50)
(23, 98,84)(24, 94, 85)(25, 104, 92)(26, 129, 135)
(28, 90, 69)(29, 34, 115)(30,48, 43)(31, 36, 80)
(32, 136, 81)(33, 74, 128)(37, 131, 138)(38, 119,147)
(41, 66, 47)(42, 124, 137)(44, 157, 144)(45, 158, 155)
(46, 105,55)(49, 67, 76)(51, 110, 117)(52, 160, 68)
(56, 95, 71)(57, 61,111)(60,151, 75)(62, 86, 143)
(65, 148, 159)(72, 116, 96)(73, 127, 109)
(77, 83,149)(82, 102, 133)(97, 123, 99)(101, 154, 122)
(118, 121, 139)
}
*/
FPGroup(N);
/*
Finitely presented group on 2 generators
Relations
    $.1^10 = Id($)
    $.2^6 = Id($)
    ($.1 * $.2^-2 * $.1)^2 = Id($)
    ($.1 * $.2^2 * $.1^2)^2 = Id($)
    ($.2^-1 * $.1^-1)^5 = Id($)
    ($.1 * $.2^2 * $.1^-1 * $.2^-1)^2 = Id($)
    $.1^-1 * $.2^-1 * $.1^5 * $.2 * $.1^-4 = Id($)
    $.2 * $.1^-2 * $.2^-1 * $.1^3 * $.2 * $.1 * $.2^3 * $.1^-1 = Id($)
*/
/*
    x^10  ,
    y^6  ,
```

```
    (x * y^-2 * x)^2 ,
    (x * y^2 * x^2)^2 ,
    (y^-1 * x^-1)^5 ,
    (x * y^2 * x^-1 * y^-1)^2 ,
    x^-1 * y^-1 * x^5 * y * x^-4 ,
    y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1
*/
NN<a,b>:=Group<a,b|a^10, b^6,(a * b^-2 * a)^2 ,
(a * b^2 * a^2)^2 , (b^-1 * a^-1)^5 ,
(a * b^2 * a^-1 * b^-1)^2 ,
a^-1 * b^-1 * a^5 * b * a^-4 , b * a^-2 * b^-1 * a^3 * b * a * b^3 * a^-1>;
#NN;
/* 1920 */

word:=function(A)
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
for i in [2..#NN] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;
A:=N!(2, 158, 67)(3, 93, 77)(4, 106, 41)(5, 60, 148)
(6, 52, 133)(7, 138, 21)(8,63, 145)(9, 127, 101)
(10, 140, 48)(11, 27, 91)(12, 54, 115)(13, 150,49)
(14, 58, 156)(15, 96, 151)(17, 74, 128)(18, 47, 29)
(19, 32, 119)(20,107, 28)(22, 129, 110)(23, 126, 84)
(24, 154, 73)(25, 153, 102)(26, 105,117)(30, 37, 152)
(31, 112, 80)(34, 66, 88)(38, 44, 132)(39, 57, 61)
(40,86, 143)(42, 124, 114)(43, 103, 131)(45, 87, 76)
(46, 125, 51)(50, 55,135)(53, 65, 72)(59, 123, 99)
(68, 104, 82)(69, 83, 120)(70, 155,130)(71, 113, 95)
(75, 116, 159)(78, 118, 121)(79, 122, 85)(81, 157,147)
(90, 142, 149)(92, 160, 141)(94, 109, 146)(134, 144, 136);
```

```
word(A);
/*
(a * b^-1 * a^-1)^2
*/
/*
(x*y^-1*x^-1)^2
*/
B:=N!(2, 150, 130)(3, 142, 107)(4, 18, 54)(5, 53, 15)
(6, 153, 141)(7, 10, 152)(8,100, 145)(9, 79, 146)
(11, 27, 89)(12, 88, 106)(13, 87, 70)
(14, 35,156)(19, 132, 134)(20, 120, 93)(21, 103, 140)
(22, 125, 50)(23, 98,84)(24, 94, 85)(25, 104, 92)
(26, 129, 135)(28, 90, 69)(29, 34, 115)
(30,48, 43)(31, 36, 80)(32, 136, 81)(33, 74, 128)
(37, 131, 138)(38, 119,147)(41, 66, 47)(42, 124, 137)
(44, 157, 144)(45, 158, 155)
(46, 105,55)(49, 67, 76)(51, 110, 117)(52, 160, 68)
(56, 95, 71)(57, 61, 111)(60,151, 75)(62, 86, 143)
(65, 148, 159)(72, 116, 96)(73, 127, 109)
(77, 83,149)(82, 102, 133)(97, 123, 99)
(101, 154, 122)(118, 121, 139);
word(B);
/*
(b * a * b)^2
*/
/*
(y*x*y)^2
*/
G<x,y,t>:=Group<x,y,t|x^10  ,
    y^6  ,
    (x * y^-2 * x)^2 ,
    (x * y^2 * x^2)^2 ,
    (y^-1 * x^-1)^5 ,
    (x * y^2 * x^-1 * y^-1)^2 ,
    x^-1 * y^-1 * x^5 * y * x^-4 ,
    y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,
    t^2,
    (t,(x*y^-1*x^-1)^2),(t,(y*x*y)^2)>;
Orbits(N1);
/*
[
    GSet{@ 1 @},
    GSet{@ 16 @},
    GSet{@ 64 @},
```

```
    GSet{@ 108 @},
    GSet{@ 8, 63, 145, 100 @},
    GSet{@ 11, 27, 91, 89 @},
    GSet{@ 14, 58, 156, 35 @},
    GSet{@ 17, 74, 33, 128 @},
    GSet{@ 23, 126, 84, 98 @},
    GSet{@ 31, 112, 80, 36 @},
    GSet{@ 39, 57, 111, 61 @},
    GSet{@ 40, 86, 62, 143 @},
    GSet{@ 42, 124, 114, 137 @},
    GSet{@ 56, 113, 95, 71 @},
    GSet{@ 59, 123, 97, 99 @},
    GSet{@ 78, 118, 139, 121 @},
    GSet{@ 2, 158, 87, 67, 13, 76, 70, 150, 45, 130, 155, 49 @},
    GSet{@ 3, 93, 69, 77, 90, 83, 28, 142, 120, 107, 20, 149 @},
    GSet{@ 4, 106, 34, 41, 29, 66, 115, 18, 88, 54, 12, 47 @},
    GSet{@ 5, 60, 116, 148, 72, 159, 96, 53, 75, 15, 151, 65 @},
    GSet{@ 6, 52, 104, 133, 25, 82, 92, 153, 68, 141, 160, 102 @},
    GSet{@ 7, 138, 43, 21, 48, 103, 30, 10, 131, 152, 37, 140 @},
    GSet{@ 9, 127, 24, 101, 85, 154, 94, 79, 73, 146, 109, 122 @},
    GSet{@ 19, 32, 157, 119, 44, 147, 144, 132, 81, 134, 136, 38 @},
    GSet{@ 22, 129, 105, 110, 46, 117, 55, 125, 26, 50, 135, 51 @}
]
*/
temp:={16,64,108,8,11,14,17,23,31,39,40,42,56,59,78,2,3,4,5,6,7,9,19,22};
for n in N do for j in temp do if 1^n eq j then word(n),j; temp:=temp diff {j}; end i
end for;
/*

a 2
b 3
a^2 6
a * b 7
b^2 11
a^3 19
a * b * a 22
a * b * a^-1 108
b * a * b 16
b * a * b^-1 59
b^2 * a 9
a^-1 * b * a * b^2 * a^-1 23
a^-2 * b^-1 17
a^2 * b^3 * a 40
a^4 5
```

```
a^2 * b * a^2 * b^2 39
a * b * a * b^-1 31
a^-1 * b * a * b * a^-1 * b * a 42
b^-1 * a^2 * b * a 4
b * a^-2 * b^-1 * a^-2 * b 64
a^2 * b^-2 * a^-1 * b^-1 * a 14
a * b^-1 * a^-1 * b * a * b * a^-1 * b 78
b^2 * a^2 * b^-1 56
a^2 * b * a * b^-1 * a^-1 * b 8
*/
/*
x 2
y 3
x^2 6
x * y 7
y^2 11
x^3 19
x * y * x 22
x * y * x^-1 108
y * x * y 16
y * x * y^-1 59
y^2 * x 9
x^-1 * y * x * y^2 * x^-1 23
x^-2 * y^-1 17
x^2 * y^3 * x 40
x^4 5
x^2 * y * x^2 * y^2 39
x * y * x * y^-1 31
x^-1 * y * x * y * x^-1 * y * x 42
y^-1 * x^2 * y * x 4
y * x^-2 * y^-1 * x^-2 * y 64
x^2 * y^-2 * x^-1 * y^-1 * x 14
x * y^-1 * x^-1 * y * x * y * x^-1 * y 78
y^2 * x^2 * y^-1 56
x^2 * y * x * y^-1 * x^-1 * y 8
*/
/*put into cycle with t included*/
/*(t,t^(x))
(t,t^(y))
(t,t^(x^2))
(t,t^(x*y))
(t,t^(y^2))
(t,t^(x^3))
(t,t^(x * y * x))
```

```
(t,t^(x * y * x^-1))
(t,t^(y * x * y))
(t,t^(y * x * y^-1))
(t,t^(y^2 * x))
(t,t^(x^-1 * y * x * y^2 * x^-1))
(t,t^(x^-2 * y^-1))
(t,t^(x^2 * y^3 * x))
(t,t^(x^4))
(t,t^(x^2 * y * x^2 * y^2))
(t,t^(x * y * x * y^-1))
(t,t^(x^-1 * y * x * y * x^-1 * y * x))
(t,t^(y^-1 * x^2 * y * x))
(t,t^(y * x^-2 * y^-1 * x^-2 * y))
(t,t^(x^2 * y^-2 * x^-1 * y^-1 * x))
(t,t^(x * y^-1 * x^-1 * y * x * y * x^-1 * y))
(t,t^(y^2 * x^2 * y^-1))
(t,t^(x^2 * y * x * y^-1 * x^-1 * y))
*/
G<x,y,t>:=Group<x,y,t| x^10  ,
    y^6  ,
    (x * y^-2 * x)^2 ,
    (x * y^2 * x^2)^2 ,
    (y^-1 * x^-1)^5 ,
    (x * y^2 * x^-1 * y^-1)^2 ,
    x^-1 * y^-1 * x^5 * y * x^-4 ,
    y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,
    t^2,
    (t,(x*y^-1*x^-1)^2),(t,(y*x*y)^2)>;
C:=Classes(N);
 C;
/*
Conjugacy Classes of group N
----------------------------
[1]     Order 1      Length 1
        Rep Id(N)

[2]     Order 2      Length 1
        Rep (1, 16)(2, 47)(3, 48)(4, 49)(5, 50)(6, 119)(7, 120)(8, 121)(9,
        122)(10, 93)(11, 123)(12, 87)(13, 106)(14, 124)(15, 125)(17,
        126)(18, 67)(19, 52)(20, 152)(21, 83)(22, 53)(23, 128)(24, 109)(25,
        157)(26, 159)(27, 99)(28, 37)(29, 158)(30, 107)(31, 143)(32,
        133)(33, 98)(34, 155)(35, 137)(36, 62)(38, 141)(39, 113)(40,
        112)(41, 150)(42, 156)(43, 142)(44, 92)(45, 115)(46, 151)(51,
        96)(54, 76)(55, 60)(56, 111)(57, 95)(58, 114)(59, 91)(61, 71)(63,
```

78)(64, 108)(65, 129)(66, 130)(68, 134)(69, 138)(70, 88)(72,
110)(73, 94)(74, 84)(75, 105)(77, 140)(79, 101)(80, 86)(81, 102)(82,
136)(85, 127)(89, 97)(90, 131)(100, 139)(103, 149)(104, 144)(116,
117)(118, 145)(132, 160)(135, 148)(146, 154)(147, 153)

[3]   Order 2   Length 10
Rep (1, 33)(2, 7)(3, 91)(4, 14)(5, 95)(6, 90)(8, 24)(9, 23)
(10, 123)(11,93)(12, 129)(13, 114)(15, 153)(16, 98)(17, 85)
(18, 83)(19, 43)(20,136)
(21, 67)(22, 115)(25, 151)(26, 70)(27, 77)
(28, 144)(29, 69)(30,68)(31, 44)(32, 103)(34, 75)
(35, 97)(36, 56)(37, 104)(38, 80)(39,135)(40, 160)
(41, 156)(42, 150)(45, 53)(46, 157)(47, 120)
(48,59)(49, 124)(50, 57)(51, 81)(52, 142)(54, 110)
(55, 61)(58, 106)(60, 71)(62, 111)(63, 154)
(64, 100)(65, 87)(66, 116)(72, 76)(73, 145)
(74,79)(78, 146)(82, 152)(84, 101)(86, 141)
(88, 159)(89, 137)(92,143)(94, 118)(96, 102)
(99, 140)(105, 155)(107, 134)(108, 139)(109,121)
(112, 132)(113, 148)(117, 130)(119, 131)
(122, 128)(125,147)(126, 127)(133, 149)(138, 158)

[4]   Order 2   Length 120
Rep (1, 97)(2, 78)(3, 61)(4, 40)(6, 65)(7, 17)(8, 29)(9, 51)(10, 95)(11,
123)(12, 130)(13, 31)(14, 114)(15, 85)(16, 89)(18, 145)(19, 22)(20,
43)(21, 74)(23, 69)(24, 102)(25, 73)(26, 82)(27, 59)(28, 103)(30,
90)(32, 110)(33, 111)(34, 76)(35, 64)(36, 100)(37, 149)(39, 77)(41,
86)(42, 156)(44, 132)(45, 88)(46, 101)(47, 63)(48, 71)(49, 112)(52,
53)(54, 155)(55, 135)(56, 98)(57, 93)(58, 124)(60, 148)(62, 139)(66,
87)(67, 118)(68, 117)(70, 115)(72, 133)(75, 144)(79, 151)(80,
150)(81, 109)(83, 84)(91, 99)(92, 160)(94, 157)(96, 122)(104,
105)(106, 143)(107, 131)(108, 137)(113, 140)(116, 134)(119,
129)(120, 126)(121, 158)(125, 127)(128, 138)(136, 159)(142,
152)(146, 147)(153, 154)

[5]   Order 3   Length 80
Rep (1, 11, 91)(3, 33, 93)(5, 52, 85)(6, 60, 136)(8, 76, 70)(9, 32,
68)(10, 48, 98)(12, 78, 80)(13, 100, 150)(15, 102, 157)(16, 123,
59)(17, 95, 142)(18, 29, 36)(19, 127, 50)(20, 90, 71)(21, 138,
111)(22, 44, 116)(23, 103, 30)(24, 72, 26)(25, 125, 81)(28, 113,
74)(31, 66, 115)(34, 112, 118)(37, 39, 84)(38, 129, 146)(40, 145,
155)(41, 106, 139)(42, 114, 64)(43, 126, 57)(45, 143, 130)(46, 153,
96)(51, 151, 147)(53, 92, 117)(54, 88, 121)(55, 82, 119)(56, 83,
69)(58, 108, 156)(61, 152, 131)(62, 67, 158)(63, 86, 87)(65, 154,

141)(73, 105, 160)(75, 132, 94)(79, 144, 148)(101, 104, 135)(107, 128, 149)(109, 110, 159)(122, 133, 134)

[6]   Order 4   Length 20
Rep (1, 118, 16, 145)(2, 69, 47, 138)(3, 70, 48, 88)(4, 90, 49, 131)(5, 41, 50, 150)(6, 124, 119, 14)(7, 29, 120, 158)(8, 146, 121, 154)(9, 17, 122, 126)(10, 38, 93, 141)(11, 86, 123, 80)(12, 112, 87, 40)(13, 107, 106, 30)(15, 81, 125, 102)(18, 36, 67, 62)(19, 135, 52, 148)(20, 71, 152, 61)(21, 111, 83, 56)(22, 99, 53, 27)(23, 85, 128, 127)(24, 78, 109, 63)(25, 137, 157, 35)(26, 59, 159, 91)(28, 133, 37, 32)(31, 130, 143, 66)(33, 94, 98, 73)(34, 110, 155, 72)(39, 142, 113, 43)(42, 95, 156, 57)(44, 117, 92, 116)(45, 77, 115, 140)(46, 97, 151, 89)(51, 147, 96, 153)(54, 105, 76, 75)(55, 136, 60, 82)(58, 68, 114, 134)(64, 84, 108, 74)(65, 160, 129, 132)(79, 100, 101, 139)(103, 144, 149, 104)

[7]   Order 4   Length 120
Rep (1, 155, 23, 50)(2, 160, 35, 142)(3, 21, 134, 102)(4, 130, 140, 20)(5, 16, 34, 128)(6, 136, 22, 116)(7, 135, 89, 12)(8, 141, 108, 37)(9, 54, 98, 42)(10, 74, 32, 146)(11, 109, 103, 139)(13, 18, 26, 96)(14, 55, 27, 92)(15, 70, 111, 58)(17, 105, 73, 95)(19, 151, 40, 158)(24, 149, 100, 123)(25, 113, 69, 129)(28, 121, 38, 64)(29, 52, 46, 112)(30, 62, 91, 147)(31, 115, 61, 131)(33, 156, 122, 76)(36, 59, 153, 107)(39, 138, 65, 157)(41, 145, 110, 127)(43, 47, 132, 137)(44, 124, 60, 99)(45, 71, 90, 143)(48, 83, 68, 81)(49, 66, 77, 152)(51, 106, 67, 159)(53, 117, 119, 82)(56, 114, 125, 88)(57, 126, 75, 94)(63, 144, 79, 86)(72, 85, 150, 118)(78, 104, 101, 80)(84, 133, 154, 93)(87, 120, 148, 97)

[8]   Order 5   Length 192
Rep (1, 6, 5, 47, 52)(2, 19, 16, 119, 50)(3, 33, 77, 93, 137)(4, 41, 97, 106, 139)(7, 65, 64, 53, 141)(8, 71, 80, 157, 30)(9, 82, 123, 158, 135)(10, 35, 48, 98, 140)(11, 29, 148, 122, 136)(12, 63, 152, 28, 102)(13, 100, 49, 150, 89)(14, 75, 56, 159, 94)(15, 113, 145, 115, 103)(17, 66, 155, 96, 43)(18, 91, 85, 60, 133)(20, 37, 81, 87, 78)(21, 114, 154, 92, 72)(22, 38, 120, 129, 108)(23, 31, 95, 151, 70)(24, 117, 42, 138, 160)(25, 107, 121, 61, 86)(26, 73, 124, 105, 111)(27, 144, 36, 134, 79)(32, 67, 59, 127, 55)(34, 51, 142, 126, 130)(39, 118, 45, 149, 125)(40, 74, 131, 54, 153)(44, 110, 83, 58, 146)(46, 88, 128, 143, 57)(62, 68, 101, 99, 104)(69, 132, 109, 116, 156)(76, 147, 112, 84, 90)

[9]   Order 5   Length 192
Rep (1, 20, 159, 29, 93)(2, 37, 79, 71, 160)(3, 89, 42, 145, 6)(4, 58,

18, 95, 146)(5, 128, 44, 88, 125)(7, 155, 94, 31, 135)(8, 55, 30, 147, 141)(9, 131, 52, 56, 54)(10, 16, 152, 26, 158)(11, 74, 53, 13, 137)(12, 157, 75, 139, 14)(15, 50, 23, 92, 70)(17, 82, 39, 102, 133)(19, 111, 76, 122, 90)(21, 86, 127, 140, 59)(22, 106, 35, 123, 84)(24, 45, 129, 36, 149)(25, 105, 100, 124, 87)(27, 142, 151, 144, 33)(28, 101, 61, 132, 47)(32, 126, 136, 113, 81)(34, 73, 143, 148, 120)(38, 121, 60, 107, 153)(40, 51, 110, 78, 117)(41, 108, 66, 134, 69)(43, 46, 104, 98, 99)(48, 97, 156, 118, 119)(49, 114, 67, 57, 154)(62, 103, 109, 115, 65)(63, 116, 112, 96, 72)(64, 130, 68, 138, 150)(77, 91, 83, 80, 85)

[10]   Order 6   Length 80
Rep (1, 36, 143, 16, 62, 31)(2, 49, 9, 47, 4, 122)(3, 12, 30, 48, 87, 107)(5, 105, 123, 50, 75, 11)(6, 84, 25, 119, 74, 157)(7, 140, 24, 120, 77, 109)(8, 89, 60, 121, 97, 55)(10, 34, 149, 93, 155, 103)(13, 43, 88, 106, 142, 70)(14, 98, 153, 124, 33, 147)(15, 27, 100, 125, 99, 139)(17, 81, 82, 126, 102, 136)(18, 90, 146, 67, 131, 154)(19, 58, 132, 52, 114, 160)(20, 94, 29, 152, 73, 158)(21, 45, 127, 83, 115, 85)(22, 145, 46, 53, 118, 151)(23, 35, 44, 128, 137, 92)(26, 159)(28, 54, 41, 37, 76, 150)(32, 133)(38, 104, 42, 141, 144, 156)(39, 113)(40, 112)(51, 116, 78, 96, 117, 63)(56, 61, 64, 111, 71, 108)(57, 95)(59, 135, 65, 91, 148, 129)(66, 79, 69, 130, 101, 138)(68, 134)(72, 110)(80, 86)

[11]   Order 6   Length 160
Rep (1, 3, 11, 33, 91, 93)(2, 7)(4, 14)(5, 17, 52, 95, 85, 142)(6, 20, 60, 90, 136, 71)(8, 26, 76, 24, 70, 72)(9, 30, 32, 23, 68, 103)(10, 16, 48, 123, 98, 59)(12, 38, 78, 129, 80, 146)(13, 42, 100, 114, 150, 64)(15, 46, 102, 153, 157, 96)(18, 56, 29, 83, 36, 69)(19, 57, 127, 43, 50, 126)(21, 62, 138, 67, 111, 158)(22, 66, 44, 115, 116, 31)(25, 51, 125, 151, 81, 147)(27, 77)(28, 79, 113, 144, 74, 148)(34, 94, 112, 75, 118, 132)(35, 97)(37, 101, 39, 104, 84, 135)(40, 105, 145, 160, 155, 73)(41, 108, 106, 156, 139, 58)(45, 117, 143, 53, 130, 92)(47, 120)(49, 124)(54, 109, 88, 110, 121, 159)(55, 131, 82, 61, 119, 152)(63, 65, 86, 154, 87, 141)(89, 137)(99, 140)(107, 133, 128, 134, 149, 122)

[12]   Order 6   Length 160
Rep (1, 93, 91, 33, 11, 3)(2, 7)(4, 14)(5, 142, 85, 95, 52, 17)(6, 71, 136, 90, 60, 20)(8, 72, 70, 24, 76, 26)(9, 103, 68, 23, 32, 30)(10, 59, 98, 123, 48, 16)(12, 146, 80, 129, 78, 38)(13, 64, 150, 114, 100, 42)(15, 96, 157, 153, 102, 46)(18, 69, 36, 83, 29, 56)(19, 126, 50, 43, 127, 57)(21, 158, 111, 67, 138, 62)(22, 31, 116, 115, 44, 66)(25, 147, 81, 151, 125, 51)(27, 77)(28, 148, 74, 144, 113,

79)(34, 132, 118, 75, 112, 94)(35, 97)(37, 135, 84, 104, 39, 101)(40, 73, 155, 160, 145, 105)(41, 58, 139, 156, 106, 108)(45, 92, 130, 53, 143, 117)(47, 120)(49, 124)(54, 159, 121, 110, 88, 109)(55, 152, 119, 61, 82, 131)(63, 141, 87, 154, 86, 65)(89, 137)(99, 140)(107, 122, 149, 134, 128, 133)

[13]   Order 8   Length 240
Rep (1, 150, 118, 5, 16, 41, 145, 50)(2, 153, 69, 51, 47, 147, 138, 96)(3, 152, 70, 61, 48, 20, 88, 71)(4, 87, 90, 40, 49, 12, 131, 112)(6, 132, 124, 65, 119, 160, 14, 129)(7, 125, 29, 102, 120, 15, 158, 81)(8, 104, 146, 103, 121, 144, 154, 149)(9, 34, 17, 110, 122, 155, 126, 72)(10, 74, 38, 64, 93, 84, 141, 108)(11, 79, 86, 100, 123, 101, 80, 139)(13, 66, 107, 31, 106, 130, 30, 143)(18, 46, 36, 97, 67, 151, 62, 89)(19, 53, 135, 27, 52, 22, 148, 99)(21, 157, 111, 35, 83, 25, 56, 137)(23, 105, 85, 76, 128, 75, 127, 54)(24, 28, 78, 133, 109, 37, 63, 32)(26, 60, 59, 82, 159, 55, 91, 136)(33, 156, 94, 57, 98, 42, 73, 95)(39, 140, 142, 45, 113, 77, 43, 115)(44, 114, 117, 134, 92, 58, 116, 68)

[14]   Order 10   Length 192
Rep (1, 2, 6, 19, 5, 16, 47, 119, 52, 50)(3, 10, 33, 35, 77, 48, 93, 98, 137, 140)(4, 13, 41, 100, 97, 49, 106, 150, 139, 89)(7, 22, 65, 38, 64, 120, 53, 129, 141, 108)(8, 25, 71, 107, 80, 121, 157, 61, 30, 86)(9, 29, 82, 148, 123, 122, 158, 136, 135, 11)(12, 37, 63, 81, 152, 87, 28, 78, 102, 20)(14, 26, 75, 73, 56, 124, 159, 105, 94, 111)(15, 45, 113, 149, 145, 125, 115, 39, 103, 118)(17, 51, 66, 142, 155, 126, 96, 130, 43, 34)(18, 55, 91, 32, 85, 67, 60, 59, 133, 127)(21, 44, 114, 110, 154, 83, 92, 58, 72, 146)(23, 46, 31, 88, 95, 128, 151, 143, 70, 57)(24, 69, 117, 132, 42, 109, 138, 116, 160, 156)(27, 68, 144, 101, 36, 99, 134, 104, 79, 62)(40, 76, 74, 147, 131, 112, 54, 84, 153, 90)

[15]   Order 10   Length 192
Rep (1, 19, 47, 50, 6, 16, 52, 2, 5, 119)(3, 35, 93, 140, 33, 48, 137, 10, 77, 98)(4, 100, 106, 89, 41, 49, 139, 13, 97, 150)(7, 38, 53, 108, 65, 120, 141, 22, 64, 129)(8, 107, 157, 86, 71, 121, 30, 25, 80, 61)(9, 148, 158, 11, 82, 122, 135, 29, 123, 136)(12, 81, 28, 20, 63, 87, 102, 37, 152, 78)(14, 73, 159, 111, 75, 124, 94, 26, 56, 105)(15, 149, 115, 118, 113, 125, 103, 45, 145, 39)(17, 142, 96, 34, 66, 126, 43, 51, 155, 130)(18, 32, 60, 127, 91, 67, 133, 55, 85, 59)(21, 110, 92, 146, 114, 83, 72, 44, 154, 58)(23, 88, 151, 57, 31, 128, 70, 46, 95, 143)(24, 132, 138, 156, 117, 109, 160, 69, 42, 116)(27, 101, 134, 62, 144, 99, 79, 68, 36, 104)(40, 147, 54, 90, 74, 112, 153, 76, 131, 84)

```
[16]     Order 12     Length 160
         Rep (1, 140, 36, 24, 143, 120, 16, 77, 62, 109, 31, 7)(2, 108, 49, 56,
         9, 61, 47, 64, 4, 111, 122, 71)(3, 135, 12, 65, 30, 91, 48, 148, 87,
         129, 107, 59)(5, 34, 105, 149, 123, 93, 50, 155, 75, 103, 11, 10)(6,
         102, 84, 136, 25, 17, 119, 81, 74, 82, 157, 126)(8, 14, 89, 98, 60,
         153, 121, 124, 97, 33, 55, 147)(13, 132, 43, 52, 88, 114, 106, 160,
         142, 19, 70, 58)(15, 23, 27, 35, 100, 44, 125, 128, 99, 137, 139,
         92)(18, 73, 90, 158, 146, 20, 67, 94, 131, 29, 154, 152)(21, 79, 45,
         69, 127, 130, 83, 101, 115, 138, 85, 66)(22, 51, 145, 116, 46, 78,
         53, 96, 118, 117, 151, 63)(26, 40, 159, 112)(28, 104, 54, 42, 41,
         141, 37, 144, 76, 156, 150, 38)(32, 57, 133, 95)(39, 134, 113,
         68)(72, 86, 110, 80)
*/
for i in [2..#C] do i, C[i][3]; word(C[i][3]);
Orbits(Centraliser(N,C[i][3]));end for;
/*
2 (1, 16)(2, 47)(3, 48)(4, 49)(5, 50)(6, 119)(7, 120)(8, 121)(9, 122)(10, 93)(11, 123
152)(21, 83)(22, 53)(23, 128)(24, 109)(25, 157)(26, 159)(27, 99)(28, 37)(29,158)(30,
    151)(51, 96)(54, 76)(55, 60)(56, 111)(57, 95)(58, 114)(59, 91)(61, 71)(63,
    78)(64, 108)(65, 129)(66, 130)(68, 134)(69, 138)(70, 88)(72, 110)(73,
    94)(74, 84)(75, 105)(77, 140)(79, 101)(80, 86)(81, 102)(82, 136)(85,
    127)(89, 97)(90, 131)(100, 139)(103, 149)(104, 144)(116, 117)(118, 145)(132,
    160)(135, 148)(146, 154)(147, 153)
a^5
/* x^5 */
[
    GSet{@ 1, 2, 3, 6, 7, 10, 11, 19, 20, 22, 33, 16, 9, 5, 57, 12, 60, 65, 66,
    35, 91, 47, 48, 29, 30, 17, 23, 127, 37, 38, 59, 90, 86, 142, 44, 77, 97,
    32, 93, 119, 120, 123, 82, 83, 51, 52, 46, 68, 18, 43, 63, 101, 64, 78, 133,
    40, 136, 8, 154, 155, 114, 115, 27, 49, 85, 98, 152, 53, 122, 148, 61, 92,
    36, 125, 50, 95, 31, 102, 144, 103, 55, 56, 34, 81, 39, 13, 129, 128, 76,
    105, 135, 71, 25, 26, 87, 126, 73, 110, 150, 116, 106, 124, 67, 137, 130,
    158, 107, 28, 58, 45, 99, 69, 151, 88, 153, 74, 118, 131, 94, 147, 104, 41,
    42, 141, 80, 134, 24, 145, 75, 96, 121, 139, 160, 156, 159, 111, 140, 89,
    21, 79, 72, 113, 117, 143, 157, 15, 132, 112, 84, 100, 108, 109, 146, 149,
    70, 54, 14, 4, 62, 138 @}
]
3 (1, 33)(2, 7)(3, 91)(4, 14)(5, 95)(6, 90)(8, 24)(9, 23)(10, 123)(11, 93)(12,
    129)(13, 114)(15, 153)(16, 98)(17, 85)(18, 83)(19, 43)(20, 136)(21, 67)(22,
    115)(25, 151)(26, 70)(27, 77)(28, 144)(29, 69)(30, 68)(31, 44)(32, 103)(34,
    75)(35, 97)(36, 56)(37, 104)(38, 80)(39, 135)(40, 160)(41, 156)(42, 150)(45,
    53)(46, 157)(47, 120)(48, 59)(49, 124)(50, 57)(51, 81)(52, 142)(54, 110)(55,
    61)(58, 106)(60, 71)(62, 111)(63, 154)(64, 100)(65, 87)(66, 116)(72, 76)(73,
```

```
    145)(74, 79)(78, 146)(82, 152)(84, 101)(86, 141)(88, 159)(89, 137)(92,
    143)(94, 118)(96, 102)(99, 140)(105, 155)(107, 134)(108, 139)(109, 121)(112,
    132)(113, 148)(117, 130)(119, 131)(122, 128)(125, 147)(126, 127)(133,
    149)(138, 158)
b^3
/* y^3 */
[
    GSet{@ 2, 158, 7, 120, 111, 67, 138, 69, 47, 62, 21, 36, 83, 29, 56, 18 @},
    GSet{@ 15, 96, 153, 147, 137, 151, 102, 81, 125, 25, 89, 97, 51, 157, 35, 46
    @},
    GSet{@ 1, 33, 98, 146, 160, 16, 78, 40, 63, 112, 94, 24, 26, 141, 34, 86,
    154, 132, 118, 8, 70, 75, 145, 121, 88, 80, 105, 38, 109, 91, 65, 93, 92,
    115, 66, 27, 143, 22, 73, 159, 155, 3, 87, 11, 116, 77, 48, 12, 123, 31, 53,
    117, 140, 44, 45, 129, 54, 72, 59, 130, 10, 99, 110, 76 @},
    GSet{@ 4, 106, 14, 124, 39, 82, 41, 58, 114, 133, 49, 135, 152, 148, 20, 57,
    144, 50, 68, 19, 90, 156, 42, 17, 13, 149, 103, 6, 71, 79, 113, 136, 28, 30,
    43, 5, 37, 95, 107, 142, 119, 61, 139, 9, 55, 108, 104, 32, 150, 85, 127,
    74, 23, 60, 131, 84, 52, 122, 134, 64, 128, 100, 101, 126 @}
]
4 (1, 97)(2, 78)(3, 61)(4, 40)(6, 65)(7, 17)(8, 29)(9, 51)(10, 95)(11, 123)(12,
    130)(13, 31)(14, 114)(15, 85)(16, 89)(18, 145)(19, 22)(20, 43)(21, 74)(23,
    69)(24, 102)(25, 73)(26, 82)(27, 59)(28, 103)(30, 90)(32, 110)(33, 111)(34,
    76)(35, 64)(36, 100)(37, 149)(39, 77)(41, 86)(42, 156)(44, 132)(45, 88)(46,
    101)(47, 63)(48, 71)(49, 112)(52, 53)(54, 155)(55, 135)(56, 98)(57, 93)(58,
    124)(60, 148)(62, 139)(66, 87)(67, 118)(68, 117)(70, 115)(72, 133)(75,
    144)(79, 151)(80, 150)(81, 109)(83, 84)(91, 99)(92, 160)(94, 157)(96,
    122)(104, 105)(106, 143)(107, 131)(108, 137)(113, 140)(116, 134)(119,
    129)(120, 126)(121, 158)(125, 127)(128, 138)(136, 159)(142, 152)(146,
    147)(153, 154)
b * a^3 * b
/* y*x^3*y */
[
    GSet{@ 5, 38, 50, 141 @},
    GSet{@ 11, 123, 42, 156 @},
    GSet{@ 32, 110, 72, 133 @},
    GSet{@ 75, 144, 105, 104 @},
    GSet{@ 1, 97, 142, 27, 89, 152, 59, 16, 83, 20, 91, 84, 43, 99, 74, 21 @},
    GSet{@ 2, 78, 116, 40, 63, 134, 4, 47, 122, 68, 49, 96, 117, 112, 51, 9 @},
    GSet{@ 3, 61, 146, 69, 71, 147, 23, 48, 53, 153, 128, 52, 154, 138, 19, 22
    @},
    GSet{@ 6, 65, 158, 127, 129, 121, 125, 119, 13, 8, 15, 31, 29, 85, 143, 106
    @},
    GSet{@ 7, 17, 136, 39, 126, 159, 77, 120, 109, 26, 140, 81, 82, 113, 102, 24
    @},
```

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    GSet{@ 10, 95, 76, 37, 57, 34, 149, 93, 86, 155, 103, 41, 54, 28, 150, 80
    @},
    GSet{@ 12, 130, 108, 18, 66, 137, 145, 87, 14, 35, 118, 114, 64, 67, 58, 124
    @},
    GSet{@ 25, 73, 45, 148, 94, 88, 60, 157, 100, 70, 55, 36, 115, 135, 62, 139
    @},
    GSet{@ 30, 90, 79, 111, 131, 151, 33, 107, 92, 46, 98, 160, 101, 56, 132, 44
    @}
]
5 (1, 11, 91)(3, 33, 93)(5, 52, 85)(6, 60, 136)(8, 76, 70)(9, 32, 68)(10, 48,
    98)(12, 78, 80)(13, 100, 150)(15, 102, 157)(16, 123, 59)(17, 95, 142)(18,
    29, 36)(19, 127, 50)(20, 90, 71)(21, 138, 111)(22, 44, 116)(23, 103, 30)(24,
    72, 26)(25, 125, 81)(28, 113, 74)(31, 66, 115)(34, 112, 118)(37, 39, 84)(38,
    129, 146)(40, 145, 155)(41, 106, 139)(42, 114, 64)(43, 126, 57)(45, 143,
    130)(46, 153, 96)(51, 151, 147)(53, 92, 117)(54, 88, 121)(55, 82, 119)(56,
    83, 69)(58, 108, 156)(61, 152, 131)(62, 67, 158)(63, 86, 87)(65, 154,
    141)(73, 105, 160)(75, 132, 94)(79, 144, 148)(101, 104, 135)(107, 128,
    149)(109, 110, 159)(122, 133, 134)
b^2
/* y^2 */
[
    GSet{@ 2, 7, 137, 97, 89, 35, 120, 47 @},
    GSet{@ 4, 14, 77, 99, 27, 140, 124, 49 @},
    GSet{@ 1, 11, 3, 149, 91, 33, 107, 9, 122, 93, 128, 32, 133, 30, 10, 68,
    134, 23, 48, 16, 103, 98, 123, 59 @},
    GSet{@ 5, 52, 17, 40, 85, 95, 145, 75, 105, 142, 155, 132, 160, 118, 43, 94,
    73, 34, 126, 50, 112, 57, 19, 127 @},
    GSet{@ 6, 60, 20, 92, 136, 90, 117, 115, 45, 71, 53, 31, 143, 116, 61, 66,
    130, 22, 152, 119, 44, 131, 55, 82 @},
    GSet{@ 8, 76, 26, 42, 70, 24, 114, 139, 100, 72, 64, 41, 150, 58, 110, 106,
    13, 108, 159, 121, 156, 109, 54, 88 @},
    GSet{@ 12, 78, 38, 79, 80, 129, 144, 39, 113, 146, 148, 84, 74, 104, 154,
    37, 28, 135, 141, 87, 101, 65, 63, 86 @},
    GSet{@ 15, 102, 46, 83, 157, 153, 69, 62, 36, 96, 56, 67, 18, 138, 51, 158,
    29, 111, 151, 125, 21, 147, 81, 25 @}
]
6 (1, 118, 16, 145)(2, 69, 47, 138)(3, 70, 48, 88)(4, 90, 49, 131)(5, 41, 50,
    150)(6, 124, 119, 14)(7, 29, 120, 158)(8, 146, 121, 154)(9, 17, 122,
    126)(10, 38, 93, 141)(11, 86, 123, 80)(12, 112, 87, 40)(13, 107, 106,
    30)(15, 81, 125, 102)(18, 36, 67, 62)(19, 135, 52, 148)(20, 71, 152, 61)(21,
    111, 83, 56)(22, 99, 53, 27)(23, 85, 128, 127)(24, 78, 109, 63)(25, 137,
    157, 35)(26, 59, 159, 91)(28, 133, 37, 32)(31, 130, 143, 66)(33, 94, 98,
    73)(34, 110, 155, 72)(39, 142, 113, 43)(42, 95, 156, 57)(44, 117, 92,
    116)(45, 77, 115, 140)(46, 97, 151, 89)(51, 147, 96, 153)(54, 105, 76,
```

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    75)(55, 136, 60, 82)(58, 68, 114, 134)(64, 84, 108, 74)(65, 160, 129,
    132)(79, 100, 101, 139)(103, 144, 149, 104)
(a^-1 * b)^3
/* (x^-1*y)^3 */
[
    GSet{@ 1, 118, 156, 127, 16, 57, 23, 145, 105, 122, 75, 42, 85, 76, 126, 54,
    95, 128, 110, 94, 5, 73, 72, 9, 155, 98, 41, 33, 34, 17, 50, 150 @},
    GSet{@ 8, 63, 146, 149, 84, 100, 24, 93, 139, 121, 104, 108, 28, 154, 86,
    80, 101, 78, 141, 79, 109, 37, 103, 74, 133, 123, 11, 10, 144, 64, 32, 38
    @},
    GSet{@ 2, 45, 69, 135, 157, 13, 77, 117, 4, 47, 52, 35, 125, 138, 40, 119,
    112, 107, 91, 115, 92, 90, 140, 60, 88, 55, 148, 25, 102, 12, 14, 87, 30,
    83, 19, 137, 62, 134, 31, 99, 36, 106, 26, 89, 21, 116, 49, 82, 3, 136, 44,
    131, 65, 29, 51, 68, 129, 15, 6, 56, 18, 58, 130, 53, 67, 81, 159, 20, 124,
    143, 152, 59, 114, 96, 46, 111, 27, 43, 158, 70, 160, 120, 147, 132, 48, 7,
    142, 71, 66, 61, 153, 151, 22, 97, 39, 113 @}
]
7 (1, 155, 23, 50)(2, 160, 35, 142)(3, 21, 134, 102)(4, 130, 140, 20)(5, 16, 34,
    128)(6, 136, 22, 116)(7, 135, 89, 12)(8, 141, 108, 37)(9, 54, 98, 42)(10,
    74, 32, 146)(11, 109, 103, 139)(13, 18, 26, 96)(14, 55, 27, 92)(15, 70, 111,
    58)(17, 105, 73, 95)(19, 151, 40, 158)(24, 149, 100, 123)(25, 113, 69,
    129)(28, 121, 38, 64)(29, 52, 46, 112)(30, 62, 91, 147)(31, 115, 61,
    131)(33, 156, 122, 76)(36, 59, 153, 107)(39, 138, 65, 157)(41, 145, 110,
    127)(43, 47, 132, 137)(44, 124, 60, 99)(45, 71, 90, 143)(48, 83, 68, 81)(49,
    66, 77, 152)(51, 106, 67, 159)(53, 117, 119, 82)(56, 114, 125, 88)(57, 126,
    75, 94)(63, 144, 79, 86)(72, 85, 150, 118)(78, 104, 101, 80)(84, 133, 154,
    93)(87, 120, 148, 97)
a * b^-1 * a * b^-2 * a * b
/* x*y^-1*x*y^-2*x*y */
[
    GSet{@ 1, 155, 150, 23, 118, 128, 50, 72, 5, 110, 85, 16, 127, 34, 41, 145
    @},
    GSet{@ 2, 160, 113, 35, 69, 137, 142, 129, 43, 65, 25, 47, 157, 132, 39, 138
    @},
    GSet{@ 3, 21, 15, 134, 70, 68, 102, 111, 81, 56, 58, 48, 114, 83, 125, 88
    @},
    GSet{@ 4, 130, 71, 140, 90, 77, 20, 143, 152, 31, 45, 49, 115, 66, 61, 131
    @},
    GSet{@ 6, 136, 44, 22, 124, 53, 116, 60, 117, 55, 99, 119, 27, 82, 92, 14
    @},
    GSet{@ 7, 135, 112, 89, 29, 97, 12, 52, 87, 19, 46, 120, 151, 148, 40, 158
    @},
    GSet{@ 8, 141, 32, 108, 146, 64, 37, 10, 28, 93, 74, 121, 84, 38, 133, 154
    @},
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    GSet{@ 9, 54, 95, 98, 17, 33, 42, 105, 156, 75, 73, 122, 94, 76, 57, 126 @},
    GSet{@ 11, 109, 79, 103, 86, 149, 139, 63, 100, 78, 144, 123, 104, 24, 101,
    80 @},
    GSet{@ 13, 18, 153, 26, 107, 159, 96, 36, 51, 62, 59, 106, 91, 67, 147, 30
    @}
]
8 (1, 6, 5, 47, 52)(2, 19, 16, 119, 50)(3, 33, 77, 93, 137)(4, 41, 97, 106,
    139)(7, 65, 64, 53, 141)(8, 71, 80, 157, 30)(9, 82, 123, 158, 135)(10, 35,
    48, 98, 140)(11, 29, 148, 122, 136)(12, 63, 152, 28, 102)(13, 100, 49, 150,
    89)(14, 75, 56, 159, 94)(15, 113, 145, 115, 103)(17, 66, 155, 96, 43)(18,
    91, 85, 60, 133)(20, 37, 81, 87, 78)(21, 114, 154, 92, 72)(22, 38, 120, 129,
    108)(23, 31, 95, 151, 70)(24, 117, 42, 138, 160)(25, 107, 121, 61, 86)(26,
    73, 124, 105, 111)(27, 144, 36, 134, 79)(32, 67, 59, 127, 55)(34, 51, 142,
    126, 130)(39, 118, 45, 149, 125)(40, 74, 131, 54, 153)(44, 110, 83, 58,
    146)(46, 88, 128, 143, 57)(62, 68, 101, 99, 104)(69, 132, 109, 116, 156)(76,
    147, 112, 84, 90)
a^2
/* x^2 */
[
    GSet{@ 1, 6, 2, 5, 19, 47, 16, 52, 119, 50 @},
    GSet{@ 3, 33, 10, 77, 35, 93, 48, 137, 98, 140 @},
    GSet{@ 4, 41, 13, 97, 100, 106, 49, 139, 150, 89 @},
    GSet{@ 7, 65, 22, 64, 38, 53, 120, 141, 129, 108 @},
    GSet{@ 8, 71, 25, 80, 107, 157, 121, 30, 61, 86 @},
    GSet{@ 9, 82, 29, 123, 148, 158, 122, 135, 136, 11 @},
    GSet{@ 12, 63, 37, 152, 81, 28, 87, 102, 78, 20 @},
    GSet{@ 14, 75, 26, 56, 73, 159, 124, 94, 105, 111 @},
    GSet{@ 15, 113, 45, 145, 149, 115, 125, 103, 39, 118 @},
    GSet{@ 17, 66, 51, 155, 142, 96, 126, 43, 130, 34 @},
    GSet{@ 18, 91, 55, 85, 32, 60, 67, 133, 59, 127 @},
    GSet{@ 21, 114, 44, 154, 110, 92, 83, 72, 58, 146 @},
    GSet{@ 23, 31, 46, 95, 88, 151, 128, 70, 143, 57 @},
    GSet{@ 24, 117, 69, 42, 132, 138, 109, 160, 116, 156 @},
    GSet{@ 27, 144, 68, 36, 101, 134, 99, 79, 104, 62 @},
    GSet{@ 40, 74, 76, 131, 147, 54, 112, 153, 84, 90 @}
]
9 (1, 20, 159, 29, 93)(2, 37, 79, 71, 160)(3, 89, 42, 145, 6)(4, 58, 18, 95,
    146)(5, 128, 44, 88, 125)(7, 155, 94, 31, 135)(8, 55, 30, 147, 141)(9, 131,
    52, 56, 54)(10, 16, 152, 26, 158)(11, 74, 53, 13, 137)(12, 157, 75, 139,
    14)(15, 50, 23, 92, 70)(17, 82, 39, 102, 133)(19, 111, 76, 122, 90)(21, 86,
    127, 140, 59)(22, 106, 35, 123, 84)(24, 45, 129, 36, 149)(25, 105, 100, 124,
    87)(27, 142, 151, 144, 33)(28, 101, 61, 132, 47)(32, 126, 136, 113, 81)(34,
    73, 143, 148, 120)(38, 121, 60, 107, 153)(40, 51, 110, 78, 117)(41, 108, 66,
    134, 69)(43, 46, 104, 98, 99)(48, 97, 156, 118, 119)(49, 114, 67, 57,
```

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    154)(62, 103, 109, 115, 65)(63, 116, 112, 96, 72)(64, 130, 68, 138, 150)(77,
    91, 83, 80, 85)
a * b^-2 * a * b
/* x*y^-2*x*y */
[
    GSet{@ 1, 20, 158, 159, 10, 29, 16, 93, 152, 26 @},
    GSet{@ 2, 37, 61, 79, 132, 71, 47, 160, 28, 101 @},
    GSet{@ 3, 89, 118, 42, 119, 145, 48, 6, 97, 156 @},
    GSet{@ 4, 58, 57, 18, 154, 95, 49, 146, 114, 67 @},
    GSet{@ 5, 128, 70, 44, 15, 88, 50, 125, 23, 92 @},
    GSet{@ 7, 155, 143, 94, 148, 31, 120, 135, 34, 73 @},
    GSet{@ 8, 55, 153, 30, 38, 147, 121, 141, 60, 107 @},
    GSet{@ 9, 131, 111, 52, 76, 56, 122, 54, 90, 19 @},
    GSet{@ 11, 74, 106, 53, 35, 13, 123, 137, 84, 22 @},
    GSet{@ 12, 157, 100, 75, 124, 139, 87, 14, 25, 105 @},
    GSet{@ 17, 82, 81, 39, 32, 102, 126, 133, 136, 113 @},
    GSet{@ 21, 86, 77, 127, 91, 140, 83, 59, 80, 85 @},
    GSet{@ 24, 45, 62, 129, 103, 36, 109, 149, 115, 65 @},
    GSet{@ 27, 142, 104, 151, 98, 144, 99, 33, 43, 46 @},
    GSet{@ 40, 51, 63, 110, 116, 78, 112, 117, 96, 72 @},
    GSet{@ 41, 108, 68, 66, 138, 134, 150, 69, 64, 130 @}
]
10 (1, 36, 143, 16, 62, 31)(2, 49, 9, 47, 4, 122)(3, 12, 30, 48, 87, 107)(5,
    105, 123, 50, 75, 11)(6, 84, 25, 119, 74, 157)(7, 140, 24, 120, 77, 109)(8,
    89, 60, 121, 97, 55)(10, 34, 149, 93, 155, 103)(13, 43, 88, 106, 142,
    70)(14, 98, 153, 124, 33, 147)(15, 27, 100, 125, 99, 139)(17, 81, 82, 126,
    102, 136)(18, 90, 146, 67, 131, 154)(19, 58, 132, 52, 114, 160)(20, 94, 29,
    152, 73, 158)(21, 45, 127, 83, 115, 85)(22, 145, 46, 53, 118, 151)(23, 35,
    44, 128, 137, 92)(26, 159)(28, 54, 41, 37, 76, 150)(32, 133)(38, 104, 42,
    141, 144, 156)(39, 113)(40, 112)(51, 116, 78, 96, 117, 63)(56, 61, 64, 111,
    71, 108)(57, 95)(59, 135, 65, 91, 148, 129)(66, 79, 69, 130, 101, 138)(68,
    134)(72, 110)(80, 86)
(b * a^-1)^2
/* (y*x^-1)^2 */
[
    GSet{@ 26, 159, 113, 40, 39, 112, 68, 134 @},
    GSet{@ 32, 133, 72, 57, 110, 95, 86, 80 @},
    GSet{@ 1, 36, 157, 140, 143, 6, 24, 126, 17, 16, 84, 120, 102, 81, 7, 119, 62, 25
    GSet{@ 2, 49, 116, 108, 9, 78, 56, 46, 151, 47, 96, 61, 53, 22, 71, 63, 4, 117, 6
    GSet{@ 3, 12, 58, 135, 30, 132, 65, 13, 106, 48, 52, 91, 43, 142, 59, 160, 87, 11
    GSet{@ 5, 105, 144, 34, 123, 156, 149, 76, 54, 50, 38, 93, 150, 41, 10, 42, 75, 1
    GSet{@ 8, 89, 83, 14, 60, 115, 98, 101, 79, 121, 85, 153, 138, 69, 147, 45, 97, 2
    GSet{@ 15, 27, 20, 23, 100, 94, 35, 67, 18, 125, 29, 44, 131, 90, 92, 73, 99, 152
]
```

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11 (1, 3, 11, 33, 91, 93)(2, 7)(4, 14)(5, 17, 52, 95, 85, 142)(6, 20, 60, 90, 136, 71
    78, 129, 80, 146)(13, 42, 100, 114, 150, 64)(15, 46, 102, 153, 157, 96)(18, 56, 2
    115, 116, 31)(25, 51, 125, 151, 81, 147)(27, 77)(28, 79, 113, 144, 74, 148)(34, 9
    73)(41, 108, 106, 156, 139, 58)(45, 117, 143, 53, 130, 92)(47, 120)(49, 124)(54,
    137)(99, 140)(107, 133, 128, 134, 149, 122)
b
/* y */
[
    GSet{@ 2, 7, 47, 120 @},
    GSet{@ 4, 14, 49, 124 @},
    GSet{@ 27, 77, 99, 140 @},
    GSet{@ 35, 97, 137, 89 @},
    GSet{@ 1, 3, 123, 11, 98, 91, 33, 59, 93, 16, 10, 48 @},
    GSet{@ 5, 17, 19, 52, 57, 85, 95, 127, 142, 50, 43, 126 @},
    GSet{@ 6, 20, 55, 60, 131, 136, 90, 82, 71, 119, 61, 152 @},
    GSet{@ 8, 26, 54, 76, 109, 70, 24, 88, 72, 121, 110, 159 @},
    GSet{@ 9, 30, 133, 32, 128, 68, 23, 134, 103, 122, 149, 107 @},
    GSet{@ 12, 38, 63, 78, 65, 80, 129, 86, 146, 87, 154, 141 @},
    GSet{@ 13, 42, 139, 100, 58, 150, 114, 41, 64, 106, 108, 156 @},
    GSet{@ 15, 46, 81, 102, 147, 157, 153, 25, 96, 125, 51, 151 @},
    GSet{@ 18, 56, 158, 29, 21, 36, 83, 62, 69, 67, 138, 111 @},
    GSet{@ 22, 66, 92, 44, 45, 116, 115, 117, 31, 53, 143, 130 @},
    GSet{@ 28, 79, 39, 113, 104, 74, 144, 84, 148, 37, 135, 101 @},
    GSet{@ 34, 94, 40, 112, 105, 118, 75, 145, 132, 155, 160, 73 @}
]
12 (1, 93, 91, 33, 11, 3)(2, 7)(4, 14)(5, 142, 85, 95, 52, 17)(6, 71, 136, 90, 60, 20
    80, 129, 78, 38)(13, 64, 150, 114, 100, 42)(15, 96, 157, 153, 102, 46)(18, 69, 36
    115, 44, 66)(25, 147, 81, 151, 125, 51)(27, 77)(28, 148, 74, 144, 113, 79)(34, 13
    105)(41, 58, 139, 156, 106, 108)(45, 92, 130, 53, 143, 117)(47, 120)(49, 124)(54,
    137)(99, 140)(107, 122, 149, 134, 128, 133)
b^-1
/* y^-1 */
[
    GSet{@ 2, 7, 47, 120 @},
    GSet{@ 4, 14, 49, 124 @},
    GSet{@ 27, 77, 99, 140 @},
    GSet{@ 35, 97, 137, 89 @},
    GSet{@ 1, 93, 123, 91, 48, 33, 16, 11, 10, 3, 59, 98 @},
    GSet{@ 5, 142, 19, 85, 126, 95, 50, 52, 43, 17, 127, 57 @},
    GSet{@ 6, 71, 55, 136, 152, 90, 119, 60, 61, 20, 82, 131 @},
    GSet{@ 8, 72, 54, 70, 159, 24, 121, 76, 110, 26, 88, 109 @},
    GSet{@ 9, 103, 133, 68, 107, 23, 122, 32, 149, 30, 134, 128 @},
    GSet{@ 12, 146, 63, 80, 141, 129, 87, 78, 154, 38, 86, 65 @},
    GSet{@ 13, 64, 139, 150, 156, 114, 106, 100, 108, 42, 41, 58 @},
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```
        GSet{@ 15, 96, 81, 157, 151, 153, 125, 102, 51, 46, 25, 147 @},
        GSet{@ 18, 69, 158, 36, 111, 83, 67, 29, 138, 56, 62, 21 @},
        GSet{@ 22, 31, 92, 116, 130, 115, 53, 44, 143, 66, 117, 45 @},
        GSet{@ 28, 148, 39, 74, 101, 144, 37, 113, 135, 79, 84, 104 @},
        GSet{@ 34, 132, 40, 118, 73, 75, 155, 112, 160, 94, 145, 105 @}
]
13 (1, 150, 118, 5, 16, 41, 145, 50)(2, 153, 69, 51, 47, 147, 138, 96)(3, 152, 70, 61
    129)(7, 125, 29, 102, 120, 15, 158, 81)(8, 104, 146, 103, 121, 144, 154, 149)(9,
    123, 101, 80, 139)(13, 66, 107, 31, 106, 130, 30, 143)(18, 46, 36, 97, 67, 151, 6
    105, 85, 76, 128, 75, 127, 54)(24, 28, 78, 133, 109, 37, 63, 32)(26, 60, 59, 82,
    115)(44, 114, 117, 134, 92, 58, 116, 68)
a * b * a * b^2 * a * b
/* x*y*x*y^2*x*y */
[
        GSet{@ 1, 150, 118, 5, 16, 41, 145, 50 @},
        GSet{@ 2, 153, 69, 51, 47, 147, 138, 96 @},
        GSet{@ 3, 152, 70, 61, 48, 20, 88, 71 @},
        GSet{@ 4, 87, 90, 40, 49, 12, 131, 112 @},
        GSet{@ 6, 132, 124, 65, 119, 160, 14, 129 @},
        GSet{@ 7, 125, 29, 102, 120, 15, 158, 81 @},
        GSet{@ 8, 104, 146, 103, 121, 144, 154, 149 @},
        GSet{@ 9, 34, 17, 110, 122, 155, 126, 72 @},
        GSet{@ 10, 74, 38, 64, 93, 84, 141, 108 @},
        GSet{@ 11, 79, 86, 100, 123, 101, 80, 139 @},
        GSet{@ 13, 66, 107, 31, 106, 130, 30, 143 @},
        GSet{@ 18, 46, 36, 97, 67, 151, 62, 89 @},
        GSet{@ 19, 53, 135, 27, 52, 22, 148, 99 @},
        GSet{@ 21, 157, 111, 35, 83, 25, 56, 137 @},
        GSet{@ 23, 105, 85, 76, 128, 75, 127, 54 @},
        GSet{@ 24, 28, 78, 133, 109, 37, 63, 32 @},
        GSet{@ 26, 60, 59, 82, 159, 55, 91, 136 @},
        GSet{@ 33, 156, 94, 57, 98, 42, 73, 95 @},
        GSet{@ 39, 140, 142, 45, 113, 77, 43, 115 @},
        GSet{@ 44, 114, 117, 134, 92, 58, 116, 68 @}
]
14 (1, 2, 6, 19, 5, 16, 47, 119, 52, 50)(3, 10, 33, 35, 77, 48, 93, 98, 137, 140)(4,
    108)(8, 25, 71, 107, 80, 121, 157, 61, 30, 86)(9, 29, 82, 148, 123, 122, 158, 136
    105, 94, 111)(15, 45, 113, 149, 145, 125, 115, 39, 103, 118)(17, 51, 66, 142, 155
    154, 83, 92, 58, 72, 146)(23, 46, 31, 88, 95, 128, 151, 143, 70, 57)(24, 69, 117,
    76, 74, 147, 131, 112, 54, 84, 153, 90)
a
/* x */
[
        GSet{@ 1, 2, 6, 19, 5, 16, 47, 119, 52, 50 @},
```

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        GSet{@ 3, 10, 33, 35, 77, 48, 93, 98, 137, 140 @},
        GSet{@ 4, 13, 41, 100, 97, 49, 106, 150, 139, 89 @},
        GSet{@ 7, 22, 65, 38, 64, 120, 53, 129, 141, 108 @},
        GSet{@ 8, 25, 71, 107, 80, 121, 157, 61, 30, 86 @},
        GSet{@ 9, 29, 82, 148, 123, 122, 158, 136, 135, 11 @},
        GSet{@ 12, 37, 63, 81, 152, 87, 28, 78, 102, 20 @},
        GSet{@ 14, 26, 75, 73, 56, 124, 159, 105, 94, 111 @},
        GSet{@ 15, 45, 113, 149, 145, 125, 115, 39, 103, 118 @},
        GSet{@ 17, 51, 66, 142, 155, 126, 96, 130, 43, 34 @},
        GSet{@ 18, 55, 91, 32, 85, 67, 60, 59, 133, 127 @},
        GSet{@ 21, 44, 114, 110, 154, 83, 92, 58, 72, 146 @},
        GSet{@ 23, 46, 31, 88, 95, 128, 151, 143, 70, 57 @},
        GSet{@ 24, 69, 117, 132, 42, 109, 138, 116, 160, 156 @},
        GSet{@ 27, 68, 144, 101, 36, 99, 134, 104, 79, 62 @},
        GSet{@ 40, 76, 74, 147, 131, 112, 54, 84, 153, 90 @}
]
15 (1, 19, 47, 50, 6, 16, 52, 2, 5, 119)(3, 35, 93, 140, 33, 48, 137, 10, 77, 98)(4,
    129)(8, 107, 157, 86, 71, 121, 30, 25, 80, 61)(9, 148, 158, 11, 82, 122, 135, 29,
    26, 56, 105)(15, 149, 115, 118, 113, 125, 103, 45, 145, 39)(17, 142, 96, 34, 66,
    114, 83, 72, 44, 154, 58)(23, 88, 151, 57, 31, 128, 70, 46, 95, 143)(24, 132, 138
    147, 54, 90, 74, 112, 153, 76, 131, 84)
a^3
/* x^3 */
[
        GSet{@ 1, 19, 47, 50, 6, 16, 52, 2, 5, 119 @},
        GSet{@ 3, 35, 93, 140, 33, 48, 137, 10, 77, 98 @},
        GSet{@ 4, 100, 106, 89, 41, 49, 139, 13, 97, 150 @},
        GSet{@ 7, 38, 53, 108, 65, 120, 141, 22, 64, 129 @},
        GSet{@ 8, 107, 157, 86, 71, 121, 30, 25, 80, 61 @},
        GSet{@ 9, 148, 158, 11, 82, 122, 135, 29, 123, 136 @},
        GSet{@ 12, 81, 28, 20, 63, 87, 102, 37, 152, 78 @},
        GSet{@ 14, 73, 159, 111, 75, 124, 94, 26, 56, 105 @},
        GSet{@ 15, 149, 115, 118, 113, 125, 103, 45, 145, 39 @},
        GSet{@ 17, 142, 96, 34, 66, 126, 43, 51, 155, 130 @},
        GSet{@ 18, 32, 60, 127, 91, 67, 133, 55, 85, 59 @},
        GSet{@ 21, 110, 92, 146, 114, 83, 72, 44, 154, 58 @},
        GSet{@ 23, 88, 151, 57, 31, 128, 70, 46, 95, 143 @},
        GSet{@ 24, 132, 138, 156, 117, 109, 160, 69, 42, 116 @},
        GSet{@ 27, 101, 134, 62, 144, 99, 79, 68, 36, 104 @},
        GSet{@ 40, 147, 54, 90, 74, 112, 153, 76, 131, 84 @}
]
16 (1, 140, 36, 24, 143, 120, 16, 77, 62, 109, 31, 7)(2, 108, 49, 56, 9, 61, 47, 64,
    149, 123, 93, 50, 155, 75, 103, 11, 10)(6, 102, 84, 136, 25, 17, 119, 81, 74, 82,
    114, 106, 160, 142, 19, 70, 58)(15, 23, 27, 35, 100, 44, 125, 128, 99, 137, 139,
```

```
    83, 101, 115, 138, 85, 66)(22, 51, 145, 116, 46, 78, 53, 96, 118, 117, 151, 63)(2
    95)(39, 134, 113, 68)(72, 86, 110, 80)
b * a^-1
/* y*x^-1 */
[
    GSet{@ 26, 40, 159, 112 @},
    GSet{@ 32, 57, 133, 95 @},
    GSet{@ 39, 134, 113, 68 @},
    GSet{@ 72, 86, 110, 80 @},
    GSet{@ 1, 143, 140, 62, 120, 36, 109, 16, 24, 31, 77, 7 @},
    GSet{@ 2, 9, 108, 4, 61, 49, 111, 47, 56, 122, 64, 71 @},
    GSet{@ 3, 30, 135, 87, 91, 12, 129, 48, 65, 107, 148, 59 @},
    GSet{@ 5, 123, 34, 75, 93, 105, 103, 50, 149, 11, 155, 10 @},
    GSet{@ 6, 25, 102, 74, 17, 84, 82, 119, 136, 157, 81, 126 @},
    GSet{@ 8, 60, 14, 97, 153, 89, 33, 121, 98, 55, 124, 147 @},
    GSet{@ 13, 88, 132, 142, 114, 43, 19, 106, 52, 70, 160, 58 @},
    GSet{@ 15, 100, 23, 99, 44, 27, 137, 125, 35, 139, 128, 92 @},
    GSet{@ 18, 146, 73, 131, 20, 90, 29, 67, 158, 154, 94, 152 @},
    GSet{@ 21, 127, 79, 115, 130, 45, 138, 83, 69, 85, 101, 66 @},
    GSet{@ 22, 46, 51, 118, 78, 145, 117, 53, 116, 151, 96, 63 @},
    GSet{@ 28, 41, 104, 76, 141, 54, 156, 37, 42, 150, 144, 38 @}
]
*/
for j in [2..#N] do for i in [1..#Setseq(Set(N))] do if 1^Setseq(Set(N))[i] eq j then
/*
2 a
3 a^2 * b^2 * a * b * a^-1
4 b * a * b^-2 * a * b * a
5 a^-2 * b^-2
6 b^-1 * a^-1 * b^-1 * a^-3
7 b * a * b^2 * a * b * a * b
8 a^-1 * b * a * b * a^-1 * b^-1 * a^-1 * b^-1
9 b * a^2 * b^-1 * a
10 b * a
11 b^2
12 a * b * a^2 * b^-1 * a^-2
13 b^-1 * a^-1 * b^-1 * a * b^-1 * a^-1 * b
14 b^-1 * a * b * a^-2 * b * a
15 a^-3 * b^-1 * a^-1 * b^2
16 a^-2 * b^-2 * a
17 a^4 * b
18 b * a^-1 * b * a^-1 * b^2
19 b^-1 * a^-1 * b^-1 * a^-2
20 a^2 * b^-1 * a * b * a^-2 * b^-1
```

```
21 a^2 * b * a * b^-1 * a
22 a * b * a
23 b^-1 * a^-2 * b * a * b^-1
24 b * a^-2 * b * a^-1 * b^-1 * a
25 a * b^-1 * a^2 * b * a^2
26 b * a^-2 * b * a * b^-1 * a
27 a^-1 * b^3 * a * b * a^-1
28 b^2 * a^-1 * b^-2 * a * b^-1
29 b^-1 * a^-1 * b * a^-3
30 a^-3 * b^-1 * a
31 a^2 * b * a * b * a^-2 * b^-1
32 a * b * a^2 * b * a^-1 * b
33 b^-1 * a * b^-2 * a * b^-2
34 a * b * a * b * a^-3
35 a * b^-1 * a^-1 * b^-1 * a^2 * b
36 (b * a^-1)^2
37 a * b^-1 * a^3 * b^-1 * a
38 b^2 * a^-1 * b * a^-1 * b
39 a^-1 * b * a^-1 * b^-1 * a^-1 * b^-1 * a
40 b * a * b^-1 * a^-1 * b * a
41 a * b^-1 * a^4 * b * a
42 b * a^3 * b^-1 * a^-1 * b^-1
43 a * b^-2 * a^-2 * b^-1
44 b^2 * a * b^-1 * a^-2 * b^-1
45 b * a^-1 * b * a^-1 * b^-1 * a * b
46 a * b^3 * a * b^-1 * a^-1
47 b * a * b * a * b^-2
48 b * a * b^2
49 b^2 * a^2 * b^-1 * a * b
50 b * a * b * a^4
51 a^-1 * b^-1 * a^3
52 b * a * b * a^3
53 a * b^-1 * a^-4
54 b * a * b^-1 * a^-1 * b * a^-3
55 b^-1 * a * b^-2 * a * b^-1 * a^-1
56 a^3 * b^-1 * a^-1 * b * a
57 b^-2 * a * b * a^-1
58 a^-1 * b^-1 * a^-1 * b * a
59 a * b^-2 * a * b^2 * a
60 b^2 * a^-1 * b * a^-2 * b
61 b^-1 * a^-1 * b * a^-2 * b
62 a^2 * b^2 * a^-1 * b^-2
63 a^-2 * b * a^2 * b * a^-1
64 b * a^-2 * b * a^2 * b^-1
```

```
65 (a^2 * b * a^-1)^2
66 a^-2 * b^3 * a^-1
67 a^-2 * b^2 * a
68 b^-1 * a^-2 * b^-1 * a
69 a^-1 * b * a^-1 * b^-1 * a^3
70 a * b * a^-1 * b^-2 * a * b^-1
71 b * a * b * a^-3 * b^-1
72 a^2 * b^-1 * a^2 * b * a^-1
73 a^4 * b^-1 * a * b
74 a^-2 * b^-1 * a * b^-2 * a^-1
75 a^-1 * b^-2 * a * b * a^-2
76 a^-1 * b * a^-1 * b * a^-2 * b
77 b^-1 * a^-2 * b^-2
78 a * b * a^-3 * b^-1
79 a^-2 * b * a * b * a^2
80 a * b^2 * a * b * a * b^-2
81 b * a^-1 * b * a * b^-1 * a * b
82 a^2 * b^-1 * a^-1 * b^-1 * a * b
83 a * b * a^-2 * b^-2 * a
84 a^-2 * b * a * b * a * b
85 a * b^-1 * a * b * a * b^-1
86 b^2 * a^-1 * b * a * b^2
87 b^-1 * a * b^-1 * a^-1 * b * a^-1
88 b * a * b^-1 * a * b * a^-2
89 (b * a^-1 * b^-1)^2
90 a^2 * b * a^2 * b^-1 * a^-1 * b^-1
91 (b * a^-1 * b^-1)^4
92 a^-1 * b * a^2 * b
93 a * b^2 * a^-1 * b^-1
94 a^4 * b * a^-1 * b
95 a^-1 * b * a^-2 * b^-2
96 a^3 * b^-1 * a
97 (b * a * b^-1)^3
98 b * a^2 * b^2 * a
99 b * a^-2 * b^-2 * a * b
100 b^3 * a * b^-1 * a^-1
101 a * b * a^2 * b^-1 * a^-1 * b
102 a^2 * b * a^-1
103 b * a^2 * b^-1 * a * b^-1
104 b * a^-1 * b^-1 * a^2
105 b^-1 * a * b * a^-1 * b * a * b
106 b^3 * a * b^-1 * a^2
107 b * a * b^-1 * a * b^-1
108 b^3 * a * b^-1 * a^-2 * b
```

```
109 b^2 * a^-1 * b^-1 * a^-1 * b
110 a * b^-1 * a^-1 * b^-1 * a * b * a
111 b^-1 * a * b * a^-2 * b
112 b * a^2 * b * a^-1 * b * a
113 b^2 * a^-1 * b^-2 * a * b
114 a * b^-1 * a^-1 * b^-1 * a * b
115 a^2 * b * a^2 * b^2 * a^-1
116 b^-2 * a * b * a^2 * b^-1
117 b * a^-1 * b^-1 * a^-1 * b * a
118 a^2 * b * a^-1 * b^-2 * a^-1
119 a^-3
120 b * a * b * a * b
121 a^2 * b^-1 * a^3
122 (b^-1 * a)^2
123 b^3 * a^-1 * b^3
124 b^-1 * a^-3 * b * a * b
125 b * a * b * a^-1 * b * a * b
126 a * b^2 * a^2 * b^-1
127 a^-2 * b * a * b^-1 * a
128 b * a * b^-1 * a * b
129 a^-1 * b^-1 * a^-1 * b^-1 * a
130 a * b^-2 * a^-2 * b^-1 * a^-1
131 b^2 * a^3 * b^-1
132 a^-1 * b^-1 * a * b^-1
133 a^2 * b^-1 * a * b
134 b * a^-2 * b^-2 * a * b * a
135 b * a^2 * b^-1 * a^-1
136 b * a * b^-1 * a * b^-1 * a^-1 * b^-1
137 b^-1 * a^-1 * b^2 * a
138 a * b * a * b^-1 * a * b^-1 * a
139 a^-1 * b * a^-1 * b^-1 * a * b
140 a * b^2 * a^-1 * b * a^-1
141 a * b^-1 * a^-2
142 a^-1 * b * a^-2
143 b * a * b^-1 * a * b * a^2
144 b^2 * a * b^-1 * a^-1 * b^-1 * a^-1
145 a * b * a * b * a^-1 * b * a^-1
146 a^2 * b^-1 * a^2 * b
147 a^2 * b^-1 * a^-1 * b^-1
148 a^2 * b * a * b^2 * a^-1 * b^-1
149 b^2 * a^-1 * b^-2 * a^-1 * b
150 a * b * a * b^2 * a * b
151 a^-1 * b^-1 * a^-1 * b^-2 * a^-1
152 b * a * b * a^2 * b
```

```
153 a * b^2 * a * b * a^-1 * b
154 a * b^-1 * a^-2 * b^-2
155 a^4 * b^-1 * a
156 b^-1 * a^2 * b * a^-1 * b^-1
157 a^-2 * b * a^-1 * b^2 * a
158 a^2 * b^2 * a^-1 * b^2
159 b^2 * a^-1 * b^-1 * a^-1 * b^-1
160 a^-2 * b * a^-2 * b^-1 * a

*/
/*
2 x
3 x^2 * y^2 * x * y * x^-1
4 y * x * y^-2 * x * y * x
5 x^-2 * y^-2
6 y^-1 * x^-1 * y^-1 * x^-3
7 y * x * y^2 * x * y * x * y
8 x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1
9 y * x^2 * y^-1 * x
10 y * x
11 y^2
12 x * y * x^2 * y^-1 * x^-2
13 y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y
14 y^-1 * x * y * x^-2 * y * x
15 x^-3 * y^-1 * x^-1 * y^2
16 x^-2 * y^-2 * x
17 x^4 * y
18 y * x^-1 * y * x^-1 * y^2
19 y^-1 * x^-1 * y^-1 * x^-2
20 x^2 * y^-1 * x * y * x^-2 * y^-1
21 x^2 * y * x * y^-1 * x
22 x * y * x
23 y^-1 * x^-2 * y * x * y^-1
24 y * x^-2 * y * x^-1 * y^-1 * x
25 x * y^-1 * x^2 * y * x^2
26 y * x^-2 * y * x * y^-1 * x
27 x^-1 * y^3 * x * y * x^-1
28 y^2 * x^-1 * y^-2 * x * y^-1
29 y^-1 * x^-1 * y * x^-3
30 x^-3 * y^-1 * x
31 x^2 * y * x * y * x^-2 * y^-1
32 x * y * x^2 * y * x^-1 * y
33 y^-1 * x * y^-2 * x * y^-2
34 x * y * x * y * x^-3
```

```
35 x * y^-1 * x^-1 * y^-1 * x^2 * y
36 (y * x^-1)^2
37 x * y^-1 * x^3 * y^-1 * x
38 y^2 * x^-1 * y * x^-1 * y
39 x^-1 * y * x^-1 * y^-1 * x^-1 * y^-1 * x
40 y * x * y^-1 * x^-1 * y * x
41 x * y^-1 * x^4 * y * x
42 y * x^3 * y^-1 * x^-1 * y^-1
43 x * y^-2 * x^-2 * y^-1
44 y^2 * x * y^-1 * x^-2 * y^-1
45 y * x^-1 * y * x^-1 * y^-1 * x * y
46 x * y^3 * x * y^-1 * x^-1
47 y * x * y * x * y^-2
48 y * x * y^2
49 y^2 * x^2 * y^-1 * x * y
50 y * x * y * x^4
51 x^-1 * y^-1 * x^3
52 y * x * y * x^3
53 x * y^-1 * x^-4
54 y * x * y^-1 * x^-1 * y * x^-3
55 y^-1 * x * y^-2 * x * y^-1 * x^-1
56 x^3 * y^-1 * x^-1 * y * x
57 y^-2 * x * y * x^-1
58 x^-1 * y^-1 * x^-1 * y * x
59 x * y^-2 * x * y^2 * x
60 y^2 * x^-1 * y * x^-2 * y
61 y^-1 * x^-1 * y * x^-2 * y
62 x^2 * y^2 * x^-1 * y^-2
63 x^-2 * y * x^2 * y * x^-1
64 y * x^-2 * y * x^2 * y^-1
65 (x^2 * y * x^-1)^2
66 x^-2 * y^3 * x^-1
67 x^-2 * y^2 * x
68 y^-1 * x^-2 * y^-1 * x
69 x^-1 * y * x^-1 * y^-1 * x^3
70 x * y * x^-1 * y^-2 * x * y^-1
71 y * x * y * x^-3 * y^-1
72 x^2 * y^-1 * x^2 * y * x^-1
73 x^4 * y^-1 * x * y
74 x^-2 * y^-1 * x * y^-2 * x^-1
75 x^-1 * y^-2 * x * y * x^-2
76 x^-1 * y * x^-1 * y * x^-2 * y
77 y^-1 * x^-2 * y^-2
78 x * y * x^-3 * y^-1
```

```
79 x^-2 * y * x * y * x^2
80 x * y^2 * x * y * x * y^-2
81 y * x^-1 * y * x * y^-1 * x * y
82 x^2 * y^-1 * x^-1 * y^-1 * x * y
83 x * y * x^-2 * y^-2 * x
84 x^-2 * y * x * y * x * y
85 x * y^-1 * x * y * x * y^-1
86 y^2 * x^-1 * y * x * y^2
87 y^-1 * x * y^-1 * x^-1 * y * x^-1
88 y * x * y^-1 * x * y * x^-2
89 (y * x^-1 * y^-1)^2
90 x^2 * y * x^2 * y^-1 * x^-1 * y^-1
91 (y * x^-1 * y^-1)^4
92 x^-1 * y * x^2 * y
93 x * y^2 * x^-1 * y^-1
94 x^4 * y * x^-1 * y
95 x^-1 * y * x^-2 * y^-2
96 x^3 * y^-1 * x
97 (y * x * y^-1)^3
98 y * x^2 * y^2 * x
99 y * x^-2 * y^-2 * x * y
100 y^3 * x * y^-1 * x^-1
101 x * y * x^2 * y^-1 * x^-1 * y
102 x^2 * y * x^-1
103 y * x^2 * y^-1 * x * y^-1
104 y * x^-1 * y^-1 * x^2
105 y^-1 * x * y * x^-1 * y * x * y
106 y^3 * x * y^-1 * x^2
107 y * x * y^-1 * x * y^-1
108 y^3 * x * y^-1 * x^-2 * y
109 y^2 * x^-1 * y^-1 * x^-1 * y
110 x * y^-1 * x^-1 * y^-1 * x * y * x
111 y^-1 * x * y * x^-2 * y
112 y * x^2 * y * x^-1 * y * x
113 y^2 * x^-1 * y^-2 * x * y
114 x * y^-1 * x^-1 * y^-1 * x * y
115 x^2 * y * x^2 * y^2 * x^-1
116 y^-2 * x * y * x^2 * y^-1
117 y * x^-1 * y^-1 * x^-1 * y * x
118 x^2 * y * x^-1 * y^-2 * x^-1
119 x^-3
120 y * x * y * x * y
121 x^2 * y^-1 * x^3
122 (y^-1 * x)^2
```

```
123 y^3 * x^-1 * y^3
124 y^-1 * x^-3 * y * x * y
125 y * x * y * x^-1 * y * x * y
126 x * y^2 * x^2 * y^-1
127 x^-2 * y * x * y^-1 * x
128 y * x * y^-1 * x * y
129 x^-1 * y^-1 * x^-1 * y^-1 * x
130 x * y^-2 * x^-2 * y^-1 * x^-1
131 y^2 * x^3 * y^-1
132 x^-1 * y^-1 * x * y^-1
133 x^2 * y^-1 * x * y
134 y * x^-2 * y^-2 * x * y * x
135 y * x^2 * y^-1 * x^-1
136 y * x * y^-1 * x * y^-1 * x^-1 * y^-1
137 y^-1 * x^-1 * y^2 * x
138 x * y * x * y^-1 * x * y^-1 * x
139 x^-1 * y * x^-1 * y^-1 * x * y
140 x * y^2 * x^-1 * y * x^-1
141 x * y^-1 * x^-2
142 x^-1 * y * x^-2
143 y * x * y^-1 * x * y * x^2
144 y^2 * x * y^-1 * x^-1 * y^-1 * x^-1
145 x * y * x * y * x^-1 * y * x^-1
146 x^2 * y^-1 * x^2 * y
147 x^2 * y^-1 * x^-1 * y^-1
148 x^2 * y * x * y^2 * x^-1 * y^-1
149 y^2 * x^-1 * y^-2 * x^-1 * y
150 x * y * x * y^2 * x * y
151 x^-1 * y^-1 * x^-1 * y^-2 * x^-1
152 y * x * y * x^2 * y
153 x * y^2 * x * y * x^-1 * y
154 x * y^-1 * x^-2 * y^-2
155 x^4 * y^-1 * x
156 y^-1 * x^2 * y * x^-1 * y^-1
157 x^-2 * y * x^-1 * y^2 * x
158 x^2 * y^2 * x^-1 * y^2
159 y^2 * x^-1 * y^-1 * x^-1 * y^-1
160 x^-2 * y * x^-2 * y^-1 * x

ts[2]:= f(t^(x));
ts[3]:= f(t^(x^2 * y^2 * x * y * x^-1));
ts[4]:= f(t^(y * x * y^-2 * x * y * x));
ts[5]:= f(t^(x^-2 * y^-2));
ts[6]:= f(t^(y^-1 * x^-1 * y^-1 * x^-3));
```

```
ts[7]:= f(t^(y * x * y^2 * x * y * x * y));
ts[8]:= f(t^(x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1));
ts[9]:= f(t^(y * x^2 * y^-1 * x));
ts[10]:= f(t^( y * x));
ts[11]:= f(t^(y^2));
ts[12]:= f(t^(x * y * x^2 * y^-1 * x^-2));
ts[13]:= f(t^(y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y));
ts[14]:= f(t^(y^-1 * x * y * x^-2 * y * x));
ts[15]:= f(t^(x^-3 * y^-1 * x^-1 * y^2));
ts[16]:= f(t^(x^-2 * y^-2 * x));
ts[17]:= f(t^(x^4 * y));
ts[18]:= f(t^(y * x^-1 * y * x^-1 * y^2));
ts[19]:= f(t^(y^-1 * x^-1 * y^-1 * x^-2));
ts[20]:= f(t^(x^2 * y^-1 * x * y * x^-2 * y^-1));
ts[21]:= f(t^(x^2 * y * x * y^-1 * x));
ts[22]:= f(t^(x * y * x));
ts[23]:= f(t^(y^-1 * x^-2 * y * x * y^-1));
ts[24]:= f(t^(y * x^-2 * y * x^-1 * y^-1 * x));
ts[25]:= f(t^( x * y^-1 * x^2 * y * x^2));
ts[26]:= f(t^( y * x^-2 * y * x * y^-1 * x));
ts[27]:= f(t^(x^-1 * y^3 * x * y * x^-1));
ts[28]:= f(t^(y^2 * x^-1 * y^-2 * x * y^-1));
ts[29]:= f(t^(y^-1 * x^-1 * y * x^-3));
ts[30]:= f(t^(x^-3 * y^-1 * x));
ts[31]:= f(t^(x^2 * y * x * y * x^-2 * y^-1));
ts[32]:= f(t^(x * y * x^2 * y * x^-1 * y));
ts[33]:= f(t^(y^-1 * x * y^-2 * x * y^-2));
ts[34]:= f(t^(x * y * x * y * x^-3));
ts[35]:= f(t^(x * y^-1 * x^-1 * y^-1 * x^2 * y));
ts[36]:= f(t^((y * x^-1)^2));
ts[37]:= f(t^(x * y^-1 * x^3 * y^-1 * x));
ts[38]:= f(t^( y^2 * x^-1 * y * x^-1 * y));
ts[39]:= f(t^(x^-1 * y * x^-1 * y^-1 * x^-1 * y^-1 * x));
ts[40]:= f(t^(y * x * y^-1 * x^-1 * y * x));
ts[41]:= f(t^(x * y^-1 * x^4 * y * x));
ts[42]:= f(t^(y * x^3 * y^-1 * x^-1 * y^-1));
ts[43]:= f(t^(x * y^-2 * x^-2 * y^-1));
ts[44]:= f(t^(y^2 * x * y^-1 * x^-2 * y^-1));
ts[45]:= f(t^(y * x^-1 * y * x^-1 * y^-1 * x * y));
ts[46]:= f(t^(x * y^3 * x * y^-1 * x^-1));
ts[47]:= f(t^(y * x * y * x * y^-2));
ts[48]:= f(t^( y * x * y^2));
ts[49]:= f(t^(y^2 * x^2 * y^-1 * x * y));
ts[50]:= f(t^(y * x * y * x^4));
```

```
ts[51]:= f(t^(x^-1 * y^-1 * x^3));
ts[52]:= f(t^(y * x * y * x^3));
ts[53]:= f(t^(x * y^-1 * x^-4));
ts[54]:= f(t^(y * x * y^-1 * x^-1 * y * x^-3));
ts[55]:= f(t^(y^-1 * x * y^-2 * x * y^-1 * x^-1));
ts[56]:= f(t^(x^3 * y^-1 * x^-1 * y * x));
ts[57]:= f(t^(y^-2 * x * y * x^-1));
ts[58]:= f(t^(x^-1 * y^-1 * x^-1 * y * x));
ts[59]:= f(t^(x * y^-2 * x * y^2 * x));
ts[60]:= f(t^(y^2 * x^-1 * y * x^-2 * y));
ts[61]:= f(t^(y^-1 * x^-1 * y * x^-2 * y));
ts[62]:= f(t^(x^2 * y^2 * x^-1 * y^-2));
ts[63]:= f(t^(x^-2 * y * x^2 * y * x^-1));
ts[64]:= f(t^(y * x^-2 * y * x^2 * y^-1));
ts[65]:= f(t^((x^2 * y * x^-1)^2));
ts[66]:= f(t^( x^-2 * y^3 * x^-1));
ts[67]:= f(t^(x^-2 * y^2 * x));
ts[68]:= f(t^(y^-1 * x^-2 * y^-1 * x));
ts[69]:= f(t^( x^-1 * y * x^-1 * y^-1 * x^3));
ts[70]:= f(t^(x * y * x^-1 * y^-2 * x * y^-1));
ts[71]:= f(t^(y * x * y * x^-3 * y^-1));
ts[72]:= f(t^(x^2 * y^-1 * x^2 * y * x^-1));
ts[73]:= f(t^(x^4 * y^-1 * x * y));
ts[74]:= f(t^(x^-2 * y^-1 * x * y^-2 * x^-1));
ts[75]:= f(t^(x^-1 * y^-2 * x * y * x^-2));
ts[76]:= f(t^(x^-1 * y * x^-1 * y * x^-2 * y));
ts[77]:= f(t^(y^-1 * x^-2 * y^-2));
ts[78]:= f(t^(x * y * x^-3 * y^-1));
ts[79]:= f(t^(x^-2 * y * x * y * x^2));
ts[80]:= f(t^(x * y^2 * x * y * x * y^-2));
ts[81]:= f(t^(y * x^-1 * y * x * y^-1 * x * y));
ts[82]:= f(t^(x^2 * y^-1 * x^-1 * y^-1 * x * y));
ts[83]:= f(t^(x * y * x^-2 * y^-2 * x));
ts[84]:= f(t^(x^-2 * y * x * y * x * y));
ts[85]:= f(t^(x * y^-1 * x * y * x * y^-1));
ts[86]:= f(t^(y^2 * x^-1 * y * x * y^2));
ts[87]:= f(t^(y^-1 * x * y^-1 * x^-1 * y * x^-1));
ts[88]:= f(t^(y * x * y^-1 * x * y * x^-2));
ts[89]:= f(t^((y * x^-1 * y^-1)^2));
ts[90]:= f(t^(x^2 * y * x^2 * y^-1 * x^-1 * y^-1));
ts[91]:= f(t^((y * x^-1 * y^-1)^4));
ts[92]:= f(t^(x^-1 * y * x^2 * y));
ts[93]:= f(t^(x * y^2 * x^-1 * y^-1));
ts[94]:= f(t^(x^4 * y * x^-1 * y));
```

```
ts[95]:= f(t^(x^-1 * y * x^-2 * y^-2));
ts[96]:= f(t^( x^3 * y^-1 * x));
ts[97]:= f(t^((y * x * y^-1)^3));
ts[98]:= f(t^(y * x^2 * y^2 * x));
ts[99]:= f(t^(y * x^-2 * y^-2 * x * y));
ts[100]:= f(t^(y^3 * x * y^-1 * x^-1));
ts[101]:= f(t^(x * y * x^2 * y^-1 * x^-1 * y));
ts[102]:= f(t^(x^2 * y * x^-1));
ts[103]:= f(t^( y * x^2 * y^-1 * x * y^-1));
ts[104]:= f(t^(y * x^-1 * y^-1 * x^2));
ts[105]:= f(t^(y^-1 * x * y * x^-1 * y * x * y));
ts[106]:= f(t^(y^3 * x * y^-1 * x^2));
ts[107]:= f(t^(y * x * y^-1 * x * y^-1));
ts[108]:= f(t^(y^3 * x * y^-1 * x^-2 * y));
ts[109]:= f(t^(y^2 * x^-1 * y^-1 * x^-1 * y));
ts[110]:= f(t^(x * y^-1 * x^-1 * y^-1 * x * y * x));
ts[111]:= f(t^(y^-1 * x * y * x^-2 * y));
ts[112]:= f(t^(y * x^2 * y * x^-1 * y * x));
ts[113]:= f(t^(y^2 * x^-1 * y^-2 * x * y));
ts[114]:= f(t^(x * y^-1 * x^-1 * y^-1 * x * y));
ts[115]:= f(t^(x^2 * y * x^2 * y^2 * x^-1));
ts[116]:= f(t^(y^-2 * x * y * x^2 * y^-1));
ts[117]:= f(t^(y * x^-1 * y^-1 * x^-1 * y * x));
ts[118]:= f(t^( x^2 * y * x^-1 * y^-2 * x^-1));
ts[119]:= f(t^(x^-3));
ts[120]:= f(t^(y * x * y * x * y));
ts[121]:= f(t^(x^2 * y^-1 * x^3));
ts[122]:= f(t^((y^-1 * x)^2));
ts[123]:= f(t^(y^3 * x^-1 * y^3));
ts[124]:= f(t^(y^-1 * x^-3 * y * x * y));
ts[125]:= f(t^(y * x * y * x^-1 * y * x * y));
ts[126]:= f(t^(x * y^2 * x^2 * y^-1));
ts[127]:= f(t^(x^-2 * y * x * y^-1 * x));
ts[128]:= f(t^(y * x * y^-1 * x * y));
ts[129]:= f(t^(x^-1 * y^-1 * x^-1 * y^-1 * x));
ts[130]:= f(t^(x * y^-2 * x^-2 * y^-1 * x^-1));
ts[131]:= f(t^(y^2 * x^3 * y^-1));
ts[132]:= f(t^(x^-1 * y^-1 * x * y^-1));
ts[133]:= f(t^(x^2 * y^-1 * x * y));
ts[134]:= f(t^(y * x^-2 * y^-2 * x * y * x));
ts[135]:= f(t^(y * x^2 * y^-1 * x^-1));
ts[136]:= f(t^(y * x * y^-1 * x * y^-1 * x^-1 * y^-1));
ts[137]:= f(t^(y^-1 * x^-1 * y^2 * x));
ts[138]:= f(t^(x * y * x * y^-1 * x * y^-1 * x));
```

```
ts[139]:= f(t^(x^-1 * y * x^-1 * y^-1 * x * y));
ts[140]:= f(t^(x * y^2 * x^-1 * y * x^-1));
ts[141]:= f(t^(x * y^-1 * x^-2));
ts[142]:= f(t^(x^-1 * y * x^-2));
ts[143]:= f(t^(y * x * y^-1 * x * y * x^2));
ts[144]:= f(t^(y^2 * x * y^-1 * x^-1 * y^-1 * x^-1));
ts[145]:= f(t^(x * y * x * y * x^-1 * y * x^-1));
ts[146]:= f(t^(x^2 * y^-1 * x^2 * y));
ts[147]:= f(t^(x^2 * y^-1 * x^-1 * y^-1));
ts[148]:= f(t^(x^2 * y * x * y^2 * x^-1 * y^-1));
ts[149]:= f(t^(y^2 * x^-1 * y^-2 * x^-1 * y));
ts[150]:= f(t^(x * y * x * y^2 * x * y));
ts[151]:= f(t^(x^-1 * y^-1 * x^-1 * y^-2 * x^-1));
ts[152]:= f(t^(y * x * y * x^2 * y));
ts[153]:= f(t^(x * y^2 * x * y * x^-1 * y));
ts[154]:= f(t^(x * y^-1 * x^-2 * y^-2));
ts[155]:= f(t^(x^4 * y^-1 * x));
ts[156]:= f(t^(y^-1 * x^2 * y * x^-1 * y^-1));
ts[157]:= f(t^(x^-2 * y * x^-1 * y^2 * x));
ts[158]:= f(t^(x^2 * y^2 * x^-1 * y^2));
ts[159]:= f(t^( y^2 * x^-1 * y^-1 * x^-1 * y^-1));
ts[160]:= f(t^(x^-2 * y * x^-2 * y^-1 * x));
```

We put the progenitor on the back of magma, but all the groups we got were either unfaithful meaning $k \neq 1$ or the subgroup number are not matching the subgroup number we got from x and y. Here we put some examples.

First Group

```
a:=0;b:=0;c:=2;d:=0;e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=0;l:=0;m:=0;n:=0;
G<x,y,t>:=Group<x,y,t|x^10   ,
y^6  ,
(x * y^-2 * x)^2 ,
(x * y^2 * x^2)^2 ,
(y^-1 * x^-1)^5 ,
(x * y^2 * x^-1 * y^-1)^2 ,
x^-1 * y^-1 * x^5 * y * x^-4 ,
y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,t^2,(t,(x*y^-1*x^-1)^2),
(t,(y*x*y)^2),
((y)*t^(x))^a,
((y*x^-1)*t^( x^2 * y^2 * x * y * x^-1))^b,
((y*x^3*y)*t^(y * x * y^-2 * x * y * x))^c,
((y*x^-1)*t^(x^-2 * y^-2))^d,
((y*x^3*y)*t^(y^-1 * x^-1 * y^-1 * x^-3))^e,
```

```
((y*x^-1)*t^(y * x * y^2 * x * y * x * y))^f,
((x)*t^(x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1))^g,
((y*x^-1)*t^(y * x^2 * y^-1 * x))^h,
((y*x^3*y)*t^(y * x))^i,
((y)*t^(y^2))^j,
((y*x^3*y)*t^(x * y * x^2 * y^-1 * x^-2))^k,
((y^2)*t^(y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y))^l,
((x^5)*t^(y^-1 * x * y * x^-2 * y * x))^m,
((y*x^3*y)*t^(x^-3 * y^-1 * x^-1 * y^2))^n>;
#G;
/*3840*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*1920*/
#sub<G|x,y>;
/*1920*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*2*/
CompositionFactors(G1);
/* G
    |  Cyclic(2)
    1
*/


Second Group

a:=0;b:=0;c:=3;d:=0;e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=0;l:=0;m:=0;n:=0;
G<x,y,t>:=Group<x,y,t|x^10   ,
y^6   ,
(x * y^-2 * x)^2 ,
(x * y^2 * x^2)^2 ,
(y^-1 * x^-1)^5 ,
(x * y^2 * x^-1 * y^-1)^2 ,
x^-1 * y^-1 * x^5 * y * x^-4 ,
y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,t^2,(t,(x*y^-1*x^-1)^2),
(t,(y*x*y)^2),
((y)*t^(x))^a,
((y*x^-1)*t^( x^2 * y^2 * x * y * x^-1))^b,
((y*x^3*y)*t^(y * x * y^-2 * x * y * x))^c,
((y*x^-1)*t^(x^-2 * y^-2))^d,
((y*x^3*y)*t^(y^-1 * x^-1 * y^-1 * x^-3))^e,
((y*x^-1)*t^(y * x * y^2 * x * y * x * y))^f,
((x)*t^(x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1))^g,
((y*x^-1)*t^(y * x^2 * y^-1 * x))^h,
```

```
((y*x^3*y)*t^(y * x))^i,
((y)*t^(y^2))^j,
((y*x^3*y)*t^(x * y * x^2 * y^-1 * x^-2))^k,
((y^2)*t^(y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y))^l,
((x^5)*t^(y^-1 * x * y * x^-2 * y * x))^m,
((y*x^3*y)*t^(x^-3 * y^-1 * x^-1 * y^2))^n>;
#G;
/*1451520*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*1*/
#sub<G|x,y>;
/*960*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*36*/
CompositionFactors(G1);
/*
 G
    |  C(3, 2)                        = S(6, 2)
    1
*/
```

Third Group

```
a:=3;b:=0;c:=0;d:=0;e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=0;l:=0;m:=0;n:=0;
G<x,y,t>:=Group<x,y,t|x^10   ,
y^6   ,
(x * y^-2 * x)^2 ,
(x * y^2 * x^2)^2 ,
(y^-1 * x^-1)^5 ,
(x * y^2 * x^-1 * y^-1)^2 ,
x^-1 * y^-1 * x^5 * y * x^-4 ,
y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,t^2,(t,(x*y^-1*x^-1)^2),
(t,(y*x*y)^2),
((y*x^-1)*t^(x))^a,
((x)*t^( x^2 * y^2 * x * y * x^-1))^b,
((y*x^-1)*t^(y * x * y^-2 * x * y * x))^c,
((x*y^-2*x*y)*t^(x^-2 * y^-2))^d,
((y*x^-1)*t^(y^-1 * x^-1 * y^-1 * x^-3))^e,
((y)*t^(y * x * y^2 * x * y * x * y))^f,
((y*x^-1)*t^(x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1))^g,
((x*y^-2*x*y)*t^(y * x^2 * y^-1 * x))^h,
((y*x^-1)*t^(y * x))^i,
((x^3)*t^(y^2))^j,
```

```
((x^5)*t^(x * y * x^2 * y^-1 * x^-2))^k,
((x*y^-2*x*y)*t^(y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y))^l,
((y)*t^(y^-1 * x * y * x^-2 * y * x))^m,
((x*y^-2*x*y)*t^(x^-3 * y^-1 * x^-1 * y^2))^n>;
#G;
/*466560*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*2*/
#sub<G|x,y>;
/*1920*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*7*/
CompositionFactors(G1);
/*
 G
     |  Alternating(5)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(3)
     *
     |  Cyclic(3)
     *
     |  Cyclic(3)
     *
     |  Cyclic(3)
     *
     |  Cyclic(3)
     1
*/
```

Fourth Group

```
a:=0;b:=7;c:=2;d:=0;e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=0;l:=0;m:=0;n:=0;
G<x,y,t>:=Group<x,y,t|x^10  ,
y^6  ,
(x * y^-2 * x)^2 ,
```

```
(x * y^2 * x^2)^2 ,
(y^-1 * x^-1)^5 ,
(x * y^2 * x^-1 * y^-1)^2 ,
x^-1 * y^-1 * x^5 * y * x^-4 ,
y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,t^2,(t,(x*y^-1*x^-1)^2),
(t,(y*x*y)^2),
((x^5)*t^(x))^a,
((x^3)*t^( x^2 * y^2 * x * y * x^-1))^b,
((x*y^-1*x*y^-2*x*y)*t^(y * x * y^-2 * x * y * x))^c,
((x^3)*t^(x^-2 * y^-2))^d,
((x*y^-1*x*y^-2*x*y)*t^(y^-1 * x^-1 * y^-1 * x^-3))^e,
((x^3)*t^(y * x * y^2 * x * y * x * y))^f,
(((x^-1*y)^3)*t^(x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1))^g,
((x^3)*t^(y * x^2 * y^-1 * x))^h,
((x)*t^(y * x))^i,
((y)*t^(y^2))^j,
(((x*y^-1*x^-1)^2)*t^(x * y * x^2 * y^-1 * x^-2))^k,
((y^2)*t^(y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y))^l,
(((x*y^-1*x^-1)^2)*t^(y^-1 * x * y * x^-2 * y * x))^m,
(((x^-1*y)^3)*t^(x^-3 * y^-1 * x^-1 * y^2))^n>;
#G;
/*175560*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*1*/
#sub<G|x,y>;
/*60*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*62*/
CompositionFactors(G1);
/*
 G
    |  J1
    1


*/
```

Fifth Group

```
a:=0;b:=3;c:=0;d:=0;e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=0;l:=0;m:=0;n:=0;
G<x,y,t>:=Group<x,y,t|x^10   ,
y^6  ,
(x * y^-2 * x)^2 ,
(x * y^2 * x^2)^2 ,
(y^-1 * x^-1)^5 ,
(x * y^2 * x^-1 * y^-1)^2 ,
```

```
x^-1 * y^-1 * x^5 * y * x^-4 ,
y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,t^2,(t,(x*y^-1*x^-1)^2),
(t,(y*x*y)^2),
((x^3)*t^(x))^a,
((x*y^-1*x*y^-2*x*y)*t^( x^2 * y^2 * x * y * x^-1))^b,
((x^3)*t^(y * x * y^-2 * x * y * x))^c,
((y^-1)*t^(x^-2 * y^-2))^d,
((x^3)*t^(y^-1 * x^-1 * y^-1 * x^-3))^e,
((y)*t^(y * x * y^2 * x * y * x * y))^f,
((x^3)*t^(x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1))^g,
((x*y^-1*x*y^-2*x*y)*t^(y * x^2 * y^-1 * x))^h,
((x^3)*t^(y * x))^i,
(((x*y^-1*x^-1)^2)*t^(y^2))^j,
((y^-1)*t^(x * y * x^2 * y^-1 * x^-2))^k,
((x)*t^(y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y))^l,
(((x*y^-1*x^-1)^2)*t^(y^-1 * x * y * x^-2 * y * x))^m,
((x*y^-1*x*y^-2*x*y)*t^(x^-3 * y^-1 * x^-1 * y^2))^n>;
#G;
/*604800*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*1*/
#sub<G|x,y>;
/*1920*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*6*/
CompositionFactors(G1);
/*
  G
    |  J2
    1
*/
```

Sixth Group

```
a:=3;b:=6;c:=0;d:=0;e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=0;l:=0;m:=0;n:=0;
G<x,y,t>:=Group<x,y,t|x^10   ,
y^6  ,
(x * y^-2 * x)^2 ,
(x * y^2 * x^2)^2 ,
(y^-1 * x^-1)^5 ,
(x * y^2 * x^-1 * y^-1)^2 ,
x^-1 * y^-1 * x^5 * y * x^-4 ,
y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,t^2,(t,(x*y^-1*x^-1)^2),
```

```
(t,(y*x*y)^2),
((x*y*x*y^2*x*y)*t^(x))^a,
((x)*t^( x^2 * y^2 * x * y * x^-1))^b,
((x*y*x*y^2*x*y)*t^(y * x * y^-2 * x * y * x))^c,
(((x^-1*y)^3)*t^(x^-2 * y^-2))^d,
((x*y*x*y^2*x*y)*t^(y^-1 * x^-1 * y^-1 * x^-3))^e,
((y)*t^(y * x * y^2 * x * y * x * y))^f,
((x*y*x*y^2*x*y)*t^(x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1))^g,
((y^2)*t^(y * x^2 * y^-1 * x))^h,
((x*y*x*y^2*x*y)*t^(y * x))^i,
((x*y^-2*x*y)*t^(y^2))^j,
((x^5)*t^(x * y * x^2 * y^-1 * x^-2))^k,
((x)*t^(y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y))^l,
((x*y^-2*x*y)*t^(y^-1 * x * y * x^-2 * y * x))^m,
(((x^-1*y)^3)*t^(x^-3 * y^-1 * x^-1 * y^2))^n>;
#G;
/*25920*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*1*/
#sub<G|x,y>;
/*960*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*3*/
CompositionFactors(G1);
/*
 G
    |  C(2, 3)                        = S(4, 3)
    1
*/
```

Seventh Group

```
a:=6;b:=2;c:=0;d:=0;e:=0;f:=0;g:=0;h:=0;i:=0;j:=0;k:=0;l:=0;m:=0;n:=0;
G<x,y,t>:=Group<x,y,t|x^10  ,
y^6  ,
(x * y^-2 * x)^2 ,
(x * y^2 * x^2)^2 ,
(y^-1 * x^-1)^5 ,
(x * y^2 * x^-1 * y^-1)^2 ,
x^-1 * y^-1 * x^5 * y * x^-4 ,
y * x^-2 * y^-1 * x^3 * y * x * y^3 * x^-1,t^2,(t,(x*y^-1*x^-1)^2),
(t,(y*x*y)^2),
((x^2)*t^(x))^a,
```

```
(((x^-1*y)^3)*t^( x^2 * y^2 * x * y * x^-1))^b,
((x^2)*t^(y * x * y^-2 * x * y * x))^c,
((x*y^-2*x*y)*t^(x^-2 * y^-2))^d,
((x^2)*t^(y^-1 * x^-1 * y^-1 * x^-3))^e,
((y)*t^(y * x * y^2 * x * y * x * y))^f,
((x^2)*t^(x^-1 * y * x * y * x^-1 * y^-1 * x^-1 * y^-1))^g,
(((x^-1*y)^3)*t^(y * x^2 * y^-1 * x))^h,
((x^2)*t^(y * x))^i,
((x*y^-2*x*y)*t^(y^2))^j,
((x)*t^(x * y * x^2 * y^-1 * x^-2))^k,
((x*y^-2*x*y)*t^(y^-1 * x^-1 * y^-1 * x * y^-1 * x^-1 * y))^l,
((y)*t^(y^-1 * x * y * x^-2 * y * x))^m,
(((x^-1*y)^3)*t^(x^-3 * y^-1 * x^-1 * y^2))^n>;
#G;
/*23040*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*1*/
#sub<G|x,y>;
/*960*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*6*/
CompositionFactors(G1);
/*
 G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
*/
```

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