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Symmetric Generation

Ana Gonzalez

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SYMMETRIC GENERATION

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Ana Gonzalez

May 2022

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ABSTRACT

We will examine progenitors. We start with progenitors of the form $m^{*n} : N$ where m^{*n} is a free group and N is a permutation group of degree n . But, $m^{*n} : N$ is a group of infinite order so we will factor by the necessary relations to get finite homomorphic images. These groups are constructed through the manual double coset enumeration method. We will examine how to construct progenitors for wreath products.

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Introduction

In group theory we look closely at the construction and symmetry of finite simple groups. In order to find these finite homomorphic images, we must start with progenitors such as $m^{*n} : N$ where N is transitive on n letters. Our groups are infinite therefore, to obtain finite images we must factor our progenitor by relations. Then we wish to obtain an isomorphism between our finite homomorphic image and our control group G . We use MAGMA extensively in order to accomplish this. In Chapter 1, we list our preliminaries. In Chapter 2, we show how to build a progenitor and find its first order relations. In Chapter 3, We build monomial progenitors with its first order relations. In Chapter 4, we inspect constructions of several groups and their finite homomorphic images by doing manual double coset enumeration for example, A_7 over $PSL(2, 7)$. In Chapter 5, we examine the construction of a maximal subgroup $M_{21} : 2^2$ over $PGL(2, 9) : 2$ and $PGL(2, 9)$. In Chapter 6, we obtain the wreath product and find its symmetric presentation along with building a progenitor and first order relations as shown in Chapter 2. Lastly, in Chapter 7 we find the isomorphism types of several groups. These can be either direct product, split, non-split extension, mixed or central extensions. For example, we will show the mixed extension of $(2^3 : \bullet S_4)$.

Chapter 1

Preliminaries

1.1 Definitions and Theorems

Definition 1.1.1. (Permutation) *If X is a nonempty set, a **permutation** is the bijective mapping $\alpha : X \rightarrow X$. [Rot95]*

Definition 1.1.2. (Disjoint) *Two permutations $\alpha, \beta \in S_X$ are **disjoint** if every x moved by one is fixed by the other. In symbols, if $\alpha(a) \neq a$, then $\beta(a) = a$, and if $\alpha(b) = b$, then $\beta(b) \neq b$. [Rot95]*

Theorem 1.1.3. *Every permutation $\alpha \in S_n$, is either a cycle or a product of disjoint cycles. [Rot95]*

Definition 1.1.4. (Semigroup) *A **semigroup** $(G, *)$ is a nonempty set G equipped with an associative operation $*$. [Rot95]*

Definition 1.1.5. (Symmetric Group) The *symmetric group*, denoted S_n is the set of all permutations of the nonempty set $X = \{1, 2, \dots, n\}$. S_n is a group of order $n!$ on n letters. [Rot95]

Definition 1.1.6. (Group). A *group* is a semigroup G containing an element e such that

(i) $e * a = a = a * e$ for all $a \in G$

(ii) for every $a \in G$, there is an element $b \in G$ with $a * b = e = b * a$. [Rot95]

Definition 1.1.7. (Order) If G is a group, then the *order* of G , denoted $|G|$, is the number of elements in G . [Rot95]

Definition 1.1.8. (Free Group) If X is a subset of a group F , then F is a *free group* with basis X if, for every group G and every function $f : X \rightarrow G$, there exists a unique homomorphism $\varphi : F \rightarrow G$ extending f . [Rot95]

Definition 1.1.9. (Presentation) Let X be a set and let Δ be a family of words on X . A group G has generators X and relations Δ if $G \cong F/R$, where F is the free group with basis X and R is the normal subgroup of F generated by Δ . The ordered pair $(X|\Delta)$ is called a *presentation* of G . [Rot95]

Definition 1.1.10. (Progenitor) A *progenitor* is a semi-direct product of the following form: $P \cong 2^{*n} : N = \{\pi w \mid \pi \in N, \text{ and } w \text{ is a word in the } t_i\}$, where 2^{*n}

denotes a free product of n copies of a cyclic group of order 2 generated by involutions t_i for $i=1, \dots, n$; and N is a transitive permutation group of degree n which acts on the free product by permuting the involutory generators. [Curt96]

Lemma 1.1.11. (Factoring Lemma) (Known as the Grindstaff Lemma) Factoring the progenitor $m^{*n}:N$ by (t_i, t_j) for $1 \leq i < j \leq n$ gives the group $m^n:N$. [Grind15]

Definition 1.1.12. (Character) Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m . We consider the characteristic polynomial of $A(x)$, namely

$$\det(\lambda I - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & \lambda - a_{12}(x) & \dots & \lambda - a_{1m}(x) \\ \lambda - a_{21}(x) & \lambda - a_{22}(x) & \dots & \lambda - a_{2m}(x) \\ \dots & \dots & \dots & \dots \\ \lambda - a_{m1}(x) & \lambda - a_{m2}(x) & \dots & \lambda - a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree m in λ , with the coefficient of $-\lambda^{m-1}$ is $\varphi(x) = a_{11}(x) + a_{22}(x) + \dots + a_{mm}(x)$.

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated to $\text{tr}A(x)$, so that $\varphi(x) = \text{tr}A(x)$.

We regard $\varphi(x)$ as a function on G with values in field K , and we call it the **character** of $A(x)$. [Led77]

Theorem 1.1.13. The number of irreducible characters of G is equal to the number of conjugacy classes of G . [Led77]

Definition 1.1.14. (Degree of a Character) *The sum of squares of the degrees of the distinct irreducible characters of G is equal to $|G|$. The **degree of a character** χ is $\chi(1)$. Note that a character whose degree is 1 is called a **linear character**. [Led77]*

Definition 1.1.15. (Lifting Process) *Let N be a normal subgroup of G and suppose that $A_0(Nx)$ is a representation of degree m of the group G/N . Then $A(x) = A_0(Nx)$ defines a representation of G/N lifted from G/N . If $\varphi_0(Nx)$ is a character of $A_0(Nx)$, then $\varphi(x) = \varphi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = Im, \varphi(u) = m = \varphi(1)$. The lifting process preserves irreducibility. [Led77]*

Definition 1.1.16. (Induced Character) *The character of $A(x)$, which is called the **induced character** of ϕ , will be denoted by ϕ^G . Thus, $\phi^G = \text{tr}A(x) = \sum_{i=1}^n \phi(t_i x t_i^{-1})$. [Led77]*

Definition 1.1.17. (Formula for Induced Character)

$$\varphi_\alpha^G(x) = \frac{n}{h_\alpha} \sum_{\omega \in C_\alpha \cap H} \varphi(\omega), \alpha = 1, 2, 3, \dots, m$$

Definition 1.1.18. (abelian) *A pair of elements a and b in a group **commutes** if $a*b=b*a$. A group is **abelian** if every pair of its elements commutes. [Rot95]*

Definition 1.1.19. (homomorphism) *Let $(G,*)$ and (H,\circ) be groups. A function $f:G \leftrightarrow H$ is a **homomorphism** if, for all $a, b \in G$, $f(a*b) = f(a) \circ f(b)$. [Rot95]*

Definition 1.1.20. (isomorphism) An *isomorphism* is a homomorphism that is also a bijection. We say that G is *isomorphic* to H , denoted $G \cong H$, if there exists an isomorphism $f: G \leftrightarrow H$. [Rot95]

Theorem 1.1.21. Let p be a prime. A group G of order p^n is cyclic if and only if it is an abelian group having a unique subgroup of order p . [Rot95]

Definition 1.1.22. (normal subgroup) A subgroup $K \leq G$ is a *normal subgroup*, denoted by $K \trianglelefteq G$, if $gKg^{-1} = K$ for every $g \in G$. [Rot95]

Theorem 1.1.23. (First Isomorphism Theorem) Let $f: G \rightarrow H$ be a homomorphism with kernel K . Then K is a normal subgroup of G and $G/K \cong \text{im}(f)$. [Rot95]

Theorem 1.1.24. (Second Isomorphism Theorem) Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$. [Rot95]

Theorem 1.1.25. (Third Isomorphism Theorem) Let $K \leq H \leq G$, where both K and H are normal subgroups of G . Then H/K is a normal subgroup of G/K and $(G/K)(H/K) \cong G/H$. [Rot95]

Theorem 1.1.26. (Correspondence Theorem) Let $K \trianglelefteq G$ and let $\pi: G \rightarrow G/K$ be the natural map. Then $S \mapsto \pi(S) = S/K$ is a bi-jection from the family of all

those subgroups S of G which contain K to the family of all the subgroups of G/K . Moreover, if we denote S/K by S^* , then: (i) $T \leq S$ if and only if $T^* \leq S^*$, and then $[S:T] = [S^*:T^*]$; and (ii) $T \trianglelefteq S$ if and only if $T^* \trianglelefteq S^*$, and then $S/T \cong S^*/T^*$. [Rot95]

Definition 1.1.27. (maximal normal subgroup) A subgroup $H \leq G$ is a **maximal normal subgroup** of G if there is no normal subgroup N of G with $H < N < G$. [Rot95]

Definition 1.1.28. (simple) A group $G \neq 1$ is **simple** if it has no normal subgroups other than G and itself. [Rot95]

Definition 1.1.29. (direct product) If H and K are groups, then their **direct product**, denoted by $H \times K$, is the group with elements all ordered pairs (h, k) , where $h \in H$ and $k \in K$, and with the operation $(h, k)(h', k') = (hh', kk')$. [Rot95]

Theorem 1.1.30. (Jordan-Holder Theorem) Every two composition series of a group G are equivalent.

Suppose that the finite group G has two composition series

$$G = B_0 > B_1 > \dots > B_n = \{1\} \text{ and } G = C_0 > C_1 > \dots > C_m = \{1\}.$$

Then $n = m$ and the lists of composition factors for the two series are identical in the sense that if $|H| \leq |G|$ and $\Phi(H) = \{i \geq 1 : B_{i-1}/B_i \cong H\}$ and $\Psi(H) = \{i \geq 1 : C_{i-1}/C_i \cong H\}$ then $\Phi(H) = \Psi(H)$. [Rot95]

Definition 1.1.31. (semi-direct product) A group G is a *semi-direct product* of K by Q , denoted by $G=K \rtimes Q$, if $K \triangleleft G$ and K has a complement $Q_1 \cong Q$. One also says that G splits over K . [Rot95]

Definition 1.1.32. (mixed-extension) If G is an extension of an abelian group not equal to the center of G , then this is called a *mixed extension*. [Rot95]

Definition 1.1.33.(normal subgroup in composition series) A normal subgroup N of a group G is called a *maximal normal subgroup* of G if

(a) $N \neq G$

(b) whenever $N \leq M \triangleleft G$ then either $M = N$ or $M = G$.

By the Correspondence Theorem, if $N \triangleleft G$ and $N \neq G$ then every normal subgroup of G/N corresponds to a normal subgroup of G containing N . So a normal subgroup N is maximal if and only if G/N is simple.

Definition 1.1.34. (Composition series) Given a group G , a *composition series* for G of length n is a sequence of subgroups $G=B_0 > B_1 > \dots > B_n =1_G$ such that

(i) $B_i \triangleleft B_{i-1}$ for $i= 1, \dots, n$.

(ii) B_{i-1}/B_i is simple for $i=1, \dots, n$. In particular, B_i is a maximal normal subgroup of G and B_{i-1} is simple. The (isomorphism classes of the) quotient groups B_i/B_{i-1} are called *composition factors* of G .

Example 1.1.35.

S_4 has the following composition series of length 4, where K is the Klein group $\{(1), (12)(34), (13)(24), (14)(23)\}$.

$$S_4 > A_4 > K > \{(12)(34)\} > \{1\}$$

We know that $A_4 \triangleleft S_4$; the composition factor $S_4/A_4 \cong C_2$.

We have seen that $K \triangleleft A_4$; and $A_4/K \cong C_3$

All subgroups of K are normal in K , because K is abelian.

Both $K/\{(12)(34)\}$ and $\{(12)(34)\}/\{1\}$ are isomorphic to C_2 .

So the composition factors of S_4 are C_2 (three times) and C_3 (once).

Example 1.1.36.

If G is simple then its only composition series is $G > \{1\}$, of length 1.

Example 1.1.37.

$(\mathbb{Z}, +)$ has no composition series. If $H \leq \mathbb{Z}$ then H is cyclic of infinite order.

If $H = \langle x \rangle$ then $\langle 2x \rangle$ is a subgroup of H with $\{0\} \neq \langle 2x \rangle \neq H$, and $\langle 2x \rangle \triangleleft H$ because H is abelian. So H is not simple. If $B_0 > B_1 > \dots > B_n$ is a composition series then B_{n-1} is simple, so there can be no composition series.

Theorem 1.1.38: Every finite group G has a composition series.

Example 1.1.39. (Composition Series for Q_8 and D_8)

Q_8 :

All three possible composition series for Q_8 are valid and are the following:

$$1 \leq \langle -1 \rangle \leq \langle i \rangle \leq Q_8$$

$$1 \leq \langle -1 \rangle \leq \langle j \rangle \leq Q_8$$

$$1 \leq \langle -1 \rangle \leq \langle k \rangle \leq Q_8$$

The respective composition factors being $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_2$.

D_8 :

There are 7 composition series for D_8 listed below:

$$1 \leq \langle s \rangle \leq \langle s, r^2 \rangle \leq D_8$$

$$1 \leq \langle r^2 s \rangle \leq \langle s, r^2 \rangle \leq D_8$$

$$1 \leq \langle r^2 \rangle \leq \langle s, r^2 \rangle \leq D_8$$

$$1 \leq \langle r^2 \rangle \leq \langle r \rangle \leq D_8$$

$$1 \leq \langle r^2 \rangle \leq \langle rs, r^2 \rangle \leq D_8$$

$$1 \leq \langle rs \rangle \leq \langle rs, r^2 \rangle \leq D_8$$

$$1 \leq \langle r^3 s \rangle \leq \langle rs, r^2 \rangle \leq D_8$$

Although $D_8 \not\cong Q_8$, they have the same composition factors.

Definition 1.1.40. (right coset) If S is a subgroup of G and if $t \in G$, then a **right coset** of $S \in G$ is the subset of G : $St = \{st : s \in S\}$ (a **left coset** is $tS = \{ts : s \in S\}$). One calls t a **representative** of St (and also tS). [Rot95]

Theorem 1.1.41. If $S \leq G$, then any two right (or any two left) cosets of S in G are either identical or disjoint. [Rot95]

Theorem 1.1.41. If $S \leq G$, then the number right cosets of S in G is equal to the number of left cosets of S in G . [Rot95]

Definition 1.1.42. (index) If $S \leq G$, then the **index** of S in G , denoted $[G:S]$, is the number of right cosets of S in G . [Rot95]

Definition 1.1.43. (conjugate) If $x \in G$, then a **conjugate** of x in G is an element of the form axa^{-1} for some $a \in G$. [Rot95]

Definition 1.1.44. (double coset) If S and T are subgroups of G , then a **double coset** is a subset of G of the form SgT , where $g \in G$. [Rot95]

Definition 1.1.45. (G-set) If X is a set and G is a group, then X is a **G-set** if there is a function $\alpha: G \times X \rightarrow X$ (called an **action**), denoted by $\alpha: (g, x) \mapsto gx$, such that:

- (i) $1x = x$ for all $x \in X$; and
- (ii) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$. [Rot95]

Definition 1.1.46. (acts) G **acts** on X , if $|X| = n$, then n is called the **degree** of the G -set X . [Rot95]

Definition 1.1.47. (G-orbit) If X is a G -set and $x \in X$, then the **G-orbit** of x is $\vartheta(x) = \{gx : g \in G\} \subset X$, ($\vartheta(x)$ denoted Gx). [Rot95]

Definition 1.1.48. (stabilizer) If X is a G -set and $x \in X$, then the **stabilizer** of x , denoted by G_x , is the subgroup $G_x = \{g \in G : gx = x\} \leq G$. [Rot95]

Theorem 1.1.49. *If X is a G -set and $x \in X$, then $|\vartheta(x)| = [G : G_x]$. [Rot95]*

Corollary 1.1.50. *If a finite group G acts on a set X , then the number of elements in any orbit is a divisor of $|G|$. [Rot95]*

Corollary 1.1.51. *(i) If G is a finite group and $x \in G$, then the number of conjugates of $x \in G$ is $[G : C_G(x)]$ (C_G , is **centralizer**). (ii) If G is a finite group and $H \leq G$, then the number of conjugates of $H \in G$ is $[G : N_G(H)]$ (N_G , is **normalizer**). [Rot95]*

Definition 1.1.52. **(transitive)** *A G -set X is **transitive** if it has only one orbit; that is for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$. [Rot95]*

Definition 1.1.53. **(Double Coset Algorithm)** *Perform the double coset enumeration of group G over transitive group N , where double cosets take the form $NwN = \{Nwn \mid n \in N\} = \{Nw^n \mid n \in N\}$.*

(i) Compute the point-stabilizer N^w and coset stabilizer of each double coset.

(ii) Compute the number of right cosets by using the formula $\frac{|N|}{|N^{(w)}|}$, where $N^{(w)} = \{n \in N \mid Nw^n = Nw\}$ is the coset stabilizer of the right coset.

(iii) For each double coset NwN , compute the orbits of $N^{(w)}$. It suffices to determine the double coset of Nwt_i for a single representative of each orbit. Note, $N^{(w)} \geq N^w$ is always true.

(iv) Determine which double coset each coset representative Nwt_i belongs to, (repeat the process until closed by coset multiplication).

Definition 1.1.54. (Wreath Product) *Let X and Y be non-empty sets. Let H be a permutation group on X and K on Y . Let $Z = X \times Y$, then the **wreath product** is a semi-direct product of X and Y . We define a permutation group on Z such that we let $\gamma \in H$, and define a permutation of $\gamma(y)$ of Z by*

$$\gamma(y) = \begin{cases} (x, y) \mapsto (x\gamma, y), & \text{if } y_1 = y \\ (x, y_1) \mapsto (x, y_1), & y \neq y_1 \end{cases}$$

Also, for $k \in K$, define $k^*(x, y) = (x, (y)k)$; such that $B = \prod_{y \in Y} H(y)$ is a direct product of the group generated by the y 's. Thus, $G = B:k^*(x, y)$ is called a wreath product of H and K , where H is normal subgroup, denoted by $H \wr K$. [Rot95]

Chapter 2

Writing Progenitors

2.1 Permutation Progenitor of $2^{*6} : (2^2 : (3x2))$

First we will write a symmetric presentation for $G = 2^{*6} : (2^2 : (3x2))$. We begin with a transitive group on 6 letters and examine its composition factors to find the isomorphism type of our control group N .

```
>N:=TransitiveGroup(6,7);
```

```
>N;
```

```
Permutation group N acting on a set of cardinality 6
```

```
Order = 24 = 2^3 * 3
```

```
(1, 4)(2, 5)
```

```
(1, 3, 5)(2, 4, 6)
```

```
(1, 5)(2, 4)
```

```
> CompositionFactors(N);
```



```

G
| Cyclic(2)
*
| Cyclic(3)
*
| Cyclic(2)
*
| Cyclic(2)
1

```

A presentation for N is $2^2 : (3x2)$

```

>S:=Sym(6);
>xx:=S!(1, 4)(2, 5);
>yy:=S!(1, 3, 5)(2, 4, 6);
>zz:=S!(1, 5)(2, 4);
>N:=sub<S|xx,yy,zz>;
>#N;

```

24

```

>FPGroup(N);

```

Finitely presented group on 3 generators

Relations

$$$.1^2 = \text{Id}(\$)$$

$$$.2^3 = \text{Id}(\$)$$

$$$.3^2 = \text{Id}(\$)$$

$$($.2^{-1} * $.3)^2 = \text{Id}(\$)$$

```

($ .1 * $ .3)^2 = Id($)
($ .1 * $ .2^-1)^3 = Id($)

>NN<a,b,c>:=Group<a,b,c|a^2,b^3,c^2,(b^-1*c)^2,(a*c)^2,(a*b^-1)^3>;
>#NN;
24

```

```

>f, G1, k :=CosetAction(NN, sub<NN|Id(NN)>);
>IsIsomorphic(N, G1);
true Mapping from: GrpPerm: N to GrpPerm: G1
Composition of Mapping from: GrpPerm: N to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G1

```

We see here that the presentation is isomorphic to N which is generated by $x \sim (1, 4)(2, 5)$, $y \sim (1, 3, 5)(2, 4, 6)$, and $z \sim (1, 5)(2, 4)$. Next, we add t^2 to the presentation since $|t_i| = 2$, which gives the free product 2^{*6} . Therefore our progenitor becomes

$$G \langle x, y, z, t \rangle := \text{Group} \langle x, y, z, t | x^2, y^3, z^2, (y^{-1} * z)^2, (x * z)^2, (x * y^{-1})^3, t^2, (t, N^1) \rangle;$$

where $t \sim t_1$ and N^1 is the point stabilizer defined by $N^1 = \{n \in N | 1^n = 1\}$, where n are elements of N that fix 1. Now the point stabilizers are $\langle (2, 3)(5, 6), (2, 6)(3, 5) \rangle$, which we convert to words using a program that uses the Schreier System of G .

```

> Sch:=SchreierSystem(NN, sub<NN|Id(NN)>);

```

```

> ArrayP:=Id(N): i in [1..#N]];
> for i in [2..#N] do
for> P:=Id(N): I in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy^1; end if;
for|for> if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz^1; end if;
for|for> if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
>
> for i in [1..#Sch] do if ArrayP[i] eq N!(2,3)(5,6)
then print Sch[i]; end if; end for;
a * c * b * a

>for i in [1..#Sch] do if ArrayP[i] eq N!(2, 6)(3, 5) then Sch[i];
end if; end for;
c*b

```

So now, we can re-write $acba$ as $xyzx$. Hence, the progenitor becomes $Group \langle x, y, z, t | x^2, y^3, z^2, (y^{-1} * z)^2, (x * z)^2, (x * y^{-1})^3, t^2, (t, x * z * y * x), (t, z * y) \rangle$. Recall, this progenitor is infinite, therefore, we must factor by first order relations in order to see the finite homomorphic images of $2^{*6} : (2^2 : (3x2))$.

2.1.1 First Order Relations of $2^{*6} : (2^2 : (3x2))$

The first order relations we will examine are of the form $(\pi t_i^a)^b$, where π is a permutation in N . In order to find first order relations, we examine the conjugacy classes of N given in the table below.

Table 2.1: Conjugacy Classes of N

Class Number	Order	Class Representative	Length
1	1	Id(N)	1
2	2	(2, 5)(3, 6)	3
3	2	(1, 5)(2, 4)	6
4	3	(1, 3, 5)(2, 4, 6)	8
5	4	(1, 4)(2, 6, 5, 3)	6

From above, we find the centralizers of each class representative and the corresponding orbits on $\{1, 2, 3, 4, 5, 6\}$.

Class 2 has centralizer

$$\langle (2, 5)(3, 6) \rangle,$$

with orbits $\{1, 4\}$, and $\{2, 3, 5, 6\}$.

Class 3 has centralizer

$$\langle (1, 5)(2, 4) \rangle,$$

with orbits $\{1, 2, 4, 5\}$, $\{3\}$, and $\{6\}$.

Class 4 has centralizer

$$\langle (1, 3, 5)(2, 4, 6) \rangle,$$

with orbits $\{1, 3, 5\}$, and $\{2, 4, 6\}$.

Class 5 has centralizer

$$\langle (1, 4)(2, 6, 5, 3) \rangle,$$

with orbits $\{1, 4\}$, and $\{2, 3, 5, 6\}$.

With this we are given the first order relations of N which we convert to words.

$$(2, 5)(3, 6)t = (yxy^{-1})t$$

$$(2, 5)(3, 6)t = (yxy^{-1})t^{xz}$$

$$(1, 5)(2, 4)t = (z)t$$

$$(1, 5)(2, 4)t = (z)t^y$$

$$(1, 5)(2, 4)t = (z)t^{xy}$$

$$(1, 3, 5)(2, 4, 6)t = (y)t$$

$$(1, 3, 5)(2, 4, 6)t = (y)t^{xz}$$

$$(1, 4)(2, 6, 5, 3)t = (zyx)t$$

$$(1, 4)(2, 6, 5, 3)t = (zyx)t^{xz}$$

Adding these relations gives us our final representation of $G = 2^{*6} : (2^2 : (3 \times 2))$,

Group $\langle x, y, z, t \mid x^2, y^3, z^2, (y^{-1}z)^2, (xz)^2, (xy^{-1})^3, t^2,$

$(t, xzyx), (t, zy), (yxy^{-1}t)^a, (yxy^{-1}t^{xz})^b, (zt)^c, (zt^y)^d, (zt^{xy})^e, (yt)^f, (yt(xz))^g, (zyxt)^h,$

$(zyxt^{xz})^i \rangle;$

The following table gives the finite images of this progenitor $2^{*6} : (2^2 : (3 \times 2))$ when these first order relations are added.

Table 2.2: Some Finite Images of the Progenitor $2^{*6} : (2^2 : (3 \times 2))$

a	b	c	d	e	f	g	h	i	j	Index of G
0	0	0	0	0	0	0	2	5	0	20
0	0	0	0	0	0	0	4	4	0	64
0	0	0	0	0	0	0	8	4	0	512
0	0	0	0	0	0	4	6	0	0	216
0	0	0	0	0	0	4	10	0	0	1000
0	0	0	0	0	0	5	4	5	0	160
0	0	0	0	0	0	5	8	5	0	10240
0	0	0	0	0	0	6	2	8	0	32
0	0	0	0	0	0	6	2	10	0	50
0	0	0	0	0	0	6	4	6	0	73728
0	0	0	0	0	0	7	2	8	0	56
0	0	0	0	0	0	7	2	9	0	84
0	0	0	0	0	0	8	2	8	0	112
0	0	0	0	0	0	8	2	10	0	720
0	0	0	0	0	0	9	2	9	0	570
0	0	0	0	0	0	9	2	10	0	3420
0	0	0	0	0	4	6	0	6	0	54
0	0	0	0	0	4	8	0	0	0	128
0	0	0	0	0	4	10	0	0	0	250
0	0	0	0	0	5	6	0	0	0	650
0	0	0	0	0	6	6	4	6	0	1152
0	0	0	0	0	6	8	6	6	0	3456

2.2 M_{12} Permutation Progenitor of $2^{*96} : (2^3 : S_4)$

A symmetric presentation for $G \cong (2^3 : S_4)$. We start with a group on 96 letters and examine the FPGROUP to find a presentation of our group N .

```

>S:=Sym(96);
>xx:=S!(1, 86, 25, 96)(2, 51, 40, 54)(3, 73, 60, 77)(4, 78, 67, 70)
(5, 85, 7, 88)(6,87, 28, 64)(8, 17, 72, 58)(9, 90, 82, 55)(10, 14,
12, 92)(11, 89, 39, 71)(13, 22, 76, 69)(15, 75, 65, 80)(16, 41, 19,
79)(18, 74, 68, 35)(20, 33, 57, 84)(21, 48, 23, 47)(24, 95, 29, 45)
(26, 31, 62, 34)(27, 46, 66, 49)(30, 59, 32, 56)(36, 53, 38, 50)(37,
91, 81, 52)(42, 61, 44, 94)(43, 93, 83, 63);
>yy:=S!(1, 3, 14)(2, 8, 13)(4, 20, 51)(5, 15, 55)(6, 26, 71)(7, 19,
54)(9, 35, 46)(10,30, 64)(11, 41, 70)(12, 34, 49)(16, 50, 21)(17,
61, 29)(18, 63, 24)(22, 68, 89)(23, 65, 92)(25, 57, 53)(27, 59, 94)
(28, 56, 52)(31, 45, 36)(32, 76, 38)(33, 78, 43)(37, 80, 47)(39, 75,
96)(40, 74, 95)(42, 84, 85)(44, 77, 48)(58, 91, 69)(60, 90, 67)(62,
93, 66)(72, 87, 82)(73, 86, 83)(79, 88, 81);
>N:=sub<S|xx,yy>;
>#N;
192

```

To find the presentation of N:

```

>FPGGroup(N);
Finitely presented group on 2 generators
Relations
$.1^4 = Id($)
$.2^3 = Id($)
($.1 * $.2^-1 * $.1^-1 * $.2^-1)^2 = Id($)
$.1^-1 * $.2 * $.1^-2 * $.2^-1 * $.1^-1 * $.2 * $.1^2 * $.2^-1 = Id($)

```

Then, we convert this into elements of a and b to ensure we have the same group.

```

>NN<a,b>:=Group<a,b|
    a^4  ,
    b^3  ,
    (a * b^-1 * a^-1 * b^-1)^2,
    a^-1 * b * a^-2 * b^-1 * a^-1 * b * a^2 * b^-1 >;
>#NN;
192
>f, G1, k :=CosetAction(NN, sub<NN|Id(NN)>);
>w:=IsIsomorphic(N, G1);
>w;
true

```

We see here that the presentation is isomorphic to N . Next, we add t^2 to the presentation since $|t_i| = 2$, which gives the free product 2^{*96} . Therefore our progenitor becomes

$$G \langle x, y, t | x^4, y^3, (xy^{-1}x^{-1}y^{-1})^2, x^{-1}yx^{-2}y^{-1}x^{-1}yx^2y^{-1}, t^2, (t, N^1) \rangle;$$

where $t \sim t_1$ and N^1 is the point stabilizer defined by $N^1 = \{n \in N | 1^n = 1\}$. Now, the point stabilizer is $\langle (2, 42)(3, 62)(5, 25)(6, 29)(8, 74)(9, 83)(10, 44)(11, 40)(12, 39)(13, 70)(14, 93)(15, 68)(16, 26)(17, 60)(18, 20)(19, 58)(21, 67)(22, 27)(30, 34)(31, 35)(32, 72)(33, 73)(36, 43)(37, 82)(38, 81)(41, 84)(45, 86)(46, 78)(47, 76)(48, 49)(50, 61)(51, 63)(52, 92)(53, 89)(54, 91)(55, 94)(56, 65)(57, 59)(64, 96)(71, 90)(75, 77)(79, 80)(85, 95)(87, 88) \rangle$, which we convert to words using a program that uses the Schreier System.


```

>word:=function(A)
>Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
>for i in [2..#NN] do
>P:=[Id(N): 1 in [1..#Sch[i]]];
>for j in [1..#Sch[i]] do
>if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
>if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
>if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
>if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
>end for;
>PP:=Id(N);
>for k in [1..#P] do
>PP:=PP*P[k]; end for;
>if A eq PP then B:=Sch[i]; end if;
>end for;
>return B;
>end function;

>A:=N!(2, 42)(3, 62)(5, 25)(6, 29)(8, 74)(9, 83)(10, 44)(11, 40)
(12, 39)(13, 70)(14, 93)(15, 68)(16, 26)(17, 60)(18, 20)(19, 58)
(21, 67)(22, 27)(30, 34)(31, 35)(32, 72)(33, 73)(36, 43)(37, 82)
(38, 81)(41, 84)(45, 86)(46, 78)(47, 76)(48, 49)(50, 61)(51, 63)
(52, 92)(53, 89)(54, 91)(55, 94)(56, 65)(57,59)(64, 96)(71, 90)
(75, 77)(79, 80)(85, 95)(87, 88);

```

>word(A);

a*b^-1*a^-1*b^-1

So now, we can re-write $(a*b^{-1}*a^{-1}*b^{-1})$ as $(xy^{-1}x^{-1}y^{-1})$. Hence, the progenitor becomes

$G < x, y, t | x^4, y^3, (xy^{-1}x^{-1}y^{-1})^2, x^{-1}yx^{-2}y^{-1}x^{-1}yx^2y^{-1}, t^2, (t, xy^{-1}x^{-1}y^{-1}) >$;

Recall, this progenitor is infinite, therefore, we must factor by first order relations in order to see the finite homomorphic images of $2^{*96} : (2^3 : S_4)$.

2.2.1 First Order Relations of $2^{*96} : (2^3 : S_4)$

The first order relations we will examine are of the form $(\pi t_i^a)^b$, where π is a permutation in N . In order to find first order relations, we examine the conjugacy classes of $N \cong 2^3 : S_4$. We find the centralizers of each class representative and the corresponding orbits on 96 letters.

Class 2 has centralizer $< (1, 7)(2, 12)(3, 19)(4, 23)(5, 25)(6, 29)(8, 34)(9, 38)(10, 40)(11, 44)(13, 49)(14, 54)(15, 57)(16, 60)(17, 26)(18, 56)(20, 65)(21, 67)(22, 27)(24, 28)(30, 74)(31, 72)(32, 35)(33, 80)(36, 82)(37, 43)(39, 42)(41, 77)(45, 87)(46, 76)(47, 78)(48, 70)(50, 90)(51, 92)(52, 63)(53, 55)(58, 62)(59, 68)(61, 71)(64, 95)(66, 69)(73, 79)(75, 84)(81, 83)(85, 96)(86, 88)(89, 94)(91, 93) >$.

Class 3 has centralizer $< (1, 25)(2, 40)(3, 60)(4, 67)(5, 7)(6, 28)(8, 72)(9, 82)(10, 12)(11, 39)(13, 76)(14, 92)(15, 65)(16, 19)(17, 58)(18, 68)(20, 57)(21, 23)(22, 69)(24, 29)(26, 62)(27, 66)(30, 32)(31, 34)(33, 84)(35, 74)(36, 38)(37, 81)(41, 79)(42, 44)(43, 83)(45, 95)(46, 49)(47, 48)(50, 53)(51, 54)(52, 91)(55, 90)(56, 59)(61, 94)(63, 93)(64, 87)(70, 78)(71, 89)(73, 77)(75, 80)(85, 88)(86, 96) >$.

Class 4 has centralizer $\langle (1, 66)(2, 11)(4, 24)(5, 27)(6, 67)(7, 69)(8, 84)(9, 43)(10, 39)(12, 44)(13, 95)(14, 93)(15, 57)(16, 60)(17, 26)(21, 29)(22, 25)(23, 28)(30, 77)(31, 33)(32, 79)(34, 75)(35, 73)(36, 83)(37, 38)(40, 42)(41, 74)(45, 46)(47, 88)(48, 96)(49, 64)(50, 71)(51, 63)(52, 92)(53, 94)(54, 91)(55, 89)(59, 68)(61, 90)(70, 85)(72, 80)(76, 87)(78, 86)(81, 82) \rangle$.

Class 5 has centralizer $\langle (1, 42)(2, 22)(3, 87)(4, 44)(5, 81)(6, 10)(7, 39)(8, 52)(9, 69)(11, 23)(12, 27)(13, 16)(14, 80)(15, 64)(17, 88)(18, 96)(19, 45)(20, 76)(21, 37)(24, 82)(25, 83)(26, 86)(28, 36)(29, 40)(30, 91)(31, 94)(32, 71)(33, 54)(34, 63)(35, 61)(38, 66)(41, 55)(43, 67)(46, 65)(47, 59)(48, 58)(49, 60)(50, 75)(51, 79)(53, 77)(56, 85)(57, 95)(62, 70)(68, 78)(72, 89)(73, 92)(74, 93)(84, 90) \rangle$.

Class 6 has centralizer $\langle (1, 3, 14)(2, 8, 13)(4, 20, 51)(5, 15, 55)(6, 26, 71)(7, 19, 54)(9, 35, 46)(10, 30, 64)(11, 41, 70)(12, 34, 49)(16, 50, 21)(17, 61, 29)(18, 63, 24)(22, 68, 89)(23, 65, 92)(25, 57, 53)(27, 59, 94)(28, 56, 52)(31, 45, 36)(32, 76, 38)(33, 78, 43)(37, 80, 47)(39, 75, 96)(40, 74, 95)(42, 84, 85)(44, 77, 48)(58, 91, 69)(60, 90, 67)(62, 93, 66)(72, 87, 82)(73, 86, 83)(79, 88, 81) \rangle$.

Class 7 has centralizer $\langle (1, 24, 7, 28)(2, 43, 12, 37)(3, 57, 19, 15)(4, 66, 23, 69)(5, 29, 25, 6)(8, 77, 34, 41)(9, 11, 38, 44)(10, 83, 40, 81)(13, 46, 49, 76)(14, 71, 54, 61)(16, 20, 60, 65)(17, 56, 26, 18)(21, 27, 67, 22)(30, 84, 74, 75)(31, 73, 72, 79)(32, 80, 35, 33)(36, 39, 82, 42)(45, 95, 87, 64)(47, 70, 78, 48)(50, 93, 90, 91)(51, 89, 92, 94)(52, 55, 63, 53)(58, 68, 62, 59)(85, 88, 96, 86) \rangle$.

Class 8 has centralizer $\langle (1, 67, 7, 21)(2, 43, 12, 37)(3, 18, 19, 56)(4, 25, 23, 5)(6, 69, 29, 66)(8, 31, 34, 72)(9, 44, 38, 11)(10, 83, 40, 81)(13, 96, 49, 85)(14, 94, 54, 89)(15, 26, 57, 17)(16, 68, 60, 59)(20, 62, 65, 58)(22, 28, 27, 24)(30, 35, 74, 32)(33, 75, 80, 84)(36, 42, 82, 39)(41, 79, 77, 73)(45, 78, 87, 47)(46, 86, 76, 88)(48, 64, 70, 95)(50, 63, 90, 52)(51, 61, 92, 71)(53, 91, 55, 93) \rangle$.

Class 9 has centralizer $\langle (1, 86, 25, 96)(2, 51, 40, 54)(3, 73, 60, 77)(4, 78, 67, 70)(5, 85, 7, 88)(6, 87, 28, 64)(8, 17, 72, 58)(9, 90, 82, 55)(10, 14, 12, 92)(11, 89, 39, 71)(13, 22, 76, 69)(15, 75, 65, 80)(16, 41, 19, 79)(18, 74, 68, 35)(20, 33, 57, 84)(21, 48, 23, 47)(24, 95, 29, 45)(26, 31, 62, 34)(27, 46, 66, 49)(30, 59, 32, 56)(36, 53, 38, 50)(37, 91, 81, 52)(42, 61, 44, 94)(43, 93, 83, 63) \rangle$.

Class 10 has centralizer $\langle (1, 72, 4, 32)(2, 60, 9, 65)(3, 40, 15, 82)(5, 74, 21, 34)(6, 79, 27, 33)(7, 31, 23, 35)(8, 25, 30, 67)(10, 57, 36, 19)(11, 62, 37, 68)(12, 16, 38, 20)(13, 92, 45, 53)(14, 76, 50, 95)(17, 42, 56, 83)(18, 81, 26, 39)(22, 80, 29, 73)(24, 75, 69, 77)(28, 84, 66, 41)(43, 59, 44, 58)(46, 90, 64, 54)(47, 93, 85, 61)(48, 94, 88, 63)(49, 51, 87, 55)(52, 70, 89, 86)(71, 78, 91, 96) \rangle$.

Class 11 has centralizer $\langle (1, 57, 51, 7, 15, 92)(2, 74, 46, 12, 30, 76)(3, 90, 5, 19, 50, 25)(4, 60, 14, 23, 16, 54)(6, 56, 94, 29, 18, 89)(8, 87, 10, 34, 45, 40)(9, 72, 13, 38, 31, 49)(11, 75, 47, 44, 84, 78)(17, 91, 28, 26, 93, 24)(20, 53, 21, 65, 55, 67)(22, 58, 61, 27, 62, 71)(32, 64, 82, 35, 95, 36)(33, 85, 81, 80, 96, 83)(37, 79, 70, 43, 73, 48)(39, 41, 88, 42, 77, 86)(52, 69, 68, 63, 66, 59) \rangle$.

Class 12 has centralizer $\langle (1, 83, 24, 40, 7, 81, 28, 10)(2, 4, 43, 66, 12, 23, 37, 69)(3, 86, 57, 85, 19, 88, 15, 96)(5, 42, 29, 36, 25, 39, 6, 82)(8, 61, 77, 14, 34, 71, 41, 54)(9, 67, 11, 22, 38, 21, 44, 27)(13, 68, 46, 62, 49, 59, 76, 58)(16, 70, 20, 78, 60, 48, 65, 47)(17, 87, 56, 64, 26, 45, 18, 95)(30, 94, 84, 51, 74, 89, 75, 92)(31, 93, 73, 90, 72, 91, 79, 50)(32, 52, 80, 55, 35, 63, 33, 53) \rangle$.

Class 13 has centralizer $\langle (1, 73, 67, 41, 7, 79, 21, 77)(2, 20, 43, 62, 12, 65, 37, 58)(3, 83, 18, 40, 19, 81, 56, 10)(4, 33, 25, 75, 23, 80, 5, 84)(6, 72, 69, 8, 29, 31, 66, 34)(9, 60, 44, 59, 38, 16, 11, 68)(13, 89, 96, 14, 49, 94, 85, 54)(15, 39, 26, 36, 57, 42, 17, 82)(22, 32, 28, 30, 27, 35, 24, 74)(45, 63, 78, 90, 87, 52, 47, 50)(46, 93, 86, 53, 76, 91, 88, 55)(48,$

92, 64, 71, 70, 51, 95, 61) >.

With this we are given the first order relations of N which we convert to words. (1, 7)(2, 12)(3, 19)(4, 23)(5, 25)(6, 29)(8, 34)(9, 38)(10, 40)(11, 44)(13, 49)(14, 54)(15, 57)(16, 60)(17, 26)(18, 56)(20, 65)(21, 67)(22, 27)(24, 28)(30, 74)(31, 72)(32, 35)(33, 80)(36, 82)(37, 43)(39, 42)(41, 77)(45, 87)(46, 76)(47, 78)(48, 70)(50, 90)(51, 92)(52, 63)(53, 55)(58, 62)(59, 68)(61, 71)(64, 95)(66, 69)(73, 79)(75, 84)(81, 83)(85, 96)(86, 88)(89, 94)(91, 93) $t = ((xy)^4)t$.

(1, 25)(2, 40)(3, 60)(4, 67)(5, 7)(6, 28)(8, 72)(9, 82)(10, 12)(11, 39)(13, 76)(14, 92)(15, 65)(16, 19)(17, 58)(18, 68)(20, 57)(21, 23)(22, 69)(24, 29)(26, 62)(27, 66)(30, 32)(31, 34)(33, 84)(35, 74)(36, 38)(37, 81)(41, 79)(42, 44)(43, 83)(45, 95)(46, 49)(47, 48)(50, 53)(51, 54)(52, 91)(55, 90)(56, 59)(61, 94)(63, 93)(64, 87)(70, 78)(71, 89)(73, 77)(75, 80)(85, 88)(86, 96) $t = x^2t$.

(1, 66)(2, 11)(4, 24)(5, 27)(6, 67)(7, 69)(8, 84)(9, 43)(10, 39)(12, 44)(13, 95)(14, 93)(15, 57)(16, 60)(17, 26)(21, 29)(22, 25)(23, 28)(30, 77)(31, 33)(32, 79)(34, 75)(35, 73)(36, 83)(37, 38)(40, 42)(41, 74)(45, 46)(47, 88)(48, 96)(49, 64)(50, 71)(51, 63)(52, 92)(53, 94)(54, 91)(55, 89)(59, 68)(61, 90)(70, 85)(72, 80)(76, 87)(78, 86)(81, 82) $t = (xyx^{-1}y)t$.

(1, 42)(2, 22)(3, 87)(4, 44)(5, 81)(6, 10)(7, 39)(8, 52)(9, 69)(11, 23)(12, 27)(13, 16)(14, 80)(15, 64)(17, 88)(18, 96)(19, 45)(20, 76)(21, 37)(24, 82)(25, 83)(26, 86)(28, 36)(29, 40)(30, 91)(31, 94)(32, 71)(33, 54)(34, 63)(35, 61)(38, 66)(41, 55)(43, 67)(46, 65)(47, 59)(48, 58)(49, 60)(50, 75)(51, 79)(53, 77)(56, 85)(57, 95)(62, 70)(68, 78)(72, 89)(73, 92)(74, 93)(84, 90) $t = (x^2yxy^{-1})t$.

(1, 3, 14)(2, 8, 13)(4, 20, 51)(5, 15, 55)(6, 26, 71)(7, 19, 54)(9, 35, 46)(10, 30, 64)(11, 41, 70)(12, 34, 49)(16, 50, 21)(17, 61, 29)(18, 63, 24)(22, 68, 89)(23, 65, 92)(25, 57, 53)(27, 59, 94)(28, 56, 52)(31, 45, 36)(32, 76, 38)(33, 78, 43)(37, 80, 47)(39, 75, 96)(40, 74, 95)(42, 84, 85)(44, 77, 48)(58, 91, 69)(60, 90, 67)(62, 93, 66)(72, 87, 82)(73, 86, 83)(79, 88, 81) $t = yt$.

(1, 24, 7, 28)(2, 43, 12, 37)(3, 57, 19, 15)(4, 66, 23, 69)(5, 29, 25, 6)(8, 77, 34, 41)(9, 11, 38, 44)(10, 83,

40, 81)(13, 46, 49, 76)(14, 71, 54, 61)(16, 20, 60, 65)(17, 56, 26, 18)(21, 27, 67, 22)(30, 84, 74, 75)(31, 73, 72, 79)(32, 80, 35, 33)(36, 39, 82, 42)(45, 95, 87, 64)(47, 70, 78, 48)(50, 93, 90, 91)(51, 89, 92, 94)(52, 55, 63, 53)(58, 68, 62, 59)(85, 88, 96, 86) $t = (xy)^2t$.

(1, 67, 7, 21)(2, 43, 12, 37)(3, 18, 19, 56)(4, 25, 23, 5)(6, 69, 29, 66)(8, 31, 34, 72)(9, 44, 38, 11)(10, 83, 40, 81)(13, 96, 49, 85)(14, 94, 54, 89)(15, 26, 57, 17)(16, 68, 60, 59)(20, 62, 65, 58)(22, 28, 27, 24)(30, 35, 74, 32)(33, 75, 80, 84)(36, 42, 82, 39)(41, 79, 77, 73)(45, 78, 87, 47)(46, 86, 76, 88)(48, 64, 70, 95)(50, 63, 90, 52)(51, 61, 92, 71)(53, 91, 55, 93) $t = (xy^{-1})^2t$.

(1, 86, 25, 96)(2, 51, 40, 54)(3, 73, 60, 77)(4, 78, 67, 70)(5, 85, 7, 88)(6, 87, 28, 64)(8, 17, 72, 58)(9, 90, 82, 55)(10, 14, 12, 92)(11, 89, 39, 71)(13, 22, 76, 69)(15, 75, 65, 80)(16, 41, 19, 79)(18, 74, 68, 35)(20, 33, 57, 84)(21, 48, 23, 47)(24, 95, 29, 45)(26, 31, 62, 34)(27, 46, 66, 49)(30, 59, 32, 56)(36, 53, 38, 50)(37, 91, 81, 52)(42, 61, 44, 94)(43, 93, 83, 63) $t = xt$.

(1, 72, 4, 32)(2, 60, 9, 65)(3, 40, 15, 82)(5, 74, 21, 34)(6, 79, 27, 33)(7, 31, 23, 35)(8, 25, 30, 67)(10, 57, 36, 19)(11, 62, 37, 68)(12, 16, 38, 20)(13, 92, 45, 53)(14, 76, 50, 95)(17, 42, 56, 83)(18, 81, 26, 39)(22, 80, 29, 73)(24, 75, 69, 77)(28, 84, 66, 41)(43, 59, 44, 58)(46, 90, 64, 54)(47, 93, 85, 61)(48, 94, 88, 63)(49, 51, 87, 55)(52, 70, 89, 86)(71, 78, 91, 96) $t = (xy^{-1}xyxy)t$.

(1, 57, 51, 7, 15, 92)(2, 74, 46, 12, 30, 76)(3, 90, 5, 19, 50, 25)(4, 60, 14, 23, 16, 54)(6, 56, 94, 29, 18, 89)(8, 87, 10, 34, 45, 40)(9, 72, 13, 38, 31, 49)(11, 75, 47, 44, 84, 78)(17, 91, 28, 26, 93, 24)(20, 53, 21, 65, 55, 67)(22, 58, 61, 27, 62, 71)(32, 64, 82, 35, 95, 36)(33, 85, 81, 80, 96, 83)(37, 79, 70, 43, 73, 48)(39, 41, 88, 42, 77, 86)(52, 69, 68, 63, 66, 59) $t = (x^2y)t$.

(1, 83, 24, 40, 7, 81, 28, 10)(2, 4, 43, 66, 12, 23, 37, 69)(3, 86, 57, 85, 19, 88, 15, 96)(5, 42, 29, 36, 25, 39, 6, 82)(8, 61, 77, 14, 34, 71, 41, 54)(9, 67, 11, 22, 38, 21, 44, 27)(13, 68, 46, 62, 49, 59, 76, 58)(16, 70, 20, 78, 60, 48, 65, 47)(17, 87, 56, 64, 26, 45, 18, 95)(30, 94, 84, 51, 74, 89, 75, 92)(31, 93, 73, 90, 72, 91, 79, 50)(32, 52, 80, 55, 35, 63, 33, 53) $t = (xy)t$.

(1, 73, 67, 41, 7, 79, 21, 77)(2, 20, 43, 62, 12, 65, 37, 58)(3, 83, 18, 40, 19, 81, 56, 10)(4, 33, 25, 75,

23, 80, 5, 84)(6, 72, 69, 8, 29, 31, 66, 34)(9, 60, 44, 59, 38, 16, 11, 68)(13, 89, 96, 14, 49, 94, 85, 54)
 (15, 39, 26, 36, 57, 42, 17, 82)(22, 32, 28, 30, 27, 35, 24, 74)(45, 63, 78, 90, 87, 52, 47, 50)(46, 93,
 86, 53, 76, 91, 88, 55)(48, 92, 64, 71, 70, 51, 95, 61) $t = (xy^{-1})t$.

Adding these relations gives us our final representation of $G \cong 2^{*96} : (2^3 : S_4)$,

$$G < x, y, t | x^4, y^3, (xy^{-1}x^{-1}y^{-1})^2, x^{-1}yx^{-2}y^{-1}x^{-1}yx^2y^{-1}, t^2, (t, xy^{-1}x^{-1}y^{-1}), \\
 ((xy)^2t)^a, [(xy)^2t^{(x^{-1}y)^3}]^b, [(xy)^2t^{xy^{-1}x^{-1}}]^c, [(xy)^2t^{yx^{-1}y^{-1}x^{-1}y^{-1}x^{-1}}]^d, [(xy^{-1})^2t^{(x^{-1}y)^3}]^e, \\
 [(xy^{-1})^2t^{xy^{-1}x^{-1}}]^f, [(xy^{-1})^2t]^g, [(xy^{-1})^2t^{x^2yxyxy^{-1}xy^{-1}}]^h, (xt)^i, [xt^{(x^{-1}y)^3}]^j, [xt^{xy^{-1}x^{-1}}]^k, \\
 [xt^{x^2yx^{-1}yx^{-1}}]^l > .$$

The following table gives the finite images of the progenitor $2^{*96} : (2^3 : S_4)$ when these first order relations are added.

2.3 Permutation Progenitor of $2^{*72} : (3^2 : (2 : S_4))$

Now, we have a symmetric presentation for $G \cong 2^{*72} : (3^2 : (2 : S_4))$. We start with a group on 72 letters and examine the *FPGroup* to find a presentation of our group N as previously seen.

```
>S:=Sym(72);
>xx:=S!(1, 23, 49, 32, 12, 60)(2, 44, 16, 45, 17, 62)(3, 41, 6, 55,
50, 33)(4, 63, 14, 46, 8, 43)(5, 66, 26, 58, 31, 72)(7, 61, 19, 28,
42, 10)(9,20, 24, 54, 29, 38)(11, 69, 35, 70, 37, 18)(13, 21)(15,
36, 59, 53, 47, 48)(22, 67)(25, 40, 71, 27, 68, 52)(30, 64, 39, 57,
34, 56)(51, 65);
>yy:=S!(1, 35, 58, 2, 6, 54, 25, 4)(3, 40, 46, 64, 14, 21, 19, 69)
```

Table 2.3: Some Finite Images of the Progenitor $2^{*96} : (2^3 : S_4)$

a	b	c	d	e	f	g	h	i	j	k	l	Order of G
0	0	0	0	0	0	0	0	0	0	2	2	384
0	0	0	0	0	0	0	0	0	0	2	3	192
0	0	0	0	0	0	0	0	0	2	0	3	1536
0	0	0	0	0	0	0	0	0	2	3	3	768
0	0	0	0	0	0	0	0	0	3	3	3	216
0	0	0	0	0	0	0	0	0	3	3	4	24576
0	0	0	0	0	0	0	0	0	3	3	5	150
0	0	0	0	0	0	0	0	0	4	2	4	3072
0	0	0	0	0	0	0	0	0	4	2	6	10368
0	0	0	0	0	0	0	0	0	4	2	10	48000
0	0	0	0	0	0	0	0	0	5	2	5	3840
0	0	0	0	0	0	0	0	0	5	2	6	1560
0	0	0	0	0	0	0	0	0	5	3	3	600
0	0	0	0	0	0	0	0	0	10	2	4	2400
0	0	0	0	0	0	0	0	2	6	4	0	5184
0	0	0	0	0	0	0	0	3	7	3	0	1176
0	0	0	0	0	0	0	0	3	9	3	0	1944
0	0	0	0	0	0	0	2	0	3	4	3	6144
0	0	0	0	0	0	0	2	0	3	4	4	2688
0	0	0	0	0	0	0	2	0	3	6	3	864
0	0	0	0	0	0	0	2	0	5	4	3	2211840
0	0	0	0	0	0	0	3	0	5	0	2	7800

(5, 36, 38,15, 22, 60, 17, 55)(7, 43, 66, 29, 32, 53, 71, 30)(8,
23, 27, 57, 37, 39, 67, 41)(9, 13, 50, 59, 42, 26, 20, 44)(10, 24,
12, 31, 45, 48, 16, 65)(11, 63, 18, 51, 49, 34, 61, 52)(28, 62,
68, 70, 33, 56, 72, 47);

>N:=sub<S|xx,yy>;

>#N;

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To find the presentation of N we find the FPGGroup first:

```
>FPGGroup(N);
```

```
Finitely presented group on 2 generators
```

```
Relations
```

$$$.1^6 = \text{Id}(\$)$$

$$$.2^8 = \text{Id}(\$)$$

$$$.2^2 * $.1^{-2} * $.2^{-1} * $.1 * $.2 = \text{Id}(\$)$$

$$$.1^{-1} * $.2^{-1} * $.1 * $.2^{-1} * $.1^{-1} * $.2^{-1} * $.1^3 * $.2 = \text{Id}(\$)$$

Next, we add t^2 to the presentation since $|t_i| = 2$, which gives the free product 2^{*72} . Therefore our progenitor becomes

$$G \langle x, y, t \rangle := \text{Group} \langle x, y, t | x^6, y^8, y^2 x^{-2} y^{-1} x y, x^{-1} y^{-1} x y^{-1} x^{-1} y^{-1} x^3 y, t^2, \\ (t, x y x y x^{-1} y^{-1}), (t, x y^{-1} x^2 y) \rangle;$$

where $t \sim t_1$ and N^1 is the point stabilizer defined by $N^1 = \{n \in N | 1^n = 1\}$.

Now, the point stabilizers are

$$(2, 58)(3, 55)(4, 25)(5, 45)(8, 44)(9, 18)(10, 22)(11, 40)(12, 23)(13, 60)(14, \\ 15)(16, 39)(17, 63)(19, 67)(20, 31)(21, 49)(24, 64)(26, 36)(27, 46)(29, 66)(30, 71)(34, \\ 38)(35, 54)(37, 59)(41, 51)(42, 61)(43, 53)(47, 72)(48, 69)(50, 65)(52, 57)(56, 62)(68, 70),$$

and

$$(2, 25, 54)(3, 41, 65)(4, 58, 35)(5, 18, 46)(8, 69, 34)(9, 45, 27)(10, 67, 42)(11, \\ 26, 39)(12, 60, 21)(13, 23, 49)(14, 57, 31)(15, 20, 52)(16, 36, 40)(17, 24, 59)(19, 22, 61) \\ (29, 71, 53)(30, 66, 43)(37, 64, 63)(38, 48, 44)(47, 68, 62)(50, 51, 55)(56, 70, 72),$$

which we convert to words using the Schreier System, as previously seen. So, we can re-write $(ababa^{-1}b^{-1})$ as $(xyxyx^{-1}y^{-1})$ and $(ab^{-1}a^2b)$ as $(xy^{-1}x^2y)$. Hence, the

progenitor becomes

$$G < x, y, t | x^6, y^8, y^2 x^{-2} y^{-1} x y, x^{-1} y^{-1} x y^{-1} x^{-1} y^{-1} x^3 y, t^2, \\ (t, x y x y x^{-1} y^{-1}), (t, x y^{-1} x^2 y) > .$$

Recall, this progenitor is infinite, therefore, we must factor by first order relations in order to see the finite homomorphic images of $2^{*72} : (3^2 : (2 : S_4))$.

2.3.1 First Order Relations of $2^{*72} : (3^2 : (2 : S_4))$

The first order relations we will examine are of the form $(\pi t_i^a)^b$, where π is a permutation in N . In order to find first order relations, we examine the conjugacy classes of $N \cong (3^2 : (2 : S_4))$. As previously seen, we find the centralizers of each class representative and the corresponding orbits on 72 letters. Thus, we are given the first order relations of N which we convert to words.

$$(1, 32)(2, 45)(3, 55)(4, 46)(5, 58)(6, 33)(7, 28)(8, 63)(9, 54)(10, 19)(11, 70)(12, 23)(13, 21) \\ (14, 43)(15, 53)(16, 62)(17, 44)(18, 35)(20, 29)(22, 67)(24, 38)(25, 27)(26, 72)(30, \\ 57)(31, 66)(34, 64)(36, 47)(37, 69)(39, 56)(40, 68)(41, 50)(42, 61)(48, 59)(49, \\ 60)(51, 65)(52, 71)t = x^3 t..$$

$$(1, 34)(2, 30)(3, 5)(4, 47)(6, 59)(7, 69)(8, 33)(9, 68)(10, 41)(11, 45)(12, 27)(13, 53)(14, 72) \\ (15, 25)(16, 71)(17, 32)(18, 66)(19, 60)(20, 46)(21, 35)(22, 62)(23, 31)(24, 28)(26, \\ 61)(29, 70)(36, 57)(39, 58)(40, 55)(42, 52)(43, 67)(49, 50)(51, 56)(54, 65)t = \\ (xy^{-1})^3 t.$$

$$(1, 7, 33)(2, 10, 46)(3, 15, 43)(4, 19, 45)(5, 25, 67)(6, 28, 32)(8, 34, 69)(9, 35, 61)(11, 47, 60)$$

$$(12, 39, 62)(13, 40, 72)(14, 53, 55)(16, 56, 23)(17, 59, 24)(18, 54, 42)(20, 30, 41)(21, 26, 68)(22, 27, 58)(29, 50, 57)(31, 71, 51)(36, 70, 49)(37, 64, 63)(38, 48, 44)(52, 66, 65)t = (xy^{-1})^2t.$$

$$(1, 49, 12)(2, 16, 17)(3, 6, 50)(4, 14, 8)(5, 26, 31)(7, 19, 42)(9, 24, 29)(10, 61, 28)(11, 35, 37)(15, 59, 47)(18, 69, 70)(20, 54, 38)(23, 32, 60)(25, 71, 68)(27, 52, 40)(30, 39, 34)(33, 41, 55)(36, 53, 48)(43, 63, 46)(44, 45, 62)(56, 64, 57)(58, 72, 66)t = x^2t.$$

$$(1, 61, 68)(2, 10, 11)(3, 25, 7)(4, 19, 20)(5, 44, 16)(6, 42, 66)(8, 35, 67)(9, 31, 38)(12, 58, 28)(13, 59, 62)(14, 27, 63)(15, 26, 48)(17, 54, 22)(18, 40, 37)(21, 69, 56)(23, 71, 33)(24, 53, 65)(29, 50, 46)(30, 41, 36)(32, 55, 72)(34, 43, 51)(39, 52, 64)(45, 70, 49)(47, 60, 57)t = (x^2y^2)t.$$

$$(1, 58, 6, 25)(2, 54, 4, 35)(3, 46, 14, 19)(5, 38, 22, 17)(7, 66, 32, 71)(8, 27, 37, 67)(9, 50, 42, 20)(10, 12, 45, 16)(11, 18, 49, 61)(13, 59, 26, 44)(15, 60, 55, 36)(21, 69, 40, 64)(23, 57, 39, 41)(24, 31, 48, 65)(28, 68, 33, 72)(29, 53, 30, 43)(34, 52, 63, 51)(47, 62, 70, 56)t = y^2t.$$

$$(1, 23, 49, 32, 12, 60)(2, 44, 16, 45, 17, 62)(3, 41, 6, 55, 50, 33)(4, 63, 14, 46, 8, 43)(5, 66, 26, 58, 31, 72)(7, 61, 19, 28, 42, 10)(9, 20, 24, 54, 29, 38)(11, 69, 35, 70, 37, 18)(13, 21)(15, 36, 59, 53, 47, 48)(22, 67)(25, 40, 71, 27, 68, 52)(30, 64, 39, 57, 34, 56)(51, 65)t = xt.$$

$$(1, 8, 7, 34, 33, 69)(2, 20, 10, 30, 46, 41)(3, 67, 15, 5, 43, 25)(4, 11, 19, 47, 45, 60)(6, 17, 28, 59, 32, 24)(9, 26, 35, 68, 61, 21)(12, 22, 39, 27, 62, 58)(13, 14, 40, 53, 72, 55)(16, 31, 56, 71, 23, 51)(18, 52, 54, 66, 42, 65)(29, 36, 50, 70, 57, 49)(37, 63, 64)(38, 44, 48)t = (xy^{-1})t.$$

$$(1, 35, 58, 2, 6, 54, 25, 4)(3, 40, 46, 64, 14, 21, 19, 69)(5, 36, 38, 15, 22, 60, 17, 55)(7, 43, 66, 29, 32, 53, 71, 30)(8, 23, 27, 57, 37, 39, 67, 41)(9, 13, 50, 59, 42, 26, 20, 44)(10, 24, 12, 31,$$

45, 48, 16, 65)(11, 63, 18, 51, 49, 34, 61, 52)(28, 62, 68, 70, 33, 56, 72, 47)t = yt.
 (1, 54, 58, 4, 6, 35, 25, 2)(3, 21, 46, 69, 14, 40, 19, 64)(5, 60, 38, 55, 22, 36, 17, 15)(7, 53, 66, 30, 32, 43,
 71, 29)(8, 39, 27, 41, 37, 23, 67, 57)(9, 26, 50, 44, 42, 13, 20, 59)(10, 48, 12, 65, 45,
 24, 16, 31)(11, 34, 18, 52, 49, 63, 61, 51)(28, 56, 68, 47, 33, 62, 72, 70)t = y⁻³t.

Adding these relations gives us our final representation of $G \cong 2^{*72} : (3^2 : (2 : S_4))$,

$$G < x, y, t | x^6, y^8, y^2 x^{-2} y^{-1} x y, x^{-1} y^{-1} x y^{-1} x^{-1} y^{-1} x^3 y, t^2, \\
 (t, x y x y x^{-1} y^{-1}), (t, x y^{-1} x^2 y), (x^3 t y^{-2} x y^{-1})^a, (x^3 t y x^{-1} y^{-1} x^{-2})^b, ((x y^{-1})^3 t y x^{-2} y x^{-1})^c, \\
 ((x y^{-1})^3 t y^2 x y^{-2} x^2)^d, ((x y^{-1})^3 t y, x^{-1})^e, ((x y^{-1})^3 t y x y x^{-1} y^{-2})^f, ((x y^{-1})^3 t y x^{-1} y^{-2} x^{-1})^g, \\
 ((x y^{-1})^2 t x y x y^{-1} x y)^h, ((x y^{-1})^2 t x y x y^{-1} x)^i, (x^2 t x y^{-1} x y x^{-1})^j, (x^2 t y x^{-1} y^{-2} x^{-1})^k, (x^2 t x y^{-1} x^2 y^{-1})^l > .$$

The following table gives the finite images of the progenitor $2^{*72} : (3^2 : (2 : S_4))$. Now, we look at finite images of the progenitor $2^{*72} : (3^2 : (2 : S_4))$ with

Table 2.4: Some Finite Images of the Progenitor $2^{*72} : (3^2 : (2 : S_4))$

a	b	c	d	e	f	g	h	i	j	k	l	Order of G	Shape of G
0	0	0	0	0	0	0	0	0	0	0	3	120	
0	0	0	0	0	0	0	0	0	0	2	2	864	
0	0	0	0	0	0	0	0	0	0	2	4	221184	

different First Order Relations (FOR).

$$G < x, y, t | x^6, y^8, y^2 x^{-2} y^{-1} x y, x^{-1} y^{-1} x y^{-1} x^{-1} y^{-1} x^3 y, t^2, \\
 (t, x y x y x^{-1} y^{-1}), (t, x y^{-1} x^2 y), (x^3 t y x^{-1} y^{-1} x^{-2})^a, ((x y^{-1})^3 t y x^{-2} y x^{-1})^b, ((x y^{-1})^2 t x y x y^{-1} x y)^c, \\
 (x^2 t x y^{-1} x y x^{-1})^d, ((x^2 y^2) t x^{-2})^e, (y^2 t x y)^f, (x t x^2 y^{-1} x^{-2})^g, ((x y^{-1}) t y x^3)^h, \\
 (y t y^{-2} x y^{-1})^i, (y^{-3} t (y^{-1} x^{-1})^2)^j, (x^2 t x y^{-1} x^2 y^{-1})^k, (x t x y x^{-1} y)^l > .$$

Table 2.5: Some Finite Images of the Progenitor $2^{*72} : (3^2 : (2 : S_4))$ with different FOR

a	b	c	d	e	f	g	h	i	j	k	l	Order of G	Shape of G
0	0	0	0	0	0	0	0	0	0	0	3	11664	
0	0	0	0	0	0	0	2	0	0	0	3	432	

Chapter 3

Monomial Progenitors

3.1 Monomial Progenitor $3^{*7} :_m PSL(2, 7)$

We begin with a group G generated by xx and yy , where $xx = (1, 2, 3, 4, 5, 6, 7)$, and $yy = (1, 2)(3, 6)$. The conjugacy classes of group G are listed below.

Conjugacy Classes of G :

$$C1 = \{e\},$$

$$C2 = \{(1, 2)(3, 6), (2, 5)(4, 6), (1, 6)(2, 3), (2, 3)(4, 7), (2, 6)(4, 5), (3, 6)(5, 7), (3, 5)(6, 7), (2, 4)(5, 6), (1, 5)(2, 7), (1, 3)(4, 5), (1, 7)(2, 5), (1, 6)(4, 7), (1, 2)(5, 7), (1, 4)(3, 5), (1, 3)(2, 6), (3, 7)(5, 6), (1, 7)(4, 6), (2, 4)(3, 7), (1, 4)(6, 7), (1, 5)(3, 4), (2, 7)(3, 4)\},$$

$$C3 = \{(1, 5, 6)(3, 7, 4), (1, 4, 7)(2, 5, 3), (2, 6, 3)(4, 5, 7), (1, 2, 5)(3, 6, 4), (1, 4, 2)(3, 6, 7), (1, 2, 7)(3, 4, 6), (1, 4, 6)(2, 3, 5), (1, 7, 3)(4, 6, 5), (1, 2, 4)(3, 7, 6), (1, 6, 7)(2, 3, 5), (1, 6, 5)(2, 4, 3), (1, 2, 4)(5, 6, 7), (2, 7, 5)(3, 6, 4), (1, 5, 6)(2, 4, 7), (1, 6, 5)(2, 7, 4), (1, 6, 4)(2, 5, 3), (1, 6, 5)(3, 4, 7), (1, 2, 6)(4, 7, 5), (1, 3,$$

5)(2, 6, 7), (1, 4, 2)(3, 7, 5), (1, 4, 2)(5, 7, 6), (2, 5, 3)(4, 6, 7), (2, 3, 5)(4, 7, 6)
 (1, 7, 3)(2, 5, 6), (1, 5, 4)(2, 6, 7), (2, 6, 7)(3, 4, 5), (1, 2, 3)(4, 5, 7), (1, 3, 7)(2,
 4, 6), (1, 3, 2)(4, 7, 5), (1, 2, 4)(3, 5, 7), (1, 4, 5)(2, 7, 6), (1, 6, 3)(4, 7, 5), (1, 7,
 3)(2, 6, 4), (1, 3, 6)(4, 5, 7), (1, 6, 5)(2, 3, 7), (2, 3, 6)(4, 7, 5), (1, 5, 3)(2, 7, 6),
 (1, 5, 7)(3, 6, 4), (1, 3, 7)(2, 5, 4), (1, 7, 2)(3, 6, 4), (1, 5, 2)(3, 4, 6), (1, 7, 4)(2,
 3, 5), (1, 7, 6)(2, 5, 3), (1, 3, 4)(2, 7, 6), (1, 3, 7)(2, 6, 5), (1, 3, 7)(4, 5, 6), (1, 5,
 6)(2, 7, 3), (2, 5, 7)(3, 4, 6), (1, 7, 3)(2, 4, 5), (1, 7, 5)(3, 4, 6), (1, 4, 2)(3, 5, 6),
 (1, 2, 4)(3, 6, 5), (2, 7, 6)(3, 5, 4), (1, 4, 3)(2, 6, 7), (1, 6, 2)(4, 5, 7), (1, 5, 6)(2,
 3, 4)},

$C4 = \{(1, 5)(2, 3, 7, 4), (1, 6)(2, 4, 3, 7), (1, 4, 3, 5)(2, 6), (1, 4, 6, 7)(2, 3), (1, 6,$
 $3, 2)(4, 5), (1, 5, 4, 3)(6, 7), (1, 2, 6, 3)(4, 7), (1, 2)(3, 5, 6, 7), (2, 3, 4, 7)(5, 6),$
 $(1, 6, 2, 3)(5, 7), (1, 7, 6, 4)(2, 3), (1, 2, 7, 5)(4, 6), (1, 2)(3, 7, 6, 5), (1, 7, 5, 2)(3,$
 $4), (1, 7)(2, 4, 5, 6), (1, 4)(3, 7, 5, 6), (2, 6, 4, 5)(3, 7), (1, 6, 4, 7)(3, 5), (1, 3)(2,$
 $4, 6, 5), (1, 2, 5, 7)(3, 4), (1, 5, 2, 7)(3, 6), (1, 4)(3, 6, 5, 7), (1, 3, 6, 2)(4, 7), (1,$
 $6)(2, 7, 3, 4), (1, 3)(2, 5, 6, 4), (2, 4)(3, 5, 7, 6), (1, 5, 3, 4)(2, 6), (1, 6, 7, 4)(2, 5),$
 $(1, 3, 5, 4)(2, 7), (1, 4, 5, 3)(2, 7), (2, 7, 4, 3)(5, 6), (1, 3, 4, 5)(6, 7), (1, 7, 2, 5)(3,$
 $6), (1, 4, 7, 6)(2, 5), (1, 5)(2, 4, 7, 3), (1, 2, 3, 6)(4, 5), (1, 7, 4, 6)(3, 5), (2, 4)(3,$
 $6, 7, 5), (1, 5, 7, 2)(4, 6), (1, 7)(2, 6, 5, 4), (2, 5, 4, 6)(3, 7), (1, 3, 2, 6)(5, 7)\},$

$C5 = \{(1, 5, 3, 6, 2, 4, 7), (1, 6, 4, 5, 3, 7, 2), (1, 4, 3, 2, 6, 5, 7), (1, 6, 3, 5, 4, 2,$
 $7), (1, 7, 4, 3, 5, 6, 2), (1, 6, 7, 5, 2, 4, 3), (1, 5, 4, 6, 7, 3, 2), (1, 7, 2, 3, 6, 5, 4),$
 $(1, 3, 5, 7, 2, 4, 6), (1, 4, 6, 2, 3, 7, 5), (1, 7, 5, 3, 4, 2, 6), (1, 3, 6, 7, 4, 2, 5), (1,$
 $2, 3, 4, 5, 6, 7), (1, 2, 6, 4, 7, 3, 5), (1, 4, 7, 2, 5, 6, 3), (1, 5, 7, 6, 4, 2, 3), (1, 5,$
 $2, 6, 3, 7, 4), (1, 2, 7, 4, 6, 5, 3), (1, 6, 2, 5, 7, 3, 4), (1, 7, 6, 3, 2, 4, 5), (1, 2, 5,$
 $4, 3, 7, 6), (1, 4, 5, 2, 7, 3, 6), (1, 3, 4, 7, 6, 5, 2), (1, 3, 2, 7, 5, 6, 4)\},$ and

$C6 = \{(1, 4, 7, 3, 6, 2, 5), (1, 5, 4, 2, 3, 6, 7), (1, 5, 3, 7, 4, 6, 2), (1, 7, 5, 6, 2, 3, 4), (1, 5, 2, 4, 7, 6, 3), (1, 4, 5, 6, 3, 2, 7), (1, 6, 7, 3, 4, 5, 2), (1, 3, 4, 2, 5, 7, 6), (1, 7, 6, 5, 4, 3, 2), (1, 2, 3, 7, 6, 4, 5), (1, 4, 6, 5, 7, 2, 3), (1, 2, 6, 5, 3, 4, 7), (1, 6, 4, 2, 7, 5, 3), (1, 3, 5, 6, 4, 7, 2), (1, 4, 3, 7, 5, 2, 6), (1, 3, 2, 4, 6, 7, 5), (1, 2, 7, 3, 5, 4, 6), (1, 5, 7, 3, 2, 6, 4), (1, 2, 5, 6, 7, 4, 3), (1, 7, 2, 4, 5, 3, 6), (1, 6, 2, 4, 3, 5, 7), (1, 7, 4, 2, 6, 3, 5), (1, 3, 6, 5, 2, 7, 4), (1, 6, 3, 7, 2, 5, 4).\}$

Similarly, consider the subgroup H of G generated by $\langle(1, 5)(2, 7), (1, 2, 4)(5, 6, 7)\rangle$, has the following conjugacy classes.

The conjugacy classes of H are:

$$D1 = \{e\},$$

$$D2 = \{(1, 7)(4, 6), (1, 7)(2, 5), (2, 5)(4, 6)\},$$

$$D3 = \{(2, 6)(4, 5), (1, 5)(2, 7), (1, 4)(6, 7), (1, 6)(4, 7), (2, 4)(5, 6), (1, 2)(5, 7)\},$$

$$D4 = \{(1, 6, 5)(2, 7, 4), (1, 2, 6)(4, 7, 5), (1, 4, 5)(2, 7, 6), (1, 6, 2)(4, 5, 7), (1, 5, 6)(2, 4, 7), (1, 4, 2)(5, 7, 6), (1, 5, 4)(2, 6, 7), (1, 2, 4)(5, 6, 7)\}, \text{ and}$$

$$D5 = \{(1, 4, 7, 6)(2, 5), (1, 2, 7, 5)(4, 6), (1, 7)(2, 4, 5, 6), (1, 5, 7, 2)(4, 6), (1, 7)(2, 6, 5, 4), (1, 6, 7, 4)(2, 5)\}.$$

We also consider the irreducible characters ϕ (of H) and ϕ^G (of G) given below.

Table 3.1: Irreducible Characters ϕ

Class	$D1$	$D2$	$D3$	$D4$	$D5$
Size	1	3	6	8	6
Representative	e	$(2, 5)(4, 6)$	$(1, 5)(2, 7)$	$(1, 2, 4)(5, 6, 7)$	$(1, 7)(2, 4, 5, 6)$
ϕ	1	1	-1	1	-1

Table 3.2: Induced Characters ϕ^G

Class	C_1	C_2	C_3	C_4	C_5	C_6
Size	1	21	56	42	24	24
Representative	e	$(1, 2)(3, 6)$	$(1, 3, 7)(4, 5, 6)$	$(1, 5)(2, 4, 7, 3)$	$(1, 2, 3, 4, 5, 6, 7)$	$(1, 4, 7, 3, 6, 2, 5)$
ϕ^G	7	-1	1	-1	0	0

We induce the character ϕ of H up to G to obtain the character ϕ^G of G ($\phi \uparrow_H^G$) as follows.

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$$

, where $n = \frac{|G|}{|H|} = \frac{168}{24} = 7$.

$$\phi_1^G = \frac{7}{1} \sum_{w \in H \cap C_1} \phi(w)$$

So, $\phi_1^G = 7(\phi(1)) = 7(1) = 7$.

$$\phi_2^G = \frac{7}{21} \sum_{w \in H \cap C_2} \phi(w)$$

So, $\phi_2^G = \frac{1}{3}(\phi[(1, 7)(4, 6) + (1, 7)(2, 5) + (2, 5)(4, 6) + (2, 6)(4, 5) + (1, 5)(2, 7) + (1, 4)(6, 7) + (1, 6)(4, 7) + (2, 4)(5, 6) + (1, 2)(5, 7)]) = \frac{1}{3}(1 + 1 + 1 + -1 + -1 + -1 + -1 + -1 + -1) = \frac{1}{3}(-3) = -1$.

$$\phi_3^G = \frac{7}{56} \sum_{w \in H \cap C_3} \phi(w)$$

So, $\phi_3^G = \frac{7}{56}(\phi[(1, 6, 5)(2, 7, 4) + (1, 2, 6)(4, 7, 5) + (1, 4, 5)(2, 7, 6) + (1, 6, 2)(4, 5, 7) + (1, 5, 6)(2, 4, 7) + (1, 4, 2)(5, 7, 6) + (1, 5, 4)(2, 6, 7) + (1, 2, 4)(5, 6, 7)]) = \frac{1}{8}(1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) = \frac{1}{8}(8) = 1$.

$$\phi_4^G = \frac{7}{42} \sum_{w \in H \cap C_\alpha} \phi(w)$$

So, $\phi_4^G = \frac{1}{6}(\phi[(1, 2, 7, 5)(4, 6) + (1, 7)(2, 4, 5, 6) + (1, 6, 7, 4)(2, 5) + (1, 4, 7, 6)(2, 5) + (1, 5, 7, 2)(4, 6) + (1, 7)(2, 6, 5, 4)]) = \frac{1}{6}(-1 + -1 + -1 + -1 + -1 + -1) = \frac{1}{6}(-6) = -1$.

$$\phi_5^G = \frac{7}{24} \sum_{w \in H \cap C_\alpha} \phi(w)$$

So, $\phi_5^G = \frac{7}{24}(\phi(\emptyset)) = \frac{7}{24}(0) = 0$.

$$\phi_6^G = \frac{7}{24} \sum_{w \in H \cap C_\alpha} \phi(w)$$

So, $\phi_6^G = \frac{7}{24}(\phi(\emptyset)) = \frac{7}{24}(0) = 0$.

Thus, $\phi \uparrow_H^G = 7 \quad -1 \quad 1 \quad -1 \quad 0 \quad 0$.

Now, we find the right coset representation of G over H (transversal of H in G), such that, $G = He \cup H(1, 2, 3, 4, 5, 6, 7) \cup H(1, 4, 7, 3, 6, 2, 5) \cup H(1, 3, 5, 7, 2, 4, 6) \cup H(1, 3, 7)(2, 4, 6) \cup H(1, 3)(2, 4, 6, 5) \cup H(1, 7, 6, 3, 2, 4, 5)$ and let $t_1 = e$, $t_2 = (1, 2, 3, 4, 5, 6, 7)$, $t_3 = (1, 4, 7, 3, 6, 2, 5)$, $t_4 = (1, 3, 5, 7, 2, 4, 6)$, $t_5 = (1, 3, 7)(2, 4, 6)$,

$t_6 = (1, 3)(2, 4, 6, 5)$, and $t_7 = (1, 7, 6, 3, 2, 4, 5)$.

The monomial representation has generators

$$\begin{aligned}
 & A(xx) = \\
 & \begin{bmatrix}
 \phi(t_1xxt_1^{-1}) & \phi(t_1xxt_2^{-1}) & \phi(t_1xxt_3^{-1}) & \phi(t_1xxt_4^{-1}) & \phi(t_1xxt_5^{-1}) & \phi(t_1xxt_6^{-1}) & \phi(t_1xxt_7^{-1}) \\
 \phi(t_2xxt_1^{-1}) & \phi(t_2xxt_2^{-1}) & \phi(t_2xxt_3^{-1}) & \phi(t_2xxt_4^{-1}) & \phi(t_2xxt_5^{-1}) & \phi(t_2xxt_6^{-1}) & \phi(t_2xxt_7^{-1}) \\
 \phi(t_3xxt_1^{-1}) & \phi(t_3xxt_2^{-1}) & \phi(t_3xxt_3^{-1}) & \phi(t_3xxt_4^{-1}) & \phi(t_3xxt_5^{-1}) & \phi(t_3xxt_6^{-1}) & \phi(t_3xxt_7^{-1}) \\
 \phi(t_4xxt_1^{-1}) & \phi(t_4xxt_2^{-1}) & \phi(t_4xxt_3^{-1}) & \phi(t_4xxt_4^{-1}) & \phi(t_4xxt_5^{-1}) & \phi(t_4xxt_6^{-1}) & \phi(t_4xxt_7^{-1}) \\
 \phi(t_5xxt_1^{-1}) & \phi(t_5xxt_2^{-1}) & \phi(t_5xxt_3^{-1}) & \phi(t_5xxt_4^{-1}) & \phi(t_5xxt_5^{-1}) & \phi(t_5xxt_6^{-1}) & \phi(t_5xxt_7^{-1}) \\
 \phi(t_6xx_1^{-1}) & \phi(t_6xxt_2^{-1}) & \phi(t_6xxt_3^{-1}) & \phi(t_6xxt_4^{-1}) & \phi(t_6xxt_5^{-1}) & \phi(t_6xxt_6^{-1}) & \phi(t_6xxt_7^{-1}) \\
 \phi(t_7xxt_1^{-1}) & \phi(t_7xxt_2^{-1}) & \phi(t_7xxt_3^{-1}) & \phi(t_7xxt_4^{-1}) & \phi(t_7xxt_5^{-1}) & \phi(t_7xxt_6^{-1}) & \phi(t_7xxt_7^{-1})
 \end{bmatrix} \\
 & = \begin{bmatrix}
 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 -1 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \end{aligned}$$

and

$$A(yy) =$$

$$\begin{bmatrix}
\phi(t_1yyt_1^{-1}) & \phi(t_1yyt_2^{-1}) & \phi(t_1yyt_3^{-1}) & \phi(t_1yyt_4^{-1}) & \phi(t_1yyt_5^{-1}) & \phi(t_1yyt_6^{-1}) & \phi(t_1yyt_7^{-1}) \\
\phi(t_2yyt_1^{-1}) & \phi(t_2yyt_2^{-1}) & \phi(t_2yyt_3^{-1}) & \phi(t_2yyt_4^{-1}) & \phi(t_2yyt_5^{-1}) & \phi(t_2yyt_6^{-1}) & \phi(t_2yyt_7^{-1}) \\
\phi(t_3yyt_1^{-1}) & \phi(t_3yyt_2^{-1}) & \phi(t_3yyt_3^{-1}) & \phi(t_3yyt_4^{-1}) & \phi(t_3yyt_5^{-1}) & \phi(t_3yyt_6^{-1}) & \phi(t_3yyt_7^{-1}) \\
\phi(t_4yyt_1^{-1}) & \phi(t_4yyt_2^{-1}) & \phi(t_4yyt_3^{-1}) & \phi(t_4yyt_4^{-1}) & \phi(t_4yyt_5^{-1}) & \phi(t_4yyt_6^{-1}) & \phi(t_4yyt_7^{-1}) \\
\phi(t_5yyt_1^{-1}) & \phi(t_5yyt_2^{-1}) & \phi(t_5yyt_3^{-1}) & \phi(t_5yyt_4^{-1}) & \phi(t_5yyt_5^{-1}) & \phi(t_5yyt_6^{-1}) & \phi(t_5yyt_7^{-1}) \\
\phi(t_6yy_1^{-1}) & \phi(t_6yyt_2^{-1}) & \phi(t_6yyt_3^{-1}) & \phi(t_6yyt_4^{-1}) & \phi(t_6yyt_5^{-1}) & \phi(t_6yyt_6^{-1}) & \phi(t_6yyt_7^{-1}) \\
\phi(t_7yyt_1^{-1}) & \phi(t_7yyt_2^{-1}) & \phi(t_7yyt_3^{-1}) & \phi(t_7yyt_4^{-1}) & \phi(t_7yyt_5^{-1}) & \phi(t_7yyt_6^{-1}) & \phi(t_7yyt_7^{-1})
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}$$

Since G has square root of unity we find that \mathbb{Z}_3 is the smallest finite field of square root unity. Thus, the primitive root of \mathbb{Z}_3 is -1 , and the entries of the matrix $A(xx)$ and $A(yy)$ are powers of 3 , namely, $-1 \equiv_3 2$.

The permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation is

$$\begin{aligned}
A(xx) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

where $a_{12} = 1$, $a_{24} = 1$, $a_{35} = 1$, $a_{43} = 1$, $a_{56} = 1$, $a_{67} = 2$, $a_{71} = 2$.

Therefore,

$$t_1 \rightarrow t_2,$$

$$t_2 \rightarrow t_4,$$

$$t_3 \rightarrow t_5,$$

$$t_4 \rightarrow t_3,$$

$$t_5 \rightarrow t_6,$$

$$t_6 \rightarrow t_7^2,$$

$$t_7 \rightarrow t_1^2.$$

Since, we have 7 t'_i s of order 2 we label them accordingly:

Table 3.3: Labeling for A(xx)

1	2	3	4	5	6	7	8	9	10	11	12	13	14
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_7^2
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
t_2	t_4	t_5	t_3	t_6	t_7^2	t_1^2	t_2^2	t_4^2	t_5^2	t_3^2	t_6^2	t_7	t_1
2	4	5	3	6	14	8	9	11	12	10	13	7	1

Therefore, $A(xx) = (1, 2, 4, 3, 5, 6, 14)(7, 8, 9, 11, 10, 12, 13)$. Similarly,

$$\begin{aligned}
 A(yy) &= \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix},
 \end{aligned}$$

where $a_{13} = 2$, $a_{22} = 1$, $a_{31} = 2$, $a_{44} = 2$, $a_{55} = 2$, $a_{67} = 2$, $a_{76} = 2$. Therefore,

Table 3.4: Labeling for A(yy)

1	2	3	4	5	6	7	8	9	10	11	12	13	14
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_7^2
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
t_3^2	t_2	t_1^2	t_4^2	t_5^2	t_7^2	t_6^2	t_3	t_2^2	t_1	t_4	t_5	t_7	t_6
10	2	8	11	12	14	13	3	9	1	4	5	7	6

$$t_1 \rightarrow t_3^2,$$

$$t_2 \rightarrow t_2,$$

$$t_3 \rightarrow t_1^2,$$

$$t_4 \rightarrow t_4^2,$$

$$t_5 \rightarrow t_5^2,$$

$$t_6 \rightarrow t_7^2,$$

$$t_7 \rightarrow t_6^2.$$

Therefore, $A(yy) = (1, 10)(3, 8)(4, 11)(5, 12)(6, 14)(7, 13)$. In MAGMA we confirm if A(xx) and A(yy) generate a group isomorphic to the group G.

```

>S:=Sym(14);
>Aper:=S!(1, 2, 4, 3, 5, 6, 14)(7, 8, 9, 11, 10, 12, 13);
>Bper:=S!(1,10)(3,8)(4,11)(5,12)(6,14)(7,13);
>HH:=sub<S|Aper,Bper>;
>IsIsomorphic(HH,G);
true Mapping from: GrpPerm: HH to GrpPerm: G
Composition of Mapping from: GrpPerm: HH to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G

```



```

>FPGroup(G);
Finitely presented group on 2 generators
Relations
$.1^7 = Id($)
$.2^2 = Id($)
($.2 * $.1^-1)^3 = Id($)
($.1 * $.2 * $.1^2)^4 = Id($)

>NN<a,b>:=Group<a,b | a^7, b^2, (b*a)^3, (a*b*a^2)^4 >;
>#NN;
168

```

Hence, our work is correct.

Let $G = \langle x, y \rangle$ such that $x = A(xx)$ and $y = A(yy)$ (as above) then the presentation for the monomial matrix is given by $G = \langle x, y | x^7, y^2, (yx)^3, (xyx^2)^4 \rangle$ with order 168. In order to build a monomial progenitor from this presentation we proceed as shown in the following steps.

Step 1: Find the Normalizer in G

We find $\langle t_1 \rangle = \{t_1, t_1^2, e\}$. The normalizer of this group in G is $\{n \in G | \{t_1, t_1^2, e\}^n = \{t_1, t_1^2, e\}\}$. We compute the generators of the normalizer of $\langle t_1 \rangle$ in G .

```

>Nt:=Stabiliser(G,{1,8}); Nt;
Permutation group Nt acting on a set of cardinality 14

```

```

Order = 24 = 2^3 * 3
      (2, 7, 12)(3, 11, 13)(4, 6, 10)(5, 9, 14)
      (2, 10)(3, 9)(4, 14)(5, 12)(6, 13)(7, 11)
      (1, 8)(2, 6, 3, 12)(4, 7, 11, 14)(5, 9, 13, 10)
>yy*xx^-3;
(1, 8)(2, 6, 3, 12)(4, 7, 11, 14)(5, 9, 13, 10)

```

We find the single permutation, $(1, 8)(2, 6, 3, 12)(4, 7, 11, 14)(5, 9, 13, 10)$, generates the normalizer of G in $\langle t_1 \rangle$ and notice that the permutation is represented as $yy * xx^{-3}$. This permutation takes 1 to 8. So, by the labeling of the automorphisms, we have $t_1^{yx^{-3}} = t_1^2$ (note $yy \sim y$).

Step 2: Find the point stabilizer of 1 in G , along with its orbits.

```

>Stabilizer(G,1);
Permutation group acting on a set of cardinality 14
Order = 12 = 2^2 * 3
      (2, 13, 14)(3, 12, 4)(5, 11, 10)(6, 7, 9)
      (2, 5, 4)(3, 6, 14)(7, 10, 13)(9, 12, 11)

>Orbits(Stabilizer(G,1));
[
  GSet{@ 1 @},
  GSet{@ 8 @},
  GSet{@ 2, 13, 11, 14, 9, 10, 6, 12, 4, 5, 7, 3 @}
]

```

Thus, there are 2 generators and 3 orbits. The permutations in G which take 1 to 1, 1 to 8, and 1 to 2 are given in words in the following code.

```
>1^xx;
2
>1^(xx^-3 * yy * xx^3);
8
```

Therefore, the permutation represented by xx is a permutation which takes t_1 to t_2 .

Step 3: Build Progenitor & Test by Grindstaff Lemma(**(Section 1.1)**)

```
>FPGGroup(G);
Finitely presented group on 2 generators
Relations
$.1^7 = Id($)
$.2^2 = Id($)
($.2 * $.1^-1)^3 = Id($)
($.1 * $.2 * $.1^2)^4 = Id($)

>G<x,y,t>:=Group<x,y ,t| x^7, y^2, (y*x^-1)^3, (x*y*x^2)^4,
t^3, (t,x * y * x^-3 * y * x), (t,y * x^-2 * y * x^2 * y), t^(y * x^-3)=t^2,
(t, t^x), (t, t^(x^-3 * y * x^3)) >;
> #G;
367416
```

>3^7*168;

367416

Hence, a symmetric presentation for the progenitor $3^{*7} :_m PSL(2, 7)$ is $\langle x, y, t | x^7, y^2, (yx)^3, (xyx^2)^4, t^3, (t, xyx^{-3}yx), (t, yx^{-2}yx^2y), t^{yx^{-3}} = t^2 \rangle$. We obtain our first order relations as previously done in **Section 2.1**. So, the first order relations are $(xyx^{-1}t)^a, (xyt^{xyx^{-2}yxy})^b, (yx^3t^{yx^{-3}y})^c, (xt)^d, (x^3t^{xyx^2})^e, (xyx^{-1}t^{x^2yx^2yx^{-2}})^f, (xyt^{x^{-3}yx})^g, (yx^3t^{x^2yx^2yx^{-2}})^h, (xt^{xyx^2})^i, (xyt^{yx^{-2}yx^3y})^j$. We factor the progenitor G by these relations to obtain finite images (see **Chapter 4**). Thus, the final symmetric presentation is given by

$$\begin{aligned} G < x, y, t | x^7, y^2, (yx^{-1})^3, (xyx^2)^4, t^3, (t, xyx^{-3}yx), (t, yx^{-2}yx^2y), t^{yx^{-3}} = t^2, \\ (t, tx), (t, t^{x^{-3}yx^3}), (xyx^{-1}t)^a, (xyt^{xyx^{-2}yxy})^b, (yx^3t^{yx^{-3}y})^c, (xt)^d, (x^3t^{xyx^2})^e, \\ (xyx^{-1}t^{x^2yx^2yx^{-2}})^f, (xyt^{x^{-3}yx})^g, (yx^3t^{x^2yx^2yx^{-2}})^h, (xt^{xyx^2})^i, (xyt^{yx^{-2}yx^3y})^j >; \end{aligned}$$

3.2 Monomial Progenitor $3^{*4} :_m (2^4 : 4)$

We begin with a group G generated by xx and yy , where $xx \sim (4, 8)$, and $yy \sim (1, 2, 3, 8)(4, 5, 6, 7)$. The conjugacy classes of group G are listed below.

Conjugacy Classes of G :

$$C1 = \{e\},$$

$$C2 = \{(1, 5)(2, 6)(3, 7)(4, 8)\},$$

$$C3 = \{(1, 5)(3, 7)\},$$

$$C4 = \{(1, 3)(2, 8)(4, 6)(5, 7)\},$$

$$C5 = \{(1, 5)\},$$

$$C6 = \{(1, 5)(4, 8)\},$$

$$C7 = \{(1, 5)(3, 7)(4, 8)\},$$

$$C8 = \{(1, 7, 5, 3)(2, 4, 6, 8)\},$$

$$C9 = \{(1, 2, 3, 8)(4, 5, 6, 7)\},$$

$$C10 = \{(1, 8, 3, 2)(4, 7, 6, 5)\},$$

$$C11 = \{(1, 7, 5, 3)(2, 8)(4, 6)\},$$

$$C12 = \{(1, 4, 7, 6, 5, 8, 3, 2)\}, \text{ and}$$

$$C13 = \{1, 6, 3, 4, 5, 2, 7, 8\}.$$

Similarly, consider the subgroup H of G generated by $\langle (2, 6), (1, 3)(2, 8)(4, 6)(5, 7), (1, 5)(2, 6)(3, 7)(4, 8), (2, 6)(4, 8) \rangle$, has the following conjugacy classes.

The conjugacy classes of H are:

$$D1 = \{e\},$$

$$D2 = \{(1, 5)(2, 6)(3, 7)(4, 8)\},$$

$$D3 = \{(2, 6)(4, 8)\},$$

$$D4 = \{(1, 5)(3, 7)\},$$

$$D5 = \{(1, 3)(2, 8)(4, 6)(5, 7)\},$$

$$D6 = \{(1, 7)(2, 4)(3, 5)(6, 8)\},$$

$$D7 = \{(2, 6)\},$$

$$D8 = \{(1, 5)(3, 7)(4, 8)\},$$

$$D9 = \{(1, 3)(2, 4, 6, 8)(5, 7)\},$$

$$D10 = \{(1, 7)(2, 8, 6, 4)(3, 5)\}.$$

We also consider the irreducible characters ϕ (of H) and ϕ^G (of G) given below.

Table 3.5: Irreducible Characters ϕ

Class	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$
Size	1	1	1	1	2	2
Representative	e	$(1, 5)(2, 6)(3, 7)(4, 8)$	$(2, 6)(4, 8)$	$(1, 5)(3, 7)$	$(1, 3)(2, 8)(4, 6)(5, 7)$	$(1, 7)(2, 4)$
ϕ	1	-1	1	-1	-1	1

We induce the character ϕ of H up to G to obtain the character ϕ^G of G ($\phi \uparrow_H^G$) as example 3.2.

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$$

, where $n = \frac{|G|}{|H|} = \frac{64}{16} = 4$.

$$\phi_1^G = \frac{4}{1} \sum_{w \in H \cap C_1} \phi(w)$$

Table 3.6: Induced Characters ϕ^G

Class	C_1	C_2	C_3	C_4	C_5	C_6
Size	1	1	2	4	4	4
Representative	e	$(1, 5)(2, 6)(3, 7)(4, 8)$	$(1, 5)(3, 7)$	$(1, 3)(2, 8)(4, 6)(5, 7)$	$(1, 5)$	$(1, 5)(4, 8)$
ϕ^G	4	-4	0	0	-2	0

So, $\phi_1^G = 4(\phi(e)) = 4(1) = 4$.

$$\phi_2^G = \frac{4}{1} \sum_{w \in H \cap C_2} \phi(w)$$

So, $\phi_2^G = 4(\phi((1, 5)(2, 6)(3, 7)(4, 8))) = 4(-1) = -4$.

$$\phi_3^G = \frac{4}{2} \sum_{w \in H \cap C_3} \phi(w)$$

So, $\phi_3^G = 2(\phi((2, 6)(4, 8) + (1, 5)(3, 7))) = 2(1 + -1) = 0$.

$$\phi_4^G = \frac{4}{4} \sum_{w \in H \cap C_4} \phi(w)$$

So, $\phi_4^G = 1(\phi((1, 3)(2, 8)(4, 6)(5, 7) + (1, 7)(2, 4)(3, 5)(6, 8) + (1, 7)(2, 8)(3, 5)(4, 6) + (1, 3)(2, 4)(5, 7)(6, 8))) = 1(-1 + 1 + 1 + -1) = 0$.

$$\phi_5^G = \frac{4}{4} \sum_{w \in H \cap C_5} \phi(w)$$

So, $\phi_5^G = 1(\phi((2, 6) + (4, 8))) = 1(-1 + -1) = 1(-2) = -2$.

$$\phi_6^G = \frac{4}{4} \sum_{w \in H \cap C_6} \phi(w)$$

So, $\phi_6^G = 1(\phi(\emptyset)) = 1(0) = 0$.

$$\phi_7^G = \frac{4}{4} \sum_{w \in H \cap C_7} \phi(w)$$

So, $\phi_7^G = 1(\phi((1, 5)(2, 6)(3, 7) + (1, 5)(3, 7)(4, 8))) = 1(1 + 1) = 1(2) = 2$.

$$\phi_8^G = \frac{4}{4} \sum_{w \in H \cap C_8} \phi(w)$$

So, $\phi_8^G = 1(\phi(\emptyset)) = 1(0) = 0$.

$$\phi_9^G = \frac{4}{8} \sum_{w \in H \cap C_9} \phi(w)$$

So, $\phi_9^G = \frac{1}{2}(\phi(\emptyset)) = \frac{1}{2}(0) = 0$.

$$\phi_{10}^G = \frac{4}{8} \sum_{w \in H \cap C_{10}} \phi(w)$$

So, $\phi_{10}^G = \frac{1}{2}(\phi(\emptyset)) = \frac{1}{2}(0) = 0$.

$$\phi_{11}^G = \frac{4}{8} \sum_{w \in H \cap C_{11}} \phi(w)$$

So, $\phi_{11}^G = \frac{1}{2}(\phi((1, 3)(2, 8, 6, 4)(5, 7) + (1, 3)(2, 4, 6, 8)(5, 7) + (1, 7)(2, 8, 6, 4)(3, 5) + (1, 7)(2, 4, 6, 8)(3, 5))) = \frac{1}{2}(1 + 1 + -1 + -1) = \frac{1}{2}(0) = 0$.

$$\phi_{12}^G = \frac{4}{8} \sum_{w \in H \cap C_{12}} \phi(w)$$

So, $\phi_{12}^G = \frac{1}{2}(\phi(\emptyset)) = \frac{1}{2}(0) = 0$.

$$\phi_{13}^G = \frac{4}{8} \sum_{w \in H \cap C_{13}} \phi(w)$$

So, $\phi_{13}^G = \frac{1}{2}(\phi(\emptyset)) = \frac{1}{2}(0) = 0$.

Thus, $\phi \uparrow_H^G = 4 \quad -4 \quad 0 \quad 0 \quad -2 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$.

Now, we find the right coset representation of G over H (transversal of H in G), such that, $G = He \cup (1, 2, 3, 8)(4, 5, 6, 7) \cup (1, 2, 3, 4, 5, 6, 7, 8) \cup (1, 3, 5, 7)(2, 8)(4, 6)$ and let $t_1 = e$, $t_2 = (1, 2, 3, 8)(4, 5, 6, 7)$, $t_3 = (1, 2, 3, 4, 5, 6, 7, 8)$, and $t_4 = (1, 3, 5, 7)(2, 8)(4, 6)$.

The monomial representation has generators

$$A(xx) = \begin{bmatrix} \phi(t_1 x t_1^{-1}) & \phi(t_1 x t_2^{-1}) & \phi(t_1 x t_3^{-1}) & \phi(t_1 x t_4^{-1}) \\ \phi(t_2 x t_1^{-1}) & \phi(t_2 x t_2^{-1}) & \phi(t_2 x t_3^{-1}) & \phi(t_2 x t_4^{-1}) \\ \phi(t_3 x t_1^{-1}) & \phi(t_3 x t_2^{-1}) & \phi(t_3 x t_3^{-1}) & \phi(t_3 x t_4^{-1}) \\ \phi(t_4 x t_1^{-1}) & \phi(t_4 x t_2^{-1}) & \phi(t_4 x t_3^{-1}) & \phi(t_4 x t_4^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$A(yy) = \begin{bmatrix} \phi(t_1yyt_1^{-1}) & \phi(t_1yyt_2^{-1}) & \phi(t_1yyt_3^{-1}) & \phi(t_1yyt_4^{-1}) \\ \phi(t_2yyt_1^{-1}) & \phi(t_2yyt_2^{-1}) & \phi(t_2yyt_3^{-1}) & \phi(t_2yyt_4^{-1}) \\ \phi(t_3yyt_1^{-1}) & \phi(t_3yyt_2^{-1}) & \phi(t_3yyt_3^{-1}) & \phi(t_3yyt_4^{-1}) \\ \phi(t_4yyt_1^{-1}) & \phi(t_4yyt_2^{-1}) & \phi(t_4yyt_3^{-1}) & \phi(t_4yyt_4^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since G has square root of unity we find that \mathbb{Z}_3 is the smallest finite field of square root unity. Thus, the primitive root of \mathbb{Z}_3 is -1 , and the entries of the matrix $A(xx)$ and $A(yy)$ are powers of 3, namely, $-1 \equiv_3 2$.

The permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation is

$$A(xx) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

where $a_{11} = 2$, $a_{23} = 1$, $a_{32} = 1$, $a_{44} = 1$.

Therefore,

$$t_1 \rightarrow t_1^2,$$

$$t_2 \rightarrow t_3,$$

$$t_3 \rightarrow t_2,$$

$$t_4 \rightarrow t_4^2.$$

Since, we have 4 t_i 's of order 3 we label them accordingly:

Table 3.7: Labeling for $A(xx)$

1	2	3	4	5	6	7	8
t_1	t_2	t_3	t_4	t_1^2	t_2^2	t_3^2	t_4^2
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
t_1^2	t_3	t_2	t_4^2	t_1	t_3^2	t_2^2	t_4
5	3	2	8	1	7	6	4

Therefore, $A(xx) = (1, 5)(2, 3)(4, 8)(6, 7)$. Similarly,

$$A(yy) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $a_{12} = 1$, $a_{21} = 2$, $a_{34} = 1$, $a_{43} = 1$.

Therefore,

$$t_1 \rightarrow t_2,$$

$$t_2 \rightarrow t_4^2,$$

$$t_3 \rightarrow t_4,$$

$$t_4 \rightarrow t_3.$$

Table 3.8: Labeling for $A(yy)$

1	2	3	4	5	6	7	8
t_1	t_2	t_3	t_4	t_1^2	t_2^2	t_3^2	t_4^2
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
t_2	t_1^2	t_4	t_3	t_2^2	t_1	t_4^2	t_3^2
2	5	4	3	6	1	8	7

Therefore, $A(yy) = (1, 2, 5, 6)(3, 4)(7, 8)$. In MAGMA we confirm if $A(xx)$ and $A(yy)$ generate a group isomorphic to the group G .

`>S:=Sym(8);`

```

>Aper:=S!(1,5)(2,3)(4,8)(6,7);
>Bper:=S!(1,2,5,6)(3,4)(7,8);
>HH:=sub<S|Aper,Bper>;
>IsIsomorphic(HH,G);
true Mapping from: GrpPerm: HH to GrpPerm: G
Composition of Mapping from: GrpPerm: HH to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G

>FPGGroup(G);
Finitely presented group on 2 generators
Relations
$.1^2 = Id($)
$.2^4 = Id($)
($.2 * $.1 * $.2^-1 * $.1)^2 = Id($)
$.2^-1 * $.1 * $.2^-2 * $.1 * $.2^2 * $.1 * $.2^2 * $.1 * $.2^-1 = Id($)

>NN<a,b>:=Group<a,b|a^2,b^4,(b*a*b^-1*a)^2,
(b^-1 * a * b^-2 * a * b^2 * a * b^2 * a * b^-1)>;
>#NN;

```

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Hence, our work is correct and we have a monomial progenitor $3^{*4} : (2^4 : 4)$. Hence, a symmetric presentation for the progenitor $3^{*4} ;_m (2^4 : 4)$ is $\langle x, y, t | x^2, y^4, (y * x * y^{-1} * x)^2, (y^{-1} * x * y^{-2} * x * y^2 * x * y^2 * x * y^{-1}), t^3, (t, x * y^2), (t, x * y * x * y * x * y * x * y^{-1}) \rangle$.

$x * y * x * y^{-1}$), $t^x = t^2 >$. We obtain our first order relations as previously done in **Section 2.1**. So, the first order relations are $((xy)^4t)^a$, $((yxy)^2ty^2xy)^b$, $(xt^{y^2xy})^c$, $(y^2txy^{-1}xy^2)^d$, $((xy^{-1}xy)t^{y^2xy})^e$, $((xy^2xyxy)t^{y^2xy})^f$, $((yx)^2t)^g$, $(yt^{xy^{-1}xy^2})^h$, $((y^{-1})t^{xy^{-1}xy^2})^i$, $((yxy)t^{xy^{-1}xy^2})^j$, $(yx)^3t)^k$. We factor the progenitor G by these relations to obtain finite images (see **Chapter 4**). Thus, the final symmetric presentation is given by

$$\begin{aligned}
G < x, y, t | x^2, y^4, (y * x * y^{-1} * x)^2, (y^{-1} * x * y^{-2} * x * y^2 * x * y^2 * x * \\
& y^{-1}), t^3, (t, x * y^2), (t, x * y * x * y * x * y * x * y^{-1}), t^x = t^2, ((xy)^4t)^a, \\
& ((yxy)^2ty^2xy)^b, (xt^{y^2xy})^c, (y^2txy^{-1}xy^2)^d, ((xy^{-1}xy)t^{y^2xy})^e, ((xy^2xyxy)t^{y^2xy})^f, \\
& ((yx)^2t)^g, (yt^{xy^{-1}xy^2})^h, ((y^{-1})t^{xy^{-1}xy^2})^i, ((yxy)t^{xy^{-1}xy^2})^j, (yx)^3t)^k >.
\end{aligned}$$

Chapter 4

Double Coset Enumeration

4.1 Construction of $2^4 : (3^2 : 4)$ over $(3^2 : 4)$

Consider the progenitor $2^4 : (3^2 : 4) \cong \langle x, y, z, t \mid x^3, y^2, z^4, (x^{-1} * y)^2, z^{-1} * y * z^{-1}, x^{-1} * z^{-1} * x^{-1} * z^{-1} * x^{-1} * z * x * z, t^2, (t, x), (t, z * x^{-1} * z * x) \rangle$, where $N = \langle x, y, z \rangle \cong (3^2 : 4)$ and the action on the symmetric generators are given by with $x \sim (2, 4, 6)$, $y \sim (1, 5)(2, 4)$, $z \sim (1, 4, 5, 2)(3, 6)$, and t_1 , and $2^{*6} : (3^2 : 4)$ is factored by the relations $(xzxz^{-1}t)^6$, $(xzxz^{-1}t^{-1})^3$, and $(zt)^8$. We begin by expanding these relations starting with $(xzxz^{-1}t)^6 = 1$.

$$\begin{aligned}
 & (xzxz^{-1}t)^6 = 1 \\
 \implies & [(2, 4, 6) * (1, 4, 5, 2)(3, 6) * (2, 4, 6) * (1, 2, 5, 4)(3, 6)t_1]^6 \\
 & \implies [(1, 3, 5)(2, 4, 6)t_1]^6 \\
 \implies & (\pi t_1)^6 \text{ (we let } (1, 3, 5)(2, 4, 6) = \pi \text{)}
 \end{aligned}$$

$$\begin{aligned}
&\implies \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \\
&\implies \pi \pi \pi^{-1} t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \\
&\implies \pi^2 \pi \pi^{-1} t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \\
&\implies \pi^3 \pi \pi^{-1} t_1 \pi^2 t_1 \pi t_1 \pi t_1 \pi t_1 \\
&\implies \pi^4 \pi \pi^{-1} t_1 \pi^3 t_1 \pi^2 t_1 \pi t_1 \pi t_1 \\
&\implies \pi^5 \pi \pi^{-1} t_1 \pi^4 t_1 \pi^3 t_1 \pi^2 t_1 \pi t_1 \\
&\implies \pi^6 t_1 \pi^5 t_1 \pi^4 t_1 \pi^3 t_1 \pi^2 t_1 \pi t_1 \\
&\implies et_5 t_3 t_1 t_5 t_3 t_1 = 1 \\
&\implies t_5 t_3 t_1 = t_1 t_3 t_5 \\
&\implies Nt_5 t_3 t_1 = Nt_1 t_3 t_5.
\end{aligned} \tag{4.1}$$

Now we expand our second relation, $(xzxz^{-1}tz^{-1})^3 = 1$, as follows

$$\begin{aligned}
&(xzxz^{-1}tz^{-1})^3 = 1 \\
&\implies [(2, 4, 6) * (1, 4, 5, 2)(3, 6) * (2, 4, 6) * (1, 2, 5, 4)(3, 6)t_1^{(1,2,5,4)(3,6)}]^3 \\
&\implies [(1, 3, 5)(2, 4, 6)t_2]^3 \\
&\implies (\pi t_2)^3(\text{let}(1, 3, 5)(2, 4, 6) = \pi) \\
&\implies \pi t_2 \pi t_2 \pi t_2 \\
&\implies \pi^3 t_2 \pi^2 t_2 \pi t_2 \\
&\implies et_6 t_4 t_2 = 1 \\
&\implies et_6 t_4 t_2 = 1
\end{aligned} \tag{4.2}$$

$$\implies Nt_6t_4 = Nt_2.$$

Also, $(zt)^8 = 1$, (let $z = (1, 4, 5, 2)(3, 6) = \pi$)

$$\implies (\pi t_1)^8$$

$$\implies \pi^8 t_1^{\pi^7} t_1^{\pi^6} t_1^{\pi^5} t_1^{\pi^4} t_1^{\pi^3} t_1^{\pi^2} t_1^{\pi} t_1$$

$$\implies et_2t_5t_4t_1t_2t_5t_4t_1 = 1$$

$$\implies et_2t_5t_4t_1t_2t_5t_4t_1 = 1 \tag{4.3}$$

$$\implies Nt_2t_5t_4t_1 = Nt_1t_4t_5t_2.$$

Which gives the finite homomorphic image

$$G \cong \frac{2^{*6}:(3^2:4)}{t_5t_3t_1=t_1t_3t_5, t_6t_4t_2=1, t_2t_5t_4t_1t_2t_5t_4t_1=1}.$$

Note that each of these relations, when conjugated by N , give more relations. Our goal will be to find the index of N in G through manual double coset enumeration of G over N . In order to do this we express G as a union of double cosets NgN , where g is an element of G . So $G = NeN \cup Ng_1N \cup Ng_2N \cup \dots$ where g_i 's are words in t_i 's. We will find all double cosets $[w] = [Nw^n | n \in N]$ as well as the number of single cosets each of the double cosets contain. We take one representative from each orbit of the coset stabilising group $N^{(w)}$ in order to identify which double coset Nwt_i belongs and finish when the set of right cosets obtained is closed under right multiplication by the t_i 's.

We will now proceed our manual double coset enumeration of $2^4 : (3^2 : 4)$ over $(3^2 : 4)$.

First Double Coset $NeN = [*]$

Let NeN be denoted by $[*]$. $NeN = \{N(e)^n | n \in N\} = \{Ne^{(e)}, Ne^{(1,6)(2,3,4,5)}, Ne^{(1,2,5,4)(3,6)}, \dots, Ne^{(1,3,5)(2,6,4)} | n \in N\} = \{N\}$. Now, N is the only single coset contained in the double coset $[*]$ and is transitive on $X = \{1, 2, 3, 4, 5, 6\}$ so it has the single orbit:

$$\{1, 2, 3, 4, 5, 6\}.$$

We select 1 to be the representative from the orbit, so $Nt_1 \in Nt_1N = \{Nt_1^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\}$, forming the new double coset, $[1]$, which means that all 6 symmetric generators move forward to Nt_1N since they share the same orbit.

Second Double Coset $Nt_1N = [1]$

Now, Nt_1N is a new coset and the number of single cosets is $\frac{|N|}{|N^{(1)}|}$, where $N^{(1)}$ is the coset stabilizer of the coset Nt_1 which is greater than or equal to the point stabilizer of 1 in N , denoted by N^1 . $N^{(1)} \geq N^1 = \{e, (3, 5)(4, 6), (2, 4)(3, 5), (2, 4, 6), (2, 6, 4), (2, 6)(3, 5)\} = \langle (2, 4, 6), (2, 4)(3, 5) \rangle \cong S_3$. Therefore, the number of single cosets in the double coset Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{36}{6} = 6$ and the orbits of $N^{(1)}$ on X are:

$$\{1\}, \{2, 4, 6\}, \text{ and } \{3, 5\}$$

Selecting 1, 2, and 3 as representatives of each orbit, respectively, forming the cosets Nt_1t_1 , Nt_1t_2 , and Nt_1t_3 . We must determine where each coset representative belongs to which double coset:

$$Nt_1t_1 = ?, Nt_1t_2 = ?, \text{ and } Nt_1t_3 = ?$$

Now, $Nt_1t_1 \in [*]$, since $Nt_1t_1 = Nt_1^2 = N$. Therefore, one symmetric generator goes back to $[*]$. Next, $Nt_1t_2 \in Nt_1t_2N = [12]$ so, two symmetric generators extend to $[12]$. Also, Nt_1t_3 , we must use one of our relations to gain clarity. Using the 4.2 relation listed above, $64 = 2$ and conjugating the relation by one element of N , $(1,4,3,6)(2,5)$, we obtain

$$\begin{aligned} Nt_6t_4^{(1436)(25)} &= Nt_2^{(1436)(25)} \\ \implies Nt_1t_3 &= Nt_1 \in [1] \end{aligned}$$

. Hence, two symmetric generators loop to $[1]$.

Third Double Coset $Nt_1t_2N = [12]$

Now, Nt_1t_2N is a new coset. The point stabilizer $N^{12} = \{e, (35)(46)\}$. But, looking at our relation, $1212=e$, we can see that it gives us $Nt_1t_2 = Nt_1t_2$, conjugating by $(1,2)(3,4,5,6)$:

$$\begin{aligned} Nt_1t_2^{(12)(3,4,5,6)} &= Nt_2t_1 \\ \implies &= Nt_1t_2 \in [12]. \end{aligned}$$

So, $N^{(12)} \geq N^{12} = \langle N^{12}, (1,2)(3,4,5,6) \rangle$. Therefore, the number of single cosets in the double coset Nt_1t_2N is $\frac{|N|}{|N^{(12)}|} = \frac{36}{4} = 9$ and the orbits of $N^{(12)}$ on X are:

$$\{1, 2\}, \text{ and } \{3, 4, 5, 6\}$$

Now, selecting 2 and 3 representatives from each orbit, respectively, forming the cosets $Nt_1t_2t_2$, and $Nt_1t_2t_3$. We must determine:

$$Nt_1t_2t_2 = ?, \text{ and } Nt_1t_2t_3 = ?$$

Now, $Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$. Therefore, two symmetric generators go back to $[1]$. Now, to find $Nt_1t_2t_3$ we must start with

$$Nt_1t_2 = Nt_2t_1(\text{relation above})$$

$$\implies Nt_1t_2t_3 = Nt_2t_1t_3(\text{multiply by } 3)$$

we know

$$Nt_1t_3 = Nt_5(\text{from relation above})$$

$$\implies Nt_2t_1t_3 = Nt_2t_5(\text{multiplying by } 2 \text{ on the left})$$

So, $Nt_1t_2t_3 = Nt_2t_5 \in [1,2]$ (since $Nt_2t_5^{(1,2,5,6)(3,4)} = Nt_1t_2$). Therefore, four symmetric generators loop to $[1,2]$. We have no new double cosets, hence, we can construct the following Cayley diagram.

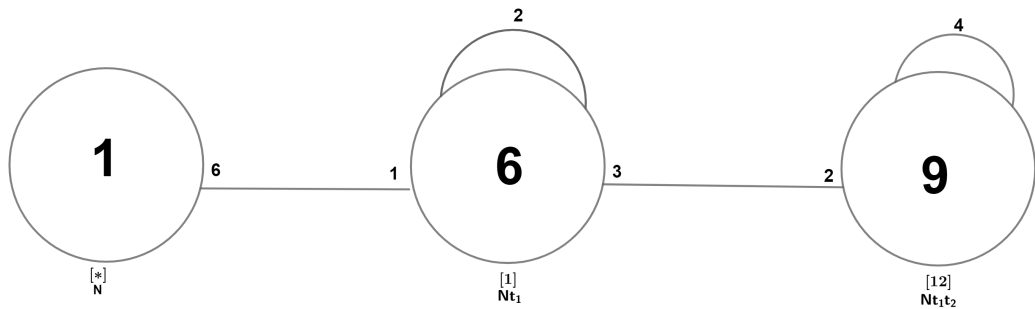


Figure 4.1: Cayley Diagram of $2^4 : (3^2 : 4)$ over $(3^2 : 4)$

4.2 Construction of A_7 over $PSL(2, 7)$

Consider the progenitor $A_7 \cong \langle x, y, t \mid x^3, y^2, (x * y)^2, t^3, t(x * y) = t^2 \rangle$, where $N = \langle x, y \rangle \sim PSL(2, 7)$ and the action on the symmetric generators are given by with $x \sim (1, 3, 5)(2, 4, 6)$, $y \sim (1, 6)(2, 5)(3, 4)$ and $t \sim t_1$, factored by the relations: $(x * t)^4$, $(x * y * t)^2$, and $(x * y * t^x * t^2 * t^y * (t^x))^3$. Our expanded relations results are:

$$(135)(246)t_1t_5t_3t_1 = 1 \quad (4.4)$$

$$\implies Nt_1t_5 = Nt_1t_3$$

$$et_4t_1 = 1 \quad (4.5)$$

$$\implies Nt_4 = Nt_1$$

$$(14)(23)(56)t_3t_4t_6t_3t_2t_1t_5t_2t_3t_4t_6t_3 = 1 \quad (4.6)$$

$$\implies Nt_3t_4t_6t_3t_2t_1 = Nt_3t_6t_4t_3t_2t_5.$$

Which gives the finite homomorphic image

$$G \cong \frac{2^{*6} : (PSL(2, 7))}{t_{tt} = t_{tt}, t_{tt} = t_{tt}, t_{ttt} = t_{ttt}}.$$

We will now proceed our manual double coset enumeration of A_7 over $PSL(2, 7)$.

First Double Coset $NeN = [*]$

Let NeN be denoted by $[*]$. $NeN = \{N(e)^n \mid n \in N\} = \{N\}$. Now, N is the only single coset contained in the double coset $[*]$ and is transitive on $X = \{1, 2, 3, 4, 5, 6\}$ so it has the single orbit:

$$\{1, 2, 3, 4, 5, 6\}.$$

We select 1 to be the representative from the orbit, so $Nt_1 \in Nt_1N = \{Nt_1^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\}$, forming the new double coset, [1], which means that all 6 symmetric generators move forward to Nt_1N since they share the same orbit.

Second Double Coset $Nt_1N = [1]$

Now, Nt_1N is a new coset and the number of single cosets is $\frac{|N|}{|N^{(1)}|}$, where $N^{(1)}$ is the coset stabilizer of the coset Nt_1 which is greater than equal to the point stabilizer of 1 in N , denoted by N^1 . In this case, $N^{(1)} \geq N^1 = \{e\}$. Therefore, the number of single cosets in the double coset Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{6}{1} = 6$ and the orbits of $N^{(1)}$ on X are:

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \text{ and } \{6\}.$$

Selecting 1, 2, 3, 4, 5, and 6 as representatives of each orbit, respectively, forming the cosets $Nt_1t_1, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_1t_5$ and Nt_1t_6 . We must determine where each coset representative belongs to which double coset:

$$Nt_1t_1 = ?, Nt_1t_2 = ?, Nt_1t_3 = ?, Nt_1t_4 = ?, Nt_1t_5 = ? \text{ and } Nt_1t_6 = ?$$

Now, $Nt_1t_1 \in [1]$, since $Nt_1t_1 = Nt_1^{(x*y)} = Nt_1^{(1,4)(2,3)(5,6)} = Nt_4 \in Nt_1N$. Therefore, one symmetric generator loops to [1]. Next, $Nt_1t_2 \in Nt_1t_2N = [12]$ so, one symmetric generator extends to [12]. Similarly, $Nt_1t_3, Nt_1t_5,$ and Nt_1t_6 one symmetric generator extend to [13], [15], and [16], respectively. Now, for Nt_1t_4 we

must use one of our relations to gain clarity to conclude that $Nt_1t_4 \in [\star]$.

Third Double Coset $Nt_1t_2N = [12]$

Now, Nt_1t_2N is a new coset. The point stabilizer $N^{12} = \{e\}$. So, $N^{(12)} \geq N^{12} = \{e\}$. Therefore, the number of single cosets in the double coset Nt_1t_2N is $\frac{|N|}{|N^{(12)}|} = \frac{6}{1} = 6$ and the orbits of $N^{(12)}$ on X are:

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \text{ and } \{6\}.$$

Now, selecting one representative from each orbit, respectively, forming the cosets: $Nt_1t_2t_1$, $Nt_1t_2t_2$, $Nt_1t_2t_3$, $Nt_1t_2t_4$, $Nt_1t_2t_5$, and $Nt_1t_2t_6$. We must determine where each of these go to. Now, $Nt_1t_2t_1 \in [121]$. Therefore, one symmetric generator extends to $[121]$. Now, $Nt_1t_2t_2 \in [15]$ since $Nt_1t_2t_2 = t_2^2 = t_1t_5$ since by our t_i 's $t_5 = t_2^{-1}$. Next, $Nt_1t_2t_3 = t_1t_5^{(1,6)(2,5)(3,4)} = t_6t_2 \in [15]$. Therefore, two symmetric generators extend to $[15]$. So, $Nt_1t_2t_4 = Nt_4t_5 \in [1,6]$ (since $Nt_4t_5^{(1,4)(2,3)(5,6)} = Nt_1t_5$). Therefore, one symmetric generator goes back to $[1,6]$. Next, $Nt_1t_2t_5 \in [1]$. Since $Nt_1t_2t_5 = Nt_1^e = Nt_1$. Lastly, $Nt_1t_2t_6 \in [12]$. since $Nt_1t_2t_6 = Nt_1t_2^{(1,2)(3,6)(4,5)} = Nt_2t_1$.

Fourth Double Coset $Nt_1t_3N = [13]$

Now, Nt_1t_3N is a new coset. The point stabilizer $N^{13} = \{e\}$. So, $N^{(13)} \geq N^{13} = \{e\}$.

Lemma: $13=42$

Proof:

Now, $N(t_1t_3)^{(1,4)(2,3)(5,6)} = Nt_4t_2 = Nt_1t_3$ implies $(1, 4)(2, 3)(5, 6) \in N^{(13)}$. So, $N^{(13)} \geq \langle N^{13}, (1, 4)(2, 3)(5, 6) \rangle$ and $[1, 3]^{N^{(13)}}$ gives $13 \sim 42$.

Therefore, the number of single cosets in the double coset Nt_1t_3N is $\frac{|N|}{|N^{(13)}|} = \frac{6}{2} = 3$. and the orbits of $N^{(13)}$ on X are:

$$\{1, 4\}, \{2, 3\}, \text{ and } \{5, 6\}.$$

Now, selecting one representative from each orbit, respectively, forming the cosets: $Nt_1t_3t_1$, $Nt_1t_3t_2$, and $Nt_1t_3t_5$. We must determine where each of these go to. Now, $Nt_1t_3t_1 = t_1t_6^{(1,2)(3,6)(4,5)} = t_6t_2 \in [16]$. Therefore, two symmetric generators extend to [16]. Next, $Nt_1t_3t_2 = t_1t_6^{(1,4)(2,3)(5,6)} = t_4t_5 \in [16]$. Therefore, two symmetric generators extend to [16]. Now, $Nt_1t_3t_5 = Nt_1^{(1,4)(2,3)(5,6)} = Nt_4 \in [1]$. Therefore, two symmetric generators go back to [1].

Fifth Double Coset $Nt_1t_5N = [15]$

Now, Nt_1t_5N is a new coset. The point stabilizer $N^{15} = \{e\}$. So, $N^{(15)} \geq N^{15} = \{e\}$.

Lemma: 15=46

Proof:

Now, $N(t_1t_5)^{(1,4)(2,3)(5,6)} = Nt_4t_6 = Nt_1t_5$ implies $(1, 4)(2, 3)(5, 6) \in N^{(15)}$. So, $N^{(15)} \geq \langle N^{15}, (1, 4)(2, 3)(5, 6) \rangle$ and $[1, 5]^{N^{(15)}}$ gives $15 \sim 46$. Therefore, the number of single cosets in the double coset Nt_1t_5N is $\frac{|N|}{|N^{(13)}|} = \frac{6}{2} = 3$. and the orbits of $N^{(15)}$ on X are:

$$\{1, 4\}, \{2, 3\}, \text{ and } \{5, 6\}.$$

Now, selecting one representative from each orbit, respectively, forming the cosets: $Nt_1t_5t_1$, $Nt_1t_5t_2$, and $Nt_1t_5t_5$. We must determine where each of these go to. Now, $Nt_1t_5t_1 = t_1t_2^{(1,6)(2,5)(3,4)} = t_6t_5 \in [12]$. Therefore, two symmetric generators extend to [12]. Next, $Nt_1t_5t_2 = t_1^{(e)} = t_1 \in [1]$. Therefore, two symmetric generators extend to [1]. Now, $Nt_1t_5t_5 = Nt_1t_2^{(e)} = Nt_{12} \in [12]$. Therefore, two symmetric generators extend to [12].

Sixth Double Coset $Nt_1t_6N = [16]$

Now, Nt_1t_6N is a new coset. The point stabilizer $N^{16} = \{e\}$. So, $N^{(16)} \geq N^{16} = \{e\}$. Therefore, the number of single cosets in the double coset Nt_1t_6N is $\frac{|N|}{|N^{(16)}|} = \frac{6}{1} = 6$ and the orbits of $N^{(16)}$ on X are:

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \text{ and } \{6\}.$$

Now, selecting one representative from each orbit, respectively, forming the cosets: $Nt_1t_6t_1$, $Nt_1t_6t_2$, $Nt_1t_6t_3$, $Nt_1t_6t_4$, $Nt_1t_6t_5$ and $Nt_1t_6t_6$. We must determine where each of these go to. Now, $Nt_1t_6t_1 = t_1t_2t_1^{(1,5,3)(2,6,4)} = t_5t_6t_5 \in [121]$. Therefore, one symmetric generator extends to [121]. Next, $Nt_1t_6t_2 = t_1t_6^{(1,6)(2,5)(3,4)} = t_6t_1 \in [16]$. Therefore, two symmetric generators loop to [16]. Now, $Nt_1t_6t_3 = Nt_1^{(e)} = Nt_1 \in [1]$. Therefore, one symmetric generator extends to [1]. Now, $Nt_1t_6t_4 = t_1t_2^{(1,4)(2,3)(5,6)} = t_4t_3 \in [12]$. Therefore, one symmetric generator extends to [12]. Now, $Nt_1t_6t_5 = t_1t_3^{(1,2)(3,6)(4,5)} = t_2t_6 \in [13]$. Therefore, one symmetric generator extends to [13]. Lastly, $Nt_1t_6t_6 = t_1t_3^{(e)} = t_1t_3 \in [13]$. Therefore, one symmetric generator extends to [13].

Seventh Double Coset $Nt_1t_21N = [121]$

Now, Nt_1t_21N is a new coset. The point stabilizer $N^{121} = \{e\}$. So, $N^{(121)} \geq N^{121} = \{e\}$.

Lemma: 121=656

Proof:

Now, $N(t_1t_2t_1)^{(1,6)(2,5)(3,4)} = Nt_6t_5t_6 = Nt_1t_2t_1$ implies $(1, 6)(2, 5)(3, 4) \in N^{(121)}$. So, $N^{(121)} \geq \langle N^{121}, (1, 6)(2, 5)(3, 4) \rangle$ and $[1, 2, 1]^{N^{(121)}}$ gives $121 \sim 656$.

Therefore, the number of single cosets in the double coset $Nt_1t_2t_1N$ is $\frac{|N|}{|N^{(121)}|} = \frac{6}{2} = 3$ and the orbits of $N^{(121)}$ on X are:

$$\{1, 6\}, \{2, 5\}, \text{ and } \{3, 4\}.$$

Now, selecting one representative from each orbit, respectively, forming the cosets: $Nt_1t_2t_1t_1$, $Nt_1t_2t_1t_2$, and $Nt_1t_2t_1t_3$. We must determine where each of these go to. Now, $Nt_1t_2t_1t_1 = t_1t_6^{(1,4)(2,3)(5,6)} = t_4t_5 \in [16]$. Therefore, two symmetric generators extend to $[16]$. Next, $Nt_1t_2t_1t_2 = t_1t_2t_1^{(e)} = t_1t_2t_1 \in [121]$. Therefore, two symmetric generators extend to $[121]$. Now, $Nt_1t_2t_1t_3 = Nt_1t_2^{(1,6)(2,5)(3,4)} = Nt_6t_5 \in [12]$. Therefore, two symmetric generators extend to $[12]$. A cayley diagram is shown below.

$$\{1, 2, 3, 4, 5, 6, 7, 8\}.$$

Based on the relations (see appendix), we see that $t_1 = t_5, t_2 = t_6, t_3 = t_7, t_4 = t_8$, which means that only four symmetric generators move forward to Nt_1N . We select 1 to be the representative from the orbit, so $Nt_1 \in Nt_1N = \{Nt_1^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4\}$, forming the new double coset, [1].

Second Double Coset $Nt_1N = [1]$

Now, Nt_1N is a new coset and the number of single cosets is $\frac{|N|}{|N^{(1)}|}$, where $N^{(1)}$ is the coset stabilizer of the coset Nt_1 which is greater than equal to the point stabilizer of 1 in N, denoted by N^1 . In this case, $N^{(1)} \geq N^1 = \{e, (2, 3, 8)(4, 6, 7)\}$. Therefore, the number of single cosets in the double coset Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{24}{6} = 4$ and the orbits of $N^{(1)}$ on X are:

$$\{1\}, \{5\}, \{2, 3, 8\}, \text{ and } \{4, 6, 7\}.$$

Selecting 1, 5, 2, and 6 as representatives of each orbit, respectively, forming the cosets $Nt_1t_1, Nt_1t_5, Nt_1t_2$, and Nt_1t_6 . We must determine where each coset representative belongs to which double coset:

$$Nt_1t_1 = ?, Nt_1t_5 = ?, Nt_1t_2 = ?, \text{ and } Nt_1t_6 = ?$$

Now, $Nt_1t_1 \in [1]$, since $Nt_1t_1 = Nt_1^2 = Ne \in N$. Therefore, one symmetric generator goes back [*]. Next, $Nt_1t_5 = Nt_1t_1$ since $t_1 = t_5$ which belongs to N . So, one symmetric generator goes back to [*]. Now, Nt_1t_2 extends three symmetric

generators to [12]. For $Nt_1t_6 = Nt_1t_2$ since $t_2 = t_6$, hence, three symmetric generators extend to [12].

Third Double Coset $Nt_1t_2N = [12]$

Now, Nt_1t_2N is a new coset. The point stabilizer $N^{12} = \{e\}$. So, $N^{(12)} \geq N^{12} = \langle (1,5)(2,6)(3,4)(7,8) \rangle$. Lemma 12 \sim 56. Therefore, the number of single cosets in the double coset Nt_1t_2N is $\frac{|N|}{|N^{(12)}|} = \frac{24}{2} = 12$ and the orbits of $N^{(12)}$ on X are:

$$\{1, 5\}, \{2, 6\}, \{3, 4\}, \text{ and } \{7, 8\}.$$

Selecting 1, 2, 3 and 7 as representatives of each orbit, respectively, forming the cosets $Nt_1t_2t_1$, $Nt_1t_2t_2$, $Nt_1t_2t_3$, and $Nt_1t_2t_7$. We must determine where each coset representative belongs to which double coset:

$$Nt_1t_2t_1 = ?, Nt_1t_2t_2 = ?, Nt_1t_2t_3 = ?, \text{ and } Nt_1t_2t_7 = ?$$

Now, $Nt_1t_2t_1 = Nt_1t_2 \in [12]$. Therefore, two symmetric generators loop to [12]. Next, $Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$ So, two symmetric generators go back to [1]. Now, $Nt_1t_2t_3$ extends two symmetric generators to [123]. For $Nt_1t_2t_7 = Nt_1t_2t_3$ since $t_3 = t_7$, hence, two symmetric generators extend to [123].

Fourth Double Coset $Nt_1t_2t_3N = [123]$

Now, $Nt_1t_2t_3N$ is a new coset. The point stabilizer $N^{123} = \{e\}$. So, $N^{(123)} \geq N^{123} = \{e\}$. Therefore, the number of single cosets in the double coset $Nt_1t_2t_3N$ is $\frac{|N|}{|N^{(123)}|} = \frac{24}{1} = 24$ and the orbits of $N^{(123)}$ on X are:

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \text{ and } \{8\}.$$

Selecting 1, 2, 3, 4, 5, 6, 7 and 8 as representatives of each orbit, respec-

tively, forming the cosets $Nt_1t_2t_3t_1$, $Nt_1t_2t_3t_2$, $Nt_1t_2t_3t_3$, $Nt_1t_2t_3t_4$, $Nt_1t_2t_3t_5$, $Nt_1t_2t_3t_6$, $Nt_1t_2t_3t_7$ and $Nt_1t_2t_3t_8$. We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_2t_3t_1 \in [1231]$. Therefore, one symmetric generator extends to $[1231]$. Next, $Nt_1t_2t_3t_2 = Nt_1t_2t_3 \in [123]$. So, one symmetric generator loops to $[123]$. Now, $Nt_1t_2t_3t_3 = Nt_1t_2t_3^2 = Nt_1t_2 \in [12]$ so, one symmetric generator goes back to $[12]$ $Nt_1t_2t_3t_4 \in [1234]$, so one symmetric generator extends to $[1234]$. For $Nt_1t_2t_3t_5 = Nt_1t_2t_3t_1$ since $t_1 = t_5$ so one symmetric generator goes to $[1231]$. $Nt_1t_2t_3t_6 = Nt_1t_2t_3 \in [123]$ so one generator loops. Also, $Nt_1t_2t_3t_7 = Nt_1t_2t_3t_3 = Nt_1t_2t_3^2 = Nt_1t_2 \in [12]$ and one generator goes back to $[12]$. Lastly, $Nt_1t_2t_3t_8 = Nt_1t_2t_3t_4 \in [1234]$ so one generator extends to $[1234]$.

Fifth Double Coset $Nt_1t_2t_3t_4N = [1234]$

Now, $Nt_1t_2t_3t_4N$ is a new coset. The point stabilizer $N^{1234} = \{e\}$. So, $N^{(1234)} \geq N^{1234} = \langle (1, 8, 2)(4, 6, 5) \rangle$. Therefore, the number of single cosets in the double coset $Nt_1t_2t_3t_4N$ is $\frac{|N|}{|N^{(1234)}|} = \frac{24}{3} = 8$ and the orbits of $N^{(1234)}$ on X are:

$$\{3\}, \{7\}, \{1, 8, 2\}, \text{ and } \{4, 6, 5\}.$$

Selecting 3, 7, 8, and 4 as representatives of each orbit, respectively, forming the cosets $Nt_1t_2t_3t_4t_3$, $Nt_1t_2t_3t_4t_7$, $Nt_1t_2t_3t_4t_8$, and $Nt_1t_2t_3t_4t_4$. We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_2t_3t_4t_3 \in [1234]$, so, one symmetric generator loops to $[1234]$. Next, $Nt_1t_2t_3t_4t_7 = Nt_1t_2t_3t_4t_3 \in [1234]$. So, one symmetric generator loops to $[1234]$. Now, $Nt_1t_2t_3t_4t_8 = Nt_1t_2t_3t_4t_4 = Nt_1t_2t_3t_4^2 = Nt_1t_2t_3 \in [123]$ so, three symmetric generator go back to $[123]$. Lastly, $Nt_1t_2t_3t_4t_4 = Nt_1t_2t_3 \in [123]$, so

three symmetric generators go back to [123].

Sixth Double Coset $Nt_1t_2t_3t_1N = [1231]$

Now, $Nt_1t_2t_3t_1N$ is a new coset. The point stabilizer $N^{1231} = \{e\}$. So, $N^{(1231)} \geq N^{1231} = \langle (1, 4, 3, 6)(2, 5, 8, 7) \rangle$. Therefore, the number of single cosets in the double coset $Nt_1t_2t_3t_1N$ is $\frac{|N|}{|N^{(1231)}|} = \frac{24}{4} = 6$ and the orbits of $N^{(1231)}$ on X are:

$$\{1, 4, 3, 6\}, \text{ and } \{2, 5, 8, 7\}.$$

Selecting 1, and 5 as representatives of each orbit, respectively, forming the cosets $Nt_1t_2t_3t_1t_1$, and $Nt_1t_2t_3t_1t_5$. We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_2t_3t_1t_1 = Nt_1t_2t_3t_1^2 = Nt_1t_2t_3 \in [123]$, so, four symmetric generators go back to [123]. Next, $Nt_1t_2t_3t_1t_5 = Nt_1t_2t_3t_1t_5$ since $t_1 = t_5$. Therefore, $Nt_1t_2t_3t_1^2 = Nt_1t_2t_3 \in [123]$. So, four symmetric generators go back to [123]. A cayley diagram for this manual double coset enuermation is shown below.

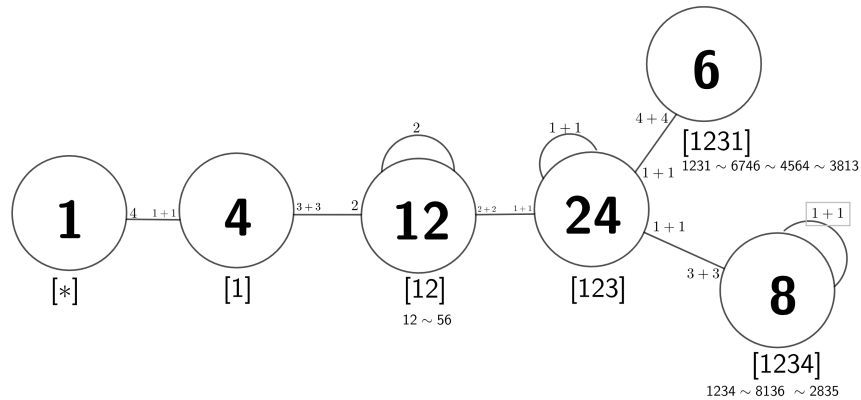


Figure 4.3: Cayley Diagram of $L(2, 11)$ over N

4.4 Construction of $2M_{22}$

Consider the progenitor $2M_{22} \cong \langle x, y, t \mid x^6, y^4, xy^{-2}x^2y^2x, x^{-1}y^{-2}x^{-1}y^{-1}xy^2xy^{-1}, x^{-1}y^{-1}x^{-3}y^{-1}x^3y^{-1}x^{-2}, (y^{-1}x^{-1})^4(yx)^2yx^{-1}, (y^{-1}x^{-2})^2y^{-1}x^2yxy^{-1}x^3yx^{-1}, t^2, (t, yx^2yxyx^{-1}yx), (t, x^2(yx)^2yx^{-2}), (t, xyx^2yx^{-1}yxy^{-1}x) \rangle$, where $N = \langle x, y \rangle \cong A_7$ and the action on the symmetric generators are given by with $x \sim (1, 3, 7, 15, 29, 45)(2, 5, 11, 22, 42, 66)(4, 9, 18, 34, 26, 46)(6, 13, 25)(8, 16, 31, 52, 39, 61)(10, 20, 38, 58, 78, 75)(12, 24, 14, 27, 47, 53)(17, 33, 48, 32, 54, 73)(19, 36, 30, 50, 63, 85)(21, 40, 62, 83, 23, 43)(28, 41, 64, 87, 69, 90)(35, 57, 76, 97, 80, 100)(37, 59, 79, 70, 92, 86)(44, 67, 88, 94, 91, 49)(51, 71, 65, 77, 98, 104)(55, 74, 56, 72, 95, 84)(60, 81, 101)(68, 89, 82)(93, 103, 99, 102, 105, 96)$ and $y \sim (1, 2, 4, 8)(3, 6, 12, 23)(5, 10, 19, 35)(7, 14, 26, 45)(9, 17, 32, 53)(11, 21, 39, 13)(15, 28, 47, 69)(16, 30, 49, 66)(18, 27, 29, 48)(20, 37, 58, 77)(22, 41, 63, 84)(24, 44, 54, 36)(25, 43, 62, 82)(31, 51, 42, 65)(33, 55, 34, 56)(38, 60, 80, 81)(40, 46, 68, 73)(50, 70, 91, 59)(52, 72, 94, 87)(57, 75)(61, 76, 67, 78)(64, 86)(71, 93, 79, 99)(74, 96, 90, 102)(83, 88, 89, 85)(92, 100, 104, 97)(95, 98)(101, 103)$. and t_1 , and $2^{*(105)} : A_7$ is factored by the relation $((yx)^2yx^{-2}y^{-1}xt^{xyx^2yx^3})^3$.

The relation is expanded as shown in Chapter 4.1:

let $xyx^2yx^3 = (1, 62, 92)(2, 60, 31)(3, 41, 19)(4, 46, 77)(5, 82, 72)(6, 104, 34)(7, 79, 33)(8, 56, 11)(9, 26, 25)(10, 96, 76)(12, 89, 66)(13, 18, 71)(14, 22, 74)(15, 63, 43)(16, 61, 53)(17, 54, 49)(20, 105, 36)(21, 32, 37)(23, 29, 73)(24, 52, 27)(28, 97, 57)(30, 90, 40)(35, 70, 38)(39, 81, 95)(42, 55, 101)(44, 50, 99)(45, 83, 87)(47, 84, 68)(48, 94, 67)(51, 98, 65)(58, 85, 91)(59, 103, 78)(64, 100, 69)(75, 88, 93)(80, 102, 86) = a$ and $(yx)^2yx^{-2}y^{-1}x = (1, 80, 35, 74, 23)(2, 104, 56, 48, 97)(3, 93, 55, 71, 8)(4, 49, 67, 50, 44)(5, 73, 33, 38, 102)(6, 101, 9, 28, 30)(7, 16, 51, 84, 76)(10, 32, 22, 98, 52)(11, 40, 20, 39, 77)(12, 14, 79, 89, 59)(13, 63, 90, 91, 85)(15, 17, 95, 99, 105)(18, 60, 25, 36, 94)(19, 64, 69, 34, 87)(21, 103, 78, 61, 29)(24, 86, 68, 82, 92)(26, 88, 41, 46, 81)(27, 53, 37, 47, 70)(31, 58, 57, 54, 45)(42,$

$100, 75, 83, 43)(62, 72, 65, 66, 96) = b$. Now,

$$((yx)^2yx^{-2}y^{-1}xt^{xyx^2yx^3})^3$$

$$\implies (bt_1^a)^3$$

$$\implies (bt_{62})^3$$

$$\implies b^3t_{62}^{b^2}t_{62}^bt_{62}$$

$$\begin{aligned} \implies & (1, 74, 80, 23, 35)(2, 48, 104, 97, 56)(3, 71, 93, 8, 55)(4, 50, 49, 44, 67)(5, 38, 73, 102, 33) \\ & (6, 28, 101, 30, 9)(7, 84, 16, 76, 51)(10, 98, 32, 52, 22)(11, 39, 40, 77, 20)(12, 89, 14, 59, 79)(13, \\ & 91, 63, 85, 90)(15, 99, 17, 105, 95)(18, 36, 60, 94, 25)(19, 34, 64, 87, 69)(21, 61, 103, 29, 78)(24, \\ & 82, 86, 92, 68)(26, 46, 88, 81, 41)(27, 47, 53, 70, 37)(31, 54, 58, 45, 57)(42, 83, 100, 43, 75)(62, \\ & 66, 72, 96, 65)t_{35}t_{80}t_{62} = 1 \end{aligned}$$

$$\begin{aligned} \implies & (1, 74, 80, 23, 35)(2, 48, 104, 97, 56)(3, 71, 93, 8, 55)(4, 50, 49, 44, 67)(5, 38, 73, 102, 33)(6, \\ & 28, 101, 30, 9)(7, 84, 16, 76, 51)(10, 98, 32, 52, 22)(11, 39, 40, 77, 20)(12, 89, 14, 59, 79)(13, 91, 63, \\ & 85, 90)(15, 99, 17, 105, 95)(18, 36, 60, 94, 25)(19, 34, 64, 87, 69)(21, 61, 103, 29, 78)(24, 82, 86, 92, \\ & 68)(26, 46, 88, 81, 41)(27, 47, 53, 70, 37)(31, 54, 58, 45, 57)(42, 83, 100, 43, 75)(62, 66, 72, \\ & 96, 65)t_{35}t_{80} = t_{62} \quad (4.10) \end{aligned}$$

$$\implies Nt_{35}t_{80} = Nt_{62}.$$

Which gives the finite homomorphic image

$$G \cong \frac{2^{*105}:A_7}{t_{35}t_{80}=t_{62}}.$$

Now we can proceed our manual double coset enumeration of $2 \cdot M_{22}$ over A_7 .

First Double Coset $NeN = [*]$

Let NeN be denoted by $[*]$. $NeN = \{N(e)^n | n \in N\} = \{N\}$. Now, N is the only single coset contained in the double coset $[*]$ and is transitive on $X = \{1, 2, 3, 4, 5, 6, \dots, 103, 104, 105\}$ so it has the single orbit:

$$\mathcal{O}\{1, 2, 3, 4, 5, 6, \dots, 103, 104, 105\}.$$

We select 1 to be the representative from the orbit, so $Nt_1 \in Nt_1N = \{Nt_1^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, \dots, Nt_{105}\}$, forming the new double coset, $[1]$, which means that all 105 symmetric generators move forward to Nt_1N since they share the same orbit.

Second Double Coset $Nt_1N = [1]$

Now, Nt_1N is a new coset and the number of single cosets is $\frac{|N|}{|N^{(1)}|}$. The number of single cosets in the double coset Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{2520}{24} = 105$ and the orbits of $N^{(1)}$ on X are:

$\mathcal{O}\{1\}, \mathcal{O}\{21, 33\}, \mathcal{O}\{5, 20, 63, 59, 55, 26\}, \mathcal{O}\{2, 74, 93, 97, 87, 86, 39, 70, 49, 25, 9, 61\}, \mathcal{O}\{3, 67, 31, 95, 45, 19, 48, 43, 42, 17, 40, 90\}, \mathcal{O}\{4, 103, 73, 85, 98, 28, 52, 89, 32, 75, 72, 37\}, \mathcal{O}\{6, 83, 96, 41, 99, 91, 60, 14, 35, 68, 8, 77\}, \mathcal{O}\{7, 105, 80, 38, 16, 79, 88, 65, 69, 23, 51, 44, 29, 36, 92, 47, 66, 24, 13, 22, 84, 57, 34, 62\}$, and $\mathcal{O}\{10, 46, 81, 53, 54, 76, 30, 12, 18, 104, 50, 56, 27, 58, 64, 101, 82, 78, 100, 102, 15, 94, 11, 71\}$.

Selecting 1, 21, 5, 2, 3, 4, 6, 7, and 10 as representatives of each orbit, respectively, forming the cosets Nt_1t_1 , Nt_1t_{21} , Nt_1t_5 , Nt_1t_2 , Nt_1t_3 , Nt_1t_4 , Nt_1t_6 , Nt_1t_7 , and Nt_1t_{10} . We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_1 = N \in [*]$. Therefore, one symmetric generator goes back to $[*]$. Next, Nt_1t_{21} and $Nt_1t_4 \in Nt_1N = [1]$ so, two and twelve symmetric generators loop to $[1]$, respectively. Also, Nt_1t_5 and $Nt_1t_7 \in Nt_1t_5N = [15]$ so, six and twenty-four symmetric generators extend to $[15]$, respectively. Also, Nt_1t_2 , Nt_1t_3 , and $Nt_1t_6 \in [12]$ so twelve symmetric generators extend to $[12]$, respectively.

Third Double Coset $Nt_1t_2N = [12]$

Now, Nt_1t_2N is a new coset. The number of single cosets in the double coset Nt_1t_2N is $\frac{|N|}{|N^{(12)}|} = \frac{2520}{18} = 140$ and the orbits of $N^{(12)}$ on X are:

$\mathcal{O}\{19, 67, 90\}$, $\mathcal{O}\{37, 92, 79\}$, $\mathcal{O}\{1, 24, 47, 89, 105, 21, 75, 57, 33\}$, $\mathcal{O}\{2, 36, 85, 34, 38, 15, 84, 72, 11\}$, $\mathcal{O}\{3, 42, 50, 46, 9, 10, 78, 74, 45\}$, $\mathcal{O}\{4, 52, 22, 54, 56, 80, 39, 69, 88\}$, $\mathcal{O}\{7, 51, 70, 53, 12, 97, 58, 71, 73\}$, $\mathcal{O}\{8, 83, 13, 44, 28, 66, 16, 103, 55\}$, $\mathcal{O}\{30, 81, 60, 49, 101, 41, 63, 87, 94\}$, $\mathcal{O}\{5, 26, 17, 61, 76, 31, 40, 102, 43, 25, 6, 18, 64, 91, 96, 95, 48, 35\}$, and $\mathcal{O}\{14, 99, 93, 32, 23, 100, 104, 65, 86, 98, 77, 62, 29, 20, 59, 27, 82, 68\}$.

Selecting 19, 37, 1, 2, 3, 4, 7, 8, 30, 5, and 14 as representatives of each orbit, respectively, forming the cosets $Nt_1t_2t_{19}$, $Nt_1t_2t_{37}$, $Nt_1t_2t_1$, $Nt_1t_2t_2$, $Nt_1t_2t_3$, $Nt_1t_2t_4$, $Nt_1t_2t_7$, $Nt_1t_2t_8$, $Nt_1t_2t_{30}$, $Nt_1t_2t_5$, and $Nt_1t_2t_{14}$. We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_2t_{19}$ and $Nt_1t_2t_{37}$; $Nt_1t_2t_3$ and $Nt_1t_2t_8$; $Nt_1t_2t_5$ and $Nt_1t_2t_{14}$

$\in [12]$. Therefore, three, nine, and eight-teen symmetric generators loop to $[12]$, respectively. Next, $Nt_1t_2t_1$ and $Nt_1t_2t_4 \in Nt_1t_5N$ so, nine symmetric generators extend to $[15]$, respectively. Also, $Nt_1t_2t_2$, $Nt_1t_2t_7$, and $Nt_1t_2t_{30} \in Nt_1N$ so, nine symmetric generators go back to $[1]$, respectively.

Fourth Double Coset $Nt_1t_5N = [15]$

Now, Nt_1t_5N is a new coset. The number of single cosets in the double coset Nt_1t_5N is $\frac{|N|}{|N^{(15)}|} = \frac{2520}{24} = 105$ and the orbits of $N^{(15)}$ on X are:

$\mathcal{O}\{59\}$, $\mathcal{O}\{70, 86\}$, $\mathcal{O}\{1, 26, 20, 55, 63, 5\}$, $\mathcal{O}\{2, 25, 93, 39, 61, 21, 74, 87, 49, 9, 97, 33\}$, $\mathcal{O}\{3, 17, 47, 27, 10, 80, 51, 92, 77, 102, 82, 83\}$, $\mathcal{O}\{4, 75, 29, 58, 18, 7, 22, 66, 19, 30, 95, 56\}$, $\mathcal{O}\{28, 85, 64, 44, 88, 50, 81, 101, 60, 69, 36, 91\}$, $\mathcal{O}\{6, 8, 41, 62, 72, 105, 42, 13, 23, 52, 67, 99, 84, 16, 94, 90, 11, 31, 73, 32, 46, 34, 100, 57\}$, and $\mathcal{O}\{12, 78, 53, 45, 54, 38, 35, 24, 48, 15, 14, 76, 43, 40, 89, 68, 103, 96, 104, 71, 37, 79, 98, 65\}$.

Selecting 59, 70, 5, 2, 3, 4, 28, 6, and 12 as representatives of each orbit, respectively, forming the cosets $Nt_1t_5t_{59}$, $Nt_1t_5t_{70}$, $Nt_1t_5t_5$, $Nt_1t_5t_2$, $Nt_1t_5t_3$, $Nt_1t_5t_4$, $Nt_1t_5t_{28}$, $Nt_1t_5t_6$, and $Nt_1t_5t_{12}$. We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_5t_{59} \in [1 \ 5 \ 59]$, so one symmetric generator extends to $[1 \ 5 \ 59]$, respectively. Next, $Nt_1t_5t_{70}$, $Nt_1t_5t_3$ and $Nt_1t_5t_6N \in Nt_1t_5$, so two, twelve, and twenty-four symmetric generators loop to $[15]$, respectively. Also, $Nt_1t_5t_5$ and $Nt_1t_5t_{12} \in Nt_1N$ so, six and twenty-four symmetric generators go back to $[1]$, respectively. Also, $Nt_1t_5t_2, Nt_1t_5t_4$ and $Nt_1t_5t_{28} \in Nt_1t_2N$ so, twelve symmetric generators extend to $[12]$, respectively.

Fifth Double Coset $Nt_1t_5t_{59}N = [1\ 5\ 59]$

Now, $Nt_1t_5t_{59}N$ is a new coset. The number of single cosets in the double coset $Nt_1t_5t_{59}N$ is $\frac{|N|}{|N^{(1559)}|} = \frac{2520}{2520} = 1$ and the orbits of $N^{(1559)}$ on X are:

$\mathcal{O}\{1\}, \mathcal{O}\{5\}, \mathcal{O}\{59\}, \mathcal{O}\{3, 17\}, \mathcal{O}\{4, 75\}, \mathcal{O}\{19, 95\}, \mathcal{O}\{21, 33\}, \mathcal{O}\{28, 85\},$
 $\mathcal{O}\{60, 91\}, \mathcal{O}\{61, 74\}, \mathcal{O}\{70, 86\}, \mathcal{O}\{77, 83\}, \mathcal{O}\{2, 25, 93, 49\}, \mathcal{O}\{6, 8, 41, 99\}, \mathcal{O}\{7,$
 $66, 22, 29\}, \mathcal{O}\{9, 39, 87, 97\}, \mathcal{O}\{10, 82, 102, 27\}, \mathcal{O}\{11, 46, 100, 94\}, \mathcal{O}\{12, 78, 53, 76\},$
 $\mathcal{O}\{13, 57, 34, 105\}, \mathcal{O}\{14, 96, 68, 35\}, \mathcal{O}\{15, 71, 104, 54\}, \mathcal{O}\{16, 23, 62, 84\}, \mathcal{O}\{18, 56,$
 $30, 58\}, \mathcal{O}\{20, 63, 55, 26\}, \mathcal{O}\{24, 65, 79, 38\}, \mathcal{O}\{31, 67, 42, 90\}, \mathcal{O}\{32, 52, 72, 73\},$
 $\mathcal{O}\{36, 88, 44, 69\}, \mathcal{O}\{37, 103, 89, 98\}, \mathcal{O}\{40, 48, 45, 43\}, \mathcal{O}\{47, 51, 92, 80\},$ and
 $\mathcal{O}\{50, 101, 81, 64\}.$

Selecting 1, 5, 59, 3, 4, 19, 21, 28, 60, 61, 70, 77, 2, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 24, 31, 32, 36, 37, 40, 47, and 50 as representatives of each orbit, respectively, forming the cosets $Nt_1t_5t_{59}t_1, Nt_1t_5t_{59}t_5, Nt_1t_5t_{59}t_{59}, Nt_1t_5t_{59}t_3,$
 $Nt_1t_5t_{59}t_4, Nt_1t_5t_{59}t_{19}, Nt_1t_5t_{59}t_{21}, Nt_1t_5t_{59}t_{28}, Nt_1t_5t_{59}t_{60}, Nt_1t_5t_{59}t_{61}, Nt_1t_5t_{59}t_{70},$
 $Nt_1t_5t_{59}t_{77}, Nt_1t_5t_{59}t_2, Nt_1t_5t_{59}t_6, Nt_1t_5t_{59}t_7, Nt_1t_5t_{59}t_9, Nt_1t_5t_{59}t_{10}, Nt_1t_5t_{59}t_{11},$
 $Nt_1t_5t_{59}t_{12}, Nt_1t_5t_{59}t_{13}, Nt_1t_5t_{59}t_{14}, Nt_1t_5t_{59}t_{15}, Nt_1t_5t_{59}t_{16}, Nt_1t_5t_{59}t_{18}, Nt_1t_5t_{59}t_{20},$
 $Nt_1t_5t_{59}t_{24}, Nt_1t_5t_{59}t_{31}, Nt_1t_5t_{59}t_{32}, Nt_1t_5t_{59}t_{36}, Nt_1t_5t_{59}t_{37}, Nt_1t_5t_{59}t_{40}, Nt_1t_5t_{59}t_{47}$
and $Nt_1t_5t_{59}t_{50}$. We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_5t_{59}t_1, Nt_1t_5t_{59}t_5, Nt_1t_5t_{59}t_{59}; Nt_1t_5t_{59}t_3, Nt_1t_5t_{59}t_4, Nt_1t_5t_{59}t_{19},$
 $Nt_1t_5t_{59}t_{21}, Nt_1t_5t_{59}t_{28}, Nt_1t_5t_{59}t_{60}, Nt_1t_5t_{59}t_{61}, Nt_1t_5t_{59}t_{70}, Nt_1t_5t_{59}t_{77}; Nt_1t_5t_{59}t_2,$
 $Nt_1t_5t_{59}t_6, Nt_1t_5t_{59}t_7, Nt_1t_5t_{59}t_9, Nt_1t_5t_{59}t_{10}, Nt_1t_5t_{59}t_{11}, Nt_1t_5t_{59}t_{12}, Nt_1t_5t_{59}t_{13},$

$Nt_1t_5t_{59}t_{14}, Nt_1t_5t_{59}t_{15}, Nt_1t_5t_{59}t_{16}, Nt_1t_5t_{59}t_{18}, Nt_1t_5t_{59}t_{20}, Nt_1t_5t_{59}t_{24}, Nt_1t_5t_{59}t_{31},$
 $Nt_1t_5t_{59}t_{32}, Nt_1t_5t_{59}t_{36}, Nt_1t_5t_{59}t_{37}, Nt_1t_5t_{59}t_{40}, Nt_1t_5t_{59}t_{47}, Nt_1t_5t_{59}t_{50} \in [1\ 5],$
 so one, two, and four symmetric generators go back to $[1\ 5]$, respectively. A Cayley Diagram for this manual double coset enumeration is shown below.

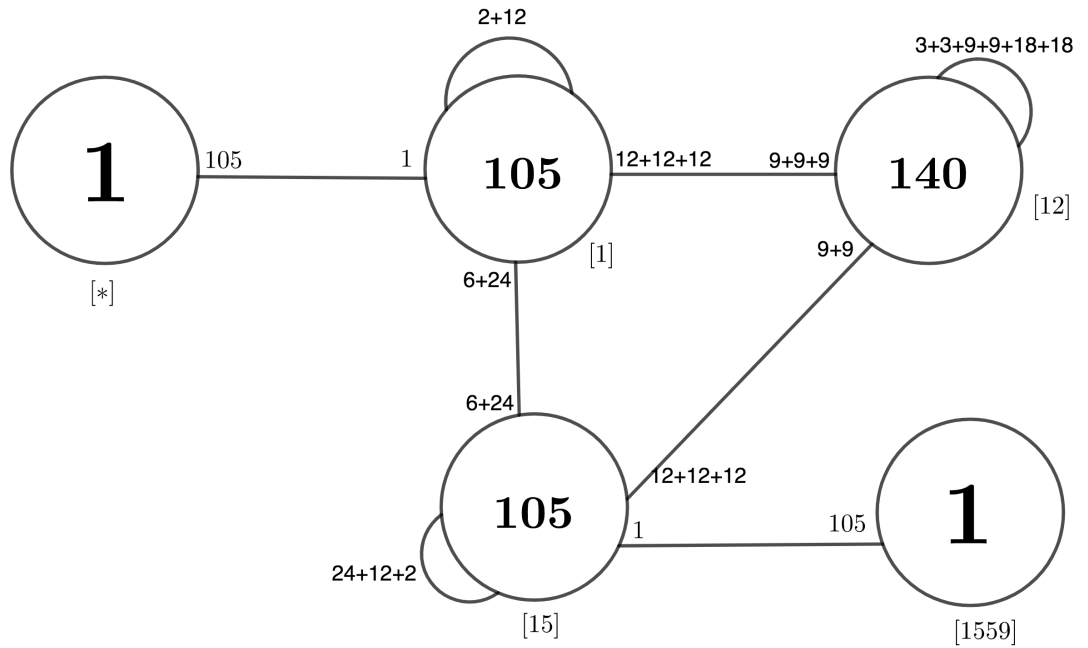


Figure 4.4: Cayley Diagram of $2M_{22}$ over A_7

4.5 Construction of M_{22} over A_7

Consider the progenitor $M_{22} \cong \langle x, y, t \mid x^6, y^4, xy^{-2}x^2y^2x, x^{-1}y^{-2}x^{-1}y^{-1}xy^2xy^{-1}, x^{-1}y^{-1}x^{-3}y^{-1}x^3y^{-1}x^{-2}, (y^{-1}x^{-1})^4(yx)^2yx^{-1}, (y^{-1}x^{-2})^2y^{-1}x^2yxy^{-1}x^3yx^{-1}, t^2, (t, yx^2yxyx^{-1}yx), (t, x^2(yx)^2yx^{-2}), (t, xyx^2yx^{-1}yxy^{-1}x) \rangle$, where $N = \langle x, y \rangle \cong A_7$ and the action on the symmetric generators are given by with $x \sim (1, 3, 7, 15, 29, 45)(2, 5, 11, 22, 42, 66)(4, 9, 18, 34, 26, 46)(6, 13, 25)(8, 16, 31, 52, 39, 61)(10, 20, 38, 58, 78, 75)(12, 24, 14, 27, 47, 53)(17, 33, 48, 32, 54, 73)(19, 36, 30, 50, 63, 85)(21, 40, 62, 83, 23, 43)(28, 41, 64, 87, 69, 90)(35, 57, 76, 97, 80, 100)(37, 59, 79, 70, 92, 86)(44, 67, 88, 94, 91, 49)(51, 71, 65, 77, 98, 104)(55, 74, 56, 72, 95, 84)(60, 81, 101)(68, 89, 82)(93, 103, 99, 102, 105, 96)$ and $y \sim (1, 2, 4, 8)(3, 6, 12, 23)(5, 10, 19, 35)(7, 14, 26, 45)(9, 17, 32, 53)(11, 21, 39, 13)(15, 28, 47, 69)(16, 30, 49, 66)(18, 27, 29, 48)(20, 37, 58, 77)(22, 41, 63, 84)(24, 44, 54, 36)(25, 43, 62, 82)(31, 51, 42, 65)(33, 55, 34, 56)(38, 60, 80, 81)(40, 46, 68, 73)(50, 70, 91, 59)(52, 72, 94, 87)(57, 75)(61, 76, 67, 78)(64, 86)(71, 93, 79, 99)(74, 96, 90, 102)(83, 88, 89, 85)(92, 100, 104, 97)(95, 98)(101, 103)$. and t_1 , and $2^{*(105)} : A_7$ is factored by the relations $((yx)^2yx^{-2}y^{-1}xtxyx^2yx^3)^3$ and $x^{-1}yx^{-1}yxyxyx^{-1}xtxtx^{-1}y^{-1}tyt$. For the calculation of the former relation please refer to Chapter 4 (4.10). The explanation for the latter relation can be accomplished by similar fashion.

Now we can proceed our manual double coset enumeration of M_{22} over A_7 .

First Double Coset $NeN = [*]$

Let NeN be denoted by $[*]$. $NeN = \{N(e)^n \mid n \in N\} = \{N\}$. Now, N is the only single coset contained in the double coset $[*]$ and is transitive on $X = \{1, 2, 3, 4, 5, 6, \dots, 103, 104, 105\}$ so it has the single orbit:

$$\mathcal{O}\{1, 2, 3, 4, 5, 6, \dots, 103, 104, 105\}.$$

We select 1 to be the representative from the orbit, so $Nt_1 \in Nt_1N = \{Nt_1^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, \dots, Nt_{105}\}$, forming the new double coset, [1], which means that all 105 symmetric generators extend to Nt_1N since they share the same orbit.

Second Double Coset $Nt_1N = [1]$

Now, Nt_1N is a new coset and the number of single cosets is $\frac{|N|}{|N^{(1)}|}$. The number of single cosets in the double coset Nt_1N is $\frac{|N|}{|N^{(1)}|} = \frac{2520}{24} = 105$ and the orbits of $N^{(1)}$ on X are:

$$\begin{aligned} &\mathcal{O}\{1\}, \mathcal{O}\{21, 33\}, \mathcal{O}\{5, 20, 55, 59, 63, 26\}, \mathcal{O}\{2, 49, 39, 25, 97, 87, 9, 74, 70, 61, \\ &86, 93\}, \mathcal{O}\{3, 67, 40, 90, 95, 17, 19, 31, 45, 42, 43, 48\}, \mathcal{O}\{4, 72, 103, 32, 28, 75, 85, 37, 89, \\ &52, 98, 73\}, \mathcal{O}\{6, 14, 35, 68, 99, 41, 8, 77, 91, 83, 96, 60\}, \mathcal{O}\{7, 92, 62, 47, 65, 29, 38, 79, 24, \\ &34, 44, 13, 23, 51, 84, 16, 69, 88, 80, 22, 66, 57, 105, 36\}, \text{ and } \mathcal{O}\{10, 64, 71, 101, 12, 27, 53, \\ &76, 78, 11, 56, 100, 104, 50, 15, 54, 18, 30, 81, 102, 82, 94, 46, 58\}. \end{aligned}$$

Selecting 1, 21, 5, 2, 3, 4, 6, 7, and 10 as representatives of each orbit, respectively, forming the cosets $Nt_1t_1, Nt_1t_{21}, Nt_1t_5, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_1t_6, Nt_1t_7$, and Nt_1t_{10} . We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_1 = N \in [*]$. Therefore, one symmetric generator goes back to [*]. Next, $Nt_1t_{21}, Nt_1t_5, Nt_1t_4, Nt_1t_7$, and $Nt_1t_{10} \in Nt_1N = [1]$ so, two, six,

twelve, twenty-four, and twenty-four symmetric generators loop to [1], respectively. Also, Nt_1t_2 , Nt_1t_3 , and $Nt_1t_6 \in Nt_1t_2N = [12]$ so, twelve symmetric generators extend to [12], respectively.

Third Double Coset $Nt_1t_2N = [12]$

Now, Nt_1t_2N is a new coset. The number of single cosets in the double coset Nt_1t_2N is $\frac{|N|}{|N^{(12)}|} = \frac{2520}{36} = 70$ and the orbits of $N^{(12)}$ on X are:

$\mathcal{O}\{19, 67, 92, 37, 79, 90\}$, $\mathcal{O}\{8, 83, 13, 44, 28, 66, 16, 55, 103\}$, $\mathcal{O}\{1, 24, 41, 87, 105, 101, 60, 81, 75, 57, 49, 30, 21, 89, 94, 63, 47, 33\}$, $\mathcal{O}\{2, 36, 56, 69, 38, 80, 54, 4, 84, 72, 22, 52, 15, 34, 88, 39, 85, 11\}$, $\mathcal{O}\{3, 42, 50, 46, 12, 97, 58, 73, 10, 78, 70, 51, 71, 74, 53, 7, 45, 9\}$, $\mathcal{O}\{5, 26, 17, 61, 77, 14, 32, 104, 43, 25, 62, 82, 68, 23, 18, 48, 99, 93, 95, 64, 98, 86, 96, 102, 27, 29, 6, 40, 65, 59, 76, 35, 100, 20, 31, 91\}$.

Selecting 19, 8, 1, 2, 3, and 5 as representatives of each orbit, respectively, forming the cosets $Nt_1t_2t_{19}$, $Nt_1t_2t_8$, $Nt_1t_2t_1$, $Nt_1t_2t_2$, $Nt_1t_2t_3$, and $Nt_1t_2t_5$. We must determine where each coset representative belongs to which double coset:

Now, $Nt_1t_2t_{19}$, $Nt_1t_2t_8$, $Nt_1t_2t_3$ and $Nt_1t_2t_5 \in [12]$. Therefore, six, nine, and thirty-six symmetric generators loop to [12], respectively. Next, $Nt_1t_2t_1$, $Nt_1t_2t_2$, and $Nt_1t_2t_3 \in Nt_1N$ so, eight-teen symmetric generators go back to [1], respectively. A cayley diagram for this manual double coset enumeration is shown below.

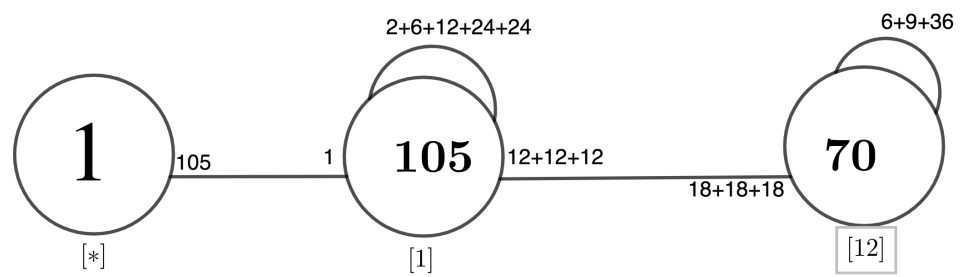


Figure 4.5: Cayley Diagram of M_{22} over A_7

Chapter 5

Double Coset Enumeration over a Maximal Subgroup

5.1 Construction of $M_{21} : 2^2$ over $PGL(2, 9) : 2$ and $PGL(2, 9)$

Begin with a progenitor $G \cong \frac{2^{*10}:PGL(2,9)}{(zxyt)^6, (zt)^8, (tt^x)^2=y^4z}$.

A symmetric presentation of the progenitor $M_{21} : 2^2$ is
 $\langle x, y, z, t | x^3, y^8, z^2, (y^{-1}z)^2, (zx^{-1})^3, (y^{-1}x^{-1}y^{-1}x^{-1}y^2x), (xy^{-2}x^{-1}yx^{-1}y), t^2,$
 $(t, y^{-1}xzy^2), (t, y^{-2}xy^{-1}) \rangle,$

where $N \cong PGL(2, 9)$ is of order 720, with

$x \sim (1,2,10)(3,4,5)(6,7,8)$, $y \sim (1,7,3,4,2,5,6,8)$, $z \sim (1,2)(4,7)(5,8)(9,10)$, and $t \sim t_1$.

We prove the above progenitor factored by $(zxyt)^6$, $(zt)^8$, and $(tt^x)^2 = y^4z$ is isomorphic to $M_{21} : 2^2$. We expand our relations as follows:

(1) Relation $(zxyt)^6=1$

$$(zxyt)^6=1$$

$$\implies [(1,2)(4,7)(5,8)(9,10) * (1,2,10)(3,4,5)(6,7,8) * (1,7,3,4,2,5,6,8) * t_1]^6$$

$$\implies [(1,10,9,7,6,3,2,5,8,4) * t_1]^6$$

$$\implies (\pi t_1)^6 \text{ (we let } (1,10,9,7,6,3,2,5,8,4) = \pi \text{)}$$

$$\implies \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi \pi \pi^{-1} t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi^2 \pi \pi^{-1} t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi^3 \pi \pi^{-1} t_1 \pi^2 t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi^4 \pi \pi^{-1} t_1 \pi^3 t_1 \pi^2 t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi^5 \pi \pi^{-1} t_1 \pi^4 t_1 \pi^3 t_1 \pi^2 t_1 \pi t_1 \pi t_1$$

$$\implies \pi^6 t_1 \pi^5 t_1 \pi^4 t_1 \pi^3 t_1 \pi^2 t_1 \pi t_1$$

$$\implies (1,2,9,8,6)(3,10,5,7,4)t_3 t_6 t_7 t_9 t_{10} t_1 = 1.$$

(2) Relation $(zt)^8=1$

$$\implies ((1,2)(4,7)(5,8)(9,10) * t_1)^8$$

$$\implies (\pi t_1)^8 \text{ (we let } (1,2)(4,7)(5,8)(9,10) = \pi \text{)}$$

$$\implies \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1$$

$$\implies \pi^8 t_1 \pi^7 t_1 \pi^6 t_1 \pi^5 t_1 \pi^4 t_1 \pi^3 t_1 \pi^2 t_1 \pi t_1$$

$$\implies e t_2 t_1 t_2 t_1 t_2 t_1 t_2 t_1 = 1.$$

(3) Relation $(t * t^x)^2 = y^4 * z$

$$(t * t^x)^2$$

$$\implies t * t^x * t * t^x$$

$$\implies t * t(1, 2, 10)(3, 4, 5)(6, 7, 8) * t * t(1, 2, 10)(3, 4, 5)(6, 7, 8)$$

$$\implies t_1 t_2 t_1 t_2 = y^4 * z.$$

(4) H Relation $z * y * t * x * t * y * t * x^{-1} * t * y * t = Id(H)$

$$\implies zy(y) * tx * ty * tx^{-1} * t^y * t$$

$$\implies zy(yx^{-1}) * tx * ty * t^{x^{-1}y} * t^y * t$$

$$\implies zy(y(yx^{-1})) * tx * t^{y(x^{-1}y)} * t^{x^{-1}y} * t^y * t$$

$$\implies zy(yx(x^{-1}y)) * t^{xy(x^{-1}y)} * t^{y(x^{-1}y)} * t^{x^{-1}y} * t^y * t$$

$$\implies zy(yx(x^{-1}y)) * t_2 t_8 t_{10} t_7 t_1 = 1.$$

Now we proceed with the Double Coset Enumeration (DCE) of G over H and N.

First Double Coset HeN = [*]

We let HeN be denoted by $[*]$. $HeN = \{H(e)^n \mid n \in \mathbb{N}\} = \{H\}$. H is the only single coset contained in the double coset $[*]$. The coset representative of the double coset $[*]$ is H. The coset stabilizer in H of the coset H is $\langle H \rangle$. Therefore, the number of single cosets of H is $\frac{|N|}{|H|} = \frac{720}{720} = 1$.

H is transitive on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, so it has the single orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Select a t_i representative from the single orbit, say 1, and form the new double coset, $[1]$.

Second Double Coset $Ht_1N = [1]$

$Ht_1N = \{Ht_1, Ht_2, Ht_3, Ht_4, Ht_5, Ht_6, Ht_7, Ht_8, Ht_9, Ht_{10}\}$, with $Ht_1 \in [1]$ which means that all 10 symmetric generators move forward to Ht_1N since they share the same orbit. The number of single cosets contained in this double coset is given by $\frac{|N|}{|N^{(1)}|}$, where $N^{(1)}$ is the coset stabilizer of the coset Ht_1 which is greater than or equal to the point stabilizer of 1 in N , denoted by $Ht_1 \geq N^1$. $N^{(1)} \geq N^1 = \langle (2,7)(3,9)(4,6)(5,10), (2,10,3,6,7,5,9,4), (2,3,4,6,10,8,7,5), (2,7,10,4)(3,5,8,6) \rangle$. Therefore, the number of single cosets in the double coset Ht_1N is

$$\frac{|N|}{|N^{(1)}|} = \frac{720}{72} = 10.$$

The orbits of $N^{(1)}$ on X are $\{1\}$ and $\{2,5,6,4,7,9,8,3,10\}$. Now, selecting a t_i representative from each orbit, say 1 and 2, respectively, forming the cosets Ht_1t_1 and Ht_1t_2 . We must determine where each coset representative belongs to which double coset:

$$Ht_1t_1 =? \text{ and } Ht_1t_2 =?$$

Now, $Ht_1t_1 \in [*]$, since $Ht_1t_1 = Ht_1^2 = H$. Therefore, one symmetric generator goes back to $[*]$. Now, Ht_1t_2 forms a new double coset so nine symmetric generators move forward to $[12]$.

Third Double Coset $Ht_1t_2N = [12]$

The number of single cosets contained in this double coset is given by $\frac{|N|}{|N^{(12)}|}$, where $N^{(12)}$ is the coset stabilizer of the coset Ht_1t_2 which is greater

than or equal to the point stabilizer of 12 in N , denoted by N^{12} ; $N^{12} = \langle (3, 4, 9, 8, 6, 5, 10, 7), (3, 9, 6, 10)(4, 8, 5, 7) \rangle$. Therefore, the number of single cosets in the double coset Ht_1t_2N is $\frac{|N|}{|N^{(12)}|} = \frac{720}{16} = 45$.

The orbits of $N^{(12)}$ on X are $\{1\}, \{2\}$, and $\{3, 8, 10, 4, 6, 7, 9, 5\}$. Now, selecting a t_i representative from each orbit, say 1, 2, and 3, respectively, forming the cosets $Ht_1t_2t_1$, $Ht_1t_2t_2$, and $Ht_1t_2t_3$. We must determine where each coset representative belongs to which double coset:

$$Ht_1t_2t_1 = ?, Ht_1t_2t_2 = ?, \text{ and } Ht_1t_2t_3 = ?$$

$Ht_1t_2t_2 = Ht_1 \in [1]$, since $Ht_1t_2t_2 = Ht_1t_2^2 = Ht_1$. Therefore, one symmetric generator goes back to $[1]$.

Proof of $Ht_1t_2t_1 = Ht_1 \in [1]$. By relation (3), we have $t_1t_2t_1 = t_2$. Conjugating by H on the left, we obtain $Ht_1t_2t_1 = Ht_2$. Now, conjugating Ht_1 by the permutation x , $Ht_1^{(1,2,10)(3,4,5)(6,7,8)} = Ht_2 \in Ht_1N = [1]$. Therefore, one symmetric generator goes back to $[1]$.

Now, to prove $Ht_1t_2t_3 = Ht_1t_2 \in [12]$ we use our H relation,

$$z * y(x * y(x^{-1} * y))t_2t_8t_{10}t_7t_1 = Id(H)$$

$$\implies z * y(x * y(x^{-1} * y))t_2t_8t_{10} = t_7t_1$$

$$\implies Hz * y(x * y(x^{-1} * y))t_2t_8t_{10} = Ht_7t_1 \text{ (since } \in H)$$

$$\implies Ht_2t_8t_{10} = Ht_7t_1. \text{ Now, conjugating by } (1, 10, 3, 7, 9, 4, 8, 2), \text{ we have}$$

$$\implies Ht_1t_2t_3 = Ht_{10}t_9.$$

But, $H(t_1t_2)^{(1,10,7,2,9,5,6,4,3,8)} = Ht_{10}t_9 \in [12]$. Therefore, eight symmetric generators loop back to $[1, 2]$.

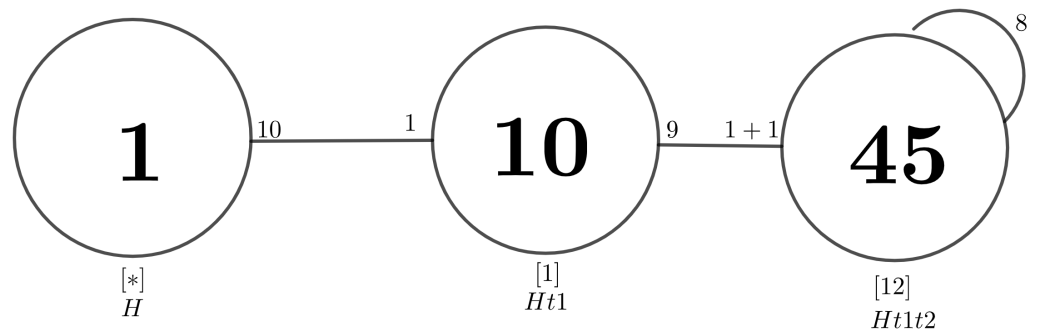


Figure 5.1: Cayley diagram of $M_{21} : 2^2$ over $PGL(2, 9) : 2$ and $PGL(2, 9)$

Chapter 6

Wreath Products

6.1 Wreath Products

Given two groups, H and K . The wreath product of H by K , denoted by $H \wr K$ is a semi-direct product composed of as many copies of H as the number of letters on which the permutation group K acts on. We define the wreath product below.

Definition 5.1.1. (Wreath Product) *Let X and Y be non-empty sets. Let H be a permutation group on X and K on Y . Let $Z = X \times Y$, then the **wreath product** is a semi-direct product of X and Y . We define a permutation group on Z such that we let $\gamma \in H$, and define a permutation of $\gamma(y)$ of Z by*

$$\gamma(y) = \begin{cases} (x, y) \mapsto (x\gamma, y), & \text{if } y_1 = y \\ (x, y_1) \mapsto (x, y_1), & y \neq y_1 \end{cases}$$

Also, for $k \in K$, define $k^(x, y) = (x, (y)k)$; such that $B = \prod_{y \in Y} H(y)$ is a direct product of the group generated by the y 's. Thus, $G = B:k^*$ is called a wreath*

product of H and K , where H is normal subgroup, denoted by $H \wr K$. [Rot95]

We look at a few examples to help us better understand how wreath products are generated.

6.1.1 Example: $\mathbb{Z}_5 \wr \mathbb{Z}_5$

Let $H = \langle (1, 2, 3, 4, 5) \rangle \cong \mathbb{Z}_5$ and $K = \langle (6, 7, 8, 9, 10) \rangle \cong \mathbb{Z}_5$ be permutation groups on $X = \{1, 2, 3, 4, 5\}$ and $Y = \{6, 7, 8, 9, 10\}$, respectively. We construct, by hand, the permutation generators of the wreath product, $H \wr K$.

Let $Z = X \times Y = \{ (1,6), (1,7), (1,8), (1,9), (1,10), (2,6), (2,7), (2,8), (2,9), (2,10), (3,6), (3,7), (3,8), (3,9), (3,10), (4,6), (4,7), (4,8), (4,9), (4,10), (5,6), (5,7), (5,8), (5,9), (5,10), \}$. We consider the labeling for the elements of Z :

By the wreath product definition, we let $\gamma_1 = (1,2,3,4,5) \in H$ and $y \in Y$. So, compute $\gamma_1(6), \gamma_1(7), \gamma_1(8), \gamma_1(9), \gamma_1(10)$. We compute the action of $\gamma_1=(1,2,3,4,5)$ on K . Note, γ will only change elements which contain 1, 2, 3, 4, and 5 in the x -coordinate and 6 in the y -coordinate.

Table 6.1: Labeling of Z

Labeling	Element
11	(1,6)
12	(1,7)
13	(1,8)
14	(1,9)
15	(1,10)
16	(2,6)
17	(2,7)
18	(2,8)
19	(2,9)
20	(2,10)
21	(3,6)
22	(3,7)
23	(3,8)
24	(3,9)
25	(3,10)
26	(4,6)
27	(4,7)
28	(4,8)
29	(4,9)
30	(4,10)
31	(5,6)
32	(5,7)
33	(5,8)
34	(5,9)
35	(5,10)

From **Table 2**, $\gamma_1(6) = ((1,6), (2,6), (3,6), (4,6), (5,6)) = (11,16, 21, 26, 31)$. Similarly, we obtain $\gamma_1(7) = (12,17,22, 27,32)$, $\gamma_1(8) = (13,18,23,28,33)$, $\gamma_1(9) = (14,19,24,29,34)$, and $\gamma_1(10) = (15,20,25,30,35)$.

We have formed $B = H(6) \times H(7) \times H(8) \times H(9) \times H(10)$, and obtain five copies of \mathbb{Z}_5 . This can be expressed as $B = \langle \gamma_1(6) \rangle \times \langle \gamma_1(7) \rangle \times$

Table 6.2: Labeling of $\gamma_1(6)$

Labeling	Element		Element	Labeling
11	(1,6)	→	($\gamma(1),6$)	16
12	(1,7)	→	($\gamma(1),7$)	12
13	(1,8)	→	($\gamma(1),8$)	13
14	(1,9)	→	($\gamma(1),9$)	14
15	(1,10)	→	($\gamma(1),10$)	15
16	(2,6)	→	($\gamma(2),6$)	21
17	(2,7)	→	($\gamma(2),7$)	17
18	(2,8)	→	($\gamma(2),8$)	18
19	(2,9)	→	($\gamma(2),9$)	19
20	(2,10)	→	($\gamma(2),10$)	20
21	(3,6)	→	($\gamma(3),6$)	26
22	(3,7)	→	($\gamma(3),7$)	22
23	(3,8)	→	($\gamma(3),8$)	23
24	(3,9)	→	($\gamma(3),9$)	24
25	(3,10)	→	($\gamma(3),10$)	25
26	(4,6)	→	($\gamma(4),6$)	31
27	(4,7)	→	($\gamma(4),7$)	27
28	(4,8)	→	($\gamma(4),8$)	28
29	(4,9)	→	($\gamma(4),9$)	29
30	(4,10)	→	($\gamma(4),10$)	30
31	(5,6)	→	($\gamma(5),6$)	11
32	(5,7)	→	($\gamma(5),7$)	32
33	(5,8)	→	($\gamma(5),8$)	33
34	(5,9)	→	($\gamma(5),9$)	34
35	(5,10)	→	($\gamma(5),10$)	35

$\langle \gamma_1(8) \rangle \times \langle \gamma_1(9) \rangle \times \langle \gamma_1(10) \rangle$. We label each as $a = (11,16, 21, 26, 31)$, $b = (12,17,22, 27,32)$, $c = (13,18,23,28,33)$, $d = (14,19,24,29,34)$, and $e = (15,20,25,30,35)$, such that $B = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$.

We now compute k^* , where $K = \langle (6,7,8,9,10) \rangle = \mathbb{Z}_5$. Since K is transitive, we compute $k_1^* = (6, 7, 8, 9, 10)^*$. We proceed as shown above.

Table 6.3: Labeling of $\gamma_1(7)$

Labeling	Element		Element	Labeling
11	(1,6)	→	($\gamma(1)$,6)	11
12	(1,7)	→	($\gamma(1)$,7)	17
13	(1,8)	→	($\gamma(1)$,8)	13
14	(1,9)	→	($\gamma(1)$,9)	14
15	(1,10)	→	($\gamma(1)$,10)	15
16	(2,6)	→	($\gamma(2)$,6)	16
17	(2,7)	→	($\gamma(2)$,7)	22
18	(2,8)	→	($\gamma(2)$,8)	18
19	(2,9)	→	($\gamma(2)$,9)	19
20	(2,10)	→	($\gamma(2)$,10)	20
21	(3,6)	→	($\gamma(3)$,6)	21
22	(3,7)	→	($\gamma(3)$,7)	27
23	(3,8)	→	($\gamma(3)$,8)	23
24	(3,9)	→	($\gamma(3)$,9)	24
25	(3,10)	→	($\gamma(3)$,10)	25
26	(4,6)	→	($\gamma(4)$,6)	26
27	(4,7)	→	($\gamma(4)$,7)	32
28	(4,8)	→	($\gamma(4)$,8)	28
29	(4,9)	→	($\gamma(4)$,9)	29
30	(4,10)	→	($\gamma(4)$,10)	30
31	(5,6)	→	($\gamma(5)$,6)	31
32	(5,7)	→	($\gamma(5)$,7)	12
33	(5,8)	→	($\gamma(5)$,8)	33
34	(5,9)	→	($\gamma(5)$,9)	34
35	(5,10)	→	($\gamma(5)$,10)	35

From Table 8 labeling, we obtain $k_1^* = (11,12,13,14,15)(16,17,18,19,20)(21,22,23,24,25)$
 $(26,27,28,29,30)(31,32,33,34,35)$, denoted by

$$f = (11,12,13,14,15)(16,17,18,19,20)(21,22,23,24,25) (26,27,28,29,30)(31,32,33,34,35).$$

So, $k^* = \langle k_1^* \rangle = \langle f \rangle$. Thus, the wreath product $H \wr K$, such that,

$$G = B : k^* = \langle a, b, c, d, e \rangle : \langle f \rangle \text{ is } \mathbb{Z}_5^5 \wr \mathbb{Z}_5.$$

Table 6.4: Labeling of $\gamma_1(8)$

Labeling	Element		Element	Labeling
11	(1,6)	→	($\gamma(1),6$)	11
12	(1,7)	→	($\gamma(1),7$)	12
13	(1,8)	→	($\gamma(1),8$)	18
14	(1,9)	→	($\gamma(1),9$)	14
15	(1,10)	→	($\gamma(1),10$)	15
16	(2,6)	→	($\gamma(2),6$)	16
17	(2,7)	→	($\gamma(2),7$)	17
18	(2,8)	→	($\gamma(2),8$)	23
19	(2,9)	→	($\gamma(2),9$)	19
20	(2,10)	→	($\gamma(2),10$)	20
21	(3,6)	→	($\gamma(3),6$)	21
22	(3,7)	→	($\gamma(3),7$)	22
23	(3,8)	→	($\gamma(3),8$)	28
24	(3,9)	→	($\gamma(3),9$)	24
25	(3,10)	→	($\gamma(3),10$)	25
26	(4,6)	→	($\gamma(4),6$)	26
27	(4,7)	→	($\gamma(4),7$)	27
28	(4,8)	→	($\gamma(4),8$)	33
29	(4,9)	→	($\gamma(4),9$)	29
30	(4,10)	→	($\gamma(4),10$)	30
31	(5,6)	→	($\gamma(5),6$)	31
32	(5,7)	→	($\gamma(5),7$)	32
33	(5,8)	→	($\gamma(5),8$)	13
34	(5,9)	→	($\gamma(5),9$)	34
35	(5,10)	→	($\gamma(5),10$)	35

6.1.2 A Presentation of $\mathbb{Z}_5^5 \wr \mathbb{Z}_5$

Now, we write a presentation for the previous example. To accomplish this, we compute the action of $K=\mathbb{Z}_5$ on $H=\mathbb{Z}_5^5$. H is composed of 5 copies of \mathbb{Z}_5 which is generated by $\langle a, b, c, d, e \rangle$ and $K = \mathbb{Z}_5 = \langle f \rangle$. We compute:

Table 6.5: Labeling of $\gamma_1(9)$

Labeling	Element		Element	Labeling
11	(1,6)	→	($\gamma(1)$,6)	11
12	(1,7)	→	($\gamma(1)$,7)	12
13	(1,8)	→	($\gamma(1)$,8)	13
14	(1,9)	→	($\gamma(1)$,9)	19
15	(1,10)	→	($\gamma(1)$,10)	15
16	(2,6)	→	($\gamma(2)$,6)	16
17	(2,7)	→	($\gamma(2)$,7)	17
18	(2,8)	→	($\gamma(2)$,8)	18
19	(2,9)	→	($\gamma(2)$,9)	24
20	(2,10)	→	($\gamma(2)$,10)	20
21	(3,6)	→	($\gamma(3)$,6)	21
22	(3,7)	→	($\gamma(3)$,7)	22
23	(3,8)	→	($\gamma(3)$,8)	23
24	(3,9)	→	($\gamma(3)$,9)	29
25	(3,10)	→	($\gamma(3)$,10)	25
26	(4,6)	→	($\gamma(4)$,6)	26
27	(4,7)	→	($\gamma(4)$,7)	27
28	(4,8)	→	($\gamma(4)$,8)	28
29	(4,9)	→	($\gamma(4)$,9)	34
30	(4,10)	→	($\gamma(4)$,10)	30
31	(5,6)	→	($\gamma(5)$,6)	31
32	(5,7)	→	($\gamma(5)$,7)	32
33	(5,8)	→	($\gamma(5)$,8)	33
34	(5,9)	→	($\gamma(5)$,9)	14
35	(5,10)	→	($\gamma(5)$,10)	35

$$a^f = (11, 16, 21, 26, 31)^{(11,12,13,14,15)(16,17,18,19,20)(21,22,23,24,25)(26,27,28,29,30)(31,32,33,34,35)}$$

$$= (12, 17, 22, 27, 32) = b,$$

$$b^f = (12, 17, 22, 27, 32)^{(11,12,13,14,15)(16,17,18,19,20)(21,22,23,24,25)(26,27,28,29,30)(31,32,33,34,35)}$$

$$= (13, 18, 23, 28, 33) = c,$$

$$c^f = d,$$

$$d^f = e,$$

Table 6.6: Labeling of $\gamma_1(10)$

Labeling	Element		Element	Labeling
11	(1,6)	→	($\gamma(1),6$)	11
12	(1,7)	→	($\gamma(1),7$)	12
13	(1,8)	→	($\gamma(1),8$)	13
14	(1,9)	→	($\gamma(1),9$)	14
15	(1,10)	→	($\gamma(1),10$)	20
16	(2,6)	→	($\gamma(2),6$)	16
17	(2,7)	→	($\gamma(2),7$)	17
18	(2,8)	→	($\gamma(2),8$)	18
19	(2,9)	→	($\gamma(2),9$)	19
20	(2,10)	→	($\gamma(2),10$)	25
21	(3,6)	→	($\gamma(3),6$)	21
22	(3,7)	→	($\gamma(3),7$)	22
23	(3,8)	→	($\gamma(3),8$)	23
24	(3,9)	→	($\gamma(3),9$)	24
25	(3,10)	→	($\gamma(3),10$)	30
26	(4,6)	→	($\gamma(4),6$)	26
27	(4,7)	→	($\gamma(4),7$)	27
28	(4,8)	→	($\gamma(4),8$)	28
29	(4,9)	→	($\gamma(4),9$)	29
30	(4,10)	→	($\gamma(4),10$)	35
31	(5,6)	→	($\gamma(5),6$)	31
32	(5,7)	→	($\gamma(5),7$)	32
33	(5,8)	→	($\gamma(5),8$)	33
34	(5,9)	→	($\gamma(5),9$)	34
35	(5,10)	→	($\gamma(5),10$)	15

$$e^f = a,$$

Hence, a presentation for $\mathbb{Z}_5^5 \wr \mathbb{Z}_5$ is given by:

$$\langle a, b, c, d, e, f, \mid a^5, b^5, c^5, d^5, e^5, (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e), f^5, a^f = b, b^f = c, c^f = d, d^f = e, e^f = a \rangle.$$

Table 6.7: Labeling of $\gamma_1(7)$

Labeling	Element		Element	Labeling
8	(1,4)	\longrightarrow	$(\gamma(1),4)$	8
9	(1,5)	\longrightarrow	$(\gamma(1),5)$	9
10	(1,6)	\longrightarrow	$(\gamma(1),6)$	10
11	(1,7)	\longrightarrow	$(\gamma(1),7)$	15
12	(2,4)	\longrightarrow	$(\gamma(2),4)$	12
13	(2,5)	\longrightarrow	$(\gamma(2),5)$	13
14	(2,6)	\longrightarrow	$(\gamma(2),6)$	14
15	(2,7)	\longrightarrow	$(\gamma(2),7)$	19
16	(3,4)	\longrightarrow	$(\gamma(3),4)$	16
17	(3,5)	\longrightarrow	$(\gamma(3),5)$	17
18	(3,6)	\longrightarrow	$(\gamma(3),6)$	18
19	(3,7)	\longrightarrow	$(\gamma(3),7)$	11

6.1.3 Verifying the Wreath Product Presentation for $\mathbb{Z}_5^5 \wr \mathbb{Z}_5$

In order to build a successful progenitor with MAGMA we need to confirm that H and K are transitive. We verify to see if the presentation for the wreath product is correct by doing the following MAGMA code:

```

W:=WreathProduct(CyclicGroup(5),CyclicGroup(5));
G<a,b,c,d,e,f>:=Group<a,b,c,d,e,f| a^5, b^5, c^5, d^5, e^5, (a,b), (a,c),
(a,d), (a,e), (b,c), (b,d), (b,e), (c,d), (c,e), (d,e),
f^5, a^f = b, b^f = c, c^f = d, d^f = e, e^f = a >;
f,G1,k:=CosetAction(G,sub<G| Id(G)>);
#W;
15625
#G1;
15625

```

Table 6.8: Labeling of $k^* = (6, 7, 8, 9, 10)^*$

Labeling	Element		Element	Labeling
11	(1,6)	→	(1,7)	12
12	(1,7)	→	(1,8)	13
13	(1,8)	→	(1,9)	14
14	(1,9)	→	(1,10)	15
15	(1,10)	→	(1,6)	11
16	(2,6)	→	(2,7)	17
17	(2,7)	→	(2,8)	18
18	(2,8)	→	(2,9)	19
19	(2,9)	→	(2,10)	20
20	(2,10)	→	(2,6)	16
21	(3,6)	→	(3,7)	22
22	(3,7)	→	(3,8)	23
23	(3,8)	→	(3,9)	24
24	(3,9)	→	(3,10)	25
25	(3,10)	→	(3,6)	21
26	(4,6)	→	(4,7)	27
27	(4,7)	→	(4,8)	28
28	(4,8)	→	(4,9)	29
29	(4,9)	→	(4,10)	30
30	(4,10)	→	(4,6)	26
31	(5,6)	→	(5,7)	32
32	(5,7)	→	(5,8)	33
33	(5,8)	→	(5,9)	34
34	(5,9)	→	(5,10)	35
35	(5,10)	→	(5,6)	31

IsIsomorphic(G1,W);

true Mapping from: GrpPerm: G1 to GrpPerm: W

Composition of Mapping from: GrpPerm: G1 to GrpPC and

Mapping from: GrpPC to GrpPC and

Mapping from: GrpPC to GrpPerm: W

Chapter 7

Isomorphism Types

7.1 Mixed Extension ($2^3 : \bullet S_4$)

We find the isomorphism type of the group N on 96 letters given below. $xx \sim (1, 86, 25, 96)(2, 51, 40, 54)(3, 73, 60, 77)(4, 78, 67, 70)(5, 85, 7, 88)(6, 87, 28, 64)(8, 17, 72, 58)(9, 90, 82, 55)(10, 14, 12, 92)(11, 89, 39, 71)(13, 22, 76, 69)(15, 75, 65, 80)(16, 41, 19, 79)(18, 74, 68, 35)(20, 33, 57, 84)(21, 48, 23, 47)(24, 95, 29, 45)(26, 31, 62, 34)(27, 46, 66, 49)(30, 59, 32, 56)(36, 53, 38, 50)(37, 91, 81, 52)(42, 61, 44, 94)(43, 93, 83, 63)$ and $yy \sim (1, 3, 14)(2, 8, 13)(4, 20, 51)(5, 15, 55)(6, 26, 71)(7, 19, 54)(9, 35, 46)(10, 30, 64)(11, 41, 70)(12, 34, 49)(16, 50, 21)(17, 61, 29)(18, 63, 24)(22, 68, 89)(23, 65, 92)(25, 57, 53)(27, 59, 94)(28, 56, 52)(31, 45, 36)(32, 76, 38)(33, 78, 43)(37, 80, 47)(39, 75, 96)(40, 74, 95)(42, 84, 85)(44, 77, 48)(58, 91, 69)(60, 90, 67)(62, 93, 66)(72, 87, 82)(73, 86, 83)(79, 88, 81)$ with the order of N equal to 192.

We examine the composition factors of N , which show us the simple groups.

`CompositionFactors(N);`

```

>G
  | Cyclic(2)
  *
  | Cyclic(3)
  *
  | Cyclic(2)
  *
  | Cyclic(2)
  *
  | Cyclic(2)
  *
  | Cyclic(2)
  *
  | Cyclic(2)
  1

```

Now, we look at the normal lattice of N to see how our group looks like. Finding the largest abelian normal subgroup can help us determine whether this a direct product, split, nonsplit, mixed or central extension. In this case, $NL[4]$ is our largest normal abelian subgroup.

```
NL:=NormalLattice(N);
```

```
NL;
```

```
>Normal subgroup lattice
```

```
-----
```

```
[8] Order 192 Length 1 Maximal Subgroups: 7
```

```

---
[7] Order 96   Length 1   Maximal Subgroups: 6
---
[6] Order 32   Length 1   Maximal Subgroups: 3 4 5
---
[5] Order 8    Length 1   Maximal Subgroups: 2
[4] Order 8    Length 1   Maximal Subgroups: 2
[3] Order 8    Length 1   Maximal Subgroups: 2
---
[2] Order 2    Length 1   Maximal Subgroups: 1
---
[1] Order 1    Length 1   Maximal Subgroups:

```

We proceed by determining the isomorphism type of $NL[4]$. We see that $NL[4]$ is either Z_8 , $4x2$ or $2x2x2$. Our guess is that $NL[4]$ is $2x2x2$ since $NL[4]$ has three generators of order 2 as shown below.

$NL[4]$;

>Permutation group acting on a set of cardinality 96

Order = 8 = 2^3

```

(1, 5)(2, 10)(3, 16)(4, 21)(6, 24)(7, 25)(8, 31)(9, 36)(11, 42)(12, 40)(13,
46)(14, 51)(15, 20)(17, 62)(18, 59)(19, 60)(22, 66)(23, 67)(26, 58)(27,
69)(28, 29)(30, 35)(32, 74)(33, 75)(34, 72)(37, 83)(38, 82)(39, 44)(41,
73)(43, 81)(45, 64)(47, 70)(48, 78)(49, 76)(50, 55)(52, 93)(53, 90)(54,
92)(56, 68)(57, 65)(61, 89)(63, 91)(71, 94)(77, 79)(80, 84)(85, 86)(87,
95)(88, 96)

```


(1, 23)(2, 38)(3, 57)(4, 7)(5, 67)(6, 22)(8, 74)(9, 12)(10, 82)(11, 43)(13, 87)(14, 90)(15, 19)(16, 65)(17, 18)(20, 60)(21, 25)(24, 66)(26, 56)(27, 29)(28, 69)(30, 34)(31, 32)(33, 73)(35, 72)(36, 40)(37, 44)(39, 83)(41, 75)(42, 81)(45, 49)(46, 95)(47, 96)(48, 86)(50, 54)(51, 53)(52, 94)(55, 92)(58, 68)(59, 62)(61, 91)(63, 89)(64, 76)(70, 88)(71, 93)(77, 84)(78, 85)(79, 80)

(1, 7)(2, 12)(3, 19)(4, 23)(5, 25)(6, 29)(8, 34)(9, 38)(10, 40)(11, 44)(13, 49)(14, 54)(15, 57)(16, 60)(17, 26)(18, 56)(20, 65)(21, 67)(22, 27)(24, 28)(30, 74)(31, 72)(32, 35)(33, 80)(36, 82)(37, 43)(39, 42)(41, 77)(45, 87)(46, 76)(47, 78)(48, 70)(50, 90)(51, 92)(52, 63)(53, 55)(58, 62)(59, 68)(61, 71)(64, 95)(66, 69)(73, 79)(75, 84)(81, 83)(85, 96)(86, 88)(89, 94)(91, 93)

We do the following to verify our initial suspicion.

```
X:=AbelianGroup(GrpPerm,[2,2,2]);
IsIsomorphic(NL[4],X);
>true
```

Thus, we can say that $NL[4]$ is 2^3 . So far, our presentation for $NL[4]$ is as follows:

```
H<a,b,c>:=Group<a,b,c|a^2,b^2,c^2,(a,b),(a,c),(b,c)>;
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
w:=IsIsomorphic(H1,NL[4]);
w;
>true
```

Next, we want to factor out N by $NL[4]$. So we examine the normal lattice of q to determine the isomorphism type of q .

```
q,ff:=quo<N|NL[4]>;
q;
>Permutation group q acting on a set of cardinality 4
Order = 24 = 2^3 * 3
(1, 2)
(2, 3, 4)
```

Since q is of order 24, we can quickly determine that $q \cong S_4$. Note: full explanation of finding isomorphism type of q on appendix. Therefore, a presentation for q is as follows: $Q \langle d, e | d^2, e^3, (e^{-1}d)^4 \rangle$. Now, we conjugate q by $N/NL[4] = \langle NL[4]D, NL[4]E \rangle = (\langle A \rangle D, \langle B \rangle D, \langle C \rangle D, \langle A \rangle E, \langle B \rangle E, \langle C \rangle E)$ to determine the action of Q on H by labeling the generators of $NL[4]$ and the generators of q as follows:

```
>T:=Transversal(N,NL[4]);
>for i in [1..#T] do if ff(T[i]) eq q.1 then i; end if; end for;
2
>for i in [1..#T] do if ff(T[i]) eq q.2 then i; end if; end for;
3

>A:=NL[4].1; B:=NL[4].2; C:=NL[4].3;
>D:=T[2]; E:=T[3];
```

The reader should note that the generators were saved as permutations, but for

simplicity we do not show here. If the reader wishes to see how this was done, please refer to the appendix. As previously shown, we will conjugate the generators of H by the generators of Q , hence, give us a word that is equal in terms of H .

```
>for i,j,k in [0..1] do if A^T2 eq A^i*B^j*C^k then i,j,k; end if; end for;
1 0 0
>for i,j,k in [0..1] do if A^T3 eq A^i*B^j*C^k then i,j,k; end if; end for;
0 1 1
>for i,j,k in [0..1] do if B^T2 eq A^i*B^j*C^k then i,j,k; end if; end for;
1 1 1
>for i,j,k in [0..1] do if B^T3 eq A^i*B^j*C^k then i,j,k; end if; end for;
1 1 0
>for i,j,k in [0..1] do if C^T2 eq A^i*B^j*C^k then i,j,k; end if; end for;
0 0 1
>for i,j,k in [0..1] do if C^T3 eq A^i*B^j*C^k then i,j,k; end if; end for;
0 0 1
```

Now, this indicates that $a^d = a$, $a^e = b * c$, $b^d = a * b * c$, $b^e = a * b$, $c^d = c$, and $c^e = c$, we will add them to the overall presentation K which contains all the elements from H and Q .

```
>K<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^2,c^2, d^2,e^3,(e^-1*d)^4,
(a,b),(a,c),(b,c), a^d=a,a^e=b*c,b^d=a*b*c,b^e=a*b,c^d=c,c^e=c>;
>#K;
192
>f,K1,k:=CosetAction(K,sub<K|Id(K)>);
```

```
>IsIsomorphic(N,K1);
```

```
false
```

This tells us that N is not a semi-direct $2^3 : S_4$ but the mixed extension of 2^3 by S_4 . That means, one of the generators of q must be further investigated. More specifically, we look at $(e^{-1}d)^4$.

```
>(T[3]^-1*T[2])^4;
```

```
(1, 7)(2, 12)(3, 19)(4, 23)(5, 25)(6, 29)(8, 34)(9, 38)(10, 40)(11,
  44)(13, 49)(14, 54)(15, 57)(16, 60)(17, 26)(18, 56)(20, 65)(21,
  67)(22, 27)(24, 28)(30, 74)(31, 72)(32, 35)(33, 80)(36, 82)(37,
  43)(39, 42)(41, 77)(45, 87)(46, 76)(47, 78)(48, 70)(50, 90)(51,
  92)(52, 63)(53, 55)(58, 62)(59, 68)(61, 71)(64, 95)(66, 69)(73,
  79)(75, 84)(81, 83)(85, 96)(86, 88)(89, 94)(91, 93)
```

```
> C eq (T[3]^-1*T[2])^4;
```

```
true
```

Thus, we can see that $(e^{-1}d)^4 = C$. Hence, our final presentation is:

$K \langle a, b, c, d, e \rangle := \text{Group} \langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, e^3, (e^{-1}d)^4 = c, (a, b), (a, c), (b, c), a^d = a, a^e = b * c, b^d = a * b * c, b^e = a * b, c^d = c, c^e = c \rangle$. To verify that this is correct we check that our order is still 192.

```
>K<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^2,c^2, d^2,e^3,(e^-1*d)^4=c,
```

```
(a,b),(a,c),(b,c), a^d=a,a^e=b*c,b^d=a*b*c,b^e=a*b,c^d=c,c^e=c>;
```

```
>#K;
```

```
192
```

```
>f,K1,k:=CosetAction(K,sub<K|Id(K)>);
```

```
>IsIsomorphic(N,K1);
true
```

Thus, we verified our final presentation and can conclude that the isomorphism type of N is the mixed extension $(2^3 : \bullet S_4)$.

7.2 Semi-direct $(3^2 : (2 : S_4))$

We begin with a transitive group on 72 letters.

```
>S:=Sym(72);
>xx:=S!(1, 23, 49, 32, 12, 60)(2, 44, 16, 45, 17, 62)(3, 41, 6, 55, 50, 33)(4,
63, 14, 46, 8, 43)(5, 66, 26, 58, 31, 72)(7, 61, 19, 28, 42, 10)(9, 20, 24,
54, 29, 38)(11, 69, 35, 70, 37, 18)(13, 21)(15, 36, 59, 53, 47, 48)(22,
67)(25, 40, 71, 27, 68, 52)(30, 64, 39, 57, 34, 56)(51, 65);
>yy:=S!(1, 35, 58, 2, 6, 54, 25, 4)(3, 40, 46, 64, 14, 21, 19, 69)(5, 36,
38, 15, 22, 60, 17, 55)(7, 43, 66, 29, 32, 53, 71, 30)(8, 23, 27, 57, 37,
39, 67, 41)(9, 13, 50, 59, 42, 26, 20, 44)(10, 24, 12, 31, 45, 48, 16,
65)(11, 63, 18, 51, 49, 34, 61, 52)(28, 62, 68, 70, 33, 56, 72, 47);
>N:=sub<S|xx,yy>;
>#N;
432
```

Next we check the composition factors of N to see what kind of simple groups it consists of.

```
>CompositionFactors(N);
```

G
 | $\text{Cyclic}(2)$
 *
 | $\text{Cyclic}(3)$
 *
 | $\text{Cyclic}(2)$
 *
 | $\text{Cyclic}(2)$
 *
 | $\text{Cyclic}(2)$
 *
 | $\text{Cyclic}(3)$
 *
 | $\text{Cyclic}(3)$
 1

The normal lattice of N is as follows

Normal Subgroup	Order
[6]	432
[5]	216
[4]	72
[3]	18
[2]	9
[1]	1

In this example, the largest abelian normal subgroup is $NL[2]$. We can see that the order of this is 9 so it can either be Z_9 or Z_3Z_3 . Magma verifies that it is in

fact, Z_3Z_3 . And a presentation of $NL[2]$ is shown in H.

```
>X:=AbelianGroup(GrpPerm,[3,3]);
>s:=IsIsomorphic(NL[2],X);
>s;
true

>H<a,b>:=Group<a,b|a^3,b^3,(a,b)>;
>#H;
9
>f,H1,k:=CosetAction(H,sub<H|Id(H)>);
>w1:=IsIsomorphic(H1,NL[2]);
>w1;
true
```

Now, we because there is no normal subgroup of order 48 outside of $NL[2]$. So, N cannot be a direct product. Hence, we need to investigate what the isomorphism of q is to helps us determine the type for our group.

7.3 Semi-direct $(2^3 : 4)$

We begin with a transitive group on eight letters.

```
>S:=Sym(8);
>xx:=S!(1, 8)(2, 3)(4, 5)(6, 7);
>yy:=S!(1, 3)(2, 8)(4, 6)(5, 7);
>zz:=S!(1, 5)(2, 6)(3, 7)(4, 8);
```

```
>tt:=S!(1, 3)(4, 5, 6, 7);
```

```
>N:=sub<S|xx,yy,zz,tt>;
```

```
>#N;
```

```
32
```

Next, we examine the composition factors of N to see the simple groups of N.

```
>CompositionFactors(N);
```

```
G
```

```
| Cyclic(2)
```

```
*
```

```
| Cyclic(2)
```

```
*
```

```
| Cyclic(2)
```

```
*
```

```
| Cyclic(2)
```

```
*
```

```
| Cyclic(2)
```

```
1
```

As previously seen, we examine the normal lattice of N to get an idea of how our N will look like, i.e. direct product, split, non-split, central or mixed extension.

```
NL:=NormalLattice(N);
```

```
NL;
```


Normal subgroup lattice

[12] Order 32 Length 1 Maximal Subgroups: 9 10 11

[11] Order 16 Length 1 Maximal Subgroups: 7

[10] Order 16 Length 1 Maximal Subgroups: 6 7 8

[9] Order 16 Length 1 Maximal Subgroups: 7

[8] Order 8 Length 1 Maximal Subgroups: 3

[7] Order 8 Length 1 Maximal Subgroups: 3 4 5

[6] Order 8 Length 1 Maximal Subgroups: 3

[5] Order 4 Length 1 Maximal Subgroups: 2

[4] Order 4 Length 1 Maximal Subgroups: 2

[3] Order 4 Length 1 Maximal Subgroups: 2

[2] Order 2 Length 1 Maximal Subgroups: 1

[1] Order 1 Length 1 Maximal Subgroups:

In this case, the largest normal abelian subgroup of N is $NL[8]$ with order 8. Thus, $NL[8]$ is isomorphic to either Z_8 , $Z_4 \times Z_2$ or $Z_2 \times Z_2 \times Z_2$. $NL[8]$ has three generators of order 2 so our initial suspicion is that it is isomorphic to Z_8 .

```
>X:=AbelianGroup(GrpPerm,[2,2,2]);
```

```
>IsIsomorphic(NL[8],X);
```

```
true
```

Hence, a presentation for $NL[8]$ is as follows

$H \langle a, b, c \rangle := \text{Group} \langle a, b, c | a^2, b^2, c^2, (a, b), (b, c), (a, c) \rangle$. We verify that this is correct:

```
>#H;
```

```
8
```

```
>f,H1,k:=CosetAction(H,sub<H|Id(H)>);
```

```
>IsIsomorphic(H1,NL[8]);
```

```
true
```

We know that N cannot be a direct product since there is no other normal subgroup of order 4 that is not contained in $NL[8]$. Factoring N by $NL[8]$:

```
>q,ff:=quo<N|NL[8]>;
```

```
>q;
```

```
Permutation group q acting on a set of cardinality 4
```

```
Order = 4 = 2^2
```

```
Id(q)
```

```
Id(q)
```

```
Id(q)
```

```
(1, 2, 3, 4)
```

```
> IsCyclic(q);
```

```
true
```

Therefore, isomorphism type of q is $Q < d|d^4 >$. Now to find the action of q on H . Since q is abelian, we can find it by doing the following loops:

```
>T:=Transversal(N,NL[8]);
>for i in [1..#T] do if ff(T[i]) eq q.1 then i; end if; end for;
1
>for i in [1..#T] do if ff(T[i]) eq q.2 then i; end if; end for;
1
>for i in [1..#T] do if ff(T[i]) eq q.3 then i; end if; end for;
1
>for i in [1..#T] do if ff(T[i]) eq q.4 then i; end if; end for;
2
```

The generators for $NL[8]$ and q are labeled as such:

```
>A:=N!(1, 5)(2, 6)(3, 7)(4, 8);
>B:=N!(1, 3)(2, 8)(4, 6)(5, 7);
>C:=N!(1, 8)(2, 3)(4, 5)(6, 7);
>D:=N!(1, 3)(4, 5, 6, 7);
```

Now, we conjugate every generator of $NL[8]$ by D , such that, $N/NL[8]=< NL[8]D >$. $N/NL[8]=(< A > D, < B > D, < C > D)$.

```
>for q,r,s in [0..1] do if A^D eq A^q*B^r*C^s then q,r,s; end
if; end for;
1 0 1
>for q,r,s in [0..1] do if B^D eq A^q*B^r*C^s then q,r,s; end
if; end for;
```

```

0 1 0
>for q,r,s in [0..1] do if C^D eq A^q*B^r*C^s then q,r,s; end
if; end for;
0 1 1
>A^D eq A*C;
true
>B^D eq B;
true
>C^D eq B*C;
true

```

Thus, a final presentation of N is the following

```

>Z<a,b,c,d>:=Group<a,b,c,d|a^2,b^2,c^2,(a,b),(a,c),(b,c),
d^4,a^d=a*c,(b,d),c^d=b*c>;
>#Z;
32
>f,Z1,k:=CosetAction(Z,sub<Z|Id(Z)>);
>#Z1;
32
>w:=IsIsomorphic(Z1,N);
>w;
true

```

Hence, our presentation of $N \cong (2^3 : 4)$.

7.4 Semi-direct $(7^4 : (2^3 : 2))$

We start with a transitive group on 28 letters. N is of order 38416 with $x \sim (1, 14, 17, 27, 7, 9, 20, 22, 6, 11, 16, 24, 5, 13, 19, 26, 4, 8, 15, 28, 3, 10, 18, 23, 2, 12, 21, 25)$ and $y \sim (1, 28, 5, 22)(2, 23, 4, 27)(3, 25)(6, 24, 7, 26)(8, 16)(9, 21, 14, 18)(10, 19, 13, 20)(11, 17, 12, 15)$.

In this case, the largest abelian normal subgroup is $NL[7]$ and the order is 2401. So the isomorphism type is either Z_{2401} , or a combination of multiple of 7. We know that it cannot be a direct product since there does not exist a normal subgroup of order 16 outside of $NL[7]$. Thus, a presentation for $NL[7]$ is shown by E. Since $NL[7]$ has four generators of order 7 we can assume it is 7^4 , Magma can confirm that we have 7^4 . And a presentation for $NL[7]$ follows.

```
>IsIsomorphic(NL[7],AbelianGroup(GrpPerm,[7,7,7,7]));
true
>E<a,b,c,d>:=Group<a,b,c,d|a^7,b^7,c^7,d^7, (a,b),
(a,c), (b,c), (a,d), (b,d), (c,d)>;
>#E;
2401
```

Now we must factor out N by $NL[7]$ and $NL[7]$ is an extension of $q = N/NL[7]$. The generators of q are $E = (1, 2, 4, 7)(3, 5, 8, 6)$ and $F = (1, 3, 6, 2)(4, 8, 5, 7)$. Hence, a presentation for q is

$$F \langle e, f \rangle := \text{Group} \langle e, f | e^4, f^4, (e^{-1} * f^{-1})^2, (e * f^{-1})^2 \rangle$$

After a series of transversal computations $T[i]$ of $NL[7]$ in N , such that $T[2] \sim E$ and $T[3] \sim F$. We find that

$$\begin{aligned}
A^{T[2]} &= A * B^5 * C * D^6, \\
B^{T[2]} &= A * B^6 * C^4 * D^3, \\
C^{T[2]} &= C^2 * D^5, \\
D^{T[2]} &= C^5 * D, \\
A^{T[3]} &= A * B^5 * C * D^6, \\
B^{T[3]} &= A * B^6 * C^2 * D^3, \\
C^{T[3]} &= C^2 * D^5, \text{ and} \\
D^{T[3]} &= C^6 * D^5
\end{aligned}$$

Therefore, a final presentation of N is $J \langle a, b, c, d, e, f \mid a^7, b^7, c^7, d^7, (a, b), (a, c), (b, c), (a, d), (b, d), (c, d), e^4, f^4, (e^{-1}f^{-1})^2, (ef^{-1})^2, a^e = ab^5cd^6, b^e = ab^6c^4d^3, c^e = c^2d^5, d^e = c^5d^5, a^f = ab^5cd^6, b^f = ab^6c^2d^3, c^f = c^2d^5, d^f = c^6d^5 \rangle$. And we can confirm via Magma that this is correct.

```

> #J; #N;
38416
38416
> f, J1, k := CosetAction(J, sub<J | Id(J)>);
> IsIsomorphic(J1, N);
true

```

Thus, we can conclude that N is a semi-direct of $(7^4 : (2^3 : 2))$.

Appendix A

MAGMA CODE Construction of $2^4 : (3^2 : 4)$ over $(3^2 : 4)$

```
N:=TransitiveGroup(6,10);
```

```
N;
```

```
S:=Sym(6);
```

```
xx:=S!(2, 4, 6);
```

```
yy:=S!(1, 5)(2, 4);
```

```
zz:=S!(1, 4, 5, 2)(3, 6);
```

```
N:=sub<S|xx,yy,zz>;
```

```
#N;
```

```
FPGroup(N);
```

```

NN<a,b,c>:=Group<a,b,c|a^3,b^2,c^4,(a^-1*b)^2,
c^-1*b*c^-1,a^-1*c^-1*a^-1*c^-1*a^-1*c*a*c>;
#NN;

Set(N);

SchreierSystem*/
SchreierSystem(NN,sub<NN|Id(NN)>);

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);

ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
/* we put -1 if our G contains powers to the negative*/
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;

end for;
PP:=Id(N);
for k in [1..#P] do

```



```

PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabilizer(N,1);
N1;

for i in [1..#Sch] do if ArrayP[i] eq N!(2,4,6) then Sch[i]; end if; end for;

for i in [1..#Sch] do if ArrayP[i] eq N!(2,6)(3,5) then Sch[i]; end if; end for;

xx;
zz*xx^-1*zz*xx;

N1 eq sub<N|xx,zz*xx^-1*zz*xx>;
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^2,z^4,(x^-1*y)^2,
z^-1*y*z^-1,x^-1*z^-1*x^-1*z^-1*x^-1*z*x*z,t^2,
(t,x),(t,z*x^-1*z*x)>;

Orbits(N1);

for j in [1..6] do for i in [1..#N] do if 1^ArrayP[i] eq j then j,
Sch[i]; break; end if; end for; end for;

```

```
NN;
```

```
#NN;
```

```
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^2,z^4,(x^-1*y)^2,
z^-1*y*z^-1,x^-1*z^-1*x^-1*z^-1*x^-1*z*x*z,t^2,
(t,x),(t,z*x^-1*z*x),
(t,t^(z*x*z)),(t,t^(z^-1))>;
#G;
```

```
2^6*36;
```

```
C:=Classes(N);
```

```
for i in [2..#C] do i, C[i][3]; for j in [1..#N] do
if ArrayP[j] eq C[i][3] then Sch[j]; end if; end for;
Orbits(Centralizer(N,C[i][3])); end for;
```

```
for j in [1..6] do for i in [1..#N] do if 1^ArrayP[i] eq j then j,
Sch[i]; break; end if; end for; end for;
```

Appendix B

MAGMA CODE Isomorphism

Type of $2^4 : (3^2 : 4)$

```

N:=TransitiveGroup(6,10);
N;
S:=Sym(6);
xx:=S!(2, 4, 6);
yy:=S!(1, 5)(2, 4);
zz:=S!(1, 4, 5, 2)(3, 6);
N:=sub<S|xx,yy,zz>;

/* to find isom type of G */
a:=0;b:=0;c:=0;d:=0;e:=6;f:=3;g:=8;h:=0;i:=0;j:=0;
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^2,z^4,(x^-1*y)^2,
z^-1*y*z^-1,x^-1*z^-1*x^-1*z^-1*x^-1*z*x*z,

```

```

t^2, (t, x), (t, z*x^-1*z*x), (y*t)^a, (y*t^(z^-1))^b,
(z*x*z^-1*t)^c, (z*x*z^-1*t^(z^-1))^d,
(x*z*x*z^-1*t)^e, (x*z*x*z^-1*t^(z^-1))^f, (z*t)^g,
(z*t^z^-1)^h, (z^-1*t)^i, (z^-1*t^(z^-1))^j>;
#G;
> f, G1, k := CosetAction(G, sub<G|x, y, z>);
> CompositionFactors(G1);

> X := AbelianGroup(GrpPerm, [2, 2, 2, 2]);
> NL := NormalLattice(G1);
> NL;

> IsIsomorphic(X, NL[2]);

> q, ff := quo<G1|NL[2]>;
> IsIsomorphic(q, sub<G1|f(x), f(y), f(z)>);

```

Appendix C

MAGMA CODE Monomial

Progenitor $3^{*7} :_m PSL(2, 7)$

Consider the following group G and its subgroup H.

```
S:=Sym(7);
xx:=S!(1, 2, 3, 4, 5, 6, 7);
yy:=S!(1, 2)(3, 6);
G:=sub<S|xx,yy>;
#G;
CG:=CharacterTable(G);
H:=sub<G|(1, 5)(2, 7),(1, 2, 4)(5, 6, 7)>;
#H;
CH:=CharacterTable(H);

CH[2];
```

```
CG[5];

I:=Induction( CH[2],G);
I;

C:=Classes(G);
#C;

C[1];

C[2];

"C2=",Set(Class(G,C[2][3]));

C[3];
"C3=",Set(Class(G,C[3][3]));

C[4];
"C4=",Set(Class(G,C[4][3]));

C[5];
"C5=",Set(Class(G,C[5][3]));
```

```
C[6];  
"C6=",Set(Class(G,C[6][3]));
```

```
D:=Classes(H);  
D;
```

```
D[1];  
"D1=",Set(Class(H,D[1][3]));
```

```
D[2];  
"D2=",Set(Class(H,D[2][3]));
```

```
D[3];  
"D3=",Set(Class(H,D[3][3]));
```

```
D[4];  
"D4=",Set(Class(H,D[4][3]));
```

```
D[5];  
"D5=",Set(Class(H,D[5][3]));
```

```
CH:=CharacterTable(H);  
CH;
```

```

T:=Transversal(G,H);
#T; /* 7 , Will be 7 since #G/#H=7.*/
T;

/* this is the version he showed in class to find perm of A */
T[1]*xx*T[1]^-1 in H;/* false */
T[1]*xx*T[2]^-1 in H;/* true */
T[1]*xx*T[3]^-1 in H;/* false */
T[1]*xx*T[4]^-1 in H;/* false */
T[1]*xx*T[5]^-1 in H;/* false */
T[1]*xx*T[6]^-1 in H;/* false */
T[1]*xx*T[7]^-1 in H;/* false */

T[1]*xx*T[2]^-1; /* Id(S) */
CH[2](H!Id(S));/* 1 */
/* so the first row is [1,0,0,0,0,0,0]

T[2]*xx*T[1]^-1 in H;/* false */
T[2]*xx*T[2]^-1 in H;/* false */
T[2]*xx*T[3]^-1 in H;/* false */
T[2]*xx*T[4]^-1 in H;/* true*/
T[2]*xx*T[5]^-1 in H;/* false */
T[2]*xx*T[6]^-1 in H;/* false */
T[2]*xx*T[7]^-1 in H;/* false */

```



```

T[2]*xx*T[4]^-1; /*Id(S) */
CH[2](H!Id(S));/* 1 */
/* so the second row is [0,1,0,0,0,0,0]

```

```

T[3]*xx*T[1]^-1 in H;
T[3]*xx*T[2]^-1 in H;
T[3]*xx*T[3]^-1 in H;
T[3]*xx*T[4]^-1 in H;
T[3]*xx*T[5]^-1 in H; /* true */
T[3]*xx*T[6]^-1 in H;
T[3]*xx*T[7]^-1 in H;

```

```

T[3]*xx*T[5]^-1; /* Id(S) */
CH[2](H!Id(S));/* 1 */
/* so the third row is [0,0,1,0,0,0,0]

```

```

T[4]*xx*T[1]^-1 in H;
T[4]*xx*T[2]^-1 in H;
T[4]*xx*T[3]^-1 in H; /* true */
T[4]*xx*T[4]^-1 in H;
T[4]*xx*T[5]^-1 in H;
T[4]*xx*T[6]^-1 in H;
T[4]*xx*T[7]^-1 in H;

```

```

T[4]*xx*T[5]^-1; /* Id(S) */
CH[2](H!Id(S)); /* 1 */
/* so the fourth row is [0,0,0,1,0,0,0]

```

```

T[5]*xx*T[1]^-1 in H;
T[5]*xx*T[2]^-1 in H;
T[5]*xx*T[3]^-1 in H;
T[5]*xx*T[4]^-1 in H;
T[5]*xx*T[5]^-1 in H;
T[5]*xx*T[6]^-1 in H; /* true */
T[5]*xx*T[7]^-1 in H;

```

```

T[5]*xx*T[6]^-1; /*(1, 4, 5)(2, 7, 6)*/
CH[2](H!(1, 4, 5)(2, 7, 6)); /* 1 */
/* so the fifth row is [0,0,0,0,1,0,0]

```

```

T[6]*xx*T[1]^-1 in H;
T[6]*xx*T[2]^-1 in H;
T[6]*xx*T[3]^-1 in H;
T[6]*xx*T[4]^-1 in H;
T[6]*xx*T[5]^-1 in H;
T[6]*xx*T[6]^-1 in H;

```

```

T[6]*xx*T[7]^-1 in H; /* true */

T[6]*xx*T[7]^-1; /*(1, 2, 7, 5)(4, 6)*/
CH[2](H!(1, 2, 7, 5)(4, 6)); /* -1 */
/* so the sixth row is [0,0,0,0,0,-1,0]

T[7]*xx*T[1]^-1 in H; /* true */
T[7]*xx*T[2]^-1 in H;
T[7]*xx*T[3]^-1 in H;
T[7]*xx*T[4]^-1 in H;
T[7]*xx*T[5]^-1 in H;
T[7]*xx*T[6]^-1 in H;
T[7]*xx*T[7]^-1 in H;

T[7]*xx*T[1]^-1; /*(1, 4, 7, 6)(2, 5)*/
CH[2](H!(1, 4, 7, 6)(2, 5)); /* -1 */
/* so the seventh row is [-1,0,0,0,0,0,0]*/

for i in [1..7] do if T[1]*xx*T[i]^-1 in H then i; end if; end for;
/* 2 , this means we will take i=2, CH[2]xxT2 */

CH[2](T[1]*xx*T[2]^-1);
/* 1 , this command means my first row of matrix is (1,0,0,0,0,0,0)

```

```

Now , we check the second row: */
for i in [1..7] do if T[2]*xx*T[i]^-1 in H then i; end if; end for;
/* 2 , this means we will take i=4, CH[4]xxT2*/

CH[2](T[2]*xx*T[4]^-1);
/* 1 , this means the second row of the matrix is (0,1,0,0,0,0,0) .
we continue this same process for rows 3-7 */

C:=CyclotomicField(3); /* Hasan looked at my CH[2] to determine this */

A:=[[C.1,0,0,0,0,0,0]: i in [1..7]];
for i,j in [1..7] do A[i,j]:=0; end for;
for i,j in [1..7] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[2](T[i]*xx*T[j]^-1); end if; end for;
A;

/*rewrite this for yy */
for i in [1..7] do if T[1]*yy*T[i]^-1 in H then i; end if; end for;
/* 3 , this means we will take CH[2]yyT3*/

CH[2](T[1]*yy*T[3]^-1);/* -1 */

C:=CyclotomicField(3); /* Hasan looked at my CH[2] to determine this */
B:=[[C.1,0,0,0,0,0,0]: i in [1..7]];

```

```

for i,j in [1..7] do B[i,j]:=0; end for;
for i,j in [1..7] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[2](T[i]*yy*T[j]^-1); end if; end for;
B;

```

```

/*Let us now write A as A(xx) and B as A(yy).*/

```

```

GG:=GL(7,C);

```

```

GG!A;

```

```

/*

```

```

[ 0  1  0  0  0  0  0]

```

```

[ 0  0  0  1  0  0  0]

```

```

[ 0  0  0  0  1  0  0]

```

```

[ 0  0  1  0  0  0  0]

```

```

[ 0  0  0  0  0  1  0]

```

```

[ 0  0  0  0  0  0 -1]

```

```

[-1  0  0  0  0  0  0]

```

```

*/

```

```

GG!B;

```

```

/*

```

```

[ 0  0 -1  0  0  0  0]

```

```

[ 0  1  0  0  0  0  0]

```

```

[-1  0  0  0  0  0  0]

```

```

[ 0  0  0 -1  0  0  0]

```

```

[ 0 0 0 0 -1 0 0]
[ 0 0 0 0 0 0 -1]
[ 0 0 0 0 0 -1 0]
*/

/*Now we check faithfulness*/
Order(GG!A);
/* 7 */
Order(xx);
/* 7 */

Order(GG!B);
/* 2 */
Order(yy);
/*2 */
IsIsomorphic(G,sub<GG|GG!A, GG!B>);/*true,
by these calculations above, this means it is faithful*/

GG:=GL(7,GF(3));
GG!A;
/*
[0 1 0 0 0 0 0]
[0 0 0 1 0 0 0]
[0 0 0 0 1 0 0]

```

```
[0 0 1 0 0 0 0]
[0 0 0 0 0 1 0]
[0 0 0 0 0 0 2]
[2 0 0 0 0 0 0]
```

So we have $A(xx)=(1, 2, 4, 3, 5, 6, 14)(7, 8, 9, 11, 10, 12, 13)$

Now we just write $A(xx)=A$ as a permutation. */

```
S:=Sym(14); /* we write 14 since we had a total of 14 ti's */
```

```
Aper:=S!(1, 2, 4, 3, 5, 6, 14)(7, 8, 9, 11, 10, 12, 13);
```

```
GG!B;
```

```
/*
```

```
[0 0 2 0 0 0 0]
[0 1 0 0 0 0 0]
[2 0 0 0 0 0 0]
[0 0 0 2 0 0 0]
[0 0 0 0 2 0 0]
[0 0 0 0 0 0 2]
[0 0 0 0 0 2 0]
```

So we have $A(yy)= (1,10)(3,8)(4,11)(5,12)(6,14)(7,13)$

Now we just write $A(yy)=B$ as a permutation. */

```
S:=Sym(14); /* we write 14 since we had a total of 14 ti's */
Bper:=S!(1,10)(3,8)(4,11)(5,12)(6,14)(7,13);
```

```
Order(Aper) eq Order(GG!A);
```

```
Order(Aper) eq Order(xx);
```

```
Order(Bper) eq Order(GG!B);
```

```
Order(Bper) eq Order(yy);
```

```
IsIsomorphic(sub<S|Aper,Bper>,sub<GG|GG!A,GG!B>);
```

```
S:=Sym(14);
```

```
xx:=S!(1, 2, 4, 3, 5, 6, 14)(7, 8, 9, 11, 10, 12, 13);
```

```
yy:=S!(1,10)(3,8)(4,11)(5,12)(6,14)(7,13);
```

```
N:=sub<S|xx,yy>;
```

```
#N; /* 168 */
```

```
N;
```

```
FPGroup(N);
```

```
NN<a,b>:=Group<a,b | a^7, b^2, (b*a)^3, (a*b*a^2)^4 >;
```

```
#NN; /* 168 */
```

```
Nt:=Stabiliser(N,{1,8});
```



```
#Nt;
```

```
Nt;
```

```
word:=function(A)
```

```
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
```

```
for i in [2..#NN] do
```

```
P:=Id(N): 1 in [1..#Sch[i]]];
```

```
for j in [1..#Sch[i]] do
```

```
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
```

```
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
```

```
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
```

```
end for;
```

```
PP:=Id(N);
```

```
for k in [1..#P] do
```

```
PP:=PP*P[k]; end for;
```

```
if A eq PP then B:=Sch[i]; end if;
```

```
end for;
```

```
return B;
```

```
end function;
```

```
A:=N!(2, 7, 12)(3, 11, 13)(4, 6, 10)(5, 9, 14);
```

```
B:=N!(2, 10)(3, 9)(4, 14)(5, 12)(6, 13)(7, 11);
```

```
C:=N!(1, 8)(2, 6, 3, 12)(4, 7, 11, 14)(5, 9, 13, 10);
```

```
word(A);
```

```
word(B);
```

```
word(C);
```

```
W:=WordGroup(N);
```

```
rho:=InverseWordMap(N);
```

```
A:=N!(2, 7, 12)(3, 11, 13)(4, 6, 10)(5, 9, 14);
```

```
B:=N!(2, 10)(3, 9)(4, 14)(5, 12)(6, 13)(7, 11);
```

```
C:=N!(1, 8)(2, 6, 3, 12)(4, 7, 11, 14)(5, 9, 13, 10);
```

```
A@rho;
```

```
wA:= function(W)
```

```
w3 := W.1^4; w2 := W.2^-1; w4 := W.1^2; w5 := w2 * w4; w6 := w3 * w5; w8 :=
    W.2 * w6; w9 := w8 * W.2; w10 := w9 * W.1; w11 := w10 * W.1; w12 := w11 *
    W.2; w1 := W.1^-1; w13 := w12 * w1; return w13;
```

```
end function;
```

```
wA(NN);
```

```
yy * xx^4 * yy^-1 * xx^2 * yy * xx^2 * yy * xx^-1; /* from wA */
/* (2, 7, 12)(3, 11, 13)(4, 6, 10)(5, 9, 14) */
```

```
xx * yy * xx^-3 * yy * xx; /* from word(A) */
/* (2, 7, 12)(3, 11, 13)(4, 6, 10)(5, 9, 14) */
/* hence, both words have the same permutations*/
```

```
B@rho;
```

```
wB:= function(W)
    w3 := W.1^4; w2 := W.2^-1; w4 := W.1^2; w5 := w2 * w4;
    w6 := w3 * w5; w8 :=
    W.2 * w6; w9 := w8 * w6; w1 := W.1^-1; w10 := w9 * w1;
    w11 := w10 * W.2;
    return w11;
```

```
end function;
```

```
wB(NN);
```

```
/* b * a^4 * b^-1 * a^6 * b^-1 * a * b ,
```

```
this means yy * xx^4 * yy^-1 * xx^6 * yy^-1 * xx * yy */
```

```
C@rho;
```

```
wC:=function(W)
```

```
    w3 := W.1^4; w2 := W.2^-1; w4 := W.1^2; w5 := w2 * w4;
```

```

w6 := w3 * w5; w8 :=
W.2 * w6; w1 := W.1^-1; w12 := w8 * w1; w13 := w12 * w1;
w14 := w13 * W.2;
return w14;
end function;

```

```

wC(NN);
/* b * a^4,
this means yy * xx^4 */

1^A;
/* 1, so we will write (t,A) or we can say t^A=t */
1^B;
/* 1, so we will write (t,B) or we can say t^B=t */
1^C;
/* 8 so we will write t^C=t8 but t8=t1^2 when we looking
at our labeling of t_i's, so
we can say t^C=t^2 */

/* Now we can input all this new info into our NN,
NN<a,b>:=Group<a,b | a^7, b^2, (b*a)^3, (a*b*a^2)^4,
t^3, (since our field is of order 3),
(t,A), we insert the perm from word(A)

```

```

here BUT make sure it only
one x and y in each
(t,B), similarly, we insert the perm from word(B) here
t^C=t^2 similarly, we insert the perm from word(C) here>; */

G<x,y,t>:=Group<x,y ,t| x^7, y^2, (y*x)^3, (x*y*x^2)^4,
t^3,
(t,x * y * x^-3 * y * x),
(t,y * x^-2 * y * x^2 * y),
t^(y * x^-3)=t^2 >;

/* Testing the progenitor - Grindstaff Lemma.
Sym 14 , xxx, yyy come from our originals (start of problem 4 ) */

S:=Sym(14);
xxx:=S!(1, 2, 4, 3, 5, 6, 14)(7, 8, 9, 11, 10, 12, 13);
yyy:=S!(1,10)(3,8)(4,11)(5,12)(6,14)(7,13);

GGG:=sub<S | xxx,yyy>;
#GGG;/* 168 */

Stabilizer(GGG,1);

```

```

Orbits(Stabilizer(GGG,1));

/* we do this loop to see what our perms are: */
temp:={2,8};
for n in N do for j in temp do if 1^n eq j then word(n),j;
temp:=temp diff {j}; end if; end for;
end for;

/*
a 2
a^-3 * b * a^3 8

this implies that 1^xx eq 2 */

1^xx;
/* 2 , sp we can rewrite as (t, t^x) */

1^(xx^-3 * yy * xx^3) ;
/* 8, so we can rewrite as (t, t^x^-3 * y * x^3) */

G<x,y,t>:=Group<x,y ,t| x^7, y^2, (y*x)^3, (x*y*x^2)^4,
t^3, /* NOTE: comes from #of field in this case its
Z3*/
(t,x * y * x^-3 * y * x), /* NOTE: comes from word of

```

```

generator of Nt:=Stabilizer*/

(t,y * x^-2 * y * x^2 * y), /* NOTE: comes from word
of generator of Nt:=Stabilizer*/

t^(y * x^-3)=t^2, /* NOTE: comes from word of
generator of Nt:=Stabilizer*/

(t, t^x), /* NOTE: comes from word of orbits
of Stabilizer*/

(t, t^(x^-3 * y * x^3)) >;/* NOTE: comes from
word of orbits of Stabilizer*/

/* #G eq Index=3^7 * #N=168 */
Index(G,sub<G | x, y >);
/* 2187 */
3^7;
/* 2187 */
2187 * 168;
/* 367416 */

#G;

G<x,y,t>:=Group<x,y,t|x^7, y^2, (y*x)^3, (x*y*x^2)^4,

```

```

t^3,
(t,x*y *x^-3*y*x),
(t,y*x^-2*y*x^2*y),
t^(y*x^-3)=t^2 >;

C:=Classes(N);
C;

for i in [2..#C] do i, C[i][3]; word(C[i][3]);
Orbits(Centraliser(N,C[i][3])); end for;

/* to find out what ts are we do the following loop */
for j in [2..#N] do for i in [1..#Setseq(Set(N))] do
if 1^Setseq(Set(N))[i] eq
j then j,word(Setseq(Set(N))[i]) ; break; end if; end for; end for;

```


Appendix D

MAGMA CODE Monomial

Progenitor $3^*4 :_m (2^4 : 4)$

```
T:=TransitiveGroups(8);
G:=T[27];
G;

CG:=CharacterTable(G);

S:=Subgroups(G);
H:=S[40] ' subgroup;
CH:=CharacterTable(H);
Induction(CH[2],G) eq CG[11];
/* true */
```

```

S:=Sym(8);
xx:=S!(4, 8);
yy:=S!(1, 2, 3, 8)(4, 5, 6, 7);
G:=sub<S|xx,yy>;
G eq T[27];
/* true */

H;

for g,h in H do if H eq sub<H | g,h> then g,h; end if; end for;
/* nothing came out so means we need to use all four generators */

A:=G!(2, 6);
B:=G!(1, 3)(2, 8)(4, 6)(5, 7);
C:=G!(1, 5)(2, 6)(3, 7)(4, 8);
D:=G!(2, 6)(4, 8);
S:=Subgroups(G);
H1:=sub<G|A,B,C,D>;
H1 eq S[40] 'subgroup; /* true */

CH:=CharacterTable(H);
CG:=CharacterTable(G);

CH;

```

```

CG;

for i in [2..#CH] do for j in [9..#CG] do if Induction(CH[i],G) eq CG[j]
and IsFaithful(CG[j]) then i,j; end if; end for; end for;

Induction(CH[2],G) eq CG[11]; /* true */
Induction(CH[3],G) eq CG[12];/* true */
Induction(CH[5],G) eq CG[11];/* true */
Induction(CH[6],G) eq CG[12];/* true */

T:=Transversal(H, Hprime);
#T; /* 8 , these are the 8 cosets of Hprime in H */

Ht:=[{h*T[i] : h in Hprime} : i in [1..8]];
Ht[1] eq Set(Hprime);
/* true */

q,ff:=quo<H|Hprime>;
IsAbelian(q); /* true */

q;
IsIsomorphic(q,AbelianGroup(GrpPerm,[2,2,2]));
/* true */

```

```

/* So we check the char table of Z2 X Z2 X Z2 */
ff(T[2]) eq q.1;
/* true */

ff(T[3]) eq q.2;
/* true */

ff(T[4]) eq q.3;
/* true */

/*This means,    a=T[2], b=T[3],c=T[4]*/

/* Classes of H/Hprime: */
/*  e   a   b   c   ab  ac  bc   abc */
/* the 8 elements are obtained substituting: (-1)^ (-1)^ (-1)^ */

for i in [1..8] do if T[2]*T[3] in Ht[i] then i; end if; end for;
/*5, ab=Ht[5]*/

for i in [1..8] do if T[2]*T[4] in Ht[i] then i; end if; end for;
/*6, ac=Ht[6] */

for i in [1..8] do if T[3]*T[4] in Ht[i] then i; end if; end for;
/* 7, bc=Ht[7] */

for i in [1..8] do if T[2]*T[3]*T[4]in Ht[i] then i; end if; end for;
/*8, abc=Ht[8] */

```

```
C:=Classes(H);
CC:=[Set(Class(H,C[i][3])) : i in [1..#C]];

CC[1];

CC[2];/* we do this one since we are looking at CH[2] */

#C; /* 10 *//* there are 10 since this in H*/

Hprime;

for i in [1..#C] do if Id(H) in CC[i] then i; end if; end for;

for i in [1..#C] do if H!(2,6)(4,8) in CC[i] then i; end if; end for;

for i in [1..#C] do if Ht[2] subset CC[i] then i; end if; end for;

for i in [1..#C] do if Ht[3] subset CC[i] then i; end if; end for;

for i in [1..#C] do if Ht[4] subset CC[i] then i; end if; end for;
Ht[4];

for i in [1..#C] do if H!(1, 5)(3, 7) in CC[i] then i; end if; end for;
```

```
for i in [1..#C] do if H!(1, 5)(2, 6)(3, 7)(4, 8) in CC[i] then i;
end if; end for;
```

```
for i in [1..#C] do if Ht[5] subset CC[i] then i; end if; end for;
```

```
for i in [1..#C] do if Ht[6] subset CC[i] then i; end if; end for;
```

```
for i in [1..#C] do if Ht[7] subset CC[i] then i; end if; end for;
```

```
for i in [1..#C] do if Ht[8] subset CC[i] then i; end if; end for;
```

```
CH[8];
```

```
/* ( 1, 1, 1, 1, 1, 1, -1, -1, -1, -1 ) */
```

```
for i in [1..10] do if CH[8](C[i][3]) ne Xdot[i] then i; end if; end for;
```

```
T:=TransitiveGroups(8);
```

```
G:=T[27];
```

```
G;
```

```
/*
```

```
Permutation group G acting on a set of cardinality 8
```

```
Order = 64 = 2^6
```

```

(4, 8)
(1, 2, 3, 8)(4, 5, 6, 7)
*/
CG:=CharacterTable(G);

S:=Subgroups(G);
H:=S[40] 'subgroup;
CH:=CharacterTable(H);
Induction(CH[2],G) eq CG[11];
/* true */

S:=Sym(8);
xx:=S!(4, 8);
yy:=S!(1, 2, 3, 8)(4, 5, 6, 7);
G:=sub<S|xx,yy>;
G eq T[27];
/* true */
#G;

H;

for g,h in H do if H eq sub<H | g,h> then g,h; end if; end for;

A:=G!(2, 6);

```

```

B:=G!(1, 3)(2, 8)(4, 6)(5, 7);
C:=G!(1, 5)(2, 6)(3, 7)(4, 8);
D:=G!(2, 6)(4, 8);

H1:=sub<G|A,B,C,D>;
S:=Subgroups(G);
H1 eq S[40] 'subgroup;
H1;

CH1:=CharacterTable(H1);
CH1;

/* Induce the Character CH1[2] up to the character CG[11] */
CH1[2];
/* ( 1, -1, 1, -1, -1, 1, -1, 1, 1, -1 ) */
CG[11];
/* ( 4, -4, 0, 0, -2, 0, 2, 0, 0, 0, 0 ) */

/* we can do the following command for a quicker result */
I:=Induction( CH[2],G);
I;/* ( 4, -4, 0, 0, -2, 0, 2, 0, 0, 0, 0 ) */

/* Now we can get the same result by hand:
to do by hand we do the following: */

```



```

C:=Classes(G);
#C; /* 13 */

C[1]; /* <1, 1, Id(G)> , the first # is the order,
second # is the # of elements , third # is a member of the class*/

C[2]; /* <2, 1, (1, 5)(2, 6)(3, 7)(4, 8)>, the first # is the order,
second # is the # of elements , third # is a member of the class.*/
C[3]; /* <2, 2, (1, 5)(3, 7)>, the first # is the order,
second # is the # of elements , third # is a member of the class.

In this case, we have 2 elements of order 2. one of these 2
elements is (15)(37)..we need all members of this class.

This is how we obtain them: */
"C3=",Set(Class(G,C[3][3])); /* the [3] comes from the
element being in the third slot < ?,?,(15)(37) > */
/*
C3= {
    (2, 6)(4, 8),
    (1, 5)(3, 7)
}*/

C[4];
/* <2, 4, (1, 3)(2, 8)(4, 6)(5, 7)>*/
"C4=",Set(Class(G,C[4][3]));

```

```
C[5];/* <2, 4, (1, 5)> */
"C5=",Set(Class(G,C[5][3]));

C[6];/* <2, 4, (1, 5)(4, 8)>*/
"C6=",Set(Class(G,C[6][3]));

C[7];/* <2, 4, (1, 5)(3, 7)(4, 8)>*/
"C7=",Set(Class(G,C[7][3]));

C[8];/* <4, 4, (1, 7, 5, 3)(2, 4, 6, 8)>*/
"C8=",Set(Class(G,C[8][3]));

C[9];/* <4, 8, (1, 2, 3, 8)(4, 5, 6, 7)>*/
"C9=",Set(Class(G,C[9][3]));

C[10];/* <4, 8, (1, 8, 3, 2)(4, 7, 6, 5)>*/
"C10=",Set(Class(G,C[10][3]));

C[11];/* <4, 8, (1, 7, 5, 3)(2, 8)(4, 6)>*/
"C11=",Set(Class(G,C[11][3]));

C[12];/* <8, 8, (1, 4, 7, 6, 5, 8, 3, 2)>*/
"C12=",Set(Class(G,C[12][3]));
```

```
C[13];/* <8, 8, (1, 6, 3, 4, 5, 2, 7, 8)>*/  
"C13=",Set(Class(G,C[13][3]));  
  
/*Let us now find the Classes of H.*/  
D:=Classes(H);  
D;  
  
D[1]; /* <1, 1, Id(H)> */  
"D1=",Set(Class(H,D[1][3]));  
  
D[2];/*<2, 1, (1, 5)(2, 6)(3, 7)(4, 8)>*/  
  
D[3];/*<2, 1, (2, 6)(4, 8)>*/  
  
D[4];/*<2, 1, (1, 5)(3, 7)>*/  
  
D[5];/*<2, 2, (1, 3)(2, 8)(4, 6)(5, 7)>*/  
"D5=",Set(Class(H,D[5][3]));  
  
D[6];/*<2, 2, (1, 7)(2, 4)(3, 5)(6, 8)>*/  
"D6=",Set(Class(H,D[6][3]));  
  
D[7];/* <2, 2, (2, 6)> */
```

```

"D7=",Set(Class(H,D[7][3]));

D[8];/* <2, 2, (1, 5)(3, 7)(4, 8)>*/
"D8=",Set(Class(H,D[8][3]));

D[9];/* <4, 2, (1, 3)(2, 4, 6, 8)(5, 7)>*/
"D9=",Set(Class(H,D[9][3]));

D[10];/* <4, 2, (1, 7)(2, 8, 6, 4)(3, 5)> */
"D10=",Set(Class(H,D[10][3]));

CH1:=CharacterTable(H1);
CH1;

T:=Transversal(G,H1);
#T; 4, /* 4, since #G/#H=64/16=4 , hence we will have 4x4 matrices*/
T;

/*First we find A(xx).*/
/*Our 4x4 matrix for xx (A(xx)) is:
T1xxT1^-1 T1xxT2^-1 T1xxT3^-1 T1xxT4^-1
T2xxT1^-1 T2xxT2^-1 T2xxT3^-1 T2xxT4^-1
T3xxT1^-1 T3xxT2^-1 T3xxT3^-1 T3xxT4^-1
T4xxT1^-1 T4xxT2^-1 T4xxT3^-1 T4xxT4^-1

```

```

*/
/* this is the version he showed in class to find perm of A */
T[1]*xx*T[1]^-1 in H;/* true*/
T[1]*xx*T[2]^-1 in H;/* false */
T[1]*xx*T[3]^-1 in H;/* false */
T[1]*xx*T[4]^-1 in H;/* false */

T[1]*xx*T[1]^-1; /* (4, 8) */
CH[2](H!(4, 8));/* -1 */
/* so the first row is [-1,0,0,0]

T[2]*xx*T[1]^-1 in H;/* false */
T[2]*xx*T[2]^-1 in H;/* false */
T[2]*xx*T[3]^-1 in H;/* true*/
T[2]*xx*T[4]^-1 in H;/* false */

T[2]*xx*T[3]^-1; /* Id(H) */
CH[2](H!Id(H));/* 1 */
/* so the second row is [0,1,0,0]

T[3]*xx*T[1]^-1 in H;/* false */
T[3]*xx*T[2]^-1 in H;/* true*/
T[3]*xx*T[3]^-1 in H;/* false */
T[3]*xx*T[4]^-1 in H;/* false */

```

```

T[3]*xx*T[2]^-1;/* Id(H)*/
CH[2](H!Id(H));/* 1 */
/* so the third row is [0,0,1,0]

T[4]*xx*T[1]^-1 in H;/* false */
T[4]*xx*T[2]^-1 in H;/* false */
T[4]*xx*T[3]^-1 in H;/* false */
T[4]*xx*T[4]^-1 in H;/* true*/

T[4]*xx*T[4]^-1;/* (2,6)*/
CH[2](H!(2,6));/* -1 */
/* so the third row is [0,0,0,-1]

for i in [1..4] do if T[1]*xx*T[i]^-1 in H then i; end if; end for;
/* 1 , this means we will take i=1, CH[2]xxT1. Now we figure out
what perm T[1] is */
CH[2](T[1]*xx*T[1]^-1);/* -1, this means my first row of matrix is (-1,0,0,0).

Now , we check the second row: */
for i in [1..4] do if T[2]*xx*T[i]^-1 in H then i; end if; end for;
/*3, this means we will take i=3, CH[2]xxT3*/
CH[2](T[2]*xx*T[3]^-1);/* 1 , this means the second row
of the matrix is (0,1,0,0) . we continue this same process for rows 3 &4 */

```

```

/* this is a shortcut */
CF:=CyclotomicField(3);/* cycl field is 3 since our field is 3 */
GG:=GL(4,CF); /* the four comes from our dim size */
AA:=[[CF.1,0,0,0] : i in [1..4]];
for i ,j in [1..4] do AA[i,j]:=0; end for;
for i,j in [1..4] do if T[i]*xx*T[j]^-1 in H1 then
AA[i,j]:=CH1[2](T[i]*xx*T[j]^-1); end if; end for;
BB:=[[CF.1,0,0,0] : i in [1..4]];
for i ,j in [1..4] do BB[i,j]:=0; end for;
for i,j in [1..4] do if T[i]*yy*T[j]^-1 in H1 then
BB[i,j]:=CH1[2](T[i]*yy*T[j]^-1); end if; end for;
GG!AA; GG!BB;

Order(GG!AA); Order(xx);
Order(GG!BB);Order(yy);

HH:=sub<GG|AA,BB>;
IsIsomorphic(HH,G);

GG:=GL(4,GF(3));
GG!AA;

GG!BB;

```

```

S:=Sym(8);
Aper:=S!(1, 5)(2, 3)(4, 8)(6, 7);
Bper:=S!(1,2,5,6)(3,4)(7,8);
HH:=sub<S|Aper,Bper>;
w:=IsIsomorphic(HH,G);
w;

/* need to check this once I include GG!AA & GG!BB in magma*/
Order(Aper) eq Order(GG!AA);/* true */
Order(Aper) eq Order(xx);/* true */
Order(Bper) eq Order(GG!BB);/* true */
Order(Bper) eq Order(yy);/* true */

IsIsomorphic(sub<S|Aper,Bper>,sub<GG|GG!AA,GG!BB>);/* true Mapping
from: GrpPerm: $, Degree 8, Order 2^6 to MatrixGroup(4, GF(3)) of order 2^6
Composition of Mapping from: GrpPerm: $, Degree 8, Order 2^6 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to MatrixGroup(4, GF(3)) of order 2^6 */

CompositionFactors(G);

>NL:=NormalLattice(G);
>NL;

```



```

> for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;

> X:=AbelianGroup(GrpPerm,[2,2,2,2]);
> s,t:=IsIsomorphic(NL[8],X);
> s;t;
> NL[8];
>A:=G!(2, 6);
>B:=G!(1, 5)(4, 8);
>C:=G!(2, 6)(4, 8);
>D:=G!(2, 6)(3, 7);

NL[8] eq sub<G|A,B,C,D>; /*true */
NL8:=sub<G|A,B,C,D>;
H<a,b,c,d>:=Group<a,b,c,d|a^2,b^2,c^2,d^2,(a,b),
(a,c),(a,d),(b,c),(b,d),(c,d)>;
#H;/* 16 */
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
w1:=IsIsomorphic(H1,NL[8]);
w1;
q,ff:=quo<G|NL[8]>;
q;

nl:=NormalLattice(q);

```

```

nl;

for i in [1..#nl] do if IsAbelian(nl[i]) then i; end if; end for;

S:=Sym(8);
xx:=S!(1, 5)(2, 3)(4, 8)(6, 7);
yy:=S!(1,2,5,6)(3,4)(7,8);

N:=sub<S|xx,yy>;
#N;

FPGroup(N);

NN<a,b>:=Group<a,b|a^2,b^4,(b*a*b^-1*a)^2,
(b^-1 * a * b^-2 * a * b^2 * a * b^2 * a * b^-1)>; #NN; /*64*/

Now to build progenitor:
Nt:=Stabilizer(N,{1,5});

#Nt;/*16*/

Nt;

/* Permutation group Nt acting on a set of cardinality 12

```

```

Order = 16 = 2^4
    (2, 3, 6, 7)(4, 8)
    (3, 7)(4, 8)
    (1, 5)(2, 3)(4, 8)(6, 7)
*/

word:=function(A)
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
for i in [2..#NN] do
P:=Id(N): 1 in [1..#Sch[i]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;

end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;

```

```

for g,h in N do if N eq sub<N | g,h> then g,h; break; end if;
break; end for; /* nothing came out so means we need to use all three
generators */

```

```
A:=N!(2, 3, 6, 7)(4, 8);
```

```
B:=N! (3, 7)(4, 8);
```

```
C:=N! (1, 5)(2, 3)(4, 8)(6, 7);
```

```
word(A); /* a * b^2 */ /*this means x * y^2 */
```

```
word(B);
```

```
word(C);
```

```
/* now we want to check what t commutes with A,B,C*/
```

```
1^A;
```

```
/* 1, so we will write (t,A) or we can say t^A=t */
```

```
1^B;
```

```
/* 1, so we will write (t,B) or we can say t^B=t */
```

```
1^C;
```

```
/* 5, so we will write t^C=t5 but t5=t1^2 when we looking at
our labeling of t_i's, so
```

we can say $t^C = t^2$ */

```
G<x,y,t>:=Group<x,y,t| x^2,y^4,(y*x*y^-1*x)^2,
(y^-1*x*y^-2*x*y^2*x*y^-1),
t^3,
(t, x*y^2),
(t,x*y*x*y*x*y*x*y^-1),
t^x=t^2 >;
```

Testing the mono progenitor - Grindstaff Lemma

```
S:=Sym(8);
xxx:=S!(1, 5)(2, 3)(4, 8)(6, 7);
yyy:=S!(1,2,5,6)(3,4)(7,8);
```

```
GGG:=sub<S|xxx,yyy>;
#GGG;/*64 */
```

```
Stabilizer(GGG,1);
```

```
Orbits(Stabilizer(GGG,1));
```

```
/* we do this loop to see what our perms are: */
```

```
temp:={5,4,2};
```

```
for n in N do for j in temp do if 1^n eq j then word(n),j;
```

```
temp:=temp diff {j}; end if; end for; end for;
```

```
/*
```

```
a 5
```

```
b 2
```

```
b * a * b 4
```

```
*/
```

```
this implies that  $1^{xx} \text{ eq } 5$  */
```

```
 $1^{xx}$ ;
```

```
/* 5 , so we can rewrite as  $(t, t^x)$  */
```

```
 $1^{yy}$ ;
```

```
/* 2 , so we can rewrite as  $(t, t^y)$  */
```

```
 $1^{(yy * xx * yy)}$ ; /* 4, , so we can rewrite as  $(t, t^{(y*x*y)})$  */
```

```
G<x,y,t>:=Group<x,y,t|x^2,y^4,(y*x*y^-1*x)^2,
```

```
(y^-1*x*y^-2*x*y^2*x*y^2*x*y^-1),
```

```
t^3, (t, x*y^2), (t, x*y*x*y*x*y*x*y^-1), t^x=t^2,
```

```
(t, t^x), (t, t^y),(t, t^(y*x*y)) >;
```

```
/* #Group eq Index=3^4 * #G=64 =5184*/
```

```
Index(G,sub<G|x,y>); /* 81 */
```

```
3^4; /* 81*/
```

```
81*64; /* 5184 */
```

```
#G;
```

```
C:=Classes(N);
```

```
C;
```

```
for i in [2..#C] do i, C[i][3]; word(C[i][3]);
```

```
Orbits(Centraliser(N,C[i][3])); end for;
```

```
for j in [2..#N] do for i in [1..#Setseq(Set(N))] do
```

```
if 1~Setseq(Set(N))[i] eq j then j,word(Setseq(Set(N))[i]) ;
```

```
break; end if; end for; end for;
```

Appendix E

MAGMA CODE for

Semi-direct of $(7^4 : 4^2)$

```
N:=TransitiveGroup(28, 500);
```

```
#N;
```

```
N;
```

```
S:=Sym(28);
```

```
xx:=S!(1, 14, 17, 27, 7, 9, 20, 22, 6, 11, 16, 24,
```

```
5, 13, 19, 26, 4, 8, 15, 28, 3,10, 18, 23, 2, 12, 21, 25);
```

```
yy:=S!(1, 28, 5, 22)(2, 23, 4, 27)(3, 25)(6, 24,
```

```
7, 26)(8, 16)(9, 21, 14, 18)(10,19, 13, 20)(11, 17, 12, 15);
```

```
CompositionFactors(N);
```



```

NL:=NormalLattice(N);
NL;

for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;

NL[7];

A:=N!(22,27,25,23,28,26,24);
B:=N!(15, 17, 19, 21, 16, 18, 20)(22, 28, 27, 26, 25, 24, 23);
C:=N!(8, 11, 14, 10, 13, 9, 12)(22, 25, 28, 24, 27, 23, 26);
D:=N!(1, 7, 6, 5, 4, 3, 2)(8, 11, 14, 10, 13, 9, 12)(15, 18,
21, 17, 20, 16, 19)(22, 25, 28, 24, 27, 23, 26);

IsIsomorphic(NL[7],AbelianGroup(GrpPerm,[7,7,7,7]));
X:=AbelianGroup(GrpPerm,[7,7,7,7]);
s:=IsIsomorphic(NL[7],X);
s;

FPGroup(NL[7]);

q, ff:=quo<N|NL[7]>;
q;

#q;

```

```

/* 16 */

N/NL[7];

nl:=NormalLattice(q); nl;

for i in [1..#nl] do if IsAbelian(nl[i]) then i;end if;end for;

nl[10];

FPGroup(q);

F<e,f>:=Group<e,f|e^4,f^4,(e^-1 * f^-1)^2,(e * f^-1)^2>;

f1,F1,k:=CosetAction(F,sub<F|Id(F)>);
IsIsomorphic(F1,q);

T:=Transversal(N,NL[7]);
ff(T[2]) eq q.1;
/*true */
ff(T[3]) eq q.2;
/* true */

```

```

A^T[2]; /* (1, 2, 3, 4, 5, 6, 7)*/
B^T[2]; /* (1, 5, 2, 6, 3, 7, 4)(22, 28, 27, 26, 25, 24, 23)*/
C^T[2]; /*(1, 3, 5, 7, 2, 4, 6)(15, 16, 17, 18, 19, 20, 21)*/
D^T[2]; /*(1, 3, 5, 7, 2, 4, 6)(8, 10, 12, 14, 9, 11, 13)(15,
16, 17, 18, 19, 20, 21)(22, 24, 26, 28, 23, 25, 27)*/

A^T[3]; /*(1, 2, 3, 4, 5, 6, 7)*/
B^T[3]; /*(1, 5, 2, 6, 3, 7, 4)(8, 9, 10, 11, 12, 13, 14)*/
C^T[3]; /*(1, 3, 5, 7, 2, 4, 6)(15, 16, 17, 18, 19, 20, 21)*/
D^T[3]; /*(1, 3, 5, 7, 2, 4, 6)(8, 13, 11, 9, 14, 12, 10)(15,
16, 17, 18, 19, 20, 21)(22, 27, 25, 23, 28, 26, 24)*/

> for i,j,k,l in [1..7] do if N!(1, 3, 5, 7, 2, 4, 6)
(8, 10, 12, 14, 9, 11, 13)(15,16, 17, 18, 19, 20, 21)(22, 24,
26, 28, 23, 25, 27) eq A^i*B^j*C^k*D^l then i,j,k,l; end
if; end for;

7 7 5 5

> A^T[2] eq A*B^5*C*D^6;
true

> B^T[2] eq A*B^6*C^4*D^3;
true

> C^T[2] eq C^2*D^5;
true

> D^T[2] eq C^5*D^5;

```

```

true
> for i,j,k,l in [1..7] do if N!(1, 2, 3, 4, 5, 6, 7)
eq A^i*B^j*C^k*D^l then i,j,k,l; end if; end for;
1 5 1 6
> for i,j,k,l in [1..7] do if N!(1, 5, 2, 6, 3, 7, 4)
(8, 9, 10, 11, 12, 13, 14) eq A^i*B^j*C^k*D^l then i,j,
k,l; end if; end for;
1 6 2 3
> for i,j,k,l in [1..7] do if N!(1, 3, 5, 7, 2, 4, 6)
(15, 16, 17, 18, 19, 20, 21) eq A^i*B^j*C^k*D^l then
i,j,k,l; end if; end for;
7 7 2 5
> for i,j,k,l in [1..7] do if N!(1, 3, 5, 7, 2, 4, 6)
(8, 13, 11, 9, 14, 12, 10)(15,16, 17, 18, 19, 20, 21)(22,
27, 25, 23, 28, 26, 24) eq A^i*B^j*C^k*D^l then i,j,k,l;
end if; end for;
7 7 6 5
> A^T[3] eq A*B^5*C*D^6; B^T[3] eq A*B^6*C^2*D^3; C^T[3]
eq C^2*D^5; D^T[3] eq C^6*B^5;
true
true
true
false
> A^T[3] eq A*B^5*C*D^6; B^T[3] eq A*B^6*C^2*D^3; C^T[3]

```

```

eq C^2*D^5; D^T[3] eq C^6*D^5;

true

true

true

true

> E<a,b,c,d,e,f>:=Group<a,b,c,d,e,f|a^7,b^7,c^7,d^7, (a,b),
> (a,c), (b,c), (a,d), (b,d), (c,d),e^4,f^4,(e^-1 * f^-1)^2,(e * f^-1)^2,
> a^e=a*b^5*c*d^6, b^e=a*b^6*c^4*d^3,c^e=c^2*d^5, d^e=c^5*d^5,
> a^f=a*b^5*c*d^6, b^f=a*b^6*c^2*d^3, c^f=c^2*d^5, d^f=c^6*d^5>;

> #E; #N;

38416

38416

> f,N1,k:=CosetAction(E,sub<E|Id(E)>);

> IsIsomorphic(N1,N);

```

Appendix F

MAGMA CODE for Mixed

Extension of $(2^3 : \bullet S_4)$

```

S:=Sym(96);
xx:=S!(1, 86, 25, 96)(2, 51, 40, 54)(3, 73, 60, 77)(4, 78,
67, 70)(5, 85, 7, 88)(6, 87, 28, 64)(8, 17, 72, 58)(9, 90,
82, 55)(10, 14, 12, 92)(11, 89, 39, 71)(13, 22, 76, 69)(15,
75, 65, 80)(16, 41, 19, 79)(18, 74, 68, 35)(20, 33, 57, 84)
(21, 48, 23, 47)(24, 95, 29, 45)(26, 31, 62, 34)(27, 46, 66,
49)(30, 59, 32, 56)(36, 53, 38, 50)(37, 91, 81, 52)(42, 61,
44, 94)(43, 93, 83, 63);
yy:=S!(1, 3, 14)(2, 8, 13)(4, 20, 51)(5, 15, 55)(6, 26, 71)
(7, 19, 54)(9, 35, 46)(10, 30, 64)(11, 41, 70)(12, 34, 49)(16,
50, 21)(17, 61, 29)(18, 63, 24)(22, 68, 89)(23, 65, 92)(25, 57,
53)(27, 59, 94)(28, 56, 52)(31, 45, 36)(32, 76, 38)(33, 78,

```

```

43)(37, 80, 47)(39, 75, 96)(40, 74, 95)(42, 84, 85)(44, 77, 48)
(58, 91, 69)(60, 90, 67)(62, 93, 66)(72, 87, 82)(73, 86, 83)(79,
88, 81);
N:=sub<S|xx,yy>;
#N;

CompositionFactors(N);

NL:=NormalLattice(N);
NL;

for i in [1..#NL] do if IsAbelian(NL[i]) then i;end if; end for;
NL[4];

X:=AbelianGroup(GrpPerm,[2,2,2]);
IsIsomorphic(NL[4],X);
/* true */

/* hence, a presentation for NL[4] is as follows */
H<a,b,c>:=Group<a,b,c|a^2,b^2,c^2,(a,b), (a,c), (b,c)>;
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
w:=IsIsomorphic(H1,NL[4]);
w;/* true *//* verified our presentation of NL[4]*/

```

```

/* step 6: look at NL again */
/* We factor out N by NL[4]. Thus, N is a mixed extension of NL[4] by N/NL[4]*/
q,ff:=quo<N|NL[4]>;
q;
/*
Permutation group q acting on a set of cardinality 4
Order = 24 = 2^3 * 3
(1, 2)
(2, 3, 4)
*/

FPGroup(q);
/*
Finitely presented group on 2 generators
Relations
$.1^2 = Id($)
$.2^3 = Id($)
($.2^-1 * $.1)^4 = Id($)
*/

T:=Transversal(N,NL[4]);
#T;/* 24 */

ff(T[2]) eq q.1;

```



```

/* true */
ff(T[3]) eq q.2;
/* true */
ff(T[3]^-1*T[2]) eq q.2^-1*q.1;
/* true */

/* The generators of q=NL[4]=( <A>, <B>, <C> ) */
A:=N!(1, 5)(2, 10)(3, 16)(4, 21)(6, 24)(7, 25)(8, 31)(9,
36)(11, 42)(12, 40)(13, 46)(14, 51)(15, 20)(17, 62)(18, 59)
(19, 60)(22, 66)(23, 67)(26, 58)(27, 69)(28, 29)(30, 35)(32,
74)(33, 75)(34, 72)(37, 83)(38, 82)(39, 44)(41, 73)(43, 81)
(45, 64)(47, 70)(48, 78)(49, 76)(50, 55)(52, 93)(53, 90)(54,
92)(56, 68)(57, 65)(61, 89)(63, 91)(71, 94)(77, 79)(80, 84)(85,
86)(87, 95)(88, 96);
B:=N!(1, 23)(2, 38)(3, 57)(4, 7)(5, 67)(6, 22)(8, 74)(9, 12)(10,
82)(11, 43)(13, 87)(14, 90)(15, 19)(16, 65)(17, 18)(20, 60)(21,
25)(24, 66)(26, 56)(27, 29)(28, 69)(30, 34)(31, 32)(33, 73)(35,
72)(36, 40)(37, 44)(39, 83)(41, 75)(42, 81)(45, 49)(46, 95)(47,
96)(48, 86)(50, 54)(51, 53)(52, 94)(55, 92)(58, 68)(59, 62)(61,
91)(63, 89)(64, 76)(70, 88)(71, 93)(77, 84)(78, 85)(79, 80);
C:=N!(1, 7)(2, 12)(3, 19)(4, 23)(5, 25)(6, 29)(8, 34)(9, 38)(10,
40)(11, 44)(13, 49)(14, 54)(15, 57)(16, 60)(17, 26)(18, 56)(20,
65)(21, 67)(22, 27)(24, 28)(30, 74)(31, 72)(32, 35)(33, 80)(36,

```

```

82)(37, 43)(39, 42)(41, 77)(45, 87)(46, 76)(47, 78)(48, 70)(50,
90)(51, 92)(52, 63)(53, 55)(58, 62)(59, 68)(61, 71)(64, 95)(66,
69)(73, 79)(75, 84)(81, 83)(85, 96)(86, 88)(89, 94)(91, 93);

NL4:=sub<N|A,B,C>;
X:=AbelianGroup(GrpPerm,[2,2,2]);
IsIsomorphic(NL[4],X);
/* true */

f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,NL[4]);
/* true *//* verified our presentation of NL[4]*/

for i,j,k in [0..1] do i,j,k; end for;

/* N/NL[4]=<NL[4]D, NL[4]E>. N/NL[4]=(<A>D, <B>D, <C>D, <A>E, <B>E, <C>E).
Now, conjugate every generator of NL[4] by D and E. */
/* we need this to label our T2, T3*/
T[2];

T[3];

T2:=N!(1, 86, 25, 96)(2, 51, 40, 54)(3, 73, 60, 77)(4, 78, 67, 70)(5,
85, 7,88)(6,87, 28, 64)(8, 17, 72, 58)(9, 90, 82, 55)(10, 14, 12, 92)

```

```
(11, 89, 39, 71)(13, 22, 76, 69)(15, 75, 65, 80)(16, 41, 19, 79)(18,
74, 68, 35)(20,33,57, 84)(21, 48, 23, 47)(24, 95, 29, 45)(26, 31, 62,
34)(27, 46, 66, 49)(30,59, 32, 56)(36, 53, 38, 50)(37, 91, 81, 52)(42,
61, 44, 94)(43, 93, 83,63);
```

```
T3:=N!(1, 3, 14)(2, 8, 13)(4, 20, 51)(5, 15, 55)(6, 26, 71)(7, 19, 54)
(9, 35, 46)(10,30, 64)(11, 41, 70)(12, 34, 49)(16, 50, 21)(17, 61, 29)
(18, 63, 24)(22,68,89)(23, 65, 92)(25, 57, 53)(27, 59, 94)(28, 56, 52)
(31, 45, 36)(32, 76,38)(33, 78, 43)(37, 80, 47)(39,75, 96)(40, 74, 95)
(42, 84, 85)(44, 77,48)(58, 91, 69)(60, 90, 67)(62, 93, 66)(72,87, 82)
(73, 86, 83)(79, 88,81);
```

```
for i,j,k in [0..1] do if A^T2 eq A^i*B^j*C^k then i,j,k; end if; end for;
/* 1 0 0 */
for i,j,k in [0..1] do if A^T3 eq A^i*B^j*C^k then i,j,k; end if; end for;
/* 0 1 1 */
for i,j,k in [0..1] do if B^T2 eq A^i*B^j*C^k then i,j,k; end if; end for;
/* 1 1 1 */
for i,j,k in [0..1] do if B^T3 eq A^i*B^j*C^k then i,j,k; end if; end for;
/* 1 1 0 */
for i,j,k in [0..1] do if C^T2 eq A^i*B^j*C^k then i,j,k; end if; end for;
/* 0 0 1 */
for i,j,k in [0..1] do if C^T3 eq A^i*B^j*C^k then i,j,k; end if; end for;
/* 0 0 1 */
```

```

/* hence, the presentation for N/NL[4] is: */
H<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^2,c^2,d^2,e^3,(e^-1*d)^4,
(a,b),(a,c),(b,c), a^d=a,a^e=b*c,b^d=a*b*c,b^e=a*b,c^d=c,c^e=c>;
#H;
/* 192 */

f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(N,H1);
/* false *//* this tells us N is not semi-direct 2^3:q but the mixed
extension of 2^3 by q. That means, one of our new generators (from q)
must be further investigated*/

Order(T[2]) eq Order(q.1);
/* true */
Order(T[3]) eq Order(q.2);
/* true */
Order(T[2]^-1*T[3]) eq Order(q.2^-1*q.1);
/* false */

(T[3]^-1*T[2])^4;
J<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^2,c^2,d^2,e^3,(e^-1*d)^4=c,
(a,b),(a,c),(b,c), a^d=a,a^e=b*c,b^d=a*b*c,b^e=a*b,c^d=c,c^e=c>;
#J;

```

```
/* 192*/

f,J1,k:=CosetAction(J,sub<J|Id(J)>);
wm:=IsIsomorphic(N,J1);
wm;

q,ff:=quo<N|NL[4]>;
q;

nL:=NormalLattice(q); nL;

for i in [1..#nL] do if IsAbelian(nL[i]) then i;end if; end for;
nL[2];

X:=AbelianGroup(GrpPerm,[2,2]);
s,t:=IsIsomorphic(nL[2],X);
s;
true
t;

CompositionFactors(nL[2]);

IsIsomorphic(q,Sym(4));
```

Appendix G

MAGMA CODE Construction of A_7 over $PSL(2, 7)$

```

S:=Sym(6);
xx:=S!(1, 3, 5)(2, 4, 6);
yy:=S! (1, 6)(2, 5)(3, 4);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^3,t^(x*y)=t^2,
(x*t)^4, (x*y*t)^2, (x*y*t^x*t^2*t^y*(t^x))^3>;
#N;
/* 6 */

for i in [0..2] do for j in [0..1] do printf "%o", i;j,xx^i*yy^j;
end for; end for;

```

```

f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1;
/* 168 */
CompositionFactors(G1);

IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 6] ];
ts[1]:=f(t); ts[2]:=f((t^(y))^x); ts[3]:=f(t^(x)); ts[4]:=ts[1]^-1;
ts[5]:=ts[2]^-1;ts[6]:=ts[3]^-1;

#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
/* 7 */

DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);

DC:=[ f( Id(G)), f( t), f(t * x * t), f(t * y * t ),
f( t * y * t^-1 ), f( t * x * t^-1), f(t * x * t^-1 * y * t^-1 )];

Index(G1,IN);
/* 28 */

word:=function(A)

Sch:=SchreierSystem(G,sub<G|Id(G)>);

```

```

B:=Id(G);
for i in [2..#G] do
P:=[Id(G1): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=f(t)^-1; end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;

cst := [null : i in [1 ..Index(G1,IN)]] where null is
[Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);

```



```

        end for;
return v;
end function;
for i := 1 to 6 do
    cst[prodim(1, ts, [i])] := [i];
end for;

m:=0; for i in [1..28] do if cst[i] ne [] then
m:=m+1; end if; end for; m;
/* 6 */

Orbits(N);

for i in [1..#DC] do for m,n in IN do if ts[1] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 2 */

N1:=Stabiliser(N,1);
Generators(N1);

Orbits(N1);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[1]
eq m*(DC[i])^n then i; break; end if; end for;end for;

```

```

for m,n in IN do if ts[1]*ts[1] eq m*(ts[1])^n then
word(m); word(n); end if; end for;

xx*yy;

ts[1]*ts[1] eq ts[4];
/* true */
for n in N do if 4^n eq 1 then n; end if; end for;
/* (1, 4)(2, 3)(5, 6) */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 6 */ /* new */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 5 */ /* new */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[4] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 1 */ /* goes back */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 3 */ /* new */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6] eq m*(DC[i])^n

```

```

then i; break; end if; end for;end for;
/* 4 *//* new */

/*DOING NEW COSET (Nt1t2)*/
S:={ [1,2] };
SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*
{
[ 1, 2 ]
}
*/

N12:=Stabiliser(N, [1,2]);
#N12;
/* 1 */
N12s:=N12;
for n in N do if [1,2]^n eq [1,2] then N12s:=sub<N|N12s,n>; end if; end for;

```

```

#N12s;
/* 1*/
Generators(N12s);
/* { }
*/
[1,2]^N12s;
/*
GSet{@
    [ 1, 2 ]
@}
*/

tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 12 */

Orbits(N12s);
/*

```

```

[
  GSet{@ 1 @},
  GSet{@ 2 @},
  GSet{@ 3 @},
  GSet{@ 4 @},
  GSet{@ 5 @},
  GSet{@ 6 @}
]
*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/*7 */ /* new coset*/
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 3 */ /* goes to Nt1t5*/
for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(ts[1]*ts[5] )^n then word(m);
word(n); end if; end for;
/*
Id(G)
Id(G)
/*
ts[1]*ts[2]*ts[2] eq f(Id(G))*ts[1]*ts[5];
/* true */

```

```

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;

/* 3*/* goes to Nt1t5*/
for m,n in IN do if ts[1]*ts[2]*ts[3] eq m*(ts[1]*ts[5])^n then word(m);
word(n); end if; end for;

/*
y * x
y
*/
yy;
/* (1, 6)(2, 5)(3, 4)
t1t2t3=t1t5^{(1, 6)(2, 5)(3, 4)=t6t2 */

ts[1]*ts[2]*ts[3] eq f(y*x)*(ts[6]*ts[2]);

/*true*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;

/* 4*/ /* goes to Nt1t6*/
for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(ts[1]*ts[6])^n then word(m);
word(n); end if; end for;

/*
y

```

```

x * y
*/
xx*yy;
/*
(1, 4)(2, 3)(5, 6)
t1t2t4=t1t6^(1, 4)(2, 3)(5, 6)=t4t5 */

ts[1]*ts[2]*ts[4] eq f(y)*ts[4]*ts[5];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 2*/ .* goes back to Nt1 */
for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(ts[1])^n then word(m);
word(n); end if; end for;
/*
Id(G)
Id(G)
*/
/*
Id(G)
t1t2t5=t1^Id(G)=t1 */

ts[1]*ts[2]*ts[5] eq f(Id(G))*ts[1];

```

```

/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[6] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 6 */ /* loops Nt1t2*/
for m,n in IN do if ts[1]*ts[2]*ts[6] eq m*(ts[1]*ts[2])^n then word(m)
; word(n); end if; end for;
/*
x^-1
y * x
*/
yy*xx;
/*
(1, 2)(3, 6)(4, 5)
t1t2t6=t1t2^(1, 2)(3, 6)(4, 5)=t2t1 */

ts[1]*ts[2]*ts[6] eq f(x^-1)*ts[2]*ts[1];
/*true */

/*DOING NEW COSET (Nt1t3)*/
S:={ [1, 3] };
SS:=S^N;SS;

SSS:=Setseq(SS);

```



```

for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
  then print SSS[i];
end if; end for; end for;
/*
{
  [ 1, 3 ]
}
{
  [ 4, 2 ]
}
*/

N13:=Stabiliser(N,[1,3]);
#N13;
/* 1 */
N13s:=N13;
for n in N do if [1,3]^n eq [4,2] then
N13s:=sub<N|N13s,n>; end if; end for;
#N13s;
/* 2*/
Generators(N13s);

```

```

/*
{
    (1, 4)(2, 3)(5, 6)
}
*/
[1,3]^N13s;
/*
GSet{@
    [ 1, 3 ],
    [ 4, 2 ]
@}
*/

tr1:=Transversal(N,N13s);
for i:=1 to #tr1 do
ss:=[1,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 15 */

Orbits(N13s);

```

```

/*
[
  GSet{@ 1, 4 @},
  GSet{@ 2, 3 @},
  GSet{@ 5, 6 @}
]
*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[1] eq m*(DC[i])^n
then i; break; end if; end for; end for;

/* 4 */ /* goes to Nt1t6*/
for m,n in IN do if ts[1]*ts[3]*ts[1] eq m*(ts[1]*ts[6] )^n then word(m);
word(n); end if; end for;

/*
x * y
y * x
*/
yy*xx;

/*
(1, 2)(3, 6)(4, 5)
t1t3t1=t1t6^(1, 2)(3, 6)(4, 5)=t2t3 */

ts[1]*ts[3]*ts[1] eq f(x*y)*ts[2]*ts[3];
/*true */

```

```

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[2] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 4 */ /*goes to Nt1t6*/
for m,n in IN do if ts[1]*ts[3]*ts[2] eq m*(ts[1]*ts[6] )^n then word(m);
word(n); end if; end for;
/*
x
x * y
*/
xx*yy;
/*
(1, 4)(2, 3)(5, 6)
t1t3t2=t1t6^(1, 4)(2, 3)(5, 6)=t4t5 */

ts[1]*ts[3]*ts[2] eq f(x)*ts[4]*ts[5];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[5] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 2 */ /*goes back to Nt1*/
for m,n in IN do if ts[1]*ts[3]*ts[5] eq m*(ts[1] )^n then word(m);
word(n); end if; end for;
/*
x

```

```

x * y
*/
xx*yy;
/*
(1, 4)(2, 3)(5, 6)
t1t3t5=t1^(1, 4)(2, 3)(5, 6)=t4 */

ts[1]*ts[3]*ts[5] eq f(x)*ts[4];
/*true */

/*DOING NEW COSET (Nt1t5)*/
S:={[1, 5]};
SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[5]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*
{
[ 1, 5 ]
}

```

```

{
    [ 4, 6 ]
}
*/

N15:=Stabiliser(N,[1,5]);
#N15;
/* 1 */
N15s:=N15;
for n in N do if [1,5]^n eq [4,6] then
N15s:=sub<N|N15s,n>; end if; end for;
#N15s;
/* 2*/
Generators(N15s);
/*
{
    (1, 4)(2, 3)(5, 6)
}
*/
[1,5]^N15s;
/*
GSet{@
    [ 1, 5 ],
    [ 4, 6 ]
}
*/

```

```

@}
*/

tr1:=Transversal(N,N15s);
for i:=1 to #tr1 do
ss:=[1,5]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 18 */

Orbits(N15s);
/*
[
  GSet{@ 1, 4 @},
  GSet{@ 2, 3 @},
  GSet{@ 5, 6 @}
]
*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[1] eq m*(DC[i])^n

```

```

then i; break; end if; end for; end for;
/* 6 */ /* goes to Nt1t2*/
for m,n in IN do if ts[1]*ts[5]*ts[1] eq m*(ts[1]*ts[2] )^n then word(m);
word(n); end if; end for;
/*
x * y
y
*/
yy;
/*
(1, 6)(2, 5)(3, 4)
t1t5t1=t1t2^(1, 6)(2, 5)(3, 4)=t6t5 */

ts[1]*ts[5]*ts[1] eq f(x*y)*ts[6]*ts[5];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[2] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 2 */ /* goes to Nt1*/
for m,n in IN do if ts[1]*ts[5]*ts[2] eq m*(ts[1])^n then word(m);
word(n); end if; end for;
/*
Id(G)
Id(G)

```



```

*/
/*
t1t5t2=t1^Id(G)=t1 */

ts[1]*ts[5]*ts[2] eq f(Id(G))*ts[1];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[5] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 6 */ /* goes to Nt1t2*/
for m,n in IN do if ts[1]*ts[5]*ts[5] eq m*(ts[1]*ts[2] )^n then word(m);
word(n); end if; end for;
/*
Id(G)
Id(G)
*/
/*
t1t5t5=t1t2^Id(G)=t1t2 */

ts[1]*ts[5]*ts[5] eq f(Id(G))*ts[1]*ts[2];
/*true */

/*DOING NEW COSET (Nt1t6)*/
S:={[1, 6]};

```

```

SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[6]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*
{
[ 1, 6 ]
}
*/

N16:=Stabiliser(N,[1,6]);
#N16;
/* 1 */
N16s:=N16;
for n in N do if [1,6]^n eq [1,6] then
N16s:=sub<N|N16s,n>; end if; end for;
#N16s;
/* 1 */
Generators(N16s);
/*

```

```

{ }
*/
[1,6]^N16s;
/*
GSet{@
    [ 1, 6 ]
@}
*/

tr1:=Transversal(N,N16s);
for i:=1 to #tr1 do
ss:=[1,6]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 24 */

Orbits(N16s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},

```

```

    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6]*ts[1] eq m*(DC[i])^n
then i; break; end if; end for; end for;

/* 7 */ /* goes to Nt1t2t1*/
for m,n in IN do if ts[1]*ts[6]*ts[1] eq
m*(ts[1]*ts[2]*ts[1] )^n then word(m); word(n); end if; end for;

/*
y * x
x^-1
*/
xx^-1;

/*
(1, 5, 3)(2, 6, 4)
t1t6t1=t1t2t1^(1, 5, 3)(2, 6, 4)=t5t6t5 */

ts[1]*ts[6]*ts[1] eq f(y*x)*ts[5]*ts[6]*ts[5];

/*true */

```

```

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6]*ts[2] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 4 */ /* loops to Nt1t6 */

for m,n in IN do if ts[1]*ts[6]*ts[2] eq m*(ts[1]*ts[6] )^n then word(m);
word(n); end if; end for;
/*
x
y
*/
xx^-1;
/*
(1, 6)(2, 5)(3, 4)
t1t6t2=t1t6^(1, 6)(2, 5)(3, 4)=t6t1 */

ts[1]*ts[6]*ts[2] eq f(x)*ts[6]*ts[1];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6]*ts[3] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 2*/ /* goes back to Nt1 */

for m,n in IN do if ts[1]*ts[6]*ts[3] eq m*(ts[1])^n then word(m);
word(n); end if; end for;
/*

```

```

Id(G)
Id(G)
*/
/*
t1t6t3=t1^Id(G)=t1 */

ts[1]*ts[6]*ts[3] eq f(Id(G))*ts[1];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6]*ts[4] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 6 */ /* goes to Nt1t2 */
for m,n in IN do if ts[1]*ts[6]*ts[4] eq m*(ts[1]*ts[2])^n then word(m);
word(n); end if; end for;
/*
y * x
x * y
*/
/*
xx*yy;
(1, 4)(2, 3)(5, 6)
t1t6t4=t1t2^(1, 4)(2, 3)(5, 6)=t4t3 */

ts[1]*ts[6]*ts[4] eq f(y*x)*ts[4]*ts[3];

```

```

/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6]*ts[5] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 5 */ /* goes to Nt1t3 */
for m,n in IN do if ts[1]*ts[6]*ts[5] eq m*(ts[1]*ts[3])^n then word(m);
word(n); end if; end for;
/*
y
y * x
*/
/*
yy*xx;
(1, 2)(3, 6)(4, 5)
t1t6t5=t1t3^(1, 2)(3, 6)(4, 5)=t2t6 */

ts[1]*ts[6]*ts[5] eq f(y)*ts[2]*ts[6];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6]*ts[6] eq m*(DC[i])^n
then i; break; end if; end for; end for;
/* 5 */ /* goes to Nt1t3 */
for m,n in IN do if ts[1]*ts[6]*ts[6] eq m*(ts[1]*ts[3])^n then word(m);
word(n); end if; end for;

```

```

/*
Id(G)
Id(G)
*/
/*
t1t6t6=t1t3^(Id(G)=t1t3 */

ts[1]*ts[6]*ts[6] eq f(Id(G))*ts[1]*ts[3];
/*true */

/*DOING NEW COSET (Nt1t2t1)*/
S:={ [1, 2, 1] };
SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;
/*
{
[ 1, 2, 1 ]
}

```



```

{
    [ 6, 5, 6 ]
}
*/

N121:=Stabiliser(N,[1,2,1]);
#N121;
/* 1 */
N121s:=N121;
for n in N do if [1,2,1]^n eq [6,5,6] then N121s:=sub<N|N121s,n>;
end if; end for;
#N121s;
/* 2 */
Generators(N121s);
/*
{
    (1, 6)(2, 5)(3, 4)
}
*/
[1,2,1]^N121s;
/*
GSet{@
    [ 1, 2, 1 ],
    [ 6, 5, 6 ]
}
*/

```

```

@}
*/

tr1:=Transversal(N,N121s);
for i:=1 to #tr1 do
ss:=[1,2,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1;

end if; end for;m;
/* 27 */

Orbits(N121s);
/*
[
  GSet{@ 1, 6 @},
  GSet{@ 2, 5 @},
  GSet{@ 3, 4 @}
]
*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[1] eq

```

```

m*(DC[i])^n then i; break; end if; end for; end for;
/* 4 */ /* goes to Nt1t6*/
for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[1] eq m*(ts[1]*ts[6] )^n
then word(m); word(n); end if; end for;
/*
y
x * y
*/
xx^-1;
/*
(1, 4)(2, 3)(5, 6)
t1t2t1t1=t1t6^(1, 4)(2, 3)(5, 6)=t4t5 */

ts[1]*ts[2]*ts[1]*ts[1] eq f(y)*ts[4]*ts[5];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[2] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 7 */ /* loops to Nt1t2t1*/
for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[2] eq m*(ts[1]*ts[2]*ts[1] )^n
then word(m); word(n); end if; end for;
/*x^-1
Id(G)*/
/*t1t2t1t2=t1t2t1^(x^-1)=t1t2t1 */

```

```

ts[1]*ts[2]*ts[1]*ts[2] eq f(x^-1)*ts[1]*ts[2]*ts[1];
/*true */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[3] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 6 */ /* goes to Nt1t2*/
for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[3] eq m*( ts[1]*ts[2] )^n
then word(m); word(n); end if; end for;
/*
Id(G)
y
*/
y;
/*
(1, 6)(2, 5)(3, 4)
t1t2t1t3=t1t2^(Id(G))=t6t5 */

ts[1]*ts[2]*ts[1]*ts[3] eq f(Id(G))*ts[6]*ts[5];
/*true */

```

Appendix H

MAGMA CODE for

Permutation Progenitor of

$$2^*6 : (2^2 : (3x2))$$

```
N:=TransitiveGroup(6,10);
#N;
N;
S:=Sym(6);
xx:=S!(2, 4, 6);
yy:=S!(1, 5)(2, 4);
zz:=S!(1, 4, 5, 2)(3, 6);
N:=sub<S|xx,yy,zz>;
#N;
/* 36 */
```

```

FPGroup(N);

NN<a,b,c>:=Group<a,b,c|a^3,b^2,c^4,(a^-1*b)^2,c^-1*b*c^-1,
a^-1*c^-1*a^-1*c^-1*a^-1*c^-1*c*a*c>;
#NN;

/*36 this means we have 36 elements/permutations */

Set(N); /* these are the 36 elements */

/*Rewrite the permutations of Set(N) above as \words" using SchreierSystem*/
SchreierSystem(NN,sub<NN|Id(NN)>);

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
/* we put -1 if our G contains powers to the negative*/
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;

```

```

end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabilizer(N,1);
N1;
/*Permutation group N1 acting on a set of cardinality 6
Order = 6 = 2 * 3
(2, 4, 6)
(2, 6)(3, 5) */
for i in [1..#Sch] do if ArrayP[i] eq N!(2,4,6) then
Sch[i]; end if; end for;
/* a */
for i in [1..#Sch] do if ArrayP[i] eq N!(2,6)(3,5) then
Sch[i]; end if; end for;
/* c * a^-1 * c * a */
xx;
zz*xx^-1*zz*xx;
N1 eq sub<N|xx,zz*xx^-1*zz*xx>;
/*true */
/* this is our G with the new elements */

```

```

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^2,z^4,(x^-1*y)^2,
z^-1*y*z^-1,x^-1*z^-1*x^-1*z^-1*x^-1*z*x*z,t^2,
(t,x),(t,z*x^-1*z*x)>;

Orbits(N1);

for j in [1..6] do for i in [1..#N] do if 1^ArrayP[i] eq j
then j, Sch[i]; break; end if; end for; end for;

NN;
#NN;
/* 36 */
G<x,y,z,t>:=Group<x,y,z,t|x^3,y^2,z^4,(x^-1*y)^2,
z^-1*y*z^-1,x^-1*z^-1*x^-1*z^-1*x^-1*z*x*z,t^2,
(t,x),(t,z*x^-1*z*x),(t,t^(z*x*z)),(t,t^(z^-1))>;
#G;/* 2304 */
2^6*36;/* 2304 */

/* Now find 1st order relations, that will help us figure out
which relations to do for DCE's*/

C:=Classes(N);
for i in [2..#C] do i, C[i][3]; for j in [1..#N] do
if ArrayP[j] eq C[i][3] then Sch[j]; end if; end for;

```



```
Orbits(Centralizer(N,C[i][3])); end for;
```

```
for j in [1..6] do for i in [1..#N] do if 1^ArrayP[i] eq j  
then j, Sch[i]; break; end if; end for; end for;
```

Appendix I

MAGMA CODE for

Permutation Progenitor of M_{12}

$$2^{*96} : (2^3 \wr S_4)$$

N;

S:=Sym(96);

xx:=S!(1, 86, 25, 96)(2, 51, 40, 54)(3, 73, 60, 77)(4, 78, 67, 70)

(5, 85, 7, 88)(6,87, 28, 64)(8, 17, 72, 58)(9, 90, 82, 55)

(10, 14, 12, 92)(11, 89, 39,71)(13, 22, 76, 69)(15, 75, 65, 80)

(16, 41, 19, 79)(18, 74, 68, 35)(20, 33,57, 84)(21, 48, 23, 47)

(24, 95, 29, 45)(26, 31, 62, 34)(27, 46, 66, 49)(30,59, 32, 56)

(36, 53, 38, 50)(37, 91, 81, 52)(42, 61, 44, 94)(43, 93, 83, 63);

yy:=S!(1, 3, 14)(2, 8, 13)(4, 20, 51)(5, 15, 55)(6, 26, 71)

(7, 19, 54)(9, 35, 46)(10,30, 64)(11, 41, 70)(12, 34, 49)(16, 50, 21)

```
(17, 61, 29)(18, 63, 24)(22, 68,89)(23, 65, 92)(25, 57, 53)
(27, 59, 94)(28, 56, 52)(31, 45, 36)(32, 76,38)(33, 78, 43)
(37, 80, 47)(39, 75, 96)(40, 74, 95)(42, 84, 85)(44, 77,48)
(58, 91, 69)(60, 90, 67)(62, 93, 66)(72, 87, 82)(73, 86, 83)(79, 88, 81);
```

```
N:=sub<S|xx,yy>;
```

```
#N;
```

```
/* 192*/
```

```
N1:=Stabiliser(N,1);
```

```
#N1;
```

```
/*2*/
```

```
Generators(N1);
```

```
/* to find the presentation of N */
```

```
FPGroup(N);
```

```
NN<a,b>:=Group<a,b|
```

```
  a^4  ,
```

```
  b^3  ,
```

```
  (a * b^-1 * a^-1 * b^-1)^2,
```

```
  a^-1 * b * a^-2 * b^-1 * a^-1 * b * a^2 * b^-1 >;
```

```
#NN;
```

```
/* 192*/
```

```

word:=function(A)
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
for i in [2..#NN] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;

end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;

A:=N!(2, 42)(3, 62)(5, 25)(6, 29)(8, 74)(9, 83)(10, 44)(11, 40)
(12, 39)(13, 70)(14, 93)(15, 68)(16, 26)(17, 60)(18, 20)(19, 58)
(21, 67)(22, 27)(30, 34)(31, 35)(32, 72)(33, 73)(36, 43)(37, 82)
(38,81)(41, 84)(45, 86)(46, 78)(47, 76)(48, 49)(50, 61)(51, 63)

```

```
(52, 92)(53, 89)(54, 91)(55, 94)(56, 65)(57,59)(64, 96)(71, 90)
(75, 77)(79, 80)(85, 95)(87, 88);
```

```
word(A);
```

```
/* a * b^-1 * a^-1 * b^-1 */
```

```
G<x,y,t>:=Group<x,y,t|x^4, y^3 , (x * y^-1 * x^-1 * y^-1)^2,
    x^-1 * y * x^-2 * y^-1 * x^-1 * y * x^2 * y^-1 , t^2,
(t,x * y^-1 * x^-1 * y^-1)>;
```

```
Orbits(N1);
```

```
temp:={4,7,23,24,28,66,69,2 ,3 ,5 ,6 , 8, 9 , 10 , 11, 12, 13,14,
15,16, 17, 18, 19,21,22,30,31,32,33,36,37,38,41,45,46,47,48,50,51,
52,53,54,55,56,57,64,71,75,79,85,87};
```

```
for n in N do for j in temp do if 1^n eq j then word(n),j;
```

```
temp:=temp diff {j}; end if; end for; end for;
```

```
G<x,y,t>:=Group<x,y,t|x^4, y^3 , (x * y^-1 * x^-1 * y^-1)^2,
    x^-1 * y * x^-2 * y^-1 * x^-1 * y * x^2 * y^-1 , t^2,
(t,x * y^-1 * x^-1 * y^-1),
(t,t^y ), (t,t^(y^-1 )),(t,t^(x^-1*y^-1) ), (t,t^(y^-1*x) ),
(t,t^(y^-1*x^-1)),(t,t^(x^2*y )),(t,t^(x^2*y^-1) ),
(t,t^(y*x^-1*y) ), (t,t^(y*x) ), (t,t^(x^-1*y^-1*x^-1)),
(t, t^(y^-1*x^-1*y)),(t,t^(y^-1*x^-1*y^-1)),(t,t^(x^2*y*x^-1)),
```

```

(t,t^(x^2*y^-1*x)),(t,t^(x^2*y^-1*x^-1)),(t,t^((x*y)^2)),
(t,t^(x*y*x*y^-1)),(t,t^(x*y*x^-1*y)),(t,t^(y*x^-1*y*x)),
(t,t^((y*x^-1)^2)),(t,t^(x^-1*y*x^2)),(t,t^(x^-1*y*x*y)),
(t,t^((x^-1*y)^2)),(t,t^(x^-1*y^-1*x^-1*y)),
(t,t^((x^-1*y^-1)^2)),(t,t^(y^-1*x^-1*y*x^-1)),
(t,t^((y^-1*x^-1)^2)),(t,t^(x^2*y*x*y)),
(t,t^(x^2*y^-1*x^2)),(t,t^(x^2*y^-1*x*y)),(t,t^(x^2*y^-1*x^-1*y)),
(t,t^(x^2*y^-1*x^-1*y^-1)),(t,t^(x*y*x^-1*y*x^-1)),
(t,t^(x*y^-1*x*y*x^-1)),(t,t^(y*x^-1*y*x^2)),
(t,t^(y * x^-1 * y * x^-1 * y)),(t,t^(y*x^-1*y^-1*x^-1*y^-1)),
(t,t^(x^-1*y*x^2*y)),(t,t^(x^-1*y*x*y*x)),(t,t^(x^-1*y*x^-1*y*x^-1)),
(t,t^(x^-1*y^-1*x^2*y^-1)),(t,t^(y^-1*x^-1*y*x^-1*y)),
(t,t^((x^2*y)^2)),(t,t^(x^2 * y * x^2 * y^-1)),(t,t^(x^2*y*x*y*x)),
(t,t^(x^2*y^-1*x*y^-1*x)),(t,t^((y*x^-1)^3)),
(t,t^(y*x^-1*y^-1*x^-1*y^-1*x^-1)),(t,t^(x^-1*y*x^2*y*x)),
(t,t^((x^-1*y)^3)),(t,t^(x^2*y*x*y*x*y^-1))>;

```

```
C:=Classes(N); C;
```

```
for i in [2..#C] do i, C[i][3];word(C[i][3]);
```

```
Orbits(Centraliser(N,C[i][3])); end for;
```

```

for j in [2..#N] do for i in [1..#Setseq(Set(N))] do if
1^Setseq(Set(N))[i] eq j then j,word(Setseq(Set(N))[i]);
break; end if; end for; end for;

```

Appendix J

MAGMA CODE for M_{12}

Permutation Progenerator

$$2^{*72} : (3^2 : (2 : S_4))$$

```
S:=Sym(72);
```

```
xx:=S!(1, 23, 49, 32, 12, 60)(2, 44, 16, 45, 17, 62)(3, 41, 6, 55, 50,
33)(4, 63, 14, 46, 8, 43)(5, 66, 26, 58, 31, 72)(7, 61, 19, 28, 42, 10)
(9, 20, 24, 54, 29, 38)(11, 69, 35, 70, 37, 18)(13, 21)(15, 36, 59, 53,
47, 48)(22, 67)(25, 40, 71, 27, 68, 52)(30, 64, 39, 57, 34, 56)(51, 65);
```

```
yy:=S!(1, 35, 58, 2, 6, 54, 25, 4)(3, 40, 46, 64, 14, 21, 19, 69)(5, 36,
38, 15, 22, 60, 17, 55)(7, 43, 66, 29, 32, 53, 71, 30)(8, 23, 27, 57, 37,
39, 67, 41)(9, 13, 50, 59, 42, 26, 20, 44)(10, 24, 12, 31, 45, 48, 16,
65)(11, 63, 18, 51, 49, 34, 61, 52)(28, 62, 68, 70, 33, 56, 72, 47);
```

```

N:=sub<S|xx,yy>;
#N;
/* 432 */

N1:=Stabiliser(N,1);
#N1;
/*6*/
Generators(N1);

FPGroup(N);
NN<a,b>:=Group<a,b|
    a^6 ,
    b^8 ,
    b^2 * a^-2 * b^-1 * a * b ,
    a^-1 * b^-1 * a * b^-1 * a^-1 * b^-1 * a^3 * b >;
#NN;
/* 432 */

word:=function(A)
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
for i in [2..#NN] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;

```



```

if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;
A:=N!(2, 58)(3, 55)(4, 25)(5, 45)(8, 44)(9, 18)(10, 22)(11, 40)(12,
23)(13, 60)(14, 15)(16, 39)(17, 63)(19, 67)(20, 31)(21, 49)(24, 64)
(26, 36)(27, 46)(29, 66)(30, 71)(34, 38)(35, 54)(37, 59)(41, 51)(42,
61)(43, 53)(47, 72)(48, 69)(50, 65)(52, 57)(56, 62)(68, 70);
word(A);
/* a * b * a * b * a^-1 * b^-1 */
B:=N!(2, 25, 54)(3, 41, 65)(4, 58, 35)(5, 18, 46)(8, 69, 34)(9, 45,
27)(10, 67, 42)(11, 26, 39)(12, 60, 21)(13, 23, 49)(14, 57, 31)(15,
20, 52)(16, 36, 40)(17, 24, 59)(19, 22, 61)(29, 71, 53)(30, 66, 43)
(37, 64, 63)(38, 48, 44)(47, 68, 62)(50, 51, 55)(56, 70, 72);
word(B);
/* a * b^-1 * a^2 * b */
G<x,y,t>:=Group<x,y,t|x^6, y^8 , y^2 * x^-2 * y^-1 * x * y ,

```

```

x^-1 * y^-1 * x * y^-1 * x^-1 * y^-1 * x^3 * y , t^2
(t,x * y * x * y * x^-1 * y^-1), (t, x * y^-1 * x^2 * y) >;

/* to verify progenitor is correct, do the following : */
sub<N|x * y * x * y * x^-1 * y^-1,x * y^-1 * x^2 * y> eq N1;
/* true*/ /* so our progenitor is correct; now do FOR */

Orbits(N1);

temp:={6,7,28, 32, 33, 2, 3, 5, 8, 10, 11, 12, 14, 17, 29, 47};
for n in N do for j in temp do if 1^n eq j then word(n),j;
temp:=temp diff {j}; end if; end for; end for;

C:=Classes(N); C;
for i in [2..#C] do i, C[i][3];word(C[i][3]);
Orbits(Centraliser(N,C[i][3])); end for;

for j in [2..#N] do for i in [1..#Setseq(Set(N))] do if
1^Setseq(Set(N))[i] eq j then j,word(Setseq(Set(N))[i]) ; break;
end if; end for; end for;

```

Appendix K

MAGMA CODE for

Construction of $M_{21} : 2^2$ over

$PGL(2, 9) : 2$ and $PGL(2, 9)$

```
S:=Sym(10);
xx:=S!(1,2,10)(3,4,5)(6,7,8);
yy:=S!(1,7,3,4,2,5,6,8);
zz:=S!(1,2)(4,7)(5,8)(9,10);
N:=sub<S|xx,yy,zz>;
#N;
/*720*/

G<x,y,z,t>:=Group<x,y,z,t|x^3,y^8,z^2,(y^-1*z)^2,
(z*x^-1)^3,(y^-1*x^-1*y^-1*x^-1*y^2*x),
```

```

(x*y^-2*x^-1*y*x^-1*y),t^2,(t,y^-1*x*z*y^2),(t,y^-2*x*y^-1),
(z*x*y*t)^6,(z*t)^8,(t*t^x)^2=y^4*z>;
#G;
/* 80640 */

#sub<G|x,y,z>;
/* 720 */

#DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>); /* 6 */

f,G1,k:=CosetAction(G,sub<G|x,y,z>);
M:=MaximalSubgroups(G1);
for i in [1..#M] do #M[i] 'subgroup/720; end for;
/*
1/3
2/5
14/15
2
16/15
56
56
56
*/
M4:=M[4] 'subgroup;

```

```

f(x) in M4 and f(y) in M4 and f(z) in M4;
/* false */

D:=Conjugates(G1,M4);
DD:=Setseq(D);
  for i in [1..#D] do if f(x) in DD[i] and f(y) in DD[i] and f(z) in DD[i]
then i; end if; end for;
/* 56 */
  for g in DD[56] do if DD[56] eq sub<G1|f(x),f(y),f(z),g>
then Sch:=SchreierSystem(G, sub<G|Id(G)>); for i in
[1..#Sch] do if g eq f(Sch[i]) then Sch[i]; end if;
end for; break g; end if; end for;
/* z * y * t * x * t * y * t * x^-1 * t * y * t */

H:=sub<G|x,y,z,z * y * t * x * t * y * t * x^-1 * t * y * t>;
#H;
/* 1440 */
#DoubleCosets(G,H,sub<G|x,y,z>);
/* 3 */
f,G1,k:=CosetAction(G,H);
#k;
/* 1 */
/* f is faithful - use this f*/

```

```

IH:=sub<G1| f(x), f(y), f(z), f(z * y * t * x * t * y * t * x^-1 * t * y * t)>;
CompositionFactors(IH);

IN:=sub<G1| f(x), f(y), f(z)>;

/*NOTE: if our k was not faithful we use the following code:
Index(G,sub<G|x,y,z>);*/

/* NOW WE CAN START THE DCE ALOG. */

DoubleCosets(G,H, sub<G|x,y,z>);
DC:=[Id(G1),f(t), f(t*x*t)];

NN<a,b,c>:=Group<a,b,c|a^3,b^8,c^2,(b^-1*c)^2,
(c*a^-1)^3,(b^-1*a^-1*b^-1*a^-1*b^2*a),
(a*b^-2*a^-1*b*a^-1*b)>;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
  for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
  if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;

```

```

    if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
  if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
    if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
  if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for j in [2..10] do for i in [1..#Sch] do if 1^ArrayP[i] eq j then j,Sch[i];
break;end if; end for; end for;

ts := [ Id(G1): i in [1 .. 10] ];/* use this to write my ten t's */
ts[1]:=f(t); ts[2]:=f(t^x); ts[3]:=f(t^(y^2)); ts[4]:=f(t^(x * y^-1));
ts[5]:=f(t^(x * y)); ts[6]:=f(t^(y * x^-1)); ts[7]:=f(t^y); ts[8]:=f(t^(y^-1));
ts[9]:=f(t^(x^-1 * z)); ts[10]:=f(t^(x^-1));

cst:=[null : i in [1..56]] where null is /* #Index(G,H) */
[Integers() |];

prodim := function(pt, Q, I)
v:= pt;
for i in I do

```

```

v:=v^(Q[i]);
end for;
return v;
end function;
for i :=1 to 10 do
cst[prodim(1,ts,[i])]:=i];
end for;

m:=0; for i in [1..56] do if cst[i] ne [] then m:=m+1;
end if; end for;m;

/*10 */

Orbits(N);

/*[
  GSet{@ 1, 2, 7, 10, 5, 8, 3, 4, 9, 6 @}
]*/

Generators(N);

/*
{
  (1, 2)(4, 7)(5, 8)(9, 10),
  (1, 2, 10)(3, 4, 5)(6, 7, 8),
  (1, 7, 3, 4, 2, 5, 6, 8)
}*/

```



```

for i in [1..3] do for g in IH do for h in IN do if ts[1] eq g*(DC[i])^h then i;
break;end if;end for;end for;end for;

/* 2 */ /* 10 things move forward, 0 + #N/#N1s=720/72=10 */

/*SECOND DOUBLE COSET Ht1 [1] */
S:={[1]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]
eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;
/*{
[ 1 ]
}*/
N1:=Stabiliser(N,1);
Generators(N1);
/*
{
(2, 7)(3, 9)(4, 6)(5, 10),
(2, 10, 3, 6, 7, 5, 9, 4),
(2, 3, 4, 6, 10, 8, 7, 5),

```

```

      (2, 7, 10, 4)(3, 5, 8, 6)
}*/

N1:=Stabiliser(N,[1]);
#N1;
/*72*/
N1s:=N1;
#N1s;
/*72*/
Generators(N1s);
/*{
      (2, 5, 6, 7)(4, 9, 10, 8),
      (2, 6, 7, 3, 10, 5, 4, 8)
}*/

Orbits(N1);
/*
[
      GSet{@ 1 @},
      GSet{@ 2, 5, 6, 4, 7, 9, 8, 3, 10 @}
]*//* so now we must find where each goes to:
Ht_1t_1= ? , Ht_1t_2= ?*/

```

```

for i in [1..3] do for g in IH do for h in IN do if ts[1]*ts[1]
eq g*(DC[i])^h then i; break;end if;end for;end for;end for;
/*1*/* one thing goes back */

for i in [1..3] do for g in IH do for h in IN do if ts[1]*ts[2]
eq g*(DC[i])^h then i; break;break;end if;end for;end for;end for;
/* 3 */* 9 things move to [12],Ht12 */

/* THIRD DOUBLE COSET [12] */
S:={[1,2]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*{
    [ 1, 2 ]
}
{
    [ 2, 1 ]
}*/

```

```

N12:=Stabiliser(N,[1,2]);
Generators(N12);
/*{
    (3, 4, 9, 8, 6, 5, 10, 7),
    (3, 9, 6, 10)(4, 8, 5, 7)
}*/
N12:=Stabiliser(N,[1,2]);
#N12;
/*8*/

N12s:=N12;
for n in N do if [1,2]^n eq [ 2,1] then N12s:=sub<N|N12s,n>;
end if; end for;
#N12s;
/*16*/
Generators(N12s);
/*{
    (1, 2)(4, 7)(5, 8)(9, 10),
    (1, 2)(3, 6)(4, 8)(5, 7),
    (3, 8, 10, 4, 6, 7, 9, 5),
    (1, 2)(3, 7)(4, 10)(5, 9)(6, 8),
    (1, 2)(3, 4)(5, 6)(7, 9)(8, 10),
    (1, 2)(3, 10)(4, 5)(6, 9),
    (1, 2)(3, 5)(4, 6)(7, 10)(8, 9),

```

```

(1, 2)(3, 8)(4, 9)(5, 10)(6, 7),
(1, 2)(3, 9)(6, 10)(7, 8)
}*/

[1,2]^N12s;
GSet{@
  [ 1, 2 ],
  [ 2, 1 ]
@}/* we have both things that are equal so our coset
should increase by 720/16 = 45 */

tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..56] do if cst[i] ne []
then m:=m+1;
end if; end for;m; /* 55, confirms it increased by 45 */

Orbits(N12);
/*
[
  GSet{@ 1 @},

```

```

    GSet{@ 2 @},
    GSet{@ 3, 8, 10, 4, 6, 7, 9, 5 @}
]*/

for i in [1..3] do for g in IH do for h in IN do if ts[1]*ts[2] *ts[1]
eq g*(DC[i])^h then i; break;end if;end for;end for;end for;
/* 2 *//* 1 thing goes back */

for i in [1..3] do for g in IH do for h in IN do if ts[1]*ts[2]*ts[2]
eq g*(DC[i])^h then i; break;end if;end for;end for;end for;
/* 2 *//* 1 thing goes back */

for i in [1..3] do for g in IH do for h in IN do if ts[1]*ts[2]*ts[3]
eq g*(DC[i])^h then i; break;end if;end for;end for;end for;
/* 3 *//* 8 things loop */

CompositionFactors(G1);

NL:=NormalLattice(G1);
NL;

for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if;end for;
/* 1 */

```

```
CompositionFactors(IH);

NL:=NormalLattice(IH);
NL;

for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if;end for;
/* 1 */

IsIsomorphic(NL[5],PGL(2,9));

K:=AutomorphismGroup(Alt(6));
#K;
/* 1440 */
GG:=PermutationGroup(K);
#GG;
/* 1440 */
IsIsomorphic(GG,IH);
Index(G1,IH);
/* 56 */

CompositionFactors(IN);

NL:=NormalLattice(IN);
NL;
```

Appendix L

MAGMA CODE for

Semi-direct of $2^3 : 4$

```
T:=TransitiveGroup(8,19);
S:=Sym(8);
T;
A:=S!(1, 8)(2, 3)(4, 5)(6, 7);
B:=S!(1, 3)(2, 8)(4, 6)(5, 7);
C:=S!(1, 5)(2, 6)(3, 7)(4, 8);
D:=S!(1, 3)(4, 5, 6, 7);

N:=sub<S|A,B,C,D>;

CompositionFactors(N);
```



```

NL:=NormalLattice(N); NL;

for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;

NL[8];

X:=AbelianGroup(GrpPerm,[2,2,2]);
IsIsomorphic(NL[8],X);
/*true */
H<a,b,c>:=Group<a,b,c|a^2,b^2,c^2,(a,b),(b,c),(a,c)>;
#H;
/* 8 */
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,NL[8]);
/* true *//* verified our presentation of NL[8]*/

/* We factor out N by NL[8]. Thus, N is a mixed extension of NL[8] by N/NL[8]*/
q,ff:=quo<N|NL[8]>;
q;

IsIsomorphic(N, DirectProduct(NL[8],NL[4]));
/*false */

T:=Transversal(N,NL[8]);

```

T;

NL[8];

A:=N!(1, 5)(2, 6)(3, 7)(4, 8);

B:=N!(1, 3)(2, 8)(4, 6)(5, 7);

C:=N!(1, 8)(2, 3)(4, 5)(6, 7);

#T;

T[2];

(1, 3)(4, 5, 6, 7)

D:=N!(1, 3)(4, 5, 6, 7);

A,B,C,B^2,A*B,A*B*C,A*B^2,A*B^2*C,A*C,B*C,A*B^2*C;

A^D;

/* (1, 4)(2, 7)(3, 6)(5, 8) */

A^D eq A*C;

/* true */

B^D;

/* (1, 3)(2, 8)(4, 6)(5, 7) */

B^D eq B;

```

/* true */ /* which means we can write as (bid) */

C^D;
/* (1, 2)(3, 8)(4, 7)(5, 6) */
C^D eq B*C;
/* true */

Order(D);
/* 4 */ /* so we write d^4 */

/* Hence, a presentation for N is below*/
H<a,b,c,d>:=Group<a,b,c,d|a^2,b^2,c^2,(a,b),(a,c),(b,c),
d^4, a^d=a*c,(b,d),c^d=b*c>;

#H;
/* 32 */ /* gave us the order of our N*/
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
#H1;
/* 32 */

w:=IsIsomorphic(H1,N);
w; /* true */ /* verified our presentation so we are done and
can conclude that the TransitiveGroup(8,19) is (2^3) : 4 */

```

Appendix M

MAGMA CODE for

Construction of $2 \cdot M_{22}$ over A_7

```

S:=Sym(105);
xx:=S!(1, 3, 7, 15, 29, 45)(2, 5, 11, 22,
    42, 66)(4, 9, 18, 34, 26, 46)(6, 13, 25)(8, 16, 31, 52,
    39, 61)(10, 20, 38, 58, 78, 75)(12, 24, 14, 27, 47, 53)(17,
    33, 48, 32, 54, 73)(19, 36, 30, 50, 63, 85)(21, 40, 62, 83, 23,
    43)(28, 41, 64, 87, 69, 90)(35, 57, 76, 97, 80, 100)(37, 59,
    79, 70, 92, 86)(44, 67, 88, 94, 91, 49)(51, 71, 65, 77, 98,
    104)(55, 74, 56, 72, 95, 84)(60, 81, 101)(68, 89, 82)(93, 103,
    99, 102, 105, 96);
yy:=S!(1, 2, 4, 8)(3, 6, 12, 23)(5, 10, 19, 35)(7, 14, 26, 45)
(9, 17, 32, 53)(11, 21, 39, 13)(15, 28, 47, 69)(16, 30, 49, 66)
(18, 27, 29, 48)(20, 37, 58, 77)(22, 41, 63, 84)(24, 44, 54, 36)

```

```
(25, 43, 62, 82)(31, 51, 42, 65)(33, 55, 34, 56)(38, 60, 80, 81)
(40, 46, 68, 73)(50, 70, 91, 59)(52, 72, 94, 87)(57, 75)(61, 76,
67, 78)(64, 86)(71, 93, 79, 99)(74, 96, 90, 102)(83, 88, 89, 85)
(92, 100, 104, 97)(95, 98)(101, 103);
```

```
N:=sub<S|xx,yy>;
```

```
#N; /* 2520 */
```

```
G<x,y,t>:=Group<x,y,t|
x^6 ,
y^4 ,
x * y^-2 * x^2 * y^2 * x ,
x^-1 * y^-2 * x^-1 * y^-1 * x * y^2 * x * y^-1 ,
x^-1 * y^-1 * x^-3 * y^-1 * x^3 * y^-1 * x^-2 ,
(y^-1 * x^-1)^4 * (y * x)^2 * y * x^-1 ,
(y^-1 * x^-2)^2 * y^-1 * x^2 * y * x * y^-1 * x^3 * y * x^-1, t^2,
(t, y * x^2 * y * x * y * x^-1 * y * x ), (t, x^2 * (y * x )^2 * y * x^-2),
(t,x * y * x^2 * y * x^-1 * y * x * y^-1 * x),
((y * x)^2 * y * x^-2 * y^-1 * x*t^(x * y * x^2 * y * x^3))^3>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);/*5 */
```

```
for i in [0..5] do for j in [0..3] do printf "%o", i;j,
```

```
xx^i*yy^j; end for; end for;
```

```

for i,j in [0..5] do for k in [0..3] do printf "%o",
i;j;k,xx^i*yy^k*xx^j; end for; end for;
for i,k in [0..5] do for j in [0..3] do printf "%o",
i;j;k,yy^i*xx^j*yy^k; end for; end for;
for i in [0..5] do for j in [0..3] do printf "%o", i;j,
(xx^i*yy^j)^-1; end for; end for;

for i,k in [0..5] do for j in [0..3] do printf "%o",
i;j;k,(yy^i*xx^j*yy^k)^-1; end for; end for;

#G1;
887040

CompositionFactors(G1);

IN:=sub<G1|f(x),f(y)>;
A:=[ Id(G), (y^-1 * x)^2, x * y^-1 * x * y * x^2 * y,
x^2 * y * x * y * x * y^-1 *
x^2, x^2 * y * x^-1 * y * x^-1 * y * x^2,
x * y * x^2 * y * x * y^2 * x^-1, x^-1
* y, x * y * x * y^-1 * x^2, y^2 * x * y * x^2 * y * x^-1,
x * y * x * y * x^-1
* y, x^-1 * y * x^-2 * y * x^2, x^-2 * y * x * y * x,
(y * x * y^-1)^2, x^2 *

```

$y^{-1} * x^{-1} * y^{-1} * x^{-1}, x^{-1} * y^2 * x * y * x^{-1},$
 $x^2 * y^{-1} * x * y^{-1} * x,$
 $x^{-1} * y^{-1} * x^{-1} * y * x^{-2} * y^{-1} * x^{-1},$
 $x^{-2} * y * x * y * x^{-2} * y^{-1},$
 $y^{-1} * x * y * x^{-2}, y^{-1} * x * y * x^{-2} * y^{-1} * x,$
 $x * y * x^{-1} * y * x, x^{-1}$
 $* y * x * y * x * y^{-1}, x^{-1} * y^{-1} * x * y * x^2 * y^{-1} * x^2,$
 $x * y^{-1} * x * y$
 $* x^2 * y^{-1} * x, x^{-1} * y * x * y * x * y^{-1} * x^{-1} * y^{-1} * x,$
 $x^{-1} * y^{-1}, y * x^{-1} * y * x^2 * y^{-1} * x * y^{-1} * x^{-1},$
 $x * y * x * y^{-1} * x^{-1} * y^{-1} * x^3,$
 $y * x^{-1} * y * x^{-1} * y * x^2 * y * x * y,$
 $x^{-2} * y * x * y * x^2 * y^{-1} * x,$
 $x^{-1} * y * x^{-2} * y * x^{-1} * y * x,$
 $x * y * x^{-1} * y * x^2 * y^{-1} * x * y, x^{-1} * y^{-1} * x^{-1}$
 $* y * x^{-2} * y^{-1}, y * x^3 * y^{-1} * x * y, x^{-2} * y * x^2 * y * x$
 $x^{-1} * y, x * y * x * y^{-1} * x * y * x^{-2} * y^{-1} * x, y * x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^3,$
 $y * x^{-1} * y * x^{-2} * y^{-1} * x^{-2}, x^{-1} * y * x * y^{-1} * x^{-1} * y * x, y * x^{-1} * y * x^{-1} * y^{-1} * x^{-1} * y^{-1},$
 $y^{-1} * x * y * x^2 * y^{-1}, x^{-1} * y * x^{-2} * y^{-1} * x * y^{-1} * x^{-1}, (x^{-1} * y)^3, x^2 * y^{-1} * x^{-1} * y * x^2 * y^{-1},$
 $x^3 * y * x^{-2} * y * x^{-1} * y^{-1},$
 $x^{-1} * y * x^2 * y^{-1} * x^{-1} * y * x, x^{-1} * y * x^2 * y * x * y * x^{-1}, x^2 * y * x * y^{-1} * x^{-1} * y * x^2, y * x * y *$

$x^{-1} * y * x * y * x^{-2}$, $x^3 * y * x * y * x^{-1}$, $x * y * x^{-2} * y * x^2 * y^{-1}$, $x * y * x^{-1} * y * x * y * x^{-1}$, $x * y * x^{-1} * y^{-1} * x * y * x^{-2} * y$, $(x * y^2)^2$, $x^{-2} * y * x^{-1} * y^{-1} * x^{-2}$, $x^{-2} * y * x^{-1} * y^{-1}$, $x * y * x * y^{-1} * x * y * x^{-1}$, $y * x^{-2} * y * x * y^{-1}$, $y * x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^{-2}$, $y * x * y * x * y * x^{-2} * y^{-1} * x * y^{-1}$, $x^2 * y * x^{-1} * y^{-1} * x^{-1} * y * x^2 * y^{-1}$, $x * y * x^2 * y * x^3$, $y^{-1} * x * y * x^2$, $x^{-1} * y * x^{-1} * y * x * y^{-1} * x^{-1} * y^{-1} * x^{-1}$, $y * x^{-1} * y * x^{-2} * y^{-1} * x^{-1} * y * x^{-1}$, $x^2 * y * x * y * x^2 * y * x^{-1}$, $x^2 * y * x^{-1} * y^{-1} * x^{-1} * y * x^2 * y$, $y * x^2 * y^{-1} * x * y^{-1} * x$, $x^{-2} * y^{-1} * x * y$, $x^3 * y * x^2 * y * x^{-2}$, $x * y * x^{-2} * y * x^2 * y * x^{-1}$, $x^3 * y^{-1} * x^{-1} * y^2$, $y * x * y * x^{-2} * y^{-1} * x^{-1} * y * x$, $x^{-2} * y * x * y^{-1} * x * y^{-1} * x^{-1}$, $x * y * x * y^{-1} * x * y^{-1} * x$, $y * x^{-1} * y * x^{-2} * y$, $y * x^{-1} * y * x * y * x^3$, $y^{-1} * x^{-1} * y^{-1}$, $y * x * y^{-1} * x * y * x^{-2} * y^{-1} * x^2$, $y * x * y * x^{-1} * y * x^3$, $y^{-1} * x^{-1} * y * x^2 * y^{-1} * x$, $x^2 * y * x^{-1} * y^2 * x * y^{-1} * x^{-1}$, $x * y * x^{-1} * y^2 * x$, $y * x^{-1} * y * x * y * x^{-2} * y^{-1} * x$, $x * y * x^{-1} * y^{-1} * x^{-1} * y$, $x^{-1} * y * x * y * x^2 * y$, $x * y * x^{-2} * y^{-1} * x^{-1} * y^{-1}$, (x^{-1}, y) , $x^{-1} * y^{-1} * x * y * x$, $x^{-1} * y * x * y^{-1} * x$, $x * y * x * y * x^3 * y^{-1} * x^{-1}$, $x^{-1} * y * x * y * x^2 * y^{-1} * x^{-1}$, $y * x * y * x^2 * y * x^{-1} * y * x^{-1}$, $x^{-1} * y^{-1} * x * y * x^{-2} * y^{-1} * x$, $y * x^3 * y^{-1} * x^{-1}$, $x^3 * y * x^{-1} * y^{-1}$, $x^3 * y * x^2 * y^{-1} * x^{-1} * y^{-1}$, $y * x * y * x * y^{-1} * x$,


```

y * x * y * x^2 * y^-1 * x * y^-1 * x, y * x * y * x * y * x^-2 * y,
y^-1 * x^-1 * y * x^2 * y * x, x^-1 * y * x * y * x^-1 * y * x * y *
x^-1, y * x^-1 * y * x * y * x * y^-1 * x^-1, x^-1 * y^-1 * x^-1 *
y * x^2 * y * x, y * x^-2 * y^-1 * x * y * x^-2 ];

```

```
ts:= [Id(G1) : i in [1..105]];
```

```
ts[1]:=f(t);
```

```
for i in [2..105] do ts[i]:=f(t^A[i]); end for;
```

```
DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
```

```
DC:= [ f( Id(G)), f( t), f(t * x * t), f(t * x^2 * t ),
f(t * x * t * x^-1 * y^-1 * t * y * t)];
```

```
Index(G1,IN);
```

```
352
```

```
word:=function(A)
```

```
Sch:=SchreierSystem(G,sub<G|Id(G)>);
```

```
B:=Id(G);
```

```
for i in [2..#G] do
```

```
P:= [Id(G1): l in [1..#Sch[i]]];
```

```
for j in [1..#Sch[i]] do
```

```
if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
```

```

if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x)^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y)^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=f(t)^-1; end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;

```

```

cst := [null : i in [1 ..Index(G1,IN)]] where null is [Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
    v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 105 do
    cst[prodim(1, ts, [i])] := [i];

```

```

end for;

m:=0; for i in [1..352] do if cst[i] ne [] then m:=m+1; end if; end for; m;
/* 105*/

Orbits(N);

for i in [1..#DC] do for m,n in IN do if ts[1] eq m*(DC[i])^n
then i; break; end if; end for; end for;

2

N1:=Stabiliser(N,1);
#N1;
24
N1;
Permutation group N1 acting on a set of cardinality 105
Order = 24 = 2^3 * 3
Generators(N1);

for i,j in [1..105] do if ts[i] eq ts[j] then i,j;end if; end for;

DOING ORBIT Nt1
S:={[1]};S;

```

```

SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]
eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;
{
[ 1 ]
}
N1s:=sub<N|N1>;

Orbits(N1);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[1] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 1*/ 1 goes back to [ *]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[21] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 2 */ /*2 things loops to [1] */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/*4*/ 6 things move to [15]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2] eq m*(DC[i])^n

```

```

then i; break; break;end if; end for;end for;
/*3*//*12 things move to [12] */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 3 */ so 12 things go to [12]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[4] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/*2*/ so 12 things loop to [1]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/*3 */12 things go to [12]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[7] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/*4 */24 things move to [15]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[10] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/*2 */24 things loop to [1]

/*DOING NEW COSET (Nt1t2) - THRD DOUBLE COSET*/
S:={[1,2]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]

```

```
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
  then print SSS[i];
end if; end for; end for;
```

```
{
  [ 1, 2 ]
}
{
  [ 24, 36 ]
}
{
  [ 47, 85 ]
}
{
  [ 89, 34 ]
}
{
  [ 21, 15 ]
}
{
  [ 75, 84 ]
}
{
  [ 33, 11 ]
}
```

```

}
{
    [ 105, 38 ]
}
{
    [ 57, 72 ]
}

N12:=Stabiliser(N,[1,2]);
#N12;
2

N12s:=N12;
for n in N do if [1,2]^n eq [1,2] then N12s:=sub<N|N12s,n>;
end if; end for;
#N12s;
2
for n in N do if [1,2]^n eq [24,36] then N12s:=sub<N|N12s,n>;
end if; end for;
#N12s;
6
for n in N do if [1,2]^n eq [47,85] then N12s:=sub<N|N12s,n>;
end if; end for;
#N12s;

```

18

```
Generators(N12s);
```

```
[1,2]^N12s;
```

```
tr1:=Transversal(N,N12s);
```

```
for i:=1 to #tr1 do
```

```
ss:=[1,2]^tr1[i];
```

```
cst[prodim(1,ts,ss)]:=ss;
```

```
end for;
```

```
m:=0; for i in [1..352] do if cst[i] ne []
```

```
then m:=m+1;
```

```
end if; end for;m;
```

```
245
```

```
Orbits(N12s);
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[19] eq m*(DC[i])^n
then i; break; end if; end for;end for;
```

```
/*3 */ 3 things will loop
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[37] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

```
/*3 */3 things will loop
```



```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

```
/* 4*/ 9 things move to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

```
2, so 9 things go back to [1]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

```
3, 9 things loop to [12]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

```
4, 9 things extend to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[7] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

```
2 , 9 things go back to [1]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[8] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

```
3, 9 things loop to [12]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[30] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

2, 9 things go back to [1]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[5] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

3, 18 things loop to [12]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[14] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
```

3, 18 things loops to [12]

```
/*DOING NEW COSET (Nt1t5) - Fourth DC*/
```

```
S:={[1, 5]};
```

```
SS:=S^N;SS;
```

```
SSS:=Setseq(SS);
```

```
for i in [1..#SSS] do
```

```
for g in IN do if ts[1]*ts[5]
```

```
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
```

```
then print SSS[i];
```

```
end if; end for; end for;
```

```
{
  [ 1, 5 ]
}
{
  [ 26, 63 ]
}
{
  [ 20, 55 ]
}
{
  [ 63, 26 ]
}
{
  [ 55, 20 ]
}
{
  [ 5, 1 ]
}

N15:=Stabiliser(N,[1,5]);
#N15;
/* 4 */
N15s:=N15;
for n in N do if [1,5]^n eq [26,63] then N15s:=sub<N|N15s,n>; end if; end for;
```

```

#N15s;
/* 24*/
Generators(N15s);

[1,5]^N15s;

tr1:=Transversal(N,N15s);
for i:=1 to #tr1 do
ss:=[1,5]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..352] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
350

Orbits(N15s);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]
eq m*(DC[i])^n then i; break; end if; end for; end for;
5, 1 thing goes to [1559]

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[70]
eq m*(DC[i])^n then i; break; end if; end for; end for;

```

4, 2 things loop to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[1]
eq m*(DC[i])^n then i; break; end if; end for; end for;
2, 6 things go back to [1]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[2]
eq m*(DC[i])^n then i; break; end if; end for; end for;
3, 12 things go back to [12]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[3]
eq m*(DC[i])^n then i; break; end if; end for; end for;
4, 12 things loop to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[4]
eq m*(DC[i])^n then i; break; end if; end for; end for;
3, 12 things go back to [12]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[28]
eq m*(DC[i])^n then i; break; end if; end for; end for;
3, 12 things go back to [12]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[6]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4, 24 things loop to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[12]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

2, 24 things go back to [1]

```
/*DOING NEW COSET (Nt1t5t59) - FIFTH DOUBLE COSET*/
```

```
S:={[1, 5,59]};
```

```
SS:=S^N;SS;
```

```
SSS:=Setseq(SS);
```

```
for i in [1..#SSS] do
```

```
for g in IN do if ts[1]*ts[5]*ts[59]
```

```
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
```

```
then print SSS[i];
```

```
end if; end for; end for;
```

```
N1559:=Stabiliser(N,[1,5,59]);
```

```
#N1559;
```

```
4
```

```
N1559s:=N1559;
```

```
for n in N do if [1,5,59]^n eq [3,11,79] then N1559s:=sub<N|N1559s,n>;
```

```
end if; end for;
```

```
#N1559s;
```

2520

```
N1559s:=N1559;
```

```
Generator(N1559s);
```

```
tr1:=Transversal(N,N1559s);
```

```
for i:=1 to #tr1 do
```

```
ss:=[1,5,59]^tr1[i];
```

```
cst[prodim(1,ts,ss)]:=ss;
```

```
end for;
```

```
m:=0; for i in [1..352] do if cst[i] ne []
```

```
then m:=m+1;
```

```
end if; end for;m;
```

351

```
Orbits(N1559s);
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[1]
```

```
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

```
4, 1 thing goes back to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[5]
```

```
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4, 1 thing goes back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[59]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4, 1 thing goes back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[3]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4, 2 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[4]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4, 2 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[19]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4, 2 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[21]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4, 2 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[28]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```


4, 2 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[60]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4, 2 things go back

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[61]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,2 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[70]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,2 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[77]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,2 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[2]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[6]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[7]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[9]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[10]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[11]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[12]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[13]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[14]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[15]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[16]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[18]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[20]
eq m*(DC[i])^n then i; break; end if; end for; end for;
```

4 ,4 things go back to [15]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[24]
eq m*(DC[i])^n then i; break; end if; end for; end for;
4 ,4 things go back to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[31]
eq m*(DC[i])^n then i; break; end if; end for; end for;
4 ,4 things go back to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[32]
eq m*(DC[i])^n then i; break; end if; end for; end for;
4 ,4 things go back to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[36]
eq m*(DC[i])^n then i; break; end if; end for; end for;
4 ,4 things go back to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[37]
eq m*(DC[i])^n then i; break; end if; end for; end for;
4 ,4 things go back to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[40]
eq m*(DC[i])^n then i; break; end if; end for; end for;
4 ,4 things go back to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[47]  
eq m*(DC[i])^n then i; break; end if; end for; end for;  
4 ,4 things go back to [15]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5]*ts[59]*ts[50]  
eq m*(DC[i])^n then i; break; end if; end for; end for;  
4 ,4 things go back to [15]
```

Appendix N

MAGMA CODE for

Construction of $L(2, 11)$ over N

```

S:=Sym(8);
xx:=S!(1, 4, 2, 7)(3, 5, 8, 6);
yy:=S!(1, 7)(2, 6)(3, 5)(4, 8);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^2,(t,y*x),(x^2*t)^5,
(y*t)^10,(y*t^(x^2))^2,x^2*t*x*t*x*t*x*t*y
*t*x^-1*t*x^2*t*x*t>;
#G;
/* 1320 */
#N;
/* 24 */
#sub<G|x,y>;

```

```

/* 24 */
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/* 6 */
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/* 1 */
#G1; /*1320 */
CompositionFactors(G1);

for i in [0..2] do for j in [0..1] do printf "%o", i;j,xx^i*yy^j;
end for; end for;

IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 8] ];
ts[1]:=f(t); ts[2]:=f(t^(x^2)); ts[3]:=f(t^(x^2*y)^2);
ts[4]:=f(t^x);ts[5]:=f(t^(x^2*y*x^2));
ts[6]:=f(t^(x^2*y));ts[7]:=f(t^y); ts[8]:=f(t^(x*y));

DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);

DC:=[ f( Id(G)), f( t), f(t * x * t), f(t * x * t * x * t ),
f( t * x * t * x * t * x * t ), f(t * x * t * y * t * x^-1 * t)];

Index(G, sub<G|x,y>);

```

```

/* 55 */

cst := [null : i in [1 ..55]] where null is [Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
    v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 8 do
    cst[prodim(1, ts, [i])] := [i];
end for;

m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1; end if; end for; m;
/* 4 */
> for i,j in [1..8] do if ts[i] eq ts[j] then i,j; break; end if; end for;

Orbits(N);

for i in [1..#DC] do for m,n in IN do if ts[1] eq m*(DC[i])^n then i;
break; end if; end for; end for;
/* 2 */ /* this means it goes to the second DC, as predicted */

```



```

N1:=Stabiliser(N,1);
Generators(N1);
Orbits(N1);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[1] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 1 */ /* goes back */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 1 */ /* goes back */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 3 */ /* NEW */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[4] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 3 */ /* NEW */

/*DOING NEW COSET (Nt1t2)*/
S:={[1,2]};
SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]

```

```

eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

/*
{
    [ 1, 2 ]
}
{
    [ 5, 6 ]
}
**/* this means these two things are equal */

N12:=Stabiliser(N,[1,2]);
#N12;
/* 1 */
N12s:=N12;
for n in N do if [1,2]^n eq [5,6] then N12s:=sub<N|N12s,n>;
end if; end for;
#N12s;
/* 2 */

Generators(N12s);

[1,2]^N12s;

```

```

tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 16 */

Orbits(N12s);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 3 *//* loops */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 2*//* goes back */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 4 *//* NEW */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[7] eq m*(DC[i])^n
then i; break; end if; end for;end for;

```

```

/* 4 */ /* NEW */

/*DOING NEW COSET (Nt1t2t3)*/
S:={1,2,3};
SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
/* {
    [ 1,2,3 ]
}
*/

N123:=Stabiliser(N,[1,2,3]);
#N123;
/* 1 */
N123s:=N123;
#N123s;
/* 1*/
Generators(N123s);

```

```

/* { } */

tr1:=Transversal(N,N123s);
for i:=1 to #tr1 do
ss:=[1,2,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 40*/

Orbits(N123s);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[1] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 6 */ /* NEW */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[2] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 4 */ /* loops to itself*/
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[3] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 3*/ /* goes back*/
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[4] eq

```

```

m*(DC[i])^n then i; break; end if; end for; end for;
/*5 */ /* NEW */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[5] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 6 */ /* equal to new */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[6] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 4*/ /* loops */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[7] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 3 */ /* goes back */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[8] eq
m*(DC[i])^n then i; break; end if; end for; end for;
/* 5 */ /* equal to new */

/*DOING NEW COSET (Nt1t2t3t4)*/
S:={[1, 2,3,4]};
SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*
ts[Rep(SSS[i])[4]] then print SSS[i]; end if; end for; end for;

```

```
/*
{
    [ 1, 2, 3, 4 ]
}
{
    [ 8, 1, 3, 6 ]
}
{
    [ 2, 8, 3, 5 ]
}
*/

N1234:=Stabiliser(N,[1,2,3,4]);
#N1234;
/* 1 */
N1234s:=N1234;
for n in N do if [1,2,3,4]^n eq [8,1,3,6] then N1234s:=sub<N|N1234s,n>;
end if; end for;
#N1234s;
/* 3 */

Generators(N1234s);

[1,2,3,4]^N1234s;
```

```

tr1:=Transversal(N,N1234s);
for i:=1 to #tr1 do
ss:=[1,2,3,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 48 */

Orbits(N1234s);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*
ts[3]*ts[4]*ts[3] eq m*(DC[i])^n then i; break; end if;
end for; end for;
/* 5 */ /* loops to itself */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]
*ts[4]*ts[7] eq m*(DC[i])^n then i; break; end if; end for;
end for;
/* loops to itself */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]
*ts[4]*ts[1] eq m*(DC[i])^n then i; break; end if; end for;
end for;

```



```

/* 4 */ /* goes back */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]
*ts[4]*ts[4] eq m*(DC[i])^n then i; break; end if; end for;
end for;
/* 4 */ /* goes back */

/*DOING NEW COSET (Nt1t2t3t1)*/
S:={[1, 2,3,1]};
SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*
ts[Rep(SSS[i])[1]] then print SSS[i]; end if; end for; end for;
/*
{
    [ 1, 2, 3, 1 ]
}
{
    [ 6, 7, 4, 6 ]
}
{

```

```

    [ 4, 5, 6, 4 ]
}
{
    [ 3, 8, 1, 3 ]
}
*/

N1231:=Stabiliser(N,[1,2,3,1]);
#N1231;
/* 1 */
N1231s:=N1231;
for n in N do if [1,2,3,1]^n eq [4,5,6,4] then N1231s:=sub<N|N1231s,n>;
end if; end for;
#N1231s;
/* 4 */
Generators(N1231s);

[1,2,3,1]^N1231s;

tr1:=Transversal(N,N1231s);
for i:=1 to #tr1 do
ss:=[1,2,3,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;

```

```
m:=0; for i in [1..55] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 54 */

Orbits(N1231s);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[1]*ts[1]
eq m*(DC[i])^n then i; break; end if; end for; end for;
/* 4 *//* goes back */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[1]*ts[2]
eq m*(DC[i])^n then i; break; end if; end for; end for;
/* 4 *//* goes back */
```

Appendix O

MAGMA CODE for

Construction of M_{22} over A_7

```

S:=Sym(105);
xx:=S!(1, 3, 7, 15, 29, 45)(2, 5, 11, 22,
    42, 66)(4, 9, 18, 34, 26, 46)(6, 13, 25)(8, 16, 31, 52,
    39, 61)(10, 20, 38, 58, 78, 75)(12, 24, 14, 27, 47, 53)(17,
    33, 48, 32, 54, 73)(19, 36, 30, 50, 63, 85)(21, 40, 62, 83, 23,
    43)(28, 41, 64, 87, 69, 90)(35, 57, 76, 97, 80, 100)(37, 59,
    79, 70, 92, 86)(44, 67, 88, 94, 91, 49)(51, 71, 65, 77, 98,
    104)(55, 74, 56, 72, 95, 84)(60, 81, 101)(68, 89, 82)(93, 103,
    99, 102, 105, 96);
yy:=S!(1, 2, 4, 8)(3, 6, 12, 23)(5, 10, 19, 35)(7, 14, 26, 45)
(9, 17, 32, 53)(11, 21, 39, 13)(15, 28, 47, 69)(16, 30, 49, 66)
(18, 27, 29, 48)(20, 37, 58, 77)(22, 41, 63, 84)(24, 44, 54, 36)

```

```
(25, 43, 62, 82)(31, 51, 42, 65)(33, 55, 34, 56)(38, 60, 80, 81)
(40, 46, 68, 73)(50, 70, 91, 59)(52, 72, 94, 87)(57, 75)(61, 76,
67, 78)(64, 86)(71, 93, 79, 99)(74, 96, 90, 102)(83, 88, 89, 85)
(92, 100, 104, 97)(95, 98)(101, 103);
```

```
N:=sub<S|xx,yy>;
```

```
#N;2520
```

```
G<x,y,t>:=Group<x,y,t|
    x^6 ,
    y^4 ,
    x * y^-2 * x^2 * y^2 * x ,
    x^-1 * y^-2 * x^-1 * y^-1 * x * y^2 * x * y^-1 ,
    x^-1 * y^-1 * x^-3 * y^-1 * x^3 * y^-1 * x^-2 ,
    (y^-1 * x^-1)^4 * (y * x)^2 * y * x^-1 ,
    (y^-1 * x^-2)^2 * y^-1 * x^2 * y * x * y^-1 * x^3 * y * x^-1,
    t^2, (t, y * x^2 * y * x * y * x^-1 * y * x ),
    (t, x^2 * (y * x )^2 * y * x^-2), (t,x * y * x^2 * y * x^-1 *
    y * x * y^-1 * x), ((y * x)^2 * y * x^-2 * y^-1 * x*t^(x *
    y * x^2 * y * x^3))^3, x^-1 * y * x^-1 * y * x * y * x * y
    * x * y^-1 * x*t * x * t * x^-1 * y^-1 * t * y * t>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
```

```

#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
3

#G1;
443520

for i in [0..5] do for j in [0..3] do printf "%o", i;j,xx^i*yy^j;
end for; end for;

IN:=sub<G1|f(x),f(y)>;
A:=[ Id(G), (y^-1 * x)^2, x * y^-1 * x * y * x^2 * y,
x^2 * y * x * y * x * y^-1 *
x^2, x^2 * y * x^-1 * y * x^-1 * y * x^2,
x * y * x^2 * y * x * y^2 * x^-1, x^-1
* y, x * y * x * y^-1 * x^2, y^2 * x * y * x^2 * y * x^-1,
x * y * x * y * x^-1
* y, x^-1 * y * x^-2 * y * x^2, x^-2 * y * x * y * x,
(y * x * y^-1)^2, x^2 *
y^-1 * x^-1 * y^-1 * x^-1, x^-1 * y^2 * x * y * x^-1,
x^2 * y^-1 * x * y^-1 * x,
x^-1 * y^-1 * x^-1 * y * x^-2 * y^-1 * x^-1,
x^-2 * y * x * y * x^-2 * y^-1,
y^-1 * x * y * x^-2, y^-1 * x * y * x^-2 * y^-1 * x,
x * y * x^-1 * y * x, x^-1

```

$* y * x * y * x * y^{-1}, x^{-1} * y^{-1} * x * y * x^2 * y^{-1} * x^2,$
 $x * y^{-1} * x * y$
 $* x^2 * y^{-1} * x, x^{-1} * y * x * y * x * y^{-1} * x^{-1} * y^{-1} * x,$
 $x^{-1} * y^{-1}, y * x^{-1} * y * x^2 * y^{-1} * x * y^{-1} * x^{-1},$
 $x * y * x * y^{-1} * x^{-1} * y^{-1} * x^3,$
 $y * x^{-1} * y * x^{-1} * y * x^2 * y * x * y,$
 $x^{-2} * y * x * y * x^2 * y^{-1} * x,$
 $x^{-1} * y * x^{-2} * y * x^{-1} * y * x,$
 $x * y * x^{-1} * y * x^2 * y^{-1} * x * y, x^{-1} * y^{-1} * x^{-1}$
 $* y * x^{-2} * y^{-1}, y * x^3 * y^{-1} * x * y, x^{-2} * y * x^2 * y * x^{-1} * y,$
 $x * y * x * y^{-1} * x * y * x^{-2} * y^{-1} * x, y * x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^3,$
 $y * x^{-1} * y * x^{-2} * y^{-1} * x^{-2}, x^{-1} * y * x * y^{-1} * x^{-1} * y * x, y * x^{-1} * y * x^{-1} * y^{-1} * x^{-1} * y^{-1},$
 $y^{-1} * x * y * x^2 * y^{-1}, x^{-1} * y * x^{-2} * y^{-1} * x * y^{-1} * x^{-1}, (x^{-1} * y)^3, x^2 * y^{-1} * x^{-1} * y * x^2 * y^{-1},$
 $x^3 * y * x^{-2} * y * x^{-1} * y^{-1}, x^{-1} * y * x^2 * y^{-1} * x^{-1} * y * x, x^{-1} * y * x^2 * y * x * y * x^{-1},$
 $x^2 * y * x * y^{-1} * x^{-1} * y * x^2, y * x * y * x^{-1} * y * x * y * x^{-2},$
 $x^3 * y * x * y * x^{-1}, x * y * x^{-2} * y * x^2 * y^{-1},$
 $x * y * x^{-1} * y * x * y * x^{-1}, x * y * x^{-1} * y^{-1} * x * y * x^{-2} * y,$
 $(x * y^2)^2, x^{-2} * y * x^{-1} * y^{-1} * x^{-2},$
 $x^{-2} * y * x^{-1} * y^{-1}, x * y * x * y^{-1} * x * y * x^{-1},$
 $y * x^{-2} * y * x * y^{-1}, y * x^{-1} * y^{-1} * x^{-1} * y^{-1} * x^{-2},$
 $y * x * y * x * y * x^{-2} * y^{-1} * x * y^{-1}, x^2 * y * x^{-1} * y^{-1} *$

```

x^-1 * y * x^2 * y^-1, x * y * x^2 * y * x^3, y^-1 * x * y * x^2,
x^-1 * y * x^-1 * y * x * y^-1 * x^-1 * y^-1 * x^-1,
y * x^-1 * y * x^-2 * y^-1 * x^-1 * y * x^-1, x^2 * y * x * y *
x^2 * y * x^-1, x^2 * y * x^-1 * y^-1 * x^-1 * y * x^2 * y, y *
x^2 * y^-1 * x * y^-1 * x, x^-2 * y^-1 * x * y, x^3 * y * x^2 *
y * x^-2, x * y * x^-2 * y * x^2 * y * x^-1, x^3 * y^-1 * x^-1 * y^2,
y * x * y * x^-2 * y^-1 * x^-1 * y * x, x^-2 * y * x * y^-1 * x *
y^-1 * x^-1, x * y * x * y^-1 * x * y^-1 * x, y * x^-1 * y * x^-2 * y,
y * x^-1 * y * x * y * x^3, y^-1 * x^-1 * y^-1, y * x * y^-1 * x *
y * x^-2 * y^-1 * x^2, y * x * y * x^-1 * y * x^3, y^-1 * x^-1 * y *
x^2 * y^-1 * x, x^2 * y * x^-1 * y^2 * x * y^-1 * x^-1, x * y * x^-1 *
y^2 * x, y * x^-1 * y * x * y * x^-2 * y^-1 * x, x * y * x^-1 * y^-1 *
x^-1 * y, x^-1 * y * x * y * x^2 * y, x * y * x^-2 * y^-1 * x^-1 *
y^-1, (x^-1, y), x^-1 * y^-1 * x * y * x, x^-1 * y * x * y^-1 * x,
x * y * x * y * x^3 * y^-1 * x^-1, x^-1 * y * x * y * x^2 * y^-1 *
x^-1, y * x * y * x^2 * y * x^-1 * y * x^-1, x^-1 * y^-1 * x * y
* x^-2 * y^-1 * x, y * x^3 * y^-1 * x^-1, x^3 * y * x^-1 * y^-1,
x^3 * y * x^2 * y^-1 * x^-1 * y^-1, y * x * y * x * y^-1 * x,
y * x * y * x^2 * y^-1 * x * y^-1 * x, y * x * y * x * y * x^-2 * y,
y^-1 * x^-1 * y * x^2 * y * x, x^-1 * y * x * y * x^-1 * y * x * y *
x^-1, y * x^-1 * y * x * y * x * y^-1 * x^-1, x^-1 * y^-1 * x^-1 *
y * x^2 * y * x, y * x^-2 * y^-1 * x * y * x^-2 ];
ts:=[Id(G1) : i in [1..105]];
ts[1]:=f(t);

```



```
for i in [2..105] do ts[i]:=f(t^A[i]); end for;
```

```
DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
```

```
DC:=[ f( Id(G)), f( t), f(t * x * t)];
```

```
Index(G1,IN);
```

```
176
```

```
cst := [null : i in [1 ..Index(G1,IN)]] where null is [Integers() | ];
```

```
prodim := function(pt, Q, I)
```

```
v := pt;
```

```
for i in I do
```

```
    v := v^(Q[i]);
```

```
end for;
```

```
return v;
```

```
end function;
```

```
for i := 1 to 105 do
```

```
    cst[prodim(1, ts, [i])] := [i];
```

```
end for;
```

```
m:=0; for i in [1..176] do if cst[i] ne [] then m:=m+1; end if; end for; m;
```

```
105
```

```
Orbits(N);
```

```

N1:=Stabiliser(N,1);
#N1;
24
Generators(N1);

DOING ORBIT Nt1
S:={[1]};S;
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]
eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;
{
  [ 1 ]
}

N1s:=sub<N|N1>;

Orbits(N1);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[1] eq m*(DC[i])^n

```

```

then i; break; end if; end for;end for;
/* 1*/ 1 goes back to [ *]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[21] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 2*/ /*2 things loop to [1 ] */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[5] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/*2*/ 6 things loop to [1 ]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/*3*//*12 things [12 ] */
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 3 */ so 12 things go to [ 12]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[4] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/*2*/ so 12 things loop to [1]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[6] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 3*/12 things go to [12 ]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[7] eq m*(DC[i])^n
then i; break; break;end if; end for;end for;
/* 2*/24 things loop to [1 ]
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[10] eq m*(DC[i])^n

```

```

then i; break; break;end if; end for;end for;
/*2 */24 things loop to [ 1]

```

```

DOING ORBIT Nt1t2 third double coset

```

```

S:={[1,2]};S;

```

```

SS:=S^N;SS;

```

```

SSS:=Setseq(SS);

```

```

for i in [1..#SSS] do

```

```

for g in IN do if ts[1]*ts[2]

```

```

eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]

```

```

then print SSS[i];

```

```

end if; end for; end for;

```

```

{

```

```

    [ 1, 2 ]

```

```

}

```

```

{

```

```

    [ 24, 36 ]

```

```

}

```

```

{

```

```

    [ 41, 56 ]

```

```

}

```

```

{

```

```

    [ 87, 69 ]

```

```

}

```

```
{  
  [ 47, 85 ]  
}  
{  
  [ 89, 34 ]  
}  
{  
  [ 49, 22 ]  
}  
{  
  [ 21, 15 ]  
}  
{  
  [ 63, 39 ]  
}  
{  
  [ 81, 4 ]  
}  
{  
  [ 94, 88 ]  
}  
{  
  [ 75, 84 ]  
}
```

```
{
  [ 33, 11 ]
}
{
  [ 30, 52 ]
}
{
  [ 60, 54 ]
}
{
  [ 105, 38 ]
}
{
  [ 57, 72 ]
}
{
  [ 101, 80 ]
}

N12:=Stabiliser(N,[1,2]);
#N12;
2
for n in N do if [1,2]^n eq [24,36] then N12s:=sub<N|N12s,n>; end if; end for;
#N12s;
```

6

```
for n in N do if [1,2]^n eq [41,56] then N12s:=sub<N|N12s,n>; end if; end for;
```

```
#N12s;
```

36

```
Generators(N12s);
```

```
[1,2]^N12s;
```

```
tr1:=Transversal(N,N12s);
```

```
for i:=1 to #tr1 do
```

```
ss:=[1,2]^tr1[i];
```

```
cst[prodim(1,ts,ss)]:=ss;
```

```
end for;
```

```
m:=0; for i in [1..176] do if cst[i] ne []
```

```
then m:=m+1;
```

```
end if; end for;m;
```

175

```
Orbits(N12s);
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[19] eq
```

```
m*(DC[i])^n then i; break; end if; end for;end for;
```

```
3, 6 things loop to [12]
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[8] eq
m*(DC[i])^n then i; break; end if; end for;end for;
```

3, 9 things loop to [12]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1] eq
m*(DC[i])^n then i; break; end if; end for;end for;
```

2, 18 things go back to [1]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[2] eq
m*(DC[i])^n then i; break; end if; end for;end for;
```

2, 18 things go back to [1]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3] eq
m*(DC[i])^n then i; break; end if; end for;end for;
```

2, 18 things go back to [1]

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[5] eq
m*(DC[i])^n then i; break; end if; end for;end for;
```

3, 36 things loop to [12]

Appendix P

MAGMA CODE Wreath Product Symmetric Presentation and Building Progenitor

```
N:=WreathProduct(CyclicGroup(5),CyclicGroup(5));  
#N;  
/* 15625*/  
  
/*Note: i found my xx and yy by using the code:  
Generators(N); */  
S:=Sym(25);
```

```

xx:=S!(1, 6, 11, 16, 21)(2, 7, 12, 17, 22)(3, 8, 13, 18, 23)(4, 9,
14, 19, 24)(5, 10, 15, 20, 25);
yy:=S!(1, 2, 3, 4, 5);
N:=sub<S|xx,yy>;
#N; /*15625*/

FPGroup(N);

NN<a,b>:=Group<a,b|a^5, b^5,
(b, a^-1, b) ,
a * b^-1 * a^-2 * b^-1 * a^2 * b * a^-2 * b * a >;
#NN; /* 15625*/

word:=function(A)
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
for i in [2..#NN] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);

```

```

for k in [1..#P] do
PP:=PP*P[k]; end for;
if A eq PP then B:=Sch[i]; end if;
end for;
return B;
end function;

N1:=Stabiliser(N,1);
#N1;
/* 625 */
Generators(N1);

A:=N!(11, 12, 13, 14, 15)(16, 17, 18, 19, 20);
word(A);
/*a^2 * b * a * b * a^2*/
B:=N!(11, 15, 14, 13, 12);
word(B);
/* a^-2 * b^-1 * a^2*/
C:=N!(6, 8, 10, 7, 9)(16, 17, 18, 19, 20)(21, 24, 22, 25, 23);
word(C);
/* a * b^-2 * a * b * a^2 * b^2 * a */
D:=N!(11, 13, 15, 12, 14)(16, 19, 17, 20, 18)(21, 23, 25, 22, 24);
word(D);

```

```

/* a * b^2 * a * b^-2 * a * b^2 * a^2 */

G<x,y,t>:=Group<x,y,t|x^5, y^5,
  (y, x^-1, y) ,
  x * y^-1 * x^-2 * y^-1 * x^2 * y * x^-2 * y * x, t^2,
  (t,x^2 * y * x * y * x^2), (t,x^-2 * y^-1 * x^2),
  (t,x * y^-2 * x * y * x^2 * y^2 * x), (t,x * y^2 * x * y^-2 * x * y^2 * x^2)>;

/* to verify progenitor is correct, do the following : */
sub<N|xx^2 * yy * xx * yy * xx^2, xx^-2 * yy^-1 * xx^2,
xx * yy^-2 * xx * yy * xx^2 * yy^2 * xx, xx * yy^2 * xx * yy^-2 *
xx * yy^2 * xx^2> eq N1;/* true *//* so our progenitor is correct*/

Orbits(N1);
temp:={2,3,4,5,6,11,16,21};

for n in N do for j in temp do if 1^n eq j then word(n),j;
temp:=temp diff {j}; end if; end for; end for;

C:=Classes(N); C;

for i in [2..#C] do i, C[i][3];word(C[i][3]);
Orbits(Centraliser(N,C[i][3])); end for;

```

```

for j in [2..#N] do for i in [1..#Setseq(Set(N))] do if
1^Setseq(Set(N))[i] eq j then j,word(Setseq(Set(N))[i]);
break; end if; end for; end for;

/* to find the isom type of wreath product */
> S:=Sym(35);
> s1:=S!(11,16,21,26,31);
> sv1:=S!(12,17,22, 27,32);
> e1:=S!(13,18,23,28,33);
> n1:=S!(14,19,24,29,34);
> ten1:=S!(15,20,25,30,35);
> k1s:=S!(11,12,13,14,15)(16,17,18, 19,20)(21,22,23,24,25)(26,
27,28,29, 30)(31,32,33,34,35);
> wr:=sub<S|s1,sv1,e1,n1,ten1,k1s>;
> wr;

Permutation group wr acting on a set of cardinality 35
(11, 16, 21, 26, 31)
(12, 17, 22, 27, 32)
(13, 18, 23, 28, 33)
(14, 19, 24, 29, 34)
(15, 20, 25, 30, 35)
(11, 12, 13, 14, 15)(16, 17, 18, 19, 20)(21, 22, 23, 24, 25)
(26, 27, 28, 29, 30)(31, 32, 33, 34, 35)

```

```
> CompositionFactors(wr);
```

```

G
| Cyclic(5)
*
| Cyclic(5)
*
| Cyclic(5)
*
| Cyclic(5)
*
| Cyclic(5)
*
| Cyclic(5)
1

```

```
> NL:=NormalLattice(wr);
```

```
> NL;
```

```
Normal subgroup lattice
```

```
-----
```

```
[12] Order 15625 Length 1 Maximal Subgroups: 6 7 8 9 10 11
```

```
---
```

```
[11] Order 3125 Length 1 Maximal Subgroups: 5
```

```
[10] Order 3125   Length 1   Maximal Subgroups: 5
[ 9] Order 3125   Length 1   Maximal Subgroups: 5
[ 8] Order 3125   Length 1   Maximal Subgroups: 5
[ 7] Order 3125   Length 1   Maximal Subgroups: 5
[ 6] Order 3125   Length 1   Maximal Subgroups: 5
---
[ 5] Order 625    Length 1   Maximal Subgroups: 4
---
[ 4] Order 125    Length 1   Maximal Subgroups: 3
---
[ 3] Order 25     Length 1   Maximal Subgroups: 2
---
[ 2] Order 5      Length 1   Maximal Subgroups: 1
---
[ 1] Order 1      Length 1   Maximal Subgroups:
```

```
> for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
1
2
3
4
5
7
> NL[7];
```

```

Permutation group acting on a set of cardinality 35
Order = 3125 = 5^5
  (15, 20, 25, 30, 35)
  (11, 31, 26, 21, 16)(12, 17, 22, 27, 32)
  (12, 32, 27, 22, 17)(13, 18, 23, 28, 33)
  (13, 33, 28, 23, 18)(14, 19, 24, 29, 34)
  (14, 34, 29, 24, 19)(15, 20, 25, 30, 35)
> X:=AbelianGroup(GrpPerm,[5,5,5,5,5]);
> IsIsomorphic(NL[7],X);
true Mapping from: GrpPerm: $, Degree 35, Order 5^5 to GrpPerm: X
Composition of Mapping from: GrpPerm: $, Degree 35, Order 5^5 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: X

H<a,b,c,d,e>:=Group<a,b,c,d,e|a^5,b^5,c^5,d^5, e^5,(a,b), (a,c),
(a,d), (a,e), (b,c),(b,d), (b,e), (c,d), (c,e), (d,e)>;
#H;
3125

> q,ff:=quo<wr|NL[7]>;
> q;
Permutation group q acting on a set of cardinality 5
Order = 5
  Id(q)

```



```

    Id(q)
    Id(q)
    Id(q)
    Id(q)
    (1, 2, 3, 4, 5)
> IsAbelian(q);
true
> T:=Transversal(wr,NL[7]);
> #T;
5
> T;
{@
    Id(wr),
    (11, 12, 13, 14, 15)(16, 17, 18, 19, 20)(21, 22, 23, 24, 25)
    (26, 27, 28, 29, 30)(31, 32, 33, 34, 35), (11, 13, 15, 12, 14)
    (16, 18, 20, 17, 19)(21, 23, 25, 22, 24)(26, 28, 30, 27, 29)(31,
    33, 35, 32, 34), (11, 14, 12, 15, 13)(16, 19, 17, 20, 18)(21, 24,
    22, 25, 23)(26, 29, 27, 30, 28)(31, 34, 32, 35, 33), (11, 15, 14,
    13, 12)(16, 20, 19, 18, 17)(21, 25, 24, 23, 22)(26, 30, 29, 28,
    27)(31, 35, 34, 33, 32)
@}

/* The generators of NL[7]=( <A>, <B>, <C> , <D>, <E>) */
A:=wr!(15, 20, 25, 30, 35);

```

```

B:=wr!(11, 31, 26, 21, 16)(12, 17, 22, 27, 32);
C:=wr!(12, 32, 27, 22, 17)(13, 18, 23, 28, 33);
D:=wr!(13, 33, 28, 23, 18)(14, 19, 24, 29, 34);
E:=wr!(14, 34, 29, 24, 19)(15, 20, 25, 30, 35);

for i in [1..#T] do if ff(T[i]) eq q.1 then i; end if; end for;
1
for i in [1..#T] do if ff(T[i]) eq q.2 then i; end if; end for;
1
for i in [1..#T] do if ff(T[i]) eq q.3 then i; end if; end for;
1
for i in [1..#T] do if ff(T[i]) eq q.4 then i; end if; end for;
1
for i in [1..#T] do if ff(T[i]) eq q.5 then i; end if; end for;
1
ff(T[1]) eq q.5>;
true
T[1];
Id(wr)

F:=wr!Id(wr);

for q,r,s,t,u in [0..1] do if A^F eq A^q*B^r*C^s*D^t*E^u
then q,r,s,t,u; end if; end for;

```

```

1 0 0 0 0
for q,r,s,t,u in [0..1] do if B^F eq A^q*B^r*C^s*D^t*E^u
then q,r,s,t,u; end if; end for;
0 1 0 0 0
for q,r,s,t,u in [0..1] do if C^F eq A^q*B^r*C^s*D^t*E^u
then q,r,s,t,u; end if; end for;
0 0 1 0 0
for q,r,s,t,u in [0..1] do if D^F eq A^q*B^r*C^s*D^t*E^u
then q,r,s,t,u; end if; end for;
0 0 0 1 0
for q,r,s,t,u in [0..1] do if E^F eq A^q*B^r*C^s*D^t*E^u
then q,r,s,t,u; end if; end for;
0 0 0 0 1

J<a,b,c,d,e,f>:=Group<a,b,c,d,e,f|a^5,b^5,c^5,d^5, e^5,(a,b),
(a,c), (a,d), (a,e), (b,c),(b,d), (b,e), (c,d), (c,e), (d,e),
f^5, a^f=a, b^f=b, c^f=c, d^f=d, e^f=e>;
#J;
15625

f,J1,k:=CosetAction(J,sub<J|Id(J)>);
#J1;
/* 15625 */

```

```
w:=IsIsomorphic(J1,wr);  
w; /* false */
```

since it is false, there is a possibility that this could be a mixed or central ext . so we check the permutation of Q in terms of J.

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